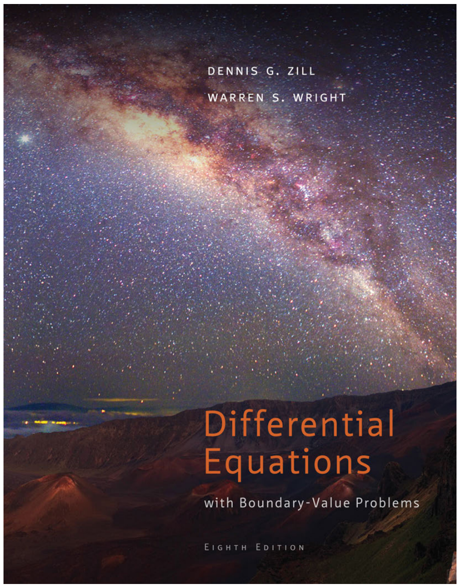
A Solution Manual For

DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.



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May 15, 2024

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1.1 problem 3. series method

Internal problem ID [6550]
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Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 3. series method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f\big|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\, f'\big|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{2}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -y\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= -y'\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= y\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= y'\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= -y\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\
&= -y'\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\
&= y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= -y(0) \\
F_5 &= -y'(0) \\
F_6 &= y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$\begin{aligned}
y = &\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8\right) y(0) \\
&+ \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) y'(0) + O(x^8)
\end{aligned}$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{(n+2)\,(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{5040}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{40320}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{362880}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{6} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 - \frac{1}{720} a_0 x^6 - \frac{1}{5040} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) a_1 + O(x^8) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8\right) y(0) \\ & + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) y'(0) + O(x^8) \end{aligned} \quad (1)$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) c_2 + O(x^8) \quad (2)$$

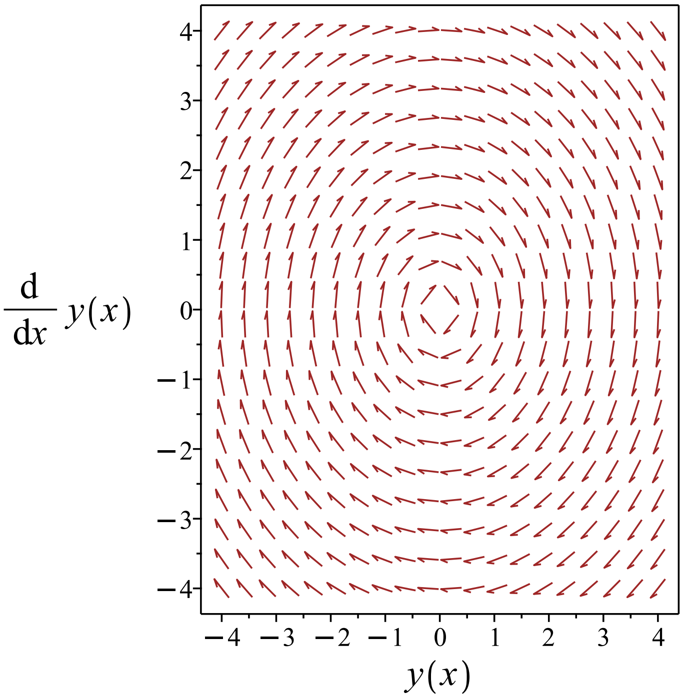

Figure 1: Slope field plot

Verification of solutions

$$y = \left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\frac{1}{40320}x^8\right)y(0) \\ + \left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7\right)y'(0)+O\left(x^8\right)$$

Verified OK.

$$y = \left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6\right)c_1+\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7\right)c_2+O\left(x^8\right)$$

Verified OK.

1.1.1 Maple step by step solution

Let's solve

$y'' = -y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y = 0$

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = \cos(x) c_1 + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{x^7}{5040}+\frac{x^5}{120}-\frac{x^3}{6}+x\right)+c_1\left(-\frac{x^6}{720}+\frac{x^4}{24}-\frac{x^2}{2}+1\right)$$

1.2 problem 3. direct method

Internal problem ID [6551]
Internal file name [`OUTPUT/5799_Sunday_June_05_2022_03_55_03_PM_58593661/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 3. direct method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

1.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) \tag{1}$$

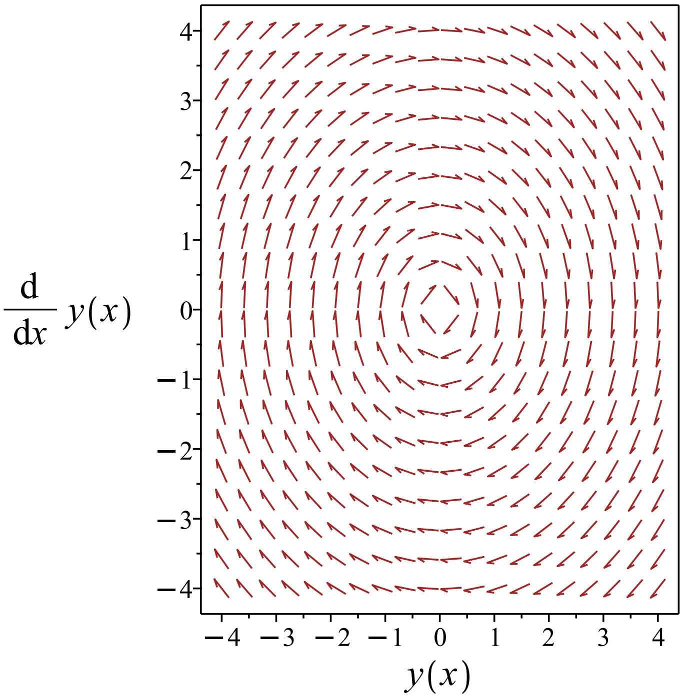

Figure 2: Slope field plot

<u>Verification of solutions</u>

$$y = \cos(x)\, c_1 + c_2 \sin(x)$$

Verified OK.

1.2.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + y'y)\, dx = 0$$

$$\frac{y'^2}{2} + \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{-y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2+2c_1}}dy = \int dx$$
$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_2 + x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2+2c_1}}dy = \int dx$$
$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_3$$

Summary

The solution(s) found are the following

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_2 + x \tag{1}$$
$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_3 \tag{2}$$

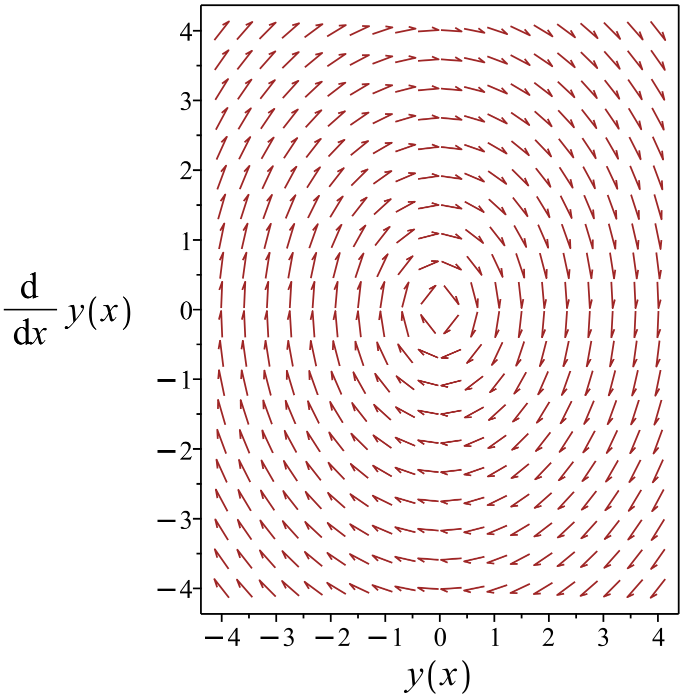

Figure 3: Slope field plot

<u>Verification of solutions</u>

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right)=c_2+x$$

Verified OK.

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right)=x+c_3$$

Verified OK.

1.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y''+y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \cos(x)\,c_1 + c_2 \sin(x) \tag{1}$$

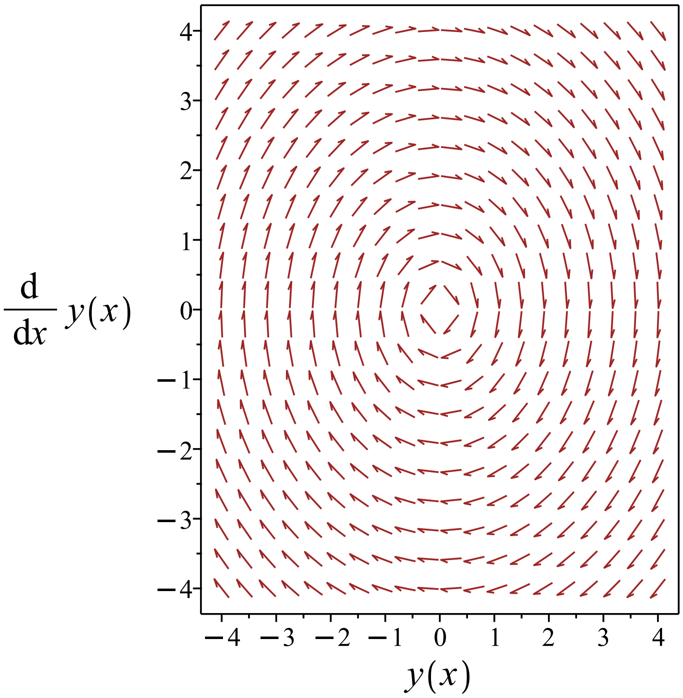

Figure 4: Slope field plot

Verification of solutions

$$y = \cos(x)\, c_1 + c_2 \sin(x)$$

Verified OK.

1.2.4 Maple step by step solution

Let's solve

$y'' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = \cos(x) c_1 + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x$2)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(x) + \cos(x) c_2$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[y''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 \cos(x) + c_2 \sin(x)$$

1.3 problem 4. series method

Internal problem ID [6552]
Internal file name [`OUTPUT/5800_Sunday_June_05_2022_03_55_04_PM_34712386/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 4. series method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} \, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} \, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (7) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (8) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= y \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= y' \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= y(0) \\
F_5 &= y'(0) \\
F_6 &= y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1+\frac{1}{2}x^{2}+\frac{1}{24}x^{4}+\frac{1}{720}x^{6}+\frac{1}{40320}x^{8}\right) y(0) \\
&+\left(x+\frac{1}{6}x^{3}+\frac{1}{120}x^{5}+\frac{1}{5040}x^{7}\right) y'(0)+O\left(x^{8}\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{(n+2)\,(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{5040}$$

For $n = 6$ the recurrence equation gives

$$56a_8 - a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{40320}$$

For $n = 7$ the recurrence equation gives

$$72a_9 - a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{362880}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 + \frac{1}{720} a_0 x^6 + \frac{1}{5040} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) a_0 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7\right) a_1 + O\left(x^8\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7\right) c_2 + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40320}x^8\right) y(0) \\ & + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7\right) y'(0) + O\left(x^8\right) \end{aligned} \quad (1)$$

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7\right) c_2 + O\left(x^8\right) \quad (2)$$

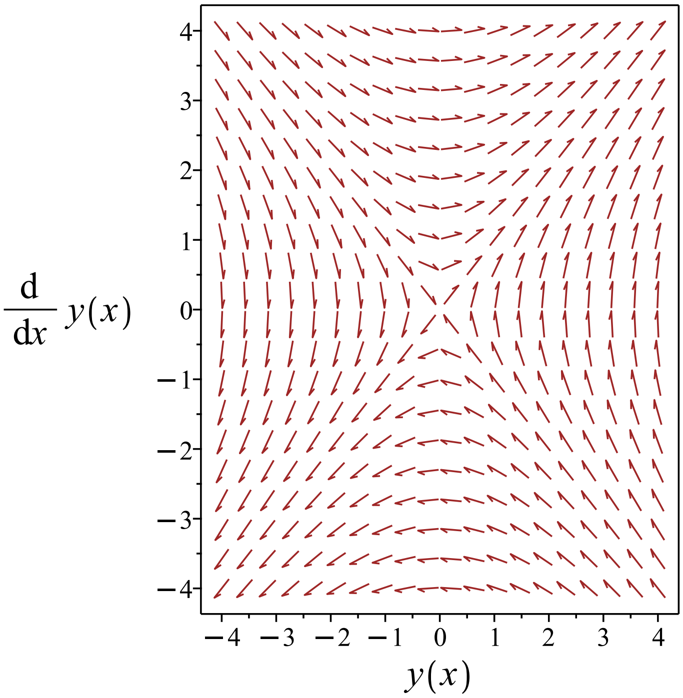

Figure 5: Slope field plot

Verification of solutions

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40320}x^8\right)y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7\right)y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right)c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7\right)c_2 + O(x^8)$$

Verified OK.

1.3.1 Maple step by step solution

Let's solve

$y'' = y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y = 0$
- Characteristic polynomial of ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x) \rightarrow c_2\left(\frac{x^7}{5040}+\frac{x^5}{120}+\frac{x^3}{6}+x\right)+c_1\left(\frac{x^6}{720}+\frac{x^4}{24}+\frac{x^2}{2}+1\right)$$

1.4 problem 4. direct method

Internal problem ID [6553]
Internal file name [`OUTPUT/5801_Sunday_June_05_2022_03_55_07_PM_20479109/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 4. direct method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

1.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} \tag{1}$$

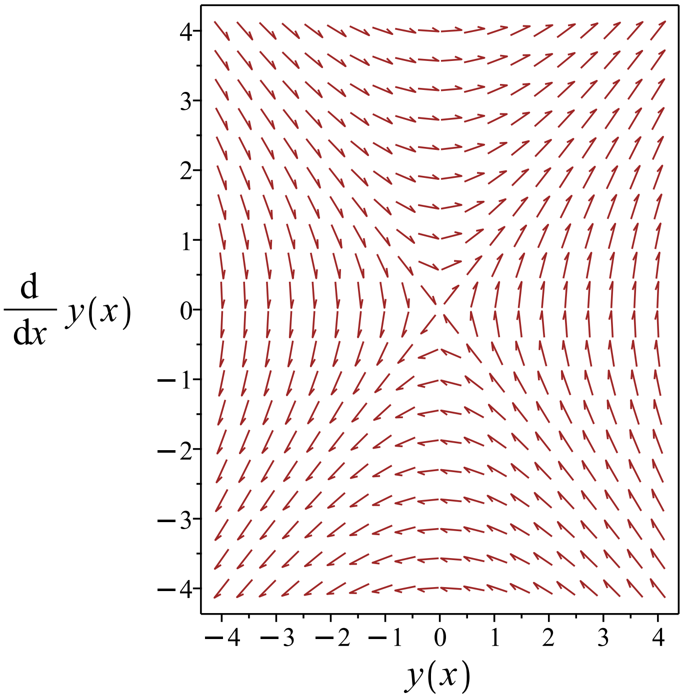

Figure 6: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{x} + c_2 e^{-x}$$

Verified OK.

1.4.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - y'y)\,dx = 0$$
$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{\sqrt{y^2+2c_1}}dy &= \int dx \\
\ln\left(y+\sqrt{y^2+2c_1}\right) &= c_2+x
\end{aligned}
$$

Raising both side to exponential gives

$$
y+\sqrt{y^2+2c_1} = \mathrm{e}^{c_2+x}
$$

Which simplifies to

$$
y+\sqrt{y^2+2c_1} = c_3\mathrm{e}^{x}
$$

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
\int -\frac{1}{\sqrt{y^2+2c_1}}dy &= \int dx \\
-\ln\left(y+\sqrt{y^2+2c_1}\right) &= x+c_4
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{y+\sqrt{y^2+2c_1}} = \mathrm{e}^{x+c_4}
$$

Which simplifies to

$$
\frac{1}{y+\sqrt{y^2+2c_1}} = c_5\mathrm{e}^{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(\mathrm{e}^{2x}c_3^2-2c_1\right)\mathrm{e}^{-x}}{2c_3} \tag{1}
$$

$$
y = -\frac{\left(2c_1c_5^2\mathrm{e}^{2x}-1\right)\mathrm{e}^{-x}}{2c_5} \tag{2}
$$

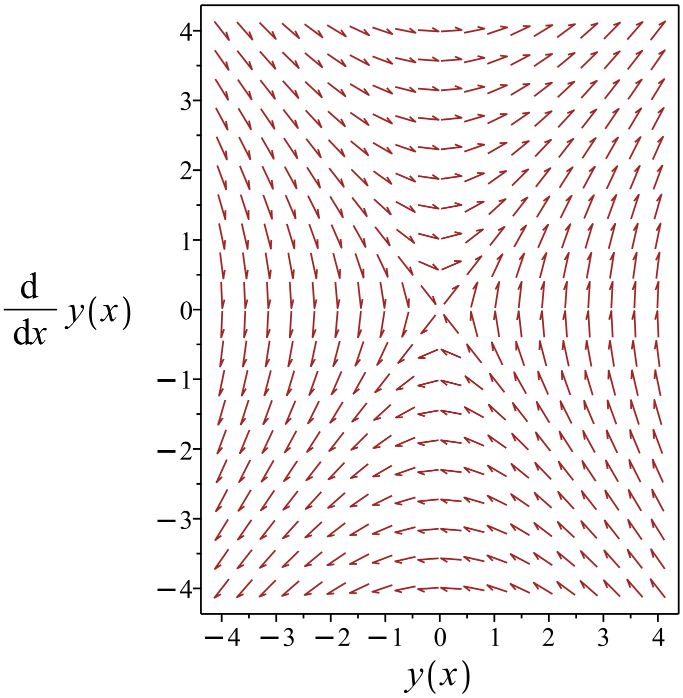

Figure 7: Slope field plot

Verification of solutions

$$y = \frac{(e^{2x}c_3^2 - 2c_1)\,e^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{(2c_1c_5^2e^{2x} - 1)\,e^{-x}}{2c_5}$$

Verified OK.

1.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 5: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t)-\deg(s)\\ &= 0-0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L=[1]$$

Since $r=1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z''=rz$ as one solution is

$$z_1(x)=\mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \mathrm{e}^{-x}\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\ &= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} \tag{1}$$

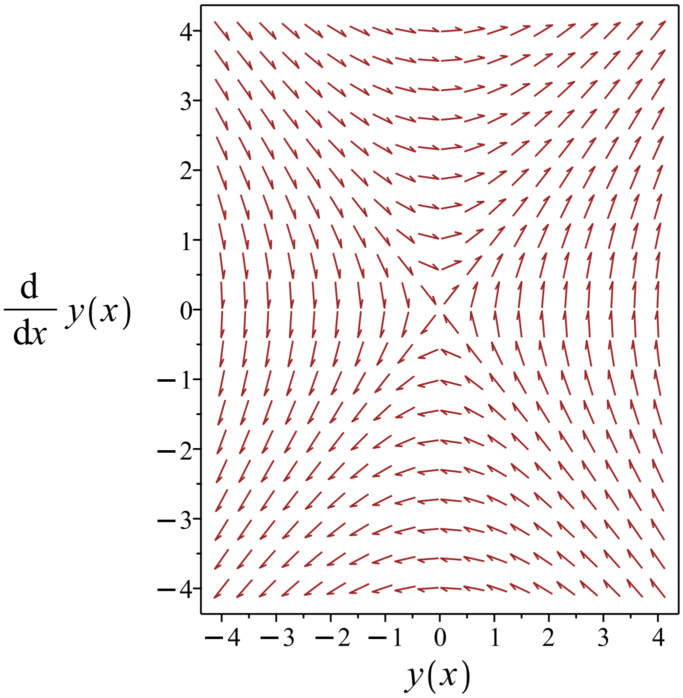

Figure 8: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2}$$

Verified OK.

1.4.4 Maple step by step solution

Let's solve

$y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the ODE

  $$y_1(x) = \mathrm{e}^{-x}$$
- 2nd solution of the ODE

  $$y_2(x) = \mathrm{e}^{x}$$
- General solution of the ODE

  $$y = c_1 y_1(x) + c_2 y_2(x)$$
- Substitute in solutions

  $$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2$$

✓ Solution by Mathematica
Time used: 0.011 (sec). Leaf size: 20

```
DSolve[y''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x} + c_2 e^{-x}$$

1.5 problem 5. series method

Internal problem ID [6554]
Internal file name [`OUTPUT/5802_Sunday_June_05_2022_03_55_08_PM_98352454/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 5. series method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots \\ &= y_0 + xy_0' + \frac{x^2}{2}\,f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\,f'|_{x_0,y_0,y_0'} + \cdots \\ &= y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y_0'} \end{aligned}$$

But

$$\begin{align} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{13} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{14} \\ \frac{d^2f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \nonumber \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2} \\ \frac{d^3f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right) \nonumber \\ &= \frac{\partial}{\partial x}\left(\frac{d^2f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2f}{dx^2}\right)f \tag{3} \\ &\vdots \nonumber \end{align}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$\begin{aligned}
F_0 &= y' \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y' \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= y' \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= y' \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= y'
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y'(0)\\
F_1 &= y'(0)\\
F_2 &= y'(0)\\
F_3 &= y'(0)\\
F_4 &= y'(0)\\
F_5 &= y'(0)\\
F_6 &= y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8\right) y'(0) + O(x^8)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}\\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=1}^{\infty} n a_n x^{n-1} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n\,=1}^{\infty} \left(-n a_n x^{n-1}\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-(n+1)\, a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} \left(-(n+1)\, a_{n+1} x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - (n+1)\, a_{n+1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n+1}}{n+2} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_1}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_1}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{5040}$$

For $n = 6$ the recurrence equation gives

$$56a_8 - 7a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_1}{40320}$$

For $n = 7$ the recurrence equation gives

$$72a_9 - 8a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{362880}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_1 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{24} a_1 x^4 + \frac{1}{120} a_1 x^5 + \frac{1}{720} a_1 x^6 + \frac{1}{5040} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = a_0 + \left(x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \frac{1}{720} x^6 + \frac{1}{5040} x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_1 + \left(x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \frac{1}{720} x^6 + \frac{1}{5040} x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = y(0) + \left(x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \frac{1}{720} x^6 + \frac{1}{5040} x^7 + \frac{1}{40320} x^8\right) y'(0) + O(x^8) \tag{1}$$

$$y = c_1 + \left(x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \frac{1}{720} x^6 + \frac{1}{5040} x^7\right) c_2 + O(x^8) \tag{2}$$

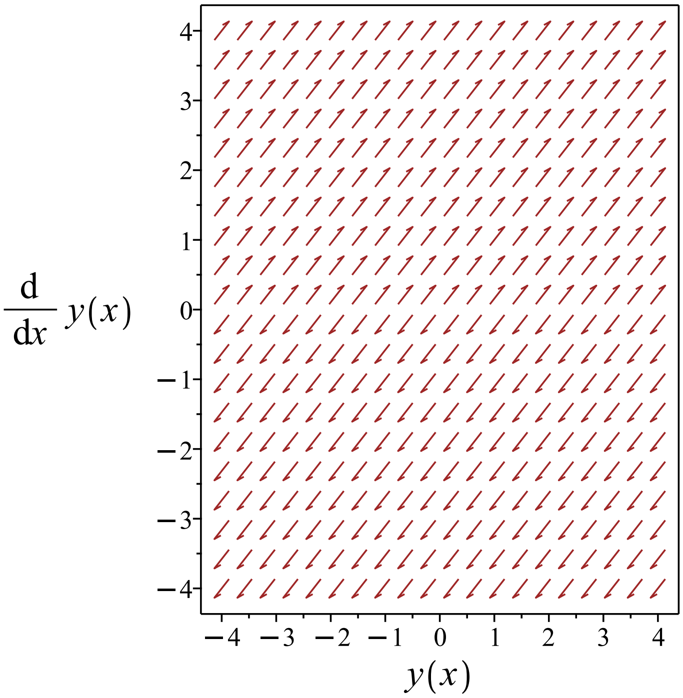

Figure 9: Slope field plot

Verification of solutions

$$y = y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8\right) y'(0) + O(x^8)$$

Verified OK.

$$y = c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7\right) c_2 + O(x^8)$$

Verified OK.

1.5.1 Maple step by step solution

Let's solve

$y'' = y'$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' = 0$

- Characteristic polynomial of ODE

  $r^2 - r = 0$
- Factor the characteristic polynomial

  $r(r-1) = 0$
- Roots of the characteristic polynomial

  $r = (0, 1)$
- 1st solution of the ODE

  $y_1(x) = 1$
- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^x$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 + c_2 \mathrm{e}^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)-diff(y(x),x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 53

```
AsymptoticDSolveValue[y''[x]-y'[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{x^7}{5040} + \frac{x^6}{720} + \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x\right) + c_1$$

1.6 problem 5. direct method

Internal problem ID [6555]
Internal file name [`OUTPUT/5803_Sunday_June_05_2022_03_55_10_PM_71178910/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 5. direct method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' = 0}$$

1.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(0)}\\ &= \frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= 0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(0)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 \tag{1}$$

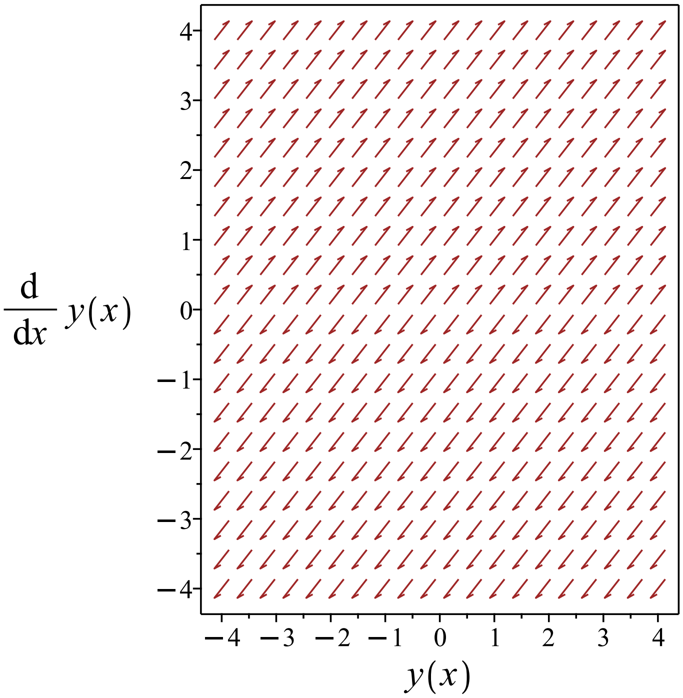

Figure 10: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2$$

Verified OK.

1.6.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - y')\,dx = 0$$
$$y' - y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{y + c_1} dy = \int dx$$
$$\ln(y + c_1) = c_2 + x$$

Raising both side to exponential gives

$$y + c_1 = e^{c_2 + x}$$

Which simplifies to

$$y + c_1 = c_3 e^x$$

Summary

The solution(s) found are the following

$$y = c_3 e^x - c_1 \tag{1}$$

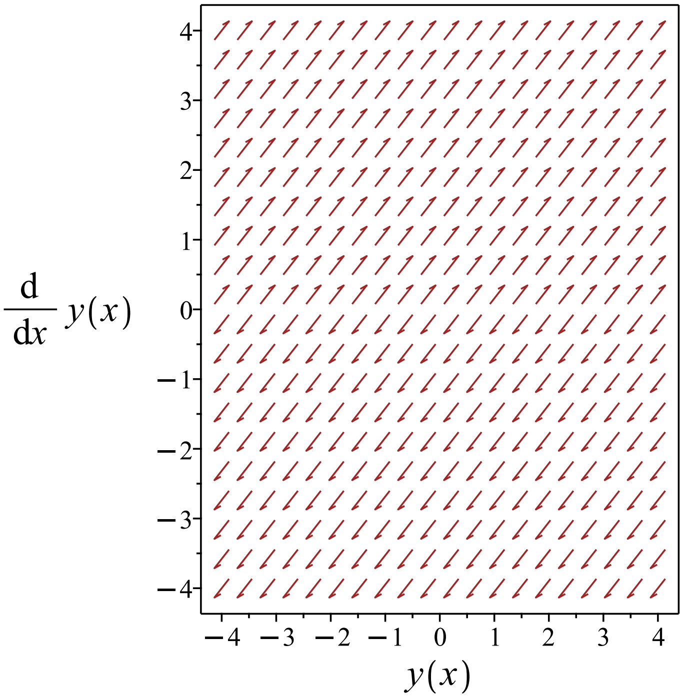

Figure 11: Slope field plot

Verification of solutions

$$y = c_3 e^x - c_1$$

Verified OK.

1.6.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned}\int \frac{1}{p} dp &= x + c_1 \\ \ln(p) &= x + c_1 \\ p &= e^{x+c_1} \\ p &= c_1 e^{x}\end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1 e^{x}$$

Integrating both sides gives

$$\begin{aligned}y &= \int c_1 e^{x} \, dx \\ &= c_1 e^{x} + c_2\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} + c_2 \tag{1}$$

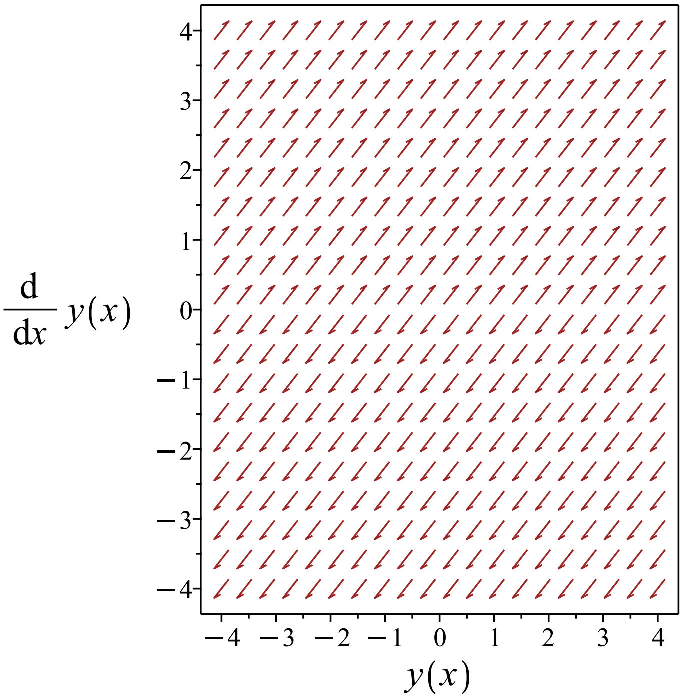

Figure 12: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2$$

Verified OK.

1.6.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - y')\,dx = 0$$
$$y' - y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{y + c_1}dy = \int dx$$
$$\ln (y + c_1) = c_2 + x$$

Raising both side to exponential gives

$$y + c_1 = \mathrm{e}^{c_2 + x}$$

Which simplifies to

$$y + c_1 = c_3 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = c_3 \mathrm{e}^{x} - c_1 \tag{1}$$

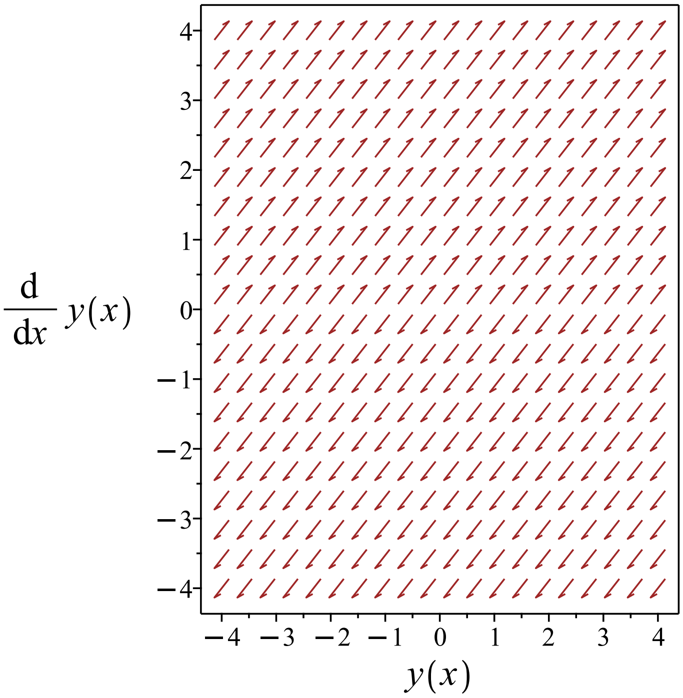

Figure 13: Slope field plot

Verification of solutions

$$y = c_3 \mathrm{e}^{x} - c_1$$

Verified OK.

1.6.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align}
y'' - y' &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align}
$$

Comparing (1) and (2) shows that

$$
\begin{align}
A &= 1 \nonumber\\
B &= -1 \tag{3}\\
C &= 0 \nonumber
\end{align}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \nonumber
\end{align}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 1\\
t &= 4
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 8: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-1}{1} \, dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\mathrm{e}^{x}))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + c_2 \mathrm{e}^{x} \tag{1}
$$

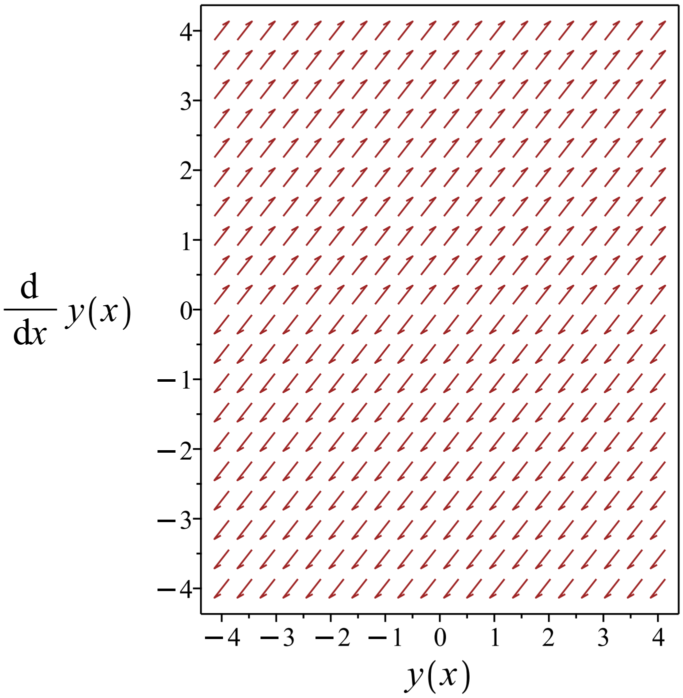

Figure 14: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 + c_2 e^x$$

Verified OK.

1.6.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -1 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$y'-y=c_1$$

We now have a first order ode to solve which is

$$y'-y=c_1$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{y+c_1}dy &= \int dx \\
\ln\left(y+c_1\right) &= c_2+x
\end{aligned}
$$

Raising both side to exponential gives

$$y+c_1=\mathrm{e}^{c_2+x}$$

Which simplifies to

$$y+c_1=c_3\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y=c_3\mathrm{e}^{x}-c_1 \tag{1}$$

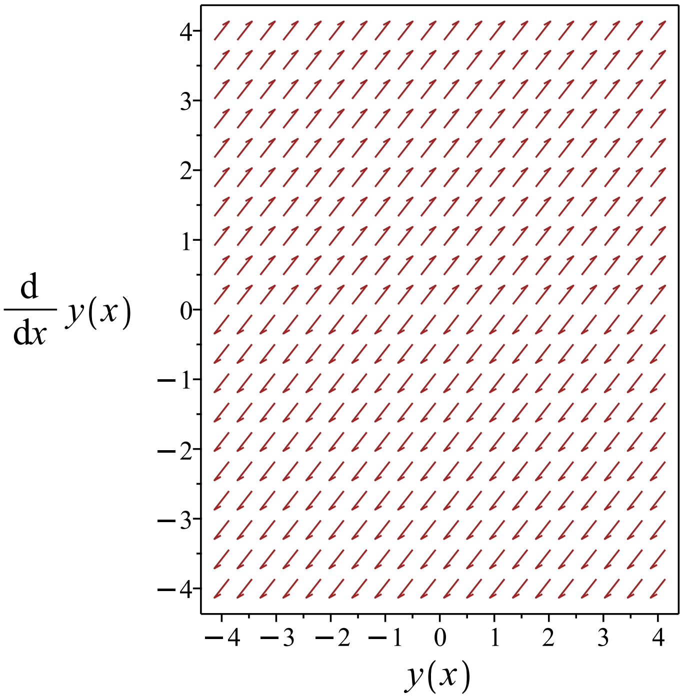

Figure 15: Slope field plot

Verification of solutions

$$y = c_3 e^x - c_1$$

Verified OK.

1.6.7 Maple step by step solution

Let's solve

$y'' - y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r = 0$

- Factor the characteristic polynomial

  $r(r-1) = 0$

- Roots of the characteristic polynomial

$r = (0, 1)$

- 1st solution of the ODE

  $y_1(x) = 1$
- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^x$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 + c_2 \mathrm{e}^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x$2)-diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^x c_1 + c_2$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 14

```
DSolve[y''[x]-y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x + c_2$$

1.7 problem 6. series method

Internal problem ID [6556]
Internal file name [`OUTPUT/5804_Sunday_June_05_2022_03_55_12_PM_64860648/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 6. series method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} \left. f \right|_{x_0, y_0, y_0'} + \frac{x^3}{3!} \left. f' \right|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (22) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (23) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2y' \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -8y' \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 16y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -32y' \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= 64y' \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -128y'
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y'(0) \\
F_1 &= 4y'(0) \\
F_2 &= -8y'(0) \\
F_3 &= 16y'(0) \\
F_4 &= -32y'(0) \\
F_5 &= 64y'(0) \\
F_6 &= -128y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = y(0) + \left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{2}{15}x^5 - \frac{2}{45}x^6 + \frac{4}{315}x^7 - \frac{1}{315}x^8\right) y'(0) + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 2(n+1)\, a_{n+1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_{n+1}}{n+2} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -a_1$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 8a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{2a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 10a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{2a_1}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 12a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{4a_1}{315}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + 14a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = -\frac{a_1}{315}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + 16a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{2a_1}{2835}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_1 x^2 + \frac{2}{3} a_1 x^3 - \frac{1}{3} a_1 x^4 + \frac{2}{15} a_1 x^5 - \frac{2}{45} a_1 x^6 + \frac{4}{315} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = a_0 + \left(x - x^2 + \frac{2}{3} x^3 - \frac{1}{3} x^4 + \frac{2}{15} x^5 - \frac{2}{45} x^6 + \frac{4}{315} x^7\right) a_1 + O\left(x^8\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_1 + \left(x - x^2 + \frac{2}{3} x^3 - \frac{1}{3} x^4 + \frac{2}{15} x^5 - \frac{2}{45} x^6 + \frac{4}{315} x^7\right) c_2 + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$y = y(0) + \left(x - x^2 + \frac{2}{3} x^3 - \frac{1}{3} x^4 + \frac{2}{15} x^5 - \frac{2}{45} x^6 + \frac{4}{315} x^7 - \frac{1}{315} x^8\right) y'(0) + O\left(x^8\right) \tag{1}$$

$$y = c_1 + \left(x - x^2 + \frac{2}{3} x^3 - \frac{1}{3} x^4 + \frac{2}{15} x^5 - \frac{2}{45} x^6 + \frac{4}{315} x^7\right) c_2 + O\left(x^8\right) \tag{2}$$

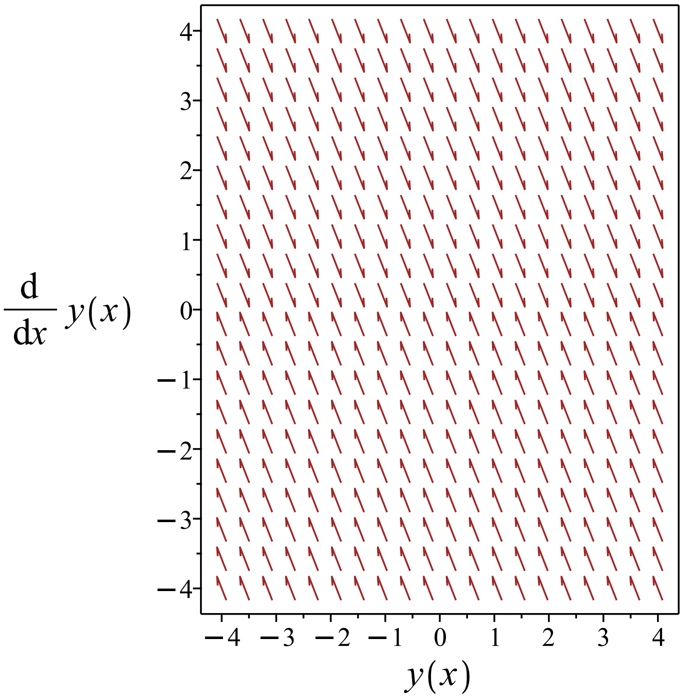

Figure 16: Slope field plot

<u>Verification of solutions</u>

$$y = y(0) + \left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{2}{15}x^5 - \frac{2}{45}x^6 + \frac{4}{315}x^7 - \frac{1}{315}x^8\right) y'(0) + O(x^8)$$

Verified OK.

$$y = c_1 + \left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{2}{15}x^5 - \frac{2}{45}x^6 + \frac{4}{315}x^7\right) c_2 + O(x^8)$$

Verified OK.

1.7.1 Maple step by step solution

Let's solve

$y'' = -2y'$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' = 0$

- Characteristic polynomial of ODE

  $r^2 + 2r = 0$
- Factor the characteristic polynomial

  $r(r+2) = 0$
- Roots of the characteristic polynomial

  $r = (-2, 0)$
- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-2x}$
- 2nd solution of the ODE

  $y_2(x) = 1$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 \mathrm{e}^{-2x} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)+2*diff(y(x),x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + \left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{2}{15}x^5 - \frac{2}{45}x^6 + \frac{4}{315}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 51

```
AsymptoticDSolveValue[y''[x]+2*y'[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{4x^7}{315} - \frac{2x^6}{45} + \frac{2x^5}{15} - \frac{x^4}{3} + \frac{2x^3}{3} - x^2 + x\right) + c_1$$

1.8 problem 6. direct method

Internal problem ID [6557]
Internal file name [`OUTPUT/5805_Sunday_June_05_2022_03_55_14_PM_79717300/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 6. direct method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' = 0}$$

1.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(0)}\\ &= -1 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 1\\ \lambda_2 &= -1 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(0)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-2x} \tag{1}$$

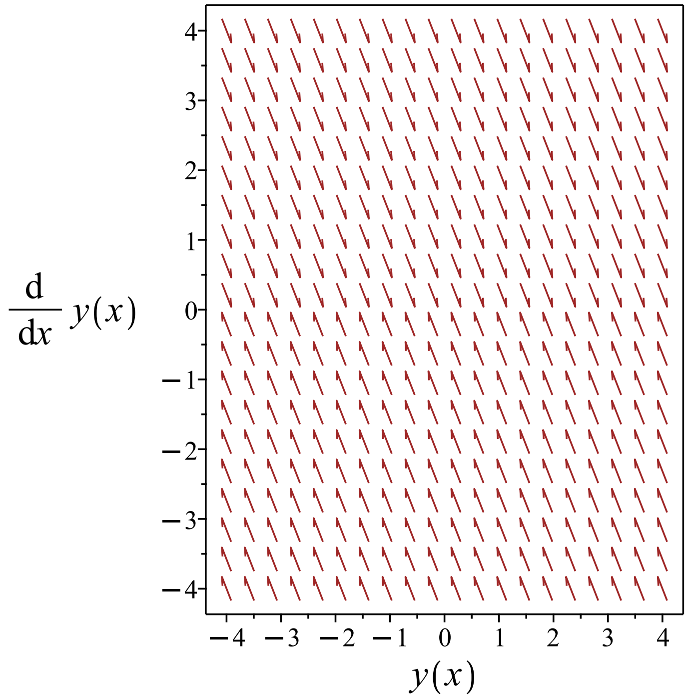

Figure 17: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-2x}$$

Verified OK.

1.8.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\, dx = 0$$
$$y' + 2y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-2y+c_1}dy = \int dx$$
$$-\frac{\ln\left(-2y+c_1\right)}{2} = c_2 + x$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2y+c_1}} = \mathrm{e}^{c_2+x}$$

Which simplifies to

$$\frac{1}{\sqrt{-2y+c_1}} = c_3\mathrm{e}^{x}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-2x}}{2c_3^2} + \frac{c_1}{2} \tag{1}$$

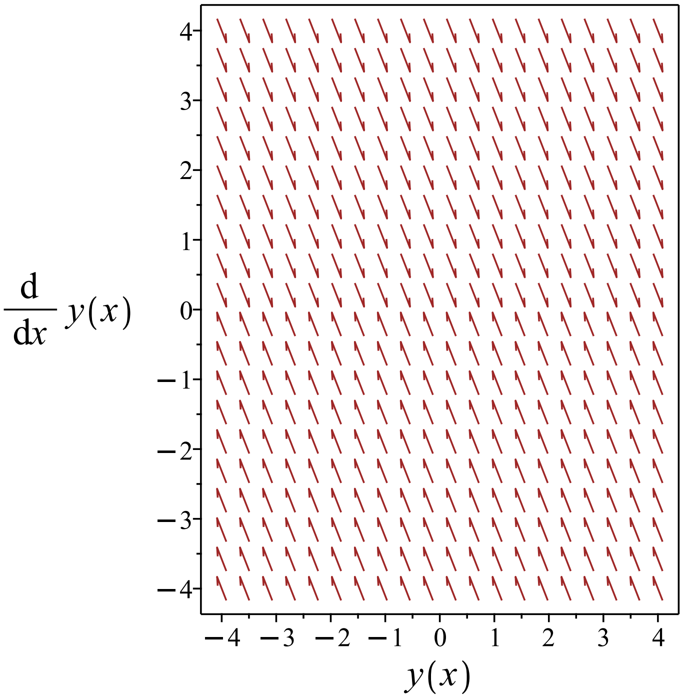

Figure 18: Slope field plot

Verification of solutions

$$y = -\frac{e^{-2x}}{2c_3^2} + \frac{c_1}{2}$$

Verified OK.

1.8.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 2p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned}\int -\frac{1}{2p}dp &= \int dx \\ -\frac{\ln(p)}{2} &= x + c_1\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{p}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{p}} = c_2 e^{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{e^{-2x}}{c_2^2}$$

Integrating both sides gives

$$\begin{aligned}y &= \int \frac{e^{-2x}}{c_2^2}\,dx \\ &= -\frac{e^{-2x}}{2c_2^2} + c_3\end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-2x}}{2c_2^2} + c_3 \tag{1}$$

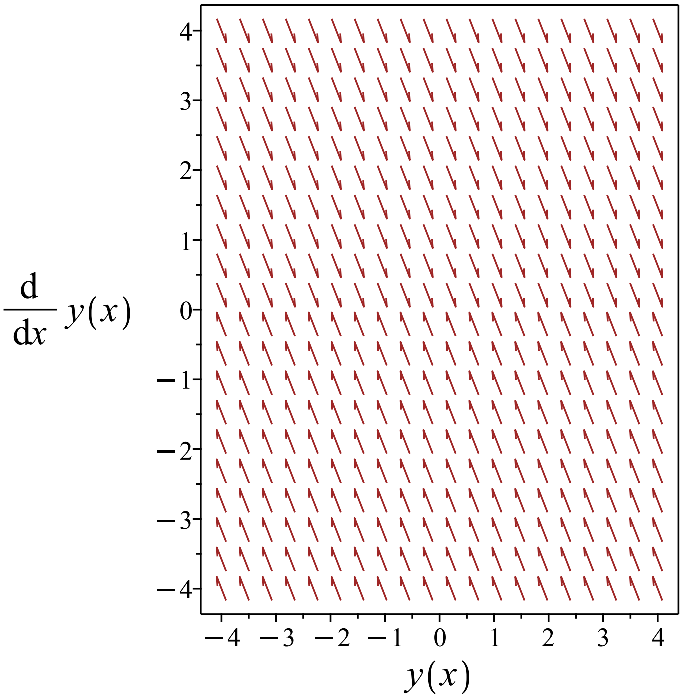

Figure 19: Slope field plot

Verification of solutions

$$y = -\frac{e^{-2x}}{2c_2^2} + c_3$$

Verified OK.

1.8.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 2y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\, dx = 0$$
$$y' + 2y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-2y + c_1} dy = \int dx$$
$$-\frac{\ln(-2y + c_1)}{2} = c_2 + x$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2y + c_1}} = \mathrm{e}^{c_2 + x}$$

Which simplifies to

$$\frac{1}{\sqrt{-2y + c_1}} = c_3 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-2x}}{2c_3^2} + \frac{c_1}{2} \tag{1}$$

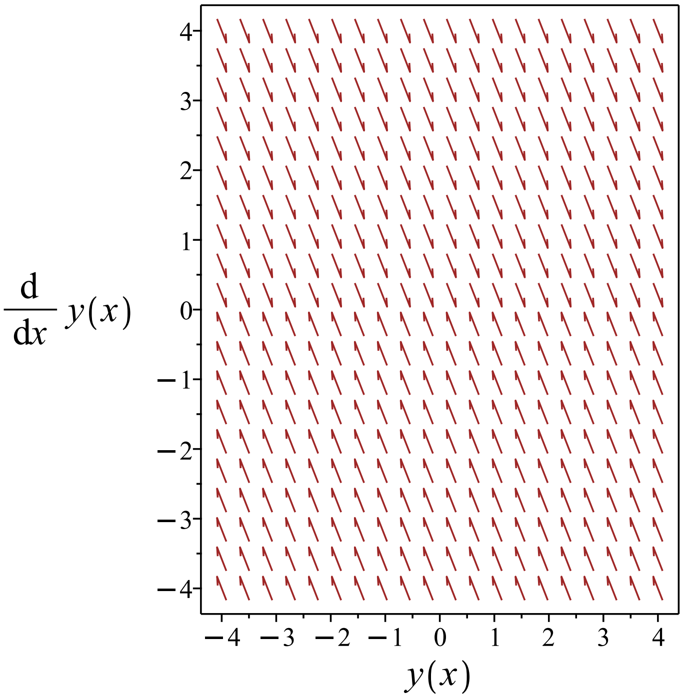

Figure 20: Slope field plot

Verification of solutions

$$y = -\frac{e^{-2x}}{2c_3^2} + \frac{c_1}{2}$$

Verified OK.

1.8.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 11: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx}\\ &= z_1 e^{-x}\\ &= z_1\left(\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{\left(y_1\right)^2}\,dx\\ &= y_1 \int \frac{e^{-2x}}{\left(y_1\right)^2}\,dx\\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \frac{c_2}{2} \tag{1}$$

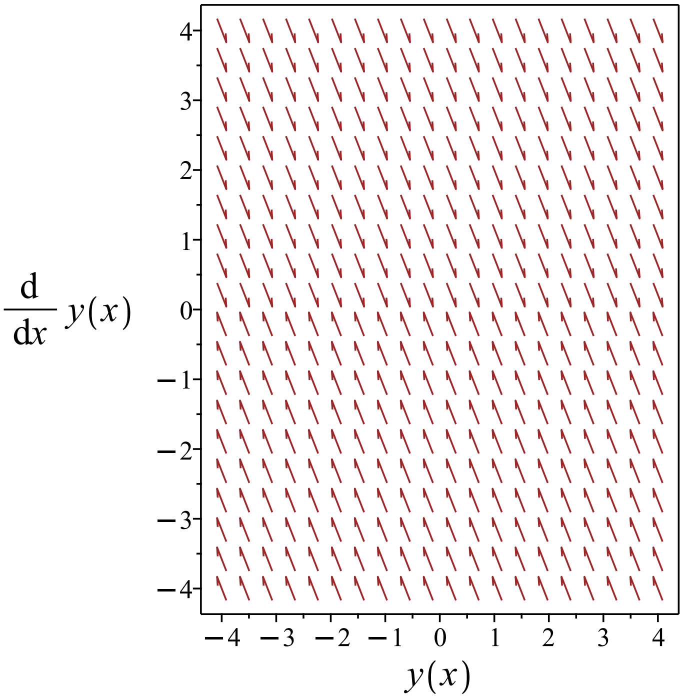

Figure 21: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + \frac{c_2}{2}$$

Verified OK.

1.8.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 2 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' + 2y = c_1$$

We now have a first order ode to solve which is

$$y' + 2y = c_1$$

Integrating both sides gives

$$\int \frac{1}{-2y+c_1}dy = \int dx$$
$$-\frac{\ln\left(-2y+c_1\right)}{2} = c_2 + x$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2y+c_1}} = \mathrm{e}^{c_2+x}$$

Which simplifies to

$$\frac{1}{\sqrt{-2y+c_1}} = c_3\mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-2x}}{2c_3^2} + \frac{c_1}{2} \tag{1}$$

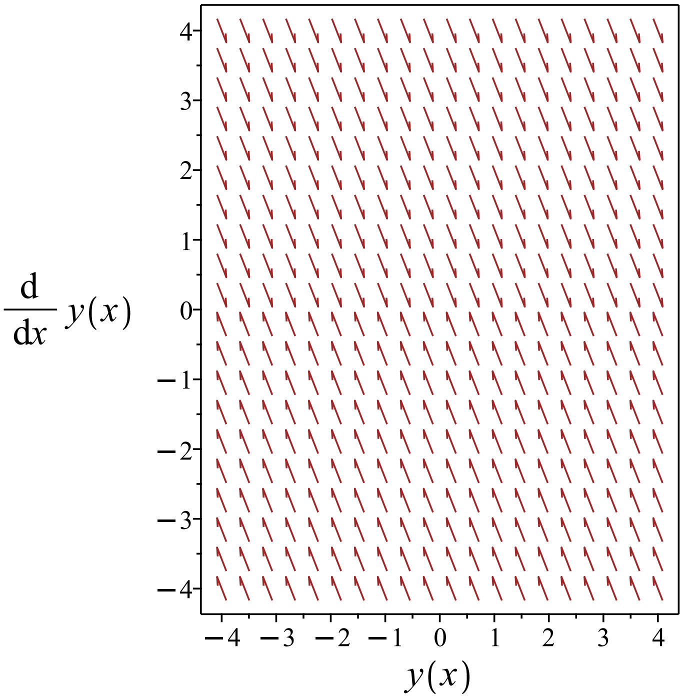

Figure 22: Slope field plot

Verification of solutions

$$y = -\frac{e^{-2x}}{2c_3^2} + \frac{c_1}{2}$$

Verified OK.

1.8.7 Maple step by step solution

Let's solve

$y'' + 2y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r = 0$

- Factor the characteristic polynomial

  $r(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 0)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 19

```
DSolve[y''[x]+2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{2} c_1 e^{-2x}$$

1.9 problem 7

Internal problem ID [6558]
Internal file name [`OUTPUT/5806_Sunday_June_05_2022_03_55_16_PM_32598388/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - xy = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let
$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\ &= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \tag{31} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{32} \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \notag \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2} \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \notag \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3} \\ &\vdots \notag \end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= xy\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= xy'+y\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= yx^2+2y'\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= x(xy'+4y)\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= yx^3+6xy'+4y\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x}+\frac{\partial F_4}{\partial y}y'+\frac{\partial F_4}{\partial y'}F_4\\
&= 9yx^2+y'x^3+10y'\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x}+\frac{\partial F_5}{\partial y}y'+\frac{\partial F_5}{\partial y'}F_5\\
&= 12x^2y'+xy\left(x^3+28\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= y(0) \\
F_2 &= 2y'(0) \\
F_3 &= 0 \\
F_4 &= 4y(0) \\
F_5 &= 10y'(0) \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7\right) y'(0) + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = x\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (1+n) x^n$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (1+n) x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2) a_{n+2} (1+n) - a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

For $n = 6$ the recurrence equation gives

$$56a_8 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$72a_9 - a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_0}{12960}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{6} a_0 x^3 + \frac{1}{12} a_1 x^4 + \frac{1}{180} a_0 x^6 + \frac{1}{504} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) a_0 + \left(x + \frac{1}{12} x^4 + \frac{1}{504} x^7\right) a_1 + O\left(x^8\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) c_1 + \left(x + \frac{1}{12} x^4 + \frac{1}{504} x^7\right) c_2 + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x + \frac{1}{12} x^4 + \frac{1}{504} x^7\right) y'(0) + O\left(x^8\right) \tag{1}$$

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) c_1 + \left(x + \frac{1}{12} x^4 + \frac{1}{504} x^7\right) c_2 + O\left(x^8\right) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x + \frac{1}{12} x^4 + \frac{1}{504} x^7\right) y'(0) + O\left(x^8\right)$$

Verified OK.

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) c_1 + \left(x + \frac{1}{12} x^4 + \frac{1}{504} x^7\right) c_2 + O\left(x^8\right)$$

Verified OK.

1.9.1 Maple step by step solution

Let's solve

$y'' = xy$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ○ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ○ Shift index using $k- >k - 1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k- >k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_{k-1} = 0$

- Shift index using $k- >k + 1$

$$\left((k+1)^2+3k+5\right)a_{k+3}-a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=\frac{a_k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=8;
dsolve(diff(y(x),x$2)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1+\frac{1}{6}x^3+\frac{1}{180}x^6\right)y(0)+\left(x+\frac{1}{12}x^4+\frac{1}{504}x^7\right)D(y)(0)+O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]-x*y[x]==0,y[x],{x,0,7}]
```

$$y(x)\to c_2\left(\frac{x^7}{504}+\frac{x^4}{12}+x\right)+c_1\left(\frac{x^6}{180}+\frac{x^3}{6}+1\right)$$

1.10 problem 8

Internal problem ID [6559]
Internal file name [`OUTPUT/5807_Sunday_June_05_2022_03_55_18_PM_34542147/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + yx^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f\big|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'\big|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (34) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (35) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\Big|_{x_0, y_0, y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -yx^2 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -x(xy' + 2y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= yx^4 - 4xy' - 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y'x^4 + 8yx^3 - 6y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12y'x^3 - x^2 y \left(x^4 - 30\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= -\left(\left(x^5 - 66x\right) y' + \left(18x^4 - 60\right) y\right) x \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(-24x^5 + 192x\right) y' + y\left(x^8 - 156x^4 + 60\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -2y(0) \\
F_3 &= -6y'(0) \\
F_4 &= 0 \\
F_5 &= 0 \\
F_6 &= 60y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8\right) y(0) + \left(x - \frac{1}{20}x^5\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-2}}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$56a_8 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{672}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{1440}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{12} a_0 x^4 - \frac{1}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x - \frac{1}{20} x^5\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{12} x^4 + \frac{1}{672} x^8\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{12} x^4 + \frac{1}{672} x^8\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^8)$$

Verified OK.

1.10.1 Maple step by step solution

Let's solve

$y'' = -y x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + yx^2 = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

☐ Rewrite ODE with series expansions

  ○ Convert $x^2 \cdot y$ to series expansion

$$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

  ○ Shift index using $k- >k-2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

  ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

  ○ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2 + 3k + 2\right) a_{k+2} + a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
Order:=8;
dsolve(diff(y(x),x$2)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) D(y)(0) + O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x^2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2 \left(x - \frac{x^5}{20}\right) + c_1 \left(1 - \frac{x^4}{12}\right)$$

1.11 problem 9

Internal problem ID [6560]
Internal file name [`OUTPUT/5808_Sunday_June_05_2022_03_55_20_PM_43448307/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{y'' - 2xy' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f\big|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'\big|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (37) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (38) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= 2xy' - y\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= 4x^2y' - 2xy + y'\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= 8y'x^3 - 4yx^2 + 8xy' - 3y\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= \left(16x^4 + 36x^2 + 5\right)y' - 8yx\left(x^2+2\right)\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= \left(32x^5 + 128x^3 + 66x\right)y' + \left(-16x^4 - 60x^2 - 21\right)y\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\
&= \left(64x^6 + 400x^4 + 456x^2 + 45\right)y' - 32y\left(x^4 + 6x^2 + \frac{93}{16}\right)x\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\
&= \left(128x^7 + 1152x^5 + 2320x^3 + 816x\right)y' - 64y\left(x^6 + \frac{35}{4}x^4 + \frac{129}{8}x^2 + \frac{231}{64}\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -y(0) \\
F_1 &= y'(0) \\
F_2 &= -3y(0) \\
F_3 &= 5y'(0) \\
F_4 &= -21y(0) \\
F_5 &= 45y'(0) \\
F_6 &= -231y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 - \frac{11}{1920}x^8\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7\right) y'(0) + O(x^8)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 2x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2na_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(2n-1)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 7a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 9a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{112}$$

For $n = 6$ the recurrence equation gives

$$56a_8 - 11a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = -\frac{11a_0}{1920}$$

For $n = 7$ the recurrence equation gives

$$72a_9 - 13a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{13a_1}{8064}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 - \frac{1}{8} a_0 x^4 + \frac{1}{24} a_1 x^5 - \frac{7}{240} a_0 x^6 + \frac{1}{112} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6\right) a_0 + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 - \frac{11}{1920}x^8\right) y(0) \\ & + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7\right) y'(0) + O(x^8) \end{aligned} \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{7}{240}x^6-\frac{11}{1920}x^8\right)y(0)+\left(x+\frac{1}{6}x^3+\frac{1}{24}x^5+\frac{1}{112}x^7\right)y'(0)+O(x^8)$$

Verified OK.

$$y = \left(1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{7}{240}x^6\right)c_1+\left(x+\frac{1}{6}x^3+\frac{1}{24}x^5+\frac{1}{112}x^7\right)c_2+O(x^8)$$

Verified OK.

1.11.1 Maple step by step solution

Let's solve

$y'' = 2xy' - y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2xy' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k->k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(2k-1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$(k^2+3k+2)\,a_{k+2}-2a_k k+a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k(2k-1)}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{7}{240}x^6\right)y(0)+\left(x+\frac{1}{6}x^3+\frac{1}{24}x^5+\frac{1}{112}x^7\right)D(y)(0)+O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]-2*x*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{x^7}{112}+\frac{x^5}{24}+\frac{x^3}{6}+x\right)+c_1\left(-\frac{7x^6}{240}-\frac{x^4}{8}-\frac{x^2}{2}+1\right)$$

1.12 problem 10

Internal problem ID [6561]
Internal file name [`OUTPUT/5809_Sunday_June_05_2022_03_55_22_PM_80074317/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Hermite]
```

$$\boxed{y'' - xy' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (40) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (41) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y'_0} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= xy' - 2y\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= x^2y' - 2xy - y'\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= \left(x^3 - x\right)y' - 2yx^2\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= y'x^4 - 2yx^3 - 2xy - y'\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= \left(\left(x^2 - 1\right)y' - 2xy\right)x\left(x^2 + 3\right)\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\
&= \left(\left(x^2 - 1\right)y' - 2xy\right)\left(x^4 + 6x^2 + 3\right)\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\
&= \left(x^4 + 10x^2 + 15\right)\left(\left(x^2 - 1\right)y' - 2xy\right)x
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0)\\
F_1 &= -y'(0)\\
F_2 &= 0\\
F_3 &= -y'(0)\\
F_4 &= 0\\
F_5 &= -3y'(0)\\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(-x^2+1\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5 - \frac{1}{1680}x^7\right) y'(0) + O\left(x^8\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}\\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n\,=1}^{\infty} \left(-n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n-2)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{1680}$$

For $n = 6$ the recurrence equation gives

$$56a_8 - 4a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$72a_9 - 5a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{a_1}{24192}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{6} a_1 x^3 - \frac{1}{120} a_1 x^5 - \frac{1}{1680} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(-x^2 + 1\right) a_0 + \left(x - \frac{1}{6} x^3 - \frac{1}{120} x^5 - \frac{1}{1680} x^7\right) a_1 + O\left(x^8\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(-x^2 + 1\right) c_1 + \left(x - \frac{1}{6} x^3 - \frac{1}{120} x^5 - \frac{1}{1680} x^7\right) c_2 + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$y = \left(-x^2 + 1\right) y(0) + \left(x - \frac{1}{6} x^3 - \frac{1}{120} x^5 - \frac{1}{1680} x^7\right) y'(0) + O\left(x^8\right) \tag{1}$$

$$y = \left(-x^2 + 1\right) c_1 + \left(x - \frac{1}{6} x^3 - \frac{1}{120} x^5 - \frac{1}{1680} x^7\right) c_2 + O\left(x^8\right) \tag{2}$$

Verification of solutions

$$y = \left(-x^2 + 1\right) y(0) + \left(x - \frac{1}{6} x^3 - \frac{1}{120} x^5 - \frac{1}{1680} x^7\right) y'(0) + O\left(x^8\right)$$

Verified OK.

$$y = \left(-x^2 + 1\right) c_1 + \left(x - \frac{1}{6} x^3 - \frac{1}{120} x^5 - \frac{1}{1680} x^7\right) c_2 + O\left(x^8\right)$$

Verified OK.

1.12.1 Maple step by step solution

Let's solve

$y'' = xy' - 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - xy' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k- > k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_k(k-2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k(k-2) = 0$

- Recursion relation; series terminates at $k = 2$

  $a_{k+2} = \frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

  $a_2 = -a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y = A_2 x^2 + A_1 x - a_0$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
Order:=8;
dsolve(diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (-x^2+1)\,y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5 - \frac{1}{1680}x^7\right) D(y)\,(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[y''[x]-x*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1(1-x^2) + c_2\left(-\frac{x^7}{1680} - \frac{x^5}{120} - \frac{x^3}{6} + x\right)$$

1.13 problem 11

Internal problem ID [6562]
Internal file name [`OUTPUT/5810_Sunday_June_05_2022_03_55_24_PM_83306450/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + x^2 y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} \left. f \right|_{x_0, y_0, y_0'} + \frac{x^3}{3!} \left. f' \right|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (43) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (44) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -x^2 y' - xy \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^4 - 3x\right) y' + y\left(x^3 - 1\right) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-x^6 + 8x^3 - 4\right) y' - yx^2\left(x^3 - 6\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\left(x^7 - 15x^4 + 34x\right) y' + y\left(x^6 - 13x^3 + 16\right)\right) x \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^{10} + 24x^7 - 122x^4 + 84x\right) y' - y\left(x^9 - 22x^6 + 86x^3 - 16\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(x^{12} - 35x^9 + 312x^6 - 658x^3 + 100\right) y' + yx^2\left(x^3 - 9\right)\left(x^6 - 24x^3 + 38\right) \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -\left(\left(x^{13} - 48x^{10} + 660x^7 - 2784x^4 + 2416x\right) y' + y\left(x^{12} - 46x^9 + 576x^6 - 1928x^3 + 784\right)\right) x
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -4y'(0) \\
F_3 &= 0 \\
F_4 &= 16y(0) \\
F_5 &= 100y'(0) \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{45}x^6\right) y(0) + \left(x - \frac{1}{6}x^4 + \frac{5}{252}x^7\right) y'(0) + O\left(x^8\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x^2 \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty} n\, x^{1+n} a_n = \sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0 \quad (3)$$

$n = 1$ gives

$$6a_3 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + (n-1)\, a_{n-1} + a_{n-1} = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-1} n}{(n+2)\,(1+n)} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{5a_1}{252}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$72a_9 + 7a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{7a_0}{3240}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{6} a_1 x^4 + \frac{1}{45} a_0 x^6 + \frac{5}{252} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{45} x^6\right) a_0 + \left(x - \frac{1}{6} x^4 + \frac{5}{252} x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{45} x^6\right) c_1 + \left(x - \frac{1}{6} x^4 + \frac{5}{252} x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{45} x^6\right) y(0) + \left(x - \frac{1}{6} x^4 + \frac{5}{252} x^7\right) y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{45} x^6\right) c_1 + \left(x - \frac{1}{6} x^4 + \frac{5}{252} x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{45} x^6\right) y(0) + \left(x - \frac{1}{6} x^4 + \frac{5}{252} x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{45} x^6\right) c_1 + \left(x - \frac{1}{6} x^4 + \frac{5}{252} x^7\right) c_2 + O(x^8)$$

Verified OK.

1.13.1 Maple step by step solution

Let's solve

$y'' = -x^2 y' - xy$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + x^2 y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ○ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ○ Shift index using $k- > k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ○ Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  ○ Shift index using $k- > k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k- > k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_{k-1}k\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + a_{k-1}k = 0$

- Shift index using $k- > k + 1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} + a_k(k+1) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k(k+1)}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
Order:=8;
dsolve(diff(y(x),x$2)+x^2*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{45}x^6\right) y(0) + \left(x - \frac{1}{6}x^4 + \frac{5}{252}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+x^2*y'[x]+x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{5x^7}{252} - \frac{x^4}{6} + x\right) + c_1\left(\frac{x^6}{45} - \frac{x^3}{6} + 1\right)$$

1.14 problem 12

Internal problem ID [6563]
Internal file name [`OUTPUT/5811_Sunday_June_05_2022_03_55_26_PM_94310891/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + 2xy' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \tag{46} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{47} \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2} \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3} \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2xy' - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4x^2 y' + 4xy - 4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -8y'x^3 - 8yx^2 + 20xy' + 12y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^4 - 72x^2 + 32\right) y' + \left(16x^3 - 56x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-32x^5 + 224x^3 - 264x\right) y' - 32y\left(x^4 - 6x^2 + \frac{15}{4}\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(64x^6 - 640x^4 + 1392x^2 - 384\right) y' + 64\left(x^4 - 9x^2 + \frac{57}{4}\right) yx \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(-128x^7 + 1728x^5 - 5920x^3 + 4464x\right) y' - 128\left(x^4 - 9x^2 + \frac{15}{4}\right) y\left(x^2 - \frac{7}{2}\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -4y'(0) \\
F_2 &= 12y(0) \\
F_3 &= 32y'(0) \\
F_4 &= -120y(0) \\
F_5 &= -384y'(0) \\
F_6 &= 1680y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -2x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n\,=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 2na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 8a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{4a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 10a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{6}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 12a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{8a_1}{105}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + 14a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{24}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + 16a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{16a_1}{945}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - a_0x^2 - \frac{2}{3}a_1x^3 + \frac{1}{2}a_0x^4 + \frac{4}{15}a_1x^5 - \frac{1}{6}a_0x^6 - \frac{8}{105}a_1x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right)a_0 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right)a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right)c_1 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right)c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8\right)y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right)y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right)c_1 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right)c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8\right)y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right)y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right)c_1 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right)c_2 + O(x^8)$$

Verified OK.

1.14.1 Maple step by step solution

Let's solve

$y'' = -2xy' - 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2xy' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\left(a_{k+2}(k+2) + 2a_k\right) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7\right) D(y)\left(0\right) + O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 54

```
AsymptoticDSolveValue[y''[x]+2*x*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{8x^7}{105} + \frac{4x^5}{15} - \frac{2x^3}{3} + x\right) + c_1\left(-\frac{x^6}{6} + \frac{x^4}{2} - x^2 + 1\right)$$

1.15 problem 13

Internal problem ID [6564]
Internal file name [`OUTPUT/5812_Sunday_June_05_2022_03_55_28_PM_79957877/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second order series method. Ordinary point", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(x-1)\,y''+y'=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (49) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (50) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$\begin{aligned}
F_0 &= -\frac{y'}{x-1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= \frac{2y'}{(x-1)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= -\frac{6y'}{(x-1)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= \frac{24y'}{(x-1)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= -\frac{120y'}{(x-1)^5}\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\
&= \frac{720y'}{(x-1)^6}\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\
&= -\frac{5040y'}{(x-1)^7}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y'(0) \\
F_1 &= 2y'(0) \\
F_2 &= 6y'(0) \\
F_3 &= 24y'(0) \\
F_4 &= 120y'(0) \\
F_5 &= 720y'(0) \\
F_6 &= 5040y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7 + \frac{1}{8}x^8\right) y'(0) + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x - 1)\, y'' + y' = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x - 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \sum_{n\,=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2} (n+1)\, x^n\right)$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2} (n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n\right) = 0 \quad (3)$$

$n = 0$ gives

$$-2a_2 + a_1 = 0$$

$$a_2 = \frac{a_1}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n - (n+2)\, a_{n+2} (n+1) + (n+1)\, a_{n+1} = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{(n+1)\, a_{n+1}}{n+2} \quad (5)$$

For $n = 1$ the recurrence equation gives

$$4a_2 - 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$9a_3 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$16a_4 - 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$25a_5 - 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_1}{6}$$

For $n = 5$ the recurrence equation gives

$$36a_6 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{7}$$

For $n = 6$ the recurrence equation gives

$$49a_7 - 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_1}{8}$$

For $n = 7$ the recurrence equation gives

$$64a_8 - 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{9}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{4} a_1 x^4 + \frac{1}{5} a_1 x^5 + \frac{1}{6} a_1 x^6 + \frac{1}{7} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = a_0 + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7 + \frac{1}{8}x^8\right) y'(0) + O(x^8) \tag{1}$$

$$y = c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7 + \frac{1}{8}x^8\right) y'(0) + O(x^8)$$

Verified OK.

$$y = c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7\right) c_2 + O(x^8)$$

Verified OK.

1.15.1 Maple step by step solution

Let's solve

$(x-1)\, y'' + y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{1}{x-1}, P_3(x) = 0\right]$

  ○ $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = 1$

  ○ $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

  $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

  ○ $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

$(x-1)\, y''+y'=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^{2}}{d u^{2}}y(u)\right)+\frac{d}{d u}y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+r}$

☐ Rewrite DE with series expansions

  ○ Convert $\frac{d}{d u}y(u)$ to series expansion

  $\frac{d}{d u}y(u)=\sum_{k=0}^{\infty} a_{k}(k+r)\, u^{k+r-1}$

  ○ Shift index using $k->k+1$

  $\frac{d}{d u}y(u)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

  ○ Convert $u\cdot\left(\frac{d^{2}}{d u^{2}}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^{2}}{d u^{2}}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, u^{k+r-1}$

  ○ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^{2}}{d u^{2}}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)^{2} u^{k+r}=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^{2}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=0$

- Recursion relation for $r=0$

  $a_{k+1}=0$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=0\right]$$

- Revert the change of variables $u=x-1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1}=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve((x-1)*diff(y(x),x$2)+diff(y(x),x)=0,y(x),type='series',x=0);
```

$$y(x)=y(0)+\left(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+\frac{1}{6}x^6+\frac{1}{7}x^7\right)D(y)(0)+O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 53

```
AsymptoticDSolveValue[(x-1)*y''[x]+y'[x]==0,y[x],{x,0,7}]
```

$$y(x)\rightarrow c_2\left(\frac{x^7}{7}+\frac{x^6}{6}+\frac{x^5}{5}+\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x\right)+c_1$$

1.16 problem 14

Internal problem ID [6565]
Internal file name [`OUTPUT/5813_Sunday_June_05_2022_03_55_29_PM_30165072/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+2)\,y''+xy'-y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (52) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (53) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= -\frac{-y + xy'}{x+2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(1+x)(-y + xy')}{(x+2)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{x(-y + xy')}{(x+2)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(-y + xy')(x-1)}{(x+2)^2} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{(-2+x)(-y + xy')}{(x+2)^2} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{(-y + xy')(x-3)}{(x+2)^2} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -\frac{(-4+x)(-y + xy')}{(x+2)^2}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)}{2} \\
F_1 &= -\frac{y(0)}{4} \\
F_2 &= 0 \\
F_3 &= \frac{y(0)}{4} \\
F_4 &= -\frac{y(0)}{2} \\
F_5 &= \frac{3y(0)}{4} \\
F_6 &= -y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \frac{1}{1440}x^6 + \frac{1}{6720}x^7 - \frac{1}{40320}x^8\right) y(0) + xy'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x + 2)\, y'' + xy' - y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x + 2)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2} (n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2} (n+1)\, x^n\right) \\ + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$4a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{4}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n + 2(n+2)\, a_{n+2} (n+1) + n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{n+1} + n a_n + n a_{n+1} - a_n}{2\,(n+2)\,(n+1)} \\ &= -\frac{(n-1)\, a_n}{2\,(n+2)\,(n+1)} - \frac{(n^2+n)\, a_{n+1}}{2\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 + 12a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{24}$$

For $n = 2$ the recurrence equation gives

$$6a_3 + 24a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$12a_4 + 40a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{480}$$

For $n = 4$ the recurrence equation gives

$$20a_5 + 60a_6 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{1440}$$

For $n = 5$ the recurrence equation gives

$$30a_6 + 84a_7 + 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{6720}$$

For $n = 6$ the recurrence equation gives

$$42a_7 + 112a_8 + 5a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = -\frac{a_0}{40320}$$

For $n = 7$ the recurrence equation gives

$$56a_8 + 144a_9 + 6a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_0}{290304}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{4} a_0 x^2 - \frac{1}{24} a_0 x^3 + \frac{1}{480} a_0 x^5 - \frac{1}{1440} a_0 x^6 + \frac{1}{6720} a_0 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \frac{1}{1440}x^6 + \frac{1}{6720}x^7\right) a_0 + a_1 x + O\left(x^8\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \frac{1}{1440}x^6 + \frac{1}{6720}x^7\right) c_1 + c_2 x + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \frac{1}{1440}x^6 + \frac{1}{6720}x^7 - \frac{1}{40320}x^8\right) y(0) + x y'(0) + O\left(x^9\right) \tag{1}$$

$$y = \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \frac{1}{1440}x^6 + \frac{1}{6720}x^7\right) c_1 + c_2 x + O\left(x^8\right) \tag{2}$$

Verification of solutions

$$y = \left(1+\frac{1}{4}x^2-\frac{1}{24}x^3+\frac{1}{480}x^5-\frac{1}{1440}x^6+\frac{1}{6720}x^7-\frac{1}{40320}x^8\right)y(0)+xy'(0)+O\left(x^8\right)$$

Verified OK.

$$y = \left(1+\frac{1}{4}x^2-\frac{1}{24}x^3+\frac{1}{480}x^5-\frac{1}{1440}x^6+\frac{1}{6720}x^7\right)c_1+c_2x+O\left(x^8\right)$$

Verified OK.

1.16.1 Maple step by step solution

Let's solve

$(x+2)\,y''+xy'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{x+2}-\frac{xy'}{x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{xy'}{x+2}-\frac{y}{x+2}=0$

☐ Check to see if $x_0=-2$ is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{x}{x+2},P_3(x)=-\frac{1}{x+2}\right]$

  ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=-2$

  ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  ○ $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

$(x+2)\,y''+xy'-y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ○ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

  ○ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-2+r)+a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-2+r)+a_k(k+r-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k(k+r-1)}{(k+1+r)(k-2+r)}$

- Recursion relation for $r=0$; series terminates at $k=1$

$a_{k+1} = -\frac{a_k(k-1)}{(k+1)(k-2)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0}{2}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$

- Revert the change of variables $u = x + 2$

  $\left[y = -\frac{a_0 x}{2}\right]$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}$

- Solution for $r = 3$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$$

- Revert the change of variables $u = x + 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+3}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = -\frac{a_0 x}{2} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+3}\right), b_{k+1} = -\frac{b_k(k+2)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
Order:=8;
dsolve((x+2)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \frac{1}{1440}x^6 + \frac{1}{6720}x^7\right) y(0) + D(y)\left(0\right) x + O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 91

```
AsymptoticDSolveValue[(x+2)*y''[x]+x*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{29x^7}{20160} - \frac{7x^6}{1440} + \frac{x^5}{240} + \frac{x^4}{24} - \frac{x^3}{6} + x\right) + c_1\left(\frac{x^7}{8064} + \frac{x^6}{576} - \frac{x^5}{96} + \frac{x^4}{48} + \frac{x^3}{24} - \frac{x^2}{4} + 1\right)$$

1.17 problem 15

Internal problem ID [6566]
Internal file name [`OUTPUT/5814_Sunday_June_05_2022_03_55_32_PM_88056441/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - (1 + x) y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (55)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (56)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= xy' + y' + y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^2 + 2x + 3\right) y' + (1 + x)\, y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 + 3x^2 + 8x + 6\right) y' + y\left(x^2 + 2x + 4\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 + 4x^3 + 15x^2 + 22x + 18\right) y' + y(1 + x) \left(x^2 + 2x + 8\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^5 + 5x^4 + 24x^3 + 52x^2 + 80x + 48\right) y' + y\left(x^4 + 4x^3 + 18x^2 + 28x + 28\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(x^6 + 6x^5 + 35x^4 + 100x^3 + 222x^2 + 260x + 156\right) y' + y(1 + x) \left(x^4 + 4x^3 + 24x^2 + 40x + 76\right) \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= (1 + x) \left(x^6 + 6x^5 + 42x^4 + 128x^3 + 362x^2 + 484x + 492\right) y' + y\left(x^4 + 4x^3 + 24x^2 + 40x + 34\right) \left(x^2 +
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= y(0) + y'(0) \\
F_1 &= 3y'(0) + y(0) \\
F_2 &= 6y'(0) + 4y(0) \\
F_3 &= 18y'(0) + 8y(0) \\
F_4 &= 48y'(0) + 28y(0) \\
F_5 &= 156y'(0) + 76y(0) \\
F_6 &= 492y'(0) + 272y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$\begin{aligned}
y = &\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{15}x^5 + \frac{7}{180}x^6 + \frac{19}{1260}x^7 + \frac{17}{2520}x^8\right) y(0) \\
&+ \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{3}{20}x^5 + \frac{1}{15}x^6 + \frac{13}{420}x^7 + \frac{41}{3360}x^8\right) y'(0) + O(x^8)
\end{aligned}$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n \\ \sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) &= \sum_{n=0}^{\infty} \left(-(n+1)\, a_{n+1} x^n\right)\end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} \left(-(n+1)\, a_{n+1} x^n\right) + \sum_{n=1}^{\infty} \left(-n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_1 - a_0 = 0$$

$$a_2 = \frac{a_0}{2} + \frac{a_1}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - (n+1)\, a_{n+1} - n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{a_n + a_{n+1}}{n+2} \\ &= \frac{a_n}{n+2} + \frac{a_{n+1}}{n+2} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 2a_2 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6} + \frac{a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_3 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{6} + \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_4 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{15} + \frac{3a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_5 - 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{7a_0}{180} + \frac{a_1}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_6 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{19a_0}{1260} + \frac{13a_1}{420}$$

For $n = 6$ the recurrence equation gives

$$56a_8 - 7a_7 - 7a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{17a_0}{2520} + \frac{41a_1}{3360}$$

For $n = 7$ the recurrence equation gives

$$72a_9 - 8a_8 - 8a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{11a_0}{4536} + \frac{29a_1}{6048}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1x + \left(\frac{a_0}{2} + \frac{a_1}{2}\right)x^2 + \left(\frac{a_0}{6} + \frac{a_1}{2}\right)x^3 + \left(\frac{a_0}{6} + \frac{a_1}{4}\right)x^4 \\ + \left(\frac{a_0}{15} + \frac{3a_1}{20}\right)x^5 + \left(\frac{7a_0}{180} + \frac{a_1}{15}\right)x^6 + \left(\frac{19a_0}{1260} + \frac{13a_1}{420}\right)x^7 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{15}x^5 + \frac{7}{180}x^6 + \frac{19}{1260}x^7\right)a_0 \\ + \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{3}{20}x^5 + \frac{1}{15}x^6 + \frac{13}{420}x^7\right)a_1 + O(x^8) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned} y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{15}x^5 + \frac{7}{180}x^6 + \frac{19}{1260}x^7\right)c_1 \\ + \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{3}{20}x^5 + \frac{1}{15}x^6 + \frac{13}{420}x^7\right)c_2 + O(x^8) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{6}x^{4}+\frac{1}{15}x^{5}+\frac{7}{180}x^{6}+\frac{19}{1260}x^{7}+\frac{17}{2520}x^{8}\right) y(0) \\
& + \left(x+\frac{1}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{3}{20}x^{5}+\frac{1}{15}x^{6}+\frac{13}{420}x^{7}+\frac{41}{3360}x^{8}\right) y'(0)+O\left(x^{8}\right)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
y = {} & \left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{6}x^{4}+\frac{1}{15}x^{5}+\frac{7}{180}x^{6}+\frac{19}{1260}x^{7}\right) c_1 \\
& + \left(x+\frac{1}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{3}{20}x^{5}+\frac{1}{15}x^{6}+\frac{13}{420}x^{7}\right) c_2+O\left(x^{8}\right)
\end{aligned} \tag{2}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{6}x^{4}+\frac{1}{15}x^{5}+\frac{7}{180}x^{6}+\frac{19}{1260}x^{7}+\frac{17}{2520}x^{8}\right) y(0) \\
& + \left(x+\frac{1}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{3}{20}x^{5}+\frac{1}{15}x^{6}+\frac{13}{420}x^{7}+\frac{41}{3360}x^{8}\right) y'(0)+O\left(x^{8}\right)
\end{aligned}
$$

Verified OK.

$$
\begin{aligned}
y = {} & \left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{6}x^{4}+\frac{1}{15}x^{5}+\frac{7}{180}x^{6}+\frac{19}{1260}x^{7}\right) c_1 \\
& + \left(x+\frac{1}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{3}{20}x^{5}+\frac{1}{15}x^{6}+\frac{13}{420}x^{7}\right) c_2+O\left(x^{8}\right)
\end{aligned}
$$

Verified OK.

1.17.1 Maple step by step solution

Let's solve

$y'' = xy' + y' + y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (-1-x)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $$x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k \, x^{k-1+m}$$

  - Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m) \, x^k$$

  - Convert y'' to series expansion

    $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

  - Shift index using $k- >k+2$

    $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1) \, x^k$$

    Rewrite DE with series expansions

    $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) - a_k(k+1)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+1)\left(a_{k+2}(k+2) - a_{k+1} - a_k\right) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_{k+1}+a_k}{k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 74

```
Order:=8;
dsolve(diff(y(x),x$2)-(x+1)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {}& \left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{6}x^4+\frac{1}{15}x^5+\frac{7}{180}x^6+\frac{19}{1260}x^7\right) y(0) \\
&+ \left(x+\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{4}x^4+\frac{3}{20}x^5+\frac{1}{15}x^6+\frac{13}{420}x^7\right) D(y)\,(0) + O\left(x^8\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 98

```
AsymptoticDSolveValue[y''[x]-(x+1)*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$
\begin{aligned}
y(x) \to {}& c_1\left(\frac{19x^7}{1260}+\frac{7x^6}{180}+\frac{x^5}{15}+\frac{x^4}{6}+\frac{x^3}{6}+\frac{x^2}{2}+1\right) \\
&+ c_2\left(\frac{13x^7}{420}+\frac{x^6}{15}+\frac{3x^5}{20}+\frac{x^4}{4}+\frac{x^3}{2}+\frac{x^2}{2}+x\right)
\end{aligned}
$$

1.18 problem 16

Internal problem ID [6567]
Internal file name [`OUTPUT/5815_Sunday_June_05_2022_03_55_34_PM_28196935/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{\left(x^2+1\right) y''-6y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (58)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (59)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\ \vdots\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{6y}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{6x^2y' - 12xy + 6y'}{\left(x^2+1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-24y'x^3 + 72yx^2 - 24xy' + 24y}{\left(x^2+1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{144((x^3+x)\, y' + (-3x^2-1)\, y)\, x}{\left(x^2+1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{1008\left(x^2 - \frac{1}{7}\right)\left((x^3+x)\, y' + (-3x^2-1)\, y\right)}{\left(x^2+1\right)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{8064\left(x^2 - \frac{3}{7}\right)\left((x^3+x)\, y' + (-3x^2-1)\, y\right) x}{\left(x^2+1\right)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -\frac{72576\left(x^4 - \frac{6}{7}x^2 + \frac{1}{21}\right)\left((x^3+x)\, y' + (-3x^2-1)\, y\right)}{\left(x^2+1\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 6y(0) \\
F_1 &= 6y'(0) \\
F_2 &= 24y(0) \\
F_3 &= 0 \\
F_4 &= -144y(0) \\
F_5 &= 0 \\
F_6 &= 3456y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \frac{3}{35}x^8\right) y(0) + \left(x^3 + x\right) y'(0) + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x^2 + 1\right) y'' - 6y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(x^2 + 1\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 6\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-6 a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} (-6a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 6a_0 = 0$$

$$a_2 = 3a_0$$

$n = 1$ gives

$$6a_3 - 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = a_1$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + (n+2)\, a_{n+2}(n+1) - 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{(n-3)\, a_n}{n+1} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-4a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = a_0$$

For $n = 3$ the recurrence equation gives

$$20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$6a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{5}$$

For $n = 5$ the recurrence equation gives

$$14a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$24a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{3a_0}{35}$$

For $n = 7$ the recurrence equation gives

$$36a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = 0$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + 3a_0x^2 + a_1x^3 + a_0x^4 - \frac{1}{5}a_0x^6 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6\right)a_0 + \left(x^3 + x\right)a_1 + O\left(x^8\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6\right)c_1 + \left(x^3 + x\right)c_2 + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \frac{3}{35}x^8\right)y(0) + \left(x^3 + x\right)y'(0) + O\left(x^8\right) \tag{1}$$

$$y = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6\right)c_1 + \left(x^3 + x\right)c_2 + O\left(x^8\right) \tag{2}$$

Verification of solutions

$$y = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \frac{3}{35}x^8\right)y(0) + \left(x^3 + x\right)y'(0) + O\left(x^8\right)$$

Verified OK.

$$y = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6\right)c_1 + \left(x^3 + x\right)c_2 + O\left(x^8\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
Order:=8;
dsolve((x^2+1)*diff(y(x),x$2)-6*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6\right) y(0) + \left(x^3 + x\right) D(y)\left(0\right) + O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 32

```
AsymptoticDSolveValue[(x^2+1)*y''[x]-6*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(x^3 + x\right) + c_1\left(-\frac{x^6}{5} + x^4 + 3x^2 + 1\right)$$

1.19 problem 17

Internal problem ID [6568]
Internal file name [`OUTPUT/5816_Sunday_June_05_2022_03_55_36_PM_14474639/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+2)\,y''+3xy'-y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (61)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (62)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \notag\\
&= \frac{dF_0}{dx} \notag\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \notag\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \notag\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \notag\\
&= \frac{d}{dx}(F_1) \notag\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \notag\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \notag\\
&\vdots \notag\\
F_n &= \frac{d}{dx}(F_{n-1}) \notag\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \notag\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3xy' - y}{x^2+2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{13x^2y' - 5xy - 4y'}{\left(x^2+2\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-70y'x^3 + 28yx^2 + 70xy' - 14y}{\left(x^2+2\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(448x^4 - 938x^2 + 112\right) y' + \left(-182x^3 + 266x\right) y}{\left(x^2+2\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(-3318x^5 + 11928x^3 - 4452x\right) y' + \left(1358x^4 - 3892x^2 + 644\right) y}{\left(x^2+2\right)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{\left(27902x^6 - 153636x^4 + 117852x^2 - 7616\right) y' - 11466yx\left(x^4 - \frac{428}{91}x^2 + \frac{30}{13}\right)}{\left(x^2+2\right)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{\left(-262584x^7 + 2055816x^5 - 2679768x^3 + 532728x\right) y' + (108164x^6 - 753648x^4 + 732480x^2 - 60536}{\left(x^2+2\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)}{2} \\
F_1 &= -y'(0) \\
F_2 &= -\frac{7y(0)}{4} \\
F_3 &= 7y'(0) \\
F_4 &= \frac{161y(0)}{8} \\
F_5 &= -119y'(0) \\
F_6 &= -\frac{7567y(0)}{16}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4 + \frac{161}{5760}x^6 - \frac{1081}{92160}x^8\right) y(0) \\
&+ \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \frac{17}{720}x^7\right) y'(0) + O\left(x^8\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x^2 + 2\right) y'' + 3xy' - y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(x^2 + 2\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 3x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} 2n(n-1)\, a_{n} x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 3n a_{n} x^{n}\right)+\sum_{n=0}^{\infty} \left(-a_{n} x^{n}\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_{n} x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^{n}\right)\\+\left(\sum_{n=1}^{\infty} 3n a_{n} x^{n}\right)+\sum_{n=0}^{\infty} \left(-a_{n} x^{n}\right)=0\end{aligned} \tag{3}$$

$n = 0$ gives

$$4a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{4}$$

$n = 1$ gives

$$12a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$n a_{n}(n-1) + 2(n+2)\, a_{n+2}(n+1) + 3n a_{n} - a_{n} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2+2n-1)}{2\,(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$7a_2 + 24a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{7a_0}{96}$$

For $n = 3$ the recurrence equation gives

$$14a_3 + 40a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$23a_4 + 60a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{161a_0}{5760}$$

For $n = 5$ the recurrence equation gives

$$34a_5 + 84a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{17a_1}{720}$$

For $n=6$ the recurrence equation gives

$$47a_6+112a_8=0$$

Which after substituting the earlier terms found becomes

$$a_8=-\frac{1081a_0}{92160}$$

For $n=7$ the recurrence equation gives

$$62a_7+144a_9=0$$

Which after substituting the earlier terms found becomes

$$a_9=\frac{527a_1}{51840}$$

And so on. Therefore the solution is

$$\begin{aligned}y&=\sum_{n=0}^{\infty}a_nx^n\\&=a_3x^3+a_2x^2+a_1x+a_0+\dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y=a_0+a_1x+\frac{1}{4}a_0x^2-\frac{1}{6}a_1x^3-\frac{7}{96}a_0x^4+\frac{7}{120}a_1x^5+\frac{161}{5760}a_0x^6-\frac{17}{720}a_1x^7+\dots$$

Collecting terms, the solution becomes

$$y=\left(1+\frac{1}{4}x^2-\frac{7}{96}x^4+\frac{161}{5760}x^6\right)a_0+\left(x-\frac{1}{6}x^3+\frac{7}{120}x^5-\frac{17}{720}x^7\right)a_1+O\left(x^8\right) \quad (3)$$

At $x=0$ the solution above becomes

$$y=\left(1+\frac{1}{4}x^2-\frac{7}{96}x^4+\frac{161}{5760}x^6\right)c_1+\left(x-\frac{1}{6}x^3+\frac{7}{120}x^5-\frac{17}{720}x^7\right)c_2+O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \left(1+\frac{1}{4}x^{2}-\frac{7}{96}x^{4}+\frac{161}{5760}x^{6}-\frac{1081}{92160}x^{8}\right) y(0) \\ &\quad + \left(x-\frac{1}{6}x^{3}+\frac{7}{120}x^{5}-\frac{17}{720}x^{7}\right) y'(0)+O\left(x^{8}\right) \end{aligned} \tag{1}$$

$$y = \left(1+\frac{1}{4}x^{2}-\frac{7}{96}x^{4}+\frac{161}{5760}x^{6}\right) c_{1}+\left(x-\frac{1}{6}x^{3}+\frac{7}{120}x^{5}-\frac{17}{720}x^{7}\right) c_{2}+O\left(x^{8}\right) \tag{2}$$

Verification of solutions

$$\begin{aligned} y &= \left(1+\frac{1}{4}x^{2}-\frac{7}{96}x^{4}+\frac{161}{5760}x^{6}-\frac{1081}{92160}x^{8}\right) y(0) \\ &\quad + \left(x-\frac{1}{6}x^{3}+\frac{7}{120}x^{5}-\frac{17}{720}x^{7}\right) y'(0)+O\left(x^{8}\right) \end{aligned}$$

Verified OK.

$$y = \left(1+\frac{1}{4}x^{2}-\frac{7}{96}x^{4}+\frac{161}{5760}x^{6}\right) c_{1}+\left(x-\frac{1}{6}x^{3}+\frac{7}{120}x^{5}-\frac{17}{720}x^{7}\right) c_{2}+O\left(x^{8}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
Order:=8;
dsolve((x^2+2)*diff(y(x),x$2)+3*x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1+\frac{1}{4}x^{2}-\frac{7}{96}x^{4}+\frac{161}{5760}x^{6}\right)y(0)+\left(x-\frac{1}{6}x^{3}+\frac{7}{120}x^{5}-\frac{17}{720}x^{7}\right)D(y)\left(0\right)+O\left(x^{8}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[(x^2+2)*y''[x]+3*x*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x)\to c_2\left(-\frac{17x^7}{720}+\frac{7x^5}{120}-\frac{x^3}{6}+x\right)+c_1\left(\frac{161x^6}{5760}-\frac{7x^4}{96}+\frac{x^2}{4}+1\right)$$

1.20 problem 18

Internal problem ID [6569]
Internal file name [`OUTPUT/5817_Sunday_June_05_2022_03_55_38_PM_71202676/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2-1)\,y''+xy'-y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (64) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (65) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \notag\\
&= \frac{dF_0}{dx} \notag\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \notag\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \notag\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \notag\\
&= \frac{d}{dx}(F_1) \notag\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \notag\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \notag\\
&\vdots \notag\\
F_n &= \frac{d}{dx}(F_{n-1}) \notag\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \notag\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{-y + xy'}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{3(-y + xy')\, x}{\left(x^2 - 1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{3(-y + xy')\,(4x^2 + 1)}{\left(x^2 - 1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{60(-y + xy')\left(x^2 + \frac{3}{4}\right) x}{\left(x^2 - 1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{45(-y + xy')\,(8x^4 + 12x^2 + 1)}{\left(x^2 - 1\right)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{315(-y + xy')\,(8x^4 + 20x^2 + 5)\, x}{\left(x^2 - 1\right)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -\frac{315(-y + xy')\,(64x^6 + 240x^4 + 120x^2 + 5)}{\left(x^2 - 1\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= 0 \\
F_2 &= -3y(0) \\
F_3 &= 0 \\
F_4 &= -45y(0) \\
F_5 &= 0 \\
F_6 &= -1575y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8\right) y(0) + xy'(0) + O(x^8)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 - 1)\, y'' + xy' - y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$(x^2 - 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n\,=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n\,=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(n+1)\, x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 - a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) - (n+2)\, a_{n+2}(n+1) + n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{(n-1)\, a_n}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$3a_2 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$8a_3 - 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$15a_4 - 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{16}$$

For $n = 5$ the recurrence equation gives

$$24a_5 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$35a_6 - 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = -\frac{5a_0}{128}$$

For $n = 7$ the recurrence equation gives

$$48a_7 - 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2}a_0 x^2 - \frac{1}{8}a_0 x^4 - \frac{1}{16}a_0 x^6 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6\right) a_0 + a_1 x + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6\right) c_1 + c_2 x + O(x^8)$$

Summary
The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8\right) y(0) + x y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6\right) c_1 + c_2 x + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8\right) y(0) + x y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6\right) c_1 + c_2 x + O(x^8)$$

Verified OK.

1.20.1 Maple step by step solution

Let's solve

$(x^2 - 1)\, y'' + x y' - y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$$y'' = -\frac{xy'}{x^2-1} + \frac{y}{x^2-1}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{xy'}{x^2-1} - \frac{y}{x^2-1} = 0$$

- Multiply by denominators of ODE

$$\left(-x^2+1\right) y'' - xy' + y = 0$$

- Make a change of variables

$$\theta = \arccos\left(x\right)$$

- Calculate y' with change of variables

$$y' = \left(\frac{d}{d\theta}y(\theta)\right) \theta'(x)$$

- Compute 1st derivative y'

$$y' = -\frac{\frac{d}{d\theta}y(\theta)}{\sqrt{-x^2+1}}$$

- Calculate y'' with change of variables

$$y'' = \left(\frac{d^2}{d\theta^2}y(\theta)\right) \theta'(x)^2 + \theta''(x) \left(\frac{d}{d\theta}y(\theta)\right)$$

- Compute 2nd derivative y''

$$y'' = \frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1} - \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\left(-x^2+1\right)^{\frac{3}{2}}}$$

- Apply the change of variables to the ODE

$$\left(-x^2+1\right)\left(\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1} - \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\left(-x^2+1\right)^{\frac{3}{2}}}\right) + \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\sqrt{-x^2+1}} + y = 0$$

- Multiply through

$$-\frac{\left(\frac{d^2}{d\theta^2}y(\theta)\right)x^2}{-x^2+1} + \frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1} + \frac{x^3\left(\frac{d}{d\theta}y(\theta)\right)}{\left(-x^2+1\right)^{\frac{3}{2}}} - \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\left(-x^2+1\right)^{\frac{3}{2}}} + \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\sqrt{-x^2+1}} + y = 0$$

- Simplify ODE

$$y + \frac{d^2}{d\theta^2}y(\theta) = 0$$

- ODE is that of a harmonic oscillator with given general solution

$$y(\theta) = c_1 \sin\left(\theta\right) + c_2 \cos\left(\theta\right)$$

- Revert back to x

$$y = c_1 \sin\left(\arccos\left(x\right)\right) + c_2 \cos\left(\arccos\left(x\right)\right)$$

- Use trig identity to simplify $\sin\left(\arccos\left(x\right)\right)$

$$\sin\left(\arccos\left(x\right)\right) = \sqrt{-x^2+1}$$

- Simplify solution to the ODE

  $y = c_1\sqrt{-x^2+1} + c_2 x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 29

```
Order:=8;
dsolve((x^2-1)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6\right) y(0) + D(y)(0)\, x + O(x^8)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[(x^2-1)*y''[x]+x*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{x^6}{16} - \frac{x^4}{8} - \frac{x^2}{2} + 1\right) + c_2 x$$

1.21 problem 19

Internal problem ID [6570]
Internal file name [`OUTPUT/5818_Sunday_June_05_2022_03_55_41_PM_53009624/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-xy'+y=0}$$

With initial conditions

$$[y(0)=-2,y'(0)=6]$$

With the expansion point for the power series method at $x=0$.

1.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}p(x) &= -\frac{x}{x-1}\\q(x) &= \frac{1}{x-1}\\F &= 0\end{aligned}$$

Hence the ode is

$$y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$$

The domain of $p(x) = -\frac{x}{x-1}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{x-1}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots\\&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots\\&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (67)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (68)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{-y + xy'}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-y + xy'}{x-1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-y + xy'}{x-1} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{-y + xy'}{x-1} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{-y + xy'}{x-1} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{-y + xy'}{x-1} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{-y + xy'}{x-1}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = -2$ and

$y'(0) = 6$ gives

$$\begin{aligned} F_0 &= -2 \\ F_1 &= -2 \\ F_2 &= -2 \\ F_3 &= -2 \\ F_4 &= -2 \\ F_5 &= -2 \\ F_6 &= -2 \end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = -x^2 + 6x - 2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360} - \frac{x^7}{2520} - \frac{x^8}{20160} + O(x^8)$$

$$y = -x^2 + 6x - 2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360} - \frac{x^7}{2520} - \frac{x^8}{20160} + O(x^8)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x-1)\,y'' - xy' + y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$(x-1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n a_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2} (n+1)\, x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2} (n+1)\, x^n\right) \\ + \sum_{n=1}^{\infty} \left(-n a_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$-2a_2 + a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n - (n+2)\, a_{n+2} (n+1) - n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_{n+1} - n a_n + n a_{n+1} + a_n}{(n+2)\,(n+1)} \\ &= \frac{(-n+1)\, a_n}{(n+2)\,(n+1)} + \frac{(n^2+n)\, a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 - 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$6a_3 - 12a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$12a_4 - 20a_5 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{120}$$

For $n = 4$ the recurrence equation gives

$$20a_5 - 30a_6 - 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$30a_6 - 42a_7 - 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{5040}$$

For $n = 6$ the recurrence equation gives

$$42a_7 - 56a_8 - 5a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{40320}$$

For $n = 7$ the recurrence equation gives

$$56a_8 - 72a_9 - 6a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_0}{362880}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_0 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_0 x^5 + \frac{1}{720} a_0 x^6 + \frac{1}{5040} a_0 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \frac{1}{720} x^6 + \frac{1}{5040} x^7\right) a_0 + a_1 x + O(x^8) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \frac{1}{720} x^6 + \frac{1}{5040} x^7\right) c_1 + c_2 x + O(x^8)$$

$$y = -2 - x^2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360} - \frac{x^7}{2520} + 6x + O(x^8)$$

Summary

The solution(s) found are the following

$$y = -x^2 + 6x - 2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360} - \frac{x^7}{2520} - \frac{x^8}{20160} + O(x^8) \tag{1}$$

$$y = -2 - x^2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360} - \frac{x^7}{2520} + 6x + O(x^8) \tag{2}$$

Verification of solutions

$$y = -x^2 + 6x - 2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360} - \frac{x^7}{2520} - \frac{x^8}{20160} + O(x^8)$$

Verified OK.

$$y = -2 - x^2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360} - \frac{x^7}{2520} + 6x + O(x^8)$$

Verified OK.

1.21.2 Maple step by step solution

Let's solve

$$\left[(x-1)\, y'' - x y' + y = 0, y(0) = -2, y'\Big|_{\{x=0\}} = 6\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{x y'}{x-1} - \frac{y}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{x y'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

  ○ $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $$\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$$

  ○ $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

$$\left((x-1)^2 \cdot P_3(x)\right)\bigg|_{x=1} = 0$$

- ○ $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- ● Multiply by denominators

  $(x-1)\,y'' - xy' + y = 0$

- ● Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- ● Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - ○ Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

  - ○ Shift index using $k- >k+1$

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$

- ● $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- ● Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k + r - 1)\left(a_{k+1}(k + 1 + r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x - 1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x - 1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(x - 1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x - 1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=8;
dsolve([(x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(0) = -2, D(y)(0) = 6],y(x),type='series
```

$$y(x) = -2 + 6x - x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 - \frac{1}{60}x^5 - \frac{1}{360}x^6 - \frac{1}{2520}x^7 + \mathrm{O}\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 48

```
AsymptoticDSolveValue[{(x-1)*y''[x]-x*y'[x]+y[x]==0,{y[0]==-2,y'[0]==6}},y[x],{x,0,7}]
```

$$y(x) \to -\frac{x^7}{2520} - \frac{x^6}{360} - \frac{x^5}{60} - \frac{x^4}{12} - \frac{x^3}{3} - x^2 + 6x - 2$$

1.22 problem 20

Internal problem ID [6571]
Internal file name [`OUTPUT/5819_Sunday_June_05_2022_03_55_44_PM_95524250/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(1+x)\,y'' - (2-x)\,y' + y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -1]$$

With the expansion point for the power series method at $x = 0$.

1.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}p(x) &= \frac{-2+x}{1+x}\\ q(x) &= \frac{1}{1+x}\\ F &= 0\end{aligned}$$

Hence the ode is

$$y''+\frac{(-2+x)\,y'}{1+x}+\frac{y}{1+x}=0$$

The domain of $p(x)=\frac{-2+x}{1+x}$ is

$$\{x<-1\vee -1<x\}$$

And the point $x_0=0$ is inside this domain. The domain of $q(x)=\frac{1}{1+x}$ is

$$\{x<-1\vee -1<x\}$$

And the point $x_0=0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\ &= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots\\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (70)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (71)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{xy' - 2y' + y}{1 + x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x^2 - 5x)\, y' + (x - 1)\, y}{(1 + x)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-x^3 + 8x^2 - 3x - 6)\, y' - y(x^2 - 4x - 3)}{(1 + x)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^3 - 12x^2 + 23x + 6)\, y' + y(x^2 - 8x + 1)}{(1 + x)^3} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-x^3 + 16x^2 - 55x + 18)\, y' - y(x^2 - 12x + 17)}{(1 + x)^3} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{(x^3 - 20x^2 + 99x - 90)\, y' + y(x^2 - 16x + 45)}{(1 + x)^3} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{(-x^4 + 23x^3 - 131x^2 + 79x + 234)\, y' - y(x^3 - 19x^2 + 65x + 61)}{(1 + x)^4}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 2$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= -4 \\
F_1 &= -2 \\
F_2 &= 12 \\
F_3 &= -4 \\
F_4 &= -52 \\
F_5 &= 180 \\
F_6 &= -356
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -2x^2 - x + 2 - \frac{x^3}{3} + \frac{x^4}{2} - \frac{x^5}{30} - \frac{13x^6}{180} + \frac{x^7}{28} - \frac{89x^8}{10080} + O(x^8)
$$

$$
y = -2x^2 - x + 2 - \frac{x^3}{3} + \frac{x^4}{2} - \frac{x^5}{30} - \frac{13x^6}{180} + \frac{x^7}{28} - \frac{89x^8}{10080} + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(1 + x)\, y'' + (-2 + x)\, y' + y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(1 + x) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \right) + (-2 + x) \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) + \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0 \quad (1)
$$

Which simplifies to

$$
\begin{gathered}
\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \\
+\sum_{n=1}^{\infty}\left(-2 n a_n x^{n-1}\right)+\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0
\end{gathered} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) &= \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n \\
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n \\
\sum_{n=1}^{\infty} \left(-2 n a_n x^{n-1}\right) &= \sum_{n=0}^{\infty} \left(-2(n+1)\, a_{n+1} x^n\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\begin{gathered}
\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n\right)+\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n\right) \\
+\sum_{n=0}^{\infty}\left(-2(n+1)\, a_{n+1} x^n\right)+\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0
\end{gathered} \tag{3}
$$

$n=0$ gives

$$
2 a_2 - 2 a_1 + a_0 = 0
$$

$$
a_2 = -\frac{a_0}{2} + a_1
$$

For $1 \leq n$, the recurrence equation is

$$
(n+1)\, a_{n+1} n + (n+2)\, a_{n+2} (n+1) - 2(n+1)\, a_{n+1} + n a_n + a_n = 0 \tag{4}
$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n a_{n+1} + a_n - 2a_{n+1}}{n+2} \\
&= -\frac{a_n}{n+2} - \frac{(n-2)\, a_{n+1}}{n+2}
\end{aligned} \tag{5}
$$

For $n = 1$ the recurrence equation gives

$$
-2a_2 + 6a_3 + 2a_1 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_3 = -\frac{a_0}{6}
$$

For $n = 2$ the recurrence equation gives

$$
12a_4 + 3a_2 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_4 = \frac{a_0}{8} - \frac{a_1}{4}
$$

For $n = 3$ the recurrence equation gives

$$
4a_4 + 20a_5 + 4a_3 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_5 = \frac{a_0}{120} + \frac{a_1}{20}
$$

For $n = 4$ the recurrence equation gives

$$
10a_5 + 30a_6 + 5a_4 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_6 = -\frac{17a_0}{720} + \frac{a_1}{40}
$$

For $n = 5$ the recurrence equation gives

$$18a_6 + 42a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{112} - \frac{a_1}{56}$$

For $n = 6$ the recurrence equation gives

$$28a_7 + 56a_8 + 7a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = -\frac{61a_0}{40320} + \frac{13a_1}{2240}$$

For $n = 7$ the recurrence equation gives

$$40a_8 + 72a_9 + 8a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{11a_0}{72576} - \frac{5a_1}{4032}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1x + \left(-\frac{a_0}{2} + a_1\right)x^2 - \frac{a_0x^3}{6} + \left(\frac{a_0}{8} - \frac{a_1}{4}\right)x^4 \\ + \left(\frac{a_0}{120} + \frac{a_1}{20}\right)x^5 + \left(-\frac{17a_0}{720} + \frac{a_1}{40}\right)x^6 + \left(\frac{a_0}{112} - \frac{a_1}{56}\right)x^7 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{8}x^{4}+\frac{1}{120}x^{5}-\frac{17}{720}x^{6}+\frac{1}{112}x^{7}\right) a_{0} \\
&+\left(x+x^{2}-\frac{1}{4}x^{4}+\frac{1}{20}x^{5}+\frac{1}{40}x^{6}-\frac{1}{56}x^{7}\right) a_{1}+O\left(x^{8}\right)
\end{aligned} \tag{3}
$$

At $x=0$ the solution above becomes

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{8}x^{4}+\frac{1}{120}x^{5}-\frac{17}{720}x^{6}+\frac{1}{112}x^{7}\right) c_{1} \\
&+\left(x+x^{2}-\frac{1}{4}x^{4}+\frac{1}{20}x^{5}+\frac{1}{40}x^{6}-\frac{1}{56}x^{7}\right) c_{2}+O\left(x^{8}\right)
\end{aligned}
$$

$$
y=2-2x^{2}-\frac{x^{3}}{3}+\frac{x^{4}}{2}-\frac{x^{5}}{30}-\frac{13x^{6}}{180}+\frac{x^{7}}{28}-x+O\left(x^{8}\right)
$$

Summary

The solution(s) found are the following

$$
y=-2x^{2}-x+2-\frac{x^{3}}{3}+\frac{x^{4}}{2}-\frac{x^{5}}{30}-\frac{13x^{6}}{180}+\frac{x^{7}}{28}-\frac{89x^{8}}{10080}+O\left(x^{8}\right) \tag{1}
$$

$$
y=2-2x^{2}-\frac{x^{3}}{3}+\frac{x^{4}}{2}-\frac{x^{5}}{30}-\frac{13x^{6}}{180}+\frac{x^{7}}{28}-x+O\left(x^{8}\right) \tag{2}
$$

Verification of solutions

$$
y=-2x^{2}-x+2-\frac{x^{3}}{3}+\frac{x^{4}}{2}-\frac{x^{5}}{30}-\frac{13x^{6}}{180}+\frac{x^{7}}{28}-\frac{89x^{8}}{10080}+O\left(x^{8}\right)
$$

Verified OK.

$$
y=2-2x^{2}-\frac{x^{3}}{3}+\frac{x^{4}}{2}-\frac{x^{5}}{30}-\frac{13x^{6}}{180}+\frac{x^{7}}{28}-x+O\left(x^{8}\right)
$$

Verified OK.

1.22.2 Maple step by step solution

Let's solve

$$\left[(1+x)\,y''+(-2+x)\,y'+y=0, y(0)=2, y'\Big|_{\{x=0\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{1+x}-\frac{(-2+x)y'}{1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(-2+x)y'}{1+x}+\frac{y}{1+x}=0$

☐ Check to see if $x_0=-1$ is a regular singular point

  - ○ Define functions

    $\left[P_2(x)=\frac{-2+x}{1+x}, P_3(x)=\frac{1}{1+x}\right]$

  - ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $$\left((1+x)\cdot P_2(x)\right)\Bigg|_{x=-1}=-3$$

  - ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $$\left((1+x)^2\cdot P_3(x)\right)\Bigg|_{x=-1}=0$$

  - ○ $x=-1$is a regular singular point

  Check to see if $x_0=-1$ is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(1+x)\,y''+(-2+x)\,y'+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(-3+u)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum\limits_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum\limits_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, u^{k+r}$
- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum\limits_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$
- Shift index using $k- >k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum\limits_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-4+r)\, u^{-1+r} + \left(\sum\limits_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-3+r) + a_k(k+1+r)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-4+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 4\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)\,(a_{k+1}(k-3+r) + a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k-3+r}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{k-3}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 3$

  $a_{k+1} = -\frac{a_k}{k-3}$
- Recursion relation for $r = 4$

  $a_{k+1} = -\frac{a_k}{k+1}$
- Solution for $r = 4$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+4}, a_{k+1}=-\frac{a_k}{k+1}\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (1+x)^{k+4}, a_{k+1}=-\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=8;
dsolve([(x+1)*diff(y(x),x$2)-(2-x)*diff(y(x),x)+y(x)=0,y(0) = 2, D(y)(0) = -1],y(x),type='se
```

$$y(x)=2-x-2x^2-\frac{1}{3}x^3+\frac{1}{2}x^4-\frac{1}{30}x^5-\frac{13}{180}x^6+\frac{1}{28}x^7+\mathrm{O}\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 48

```
AsymptoticDSolveValue[{(x+1)*y''[x]-(2-x)*y'[x]+y[x]==0,{y[0]==2,y'[0]==-1}},y[x],{x,0,7}]
```

$$y(x)\to \frac{x^7}{28}-\frac{13x^6}{180}-\frac{x^5}{30}+\frac{x^4}{2}-\frac{x^3}{3}-2x^2-x+2$$

1.23 problem 21

Internal problem ID [6572]
Internal file name [`OUTPUT/5820_Sunday_June_05_2022_03_55_47_PM_34181718/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2xy' + 8y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

1.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 8 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2xy' + 8y = 0$$

The domain of $p(x) = -2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 8$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (73)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (74)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{align*}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align*}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2xy' - 8y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= 4x^2y' - 16xy - 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= 8y'x^3 - 32yx^2 - 20xy' + 32y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(16x^4 - 48x^2 + 12\right)y' + \left(-64x^3 + 96x\right)y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= 32x\left(\left(x^4 - 3x^2 + \frac{3}{4}\right)y' + \left(-4x^3 + 6x\right)y\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4 \\
&= 64\left(x^2 + \frac{1}{2}\right)\left(\left(x^4 - 3x^2 + \frac{3}{4}\right)y' + \left(-4x^3 + 6x\right)y\right) \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5 \\
&= 128\left(x^2 + \frac{3}{2}\right)x\left(\left(x^4 - 3x^2 + \frac{3}{4}\right)y' + \left(-4x^3 + 6x\right)y\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 3$ and

$y'(0) = 0$ gives

$$\begin{aligned}F_0 &= -24\\F_1 &= 0\\F_2 &= 96\\F_3 &= 0\\F_4 &= 0\\F_5 &= 0\\F_6 &= 0\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = 4x^4 - 12x^2 + 3 + O(x^8)$$

$$y = 4x^4 - 12x^2 + 3 + O(x^8)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}\\y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 8\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 8 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 8 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 8a_0 = 0$$

$$a_2 = -4a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2n a_n + 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2a_n(n-4)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 6a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -a_1$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{4a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{10}$$

For $n = 4$ the recurrence equation gives

$$30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{210}$$

For $n = 6$ the recurrence equation gives

$$56a_8 - 4a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$72a_9 - 6a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{2520}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 4a_0 x^2 - a_1 x^3 + \frac{4}{3} a_0 x^4 + \frac{1}{10} a_1 x^5 + \frac{1}{210} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - 4x^2 + \frac{4}{3} x^4\right) a_0 + \left(x - x^3 + \frac{1}{10} x^5 + \frac{1}{210} x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - 4x^2 + \frac{4}{3} x^4\right) c_1 + \left(x - x^3 + \frac{1}{10} x^5 + \frac{1}{210} x^7\right) c_2 + O(x^8)$$

$$y = 4x^4 - 12x^2 + 3 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = 4x^4 - 12x^2 + 3 + O(x^8) \tag{1}$$
$$y = 4x^4 - 12x^2 + 3 + O(x^8) \tag{2}$$

Verification of solutions

$$y = 4x^4 - 12x^2 + 3 + O(x^8)$$

Verified OK.

$$y = 4x^4 - 12x^2 + 3 + O(x^8)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=8;
dsolve([diff(y(x),x$2)-2*x*diff(y(x),x)+8*y(x)=0,y(0) = 3, D(y)(0) = 0],y(x),type='series',x
```

$$y(x) = 4x^4 - 12x^2 + 3$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 15

```
AsymptoticDSolveValue[{y''[x]-2*x*y'[x]+8*y[x]==0,{y[0]==3,y'[0]==0}},y[x],{x,0,7}]
```

$$y(x) \to 4x^4 - 12x^2 + 3$$

1.24 problem 22

Internal problem ID [6573]
Internal file name [`OUTPUT/5821_Sunday_June_05_2022_03_55_51_PM_93001694/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(x^2+1)\,y''+2xy'=0}$$

With initial conditions

$$[y(0)=0, y'(0)=1]$$

With the expansion point for the power series method at $x=0$.

1.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{2x}{x^2+1}\\q(x)&=0\\F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{2xy'}{x^2+1} = 0$$

The domain of $p(x) = \frac{2x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (76)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (77)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \notag\\
&= \frac{dF_0}{dx} \notag\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \notag\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \notag\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \notag\\
&= \frac{d}{dx}(F_1) \notag\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \notag\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \notag\\
&\vdots \notag\\
F_n &= \frac{d}{dx}(F_{n-1}) \notag\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \notag\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2xy'}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(6x^2-2\right) y'}{\left(x^2+1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\left(-24x^3+24x\right) y'}{\left(x^2+1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{120y'\left(x^4-2x^2+\frac{1}{5}\right)}{\left(x^2+1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(-720x^5+2400x^3-720x\right) y'}{\left(x^2+1\right)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{5040y'\left(x^6-5x^4+3x^2-\frac{1}{7}\right)}{\left(x^2+1\right)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -\frac{40320xy'(x^6-7x^4+7x^2-1)}{\left(x^2+1\right)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$\begin{aligned}
F_0 &= 0\\
F_1 &= -2\\
F_2 &= 0\\
F_3 &= 24\\
F_4 &= 0\\
F_5 &= -720\\
F_6 &= 0
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + O(x^8)$$

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + O(x^8)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 1)\, y'' + 2xy' = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}\\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$(x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 2x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 2 n a_{n} x^{n}\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}=\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^{n}\right)+\left(\sum_{n=1}^{\infty} 2 n a_{n} x^{n}\right)=0 \tag{3}$$

$n=1$ gives

$$6 a_{3}+2 a_{1}=0$$

Which after substituting earlier equations, simplifies to

$$a_{3}=-\frac{a_{1}}{3}$$

For $2 \leq n$, the recurrence equation is

$$n a_{n}(n-1)+(n+2)\, a_{n+2}(n+1)+2 n a_{n}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2}=-\frac{n a_{n}}{n+2} \tag{5}$$

For $n=2$ the recurrence equation gives

$$6 a_{2}+12 a_{4}=0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$12a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$20a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$30a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{7}$$

For $n = 6$ the recurrence equation gives

$$42a_6 + 56a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$56a_7 + 72a_9 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{9}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{3} a_1 x^3 + \frac{1}{5} a_1 x^5 - \frac{1}{7} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = a_0 + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7\right) c_2 + O(x^8)$$

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + O(x^8)$$

Summary

The solution(s) found are the following

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + O(x^8) \tag{1}$$

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + O(x^8) \tag{2}$$

Verification of solutions

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + O(x^8)$$

Verified OK.

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=8;
dsolve([(x^2+1)*diff(y(x),x$2)+2*x*diff(y(x),x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='series',
```

$$y(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \mathrm{O}\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 26

```
AsymptoticDSolveValue[{(x^2+1)*y''[x]+2*x*y'[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,7}]
```

$$y(x) \to -\frac{x^7}{7} + \frac{x^5}{5} - \frac{x^3}{3} + x$$

1.25 problem 23

Internal problem ID [6574]
Internal file name [`OUTPUT/5822_Sunday_June_05_2022_03_55_54_PM_95651441/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \sin(x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (79)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (80)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\sin\left(x\right) y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -\sin\left(x\right) y' - y \cos\left(x\right) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2\cos\left(x\right) y' + \sin\left(x\right) y(1 + \sin\left(x\right)) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\sin\left(x\right)^2 + 3\sin\left(x\right)\right) y' + 4\cos\left(x\right) y \left(\sin\left(x\right) + \frac{1}{4}\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \cos\left(x\right) \left(6\sin\left(x\right) + 4\right) y' + \left(\left(\sin\left(x\right) + 11\right) \cos\left(x\right)^2 - 2\sin\left(x\right) - 7\right) y \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(\left(\sin\left(x\right) + 23\right) \cos\left(x\right)^2 - 6\sin\left(x\right) - 13\right) y' + 9\cos\left(x\right) y \left(\cos\left(x\right)^2 - \frac{26\sin\left(x\right)}{9} - \frac{10}{9}\right) \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(12\cos\left(x\right)^3 + \left(-72\sin\left(x\right) - 18\right) \cos\left(x\right)\right) y' + y\left(\cos\left(x\right)^4 + \left(-50\sin\left(x\right) - 59\right) \cos\left(x\right)^2 + 23\sin\left(x\right) +
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -2y'(0) \\
F_3 &= y(0) \\
F_4 &= 4y(0) + 4y'(0) \\
F_5 &= -y(0) + 10y'(0) \\
F_6 &= -26y(0) - 6y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 - \frac{1}{5040}x^7 - \frac{13}{20160}x^8\right) y(0) \\
&+ \left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7 - \frac{1}{6720}x^8\right) y'(0) + O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\sin(x) \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
\sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \dots \\
&= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the second term in (1) gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^5}{120} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^7}{5040} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^9}{362880} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+5} a_n}{120}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+7} a_n}{5040}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+9} a_n}{362880}\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) = \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+5} a_n}{120} = \sum_{n=5}^{\infty} \frac{a_{n-5} x^n}{120}$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+7} a_n}{5040}\right) = \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} x^n}{5040}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+9} a_n}{362880} = \sum_{n=9}^{\infty} \frac{a_{n-9} x^n}{362880}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^{n}\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^{n}\right)+\sum_{n\, =3}^{\infty} \left(-\frac{a_{n-3} x^{n}}{6}\right) \\ +\left(\sum_{n=5}^{\infty} \frac{a_{n-5} x^{n}}{120}\right)+\sum_{n\, =7}^{\infty} \left(-\frac{a_{n-7} x^{n}}{5040}\right)+\left(\sum_{n=9}^{\infty} \frac{a_{n-9} x^{n}}{362880}\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6}$$

$n = 2$ gives

$$12a_4 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{a_1}{12}$$

$n = 3$ gives

$$20a_5 + a_2 - \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_0}{120}$$

$n = 4$ gives

$$30a_6 + a_3 - \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{a_0}{180} + \frac{a_1}{180}$$

$n = 5$ gives

$$42a_7 + a_4 - \frac{a_2}{6} + \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{a_0}{5040} + \frac{a_1}{504}$$

$n = 6$ gives

$$56a_8 + a_5 - \frac{a_3}{6} + \frac{a_1}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_8 = -\frac{13a_0}{20160} - \frac{a_1}{6720}$$

$n = 7$ gives

$$72a_9 + a_6 - \frac{a_4}{6} + \frac{a_2}{120} - \frac{a_0}{5040} = 0$$

Which after substituting earlier equations, simplifies to

$$a_9 = -\frac{a_0}{13440} - \frac{7a_1}{25920}$$

For $9 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} - \frac{a_{n-3}}{6} + \frac{a_{n-5}}{120} - \frac{a_{n-7}}{5040} + \frac{a_{n-9}}{362880} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{362880a_{n-1} - 60480a_{n-3} + 3024a_{n-5} - 72a_{n-7} + a_{n-9}}{362880\,(n+2)\,(1+n)} \\ &= -\frac{a_{n-9}}{362880\,(n+2)\,(1+n)} + \frac{a_{n-7}}{5040\,(n+2)\,(1+n)} \\ &\quad - \frac{a_{n-5}}{120\,(n+2)\,(1+n)} + \frac{a_{n-3}}{6\,(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{a_0 x^3}{6} - \frac{a_1 x^4}{12} + \frac{a_0 x^5}{120} + \left(\frac{a_0}{180} + \frac{a_1}{180}\right) x^6 + \left(-\frac{a_0}{5040} + \frac{a_1}{504}\right) x^7 + \dots$$

Collecting terms, the solution becomes

$$\begin{aligned} y = {} & \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 - \frac{1}{5040}x^7\right) a_0 \\ & + \left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7\right) a_1 + O(x^8) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 - \frac{1}{5040}x^7\right) c_1 + \left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7\right) c_2 + O(x^8)$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 - \frac{1}{5040}x^7 - \frac{13}{20160}x^8\right) y(0) \\ & + \left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7 - \frac{1}{6720}x^8\right) y'(0) + O(x^8) \end{aligned} \tag{1}$$

$$\begin{aligned} y = {} & \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 - \frac{1}{5040}x^7\right) c_1 \\ & + \left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7\right) c_2 + O(x^8) \end{aligned} \tag{2}$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{6}x^{3}+\frac{1}{120}x^{5}+\frac{1}{180}x^{6}-\frac{1}{5040}x^{7}-\frac{13}{20160}x^{8}\right) y(0) \\
& + \left(x-\frac{1}{12}x^{4}+\frac{1}{180}x^{6}+\frac{1}{504}x^{7}-\frac{1}{6720}x^{8}\right) y'(0)+O\left(x^{8}\right)
\end{aligned}
$$

Verified OK.

$$
y=\left(1-\frac{1}{6}x^{3}+\frac{1}{120}x^{5}+\frac{1}{180}x^{6}-\frac{1}{5040}x^{7}\right) c_{1}+\left(x-\frac{1}{12}x^{4}+\frac{1}{180}x^{6}+\frac{1}{504}x^{7}\right) c_{2}+O\left(x^{8}\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         Equivalence transformation and function parameters: {t = 1/2*t+1/2}, {kappa = -20,
         <- Equivalence to the rational form of Mathieu ODE successful
      <- Mathieu successful
   <- special function solution successful
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      (-t^2+1)^(1/2)*u(t)-t*diff(u(t),t)+(-t^2+1)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=8;
dsolve(diff(y(x),x$2)+sin(x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{1}{6}x^3+\frac{1}{120}x^5+\frac{1}{180}x^6-\frac{1}{5040}x^7\right)y(0)+\left(x-\frac{1}{12}x^4+\frac{1}{180}x^6+\frac{1}{504}x^7\right)D(y)(0)+O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[y''[x]+Sin[x]*y[x]==0,y[x],{x,0,7}]
```

$$y(x)\to c_2\left(\frac{x^7}{504}+\frac{x^6}{180}-\frac{x^4}{12}+x\right)+c_1\left(-\frac{x^7}{5040}+\frac{x^6}{180}+\frac{x^5}{120}-\frac{x^3}{6}+1\right)$$

1.26 problem 24

Internal problem ID [6575]
Internal file name [`OUTPUT/5823_Sunday_June_05_2022_03_55_59_PM_86649983/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' \mathrm{e}^{x} - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (82)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (83)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y' \mathrm{e}^{x} + y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y' \mathrm{e}^{2x} - y' \mathrm{e}^{x} - \mathrm{e}^{x} y + y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= (y + 3y') \,\mathrm{e}^{2x} - y' \mathrm{e}^{3x} - 3y' \mathrm{e}^{x} + (-2\,\mathrm{e}^{x} + 1)\, y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 5(y + 2y') \,\mathrm{e}^{2x} + (-y - 6y') \,\mathrm{e}^{3x} + y' \mathrm{e}^{4x} + (-5\,\mathrm{e}^{x} + 1)\, y' - 5\,\mathrm{e}^{x} y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= (20y + 30y') \,\mathrm{e}^{2x} + (-9y - 29y') \,\mathrm{e}^{3x} + (y + 10y') \,\mathrm{e}^{4x} - y' \mathrm{e}^{5x} - 11y' \mathrm{e}^{x} + (-10\,\mathrm{e}^{x} + 1)\, y \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= (70y + 91y') \,\mathrm{e}^{2x} + (-56y - 126y') \,\mathrm{e}^{3x} + (14y + 70y') \,\mathrm{e}^{4x} + (-y - 15y') \,\mathrm{e}^{5x} + y' \mathrm{e}^{6x} + (-21\,\mathrm{e}^{x} + 1)\, y' \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= (273y' + 231y) \,\mathrm{e}^{2x} + (-525y' - 294y) \,\mathrm{e}^{3x} + (420y' + 126y) \,\mathrm{e}^{4x} + (-146y' - 20y) \,\mathrm{e}^{5x} + (21y' + y) \,\mathrm{e}^{6x}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) - y'(0) \\
F_1 &= -y(0) + y'(0) \\
F_2 &= -y'(0) \\
F_3 &= -y(0) + y'(0) \\
F_4 &= 3y(0) - y'(0) \\
F_5 &= y'(0) + 6y(0) \\
F_6 &= 3y(0) - y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{240}x^6 + \frac{1}{840}x^7 + \frac{1}{13440}x^8\right) y(0) \\
&+ \left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{1}{5040}x^7 - \frac{1}{40320}x^8\right) y'(0) + O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \mathrm{e}^x + \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Expanding e^x as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
\mathrm{e}^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \dots \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \\
&+\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6+\frac{1}{5040}x^7+\frac{1}{40320}x^8\right)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \\
&-\left(\sum_{n=0}^{\infty} a_n x^n\right)=0
\end{aligned}
$$

Expanding the second term in (1) gives

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+1\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+x\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\frac{x^2}{2} \\
&\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\frac{x^3}{6}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\frac{x^4}{24}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \\
&+\frac{x^5}{120}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\frac{x^6}{720}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\frac{x^7}{5040} \\
&\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\frac{x^8}{40320}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-\left(\sum_{n=0}^{\infty} a_n x^n\right)=0
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\left(\sum_{n=1}^{\infty} n a_n x^{n}\right) \\
&+\left(\sum_{n=1}^{\infty} \frac{n\, x^{1+n} a_n}{2}\right)+\left(\sum_{n=1}^{\infty} \frac{n\, x^{n+2} a_n}{6}\right)+\left(\sum_{n=1}^{\infty} \frac{n\, x^{n+3} a_n}{24}\right)+\left(\sum_{n=1}^{\infty} \frac{n\, x^{n+4} a_n}{120}\right) \\
&+\left(\sum_{n=1}^{\infty} \frac{n\, x^{n+5} a_n}{720}\right)+\left(\sum_{n=1}^{\infty} \frac{n\, x^{n+6} a_n}{5040}\right)+\left(\sum_{n=1}^{\infty} \frac{n\, x^{n+7} a_n}{40320}\right)+\sum_{n=0}^{\infty}\left(-a_n x^n\right)=0
\end{aligned}
\tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n)\, a_{1+n} x^n$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{1+n} a_n}{2} = \sum_{n=2}^{\infty} \frac{(n-1)\, a_{n-1} x^n}{2}$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+2} a_n}{6} = \sum_{n=3}^{\infty} \frac{(n-2)\, a_{n-2} x^n}{6}$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+3} a_n}{24} = \sum_{n=4}^{\infty} \frac{(n-3)\, a_{n-3} x^n}{24}$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+4} a_n}{120} = \sum_{n=5}^{\infty} \frac{(n-4)\, a_{n-4} x^n}{120}$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+5} a_n}{720} = \sum_{n=6}^{\infty} \frac{(n-5)\, a_{n-5} x^n}{720}$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+6} a_n}{5040} = \sum_{n=7}^{\infty} \frac{(n-6)\, a_{n-6} x^n}{5040}$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+7} a_n}{40320} = \sum_{n=8}^{\infty} \frac{(n-7)\, a_{n-7} x^n}{40320}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(1+n)\, x^{n}\right)+\left(\sum_{n=0}^{\infty}(1+n)\, a_{1+n} x^{n}\right) \\
&+\left(\sum_{n=1}^{\infty} n a_{n} x^{n}\right)+\left(\sum_{n=2}^{\infty} \frac{(n-1)\, a_{n-1} x^{n}}{2}\right)+\left(\sum_{n=3}^{\infty} \frac{(n-2)\, a_{n-2} x^{n}}{6}\right) \\
&+\left(\sum_{n=4}^{\infty} \frac{(n-3)\, a_{n-3} x^{n}}{24}\right)+\left(\sum_{n=5}^{\infty} \frac{(n-4)\, a_{n-4} x^{n}}{120}\right)+\left(\sum_{n=6}^{\infty} \frac{(n-5)\, a_{n-5} x^{n}}{720}\right) \\
&+\left(\sum_{n=7}^{\infty} \frac{(n-6)\, a_{n-6} x^{n}}{5040}\right)+\left(\sum_{n=8}^{\infty} \frac{(n-7)\, a_{n-7} x^{n}}{40320}\right)+\sum_{n=0}^{\infty}\left(-a_{n} x^{n}\right)=0
\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 - a_0 = 0$$

$$a_2 = \frac{a_0}{2} - \frac{a_1}{2}$$

$n = 1$ gives

$$6a_3 + 2a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$6a_3 + a_0 - a_1 = 0$$

Or

$$a_3 = -\frac{a_0}{6} + \frac{a_1}{6}$$

$n = 2$ gives

$$12a_4 + 3a_3 + a_2 + \frac{a_1}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{a_1}{24}$$

$n = 3$ gives

$$20a_5 + 4a_4 + 2a_3 + a_2 + \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = -\frac{a_0}{120} + \frac{a_1}{120}$$

$n = 4$ gives

$$30a_6 + 5a_5 + 3a_4 + \frac{3a_3}{2} + \frac{a_2}{3} + \frac{a_1}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{a_0}{240} - \frac{a_1}{720}$$

$n = 5$ gives

$$42a_7 + 6a_6 + 4a_5 + 2a_4 + \frac{a_3}{2} + \frac{a_2}{12} + \frac{a_1}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = \frac{a_0}{840} + \frac{a_1}{5040}$$

$n = 6$ gives

$$56a_8 + 7a_7 + 5a_6 + \frac{5a_5}{2} + \frac{2a_4}{3} + \frac{a_3}{8} + \frac{a_2}{60} + \frac{a_1}{720} = 0$$

Which after substituting earlier equations, simplifies to

$$a_8 = \frac{a_0}{13440} - \frac{a_1}{40320}$$

$n = 7$ gives

$$72a_9 + 8a_8 + 6a_7 + 3a_6 + \frac{5a_5}{6} + \frac{a_4}{6} + \frac{a_3}{40} + \frac{a_2}{360} + \frac{a_1}{5040} = 0$$

Which after substituting earlier equations, simplifies to

$$a_9 = -\frac{53a_0}{362880} + \frac{a_1}{362880}$$

For $8 \leq n$, the recurrence equation is

$$\begin{aligned}
&(n+2)\, a_{n+2}(1+n) + (1+n)\, a_{1+n} + n a_n + \frac{(n-1)\, a_{n-1}}{2} \\
&+ \frac{(n-2)\, a_{n-2}}{6} + \frac{(n-3)\, a_{n-3}}{24} + \frac{(n-4)\, a_{n-4}}{120} \\
&+ \frac{(n-5)\, a_{n-5}}{720} + \frac{(n-6)\, a_{n-6}}{5040} + \frac{(n-7)\, a_{n-7}}{40320} - a_n = 0
\end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned}
a_{n+2} &= \\
&-\frac{40320 n a_n + 40320 n a_{1+n} + n a_{n-7} + 8 n a_{n-6} + 56 n a_{n-5} + 336 n a_{n-4} + 1680 n a_{n-3} + 6720 n a_{n-2} + 2016}{4032}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{(40320n - 40320)\, a_n}{40320\,(n+2)\,(1+n)} - \frac{(40320n + 40320)\, a_{1+n}}{40320\,(n+2)\,(1+n)} - \frac{(n-7)\, a_{n-7}}{40320\,(n+2)\,(1+n)} \\
&- \frac{(8n - 48)\, a_{n-6}}{40320\,(n+2)\,(1+n)} - \frac{(56n - 280)\, a_{n-5}}{40320\,(n+2)\,(1+n)} - \frac{(336n - 1344)\, a_{n-4}}{40320\,(n+2)\,(1+n)} \\
&- \frac{(1680n - 5040)\, a_{n-3}}{40320\,(n+2)\,(1+n)} - \frac{(6720n - 13440)\, a_{n-2}}{40320\,(n+2)\,(1+n)} - \frac{(20160n - 20160)\, a_{n-1}}{40320\,(n+2)\,(1+n)}
\end{aligned} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned}
y = a_0 + a_1 x &+ \left(\frac{a_0}{2} - \frac{a_1}{2}\right) x^2 + \left(-\frac{a_0}{6} + \frac{a_1}{6}\right) x^3 - \frac{a_1 x^4}{24} \\
&+ \left(-\frac{a_0}{120} + \frac{a_1}{120}\right) x^5 + \left(\frac{a_0}{240} - \frac{a_1}{720}\right) x^6 + \left(\frac{a_0}{840} + \frac{a_1}{5040}\right) x^7 + \dots
\end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned}
y = &\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{240}x^6 + \frac{1}{840}x^7\right) a_0 \\
&+ \left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{1}{5040}x^7\right) a_1 + O(x^8)
\end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned}y &= \left(1+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{120}x^{5}+\frac{1}{240}x^{6}+\frac{1}{840}x^{7}\right) c_{1}\\ &+\left(x-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{24}x^{4}+\frac{1}{120}x^{5}-\frac{1}{720}x^{6}+\frac{1}{5040}x^{7}\right) c_{2}+O\left(x^{8}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y &= \left(1+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{120}x^{5}+\frac{1}{240}x^{6}+\frac{1}{840}x^{7}+\frac{1}{13440}x^{8}\right) y(0)\\ &+\left(x-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{24}x^{4}+\frac{1}{120}x^{5}-\frac{1}{720}x^{6}+\frac{1}{5040}x^{7}-\frac{1}{40320}x^{8}\right) y'(0)+O\left(x^{8}\right)\end{aligned} \tag{1}$$

$$\begin{aligned}y &= \left(1+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{120}x^{5}+\frac{1}{240}x^{6}+\frac{1}{840}x^{7}\right) c_{1}\\ &+\left(x-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{24}x^{4}+\frac{1}{120}x^{5}-\frac{1}{720}x^{6}+\frac{1}{5040}x^{7}\right) c_{2}+O\left(x^{8}\right)\end{aligned} \tag{2}$$

Verification of solutions

$$\begin{aligned}y &= \left(1+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{120}x^{5}+\frac{1}{240}x^{6}+\frac{1}{840}x^{7}+\frac{1}{13440}x^{8}\right) y(0)\\ &+\left(x-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{24}x^{4}+\frac{1}{120}x^{5}-\frac{1}{720}x^{6}+\frac{1}{5040}x^{7}-\frac{1}{40320}x^{8}\right) y'(0)+O\left(x^{8}\right)\end{aligned}$$

Verified OK.

$$\begin{aligned}y &= \left(1+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{120}x^{5}+\frac{1}{240}x^{6}+\frac{1}{840}x^{7}\right) c_{1}\\ &+\left(x-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{24}x^{4}+\frac{1}{120}x^{5}-\frac{1}{720}x^{6}+\frac{1}{5040}x^{7}\right) c_{2}+O\left(x^{8}\right)\end{aligned}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
   Change of variables used:
      [x = ln(t)]
   Linear ODE actually solved:
      -u(t)+(t^2+t)*diff(u(t),t)+t^2*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 69

```
Order:=8;
dsolve(diff(y(x),x$2)+exp(x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{120}x^5+\frac{1}{240}x^6+\frac{1}{840}x^7\right)y(0) + \left(x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{120}x^5-\frac{1}{720}x^6+\frac{1}{5040}x^7\right)D(y)(0)+O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 91

```
AsymptoticDSolveValue[y''[x]+Exp[x]*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^7}{840}+\frac{x^6}{240}-\frac{x^5}{120}-\frac{x^3}{6}+\frac{x^2}{2}+1\right)+c_2\left(\frac{x^7}{5040}-\frac{x^6}{720}+\frac{x^5}{120}-\frac{x^4}{24}+\frac{x^3}{6}-\frac{x^2}{2}+x\right)$$

1.27 problem 25 expansion at 0

Internal problem ID [6576]
Internal file name [`OUTPUT/5824_Sunday_June_05_2022_03_56_01_PM_62230208/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 25 expansion at 0.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\cos(x)\,y'' + y' + 5y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (85)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (86)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$F_0 = -\frac{y' + 5y}{\cos(x)}$$

$$\begin{aligned}F_1 &= \frac{dF_0}{dx}\\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\ &= -5\left(\left(\cos(x) + \frac{\sin(x)}{5} - \frac{1}{5}\right)y' + y(-1 + \sin(x))\right)\sec(x)^2\end{aligned}$$

$$\begin{aligned}F_2 &= \frac{dF_1}{dx}\\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\ &= \frac{5\left(\frac{\left(-3+11\cos(x)^2+(8-9\sin(x))\cos(x)+3\sin(x)\right)y'}{5} + y\left(\cos(x)^2 + (\sin(x) + 3)\cos(x) + 8\sin(x) + 2\right)\right)\sec(x)}{\cos(x) + 1 + \sin(x)}\end{aligned}$$

$$\begin{aligned}F_3 &= \frac{dF_2}{dx}\\ &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\ &= -\frac{5\sec(x)^3 \ (\frac{14\left(\frac{8\sin(x)^2}{7}+(\cos(x)-\frac{6}{7})\sin(x)-\frac{32\cos(x)}{7}-\frac{2}{7}\right)\sin(x)y'}{5} + y\big(-22 + (-\cos(x)^2 - 14\cos(x) + 2)\sin}{\cos(x) + 1 + \sin(x)}\end{aligned}$$

$$\begin{aligned}F_4 &= \frac{dF_3}{dx}\\ &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\ &= -\frac{5\Big(\Big(\frac{21\cos(x)^4}{5} + \left(-\frac{19\sin(x)}{5} + \frac{276}{5}\right)\cos(x)^3 + (-3\sin(x) - 18)\cos(x)^2 + (27\sin(x) - 57)\cos(x) -}{}\end{aligned}$$

$$\begin{aligned}F_5 &= \frac{dF_4}{dx}\\ &= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\ &= \frac{5\Big(\Big(-\frac{24\cos(x)^5}{5} + \left(-\frac{26\sin(x)}{5} - \frac{194}{5}\right)\cos(x)^4 + (-184\sin(x) + 211)\cos(x)^3 + (5\sin(x) - 31)\cos(x)^2}{}\end{aligned}$$

$$\begin{aligned}F_6 &= \frac{dF_5}{dx}\\ &= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\ &= \frac{5\sec(x)^6\Big(\Big(\frac{31\cos(x)^6}{5} + \left(-\frac{29\sin(x)}{5} + \frac{2668}{5}\right)\cos(x)^5 + \left(-\frac{837\sin(x)}{5} + \frac{2067}{5}\right)\cos(x)^4 + (1086\sin(x) - 2}{}\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y'(0) - 5y(0) \\
F_1 &= -4y'(0) + 5y(0) \\
F_2 &= 15y(0) + 8y'(0) \\
F_3 &= -25y(0) \\
F_4 &= 45y(0) + 9y'(0) \\
F_5 &= -195y(0) + 11y'(0) \\
F_6 &= 415y(0) - 42y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{5}{2}x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 - \frac{5}{24}x^5 + \frac{1}{16}x^6 - \frac{13}{336}x^7 + \frac{83}{8064}x^8\right) y(0) \\
&+ \left(x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{3}x^4 + \frac{1}{80}x^6 + \frac{11}{5040}x^7 - \frac{1}{960}x^8\right) y'(0) + O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\frac{\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 5\left(\sum_{n=0}^{\infty} a_n x^n\right)}{\cos(x)} \tag{1}
$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \dots \\
&= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\frac{1}{40320}x^8\right)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 5\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the first term in (1) gives

$$1\cdot\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) - \frac{x^2}{2}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + \frac{x^4}{24}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) - \frac{x^6}{720}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + \frac{x^8}{40320}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 5\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}\frac{n\,x^{n+6}a_n(n-1)}{40320}\right) + \sum_{n=2}^{\infty}\left(-\frac{n\,x^{n+4}a_n(n-1)}{720}\right) + \left(\sum_{n=2}^{\infty}\frac{n\,x^{n+2}a_n(n-1)}{24}\right) + \sum_{n=2}^{\infty}\left(-\frac{n a_n x^n(n-1)}{2}\right) + \left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 5a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}\frac{n\,x^{n+6}a_n(n-1)}{40320} = \sum_{n=8}^{\infty}\frac{(n-6)\,a_{n-6}(n-7)\,x^n}{40320}$$

$$\sum_{n=2}^{\infty}\left(-\frac{n\,x^{n+4}a_n(n-1)}{720}\right) = \sum_{n=6}^{\infty}\left(-\frac{(n-4)\,a_{n-4}(n-5)\,x^n}{720}\right)$$

$$\sum_{n=2}^{\infty}\frac{n\,x^{n+2} a_{n} \left(n-1\right)}{24} = \sum_{n=4}^{\infty}\frac{\left(n-2\right) a_{n-2} \left(n-3\right) x^{n}}{24}$$

$$\sum_{n=2}^{\infty}n \left(n-1\right) a_{n} x^{n-2} = \sum_{n=0}^{\infty}\left(n+2\right) a_{n+2} \left(n+1\right) x^{n}$$

$$\sum_{n=1}^{\infty}n a_{n} x^{n-1} = \sum_{n=0}^{\infty}\left(n+1\right) a_{n+1} x^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{gathered}\left(\sum_{n=8}^{\infty}\frac{\left(n-6\right) a_{n-6} \left(n-7\right) x^{n}}{40320}\right)+\sum_{n=6}^{\infty}\left(-\frac{\left(n-4\right) a_{n-4} \left(n-5\right) x^{n}}{720}\right)\\ +\left(\sum_{n=4}^{\infty}\frac{\left(n-2\right) a_{n-2} \left(n-3\right) x^{n}}{24}\right)+\sum_{n=2}^{\infty}\left(-\frac{n a_{n} x^{n} \left(n-1\right)}{2}\right)\\ +\left(\sum_{n=0}^{\infty}\left(n+2\right) a_{n+2} \left(n+1\right) x^{n}\right)+\left(\sum_{n=0}^{\infty}\left(n+1\right) a_{n+1} x^{n}\right)+\left(\sum_{n=0}^{\infty}5 a_{n} x^{n}\right) = 0\end{gathered} \tag{3}$$

$n = 0$ gives

$$2 a_{2}+a_{1}+5 a_{0} = 0$$

$$a_{2} = -\frac{5 a_{0}}{2}-\frac{a_{1}}{2}$$

$n = 1$ gives

$$6 a_{3}+2 a_{2}+5 a_{1} = 0$$

Which after substituting earlier equations, simplifies to

$$a_{3} = \frac{5 a_{0}}{6}-\frac{2 a_{1}}{3}$$

$n = 2$ gives

$$4 a_{2}+12 a_{4}+3 a_{3} = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{15a_0}{2} - 4a_1 + 12a_4 = 0$$

Or

$$a_4 = \frac{5a_0}{8} + \frac{a_1}{3}$$

$n = 3$ gives

$$2a_3 + 20a_5 + 4a_4 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{25a_0}{6} + 20a_5 = 0$$

Or

$$a_5 = -\frac{5a_0}{24}$$

$n = 4$ gives

$$\frac{a_2}{12} - a_4 + 30a_6 + 5a_5 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{15a_0}{8} - \frac{3a_1}{8} + 30a_6 = 0$$

Or

$$a_6 = \frac{a_0}{16} + \frac{a_1}{80}$$

$n = 5$ gives

$$\frac{a_3}{4} - 5a_5 + 42a_7 + 6a_6 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{13a_0}{8} - \frac{11a_1}{120} + 42a_7 = 0$$

Or

$$a_7 = -\frac{13a_0}{336} + \frac{11a_1}{5040}$$

$n = 6$ gives

$$-\frac{a_2}{360} + \frac{a_4}{2} - 10a_6 + 56a_8 + 7a_7 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{83a_0}{144} + \frac{7a_1}{120} + 56a_8 = 0$$

Or

$$a_8 = \frac{83a_0}{8064} - \frac{a_1}{960}$$

$n = 7$ gives

$$-\frac{a_3}{120} + \frac{5a_5}{6} - 16a_7 + 72a_9 + 8a_8 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{25a_0}{48} - \frac{19a_1}{504} + 72a_9 = 0$$

Or

$$a_9 = -\frac{25a_0}{3456} + \frac{19a_1}{36288}$$

For $8 \leq n$, the recurrence equation is

$$\begin{aligned}\frac{(n-6)\, a_{n-6}(n-7)}{40320} - \frac{(n-4)\, a_{n-4}(n-5)}{720} + \frac{(n-2)\, a_{n-2}(n-3)}{24} \\ - \frac{n a_n(n-1)}{2} + (n+2)\, a_{n+2}(n+1) + (n+1)\, a_{n+1} + 5a_n = 0\end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{20160n^2a_n - n^2a_{n-6} + 56n^2a_{n-4} - 1680n^2a_{n-2} - 20160na_n + 13na_{n-6} - 504na_{n-4} + 8400na_{n-2} - 403}{40320\,(n+2)\,(n+1)}$$

$$(5) \quad = \frac{(20160n^2 - 20160n - 201600)\,a_n}{40320\,(n+2)\,(n+1)} + \frac{(-n^2 + 13n - 42)\,a_{n-6}}{40320\,(n+2)\,(n+1)} + \frac{(56n^2 - 504n + 1120)\,a_{n-4}}{40320\,(n+2)\,(n+1)} + \frac{(-1680n^2 + 8400n - 10080)\,a_{n-2}}{40320\,(n+2)\,(n+1)} + \frac{(-40320n - 40320)\,a_{n+1}}{40320\,(n+2)\,(n+1)}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \left(-\frac{5a_0}{2} - \frac{a_1}{2}\right)x^2 + \left(\frac{5a_0}{6} - \frac{2a_1}{3}\right)x^3 + \left(\frac{5a_0}{8} + \frac{a_1}{3}\right)x^4 - \frac{5a_0x^5}{24} + \left(\frac{a_0}{16} + \frac{a_1}{80}\right)x^6 + \left(-\frac{13a_0}{336} + \frac{11a_1}{5040}\right)x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{5}{2}x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 - \frac{5}{24}x^5 + \frac{1}{16}x^6 - \frac{13}{336}x^7\right)a_0 + \left(x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{3}x^4 + \frac{1}{80}x^6 + \frac{11}{5040}x^7\right)a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{5}{2}x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 - \frac{5}{24}x^5 + \frac{1}{16}x^6 - \frac{13}{336}x^7\right)c_1 + \left(x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{3}x^4 + \frac{1}{80}x^6 + \frac{11}{5040}x^7\right)c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$
\begin{align*}
y &= \left(1-\frac{5}{2}x^{2}+\frac{5}{6}x^{3}+\frac{5}{8}x^{4}-\frac{5}{24}x^{5}+\frac{1}{16}x^{6}-\frac{13}{336}x^{7}+\frac{83}{8064}x^{8}\right)y(0) \\
&\quad+\left(x-\frac{1}{2}x^{2}-\frac{2}{3}x^{3}+\frac{1}{3}x^{4}+\frac{1}{80}x^{6}+\frac{11}{5040}x^{7}-\frac{1}{960}x^{8}\right)y'(0)+O\left(x^{8}\right) \tag{1}
\end{align*}
$$

$$
\begin{align*}
y &= \left(1-\frac{5}{2}x^{2}+\frac{5}{6}x^{3}+\frac{5}{8}x^{4}-\frac{5}{24}x^{5}+\frac{1}{16}x^{6}-\frac{13}{336}x^{7}\right)c_{1} \\
&\quad+\left(x-\frac{1}{2}x^{2}-\frac{2}{3}x^{3}+\frac{1}{3}x^{4}+\frac{1}{80}x^{6}+\frac{11}{5040}x^{7}\right)c_{2}+O\left(x^{8}\right) \tag{2}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= \left(1-\frac{5}{2}x^{2}+\frac{5}{6}x^{3}+\frac{5}{8}x^{4}-\frac{5}{24}x^{5}+\frac{1}{16}x^{6}-\frac{13}{336}x^{7}+\frac{83}{8064}x^{8}\right)y(0) \\
&\quad+\left(x-\frac{1}{2}x^{2}-\frac{2}{3}x^{3}+\frac{1}{3}x^{4}+\frac{1}{80}x^{6}+\frac{11}{5040}x^{7}-\frac{1}{960}x^{8}\right)y'(0)+O\left(x^{8}\right)
\end{align*}
$$

Verified OK.

$$
\begin{align*}
y &= \left(1-\frac{5}{2}x^{2}+\frac{5}{6}x^{3}+\frac{5}{8}x^{4}-\frac{5}{24}x^{5}+\frac{1}{16}x^{6}-\frac{13}{336}x^{7}\right)c_{1} \\
&\quad+\left(x-\frac{1}{2}x^{2}-\frac{2}{3}x^{3}+\frac{1}{3}x^{4}+\frac{1}{80}x^{6}+\frac{11}{5040}x^{7}\right)c_{2}+O\left(x^{8}\right)
\end{align*}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5`[0, u]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 69

```
Order:=8;
dsolve(cos(x)*diff(y(x),x$2)+diff(y(x),x)+5*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{5}{2}x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 - \frac{5}{24}x^5 + \frac{1}{16}x^6 - \frac{13}{336}x^7\right) y(0) + \left(x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{3}x^4 + \frac{1}{80}x^6 + \frac{11}{5040}x^7\right) D(y)(0) + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 91

```
AsymptoticDSolveValue[Cos[x]*y''[x]+y'[x]+5*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{11x^7}{5040} + \frac{x^6}{80} + \frac{x^4}{3} - \frac{2x^3}{3} - \frac{x^2}{2} + x\right) + c_1\left(-\frac{13x^7}{336} + \frac{x^6}{16} - \frac{5x^5}{24} + \frac{5x^4}{8} + \frac{5x^3}{6} - \frac{5x^2}{2} + 1\right)$$

1.28 problem 25 expansion at 1

Internal problem ID [6577]
Internal file name [`OUTPUT/5825_Sunday_June_05_2022_03_56_05_PM_57306520/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 25 expansion at 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\cos(x)\, y'' + y' + 5y = 0}$$

With the expansion point for the power series method at $x = 1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\cos(t+1)\left(\frac{d^2}{dt^2}y(t)\right) + \frac{d}{dt}y(t) + 5y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x,y,y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\ &= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (88) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (89) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$F_0 = -\frac{5y(t) + \frac{d}{dt}y(t)}{\cos(t+1)}$$

$$\begin{aligned}
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \left(\left(-5\cos(t+1) - \sin(t+1) + 1\right)\left(\frac{d}{dt}y(t)\right) + 5\left(-\sin(t+1) + 1\right)y(t)\right)\sec(t+1)^2
\end{aligned}$$

$$\begin{aligned}
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{5\bigg(\Big(y(t) + \frac{11\frac{d}{dt}y(t)}{5}\Big)\cos(t+1)^2 + \bigg(\Big(y(t) - \frac{9\frac{d}{dt}y(t)}{5}\Big)\sin(t+1) + 3y(t) + \frac{8\frac{d}{dt}y(t)}{5}\bigg)\cos(t+1) + \Big(8y(t)}{\cos(t+1) + 1 + \sin(t+1)}
\end{aligned}$$

$$\begin{aligned}
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= -\frac{5\sec(t+1)^3\Bigg(\frac{16\left(\frac{d}{dt}y(t)\right)\sin(t+1)^3}{5} + \frac{14\left(\cos(t+1) - \frac{6}{7}\right)\left(\frac{d}{dt}y(t)\right)\sin(t+1)^2}{5} + \Big(-y(t)\cos(t+1)^2 + \Big(-14y(t) -}{\cos(t +}
\end{aligned}$$

$$\begin{aligned}
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= -\frac{5\sec(t+1)^4\bigg(\Big(y(t) + \frac{21\frac{d}{dt}y(t)}{5}\Big)\cos(t+1)^4 + \bigg(\Big(y(t) - \frac{19\frac{d}{dt}y(t)}{5}\Big)\sin(t+1) + 41y(t) + \frac{276\frac{d}{dt}y(t)}{5}\bigg)\mathrm{co}}{}
\end{aligned}$$

$$\begin{aligned}
F_5 &= \frac{dF_4}{dt} \\
&= \frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_4}{\partial \frac{d}{dt}y(t)}F_4 \\
&= \frac{5\bigg(\Big(y(t) - \frac{24\frac{d}{dt}y(t)}{5}\Big)\cos(t+1)^5 + \bigg(\Big(-y(t) - \frac{26\frac{d}{dt}y(t)}{5}\Big)\sin(t+1) + 161y(t) - \frac{194\frac{d}{dt}y(t)}{5}\bigg)\cos(t+1)^4 +}{}
\end{aligned}$$

$$\begin{aligned}
F_6 &= \frac{dF_5}{dt} \\
&= \frac{\partial F_5}{\partial t} + \frac{\partial F_5}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_5}{\partial \frac{d}{dt}y(t)}F_5 \\
&= \frac{5\bigg(\Big(\frac{31\frac{d}{dt}y(t)}{5} + y(t)\Big)\cos(t+1)^6 + \bigg(\Big(-\frac{29\frac{d}{dt}y(t)}{5} + y(t)\Big)\sin(t+1) + \frac{2668\frac{d}{dt}y(t)}{5} + 223y(t)\bigg)\cos(t+1)^5}{}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\sec(1)\left(y'(0)+5y(0)\right)\\
F_1 &= -5y(0)\sin(1)\sec(1)^2-\sin(1)\,y'(0)\sec(1)^2-5\cos(1)\,y'(0)\sec(1)^2+5\sec(1)^2\,y(0)+\sec(1)^2\,y'(0)\\
F_2 &= \frac{10y(0)\cos(1)^2+22y'(0)\cos(1)^2+80\sin(1)\,y(0)+30y(0)\cos(1)+5y(0)\sin(2)+6\sin(1)\,y'(0)+}{2\sin(1)\cos(1)^2+2\cos(1)^3+2\cos(1)^2}\\
F_3 &= \frac{-20y(0)\cos(1)^3-64\sin(1)^3\,y'(0)-56\sin(1)^2\cos(1)\,y'(0)-540y(0)\cos(1)^2+48\sin(1)^2\,y'(0)-}{4\sin(1)\cos(1)^3}\\
F_4 &= \frac{-40y(0)\cos(1)^4-1640y(0)\cos(1)^3+1600y(0)\cos(1)^2+276y'(0)\cos(2)-5y(0)\sin(4)+19\sin(}{}\\
F_5 &= \frac{80y(0)\cos(1)^5-384\cos(1)^5\,y'(0)+12880y(0)\cos(1)^4-3104y'(0)\cos(1)^4+37600y(0)\cos(1)^3+}{}\\
F_6 &= \frac{380\sec(1)^6\sin(1)\cos(1)^2\,y'(0)-8040\sec(1)^6\sin(1)\cos(1)\,y'(0)-29\sec(1)^6\sin(1)\cos(1)^5\,y'(0)-}{}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

Expression too large to display

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1}\\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2} = -\frac{5\left(\sum_{n=0}^{\infty} a_n t^n\right)+\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)}{\cos(t+1)} \tag{1}
$$

Expanding $\cos(t+1)$ as Taylor series around $t=0$ and keeping only the first 8 terms gives

$$\begin{aligned}\cos(t+1) &= \cos(1)-\sin(1)\,t-\frac{\cos(1)\,t^2}{2}+\frac{\sin(1)\,t^3}{6}+\frac{\cos(1)\,t^4}{24}-\frac{\sin(1)\,t^5}{120}-\frac{\cos(1)\,t^6}{720}+\frac{\sin(1)\,t^7}{5040}+\\ &= \cos(1)-\sin(1)\,t-\frac{\cos(1)\,t^2}{2}+\frac{\sin(1)\,t^3}{6}+\frac{\cos(1)\,t^4}{24}-\frac{\sin(1)\,t^5}{120}-\frac{\cos(1)\,t^6}{720}+\frac{\sin(1)\,t^7}{5040}+\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\cos(1)-\sin(1)\,t-\frac{\cos(1)\,t^2}{2}+\frac{\sin(1)\,t^3}{6}+\frac{\cos(1)\,t^4}{24}-\frac{\sin(1)\,t^5}{120}-\frac{\cos(1)\,t^6}{720}\right.$$
$$\left.+\frac{\sin(1)\,t^7}{5040}+\frac{\cos(1)\,t^8}{40320}\right)\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)+\left(\sum_{n=1}^{\infty}n a_n t^{n-1}\right)+5\left(\sum_{n=0}^{\infty}a_n t^n\right)=0$$

Expanding the first term in (1) gives

$$\begin{aligned}&\cos(1)\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)-\sin(1)\,t\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)\\&-\frac{\cos(1)\,t^2}{2}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)+\frac{\sin(1)\,t^3}{6}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)\\&+\frac{\cos(1)\,t^4}{24}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)-\frac{\sin(1)\,t^5}{120}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)\\&-\frac{\cos(1)\,t^6}{720}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)+\frac{\sin(1)\,t^7}{5040}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)\\&+\frac{\cos(1)\,t^8}{40320}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)+\left(\sum_{n=1}^{\infty}n a_n t^{n-1}\right)+5\left(\sum_{n=0}^{\infty}a_n t^n\right)=0\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=2}^{\infty}\frac{n\,t^{n+6}a_n(n-1)\cos(1)}{40320}\right)+\left(\sum_{n=2}^{\infty}\frac{n\,t^{n+5}a_n(n-1)\sin(1)}{5040}\right)\\&+\sum_{n=2}^{\infty}\left(-\frac{n\,t^{n+4}a_n(n-1)\cos(1)}{720}\right)+\sum_{n=2}^{\infty}\left(-\frac{n\,t^{n+3}a_n(n-1)\sin(1)}{120}\right)\\&+\left(\sum_{n=2}^{\infty}\frac{n\,t^{n+2}a_n(n-1)\cos(1)}{24}\right)+\left(\sum_{n=2}^{\infty}\frac{n\,t^{1+n}a_n(n-1)\sin(1)}{6}\right)\\&+\sum_{n=2}^{\infty}\left(-\frac{na_nt^n\cos(1)\,(n-1)}{2}\right)+\sum_{n=2}^{\infty}\left(-n\,t^{n-1}a_n(n-1)\sin(1)\right)\\&+\left(\sum_{n=2}^{\infty}n\,t^{n-2}a_n(n-1)\cos(1)\right)+\left(\sum_{n=1}^{\infty}na_nt^{n-1}\right)+\left(\sum_{n=0}^{\infty}5a_nt^n\right)=0\end{aligned}\tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{n\,t^{n+6}a_n(n-1)\cos(1)}{40320}&=\sum_{n=8}^{\infty}\frac{(n-6)\,a_{n-6}(n-7)\cos(1)\,t^n}{40320}\\\sum_{n=2}^{\infty}\frac{n\,t^{n+5}a_n(n-1)\sin(1)}{5040}&=\sum_{n=7}^{\infty}\frac{(n-5)\,a_{n-5}(n-6)\sin(1)\,t^n}{5040}\\\sum_{n=2}^{\infty}\left(-\frac{n\,t^{n+4}a_n(n-1)\cos(1)}{720}\right)&=\sum_{n=6}^{\infty}\left(-\frac{(n-4)\,a_{n-4}(n-5)\cos(1)\,t^n}{720}\right)\\\sum_{n=2}^{\infty}\left(-\frac{n\,t^{n+3}a_n(n-1)\sin(1)}{120}\right)&=\sum_{n=5}^{\infty}\left(-\frac{(n-3)\,a_{n-3}(n-4)\sin(1)\,t^n}{120}\right)\\\sum_{n=2}^{\infty}\frac{n\,t^{n+2}a_n(n-1)\cos(1)}{24}&=\sum_{n=4}^{\infty}\frac{(n-2)\,a_{n-2}(n-3)\cos(1)\,t^n}{24}\\\sum_{n=2}^{\infty}\frac{n\,t^{1+n}a_n(n-1)\sin(1)}{6}&=\sum_{n=3}^{\infty}\frac{(n-1)\,a_{n-1}(n-2)\sin(1)\,t^n}{6}\\\sum_{n=2}^{\infty}\left(-n\,t^{n-1}a_n(n-1)\sin(1)\right)&=\sum_{n=1}^{\infty}\left(-(1+n)\,a_{1+n}n\sin(1)\,t^n\right)\end{aligned}$$

$$\sum_{n=2}^{\infty} n\, t^{n-2} a_n (n-1) \cos(1) = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n) \cos(1)\, t^n$$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (1+n)\, a_{1+n} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\begin{aligned}
&\left(\sum_{n=8}^{\infty} \frac{(n-6)\, a_{n-6} (n-7) \cos(1)\, t^n}{40320}\right) \\
&+ \left(\sum_{n=7}^{\infty} \frac{(n-5)\, a_{n-5} (n-6) \sin(1)\, t^n}{5040}\right) \\
&+ \sum_{n=6}^{\infty} \left(-\frac{(n-4)\, a_{n-4} (n-5) \cos(1)\, t^n}{720}\right) \\
&\quad + \sum_{n=5}^{\infty} \left(-\frac{(n-3)\, a_{n-3} (n-4) \sin(1)\, t^n}{120}\right) \\
&\quad + \left(\sum_{n=4}^{\infty} \frac{(n-2)\, a_{n-2} (n-3) \cos(1)\, t^n}{24}\right) \\
&\quad + \left(\sum_{n=3}^{\infty} \frac{(n-1)\, a_{n-1} (n-2) \sin(1)\, t^n}{6}\right) \\
&\quad + \sum_{n=2}^{\infty} \left(-\frac{n a_n t^n \cos(1)\,(n-1)}{2}\right) + \sum_{n=1}^{\infty} \left(-(1+n)\, a_{1+n} n \sin(1)\, t^n\right) \\
&\quad + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n) \cos(1)\, t^n\right) \\
&\quad + \left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} t^n\right) + \left(\sum_{n=0}^{\infty} 5 a_n t^n\right) = 0
\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 \cos(1) + a_1 + 5a_0 = 0$$

$$a_2 = -\frac{5a_0 + a_1}{2\cos(1)}$$

$n = 1$ gives

$$-2a_2 \sin(1) + 6a_3 \cos(1) + 2a_2 + 5a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{5\sin(1)\, a_0 + \sin(1)\, a_1 + 5a_1 \cos(1) - 5a_0 - a_1}{6\cos(1)^2}$$

$n = 2$ gives

$$-a_2 \cos(1) - 6a_3 \sin(1) + 12a_4 \cos(1) + 3a_3 + 5a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{5a_0}{2} + \frac{a_1}{2} + \frac{(5\sin(1)\, a_0 + \sin(1)\, a_1 + 5a_1 \cos(1) - 5a_0 - a_1)\sin(1)}{\cos(1)^2} + 12a_4 \cos(1) - \frac{5\sin(1)\, a_0 + \sin(1)\, a_1 + 5a_1 \cos(1) - 5a_0 - a_1}{2\cos(1)^2} - \frac{5(5a_0 + a_1)}{2\cos(1)} = 0$$

Or

$$a_4 = -\frac{10\sin(1)^2 a_0 + 2\sin(1)^2 a_1 + 10\sin(1)\, a_1 \cos(1) + 5a_0 \cos(1)^2 + a_1 \cos(1)^2 - 15\sin(1)\, a_0 - 3\sin(1)}{24\cos(1)^3}$$

$n = 3$ gives

$$\frac{a_2 \sin(1)}{3} - 3a_3 \cos(1) - 12a_4 \sin(1) + 20a_5 \cos(1) + 4a_4 + 5a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{(5a_0 + a_1)\sin(1)}{6\cos(1)} + \frac{5\sin(1)\, a_0 + \sin(1)\, a_1 + 5a_1 \cos(1) - 5a_0 - a_1}{2\cos(1)} + \frac{(10\sin(1)^2 a_0 + 2\sin(1)^2 a_1 + 10\sin(1)\, a_1 \cos(1) + 5a_0 \cos(1)^2 + a_1 \cos(1)^2 - 15\sin(1)\, a_0 - 3\sin(1)\, a}{2\cos(1)^3} + 20a_5 \cos(1) - \frac{10\sin(1)^2 a_0 + 2\sin(1)^2 a_1 + 10\sin(1)\, a_1 \cos(1) + 5a_0 \cos(1)^2 + a_1 \cos(1)^2 - 15\sin(1)\, a_0 - 3\sin(1)\, a_1}{6\cos(1)^3} - \frac{5(5\sin(1)\, a_0 + \sin(1)\, a_1 + 5a_1 \cos(1) - 5a_0 - a_1)}{6\cos(1)^2} = 0$$

Or

$$a_5 = -\frac{30\sin(1)^3 a_0 + 6\sin(1)^3 a_1 + 30\sin(1)^2 a_1\cos(1) + 25\sin(1)\,a_0\cos(1)^2 + 5\sin(1)\,a_1\cos(1)^2 + 15a_1}{}$$

$n = 4$ gives

$$\frac{a_2\cos(1)}{12} + a_3\sin(1) - 6a_4\cos(1) - 20a_5\sin(1) + 30a_6\cos(1) + 5a_5 + 5a_4 = 0$$

Which after substituting earlier equations, simplifies to

$$\begin{aligned}
&-\frac{5a_0}{24} - \frac{a_1}{24} - \frac{(5\sin(1)\,a_0 + \sin(1)\,a_1 + 5a_1\cos(1) - 5a_0 - a_1)\sin(1)}{6\cos(1)^2}\\
&+\frac{10\sin(1)^2 a_0 + 2\sin(1)^2 a_1 + 10\sin(1)\,a_1\cos(1) + 5a_0\cos(1)^2 + a_1\cos(1)^2 - 15\sin(1)\,a_0 - 3\sin(1)\,a_1}{4\cos(1)^2}\\
&+\frac{\left(30\sin(1)^3 a_0 + 6\sin(1)^3 a_1 + 30\sin(1)^2 a_1\cos(1) + 25\sin(1)\,a_0\cos(1)^2 + 5\sin(1)\,a_1\cos(1)^2 + 15a_1 \mathrm{co}\right.}{}\\
&+ 30a_6\cos(1)\\
&-\frac{30\sin(1)^3 a_0 + 6\sin(1)^3 a_1 + 30\sin(1)^2 a_1\cos(1) + 25\sin(1)\,a_0\cos(1)^2 + 5\sin(1)\,a_1\cos(1)^2 + 15a_1 \mathrm{cos}}{}\\
&-\frac{5\left(10\sin(1)^2 a_0 + 2\sin(1)^2 a_1 + 10\sin(1)\,a_1\cos(1) + 5a_0\cos(1)^2 + a_1\cos(1)^2 - 15\sin(1)\,a_0 - 3\sin(1)\right.}{24\cos(1)^3}\\
&= 0
\end{aligned}$$

Or

$$a_6 = -\frac{5a_0 + a_1 - 50\sin(1)\,a_0 - 10\sin(1)\,a_1 - 20a_1\cos(1) + 120\sin(1)\,a_1\cos(1) + 175\sin(1)^2 a_0 + 35\sin(}{}$$

$n = 5$ gives

$$\begin{aligned}
-\frac{a_2\sin(1)}{60} + \frac{a_3\cos(1)}{4} + 2a_4\sin(1) - 10a_5\cos(1)\\
- 30a_6\sin(1) + 42a_7\cos(1) + 6a_6 + 5a_5 = 0
\end{aligned}$$

Which after substituting earlier equations, simplifies to

Expression too large to display

Or

$$a_7 = -\frac{-5a_0 - a_1 + 75\sin(1)\, a_0 + 15\sin(1)\, a_1 + 25a_1\cos(1) - 250\sin(1)\, a_1\cos(1) - 425\sin(1)^2 a_0 - 85\,\mathrm{si}}{}$$

$n = 6$ gives

$$-\frac{a_2\cos(1)}{360} - \frac{a_3\sin(1)}{20} + \frac{a_4\cos(1)}{2} + \frac{10a_5\sin(1)}{3} - 15a_6\cos(1) - 42a_7\sin(1) + 56a_8\cos(1) + 7a_7 + 5a_6 = 0$$

Which after substituting earlier equations, simplifies to

Expression too large to display

Or

$$a_8 = -\frac{5a_0 + a_1 - 105\sin(1)\, a_0 - 21\sin(1)\, a_1 - 30a_1\cos(1) + 450\sin(1)\, a_1\cos(1) + 875\sin(1)^2 a_0 + 175\,\mathrm{si}}{}$$

$n = 7$ gives

$$\frac{a_2\sin(1)}{2520} - \frac{a_3\cos(1)}{120} - \frac{a_4\sin(1)}{10} + \frac{5a_5\cos(1)}{6} + 5a_6\sin(1) - 21a_7\cos(1) - 56a_8\sin(1) + 72a_9\cos(1) + 8a_8 + 5a_7 = 0$$

Which after substituting earlier equations, simplifies to

Expression too large to display

Or

Expression too large to display

For $8 \leq n$, the recurrence equation is

$$\begin{aligned} &\frac{(n-6)\, a_{n-6}(n-7)\cos(1)}{40320} + \frac{(n-5)\, a_{n-5}(n-6)\sin(1)}{5040} \\ &- \frac{(n-4)\, a_{n-4}(n-5)\cos(1)}{720} - \frac{(n-3)\, a_{n-3}(n-4)\sin(1)}{120} \\ &+ \frac{(n-2)\, a_{n-2}(n-3)\cos(1)}{24} + \frac{(n-1)\, a_{n-1}(n-2)\sin(1)}{6} \\ &- \frac{n a_n(n-1)\cos(1)}{2} - (1+n)\, a_{1+n} n\sin(1) \\ &+ (n+2)\, a_{n+2}(1+n)\cos(1) + (1+n)\, a_{1+n} + 5a_n = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2}$$

$$= \frac{-201600a_n - 40320a_{1+n} - 40320a_{1+n}n - 240a_{n-5}\sin(1) + 4032a_{n-3}\sin(1) - 13440a_{n-1}\sin(1) - 42a}{}$$

$$\begin{aligned}
&= \frac{\left(20160\cos(1)\,n^2 - 20160\cos(1)\,n - 201600\right)a_n}{40320\cos(1)\left(n^2+3n+2\right)} \\
&\quad + \frac{\left(40320\sin(1)\,n^2 + 40320\sin(1)\,n - 40320n - 40320\right)a_{1+n}}{40320\cos(1)\left(n^2+3n+2\right)} \\
&\quad + \frac{\left(-\cos(1)\,n^2 + 13\cos(1)\,n - 42\cos(1)\right)a_{n-6}}{40320\cos(1)\left(n^2+3n+2\right)} \\
&\quad + \frac{\left(-8\sin(1)\,n^2 + 88\sin(1)\,n - 240\sin(1)\right)a_{n-5}}{40320\cos(1)\left(n^2+3n+2\right)} \\
&\quad + \frac{\left(56\cos(1)\,n^2 - 504\cos(1)\,n + 1120\cos(1)\right)a_{n-4}}{40320\cos(1)\left(n^2+3n+2\right)} \\
&\quad + \frac{\left(336\sin(1)\,n^2 - 2352\sin(1)\,n + 4032\sin(1)\right)a_{n-3}}{40320\cos(1)\left(n^2+3n+2\right)} \\
&\quad + \frac{\left(-1680\cos(1)\,n^2 + 8400\cos(1)\,n - 10080\cos(1)\right)a_{n-2}}{40320\cos(1)\left(n^2+3n+2\right)} \\
&\quad + \frac{\left(-6720\sin(1)\,n^2 + 20160\sin(1)\,n - 13440\sin(1)\right)a_{n-1}}{40320\cos(1)\left(n^2+3n+2\right)}
\end{aligned} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3t^3 + a_2t^2 + a_1t + a_0 + \dots
\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\text{Expression too large to display} + \dots$$

Collecting terms, the solution becomes

$$\text{Expression too large to display} \tag{3}$$

At $t = 0$ the solution above becomes

$$\text{Expression too large to display}$$

Replacing t in the above with the original independent variable $x$susing $t = x-1$ results in

Expression too large to display

Summary

The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5`[0, u]
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 860

```
Order:=8;
dsolve(cos(x)*diff(y(x),x$2)+diff(y(x),x)+5*y(x)=0,y(x),type='series',x=1);
```

Expression too large to display

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 1808

```
AsymptoticDSolveValue[Cos[x]*y''[x]+y'[x]+5*y[x]==0,y[x],{x,1,7}]
```

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1.29 problem 26 (a)

Internal problem ID [6578]
Internal file name [`OUTPUT/5826_Sunday_June_05_2022_03_56_12_PM_38211185/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 26 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - xy = 1}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy_0' + \frac{x^2}{2}\, f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (91) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (92) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= xy+1\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= xy'+y\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= yx^2+2y'+x\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= x^2y'+4xy+3\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= yx^3+6xy'+x^2+4y\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x}+\frac{\partial F_4}{\partial y}y'+\frac{\partial F_4}{\partial y'}F_4\\
&= y'x^3+9yx^2+10y'+8x\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x}+\frac{\partial F_5}{\partial y}y'+\frac{\partial F_5}{\partial y'}F_5\\
&= 12x^2y'+\left(x^4+28x\right)y+x^3+18
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= y(0) \\
F_2 &= 2y'(0) \\
F_3 &= 3 \\
F_4 &= 4y(0) \\
F_5 &= 10y'(0) \\
F_6 &= 18
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7\right) y'(0) + \frac{x^2}{2} + \frac{x^5}{40} + \frac{x^8}{2240} + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = x \left(\sum_{n=0}^{\infty} a_n x^n\right) + 1 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n\, =0}^{\infty} \left(-x^{1+n} a_n\right) = 1 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = 1 \tag{3}$$

$n = 0$ gives

$$(2a_2)\, 1 = 1$$
$$2a_2 = 1$$

Or

$$a_2 = \frac{1}{2}$$

For $1 \leq n$, the recurrence equation is

$$\left((n+2)\, a_{n+2}(1+n) - a_{n-1}\right) x^n = 1 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$(6a_3 - a_0)\, x = 0$$
$$6a_3 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$(12a_4 - a_1)\, x^2 = 0$$
$$12a_4 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(20a_5 - a_2)\, x^3 &= 0\\ 20a_5 - a_2 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{40}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(30a_6 - a_3)\, x^4 &= 0\\ 30a_6 - a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(42a_7 - a_4)\, x^5 &= 0\\ 42a_7 - a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

For $n = 6$ the recurrence equation gives

$$\begin{aligned}(56a_8 - a_5)\, x^6 &= 0\\ 56a_8 - a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{1}{2240}$$

For $n = 7$ the recurrence equation gives

$$\begin{aligned}(72a_9 - a_6)\, x^7 &= 0\\ 72a_9 - a_6 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_0}{12960}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n\\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \frac{1}{2}x^2 + \frac{1}{6}a_0x^3 + \frac{1}{12}a_1x^4 + \frac{1}{40}x^5 + \frac{1}{180}a_0x^6 + \frac{1}{504}a_1x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6\right) a_0 + \left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7\right) a_1 + \frac{x^2}{2} + \frac{x^5}{40} + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6\right) c_1 + \left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7\right) c_2 + \frac{x^2}{2} + \frac{x^5}{40} + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7\right) y'(0) + \frac{x^2}{2} + \frac{x^5}{40} + \frac{x^8}{2240} + O(x^8) \tag{1}$$

$$y = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6\right) c_1 + \left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7\right) c_2 + \frac{x^2}{2} + \frac{x^5}{40} + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1+\frac{1}{6}x^3+\frac{1}{180}x^6\right)y(0)+\left(x+\frac{1}{12}x^4+\frac{1}{504}x^7\right)y'(0)+\frac{x^2}{2}+\frac{x^5}{40}+\frac{x^8}{2240}+O\left(x^8\right)$$

Verified OK.

$$y = \left(1+\frac{1}{6}x^3+\frac{1}{180}x^6\right)c_1+\left(x+\frac{1}{12}x^4+\frac{1}{504}x^7\right)c_2+\frac{x^2}{2}+\frac{x^5}{40}+O\left(x^8\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
Order:=8;
dsolve(diff(y(x),x$2)-x*y(x)=1,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{6}x^3+\frac{1}{180}x^6\right)y(0)+\left(x+\frac{1}{12}x^4+\frac{1}{504}x^7\right)D(y)(0)+\frac{x^2}{2}+\frac{x^5}{40}+O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]-x*y[x]==1,y[x],{x,0,7}]
```

$$y(x) \to \frac{x^5}{40} + \frac{x^2}{2} + c_2\left(\frac{x^7}{504} + \frac{x^4}{12} + x\right) + c_1\left(\frac{x^6}{180} + \frac{x^3}{6} + 1\right)$$

1.30 problem 26 (b)

Internal problem ID [6579]
Internal file name [`OUTPUT/5827_Sunday_June_05_2022_03_56_14_PM_54011304/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 26 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4xy' - 4y = e^{x}}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (94) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (95) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 4xy' + 4y + \mathrm{e}^x \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 16x^2 y' + 16xy + 4x\,\mathrm{e}^x + 8y' + \mathrm{e}^x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 64y'x^3 + 64yx^2 + 16x^2\mathrm{e}^x + 80xy' + 4x\,\mathrm{e}^x + 48y + 13\,\mathrm{e}^x \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(256x^4 + 576x^2 + 128\right) y' + \left(64x^3 + 16x^2 + 116x + 17\right) \mathrm{e}^x + \left(256x^3 + 448x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(1024x^5 + 3584x^3 + 2112x\right) y' + \left(256x^4 + 64x^3 + 784x^2 + 148x + 261\right) \mathrm{e}^x + 1024y\left(x^4 + 3x^2 + \frac{15}{16}\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(4096x^6 + 20480x^4 + 22272x^2 + 3072\right) y' + \left(1024x^5 + 256x^4 + 4672x^3 + 976x^2 + 3828x + 409\right) \mathrm{e}^x + \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(16384x^7 + 110592x^5 + 189440x^3 + 71424x\right) y' + \left(4096x^6 + 1024x^5 + 25856x^4 + 5696x^3 + 37264x^2
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 1 + 4y(0) \\
F_1 &= 1 + 8y'(0) \\
F_2 &= 13 + 48y(0) \\
F_3 &= 17 + 128y'(0) \\
F_4 &= 261 + 960y(0) \\
F_5 &= 409 + 3072y'(0) \\
F_6 &= 7309 + 26880y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1 + 2x^2 + 2x^4 + \frac{4}{3}x^6 + \frac{2}{3}x^8\right) y(0) + \left(x + \frac{4}{3}x^3 + \frac{16}{15}x^5 + \frac{64}{105}x^7\right) y'(0) \\
& + \frac{x^2}{2} + \frac{x^3}{6} + \frac{13x^4}{24} + \frac{17x^5}{120} + \frac{29x^6}{80} + \frac{409x^7}{5040} + \frac{7309x^8}{40320} + O\left(x^8\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 4x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) + \mathrm{e}^x \tag{1}
$$

Expanding e^x as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
\mathrm{e}^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \ldots \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 4x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-4n\, x^n a_n) + \sum_{n=0}^{\infty} (-4a_n x^n) \tag{2}$$
$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-4n\, x^n a_n) + \sum_{n=0}^{\infty} (-4a_n x^n) \tag{3}$$
$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7$$

$n = 0$ gives

$$(2a_2 - 4a_0)\, x^0 = 1$$
$$2a_2 - 4a_0 = 1$$

$$a_2 = 2a_0 + \frac{1}{2}$$

For $1 \leq n$, the recurrence equation is

$$\left(\left(n+2\right) a_{n+2}(n+1)-4 n a_{n}-4 a_{n}\right) x^{n} = 1+x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{24}x^{4} + \frac{1}{120}x^{5}+\frac{1}{720}x^{6}+\frac{1}{5040}x^{7} \tag{4}$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned} \left(6 a_{3}-8 a_{1}\right) x &= x \\ 6 a_{3}-8 a_{1} &= 1 \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_{3} = \frac{1}{6}+\frac{4 a_{1}}{3}$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned} \left(12 a_{4}-12 a_{2}\right) x^{2} &= \frac{x^{2}}{2} \\ 12 a_{4}-12 a_{2} &= \frac{1}{2} \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_{4} = \frac{13}{24}+2 a_{0}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned} \left(20 a_{5}-16 a_{3}\right) x^{3} &= \frac{x^{3}}{6} \\ 20 a_{5}-16 a_{3} &= \frac{1}{6} \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_{5} = \frac{17}{120}+\frac{16 a_{1}}{15}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(30a_6 - 20a_4)\, x^4 &= \frac{x^4}{24} \\ 30a_6 - 20a_4 &= \frac{1}{24}\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{29}{80} + \frac{4a_0}{3}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(42a_7 - 24a_5)\, x^5 &= \frac{x^5}{120} \\ 42a_7 - 24a_5 &= \frac{1}{120}\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{409}{5040} + \frac{64a_1}{105}$$

For $n = 6$ the recurrence equation gives

$$\begin{aligned}(56a_8 - 28a_6)\, x^6 &= \frac{x^6}{720} \\ 56a_8 - 28a_6 &= \frac{1}{720}\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{7309}{40320} + \frac{2a_0}{3}$$

For $n = 7$ the recurrence equation gives

$$\begin{aligned}(72a_9 - 32a_7)\, x^7 &= \frac{x^7}{5040} \\ 72a_9 - 32a_7 &= \frac{1}{5040}\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{4363}{120960} + \frac{256a_1}{945}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1x + \left(2a_0 + \frac{1}{2}\right)x^2 + \left(\frac{1}{6} + \frac{4a_1}{3}\right)x^3 + \left(\frac{13}{24} + 2a_0\right)x^4 \\ + \left(\frac{17}{120} + \frac{16a_1}{15}\right)x^5 + \left(\frac{29}{80} + \frac{4a_0}{3}\right)x^6 + \left(\frac{409}{5040} + \frac{64a_1}{105}\right)x^7 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y = \left(1 + 2x^2 + 2x^4 + \frac{4}{3}x^6\right)a_0 + \left(x + \frac{4}{3}x^3 + \frac{16}{15}x^5 + \frac{64}{105}x^7\right)a_1 \\ + \frac{x^2}{2} + \frac{x^3}{6} + \frac{13x^4}{24} + \frac{17x^5}{120} + \frac{29x^6}{80} + \frac{409x^7}{5040} + O(x^8) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned} y = \left(1 + 2x^2 + 2x^4 + \frac{4}{3}x^6\right)c_1 + \left(x + \frac{4}{3}x^3 + \frac{16}{15}x^5 + \frac{64}{105}x^7\right)c_2 \\ + \frac{x^2}{2} + \frac{x^3}{6} + \frac{13x^4}{24} + \frac{17x^5}{120} + \frac{29x^6}{80} + \frac{409x^7}{5040} + O(x^8) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = \left(1 + 2x^2 + 2x^4 + \frac{4}{3}x^6 + \frac{2}{3}x^8\right)y(0) + \left(x + \frac{4}{3}x^3 + \frac{16}{15}x^5 + \frac{64}{105}x^7\right)y'(0) \\ + \frac{x^2}{2} + \frac{x^3}{6} + \frac{13x^4}{24} + \frac{17x^5}{120} + \frac{29x^6}{80} + \frac{409x^7}{5040} + \frac{7309x^8}{40320} + O(x^8) \end{aligned} \tag{1}$$

$$\begin{aligned} y = \left(1 + 2x^2 + 2x^4 + \frac{4}{3}x^6\right)c_1 + \left(x + \frac{4}{3}x^3 + \frac{16}{15}x^5 + \frac{64}{105}x^7\right)c_2 \\ + \frac{x^2}{2} + \frac{x^3}{6} + \frac{13x^4}{24} + \frac{17x^5}{120} + \frac{29x^6}{80} + \frac{409x^7}{5040} + O(x^8) \end{aligned} \tag{2}$$

Verification of solutions

$$y = \left(1+2x^2+2x^4+\frac{4}{3}x^6+\frac{2}{3}x^8\right)y(0)+\left(x+\frac{4}{3}x^3+\frac{16}{15}x^5+\frac{64}{105}x^7\right)y'(0)+\frac{x^2}{2}+\frac{x^3}{6}+\frac{13x^4}{24}+\frac{17x^5}{120}+\frac{29x^6}{80}+\frac{409x^7}{5040}+\frac{7309x^8}{40320}+O(x^8)$$

Verified OK.

$$y = \left(1+2x^2+2x^4+\frac{4}{3}x^6\right)c_1+\left(x+\frac{4}{3}x^3+\frac{16}{15}x^5+\frac{64}{105}x^7\right)c_2+\frac{x^2}{2}+\frac{x^3}{6}+\frac{13x^4}{24}+\frac{17x^5}{120}+\frac{29x^6}{80}+\frac{409x^7}{5040}+O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 56

```
Order:=8;
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)-4*y(x)=exp(x),y(x),type='series',x=0);
```

$$y(x) = \left(1+2x^2+2x^4+\frac{4}{3}x^6\right)y(0)+\left(x+\frac{4}{3}x^3+\frac{16}{15}x^5+\frac{64}{105}x^7\right)D(y)(0)+\frac{x^2}{2}+\frac{x^3}{6}+\frac{13x^4}{24}+\frac{17x^5}{120}+\frac{29x^6}{80}+\frac{409x^7}{5040}+O(x^8)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 94

```
AsymptoticDSolveValue[y''[x]-4*x*y'[x]-4*y[x]==Exp[x],y[x],{x,0,7}]
```

$$y(x) \to \frac{409x^7}{5040} + \frac{29x^6}{80} + \frac{17x^5}{120} + \frac{13x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + c_2\left(\frac{64x^7}{105} + \frac{16x^5}{15} + \frac{4x^3}{3} + x\right) + c_1\left(\frac{4x^6}{3} + 2x^4 + 2x^2 + 1\right)$$

1.31 problem 27

Internal problem ID [6580]
Internal file name [`OUTPUT/5828_Sunday_June_05_2022_03_56_17_PM_36089732/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + \sin(x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (97)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (98)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{\sin (x)\, y}{x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-y' \sin (x)\, x - y(\cos (x)\, x - \sin (x))}{x^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-2\cos (x)\, x^2 + 2\sin (x)\, x)\, y' + \left(\sin (x)^2 x + (x^2 - 2)\sin (x) + 2\cos (x)\, x\right) y}{x^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(\sin (x)^2 x^2 + (3x^3 - 6x)\sin (x) + 6\cos (x)\, x^2\right) y' + y\big(-4\sin (x)^2 x + (4\cos (x)\, x^2 - 3x^2 + 6)\sin (x)}{x^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{2\left(-3\sin (x)^2 x^2 + 3(\cos (x)\, x^3 - 2x^3 + 4x)\sin (x) + 2x^2\cos (x)\,(x^2 - 6)\right) y' - y\big(x^2 \sin (x)^3 + 3(x^3 -}{x^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{-5\Bigg(-\frac{23\left(x+\frac{\sin(x)}{23}\right)x^2\cos(x)^2}{5} + \left(4x^3 + \frac{46x^2\sin(x)}{5} - 24x\right)\cos (x) - \frac{36\sin(x)^2 x}{5} + \left(x^4 - \frac{59}{5}x^2 + 24\right)\sin (x) +}{} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{-6\big(-2x^3\cos (x)^3 + (36x^3 + 2x^2\sin (x))\cos (x)^2 + ((12x^3 - 56x)\sin (x) + x^4 - 18x^2 + 120)\, x\cos (x}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= \frac{4y(0)}{3} \\
F_3 &= 2y'(0) \\
F_4 &= -\frac{53y(0)}{15} \\
F_5 &= -\frac{19y'(0)}{3} \\
F_6 &= \frac{467y(0)}{35}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{1}{2}x^2 + \frac{1}{18}x^4 - \frac{53}{10800}x^6 + \frac{467}{1411200}x^8\right) y(0) \\
&+ \left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 - \frac{19}{15120}x^7\right) y'(0) + O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
xy'' + \sin(x)\, y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) x + \sin(x) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$\begin{aligned}\sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \dots \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) x + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the second term in (1) gives

$$\begin{gathered}\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) x + x \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^5}{120} \\ \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^7}{5040} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^9}{362880} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0\end{gathered}$$

Which simplifies to

$$\begin{gathered}\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) \\ + \left(\sum_{n=0}^{\infty} \frac{x^{n+5} a_n}{120}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+7} a_n}{5040}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+9} a_n}{362880}\right) = 0\end{gathered} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) = \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+5} a_n}{120} = \sum_{n=5}^{\infty} \frac{a_{n-5} x^n}{120}$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+7} a_n}{5040}\right) = \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} x^n}{5040}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+9} a_n}{362880} = \sum_{n=9}^{\infty} \frac{a_{n-9} x^n}{362880}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{multline}\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) + \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right) \\ + \left(\sum_{n=5}^{\infty} \frac{a_{n-5} x^n}{120}\right) + \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} x^n}{5040}\right) + \left(\sum_{n=9}^{\infty} \frac{a_{n-9} x^n}{362880}\right) = 0 \tag{3}\end{multline}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

$n = 2$ gives

$$12a_4 + a_2 - \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_0}{18}$$

$n = 3$ gives

$$20a_5 + a_3 - \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_1}{60}$$

$n = 4$ gives

$$30a_6 + a_4 - \frac{a_2}{6} + \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = -\frac{53a_0}{10800}$$

$n = 5$ gives

$$42a_7 + a_5 - \frac{a_3}{6} + \frac{a_1}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{19a_1}{15120}$$

$n = 6$ gives

$$56a_8 + a_6 - \frac{a_4}{6} + \frac{a_2}{120} - \frac{a_0}{5040} = 0$$

Which after substituting earlier equations, simplifies to

$$a_8 = \frac{467a_0}{1411200}$$

For $9 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} n + a_{n-1} - \frac{a_{n-3}}{6} + \frac{a_{n-5}}{120} - \frac{a_{n-7}}{5040} + \frac{a_{n-9}}{362880} = 0 \tag{4}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n\\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{6}a_1x^3 + \frac{1}{18}a_0x^4 + \frac{1}{60}a_1x^5 - \frac{53}{10800}a_0x^6 - \frac{19}{15120}a_1x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{18}x^4 - \frac{53}{10800}x^6\right)a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 - \frac{19}{15120}x^7\right)a_1 + O\left(x^8\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{18}x^4 - \frac{53}{10800}x^6\right)c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 - \frac{19}{15120}x^7\right)c_2 + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned}y = &\left(1 - \frac{1}{2}x^2 + \frac{1}{18}x^4 - \frac{53}{10800}x^6 + \frac{467}{1411200}x^8\right)y(0)\\ &+ \left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 - \frac{19}{15120}x^7\right)y'(0) + O\left(x^8\right)\end{aligned} \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{18}x^4 - \frac{53}{10800}x^6\right)c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 - \frac{19}{15120}x^7\right)c_2 + O\left(x^8\right) \tag{2}$$

Verification of solutions

$$\begin{aligned}y = &\left(1 - \frac{1}{2}x^2 + \frac{1}{18}x^4 - \frac{53}{10800}x^6 + \frac{467}{1411200}x^8\right)y(0)\\ &+ \left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 - \frac{19}{15120}x^7\right)y'(0) + O\left(x^8\right)\end{aligned}$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{18}x^4 - \frac{53}{10800}x^6\right)c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 - \frac{19}{15120}x^7\right)c_2 + O\left(x^8\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
   -> trying with_periodic_functions in the coefficients
      --- Trying Lie symmetry methods, 2nd order ---
      `, `-> Computing symmetries using: way = 5`[0, y]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(x*diff(y(x),x$2)+sin(x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{2}x^2+\frac{1}{18}x^4-\frac{53}{10800}x^6\right)y(0) + \left(x-\frac{1}{6}x^3+\frac{1}{60}x^5-\frac{19}{15120}x^7\right)D(y)(0)+O(x^8)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[x*y''[x]+Sin[x]*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{19x^7}{15120}+\frac{x^5}{60}-\frac{x^3}{6}+x\right)+c_1\left(-\frac{53x^6}{10800}+\frac{x^4}{18}-\frac{x^2}{2}+1\right)$$

1.32 problem 28

Internal problem ID [6581]
Internal file name [`OUTPUT/5829_Sunday_June_05_2022_03_56_19_PM_77522733/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + 5xy' + y\sqrt{x} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + 5xy' + y\sqrt{x} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 5x$$
$$q(x) = \sqrt{x}$$

Table 25: Table $p(x), q(x)$ singularites.

$p(x) = 5x$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \sqrt{x}$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
```

✗ Solution by Maple

```
Order:=8;
dsolve(diff(y(x),x$2)+5*x*diff(y(x),x)+sqrt(x)*y(x)=0,y(x),type='series',x=0);
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
AsymptoticDSolveValue[y''[x]+5*x*y'[x]+Sqrt[x]*y[x]==0,y[x],{x,0,7}]
```

Not solved

1.33 problem 29 (a)

Internal problem ID [6582]
Internal file name [`OUTPUT/5830_Sunday_June_05_2022_03_56_25_PM_16382805/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 29 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + xy' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (100) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (101) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y - xy' \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x^2 y' + xy - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -y' x^3 - y x^2 + 5xy' + 3y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 9x^2 + 8\right) y' + yx\left(x^2 - 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 14x^3 - 33x\right) y' - y\left(x^4 - 12x^2 + 15\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(x^6 - 20x^4 + 87x^2 - 48\right) y' + yx\left(x^4 - 18x^2 + 57\right) \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(-x^7 + 27x^5 - 185x^3 + 279x\right) y' - y\left(x^2 - 7\right)\left(x^4 - 18x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -2y'(0) \\
F_2 &= 3y(0) \\
F_3 &= 8y'(0) \\
F_4 &= -15y(0) \\
F_5 &= -48y'(0) \\
F_6 &= 105y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) - x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{105}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + 7a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{384}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + 8a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{945}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 - \frac{1}{48} a_0 x^6 - \frac{1}{105} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) a_0 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8\right) y(0) \\ & + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7\right) y'(0) + O(x^8) \end{aligned} \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7\right) c_2 + O(x^8)$$

Verified OK.

1.33.1 Maple step by step solution

Let's solve

$y'' = -y - xy'$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + xy' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$

  ○ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\left(a_{k+2}(k+2) + a_k\right) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{2}x^{2}+\frac{1}{8}x^{4}-\frac{1}{48}x^{6}\right)y(0)+\left(x-\frac{1}{3}x^{3}+\frac{1}{15}x^{5}-\frac{1}{105}x^{7}\right)D(y)\left(0\right)+O\left(x^{8}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{x^7}{105}+\frac{x^5}{15}-\frac{x^3}{3}+x\right)+c_1\left(-\frac{x^6}{48}+\frac{x^4}{8}-\frac{x^2}{2}+1\right)$$

1.34 problem 30 (b)

Internal problem ID [6583]
Internal file name [`OUTPUT/5831_Sunday_June_05_2022_03_56_27_PM_41240378/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. Section 6.2 SOLUTIONS ABOUT ORDINARY POINTS. EXERCISES 6.2. Page 246
Problem number: 30 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y\cos(x) = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

1.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \cos(x) \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y\cos(x) = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (103)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (104)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -y\cos(x)\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= -\cos(x)\,y' + \sin(x)\,y\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= 2\sin(x)\,y' + \cos(x)\,(\cos(x)+1)\,y\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= \left(\cos(x)^2 + 3\cos(x)\right)y' - 4\sin(x)\left(\cos(x)+\frac{1}{4}\right)y\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= 2\sin(x)\,(-2-3\cos(x))\,y' - y\left(\cos(x)^3 + 11\cos(x)^2 + \cos(x) - 4\right)\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\
&= \left(-\cos(x)^3 - 23\cos(x)^2 - 5\cos(x) + 10\right)y' + 9\left(\cos(x)^2 + \frac{26\cos(x)}{9} + \frac{1}{9}\right)\sin(x)\,y\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\
&= 12\sin(x)\left(\cos(x)^2 + 6\cos(x) + \frac{1}{2}\right)y' + \left(\cos(x)^4 + 50\cos(x)^3 + 57\cos(x)^2 - 27\cos(x) - 26\right)y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=1$ and

$y'(0) = 1$ gives

$$\begin{aligned}
F_0 &= -1 \\
F_1 &= -1 \\
F_2 &= 2 \\
F_3 &= 4 \\
F_4 &= -9 \\
F_5 &= -19 \\
F_6 &= 55
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{30} - \frac{x^6}{80} - \frac{19x^7}{5040} + \frac{11x^8}{8064} + O(x^8)$$

$$y = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{30} - \frac{x^6}{80} - \frac{19x^7}{5040} + \frac{11x^8}{8064} + O(x^8)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) \cos(x) \tag{1}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$\begin{aligned}
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \dots \\
&= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8
\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8\right) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the second term in (1) gives

$$\begin{gathered}\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 1 \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^4}{24} \\ \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^6}{720} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^8}{40320} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0\end{gathered}$$

Which simplifies to

$$\begin{gathered}\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+2} a_n}{2}\right) \\ + \left(\sum_{n=0}^{\infty} \frac{x^{n+4} a_n}{24}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+6} a_n}{720}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+8} a_n}{40320}\right) = 0\end{gathered} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n \\ \sum_{n=0}^{\infty} \left(-\frac{x^{n+2} a_n}{2}\right) &= \sum_{n=2}^{\infty} \left(-\frac{a_{n-2} x^n}{2}\right) \\ \sum_{n=0}^{\infty} \frac{x^{n+4} a_n}{24} &= \sum_{n=4}^{\infty} \frac{a_{n-4} x^n}{24} \\ \sum_{n=0}^{\infty} \left(-\frac{x^{n+6} a_n}{720}\right) &= \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^n}{720}\right) \\ \sum_{n=0}^{\infty} \frac{x^{n+8} a_n}{40320} &= \sum_{n=8}^{\infty} \frac{a_{n-8} x^n}{40320}\end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^{n}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)+\sum_{n =2}^{\infty}\left(-\frac{a_{n-2} x^{n}}{2}\right) \\ +\left(\sum_{n=4}^{\infty} \frac{a_{n-4} x^{n}}{24}\right)+\sum_{n =6}^{\infty}\left(-\frac{a_{n-6} x^{n}}{720}\right)+\left(\sum_{n=8}^{\infty} \frac{a_{n-8} x^{n}}{40320}\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

$n = 2$ gives

$$12a_4 + a_2 - \frac{a_0}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_0}{12}$$

$n = 3$ gives

$$20a_5 + a_3 - \frac{a_1}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_1}{30}$$

$n = 4$ gives

$$30a_6 + a_4 - \frac{a_2}{2} + \frac{a_0}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = -\frac{a_0}{80}$$

$n = 5$ gives

$$42a_7 + a_5 - \frac{a_3}{2} + \frac{a_1}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{19a_1}{5040}$$

$n = 6$ gives

$$56a_8 + a_6 - \frac{a_4}{2} + \frac{a_2}{24} - \frac{a_0}{720} = 0$$

Which after substituting earlier equations, simplifies to

$$a_8 = \frac{11a_0}{8064}$$

$n = 7$ gives

$$72a_9 + a_7 - \frac{a_5}{2} + \frac{a_3}{24} - \frac{a_1}{720} = 0$$

Which after substituting earlier equations, simplifies to

$$a_9 = \frac{29a_1}{72576}$$

For $8 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_n - \frac{a_{n-2}}{2} + \frac{a_{n-4}}{24} - \frac{a_{n-6}}{720} + \frac{a_{n-8}}{40320} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{40320a_n - 20160a_{n-2} + 1680a_{n-4} - 56a_{n-6} + a_{n-8}}{40320\,(n+2)\,(n+1)} \\
&= -\frac{a_n}{(n+2)\,(n+1)} - \frac{a_{n-8}}{40320\,(n+2)\,(n+1)} \\
&\quad + \frac{a_{n-6}}{720\,(n+2)\,(n+1)} - \frac{a_{n-4}}{24\,(n+2)\,(n+1)} + \frac{a_{n-2}}{2\,(n+2)\,(n+1)}
\end{aligned} \tag{5}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{6}a_1x^3 + \frac{1}{12}a_0x^4 + \frac{1}{30}a_1x^5 - \frac{1}{80}a_0x^6 - \frac{19}{5040}a_1x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{80}x^6\right)a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \frac{19}{5040}x^7\right)a_1 + O\left(x^8\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{80}x^6\right)c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \frac{19}{5040}x^7\right)c_2 + O\left(x^8\right)$$

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{80} + x - \frac{x^3}{6} + \frac{x^5}{30} - \frac{19x^7}{5040} + O\left(x^8\right)$$

Summary

The solution(s) found are the following

$$y = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{30} - \frac{x^6}{80} - \frac{19x^7}{5040} + \frac{11x^8}{8064} + O\left(x^8\right) \tag{1}$$

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{80} + x - \frac{x^3}{6} + \frac{x^5}{30} - \frac{19x^7}{5040} + O\left(x^8\right) \tag{2}$$

Verification of solutions

$$y = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{30} - \frac{x^6}{80} - \frac{19x^7}{5040} + \frac{11x^8}{8064} + O(x^8)$$

Verified OK.

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{80} + x - \frac{x^3}{6} + \frac{x^5}{30} - \frac{19x^7}{5040} + O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         Equivalence transformation and function parameters: {z = 1/2*t+1/2}, {kappa = -20,
         <- Equivalence to the rational form of Mathieu ODE successful
      <- Mathieu successful
   <- special function solution successful
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      t*u(t)-t*diff(u(t),t)+(-t^2+1)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=8;
dsolve([diff(y(x),x$2)+cos(x)*y(x)=0,y(0) = 1, D(y)(0) = 1],y(x),type='series',x=0);
```

$$y(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 - \frac{1}{80}x^6 - \frac{19}{5040}x^7 + \mathrm{O}\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 48

```
AsymptoticDSolveValue[{y''[x]+Cos[x]*y[x]==0,{y[0]==1,y'[0]==1}},y[x],{x,0,7}]
```

$$y(x) \to -\frac{19x^7}{5040} - \frac{x^6}{80} + \frac{x^5}{30} + \frac{x^4}{12} - \frac{x^3}{6} - \frac{x^2}{2} + x + 1$$

2 CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255

2.1 problem 1

Internal problem ID [6584]
Internal file name [`OUTPUT/5832_Sunday_June_05_2022_03_56_31_PM_17749144/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + 4x^2y' + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^3y'' + 4x^2y' + 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{3}{x^3} \end{aligned}$$

Table 27: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3}{x^3}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✗ Solution by Maple

```
Order:=8;
dsolve(x^3*diff(y(x),x$2)+4*x^2*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.089 (sec). Leaf size: 374

```
AsymptoticDSolveValue[x^3*y''[x]+4*x^2*y'[x]+3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_1 e^{-\frac{2i\sqrt{3}}{\sqrt{x}}}\left(-\frac{9447234753831875i\sqrt{3}x^{13/2}}{4611686018427387904} + \frac{3806522094375i\sqrt{3}x^{11/2}}{4503599627370496} - \frac{14315125825ix^{9/2}}{8796093022208\sqrt{3}} + \frac{8083075ix^{7/2}}{4294967296\sqrt{3}} - \frac{15015i\sqrt{3}x^{5/2}}{8388608} + 3\right.}{}$$

$$+ \frac{c_2 e^{\frac{2i\sqrt{3}}{\sqrt{x}}}\left(\frac{9447234753831875i\sqrt{3}x^{13/2}}{4611686018427387904} - \frac{3806522094375i\sqrt{3}x^{11/2}}{4503599627370496} + \frac{14315125825ix^{9/2}}{8796093022208\sqrt{3}} - \frac{8083075ix^{7/2}}{4294967296\sqrt{3}} + \frac{15015i\sqrt{3}x^{5/2}}{8388608} - 385\right.}{}$$

2.2 problem 2

Internal problem ID [6585]
Internal file name [`OUTPUT/5833_Sunday_June_05_2022_03_56_32_PM_97516078/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x+3)^2 y''-y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+6x^2+9x\right)y''-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=0\\ q(x)&=-\frac{1}{(x+3)^2 x}\end{aligned}$$

Table 28: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{1}{(x+3)^2 x}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-3, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x^2 + 6x + 9) y'' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x^2 + 6x + 9) \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) \\ &+ \left(\sum_{n=0}^{\infty} 9x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} (-a_n x^{n+r}) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) &= \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r)\, x^{n+r-1} \\
\sum_{n=0}^{\infty} 6 x^{n+r} a_n (n+r)(n+r-1) &= \sum_{n=1}^{\infty} 6 a_{n-1} (n+r-1)(n+r-2)\, x^{n+r-1} \\
\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r)\, x^{n+r-1}\right) \\
&+ \left(\sum_{n=1}^{\infty} 6 a_{n-1} (n+r-1)(n+r-2)\, x^{n+r-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} 9 x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9 x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$9 x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$9 x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$9 x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$9 r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$9x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-6r^2 + 6r + 1}{9r(1+r)}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-2}(n+r-2)(n-3+r) + 6a_{n-1}(n+r-1)(n+r-2) \\ &+ 9a_n(n+r)(n+r-1) - a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2a_{n-2} + 6n^2a_{n-1} + 2nra_{n-2} + 12nra_{n-1} + r^2a_{n-2} + 6r^2a_{n-1} - 5na_{n-2} - 18na_{n-1} - 5ra_{n-2} - 18}{9\left(n+r\right)\left(n+r-1\right)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{\left(-a_{n-2} - 6a_{n-1}\right)n^2 + \left(3a_{n-2} + 6a_{n-1}\right)n - 2a_{n-2} + a_{n-1}}{9n\left(1+n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$	$\frac{1}{18}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{27r^4 - 39r^2 + 1}{81r\left(1+r\right)^2\left(2+r\right)}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{11}{972}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$	$\frac{1}{18}$
a_2	$\frac{27r^4-39r^2+1}{81r(1+r)^2(2+r)}$	$-\frac{11}{972}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-108r^6 - 324r^5 - 18r^4 + 504r^3 + 270r^2 - 36r - 11}{729r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{277}{104976}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$	$\frac{1}{18}$
a_2	$\frac{27r^4-39r^2+1}{81r(1+r)^2(2+r)}$	$-\frac{11}{972}$
a_3	$\frac{-108r^6-324r^5-18r^4+504r^3+270r^2-36r-11}{729r(1+r)^2(2+r)^2(3+r)}$	$\frac{277}{104976}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{405r^8 + 3240r^7 + 8370r^6 + 4860r^5 - 10962r^4 - 15768r^3 - 4299r^2 + 1338r + 277}{6561r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{12539}{18895680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$	$\frac{1}{18}$
a_2	$\frac{27r^4-39r^2+1}{81r(1+r)^2(2+r)}$	$-\frac{11}{972}$
a_3	$\frac{-108r^6-324r^5-18r^4+504r^3+270r^2-36r-11}{729r(1+r)^2(2+r)^2(3+r)}$	$\frac{277}{104976}$
a_4	$\frac{405r^8+3240r^7+8370r^6+4860r^5-10962r^4-15768r^3-4299r^2+1338r+277}{6561r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{12539}{18895680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1458r^{10} - 21870r^9 - 128385r^8 - 359640r^7 - 413478r^6 + 167184r^5 + 890541r^4 + 735966r^3 + 1107}{59049r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)^2\left(5+r\right)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{893821}{5101833600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$	$\frac{1}{18}$
a_2	$\frac{27r^4-39r^2+1}{81r(1+r)^2(2+r)}$	$-\frac{11}{972}$
a_3	$\frac{-108r^6-324r^5-18r^4+504r^3+270r^2-36r-11}{729r(1+r)^2(2+r)^2(3+r)}$	$\frac{277}{104976}$
a_4	$\frac{405r^8+3240r^7+8370r^6+4860r^5-10962r^4-15768r^3-4299r^2+1338r+277}{6561r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{12539}{18895680}$
a_5	$\frac{-1458r^{10}-21870r^9-128385r^8-359640r^7-413478r^6+167184r^5+890541r^4+735966r^3+110706r^2-73206r-12539}{59049r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$\frac{893821}{5101833600}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{5103r^{12} + 122472r^{11} + 1236627r^{10} + 6769980r^9 + 21315231r^8 + 36006768r^7 + 17587773r^6 - 44341}{531441r\,(1+r)^2\,(2+r)^2\,(3+r)^2\,(4+r)^2\,(5}$$

Which for the root $r = 1$ becomes

$$a_6 = -\frac{13183337}{275499014400}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$
a_2	$\frac{27r^4-39r^2+1}{81r(1+r)^2(2+r)}$
a_3	$\frac{-108r^6-324r^5-18r^4+504r^3+270r^2-36r-11}{729r(1+r)^2(2+r)^2(3+r)}$
a_4	$\frac{405r^8+3240r^7+8370r^6+4860r^5-10962r^4-15768r^3-4299r^2+1338r+277}{6561r(1+r)^2(2+r)^2(3+r)^2(4+r)}$
a_5	$\frac{-1458r^{10}-21870r^9-128385r^8-359640r^7-413478r^6+167184r^5+890541r^4+735966r^3+110706r^2-73206r-12539}{59049r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$
a_6	$\frac{5103r^{12}+122472r^{11}+1236627r^{10}+6769980r^9+21315231r^8+36006768r^7+17587773r^6-44341668r^5-84028203r^4-50648472r^3-3083067}{531441r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-17496r^{14} - 612360r^{13} - 9369108r^{12} - 82056240r^{11} - 451125612r^{10} - 1595197800r^9 - 350174226}{4782969r\,(1+r)^2\,(}$$

Which for the root $r = 1$ becomes

$$a_7 = \frac{265861081}{19835929036800}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$
a_2	$\frac{27r^4-39r^2+1}{81r(1+r)^2(2+r)}$
a_3	$\frac{-108r^6-324r^5-18r^4+504r^3+270r^2-36r-11}{729r(1+r)^2(2+r)^2(3+r)}$
a_4	$\frac{405r^8+3240r^7+8370r^6+4860r^5-10962r^4-15768r^3-4299r^2+1338r+277}{6561r(1+r)^2(2+r)^2(3+r)^2(4+r)}$
a_5	$\frac{-1458r^{10}-21870r^9-128385r^8-359640r^7-413478r^6+167184r^5+890541r^4+735966r^3+110706r^2-73206r-12539}{59049r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$
a_6	$\frac{5103r^{12}+122472r^{11}+1236627r^{10}+6769980r^9+21315231r^8+36006768r^7+17587773r^6-44341668r^5-84028203r^4-50648472r^3-3083067}{531441r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$
a_7	$\frac{-17496r^{14}-612360r^{13}-9369108r^{12}-82056240r^{11}-451125612r^{10}-1595197800r^9-3501742266r^8-3965575320r^7+450321984r^6+795063}{4782969r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)^2(7+r)}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\big(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \ldots\big) \\ &= x\Bigg(1 + \frac{x}{18} - \frac{11x^2}{972} + \frac{277x^3}{104976} - \frac{12539x^4}{18895680} + \frac{893821x^5}{5101833600} - \frac{13183337x^6}{275499014400} + \frac{265861081x^7}{19835929036800} + \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-6r^2 + 6r + 1}{9r\,(1 + r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{-6r^2 + 6r + 1}{9r\,(1 + r)} &= \lim_{r\to 0} \frac{-6r^2 + 6r + 1}{9r\,(1 + r)} \\ &= \mathit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2) (-1 + n + r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x(x^2 + 6x + 9)\, y'' - y = 0$ gives

$$\begin{aligned}
&x(x^2 + 6x + 9) \left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2}\right)\right) - C y_1(x) \ln(x) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}
&\left(\left(x(x^2 + 6x + 9)\, y_1''(x) - y_1(x)\right) \ln(x) + x(x^2 + 6x + 9) \left(\frac{2 y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right) C \\
&+ x(x^2 + 6x + 9) \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2}\right)\right) \\
&- \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x\left(x^2+6x+9\right) y_1''(x)-y_1(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&x\left(x^2+6x+9\right)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)C\\&+x\left(x^2+6x+9\right)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\&-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned} \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2x(x+3)^2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)-(x+3)^2\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} + \frac{x^2(x+3)^2\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x}=0 \tag{9}$$

Since $r_1=1$ and $r_2=0$ then the above becomes

$$\frac{\left(2x(x+3)^2\left(\sum_{n=0}^{\infty}x^{n}a_n(1+n)\right)-(x+3)^2\left(\sum_{n=0}^{\infty}a_n x^{1+n}\right)\right)C}{x} + \frac{x^2(x+3)^2\left(\sum_{n=0}^{\infty}x^{n-2}b_n n(n-1)\right)-\left(\sum_{n=0}^{\infty}b_n x^{n}\right)x}{x}=0 \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (1+n)\right) + \left(\sum_{n=0}^{\infty} 12C\, x^{1+n} a_n (1+n)\right) \\
&+ \left(\sum_{n=0}^{\infty} 18C\, x^{n} a_n (1+n)\right) + \sum_{n\,=0}^{\infty} \left(-C\, x^{n+2} a_n\right) + \sum_{n\,=0}^{\infty} \left(-6C\, x^{1+n} a_n\right) \\
&\quad+ \sum_{n\,=0}^{\infty} \left(-9C a_n x^n\right) + \left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 6x^n b_n n(n-1)\right) \\
&\quad+ \left(\sum_{n=0}^{\infty} 9n\, x^{n-1} b_n (n-1)\right) + \sum_{n\,=0}^{\infty} \left(-b_n x^n\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\sum_{n\,=0}^{\infty} 2C\, x^{n+2} a_n (1+n) = \sum_{n=3}^{\infty} 2C a_{n-3} (n-2)\, x^{n-1}
$$

$$
\sum_{n\,=0}^{\infty} 12C\, x^{1+n} a_n (1+n) = \sum_{n=2}^{\infty} 12C a_{n-2} (n-1)\, x^{n-1}
$$

$$
\sum_{n\,=0}^{\infty} 18C\, x^{n} a_n (1+n) = \sum_{n=1}^{\infty} 18C a_{n-1} n\, x^{n-1}
$$

$$
\sum_{n\,=0}^{\infty} \left(-C\, x^{n+2} a_n\right) = \sum_{n=3}^{\infty} \left(-C a_{n-3} x^{n-1}\right)
$$

$$
\sum_{n\,=0}^{\infty} \left(-6C\, x^{1+n} a_n\right) = \sum_{n=2}^{\infty} \left(-6C a_{n-2} x^{n-1}\right)
$$

$$
\sum_{n\,=0}^{\infty} \left(-9C a_n x^n\right) = \sum_{n=1}^{\infty} \left(-9C a_{n-1} x^{n-1}\right)
$$

$$
\sum_{n\,=0}^{\infty} n\, x^{1+n} b_n (n-1) = \sum_{n=2}^{\infty} (n-2)\, b_{n-2} (n-3)\, x^{n-1}
$$

$$
\sum_{n\,=0}^{\infty} 6x^n b_n n(n-1) = \sum_{n=1}^{\infty} 6(n-1)\, b_{n-1} (n-2)\, x^{n-1}
$$

$$\sum_{n=0}^{\infty}\left(-b_n x^n\right)=\sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\left(\sum_{n=3}^{\infty} 2 C a_{n-3}(n-2)\, x^{n-1}\right)+\left(\sum_{n=2}^{\infty} 12 C a_{n-2}(n-1)\, x^{n-1}\right) \\
&+\left(\sum_{n=1}^{\infty} 18 C a_{n-1} n\, x^{n-1}\right)+\sum_{n=3}^{\infty}\left(-C a_{n-3} x^{n-1}\right) \\
&\quad+\sum_{n=2}^{\infty}\left(-6 C a_{n-2} x^{n-1}\right)+\sum_{n=1}^{\infty}\left(-9 C a_{n-1} x^{n-1}\right) \\
&\quad+\left(\sum_{n=2}^{\infty}(n-2)\, b_{n-2}(n-3)\, x^{n-1}\right)+\left(\sum_{n=1}^{\infty} 6(n-1)\, b_{n-1}(n-2)\, x^{n-1}\right) \\
&\quad+\left(\sum_{n=0}^{\infty} 9 n\, x^{n-1} b_n(n-1)\right)+\sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right)=0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$9C-1=0$$

Which is solved for C. Solving for C gives

$$C=\frac{1}{9}$$

For $n=2$, Eq (2B) gives

$$\left(6 a_0+27 a_1\right) C-b_1+18 b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{5}{6}+18 b_2=0$$

Solving the above for b_2 gives

$$b_2=-\frac{5}{108}$$

For $n = 3$, Eq (2B) gives

$$(a_0 + 18a_1 + 45a_2)\, C + 11b_2 + 54b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{167}{486} + 54b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{167}{26244}$$

For $n = 4$, Eq (2B) gives

$$(3a_1 + 30a_2 + 63a_3)\, C + 2b_2 + 35b_3 + 108b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{13583}{104976} + 108b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{13583}{11337408}$$

For $n = 5$, Eq (2B) gives

$$(5a_2 + 42a_3 + 81a_4)\, C + 6b_3 + 71b_4 + 180b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1327279}{28343520} + 180b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{1327279}{5101833600}$$

For $n = 6$, Eq (2B) gives

$$(7a_3 + 54a_4 + 99a_5)\, C + 12b_4 + 119b_5 + 270b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{21146863}{1275458400} + 270b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = -\frac{21146863}{344373768000}$$

For $n = 7$, Eq (2B) gives

$$(9a_4 + 66a_5 + 117a_6)\,C + 20b_5 + 179b_6 + 378b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{2658363911}{459165024000} + 378b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = \frac{379766273}{24794911296000}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = \frac{1}{9}$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = \frac{1}{9}\Bigg(x\Bigg(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}-\frac{13183337x^6}{275499014400}\\ +\frac{265861081x^7}{19835929036800}+O(x^8)\Bigg)\Bigg)\ln(x)+1-\frac{5x^2}{108}+\frac{167x^3}{26244}\\ -\frac{13583x^4}{11337408}+\frac{1327279x^5}{5101833600}-\frac{21146863x^6}{344373768000}+\frac{379766273x^7}{24794911296000}+O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x\Bigg(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}-\frac{13183337x^6}{275499014400}\\ &\qquad +\frac{265861081x^7}{19835929036800}+O(x^8)\Bigg)\\ &\quad +c_2\Bigg(\frac{1}{9}\Bigg(x\Bigg(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}-\frac{13183337x^6}{275499014400}+\frac{265861081x^7}{19835929036800}\\ &\qquad +1-\frac{5x^2}{108}+\frac{167x^3}{26244}-\frac{13583x^4}{11337408}+\frac{1327279x^5}{5101833600}-\frac{21146863x^6}{344373768000}\\ &\qquad +\frac{379766273x^7}{24794911296000}+O(x^8)\Bigg) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}-\frac{13183337x^6}{275499014400}\right. \\
&\qquad \left.+\frac{265861081x^7}{19835929036800}+O(x^8)\right) \\
&\quad + c_2\left(\frac{x\left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}-\frac{13183337x^6}{275499014400}+\frac{265861081x^7}{19835929036800}+O(x^8)\right)\ln(x)}{9}\right. \\
&\quad +1-\frac{5x^2}{108}+\frac{167x^3}{26244}-\frac{13583x^4}{11337408}+\frac{1327279x^5}{5101833600}-\frac{21146863x^6}{344373768000}+\frac{379766273x^7}{24794911296000} \\
&\qquad \left.+O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}-\frac{13183337x^6}{275499014400}\right. \\
&\qquad \left.+\frac{265861081x^7}{19835929036800}+O(x^8)\right) \\
&\quad + c_2\left(\frac{x\left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}-\frac{13183337x^6}{275499014400}+\frac{265861081x^7}{19835929036800}+O(x^8)\right)\ln(x)}{9}\right. \\
&\quad +1-\frac{5x^2}{108}+\frac{167x^3}{26244}-\frac{13583x^4}{11337408}+\frac{1327279x^5}{5101833600}-\frac{21146863x^6}{344373768000}+\frac{379766273x^7}{24794911296000} \\
&\qquad \left.+O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x \left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}-\frac{13183337x^6}{275499014400}+\frac{265861081x^7}{19835929036800}+O\left(x^8\right)\right) + c_2 \left(\frac{x\left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}-\frac{13183337x^6}{275499014400}+\frac{265861081x^7}{19835929036800}+O\left(x^8\right)\right)\ln\left(x\right)}{9} + 1-\frac{5x^2}{108}+\frac{167x^3}{26244}-\frac{13583x^4}{11337408}+\frac{1327279x^5}{5101833600}-\frac{21146863x^6}{344373768000}+\frac{379766273x^7}{24794911296000}+O\left(x^8\right)\right)$$

Verified OK.

2.2.1 Maple step by step solution

Let's solve

$x(x^2+6x+9)\,y''-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{x(x^2+6x+9)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{y}{x(x^2+6x+9)}=0$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x)=0, P_3(x)=-\frac{1}{x(x^2+6x+9)}\right]$

  ○ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left.\left((x+3)\cdot P_2(x)\right)\right|_{x=-3}=0$

  ○ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left.\left((x+3)^2\cdot P_3(x)\right)\right|_{x=-3}=\frac{1}{3}$

  - $x=-3$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-3$

- Multiply by denominators

  $x(x^2+6x+9)\,y''-y=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(u^3-3u^2)\left(\frac{d^2}{du^2}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum\limits_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=2..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum\limits_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum\limits_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

    Rewrite ODE with series expansions

    $-a_0(3r^2-3r+1)\,u^r+\Bigg(\sum\limits_{k=1}^{\infty}(-a_k(3k^2+6kr+3r^2-3k-3r+1)+a_{k-1}(k+r-1)\,(k-2+r$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-3r^2+3r-1=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6},\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(k+r-1)\,(k-2+r)-3\left(k^2+(2r-1)\,k+r^2-r+\frac{1}{3}\right)a_k=0$

- Shift index using $k->k+1$

  $a_k(k+r)\,(k+r-1)-3\left((k+1)^2+(2r-1)\,(k+1)+r^2-r+\frac{1}{3}\right)a_{k+1}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r)(k+r-1)}{3k^2+6kr+3r^2+3k+3r+1}$

- Recursion relation for $r = \frac{1}{2} - \frac{I\sqrt{3}}{6}$

$$a_{k+1} = \frac{a_k \left(k+\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)\left(k-\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)+3\left(\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}-\frac{I\sqrt{3}}{2}}$$

- Solution for $r = \frac{1}{2} - \frac{I\sqrt{3}}{6}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}-\frac{I\sqrt{3}}{6}}, a_{k+1} = \frac{a_k \left(k+\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)\left(k-\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)+3\left(\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}-\frac{I\sqrt{3}}{2}}\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k+\frac{1}{2}-\frac{I\sqrt{3}}{6}}, a_{k+1} = \frac{a_k \left(k+\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)\left(k-\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)+3\left(\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}-\frac{I\sqrt{3}}{2}}\right]$$

- Recursion relation for $r = \frac{1}{2} + \frac{I\sqrt{3}}{6}$

$$a_{k+1} = \frac{a_k \left(k+\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)\left(k-\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)+3\left(\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}+\frac{I\sqrt{3}}{2}}$$

- Solution for $r = \frac{1}{2} + \frac{I\sqrt{3}}{6}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}+\frac{I\sqrt{3}}{6}}, a_{k+1} = \frac{a_k \left(k+\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)\left(k-\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)+3\left(\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}+\frac{I\sqrt{3}}{2}}\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k+\frac{1}{2}+\frac{I\sqrt{3}}{6}}, a_{k+1} = \frac{a_k \left(k+\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)\left(k-\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)+3\left(\frac{1}{2}+\frac{I\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}+\frac{I\sqrt{3}}{2}}\right]$$

- Combine solutions and rename parameters

$$\Bigg[y = \left(\sum_{k=0}^{\infty} a_k (x+3)^{k+\frac{1}{2}-\frac{I\sqrt{3}}{6}}\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k+\frac{1}{2}+\frac{I\sqrt{3}}{6}}\right), a_{k+1} = \frac{a_k \left(k+\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)\left(k-\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)+3\left(\frac{1}{2}-\frac{I\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}-}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 70

```
Order:=8;
dsolve(x*(x+3)^2*diff(y(x),x$2)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x\left(1 + \frac{1}{18}x - \frac{11}{972}x^2 + \frac{277}{104976}x^3 - \frac{12539}{18895680}x^4 + \frac{893821}{5101833600}x^5 - \frac{13183337}{275499014400}x^6 + \frac{265861081}{19835929036800}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(\ln\left(x\right)\left(\frac{1}{9}x + \frac{1}{162}x^2 - \frac{11}{8748}x^3 + \frac{277}{944784}x^4 - \frac{12539}{170061120}x^5 + \frac{893821}{45916502400}x^6 - \frac{13183337}{2479491129600}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(1 - \frac{5}{108}x^2 + \frac{167}{26244}x^3 - \frac{13583}{11337408}x^4 + \frac{1327279}{5101833600}x^5 - \frac{21146863}{344373768000}x^6 + \frac{379766273}{24794911296000}x^7 + \mathrm{O}\left(x^8\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.77 (sec). Leaf size: 121

```
AsymptoticDSolveValue[x*(x+3)^2*y''[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x(893821x^5 - 3385530x^4 + 13462200x^3 - 57736800x^2 + 283435200x + 5101833600)\log(x)}{45916502400} + \frac{24742849x^6 - 74732085x^5 + 184497750x^4 + 52488000x^3 - 10628820000x^2 + 382637520000x + 6887}{688747536000}\right.$$

$$+ c_2\left(-\frac{13183337x^7}{275499014400} + \frac{893821x^6}{5101833600} - \frac{12539x^5}{18895680} + \frac{277x^4}{104976} - \frac{11x^3}{972} + \frac{x^2}{18} + x\right)$$

2.3 problem 3

Internal problem ID [6586]
Internal file name [`OUTPUT/5834_Sunday_June_05_2022_03_56_38_PM_25908479/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^{2}-9\right)^{2} y''+\left(x+3\right) y'+2y=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (107) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (108) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{xy' + 3y' + 2y}{x^4 - 18x^2 + 81} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x^3 + 9x^2 - 8x - 78)\, y' + 8y\left(x^2 - 3x + \frac{1}{4}\right)}{(x+3)^3\,(x-3)^4} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(4x^5 - 84x^4 + 139x^3 + 843x^2 - 1576x - 786)\, y' - 40y\left(x^4 - 6x^3 + \frac{56}{5}x^2 - \frac{57}{5}x + \frac{289}{20}\right)}{(x-3)^6\,(x+3)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(-60x^7 + 900x^6 - 2901x^5 - 3963x^4 + 27462x^3 - 29286x^2 + 31564x - 71520)\, y' + 240y(x^6 - 9x^5 +}{(x+3)^5\,(x-3)^8} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(600x^9 - 9720x^8 + 48444x^7 - 60612x^6 - 185109x^5 + 791229x^4 - 2239204x^3 + 4979016x^2 - 410561}{(x-3} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{(-5880x^{11} + 108360x^{10} - 735420x^9 + 2343180x^8 - 3296217x^7 - 4290909x^6 + 52805852x^5 - 19674}{} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{(60480x^{13} - 1270080x^{12} + 10921680x^{11} - 53025840x^{10} + 168880536x^9 - 308901960x^8 - 358169031}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{2y(0)}{81} - \frac{y'(0)}{27}\\
F_1 &= \frac{2y(0)}{2187} - \frac{26y'(0)}{729}\\
F_2 &= -\frac{578y(0)}{59049} - \frac{262y'(0)}{19683}\\
F_3 &= \frac{2432y(0)}{1594323} - \frac{23840y'(0)}{531441}\\
F_4 &= -\frac{779924y(0)}{43046721} - \frac{313876y'(0)}{14348907}\\
F_5 &= \frac{6931748y(0)}{1162261467} - \frac{54955484y'(0)}{387420489}\\
F_6 &= -\frac{2375994980y(0)}{31381059609} - \frac{829049284y'(0)}{10460353203}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{81}x^2+\frac{1}{6561}x^3-\frac{289}{708588}x^4+\frac{304}{23914845}x^5-\frac{194981}{7748409780}x^6+\frac{1732937}{1464449448420}x^7\right.\\
&\left.-\frac{118799749}{63264216171744}x^8\right) y(0) + \left(x-\frac{1}{54}x^2-\frac{13}{2187}x^3-\frac{131}{236196}x^4-\frac{596}{1594323}x^5\right.\\
&\left.-\frac{78469}{2582803260}x^6-\frac{13738871}{488149816140}x^7-\frac{29608903}{15062908612320}x^8\right) y'(0) + O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
y''\left(x^4 - 18x^2 + 81\right) + (x + 3)\, y' + 2y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}\\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \left(x^4 - 18x^2 + 81\right) + (x+3) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n+2} a_n (n-1)\right) + \sum_{n\,=2}^{\infty} \left(-18 x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 81 n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=1}^{\infty} 3 n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 2 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n\,=2}^{\infty} n\, x^{n+2} a_n (n-1) = \sum_{n=4}^{\infty} (n-2)\, a_{n-2} (n-3)\, x^n$$

$$\sum_{n\,=2}^{\infty} 81 n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 81(n+2)\, a_{n+2} (n+1)\, x^n$$

$$\sum_{n\,=1}^{\infty} 3 n a_n x^{n-1} = \sum_{n=0}^{\infty} 3(n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=4}^{\infty} (n-2)\, a_{n-2} (n-3)\, x^n\right) + \sum_{n\,=2}^{\infty} \left(-18 x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 81(n+2)\, a_{n+2} (n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 3(n+1)\, a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} 2 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$162 a_2 + 3 a_1 + 2 a_0 = 0$$

$$a_2 = -\frac{a_0}{81} - \frac{a_1}{54}$$

$n = 1$ gives

$$486a_3 + 3a_1 + 6a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6561} - \frac{13a_1}{2187}$$

$n = 2$ gives

$$-32a_2 + 972a_4 + 9a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{289a_0}{729} + \frac{131a_1}{243} + 972a_4 = 0$$

Or

$$a_4 = -\frac{289a_0}{708588} - \frac{131a_1}{236196}$$

$n = 3$ gives

$$-103a_3 + 1620a_5 + 12a_4 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{1216a_0}{59049} + \frac{11920a_1}{19683} + 1620a_5 = 0$$

Or

$$a_5 = \frac{304a_0}{23914845} - \frac{596a_1}{1594323}$$

For $4 \leq n$, the recurrence equation is

$$\begin{multline} (n-2)\, a_{n-2}(n-3) - 18na_n(n-1) + 81(n+2)\, a_{n+2}(n+1) \\ + na_n + 3(n+1)\, a_{n+1} + 2a_n = 0 \end{multline} \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{18n^2 a_n - n^2 a_{n-2} - 19n a_n + 5n a_{n-2} - 3n a_{n+1} - 2a_n - 6a_{n-2} - 3a_{n+1}}{81\,(n+2)\,(n+1)} \\
&= \frac{(18n^2 - 19n - 2)\,a_n}{81\,(n+2)\,(n+1)} + \frac{(-n^2 + 5n - 6)\,a_{n-2}}{81\,(n+2)\,(n+1)} + \frac{(-3n - 3)\,a_{n+1}}{81\,(n+2)\,(n+1)}
\end{aligned} \tag{5}
$$

For $n = 4$ the recurrence equation gives

$$2a_2 - 210a_4 + 2430a_6 + 15a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{194981a_0}{7748409780} - \frac{78469a_1}{2582803260}$$

For $n = 5$ the recurrence equation gives

$$6a_3 - 353a_5 + 3402a_7 + 18a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{1732937a_0}{1464449448420} - \frac{13738871a_1}{488149816140}$$

For $n = 6$ the recurrence equation gives

$$12a_4 - 532a_6 + 4536a_8 + 21a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = -\frac{118799749a_0}{63264216171744} - \frac{29608903a_1}{15062908612320}$$

For $n = 7$ the recurrence equation gives

$$20a_5 - 747a_7 + 5832a_9 + 24a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{2223425069a_0}{19216505662167240} - \frac{14828044253a_1}{6405501887389080}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x &+ \left(-\frac{a_0}{81} - \frac{a_1}{54}\right) x^2 + \left(\frac{a_0}{6561} - \frac{13a_1}{2187}\right) x^3 + \left(-\frac{289a_0}{708588} - \frac{131a_1}{236196}\right) x^4 \\ &+ \left(\frac{304a_0}{23914845} - \frac{596a_1}{1594323}\right) x^5 + \left(-\frac{194981a_0}{7748409780} - \frac{78469a_1}{2582803260}\right) x^6 \\ &+ \left(\frac{1732937a_0}{1464449448420} - \frac{13738871a_1}{488149816140}\right) x^7 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y = \left(1 - \frac{1}{81}x^2 + \frac{1}{6561}x^3 - \frac{289}{708588}x^4 + \frac{304}{23914845}x^5 - \frac{194981}{7748409780}x^6 \right. \\ \left. + \frac{1732937}{1464449448420}x^7\right) a_0 + \left(x - \frac{1}{54}x^2 - \frac{13}{2187}x^3 - \frac{131}{236196}x^4 - \frac{596}{1594323}x^5 \right. \\ \left. - \frac{78469}{2582803260}x^6 - \frac{13738871}{488149816140}x^7\right) a_1 + O(x^8) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned} y = \left(1 - \frac{1}{81}x^2 + \frac{1}{6561}x^3 - \frac{289}{708588}x^4 + \frac{304}{23914845}x^5 - \frac{194981}{7748409780}x^6 \right. \\ \left. + \frac{1732937}{1464449448420}x^7\right) c_1 + \left(x - \frac{1}{54}x^2 - \frac{13}{2187}x^3 - \frac{131}{236196}x^4 - \frac{596}{1594323}x^5 \right. \\ \left. - \frac{78469}{2582803260}x^6 - \frac{13738871}{488149816140}x^7\right) c_2 + O(x^8) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(1-\frac{1}{81}x^{2}+\frac{1}{6561}x^{3}-\frac{289}{708588}x^{4}+\frac{304}{23914845}x^{5}-\frac{194981}{7748409780}x^{6}+\frac{1732937}{1464449448420}x^{7}-\frac{118799749}{63264216171744}x^{8}\right)y(0)+\left(x-\frac{1}{54}x^{2}-\frac{13}{2187}x^{3}-\frac{131}{236196}x^{4}-\frac{596}{1594323}x^{5}-\frac{78469}{2582803260}x^{6}-\frac{13738871}{488149816140}x^{7}-\frac{29608903}{15062908612320}x^{8}\right)y'(0)+O\left(x^{8}\right) \tag{1}$$

$$y = \left(1-\frac{1}{81}x^{2}+\frac{1}{6561}x^{3}-\frac{289}{708588}x^{4}+\frac{304}{23914845}x^{5}-\frac{194981}{7748409780}x^{6}+\frac{1732937}{1464449448420}x^{7}\right)c_{1}+\left(x-\frac{1}{54}x^{2}-\frac{13}{2187}x^{3}-\frac{131}{236196}x^{4}-\frac{596}{1594323}x^{5}-\frac{78469}{2582803260}x^{6}-\frac{13738871}{488149816140}x^{7}\right)c_{2}+O\left(x^{8}\right) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{81}x^{2}+\frac{1}{6561}x^{3}-\frac{289}{708588}x^{4}+\frac{304}{23914845}x^{5}-\frac{194981}{7748409780}x^{6}+\frac{1732937}{1464449448420}x^{7}-\frac{118799749}{63264216171744}x^{8}\right)y(0)+\left(x-\frac{1}{54}x^{2}-\frac{13}{2187}x^{3}-\frac{131}{236196}x^{4}-\frac{596}{1594323}x^{5}-\frac{78469}{2582803260}x^{6}-\frac{13738871}{488149816140}x^{7}-\frac{29608903}{15062908612320}x^{8}\right)y'(0)+O\left(x^{8}\right)$$

Verified OK.

$$y = \left(1-\frac{1}{81}x^{2}+\frac{1}{6561}x^{3}-\frac{289}{708588}x^{4}+\frac{304}{23914845}x^{5}-\frac{194981}{7748409780}x^{6}+\frac{1732937}{1464449448420}x^{7}\right)c_{1}+\left(x-\frac{1}{54}x^{2}-\frac{13}{2187}x^{3}-\frac{131}{236196}x^{4}-\frac{596}{1594323}x^{5}-\frac{78469}{2582803260}x^{6}-\frac{13738871}{488149816140}x^{7}\right)c_{2}+O\left(x^{8}\right)$$

Verified OK.

2.3.1 Maple step by step solution

Let's solve

$y''(x^4 - 18x^2 + 81) + (x + 3)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x^4-18x^2+81} - \frac{y'}{x^3-3x^2-9x+27}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x^3-3x^2-9x+27} + \frac{2y}{x^4-18x^2+81} = 0$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{1}{x^3-3x^2-9x+27}, P_3(x) = \frac{2}{x^4-18x^2+81}\right]$

  ○ $(x + 3) \cdot P_2(x)$ is analytic at $x = -3$

  $\left((x + 3) \cdot P_2(x)\right)\bigg|_{x=-3} = \frac{1}{36}$

  ○ $(x + 3)^2 \cdot P_3(x)$ is analytic at $x = -3$

  $\left((x + 3)^2 \cdot P_3(x)\right)\bigg|_{x=-3} = \frac{1}{18}$

  ○ $x = -3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -3$

- Multiply by denominators

  $y''(x^3 - 3x^2 - 9x + 27)\,(x^4 - 18x^2 + 81) + y'(x^4 - 18x^2 + 81) + (2x^3 - 6x^2 - 18x + 54)\, y = 0$

- Change variables using $x = u - 3$ so that the regular singular point is at $u = 0$

  $(u^7 - 24u^6 + 216u^5 - 864u^4 + 1296u^3) \left(\frac{d^2}{du^2} y(u)\right) + (u^4 - 12u^3 + 36u^2) \left(\frac{d}{du} y(u)\right) + (2u^3 - 24u^2$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 1..3$

$$u^{m} \cdot y(u) = \sum_{k=0}^{\infty} a_{k} u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^{m} \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^{m} \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..4$

  $$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^{m} \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m = 3..7$

  $$u^{m} \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^{m} \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$36a_{0}(36r^{2}-35r+2)\, u^{1+r} + \left(36a_{1}(36r^{2}+37r+3) - 12a_{0}(72r^{2}-71r+2)\right) u^{2+r} + (36a_{2}(36r^{2} +$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$1296r^{2}-1260r+72 = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{35}{72}-\frac{\sqrt{937}}{72}, \frac{35}{72}+\frac{\sqrt{937}}{72}\right\}$$

- The coefficients of each power of u must be 0

  $$[36a_{1}(36r^{2}+37r+3) - 12a_{0}(72r^{2}-71r+2) = 0, 36a_{2}(36r^{2}+109r+76) - 12a_{1}(72r^{2}+73r +$$

- Solve for the dependent coefficient(s)

  $$\left\{a_{1} = \frac{a_{0}(72r^{2}-71r+2)}{3(36r^{2}+37r+3)}, a_{2} = \frac{a_{0}(4320r^{4}+108r^{3}-4019r^{2}+101r+6)}{12(1296r^{4}+5256r^{3}+6877r^{2}+3139r+228)}, a_{3} = \frac{a_{0}(233280r^{6}+707616r^{5}+277488r^{4}-61842}{54(46656r^{6}+423792r^{5}+1485324r^{4}+2519317}\right.$$

- Each term in the series must be 0, giving the recursion relation

  $$(a_{k-5}-24a_{k-4}+216a_{k-3}-864a_{k-2}+1296a_{k-1})\, k^{2} + (2(a_{k-5}-24a_{k-4}+216a_{k-3}-864a_{k-2} +$$

- Shift index using $k->k+5$

$$\left(a_k-24a_{k+1}+216a_{k+2}-864a_{k+3}+1296a_{k+4}\right)(k+5)^2+\left(2\left(a_k-24a_{k+1}+216a_{k+2}-864a_{k+3}\right.\right.$$

- Recursion relation that defines series solution to ODE

$$a_{k+4}=-\frac{k^2a_k-24k^2a_{k+1}+216k^2a_{k+2}-864k^2a_{k+3}+2kra_k-48kra_{k+1}+432kra_{k+2}-1728kra_{k+3}+r^2a_k-24r^2a_{k+1}+216r^2a_{k+2}}{36(36k^2+72kr+36r^2+}$$

- Recursion relation for $r=\frac{35}{72}-\frac{\sqrt{937}}{72}$

$$a_{k+4}=-\frac{k^2a_k-24k^2a_{k+1}+216k^2a_{k+2}-864k^2a_{k+3}+2k\left(\frac{35}{72}-\frac{\sqrt{937}}{72}\right)a_k-48k\left(\frac{35}{72}-\frac{\sqrt{937}}{72}\right)a_{k+1}+432k\left(\frac{35}{72}-\frac{\sqrt{937}}{72}\right)a_{k+2}-1728k\left(\right.}{}$$

- Solution for $r=\frac{35}{72}-\frac{\sqrt{937}}{72}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{35}{72}-\frac{\sqrt{937}}{72}},a_{k+4}=-\frac{k^2a_k-24k^2a_{k+1}+216k^2a_{k+2}-864k^2a_{k+3}+2k\left(\frac{35}{72}-\frac{\sqrt{937}}{72}\right)a_k-48k\left(\frac{35}{72}-\frac{\sqrt{937}}{72}\right)a_{k}}{}\right.$$

- Revert the change of variables $u=x+3$

$$\left[y=\sum_{k=0}^{\infty}a_k(x+3)^{k+\frac{35}{72}-\frac{\sqrt{937}}{72}},a_{k+4}=-\frac{k^2a_k-24k^2a_{k+1}+216k^2a_{k+2}-864k^2a_{k+3}+2k\left(\frac{35}{72}-\frac{\sqrt{937}}{72}\right)a_k-48k\left(\frac{35}{72}-\frac{\sqrt{937}}{72}\right.}{}\right.$$

- Recursion relation for $r=\frac{35}{72}+\frac{\sqrt{937}}{72}$

$$a_{k+4}=-\frac{k^2a_k-24k^2a_{k+1}+216k^2a_{k+2}-864k^2a_{k+3}+2k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_k-48k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_{k+1}+432k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_{k+2}-1728k\left(\right.}{}$$

- Solution for $r=\frac{35}{72}+\frac{\sqrt{937}}{72}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{35}{72}+\frac{\sqrt{937}}{72}},a_{k+4}=-\frac{k^2a_k-24k^2a_{k+1}+216k^2a_{k+2}-864k^2a_{k+3}+2k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_k-48k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_{k}}{}\right.$$

- Revert the change of variables $u=x+3$

$$\left[y=\sum_{k=0}^{\infty}a_k(x+3)^{k+\frac{35}{72}+\frac{\sqrt{937}}{72}},a_{k+4}=-\frac{k^2a_k-24k^2a_{k+1}+216k^2a_{k+2}-864k^2a_{k+3}+2k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_k-48k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right.}{}\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(x+3)^{k+\frac{35}{72}-\frac{\sqrt{937}}{72}}\right)+\left(\sum_{k=0}^{\infty}b_k(x+3)^{k+\frac{35}{72}+\frac{\sqrt{937}}{72}}\right),a_{k+4}=-\frac{k^2a_k-24k^2a_{k+1}+216k^2a_{k+2}-864}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 74

```
Order:=8;
dsolve((x^2-9)^2*diff(y(x),x$2)+(x+3)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{81}x^2+\frac{1}{6561}x^3-\frac{289}{708588}x^4+\frac{304}{23914845}x^5-\frac{194981}{7748409780}x^6+\frac{1732937}{1464449448420}x^7\right)y(0)+\left(x-\frac{1}{54}x^2-\frac{13}{2187}x^3-\frac{131}{236196}x^4-\frac{596}{1594323}x^5-\frac{78469}{2582803260}x^6-\frac{13738871}{488149816140}x^7\right)D(y)(0)+O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 98

```
AsymptoticDSolveValue[(x^2-9)^2*y''[x]+(x+3)*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$\begin{aligned}y(x) \to\; & c_1\left(\frac{1732937x^7}{1464449448420}-\frac{194981x^6}{7748409780}+\frac{304x^5}{23914845}-\frac{289x^4}{708588}+\frac{x^3}{6561}-\frac{x^2}{81}+1\right)\\ & +c_2\left(-\frac{13738871x^7}{488149816140}-\frac{78469x^6}{2582803260}-\frac{596x^5}{1594323}-\frac{131x^4}{236196}-\frac{13x^3}{2187}-\frac{x^2}{54}+x\right)\end{aligned}$$

2.4 problem 4

Internal problem ID [6587]
Internal file name [`OUTPUT/5835_Sunday_June_05_2022_03_56_40_PM_76442363/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \frac{y'}{x} + \frac{y}{(x-1)^3} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' - \frac{y'}{x} + \frac{y}{(x-1)^3} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{(x-1)^3} \end{aligned}$$

Table 31: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{(x-1)^3}$	
singularity	type
$x = 1$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[1]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x-1)^3 y'' - y'(x-1)^3 + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x-1)^3 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) - \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right)(x-1)^3 + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{gathered}\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)\left(n+r-1\right)\right)+\sum_{n=0}^{\infty}\left(-3 x^{1+n+r} a_n(n+r)\left(n+r-1\right)\right) \\ +\left(\sum_{n=0}^{\infty} 3 x^{n+r} a_n(n+r)\left(n+r-1\right)\right) \\ +\sum_{n=0}^{\infty}\left(-x^{n+r-1} a_n(n+r)\left(n+r-1\right)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n(n+r)\right) \\ +\left(\sum_{n=0}^{\infty} 3 x^{1+n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-3 x^{n+r} a_n(n+r)\right) \\ +\left(\sum_{n=0}^{\infty}(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right)=0\end{gathered} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)\left(n+r-1\right)=\sum_{n=3}^{\infty} a_{n-3}(n+r-3)\left(n-4+r\right) x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-3 x^{1+n+r} a_n(n+r)\left(n+r-1\right)\right)=\sum_{n=2}^{\infty}\left(-3 a_{n-2}(n+r-2)\left(n+r-3\right) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 3 x^{n+r} a_n(n+r)\left(n+r-1\right)=\sum_{n=1}^{\infty} 3 a_{n-1}(n+r-1)\left(n+r-2\right) x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n(n+r)\right)=\sum_{n=3}^{\infty}\left(-a_{n-3}(n+r-3) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 3 x^{1+n+r} a_n(n+r)=\sum_{n=2}^{\infty} 3 a_{n-2}(n+r-2) x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-3 x^{n+r} a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-3 a_{n-1}(n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n=\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned}
&\left(\sum_{n=3}^{\infty} a_{n-3}(n+r-3)\,(n-4+r)\,x^{n+r-1}\right)\\
&+\sum_{n=2}^{\infty}\left(-3a_{n-2}(n+r-2)\,(n+r-3)\,x^{n+r-1}\right)\\
&\quad+\left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r-1}\right)\\
&\quad+\sum_{n=0}^{\infty}\left(-x^{n+r-1}a_n(n+r)\,(n+r-1)\right)+\sum_{n=3}^{\infty}\left(-a_{n-3}(n+r-3)\,x^{n+r-1}\right)\\
&\quad+\left(\sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)\,x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-3a_{n-1}(n+r-1)\,x^{n+r-1}\right)\\
&\quad+\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=2}^{\infty} a_{n-2}x^{n+r-1}\right)=0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-x^{n+r-1}a_n(n+r)\,(n+r-1)+(n+r)\,a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$-x^{-1+r}a_0 r(-1+r)+r a_0 x^{-1+r}=0$$

Or

$$\left(-x^{-1+r}r(-1+r)+r\,x^{-1+r}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$r\,x^{-1+r}(2-r)=0$$

Since the above is true for all x then the indicial equation becomes

$$-r(-2+r)=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}
r_1&=2\\
r_2&=0
\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2-r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\
y_2(x) &= Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}$$

Or

$$\begin{aligned}
y_1(x) &= x^{2}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\
y_2(x) &= Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}$$

Or

$$\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2}\\
y_2(x) &= Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{3r(-2+r)}{r^2-1}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{6r^2 - 12r + 1}{r\,(2+r)}$$

For $3 \leq n$ the recursive equation is

$$\begin{aligned}
&a_{n-3}(n+r-3)\,(n-4+r) - 3a_{n-2}(n+r-2)\,(n+r-3)\\
&\quad + 3a_{n-1}(n+r-1)\,(n+r-2) - a_n(n+r)\,(n+r-1) - a_{n-3}(n+r-3)\\
&\quad + 3a_{n-2}(n+r-2) - 3a_{n-1}(n+r-1) + a_n(n+r) + a_{n-2} = 0
\end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n^2 a_{n-3} - 3n^2 a_{n-2} + 3n^2 a_{n-1} + 2nr a_{n-3} - 6nr a_{n-2} + 6nr a_{n-1} + r^2 a_{n-3} - 3r^2 a_{n-2} + 3r^2 a_{n-1} - 8na}{n^2 + 2nr + r^2 - 2n} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{(a_{n-3} - 3a_{n-2} + 3a_{n-1})\, n^2 + (-4a_{n-3} + 6a_{n-2})\, n + 3a_{n-3} + a_{n-2} - 3a_{n-1}}{n\,(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r(-2+r)}{r^2-1}$	0
a_2	$\frac{6r^2-12r+1}{r(2+r)}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{10r^4 - 20r^3 - 4r^2 + 14r - 3}{(r+3)\,(1+r)^2\,(-1+r)}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r(-2+r)}{r^2-1}$	0
a_2	$\frac{6r^2-12r+1}{r(2+r)}$	$\frac{1}{8}$
a_3	$\frac{10r^4-20r^3-4r^2+14r-3}{(r+3)(1+r)^2(-1+r)}$	$\frac{1}{5}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{15r^6 - 54r^4 + 13r^2 - 1}{(4+r)\,(2+r)^2\, r\,(-1+r)\,(1+r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{49}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r(-2+r)}{r^2-1}$	0
a_2	$\frac{6r^2-12r+1}{r(2+r)}$	$\frac{1}{8}$
a_3	$\frac{10r^4-20r^3-4r^2+14r-3}{(r+3)(1+r)^2(-1+r)}$	$\frac{1}{5}$
a_4	$\frac{15r^6-54r^4+13r^2-1}{(4+r)(2+r)^2r(-1+r)(1+r)}$	$\frac{49}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{21r^8 + 84r^7 + 14r^6 - 252r^5 - 298r^4 - 78r^3 + 47r^2 + 30r - 9}{(5 + r)(r + 3)^2 r(-1 + r)(1 + r)^2(2 + r)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{423}{1400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r(-2+r)}{r^2-1}$	0
a_2	$\frac{6r^2-12r+1}{r(2+r)}$	$\frac{1}{8}$
a_3	$\frac{10r^4-20r^3-4r^2+14r-3}{(r+3)(1+r)^2(-1+r)}$	$\frac{1}{5}$
a_4	$\frac{15r^6-54r^4+13r^2-1}{(4+r)(2+r)^2r(-1+r)(1+r)}$	$\frac{49}{192}$
a_5	$\frac{21r^8+84r^7+14r^6-252r^5-298r^4-78r^3+47r^2+30r-9}{(5+r)(r+3)^2r(-1+r)(1+r)^2(2+r)}$	$\frac{423}{1400}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{28r^{10} + 280r^9 + 966r^8 + 1008r^7 - 1767r^6 - 5898r^5 - 5979r^4 - 1620r^3 + 1148r^2 + 560r - 147}{(6 + r)(4 + r)^2(2 + r)^2(r + 3)(1 + r)^2(-1 + r)r}$$

Which for the root $r = 2$ becomes

$$a_6 = \frac{15941}{46080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r(-2+r)}{r^2-1}$	0
a_2	$\frac{6r^2-12r+1}{r(2+r)}$	$\frac{1}{8}$
a_3	$\frac{10r^4-20r^3-4r^2+14r-3}{(r+3)(1+r)^2(-1+r)}$	$\frac{1}{5}$
a_4	$\frac{15r^6-54r^4+13r^2-1}{(4+r)(2+r)^2r(-1+r)(1+r)}$	$\frac{49}{192}$
a_5	$\frac{21r^8+84r^7+14r^6-252r^5-298r^4-78r^3+47r^2+30r-9}{(5+r)(r+3)^2r(-1+r)(1+r)^2(2+r)}$	$\frac{423}{1400}$
a_6	$\frac{28r^{10}+280r^9+966r^8+1008r^7-1767r^6-5898r^5-5979r^4-1620r^3+1148r^2+560r-147}{(6+r)(4+r)^2(2+r)^2(r+3)(1+r)^2(-1+r)r}$	$\frac{15941}{46080}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{36r^{12} + 648r^{11} + 4752r^{10} + 17820r^9 + 32403r^8 + 3924r^7 - 106866r^6 - 219348r^5 - 182457r^4 - 3186}{(7+r)(5+r)^2(r+3)^2(1+r)^2(-1+r)(4+r)(2+r)^2 r}$$

Which for the root $r = 2$ becomes

$$a_7 = \frac{30511}{78400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r(-2+r)}{r^2-1}$	0
a_2	$\frac{6r^2-12r+1}{r(2+r)}$	$\frac{1}{8}$
a_3	$\frac{10r^4-20r^3-4r^2+14r-3}{(r+3)(1+r)^2(-1+r)}$	$\frac{1}{5}$
a_4	$\frac{15r^6-54r^4+13r^2-1}{(4+r)(2+r)^2r(-1+r)(1+r)}$	$\frac{49}{192}$
a_5	$\frac{21r^8+84r^7+14r^6-252r^5-298r^4-78r^3+47r^2+30r-9}{(5+r)(r+3)^2r(-1+r)(1+r)^2(2+r)}$	$\frac{423}{1400}$
a_6	$\frac{28r^{10}+280r^9+966r^8+1008r^7-1767r^6-5898r^5-5979r^4-1620r^3+1148r^2+560r-147}{(6+r)(4+r)^2(2+r)^2(r+3)(1+r)^2(-1+r)r}$	$\frac{15941}{46080}$
a_7	$\frac{36r^{12}+648r^{11}+4752r^{10}+17820r^9+32403r^8+3924r^7-106866r^6-219348r^5-182457r^4-31860r^3+40074r^2+14274r-3807}{(7+r)(5+r)^2(r+3)^2(1+r)^2(-1+r)(4+r)(2+r)^2r}$	$\frac{30511}{78400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right)\\ &= x^2\left(1 + \frac{x^2}{8} + \frac{x^3}{5} + \frac{49x^4}{192} + \frac{423x^5}{1400} + \frac{15941x^6}{46080} + \frac{30511x^7}{78400} + O(x^8)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_2\\ &= \frac{6r^2 - 12r + 1}{r(2+r)}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} \frac{6r^2 - 12r + 1}{r(2+r)} &= \lim_{r\to 0} \frac{6r^2 - 12r + 1}{r(2+r)}\\ &= \textit{undefined}\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\\ &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)\\ \frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\\ &\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\\ &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)\end{aligned}$$

Substituting these back into the given ode $x(x-1)^3 y'' - y'(x-1)^3 + xy = 0$ gives

$$\begin{aligned}
&x(x-1)^3 \left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) \\
&- \left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right)\right) (x-1)^3 \\
&+ x \left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}
&\left(\left(x(x-1)^3 y_1''(x) - y_1'(x)(x-1)^3 + x y_1(x)\right) \ln(x) \right. \\
&\left. + x(x-1)^3 \left(\frac{2 y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) - \frac{y_1(x)(x-1)^3}{x}\right) C \\
&+ x(x-1)^3 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) \\
&- \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) (x-1)^3 + x \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x(x-1)^3 y_1''(x) - y_1'(x)(x-1)^3 + x y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(x(x-1)^3 \left(\frac{2 y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) - \frac{y_1(x)(x-1)^3}{x}\right) C \\
&+ x(x-1)^3 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) \\
&- \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) (x-1)^3 + x \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2x(x-1)^3 \left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)(x-1)^3\right) C}{x} \tag{9}$$

$$+ \frac{x^2(x-1)^3 \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) - \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right)(x-1)^3 x + x^2 \Big(\sum_{n=0}^{\infty} b_n}{x}$$

$$= 0$$

Since $r_1 = 2$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2x(x-1)^3 \left(\sum_{n=0}^{\infty} x^{1+n} a_n (n+2)\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+2}\right)(x-1)^3\right) C}{x} + \frac{x^2(x-1)^3 \left(\sum_{n=0}^{\infty} x^{n-2} b_n n(n-1)\right) - \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right)(x-1)^3 x + x^2 \left(\sum_{n=0}^{\infty} b_n x^n\right)}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+2)\right) + \sum_{n=0}^{\infty} \left(-6C\, x^{n+3} a_n (n+2)\right) \\
&+ \left(\sum_{n=0}^{\infty} 6C\, x^{n+2} a_n (n+2)\right) + \sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n (n+2)\right) \\
&+ \sum_{n=0}^{\infty} \left(-2C\, x^{n+4} a_n\right) + \left(\sum_{n=0}^{\infty} 6C\, x^{n+3} a_n\right) + \sum_{n=0}^{\infty} \left(-6C\, x^{n+2} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} n\, x^{n+2} b_n (n-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-3n\, x^{1+n} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 3x^n b_n n(n-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-n\, x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-n\, x^{n+2} b_n\right) + \left(\sum_{n=0}^{\infty} 3n\, x^{1+n} b_n\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^n b_n n\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) + \left(\sum_{n=0}^{\infty} x^{1+n} b_n\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n(n+2) = \sum_{n=5}^{\infty} 2C a_{n-5}(n-3)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-6C\, x^{n+3} a_n(n+2)\right) = \sum_{n=4}^{\infty} \left(-6C a_{n-4}(n-2)\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 6C\, x^{n+2} a_n(n+2) = \sum_{n=3}^{\infty} 6C a_{n-3}(n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n(n+2)\right) = \sum_{n=2}^{\infty} \left(-2C a_{n-2} n\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2C\, x^{n+4} a_n\right) = \sum_{n=5}^{\infty} \left(-2C a_{n-5} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 6C\, x^{n+3} a_n = \sum_{n=4}^{\infty} 6C a_{n-4} x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-6C\, x^{n+2} a_n\right) = \sum_{n=3}^{\infty} \left(-6C a_{n-3} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n = \sum_{n=2}^{\infty} 2C a_{n-2} x^{n-1}$$

$$\sum_{n=0}^{\infty} n\, x^{n+2} b_n(n-1) = \sum_{n=3}^{\infty} (n-3)\, b_{n-3}(n-4)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-3n\, x^{1+n} b_n(n-1)\right) = \sum_{n=2}^{\infty} \left(-3(n-2)\, b_{n-2}(n-3)\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 3x^n b_n n(n-1) = \sum_{n=1}^{\infty} 3(n-1)\, b_{n-1}(n-2)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-n\, x^{n+2} b_n\right) = \sum_{n=3}^{\infty} \left(-(n-3)\, b_{n-3} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 3n\, x^{1+n} b_n = \sum_{n=2}^{\infty} 3(n-2)\, b_{n-2} x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-3x^n b_n n\right) = \sum_{n=1}^{\infty} \left(-3(n-1)\, b_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\left(\sum_{n=5}^{\infty} 2C a_{n-5}(n-3)\, x^{n-1}\right) + \sum_{n=4}^{\infty} \left(-6C a_{n-4}(n-2)\, x^{n-1}\right) \\
&+ \left(\sum_{n=3}^{\infty} 6C a_{n-3}(n-1)\, x^{n-1}\right) + \sum_{n=2}^{\infty} \left(-2C a_{n-2} n\, x^{n-1}\right) \\
&+ \sum_{n=5}^{\infty} \left(-2C a_{n-5} x^{n-1}\right) + \left(\sum_{n=4}^{\infty} 6C a_{n-4} x^{n-1}\right) + \sum_{n=3}^{\infty} \left(-6C a_{n-3} x^{n-1}\right) \\
&+ \left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{n-1}\right) + \left(\sum_{n=3}^{\infty} (n-3)\, b_{n-3}(n-4)\, x^{n-1}\right) \\
&+ \sum_{n=2}^{\infty} \left(-3(n-2)\, b_{n-2}(n-3)\, x^{n-1}\right) \\
&+ \left(\sum_{n=1}^{\infty} 3(n-1)\, b_{n-1}(n-2)\, x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-n\, x^{n-1} b_n (n-1)\right) \\
&+ \sum_{n=3}^{\infty} \left(-(n-3)\, b_{n-3} x^{n-1}\right) + \left(\sum_{n=2}^{\infty} 3(n-2)\, b_{n-2} x^{n-1}\right) \\
&+ \sum_{n=1}^{\infty} \left(-3(n-1)\, b_{n-1} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) + \left(\sum_{n=2}^{\infty} b_{n-2} x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$-2C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = \frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$(6a_0 - 4a_1) C + 4b_1 - 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3 - 3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 1$$

For $n = 4$, Eq (2B) gives

$$(-6a_0 + 12a_1 - 6a_2) C - b_1 + b_2 + 9b_3 - 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{45}{8} - 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{45}{64}$$

For $n = 5$, Eq (2B) gives

$$2(a_0 - 6a_1 + 9a_2 - 4a_3) C - 8b_3 + 24b_4 - 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{51}{5} - 15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{17}{25}$$

For $n = 6$, Eq (2B) gives

$$(4a_1 - 18a_2 + 24a_3 - 10a_4)\, C + 3b_3 - 23b_4 + 45b_5 - 24b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1673}{96} - 24b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{1673}{2304}$$

For $n = 7$, Eq (2B) gives

$$6(a_2 - 4a_3 + 5a_4 - 2a_5)\, C + 8b_4 - 44b_5 + 72b_6 - 35b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{313337}{11200} - 35b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = \frac{313337}{392000}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = \frac{1}{2}$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = \frac{1}{2}\left(x^2\left(1 + \frac{x^2}{8} + \frac{x^3}{5} + \frac{49x^4}{192} + \frac{423x^5}{1400} + \frac{15941x^6}{46080} + \frac{30511x^7}{78400} + O(x^8)\right)\right)\ln(x) \\ + 1 + x^3 + \frac{45x^4}{64} + \frac{17x^5}{25} + \frac{1673x^6}{2304} + \frac{313337x^7}{392000} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2\left(1 + \frac{x^2}{8} + \frac{x^3}{5} + \frac{49x^4}{192} + \frac{423x^5}{1400} + \frac{15941x^6}{46080} + \frac{30511x^7}{78400} + O(x^8)\right) \\ &\quad + c_2\left(\frac{1}{2}\left(x^2\left(1+\frac{x^2}{8}+\frac{x^3}{5}+\frac{49x^4}{192}+\frac{423x^5}{1400}+\frac{15941x^6}{46080}+\frac{30511x^7}{78400}+O(x^8)\right)\right)\ln(x) \right. \\ &\quad \left. + 1 + x^3 + \frac{45x^4}{64} + \frac{17x^5}{25} + \frac{1673x^6}{2304} + \frac{313337x^7}{392000} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1+\frac{x^2}{8}+\frac{x^3}{5}+\frac{49x^4}{192}+\frac{423x^5}{1400}+\frac{15941x^6}{46080}+\frac{30511x^7}{78400}+O(x^8)\right) \\
&\quad + c_2 \left(\frac{x^2 \left(1+\frac{x^2}{8}+\frac{x^3}{5}+\frac{49x^4}{192}+\frac{423x^5}{1400}+\frac{15941x^6}{46080}+\frac{30511x^7}{78400}+O(x^8)\right)\ln(x)}{2}+1+x^3 \right. \\
&\quad \left. +\frac{45x^4}{64}+\frac{17x^5}{25}+\frac{1673x^6}{2304}+\frac{313337x^7}{392000}+O(x^8)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^2 \left(1+\frac{x^2}{8}+\frac{x^3}{5}+\frac{49x^4}{192}+\frac{423x^5}{1400}+\frac{15941x^6}{46080}+\frac{30511x^7}{78400}+O(x^8)\right) \\
&\quad + c_2 \left(\frac{x^2 \left(1+\frac{x^2}{8}+\frac{x^3}{5}+\frac{49x^4}{192}+\frac{423x^5}{1400}+\frac{15941x^6}{46080}+\frac{30511x^7}{78400}+O(x^8)\right)\ln(x)}{2}+1 \right. \\
&\quad \left. +x^3+\frac{45x^4}{64}+\frac{17x^5}{25}+\frac{1673x^6}{2304}+\frac{313337x^7}{392000}+O(x^8)\right)
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y &= c_1 x^2 \left(1+\frac{x^2}{8}+\frac{x^3}{5}+\frac{49x^4}{192}+\frac{423x^5}{1400}+\frac{15941x^6}{46080}+\frac{30511x^7}{78400}+O(x^8)\right) \\
&\quad + c_2 \left(\frac{x^2 \left(1+\frac{x^2}{8}+\frac{x^3}{5}+\frac{49x^4}{192}+\frac{423x^5}{1400}+\frac{15941x^6}{46080}+\frac{30511x^7}{78400}+O(x^8)\right)\ln(x)}{2}+1+x^3 \right. \\
&\quad \left. +\frac{45x^4}{64}+\frac{17x^5}{25}+\frac{1673x^6}{2304}+\frac{313337x^7}{392000}+O(x^8)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a = 0, e <> 0, c <>
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 64

```
Order:=8;
dsolve(diff(y(x),x$2)-1/x*diff(y(x),x)+1/(x-1)^3*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x^2 \left(1 + \frac{1}{8}x^2 + \frac{1}{5}x^3 + \frac{49}{192}x^4 + \frac{423}{1400}x^5 + \frac{15941}{46080}x^6 + \frac{30511}{78400}x^7 + \mathrm{O}\left(x^8\right)\right) \\
& + c_2 \left(\ln\left(x\right)\left(-x^2 - \frac{1}{8}x^4 - \frac{1}{5}x^5 - \frac{49}{192}x^6 - \frac{423}{1400}x^7 + \mathrm{O}\left(x^8\right)\right)\right. \\
& \left. + \left(-2 - 2x^3 - \frac{45}{32}x^4 - \frac{34}{25}x^5 - \frac{1673}{1152}x^6 - \frac{313337}{196000}x^7 + \mathrm{O}\left(x^8\right)\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.373 (sec). Leaf size: 107

```
AsymptoticDSolveValue[y''[x]-1/x*y'[x]+1/(x-1)^3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{(245x^4+192x^3+120x^2+960)\,x^2\log(x)}{1920} + \frac{-25025x^6-16416x^5-2250x^4+28800x^3-180000x^2+28800}{28800}\right) + c_2\left(\frac{15941x^8}{46080}+\frac{423x^7}{1400}+\frac{49x^6}{192}+\frac{x^5}{5}+\frac{x^4}{8}+x^2\right)$$

2.5 problem 5

Internal problem ID [6588]
Internal file name [`OUTPUT/5836_Sunday_June_05_2022_03_56_46_PM_7970428/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^3+4x\right)y''-2xy'+6y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+4x\right)y''-2xy'+6y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=-\frac{2}{x^2+4}\\q(x)&=\frac{6}{x\left(x^2+4\right)}\end{aligned}$$

Table 32: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2}{x^2+4}$	
singularity	type
$x = -2i$	"regular"
$x = 2i$	"regular"

$q(x) = \frac{6}{x(x^2+4)}$	
singularity	type
$x = 0$	"regular"
$x = -2i$	"regular"
$x = 2i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2i, 2i, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''(x^2+4)\,x - 2xy' + 6y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right)(x^2+4)\,x \\ &- 2x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 6\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) &= \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r-1} \\
\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) &= \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r-1}\right) \\
\sum_{n=0}^{\infty} 6a_n x^{n+r} &= \sum_{n=1}^{\infty} 6a_{n-1} x^{n+r-1}
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 6a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$4x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$4x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$4x^{-1+r}r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$4x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{r-3}{2r(1+r)}$$

For $2 \leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r) + 4a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) + 6a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-2} + 2nra_{n-2} + r^2 a_{n-2} - 5na_{n-2} - 2na_{n-1} - 5ra_{n-2} - 2ra_{n-1} + 6a_{n-2} + 8a_{n-1}}{4(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-n^2 a_{n-2} + (3a_{n-2} + 2a_{n-1})n - 2a_{n-2} - 6a_{n-1}}{4n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-3}{2r(1+r)}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-r^4 + 2r^2 - 5r + 6}{4r(1+r)^2(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-3}{2r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+2r^2-5r+6}{4r(1+r)^2(2+r)}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-2r^5 + 9r^3 + 6r^2 + 17r - 6}{8r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-3}{2r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+2r^2-5r+6}{4r(1+r)^2(2+r)}$	$\frac{1}{24}$
a_3	$\frac{-2r^5+9r^3+6r^2+17r-6}{8r(1+r)^2(2+r)^2(3+r)}$	$\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^8 + 8r^7 + 19r^6 + 17r^5 + 9r^4 + 17r^3 - 5r^2 - 114r - 72}{16r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{1}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-3}{2r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+2r^2-5r+6}{4r(1+r)^2(2+r)}$	$\frac{1}{24}$
a_3	$\frac{-2r^5+9r^3+6r^2+17r-6}{8r(1+r)^2(2+r)^2(3+r)}$	$\frac{1}{48}$
a_4	$\frac{r^8+8r^7+19r^6+17r^5+9r^4+17r^3-5r^2-114r-72}{16r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{3r^9 + 33r^8 + 124r^7 + 126r^6 - 396r^5 - 1408r^4 - 2077r^3 - 1967r^2 - 798r + 360}{32r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)^2\left(5+r\right)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{5}{2304}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-3}{2r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+2r^2-5r+6}{4r(1+r)^2(2+r)}$	$\frac{1}{24}$
a_3	$\frac{-2r^5+9r^3+6r^2+17r-6}{8r(1+r)^2(2+r)^2(3+r)}$	$\frac{1}{48}$
a_4	$\frac{r^8+8r^7+19r^6+17r^5+9r^4+17r^3-5r^2-114r-72}{16r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{1}{384}$
a_5	$\frac{3r^9+33r^8+124r^7+126r^6-396r^5-1408r^4-2077r^3-1967r^2-798r+360}{32r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{5}{2304}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-r^{12} - 24r^{11} - 239r^{10} - 1290r^9 - 4120r^8 - 8034r^7 - 10042r^6 - 9933r^5 - 8898r^4 + 3021r^3 + 31580}{64r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)^2\left(5+r\right)^2\left(6+r\right)}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{5}{21504}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-3}{2r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+2r^2-5r+6}{4r(1+r)^2(2+r)}$	$\frac{1}{24}$
a_3	$\frac{-2r^5+9r^3+6r^2+17r-6}{8r(1+r)^2(2+r)^2(3+r)}$	$\frac{1}{48}$
a_4	$\frac{r^8+8r^7+19r^6+17r^5+9r^4+17r^3-5r^2-114r-72}{16r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{1}{384}$
a_5	$\frac{3r^9+33r^8+124r^7+126r^6-396r^5-1408r^4-2077r^3-1967r^2-798r+360}{32r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{5}{2304}$
a_6	$\frac{-r^{12}-24r^{11}-239r^{10}-1290r^9-4120r^8-8034r^7-10042r^6-9933r^5-8898r^4+3021r^3+31580r^2+43980r+18000}{64r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$	$\frac{5}{21504}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-4r^{13} - 120r^{12} - 1542r^{11} - 11000r^{10} - 46560r^9 - 110400r^8 - 81043r^7 + 331680r^6 + 1238434r^5 +}{128r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)^2\left(5+r\right)^2\left(6+\right.}$$

Which for the root $r = 1$ becomes

$$a_7 = \frac{15}{50176}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{r-3}{2r(1+r)}$
a_2	$\frac{-r^4+2r^2-5r+6}{4r(1+r)^2(2+r)}$
a_3	$\frac{-2r^5+9r^3+6r^2+17r-6}{8r(1+r)^2(2+r)^2(3+r)}$
a_4	$\frac{r^8+8r^7+19r^6+17r^5+9r^4+17r^3-5r^2-114r-72}{16r(1+r)^2(2+r)^2(3+r)^2(4+r)}$
a_5	$\frac{3r^9+33r^8+124r^7+126r^6-396r^5-1408r^4-2077r^3-1967r^2-798r+360}{32r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$
a_6	$\frac{-r^{12}-24r^{11}-239r^{10}-1290r^9-4120r^8-8034r^7-10042r^6-9933r^5-8898r^4+3021r^3+31580r^2+43980r+18000}{64r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$
a_7	$\frac{-4r^{13}-120r^{12}-1542r^{11}-11000r^{10}-46560r^9-110400r^8-81043r^7+331680r^6+1238434r^5+2147540r^4+2362375r^3+1656300r^2+452340r}{128r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)^2(7+r)}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\big(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\big) \\ &= x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + \frac{5x^6}{21504} + \frac{15x^7}{50176} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{r-3}{2r\left(1+r\right)} \end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2}\frac{r-3}{2r\left(1+r\right)} &= \lim_{r\to 0}\frac{r-3}{2r\left(1+r\right)}\\ &= \textit{undefined}\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln\left(x\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln\left(x\right) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\\ &= Cy_1'(x)\ln\left(x\right) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)\\ \frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln\left(x\right) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\\ &\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\\ &= Cy_1''(x)\ln\left(x\right) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\left(-1+n+r_2\right)\right)\end{aligned}$$

Substituting these back into the given ode $y''(x^2+4)\,x - 2xy' + 6y = 0$ gives

$$\begin{aligned}&\left(Cy_1''(x)\ln\left(x\right) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right.\\ &\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\left(x^2+4\right)x\\ &- 2x\left(Cy_1'(x)\ln\left(x\right) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\ &+ 6Cy_1(x)\ln\left(x\right) + 6\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0\end{aligned}$$

Which can be written as

$$\left(\left(y_1''(x)\left(x^2+4\right)x-2y_1'(x)\,x+6y_1(x)\right)\ln\left(x\right)+\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)\left(x^2+4\right)x - 2y_1(x)\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\left(x^2+4\right)x - 2x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+6\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\left(x^2+4\right)x-2y_1'(x)\,x+6y_1(x)=0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)\left(x^2+4\right)x - 2y_1(x)\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\left(x^2+4\right)x - 2x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+6\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2x(x^2+4)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1}a_n(n+r_1)\right)-(x^2+2x+4)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C}{x} + \frac{(x^4+4x^2)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2}b_n(n+r_2)\left(-1+n+r_2\right)\right)-2\left(\sum_{n=0}^{\infty} x^{-1+n+r_2}b_n(n+r_2)\right)x^2+6\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$
\begin{aligned}
&\frac{\left(2x(x^2+4)\left(\sum_{n=0}^{\infty} x^n a_n (1+n)\right) - (x^2+2x+4)\left(\sum_{n=0}^{\infty} a_n x^{1+n}\right)\right) C}{x} \\
&+ \frac{(x^4+4x^2)\left(\sum_{n=0}^{\infty} x^{n-2} b_n n(n-1)\right) - 2\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x^2 + 6\left(\sum_{n=0}^{\infty} b_n x^n\right) x}{x} \\
&= 0
\end{aligned} \tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (1+n)\right) + \left(\sum_{n=0}^{\infty} 8C\, x^n a_n (1+n)\right) + \sum_{n=0}^{\infty} \left(-C\, x^{n+2} a_n\right) \\
&+ \sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-4C a_n x^n\right) + \left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (n-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 4n\, x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-2x^n b_n n\right) + \left(\sum_{n=0}^{\infty} 6 b_n x^n\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (1+n) &= \sum_{n=3}^{\infty} 2C a_{n-3} (n-2)\, x^{n-1} \\
\sum_{n=0}^{\infty} 8C\, x^n a_n (1+n) &= \sum_{n=1}^{\infty} 8C a_{n-1} n\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-C\, x^{n+2} a_n\right) &= \sum_{n=3}^{\infty} \left(-C a_{n-3} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n\right) &= \sum_{n=2}^{\infty} \left(-2C a_{n-2} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-4C a_n x^n\right) &= \sum_{n=1}^{\infty} \left(-4C a_{n-1} x^{n-1}\right)
\end{aligned}
$$

$$\sum_{n=0}^{\infty} n\,x^{1+n} b_n (n-1) = \sum_{n=2}^{\infty} (n-2)\, b_{n-2} (n-3)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-2x^{n} b_n n\right) = \sum_{n=1}^{\infty} \left(-2(n-1)\, b_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 6 b_n x^{n} = \sum_{n=1}^{\infty} 6 b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{gathered}\left(\sum_{n=3}^{\infty} 2C a_{n-3} (n-2)\, x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 8C a_{n-1} n\, x^{n-1}\right) \\ + \sum_{n=3}^{\infty} \left(-C a_{n-3} x^{n-1}\right) + \sum_{n=2}^{\infty} \left(-2C a_{n-2} x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-4C a_{n-1} x^{n-1}\right) \\ + \left(\sum_{n=2}^{\infty} (n-2)\, b_{n-2} (n-3)\, x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 4n\, x^{n-1} b_n (n-1)\right) \\ + \sum_{n=1}^{\infty} \left(-2(n-1)\, b_{n-1} x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 6 b_{n-1} x^{n-1}\right) = 0\end{gathered} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$4C + 6 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{3}{2}$$

For $n = 2$, Eq (2B) gives

$$(-2a_0 + 12a_1)\, C + 4b_1 + 8b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12 + 8b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{2}$$

For $n = 3$, Eq (2B) gives

$$(a_0 - 2a_1 + 20a_2) C + 2b_2 + 24b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{29}{4} + 24b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{29}{96}$$

For $n = 4$, Eq (2B) gives

$$(3a_1 - 2a_2 + 28a_3) C + 2b_2 + 48b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{3}{2} + 48b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{32}$$

For $n = 5$, Eq (2B) gives

$$(5a_2 - 2a_3 + 36a_4) C + 6b_3 - 2b_4 + 80b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{105}{64} + 80b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{21}{1024}$$

For $n = 6$, Eq (2B) gives

$$(7a_3 - 2a_4 + 44a_5) C + 12b_4 - 4b_5 + 120b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{287}{768} + 120b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = -\frac{287}{92160}$$

For $n = 7$, Eq (2B) gives

$$(9a_4 - 2a_5 + 52a_6)\,C + 20b_5 - 6b_6 + 168b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{40961}{107520} + 168b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = \frac{40961}{18063360}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{3}{2}$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = &-\frac{3}{2}\left(x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + \frac{5x^6}{21504} + \frac{15x^7}{50176} + O(x^8)\right)\right)\ln(x) \\ &+ 1 - \frac{3x^2}{2} + \frac{29x^3}{96} + \frac{x^4}{32} - \frac{21x^5}{1024} - \frac{287x^6}{92160} + \frac{40961x^7}{18063360} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + \frac{5x^6}{21504} + \frac{15x^7}{50176} + O(x^8)\right) \\ &\quad + c_2\left(-\frac{3}{2}\left(x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + \frac{5x^6}{21504} + \frac{15x^7}{50176} + O(x^8)\right)\right)\ln(x)\right. \\ &\qquad \left. + 1 - \frac{3x^2}{2} + \frac{29x^3}{96} + \frac{x^4}{32} - \frac{21x^5}{1024} - \frac{287x^6}{92160} + \frac{40961x^7}{18063360} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1-\frac{x}{2}+\frac{x^2}{24}+\frac{x^3}{48}-\frac{x^4}{384}-\frac{5x^5}{2304}+\frac{5x^6}{21504}+\frac{15x^7}{50176}+O(x^8)\right) \\
&\quad + c_2\left(-\frac{3x\left(1-\frac{x}{2}+\frac{x^2}{24}+\frac{x^3}{48}-\frac{x^4}{384}-\frac{5x^5}{2304}+\frac{5x^6}{21504}+\frac{15x^7}{50176}+O(x^8)\right)\ln(x)}{2}+1-\frac{3x^2}{2}\right. \\
&\qquad \left.+\frac{29x^3}{96}+\frac{x^4}{32}-\frac{21x^5}{1024}-\frac{287x^6}{92160}+\frac{40961x^7}{18063360}+O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1-\frac{x}{2}+\frac{x^2}{24}+\frac{x^3}{48}-\frac{x^4}{384}-\frac{5x^5}{2304}+\frac{5x^6}{21504}+\frac{15x^7}{50176}+O(x^8)\right) \\
&\quad + c_2\left(-\frac{3x\left(1-\frac{x}{2}+\frac{x^2}{24}+\frac{x^3}{48}-\frac{x^4}{384}-\frac{5x^5}{2304}+\frac{5x^6}{21504}+\frac{15x^7}{50176}+O(x^8)\right)\ln(x)}{2}+1\right. \\
&\qquad \left.-\frac{3x^2}{2}+\frac{29x^3}{96}+\frac{x^4}{32}-\frac{21x^5}{1024}-\frac{287x^6}{92160}+\frac{40961x^7}{18063360}+O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1-\frac{x}{2}+\frac{x^2}{24}+\frac{x^3}{48}-\frac{x^4}{384}-\frac{5x^5}{2304}+\frac{5x^6}{21504}+\frac{15x^7}{50176}+O(x^8)\right) \\
&\quad + c_2\left(-\frac{3x\left(1-\frac{x}{2}+\frac{x^2}{24}+\frac{x^3}{48}-\frac{x^4}{384}-\frac{5x^5}{2304}+\frac{5x^6}{21504}+\frac{15x^7}{50176}+O(x^8)\right)\ln(x)}{2}+1-\frac{3x^2}{2}\right. \\
&\qquad \left.+\frac{29x^3}{96}+\frac{x^4}{32}-\frac{21x^5}{1024}-\frac{287x^6}{92160}+\frac{40961x^7}{18063360}+O(x^8)\right)
\end{aligned}
$$

Verified OK.

2.5.1 Maple step by step solution

Let's solve

$y''(x^2+4)\,x-2xy'+6y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{6y}{(x^2+4)x}+\frac{2y'}{x^2+4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2y'}{x^2+4}+\frac{6y}{(x^2+4)x}=0$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x)=-\frac{2}{x^2+4}, P_3(x)=\frac{6}{x(x^2+4)}\right]$

  ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''(x^2+4)\,x-2xy'+6y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m =1..3$

  $$x^{m}\cdot y'' = \sum_{k =0}^{\infty} a_{k} \left(k +r \right) \left(k +r -1\right) x^{k +r -2+m}$$

- Shift index using $k - > k +2-m$

  $$x^{m}\cdot y'' = \sum_{k =-2+m}^{\infty} a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) x^{k +r}$$

  Rewrite ODE with series expansions

  $$4 a_{0} r \left(-1+r \right) x^{-1+r}+\left(4 a_{1} \left(1+r \right) r -2 a_{0} \left(-3+r \right)\right) x^{r}+\Bigg(\sum_{k =1}^{\infty} \left(4 a_{k +1} \left(k +1+r \right) \left(k +r \right)-2 a_{k} (k$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$4 r \left(-1+r \right) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, 1\right\}$$

- Each term must be 0

  $$4 a_{1} \left(1+r \right) r -2 a_{0} \left(-3+r \right) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$4 a_{k +1} \left(k +1+r \right) \left(k +r \right)-2 a_{k} \left(k +r -3\right)+a_{k -1} \left(k +r -1\right) \left(k -2+r \right) = 0$$

- Shift index using $k - > k +1$

  $$4 a_{k +2} \left(k +2+r \right) \left(k +1+r \right)-2 a_{k +1} \left(k -2+r \right)+a_{k} \left(k +r \right) \left(k +r -1\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k +2} = -\frac{k^{2} a_{k}+2 k r a_{k}+r^{2} a_{k}-a_{k} k -2 k a_{k +1}-a_{k} r -2 r a_{k +1}+4 a_{k +1}}{4 \left(k +2+r \right) \left(k +1+r \right)}$$

- Recursion relation for $r = 0$

  $$a_{k +2} = -\frac{k^{2} a_{k}-a_{k} k -2 k a_{k +1}+4 a_{k +1}}{4 \left(k +2\right) \left(k +1\right)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k =0}^{\infty} a_{k} x^{k}, a_{k +2} = -\frac{k^{2} a_{k}-a_{k} k -2 k a_{k +1}+4 a_{k +1}}{4 \left(k +2\right) \left(k +1\right)}, 6 a_{0} = 0\right]$$

- Recursion relation for $r = 1$

  $$a_{k +2} = -\frac{k^{2} a_{k}+a_{k} k -2 k a_{k +1}+2 a_{k +1}}{4 \left(k +3\right) \left(k +2\right)}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k =0}^{\infty} a_{k} x^{k +1}, a_{k +2} = -\frac{k^{2} a_{k}+a_{k} k -2 k a_{k +1}+2 a_{k +1}}{4 \left(k +3\right) \left(k +2\right)}, 8 a_{1}+4 a_{0} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{k^2 a_k - k a_k - 2k a_{k+1} + 4 a_{k+1}}{4(k+2)(k+1)}, 6a_0 = 0, b_{k+2} = -\frac{k^2 b_k + k b_k - 2k b_k}{4(k+3)(k}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 72

```
Order:=8;
dsolve((x^3+4*x)*diff(y(x),x$2)-2*x*diff(y(x),x)+6*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x\left(1-\frac{1}{2}x+\frac{1}{24}x^2+\frac{1}{48}x^3-\frac{1}{384}x^4-\frac{5}{2304}x^5+\frac{5}{21504}x^6+\frac{15}{50176}x^7+\mathrm{O}\left(x^8\right)\right) \\
& +\left(-\frac{3}{2}x+\frac{3}{4}x^2-\frac{1}{16}x^3-\frac{1}{32}x^4+\frac{1}{256}x^5+\frac{5}{1536}x^6-\frac{5}{14336}x^7+\mathrm{O}\left(x^8\right)\right)\ln\left(x\right)c_2 \\
& +\left(1+\frac{1}{2}x-\frac{7}{4}x^2+\frac{31}{96}x^3+\frac{1}{24}x^4-\frac{67}{3072}x^5-\frac{43}{10240}x^6+\frac{43061}{18063360}x^7+\mathrm{O}\left(x^8\right)\right)c_2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.237 (sec). Leaf size: 121

```
AsymptoticDSolveValue[(x^3+4*x)*y''[x]-2*x*y'[x]+6*y[x]==0,y[x],{x,0,7}]
```

$$
\begin{aligned}
y(x) \to {} & c_1\left(\frac{x(5x^5+6x^4-48x^3-96x^2+1152x-2304)\log(x)}{1536}\right. \\
& \left.+\frac{-229x^6-790x^5+2240x^4+11840x^3-76800x^2+61440x+30720}{30720}\right) \\
& +c_2\left(\frac{5x^7}{21504}-\frac{5x^6}{2304}-\frac{x^5}{384}+\frac{x^4}{48}+\frac{x^3}{24}-\frac{x^2}{2}+x\right)
\end{aligned}
$$

2.6 problem 6

Internal problem ID [6589]
Internal file name [`OUTPUT/5837_Sunday_June_05_2022_03_56_52_PM_12898100/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x-5)^2 y''+4xy'+\left(x^2-25\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^4-10x^3+25x^2\right)y''+4xy'+\left(x^2-25\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x\left(x-5\right)^2} \\ q(x) &= \frac{5+x}{\left(x-5\right)x^2} \end{aligned}$$

Table 34: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4}{x(x-5)^2}$	
singularity	type
$x = 0$	"regular"
$x = 5$	"irregular"

$q(x) = \frac{5+x}{(x-5)x^2}$	
singularity	type
$x = 0$	"regular"
$x = 5$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[5]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2 - 10x + 25)\, y'' + 4xy' + (x^2 - 25)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &x^2(x^2 - 10x + 25)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ 4x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (x^2 - 25)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-10 x^{1+n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 25 x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4 x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-25 a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-10 x^{1+n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-10 a_{n-1} (n+r-1)(n+r-2) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}\right) \\
&+ \sum_{n=1}^{\infty} \left(-10 a_{n-1} (n+r-1)(n+r-2) x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 25 x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4 x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-25 a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$25 x^{n+r} a_n (n+r)(n+r-1) + 4 x^{n+r} a_n (n+r) - 25 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$25x^r a_0 r(-1+r) + 4x^r a_0 r - 25a_0 x^r = 0$$

Or

$$\left(25x^r r(-1+r) + 4x^r r - 25x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(25r^2 - 21r - 25\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$25r^2 - 21r - 25 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{21}{50} + \frac{\sqrt{2941}}{50} \\ r_2 &= \frac{21}{50} - \frac{\sqrt{2941}}{50} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(25r^2 - 21r - 25\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{\sqrt{2941}}{25}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{21}{50}+\frac{\sqrt{2941}}{50}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{21}{50}-\frac{\sqrt{2941}}{50}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{10r(-1+r)}{25r^2 + 29r - 21}$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
a_{n-2}(n+r-2)(n-3+r) - 10a_{n-1}(n+r-1)(n+r-2) \\
+ 25a_n(n+r)(n+r-1) + 4a_n(n+r) + a_{n-2} - 25a_n = 0
\end{aligned}
\tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{n^2 a_{n-2} - 10n^2 a_{n-1} + 2nr a_{n-2} - 20nr a_{n-1} + r^2 a_{n-2} - 10r^2 a_{n-1} - 5n a_{n-2} + 30n a_{n-1} - 5r a_{n-2} +}{25n^2 + 50nr + 25r^2 - 21n - 21r - 25}
\tag{4}
$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$
a_n = \frac{((-25a_{n-2} + 250a_{n-1})\, n + 52a_{n-2} - 270a_{n-1}) \sqrt{2941} + (-625a_{n-2} + 6250a_{n-1})\, n^2 + (2600a_{n-2} - 1}{625n\left(\sqrt{2941} + 25n\right)}
\tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$\frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right)}{6250+250\sqrt{2941}}$

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{75r^4 - 4r^3 - 75r^2 - 50r + 21}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)}
$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$
a_2 = \frac{\frac{717381}{15625} + \frac{7911\sqrt{2941}}{15625}}{\left(25 + \sqrt{2941}\right)\left(50 + \sqrt{2941}\right)}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$\frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right)}{6250+250\sqrt{2941}}$
a_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}+\frac{7911\sqrt{2941}}{15625}}{\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{500r^6 + 1420r^5 + 300r^4 - 2580r^3 - 2000r^2 - 40r + 420}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r^2 + 129r + 137)}$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$a_3 = \frac{\frac{906742764}{1953125} + \frac{15291084\sqrt{2941}}{1953125}}{\left(25 + \sqrt{2941}\right)\left(50 + \sqrt{2941}\right)\left(75 + \sqrt{2941}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$\frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right)}{6250+250\sqrt{2941}}$
a_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}+\frac{7911\sqrt{2941}}{15625}}{\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)}$
a_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	$\frac{\frac{906742764}{1953125}+\frac{15291084\sqrt{2941}}{1953125}}{\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3125r^8 + 24000r^7 + 61766r^6 + 33496r^5 - 104033r^4 - 168040r^3 - 59654r^2 + 22392r + 16569}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r^2 + 129r + 137)(25r^2 + 179r + 291)}$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$a_4 = \frac{\frac{1473770634612}{244140625} + \frac{26407796172\sqrt{2941}}{244140625}}{\left(25 + \sqrt{2941}\right)\left(50 + \sqrt{2941}\right)\left(75 + \sqrt{2941}\right)\left(100 + \sqrt{2941}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$\frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right)}{6250+250\sqrt{2941}}$
a_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}+\frac{7911\sqrt{2941}}{15625}}{\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)}$
a_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	$\frac{\frac{906742764}{1953125}+\frac{15291084\sqrt{2941}}{1953125}}{\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)}$
a_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$	$\frac{\frac{1473770634612}{244140625}+\frac{26407796172\sqrt{2941}}{244140625}}{\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)(100+}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{18750r^{10} + 271250r^9 + 1552980r^8 + 4225260r^7 + 4278470r^6 - 4527750r^5 - 15744100r^4 - 1351844}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r^2 + 129r + 137)(25r^2 + 179r + 291)(2}$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$a_5 = \frac{\frac{10008934775328384}{152587890625} + \frac{181292058002304\sqrt{2941}}{152587890625}}{\left(25 + \sqrt{2941}\right)\left(50 + \sqrt{2941}\right)\left(75 + \sqrt{2941}\right)\left(100 + \sqrt{2941}\right)\left(125 + \sqrt{2941}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	
a_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	
a_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	
a_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$	
a_5	$\frac{18750r^{10}+271250r^9+1552980r^8+4225260r^7+4278470r^6-4527750r^5-15744100r^4-13518440r^3-1814650r^2+2790990r+1132740}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)}$	

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{109375r^{12} + 2537500r^{11} + 24880275r^{10} + 132196686r^9 + 398346148r^8 + 597569752r^7 - 1038473r^6}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r^2 + 129r + 137)(25r^2}$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$a_6 = \frac{-\frac{13371141904684696752}{19073486328125} - \frac{2501871693105765 12\sqrt{2941}}{19073486328125}}{\left(25 + \sqrt{2941}\right)\left(50 + \sqrt{2941}\right)\left(75 + \sqrt{2941}\right)\left(100 + \sqrt{2941}\right)\left(125 + \sqrt{2941}\right)\left(150 + \sqrt{2941}\right)}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$
a_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$
a_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$
a_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$
a_5	$\frac{18750r^{10}+271250r^9+1552980r^8+4225260r^7+4278470r^6-4527750r^5-15744100r^4-13518440r^3-1814650r^2+2790990r+1132740}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)}$
a_6	$\frac{109375r^{12}+2537500r^{11}+24880275r^{10}+132196686r^9+398346148r^8+597569752r^7-1038473r^6-1572368238r^5-2496579016r^4-14344178}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)(25r^2+279r}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{625000r^{14} + 21175000r^{13} + 314237000r^{12} + 2668782440r^{11} + 14161988600r^{10} + 47582976120r^9 + 9}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r}$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$a_7 = \frac{-\frac{1986234949213345504306464}{16689300537109375} - \frac{5236390568182719530112\sqrt{2941}}{2384185791015625}}{(25 + \sqrt{2941})(50 + \sqrt{2941})(75 + \sqrt{2941})(100 + \sqrt{2941})(125 + \sqrt{2941})(150 + \sqrt{2941})(175 +}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$
a_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$
a_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$
a_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$
a_5	$\frac{18750r^{10}+271250r^9+1552980r^8+4225260r^7+4278470r^6-4527750r^5-15744100r^4-13518440r^3-1814650r^2+2790990r+1132740}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)}$
a_6	$\frac{109375r^{12}+2537500r^{11}+24880275r^{10}+132196686r^9+398346148r^8+597569752r^7-1038473r^6-1572368238r^5-2496579016r^4-14344178}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)(25r^2+279r}$
a_7	$\frac{625000r^{14}+21175000r^{13}+314237000r^{12}+2668782440r^{11}+14161988600r^{10}+47582976120r^9+94042613520r^8+68475695800r^7-14203416}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}} \left(1 + \frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right)x}{6250+250\sqrt{2941}} + \frac{9\left(79709+879\sqrt{2941}\right)x^2}{15625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)} + 1953125 \right. \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{10r(-1+r)}{25r^2 + 29r - 21}$$

For $2 \leq n$ the recursive equation is

$$\begin{gathered} b_{n-2}(n+r-2)(n-3+r) - 10b_{n-1}(n+r-1)(n+r-2) \\ + 25b_n(n+r)(n+r-1) + 4b_n(n+r) + b_{n-2} - 25b_n = 0 \end{gathered} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{n^2 b_{n-2} - 10n^2 b_{n-1} + 2nr b_{n-2} - 20nr b_{n-1} + r^2 b_{n-2} - 10r^2 b_{n-1} - 5n b_{n-2} + 30n b_{n-1} - 5r b_{n-2} + 30}{25n^2 + 50nr + 25r^2 - 21n - 21r - 25} \tag{4}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_n = \frac{\left(\left(25b_{n-2} - 250b_{n-1}\right) n - 52b_{n-2} + 270b_{n-1}\right) \sqrt{2941} + \left(-625b_{n-2} + 6250b_{n-1}\right) n^2 + \left(2600b_{n-2} - 1350}{625n \left(-\sqrt{2941} + 25n\right)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$-\frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)}{-6250+250\sqrt{2941}}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{75r^4 - 4r^3 - 75r^2 - 50r + 21}{\left(25r^2 + 29r - 21\right)\left(25r^2 + 79r + 33\right)}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_2 = \frac{\frac{717381}{15625} - \frac{7911\sqrt{2941}}{15625}}{\left(-25 + \sqrt{2941}\right)\left(-50 + \sqrt{2941}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$-\frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)}{-6250+250\sqrt{2941}}$
b_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}-\frac{7911\sqrt{2941}}{15625}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{500r^6 + 1420r^5 + 300r^4 - 2580r^3 - 2000r^2 - 40r + 420}{\left(25r^2 + 29r - 21\right)\left(25r^2 + 79r + 33\right)\left(25r^2 + 129r + 137\right)}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_3 = \frac{-\frac{906742764}{1953125} + \frac{15291084\sqrt{2941}}{1953125}}{\left(-25 + \sqrt{2941}\right)\left(-50 + \sqrt{2941}\right)\left(-75 + \sqrt{2941}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$-\frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)}{-6250+250\sqrt{2941}}$
b_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}-\frac{7911\sqrt{2941}}{15625}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)}$
b_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	$\frac{-\frac{906742764}{1953125}+\frac{15291084\sqrt{2941}}{1953125}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{3125r^8 + 24000r^7 + 61766r^6 + 33496r^5 - 104033r^4 - 168040r^3 - 59654r^2 + 22392r + 16569}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r^2 + 129r + 137)(25r^2 + 179r + 291)}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_4 = \frac{\frac{1473770634612}{244140625} - \frac{26407796172\sqrt{2941}}{244140625}}{\left(-25 + \sqrt{2941}\right)\left(-50 + \sqrt{2941}\right)\left(-75 + \sqrt{2941}\right)\left(-100 + \sqrt{2941}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$-\frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)}{-6250+250\sqrt{2941}}$
b_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}-\frac{7911\sqrt{2941}}{15625}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)}$
b_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	$\frac{-\frac{906742764}{1953125}+\frac{15291084\sqrt{2941}}{1953125}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)}$
b_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$	$\frac{\frac{1473770634612}{244140625}-\frac{26407796172\sqrt{2941}}{244140625}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)(}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{18750r^{10} + 271250r^9 + 1552980r^8 + 4225260r^7 + 4278470r^6 - 4527750r^5 - 15744100r^4 - 13518440}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r^2 + 129r + 137)(25r^2 + 179r + 291)(2}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_5 = \frac{-\frac{10008934775328384}{152587890625} + \frac{181292058002304\sqrt{2941}}{152587890625}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)\left(-125+\sqrt{2941}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$-$
b_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$($−
b_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	$($−
b_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$	$($−
b_5	$\frac{18750r^{10}+271250r^9+1552980r^8+4225260r^7+4278470r^6-4527750r^5-15744100r^4-13518440r^3-1814650r^2+2790990r+1132740}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)}$	$($−

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{109375r^{12} + 2537500r^{11} + 24880275r^{10} + 132196686r^9 + 398346148r^8 + 597569752r^7 - 1038473r^6}{\left(25r^2+29r-21\right)\left(25r^2+79r+33\right)\left(25r^2+129r+137\right)(25r^2}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_6 = \frac{-\frac{13371141904684696752}{19073486328125} + \frac{250187169310576512\sqrt{2941}}{19073486328125}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)\left(-125+\sqrt{2941}\right)(-150+\sqrt{2941}}$$

And the table now becomes

n	$b_{n,r}$
b_0	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$
b_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$
b_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$
b_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$
b_5	$\frac{18750r^{10}+271250r^9+1552980r^8+4225260r^7+4278470r^6-4527750r^5-15744100r^4-13518440r^3-1814650r^2+2790990r+1132740}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)}$
b_6	$\frac{109375r^{12}+2537500r^{11}+24880275r^{10}+132196686r^9+398346148r^8+597569752r^7-1038473r^6-1572368238r^5-2496579016r^4-14344178}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)(25r^2+279r}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{625000r^{14} + 21175000r^{13} + 314237000r^{12} + 2668782440r^{11} + 14161988600r^{10} + 47582976120r^9 +}{\left(25r^2 + 29r - 21\right)\left(25r^2 + 79r + 33\right)\left(25r}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_7 = \frac{\frac{19862349492133455504306464}{16689300537109375} - \frac{52363905681827195301 12\sqrt{2941}}{2384185791015625}}{\left(-25 + \sqrt{2941}\right)\left(-50 + \sqrt{2941}\right)\left(-75 + \sqrt{2941}\right)\left(-100 + \sqrt{2941}\right)\left(-125 + \sqrt{2941}\right)\left(-150 + \sqrt{2941}}$$

And the table now becomes

n	$b_{n,r}$
b_0	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$
b_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$
b_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$
b_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$
b_5	$\frac{18750r^{10}+271250r^9+1552980r^8+4225260r^7+4278470r^6-4527750r^5-15744100r^4-13518440r^3-1814650r^2+2790990r+1132740}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)}$
b_6	$\frac{109375r^{12}+2537500r^{11}+24880275r^{10}+132196686r^9+398346148r^8+597569752r^7-1038473r^6-1572368238r^5-2496579016r^4-14344178}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)(25r^2+279r-}$
b_7	$\frac{625000r^{14}+21175000r^{13}+314237000r^{12}+2668782440r^{11}+14161988600r^{10}+47582976120r^9+94042613520r^8+68475695800r^7-14203416}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+2}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6+b_7x^7+b_8x^8\dots\right) \\ &= x^{\frac{21}{50}-\frac{\sqrt{2941}}{50}}\left(1-\frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)x}{-6250+250\sqrt{2941}}+\frac{9\left(79709-879\sqrt{2941}\right)x^2}{15625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)}+\frac{}{195} \right. \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1y_1(x)+c_2y_2(x)$$

$$= c_1 x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}} \left(1 + \frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right)x}{6250+250\sqrt{2941}}\right.$$

$$+ \frac{9\left(79709+879\sqrt{2941}\right)x^2}{15625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)}$$

$$+ \frac{12\left(75561897+1274257\sqrt{2941}\right)x^3}{1953125\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)}$$

$$+ \frac{12\left(122814219551+2200649681\sqrt{2941}\right)x^4}{244140625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)}$$

$$+ \frac{1152\left(8688311436917+157371578127\sqrt{2941}\right)x^5}{152587890625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)\left(125+\sqrt{2941}\right)}$$

$$+ \frac{48\left(-278565456347597849-5212232693970344\sqrt{2941}\right)x^6}{19073486328125\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)\left(125+\sqrt{2941}\right)(15}$$

$$+ \frac{96\left(-206899473876390156698 59-381820145596656632404\sqrt{2941}\right)}{16689300537109375\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)\left(125+\sqrt{2941}\right)}$$

$$\left.+ O\left(x^8\right)\right) + c_2 x^{\frac{21}{50}-\frac{\sqrt{2941}}{50}} \left(1 - \frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)x}{-6250+250\sqrt{2941}}\right.$$

$$+ \frac{9\left(79709-879\sqrt{2941}\right)x^2}{15625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)}$$

$$+ \frac{12\left(-75561897+1274257\sqrt{2941}\right)x^3}{1953125\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)}$$

$$+ \frac{12\left(122814219551-2200649681\sqrt{2941}\right)x^4}{244140625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)}$$

$$+ \frac{1152\left(-8688311436917+157371578127\sqrt{2941}\right)x^5}{152587890625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)(-125+\sqrt{29}}$$

$$+ \frac{48\left(-278565456347597849+5212232693970344\sqrt{2941}\right)x^6}{19073486328125\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)(-125+\surd}$$

$$+ \frac{96(20689947387639015669859-381820145596656632404\surd}{16689300537109375\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)(-125+}$$

$$\left.+ O\left(x^8\right)\right)$$

Hence the final solution is

$$y = y_h$$

$$
\begin{aligned}
&= c_1 x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}} \Bigg(1 + \frac{(21+\sqrt{2941})(-29+\sqrt{2941})\,x}{6250+250\sqrt{2941}} \\
&\quad + \frac{9(79709+879\sqrt{2941})\,x^2}{15625\,(25+\sqrt{2941})(50+\sqrt{2941})} \\
&\quad + \frac{12(75561897+1274257\sqrt{2941})\,x^3}{1953125\,(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})} \\
&\quad + \frac{12(122814219551+2200649681\sqrt{2941})\,x^4}{244140625\,(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})(100+\sqrt{2941})} \\
&\quad + \frac{1152(8688311436917+157371578127\sqrt{2941})\,x^5}{152587890625\,(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})(100+\sqrt{2941})(125+\sqrt{2941})} \\
&\quad + \frac{48(-278565456347597849-5212232693970344\sqrt{2941})\,x^6}{19073486328125\,(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})(100+\sqrt{2941})(125+\sqrt{2941})(150+} \\
&\quad + \frac{96(-2068994738763901566985 9-3818201455966566324 04\sqrt{2941})\,x^7}{16689300537109375\,(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})(100+\sqrt{2941})(125+\sqrt{2941})(150} \\
&\quad + O(x^8)\Bigg) + c_2 x^{\frac{21}{50}-\frac{\sqrt{2941}}{50}} \Bigg(1 - \frac{(-21+\sqrt{2941})(29+\sqrt{2941})\,x}{-6250+250\sqrt{2941}} \\
&\quad + \frac{9(79709-879\sqrt{2941})\,x^2}{15625\,(-25+\sqrt{2941})(-50+\sqrt{2941})} \\
&\quad + \frac{12(-75561897+1274257\sqrt{2941})\,x^3}{1953125\,(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})} \\
&\quad + \frac{12(122814219551-2200649681\sqrt{2941})\,x^4}{244140625\,(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})} \\
&\quad + \frac{1152(-8688311436917+157371578127\sqrt{2941})\,x^5}{152587890625\,(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})(-125+\sqrt{2941})} \\
&\quad + \frac{48(-278565456347597849+5212232693970344\sqrt{2941})\,x^6}{19073486328125\,(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})(-125+\sqrt{2941}} \\
&\quad + \frac{96(20689947387639015669859-381820145596656632404\sqrt{294}}{16689300537109375\,(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})(-125+\sqrt{2}} \\
&\quad + O(x^8)\Bigg)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}} \Bigg(1 &+ \frac{(21+\sqrt{2941})(-29+\sqrt{2941})\,x}{6250+250\sqrt{2941}} \\
&+ \frac{9(79709+879\sqrt{2941})\,x^2}{15625\,(25+\sqrt{2941})(50+\sqrt{2941})} \\
&+ \frac{12(75561897+1274257\sqrt{2941})\,x^3}{1953125\,(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})} \\
&+ \frac{12(122814219551+2200649681\sqrt{2941})\,x^4}{244140625\,(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})(100+\sqrt{2941})} \\
&+ \frac{1152(8688311436917+157371578127\sqrt{2941})\,x^5}{152587890625\,(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})(100+\sqrt{2941})(125+\sqrt{2941})} \\
&+ \frac{48(-278565456347597849-5212232693970344\sqrt{2941})\,x^6}{19073486328125\,(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})(100+\sqrt{2941})(125+\sqrt{2941})(150+} \\
&+ \frac{96(-2068994738763901566985\,9-381820145596656632404\sqrt{2941})\,x^7}{16689300537109375\,(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})(100+\sqrt{2941})(125+\sqrt{2941})(150} \\
&+ O(x^8)\Bigg) + c_2 x^{\frac{21}{50}-\frac{\sqrt{2941}}{50}} \Bigg(1 - \frac{(-21+\sqrt{2941})(29+\sqrt{2941})\,x}{-6250+250\sqrt{2941}} \\
&+ \frac{9(79709-879\sqrt{2941})\,x^2}{15625\,(-25+\sqrt{2941})(-50+\sqrt{2941})} \\
&+ \frac{12(-75561897+1274257\sqrt{2941})\,x^3}{1953125\,(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})} \\
&+ \frac{12(122814219551-2200649681\sqrt{2941})\,x^4}{244140625\,(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})} \\
&+ \frac{1152(-8688311436917+157371578127\sqrt{2941})\,x^5}{152587890625\,(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})(-125+\sqrt{2941})} \\
&+ \frac{48(-278565456347597849+5212232693970344\sqrt{2941})\,x^6}{19073486328125\,(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})(-125+\sqrt{2941})} \\
&+ \frac{96(20689947387639015669859-381820145596656632404\sqrt{2941}}{16689300537109375\,(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})(-125+\sqrt{29}} \\
&+ O(x^8)\Bigg)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}} \left(1 + \frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right) x}{6250+250\sqrt{2941}} \right. \\
& + \frac{9\left(79709+879\sqrt{2941}\right) x^2}{15625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)} \\
& + \frac{12\left(75561897+1274257\sqrt{2941}\right) x^3}{1953125\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)} \\
& + \frac{12\left(122814219551+2200649681\sqrt{2941}\right) x^4}{244140625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)} \\
& + \frac{1152\left(8688311436917+157371578127\sqrt{2941}\right) x^5}{152587890625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)\left(125+\sqrt{2941}\right)} \\
& + \frac{48\left(-278565456347597849-5212232693970344\sqrt{2941}\right) x^6}{19073486328125\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)\left(125+\sqrt{2941}\right)\left(150+\surd\right.} \\
& + \frac{96\left(-20689947387639015669859-38182014559665663240 4\sqrt{2941}\right) x^7}{16689300537109375\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)\left(125+\sqrt{2941}\right)\left(150\right.} \\
& \left. + O\left(x^8\right)\right) + c_2 x^{\frac{21}{50}-\frac{\sqrt{2941}}{50}} \left(1 - \frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right) x}{-6250+250\sqrt{2941}} \right. \\
& + \frac{9\left(79709-879\sqrt{2941}\right) x^2}{15625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)} \\
& + \frac{12\left(-75561897+1274257\sqrt{2941}\right) x^3}{1953125\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)} \\
& + \frac{12\left(122814219551-2200649681\sqrt{2941}\right) x^4}{244140625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)} \\
& + \frac{1152\left(-8688311436917+157371578127\sqrt{2941}\right) x^5}{152587890625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)\left(-125+\sqrt{2941}\right)} \\
& + \frac{48\left(-278565456347597849+5212232693970344\sqrt{2941}\right) x^6}{19073486328125\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)\left(-125+\sqrt{2941}\right)} \\
& + \frac{96\left(20689947387639015669859-381820145596656632404\sqrt{2941}\right.}{16689300537109375\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)\left(-125+\sqrt{29}\right.} \\
& \left. + O\left(x^8\right)\right)
\end{aligned}
$$

Verified OK.

2.6.1 Maple step by step solution

Let's solve

$x^2(x^2-10x+25)\,y''+4xy'+(x^2-25)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(5+x)y}{(x-5)x^2}-\frac{4y'}{x(x^2-10x+25)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{4y'}{x(x^2-10x+25)}+\frac{(5+x)y}{(x-5)x^2}=0$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{4}{x(x^2-10x+25)},P_3(x)=\frac{5+x}{(x-5)x^2}\right]$

  ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{4}{25}$

  ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

  ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(x^2-10x+25)\,(x-5)+4y'x(x-5)+(x^2-10x+25)\,(5+x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m\cdot y$ to series expansion for $m=0..3$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k- >k-m$

    $$x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$$

  - Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

    $$x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

    $$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^{m}\cdot y''$ to series expansion for $m=2..5$

    $$x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k- >k+2-m$

    $$x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$-5a_{0}(25r^{2}-21r-25)\,x^{r}+\left(-5a_{1}(25r^{2}+29r-21)+a_{0}(75r^{2}-71r-25)\right)x^{1+r}+(-5a_{2}(25r^{2}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-125r^{2}+105r+125=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{\frac{21}{50}-\frac{\sqrt{2941}}{50},\frac{21}{50}+\frac{\sqrt{2941}}{50}\right\}$$

- The coefficients of each power of x must be 0

  $$[-5a_{1}(25r^{2}+29r-21)+a_{0}(75r^{2}-71r-25)=0,-5a_{2}(25r^{2}+79r+33)+a_{1}(75r^{2}+79r-2$$

- Solve for the dependent coefficient(s)

  $$\left\{a_{1}=\frac{a_{0}(75r^{2}-71r-25)}{5(25r^{2}+29r-21)},a_{2}=\frac{6a_{0}(625r^{4}+50r^{3}-989r^{2}-464r+175)}{25(625r^{4}+2700r^{3}+2591r^{2}-702r-693)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(-125a_{k}+a_{k-3}-15a_{k-2}+75a_{k-1})\,k^{2}+(2(-125a_{k}+a_{k-3}-15a_{k-2}+75a_{k-1})\,r+105a_{k}-7a_{k}$$

- Shift index using $k- >k+3$

  $$(-125a_{k+3}+a_{k}-15a_{k+1}+75a_{k+2})\,(k+3)^{2}+(2(-125a_{k+3}+a_{k}-15a_{k+1}+75a_{k+2})\,r+105a_{k-}$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = \frac{k^2a_k-15k^2a_{k+1}+75k^2a_{k+2}+2kra_k-30kra_{k+1}+150kra_{k+2}+r^2a_k-15r^2a_{k+1}+75r^2a_{k+2}-ka_k-15ka_{k+1}+229ka_{k+2}-ra}{5(25k^2+50kr+25r^2+129k+129r+137)}$$

- Recursion relation for $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$

$$a_{k+3} = \frac{k^2a_k-15k^2a_{k+1}+75k^2a_{k+2}+2k\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)a_k-30k\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)a_{k+1}+150k\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)a_{k+2}+\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)^2a_k-15(}{5(25k^2+50k(\frac{21}{50}}$$

- Solution for $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{21}{50}-\frac{\sqrt{2941}}{50}}, a_{k+3} = \frac{k^2a_k-15k^2a_{k+1}+75k^2a_{k+2}+2k\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)a_k-30k\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)a_{k+1}+150k(\frac{21}{50}-}{}\right.$$

- Recursion relation for $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$

$$a_{k+3} = \frac{k^2a_k-15k^2a_{k+1}+75k^2a_{k+2}+2k\left(\frac{21}{50}+\frac{\sqrt{2941}}{50}\right)a_k-30k\left(\frac{21}{50}+\frac{\sqrt{2941}}{50}\right)a_{k+1}+150k\left(\frac{21}{50}+\frac{\sqrt{2941}}{50}\right)a_{k+2}+\left(\frac{21}{50}+\frac{\sqrt{2941}}{50}\right)^2a_k-15(}{5(25k^2+50k(\frac{21}{50}}$$

- Solution for $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{21}{50}+\frac{\sqrt{2941}}{50}}, a_{k+3} = \frac{k^2a_k-15k^2a_{k+1}+75k^2a_{k+2}+2k\left(\frac{21}{50}+\frac{\sqrt{2941}}{50}\right)a_k-30k\left(\frac{21}{50}+\frac{\sqrt{2941}}{50}\right)a_{k+1}+150k(\frac{21}{50}+}{}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{21}{50}-\frac{\sqrt{2941}}{50}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{21}{50}+\frac{\sqrt{2941}}{50}}\right), a_{k+3} = \frac{k^2a_k-15k^2a_{k+1}+75k^2a_{k+2}+2k\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)a_k-}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a = 0, e <> 0, c <>
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 2223

```
Order:=8;
dsolve(x^2*(x-5)^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(x^2-25)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)$$

$$= x^{\frac{21}{50}}\Bigg(c_1 x^{-\frac{\sqrt{2941}}{50}}\Bigg(1+\frac{-1166-4\sqrt{2941}}{-3125+125\sqrt{2941}}x-\frac{9}{15625}\frac{879\sqrt{2941}-79709}{(-25+\sqrt{2941})(-50+\sqrt{2941})}x^2$$

$$+\frac{\frac{15291084\sqrt{2941}}{1953125}-\frac{906742764}{1953125}}{(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})}x^3$$

$$-\frac{12}{244140625}\frac{-122814219551+2200649681\sqrt{2941}}{(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})}x^4$$

$$+\frac{-\frac{10008934775328384}{152587890625}+\frac{181292058002304\sqrt{2941}}{152587890625}}{(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})(-125+\sqrt{2941})}x^5$$

$$+\frac{-\frac{133711419046846967 52}{19073486328125}+\frac{250187169310576512\sqrt{2941}}{19073486328125}}{(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})(-125+\sqrt{2941})(-150+\sqrt{2941}}$$

$$-\frac{96}{16689300537109375}\frac{-206899473876390156698 59+38182014559665663}{(-25+\sqrt{2941})(-50+\sqrt{2941})(-75+\sqrt{2941})(-100+\sqrt{2941})(-125+\sqrt{29}}$$

$$+\mathrm{O}\left(x^8\right)\Bigg)+c_2 x^{\frac{\sqrt{2941}}{50}}\Bigg(1+\frac{1166-4\sqrt{2941}}{125\sqrt{2941}+3125}x+\frac{\frac{7911\sqrt{2941}}{15625}+\frac{717381}{15625}}{(\sqrt{2941}+25)(50+\sqrt{2941})}x^2$$

$$+\frac{\frac{15291084\sqrt{2941}}{1953125}+\frac{906742764}{1953125}}{(\sqrt{2941}+25)(50+\sqrt{2941})(\sqrt{2941}+75)}x^3$$

$$+\frac{\frac{1473770634612}{244140625}+\frac{26407796172\sqrt{2941}}{244140625}}{(\sqrt{2941}+25)(50+\sqrt{2941})(\sqrt{2941}+75)(100+\sqrt{2941})}x^4$$

$$+\frac{\frac{10008934775328384}{152587890625}+\frac{181292058002304\sqrt{2941}}{152587890625}}{(\sqrt{2941}+25)(50+\sqrt{2941})(\sqrt{2941}+75)(100+\sqrt{2941})(125+\sqrt{2941})}x^5$$

$$-\frac{48}{19073486328125}\frac{278565456347597849+5212232693970344\sqrt{2941}}{(\sqrt{2941}+25)(50+\sqrt{2941})(\sqrt{2941}+75)(100+\sqrt{2941})(125+\sqrt{2941})(150+}$$

$$-\frac{96}{16689300537109375}\frac{206899473876390156698 59+381820145596656632404\sqrt{2}}{(\sqrt{2941}+25)(50+\sqrt{2941})(\sqrt{2941}+75)(100+\sqrt{2941})(125+\sqrt{2941})(150}$$

$$+\mathrm{O}\left(x^8\right)\Bigg)\Bigg)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 22488

```
AsymptoticDSolveValue[x^2*(x-5)^2*y''[x]+4*x*y'[x]+(x^2-25)*y[x]==0,y[x],{x,0,7}]
```

Too large to display

2.7 problem 7

Internal problem ID [6590]
Internal file name [`OUTPUT/5838_Sunday_June_05_2022_03_56_58_PM_28133876/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+x-6)\,y''+(x+3)\,y'+(-2+x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\ &= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (113) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (114) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{align}
F_0 &= f(x, y, y') \tag{4} \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$F_0 = -\frac{xy + xy' - 2y + 3y'}{x^2 + x - 6}$$

$$\begin{aligned}
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(-x^3 + 3x^2 + 20x + 6) y' + y(2x^2 - 3x - 2)}{(x^2 + x - 6)^2}
\end{aligned}$$

$$\begin{aligned}
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(4x^4 - 8x^3 - 88x^2 - 130x - 138) y' + y(-2 + x)(x^3 - 7x^2 - 19x - 17)}{(x^2 + x - 6)^3}
\end{aligned}$$

$$\begin{aligned}
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x + 3)(x^5 - 23x^4 + 73x^3 + 219x^2 + 704x + 468) y' - 6(-2 + x) y \left(x^4 - \frac{16}{3}x^3 - 22x^2 - \frac{140}{3}x - 42\right)}{(x^2 + x - 6)^4}
\end{aligned}$$

$$\begin{aligned}
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-9x^7 + 117x^6 + 132x^5 - 2220x^4 - 12075x^3 - 36237x^2 - 43104x - 22284) y' - y(-2 + x)(x^6 - 38}{(x^2 + x - 6)^5}
\end{aligned}$$

$$\begin{aligned}
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{-(x + 3)(x^8 - 78x^7 + 990x^6 - 1266x^5 - 6195x^4 - 85812x^3 - 202144x^2 - 268320x - 132288) y' +}{(x^2 + x - 6)^6}
\end{aligned}$$

$$\begin{aligned}
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{16(x + 3)(x^9 - \frac{177}{4}x^8 + 474x^7 - 327x^6 - 234x^5 - \frac{221553}{4}x^4 - 186109x^3 - \frac{791961}{2}x^2 - 404436x - 175}{}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{y(0)}{3}+\frac{y'(0)}{2}\\
F_1 &= -\frac{y(0)}{18}+\frac{y'(0)}{6}\\
F_2 &= -\frac{17y(0)}{108}+\frac{23y'(0)}{36}\\
F_3 &= -\frac{7y(0)}{18}+\frac{13y'(0)}{12}\\
F_4 &= -\frac{139y(0)}{162}+\frac{619y'(0)}{216}\\
F_5 &= -\frac{5377y(0)}{1944}+\frac{689y'(0)}{81}\\
F_6 &= -\frac{36547y(0)}{3888}+\frac{19493y'(0)}{648}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{6}x^{2}-\frac{1}{108}x^{3}-\frac{17}{2592}x^{4}-\frac{7}{2160}x^{5}-\frac{139}{116640}x^{6}-\frac{5377}{9797760}x^{7}-\frac{5221}{22394880}x^{8}\right) y(0)\\
& + \left(x+\frac{1}{4}x^{2}+\frac{1}{36}x^{3}+\frac{23}{864}x^{4}+\frac{13}{1440}x^{5}+\frac{619}{155520}x^{6}+\frac{689}{408240}x^{7}+\frac{19493}{26127360}x^{8}\right) y'(0)\\
& + O\left(x^{8}\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x^{2}+x-6\right) y''+\left(x+3\right) y'+\left(-2+x\right) y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1}\\
y'' &= \sum_{n=2}^{\infty} n \left(n-1\right) a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x^2+x-6)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+(x+3)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+(-2+x)\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}
$$

Which simplifies to

$$
\begin{split}
\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n(n-1)\right)+\sum_{n=2}^{\infty}\left(-6n(n-1)\, a_n x^{n-2}\right) \\
+\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=1}^{\infty} 3n a_n x^{n-1}\right)+\sum_{n=0}^{\infty}\left(-2a_n x^n\right)+\left(\sum_{n=0}^{\infty} x^{1+n} a_n\right)=0
\end{split} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n\, x^{n-1} a_n(n-1) &= \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n \\
\sum_{n=2}^{\infty}\left(-6n(n-1)\, a_n x^{n-2}\right) &= \sum_{n=0}^{\infty}\left(-6(n+2)\, a_{n+2}(1+n)\, x^n\right) \\
\sum_{n=1}^{\infty} 3n a_n x^{n-1} &= \sum_{n=0}^{\infty} 3(1+n)\, a_{1+n} x^n \\
\sum_{n=0}^{\infty} x^{1+n} a_n &= \sum_{n=1}^{\infty} a_{n-1} x^n
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\begin{split}
\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n\right) \\
+\sum_{n=0}^{\infty}\left(-6(n+2)\, a_{n+2}(1+n)\, x^n\right)+\left(\sum_{n=1}^{\infty} n a_n x^n\right) \\
+\left(\sum_{n=0}^{\infty} 3(1+n)\, a_{1+n} x^n\right)+\sum_{n=0}^{\infty}\left(-2a_n x^n\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^n\right)=0
\end{split} \tag{3}
$$

$n = 0$ gives

$$-12a_2 + 3a_1 - 2a_0 = 0$$

$$a_2 = -\frac{a_0}{6} + \frac{a_1}{4}$$

$n = 1$ gives

$$8a_2 - 36a_3 - a_1 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{108} + \frac{a_1}{36}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (1+n)\, a_{1+n}n - 6(n+2)\, a_{n+2}(1+n) + na_n + 3(1+n)\, a_{1+n} - 2a_n + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_n + n^2 a_{1+n} + 4na_{1+n} - 2a_n + 3a_{1+n} + a_{n-1}}{6\,(n+2)\,(1+n)} \\ &= \frac{(n^2-2)\, a_n}{6\,(n+2)\,(1+n)} + \frac{(n^2+4n+3)\, a_{1+n}}{6\,(n+2)\,(1+n)} + \frac{a_{n-1}}{6\,(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$2a_2 + 15a_3 - 72a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{17a_0}{2592} + \frac{23a_1}{864}$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 24a_4 - 120a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{7a_0}{2160} + \frac{13a_1}{1440}$$

For $n = 4$ the recurrence equation gives

$$14a_4 + 35a_5 - 180a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{139a_0}{116640} + \frac{619a_1}{155520}$$

For $n = 5$ the recurrence equation gives

$$23a_5 + 48a_6 - 252a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{5377a_0}{9797760} + \frac{689a_1}{408240}$$

For $n = 6$ the recurrence equation gives

$$34a_6 + 63a_7 - 336a_8 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = -\frac{5221a_0}{22394880} + \frac{19493a_1}{26127360}$$

For $n = 7$ the recurrence equation gives

$$47a_7 + 80a_8 - 432a_9 + a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{44713a_0}{423263232} + \frac{466993a_1}{1410877440}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y = a_0 + a_1 x &+ \left(-\frac{a_0}{6} + \frac{a_1}{4}\right) x^2 + \left(-\frac{a_0}{108} + \frac{a_1}{36}\right) x^3 + \left(-\frac{17 a_0}{2592} + \frac{23 a_1}{864}\right) x^4 \\
&+ \left(-\frac{7 a_0}{2160} + \frac{13 a_1}{1440}\right) x^5 + \left(-\frac{139 a_0}{116640} + \frac{619 a_1}{155520}\right) x^6 + \left(-\frac{5377 a_0}{9797760} + \frac{689 a_1}{408240}\right) x^7 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = &\left(1 - \frac{1}{6} x^2 - \frac{1}{108} x^3 - \frac{17}{2592} x^4 - \frac{7}{2160} x^5 - \frac{139}{116640} x^6 - \frac{5377}{9797760} x^7\right) a_0 \\
&+ \left(x + \frac{1}{4} x^2 + \frac{1}{36} x^3 + \frac{23}{864} x^4 + \frac{13}{1440} x^5 + \frac{619}{155520} x^6 + \frac{689}{408240} x^7\right) a_1 + O\left(x^8\right)
\end{aligned}
\tag{3}
$$

At $x = 0$ the solution above becomes

$$
\begin{aligned}
y = &\left(1 - \frac{1}{6} x^2 - \frac{1}{108} x^3 - \frac{17}{2592} x^4 - \frac{7}{2160} x^5 - \frac{139}{116640} x^6 - \frac{5377}{9797760} x^7\right) c_1 \\
&+ \left(x + \frac{1}{4} x^2 + \frac{1}{36} x^3 + \frac{23}{864} x^4 + \frac{13}{1440} x^5 + \frac{619}{155520} x^6 + \frac{689}{408240} x^7\right) c_2 + O\left(x^8\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\left(1 - \frac{1}{6} x^2 - \frac{1}{108} x^3 - \frac{17}{2592} x^4 - \frac{7}{2160} x^5 - \frac{139}{116640} x^6 - \frac{5377}{9797760} x^7 \right. \\
&\left. - \frac{5221}{22394880} x^8\right) y(0) + \left(x + \frac{1}{4} x^2 + \frac{1}{36} x^3 + \frac{23}{864} x^4 + \frac{13}{1440} x^5 + \frac{619}{155520} x^6 \right. \\
&\left. + \frac{689}{408240} x^7 + \frac{19493}{26127360} x^8\right) y'(0) + O\left(x^8\right)
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
y = &\left(1 - \frac{1}{6} x^2 - \frac{1}{108} x^3 - \frac{17}{2592} x^4 - \frac{7}{2160} x^5 - \frac{139}{116640} x^6 - \frac{5377}{9797760} x^7\right) c_1 \\
&+ \left(x + \frac{1}{4} x^2 + \frac{1}{36} x^3 + \frac{23}{864} x^4 + \frac{13}{1440} x^5 + \frac{619}{155520} x^6 + \frac{689}{408240} x^7\right) c_2 + O\left(x^8\right)
\end{aligned}
\tag{2}
$$

Verification of solutions

$$y = \left(1-\frac{1}{6}x^{2}-\frac{1}{108}x^{3}-\frac{17}{2592}x^{4}-\frac{7}{2160}x^{5}-\frac{139}{116640}x^{6}-\frac{5377}{9797760}x^{7}-\frac{5221}{22394880}x^{8}\right)y(0)$$
$$+\left(x+\frac{1}{4}x^{2}+\frac{1}{36}x^{3}+\frac{23}{864}x^{4}+\frac{13}{1440}x^{5}+\frac{619}{155520}x^{6}+\frac{689}{408240}x^{7}+\frac{19493}{26127360}x^{8}\right)y'(0)$$
$$+O(x^{8})$$

Verified OK.

$$y = \left(1-\frac{1}{6}x^{2}-\frac{1}{108}x^{3}-\frac{17}{2592}x^{4}-\frac{7}{2160}x^{5}-\frac{139}{116640}x^{6}-\frac{5377}{9797760}x^{7}\right)c_1$$
$$+\left(x+\frac{1}{4}x^{2}+\frac{1}{36}x^{3}+\frac{23}{864}x^{4}+\frac{13}{1440}x^{5}+\frac{619}{155520}x^{6}+\frac{689}{408240}x^{7}\right)c_2+O(x^{8})$$

Verified OK.

2.7.1 Maple step by step solution

Let's solve

$(x^2+x-6)\,y''+(x+3)\,y'+(-2+x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x+3}-\frac{y'}{-2+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{-2+x}+\frac{y}{x+3}=0$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{1}{-2+x}, P_3(x)=\frac{1}{x+3}\right]$

  ○ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3}=0$

  ○ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3}=0$

- ○ $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $y''(-2+x)(x+3)+(x+3)y'+(-2+x)y=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(u^2-5u)\left(\frac{d^2}{du^2}y(u)\right)+u\left(\frac{d}{du}y(u)\right)+(-5+u)y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ○ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ○ Convert $u\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)u^{k+r}$

  Rewrite ODE with series expansions

  $-5a_0r(-1+r)u^{-1+r}+\left(-5a_1(1+r)r+a_0(r^2-5)\right)u^r+\left(\sum_{k=1}^{\infty}(-5a_{k+1}(k+1+r)(k+r)+a_k$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-5r(-1+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{0, 1\}$

- Each term must be 0

  $-5a_1(1+r)\,r + a_0(r^2-5) = 0$

- Each term in the series must be 0, giving the recursion relation

  $-5a_{k+1}(k+1+r)\,(k+r) + k^2a_k + 2kra_k + r^2a_k - 5a_k + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $-5a_{k+2}(k+2+r)\,(k+1+r) + (k+1)^2\,a_{k+1} + 2(k+1)\,ra_{k+1} + r^2a_{k+1} - 5a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}+2ka_{k+1}+2ra_{k+1}+a_k-4a_{k+1}}{5(k+2+r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{k^2a_{k+1}+2ka_{k+1}+a_k-4a_{k+1}}{5(k+2)(k+1)}$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = \frac{k^2a_{k+1}+2ka_{k+1}+a_k-4a_{k+1}}{5(k+2)(k+1)}, -5a_0 = 0\right]$$

- Revert the change of variables $u = x+3$

  $$\left[y = \sum_{k=0}^{\infty} a_k(x+3)^k, a_{k+2} = \frac{k^2a_{k+1}+2ka_{k+1}+a_k-4a_{k+1}}{5(k+2)(k+1)}, -5a_0 = 0\right]$$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{k^2a_{k+1}+4ka_{k+1}+a_k-a_{k+1}}{5(k+3)(k+2)}$

- Solution for $r = 1$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = \frac{k^2a_{k+1}+4ka_{k+1}+a_k-a_{k+1}}{5(k+3)(k+2)}, -10a_1 - 4a_0 = 0\right]$$

- Revert the change of variables $u = x+3$

  $$\left[y = \sum_{k=0}^{\infty} a_k(x+3)^{k+1}, a_{k+2} = \frac{k^2a_{k+1}+4ka_{k+1}+a_k-a_{k+1}}{5(k+3)(k+2)}, -10a_1 - 4a_0 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k(x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+3)^{k+1}\right), a_{k+2} = \frac{k^2a_{k+1}+2ka_{k+1}+a_k-4a_{k+1}}{5(k+2)(k+1)}, -5a_0 = 0, b_{k+2} =\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 74

```
Order:=8;
dsolve((x^2+x-6)*diff(y(x),x$2)+(x+3)*diff(y(x),x)+(x-2)*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & \left(1-\frac{1}{6}x^{2}-\frac{1}{108}x^{3}-\frac{17}{2592}x^{4}-\frac{7}{2160}x^{5}-\frac{139}{116640}x^{6}-\frac{5377}{9797760}x^{7}\right)y(0) \\
& +\left(x+\frac{1}{4}x^{2}+\frac{1}{36}x^{3}+\frac{23}{864}x^{4}+\frac{13}{1440}x^{5}+\frac{619}{155520}x^{6}+\frac{689}{408240}x^{7}\right)D(y)\left(0\right)+O\left(x^{8}\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 98

```
AsymptoticDSolveValue[(x^2+x-6)*y''[x]+(x+3)*y'[x]+(x-2)*y[x]==0,y[x],{x,0,7}]
```

$$
\begin{aligned}
y(x) \rightarrow c_1\left(-\frac{5377x^7}{9797760}-\frac{139x^6}{116640}-\frac{7x^5}{2160}-\frac{17x^4}{2592}-\frac{x^3}{108}-\frac{x^2}{6}+1\right)\\
+c_2\left(\frac{689x^7}{408240}+\frac{619x^6}{155520}+\frac{13x^5}{1440}+\frac{23x^4}{864}+\frac{x^3}{36}+\frac{x^2}{4}+x\right)
\end{aligned}
$$

2.8 problem 8

Internal problem ID [6591]
Internal file name [`OUTPUT/5839_Sunday_June_05_2022_03_57_01_PM_30964848/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x\left(x^2+1\right)^2 y''+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^5+2x^3+x\right)y''+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=0\\ q(x)&=\frac{1}{x\left(x^2+1\right)^2}\end{aligned}$$

Table 37: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x(x^2+1)^2}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x^4 + 2x^2 + 1) y'' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x^4 + 2x^2 + 1) \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r+3} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\,(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{n+r+3} a_n (n+r)\,(n+r-1) &= \sum_{n=4}^{\infty} a_{n-4} (n+r-4)\,(n-5+r)\, x^{n+r-1} \\
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\,(n+r-1) &= \sum_{n=2}^{\infty} 2a_{n-2} (n+r-2)\,(n-3+r)\, x^{n+r-1} \\
\sum_{n=0}^{\infty} a_n x^{n+r} &= \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\left(\sum_{n=4}^{\infty} a_{n-4} (n+r-4)\,(n-5+r)\, x^{n+r-1}\right) \\
&+ \left(\sum_{n=2}^{\infty} 2a_{n-2} (n+r-2)\,(n-3+r)\, x^{n+r-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n (n+r)\,(n+r-1) = 0
$$

When $n=0$ the above becomes

$$
x^{-1+r} a_0 r(-1+r) = 0
$$

Or

$$x^{-1+r}a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{r(1+r)}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{-2r^4 + 2r^2 + 1}{r(1+r)^2(2+r)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = \frac{4r^4 + 8r^3 + 8r^2 + 4r - 1}{r(1+r)^2(2+r)^2(3+r)}$$

For $4 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-4}(n+r-4)(n-5+r) + 2a_{n-2}(n+r-2)(n-3+r) \\ &+ a_n(n+r)(n+r-1) + a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-4} + 2n^2 a_{n-2} + 2nra_{n-4} + 4nra_{n-2} + r^2 a_{n-4} + 2r^2 a_{n-2} - 9na_{n-4} - 10na_{n-2} - 9ra_{n-4} - 10r}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(-a_{n-4} - 2a_{n-2})n^2 + (7a_{n-4} + 6a_{n-2})n - 12a_{n-4} - 4a_{n-2} - a_{n-1}}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-2r^4+2r^2+1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{4r^4+8r^3+8r^2+4r-1}{r(1+r)^2(2+r)^2(3+r)}$	$\frac{23}{144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3r^8 + 24r^7 + 66r^6 + 60r^5 - 39r^4 - 108r^3 - 90r^2 - 60r - 23}{r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{167}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-2r^4+2r^2+1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{4r^4+8r^3+8r^2+4r-1}{r(1+r)^2(2+r)^2(3+r)}$	$\frac{23}{144}$
a_4	$\frac{3r^8+24r^7+66r^6+60r^5-39r^4-108r^3-90r^2-60r-23}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{167}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-10r^8 - 120r^7 - 592r^6 - 1548r^5 - 2342r^4 - 2172r^3 - 1208r^2 - 168r + 167}{r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)^2\left(5+r\right)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{7993}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-2r^4+2r^2+1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{4r^4+8r^3+8r^2+4r-1}{r(1+r)^2(2+r)^2(3+r)}$	$\frac{23}{144}$
a_4	$\frac{3r^8+24r^7+66r^6+60r^5-39r^4-108r^3-90r^2-60r-23}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{167}{2880}$
a_5	$\frac{-10r^8-120r^7-592r^6-1548r^5-2342r^4-2172r^3-1208r^2-168r+167}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{7993}{86400}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-4r^{12} - 96r^{11} - 980r^{10} - 5520r^9 - 18471r^8 - 35952r^7 - 32578r^6 + 12648r^5 + 67379r^4 + 80920r^3 +}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{23599}{518400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-2r^4+2r^2+1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{4r^4+8r^3+8r^2+4r-1}{r(1+r)^2(2+r)^2(3+r)}$	$\frac{23}{144}$
a_4	$\frac{3r^8+24r^7+66r^6+60r^5-39r^4-108r^3-90r^2-60r-23}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{167}{2880}$
a_5	$\frac{-10r^8-120r^7-592r^6-1548r^5-2342r^4-2172r^3-1208r^2-168r+167}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{7993}{86400}$
a_6	$\frac{-4r^{12}-96r^{11}-980r^{10}-5520r^9-18471r^8-35952r^7-32578r^6+12648r^5+67379r^4+80920r^3+59174r^2+30680r+7993}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$	$\frac{23599}{518400}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{20r^{12} + 600r^{11} + 7912r^{10} + 60300r^9 + 293952r^8 + 959040r^7 + 2133504r^6 + 3249060r^5 + 3370504r^4}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)^2(7+}$$

Which for the root $r = 1$ becomes

$$a_7 = \frac{1860281}{29030400}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$-\frac{1}{r(1+r)}$
a_2	$\frac{-2r^4+2r^2+1}{r(1+r)^2(2+r)}$
a_3	$\frac{4r^4+8r^3+8r^2+4r-1}{r(1+r)^2(2+r)^2(3+r)}$
a_4	$\frac{3r^8+24r^7+66r^6+60r^5-39r^4-108r^3-90r^2-60r-23}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$
a_5	$\frac{-10r^8-120r^7-592r^6-1548r^5-2342r^4-2172r^3-1208r^2-168r+167}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$
a_6	$\frac{-4r^{12}-96r^{11}-980r^{10}-5520r^9-18471r^8-35952r^7-32578r^6+12648r^5+67379r^4+80920r^3+59174r^2+30680r+7993}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$
a_7	$\frac{20r^{12}+600r^{11}+7912r^{10}+60300r^9+293952r^8+959040r^7+2133504r^6+3249060r^5+3370504r^4+2315040r^3+866288r^2-69060r-165193}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)^2(7+r)}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \ldots\right) \\
&= x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{23x^3}{144} - \frac{167x^4}{2880} - \frac{7993x^5}{86400} + \frac{23599x^6}{518400} + \frac{1860281x^7}{29030400} + O(x^8)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_1 \\
&= -\frac{1}{r(1+r)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} -\frac{1}{r(1+r)} &= \lim_{r\to 0} -\frac{1}{r(1+r)} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\\ &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)\\ \frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\\ &\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\\ &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)\end{aligned}$$

Substituting these back into the given ode $x(x^4+2x^2+1)\,y''+y=0$ gives

$$\begin{aligned}&x(x^4+2x^2+1)\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right.\\ &\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0\end{aligned}$$

Which can be written as

$$\begin{aligned}&\left(\left(x(x^4+2x^2+1)\,y_1''(x) + y_1(x)\right)\ln(x)\right.\\ &\left.+ x(x^4+2x^2+1)\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right)C\\ &+ x(x^4+2x^2+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\ &+ \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x(x^4+2x^2+1)\,y_1''(x) + y_1(x) = 0$$

Eq (7) simplifes to

$$
\begin{aligned}
&x\left(x^4+2x^2+1\right)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)C\\
&+x\left(x^4+2x^2+1\right)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\frac{\left(2x(x^2+1)^2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)-(x^2+1)^2\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}\\
&+\frac{x^2(x^2+1)^2\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)\,(-1+n+r_2)\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x}=0
\end{aligned}
\tag{9}
$$

Since $r_1=1$ and $r_2=0$ then the above becomes

$$
\begin{aligned}
&\frac{\left(2x(x^2+1)^2\left(\sum_{n=0}^{\infty}x^{n}a_n(1+n)\right)-(x^2+1)^2\left(\sum_{n=0}^{\infty}a_n x^{1+n}\right)\right)C}{x}\\
&+\frac{x^2(x^2+1)^2\left(\sum_{n=0}^{\infty}x^{n-2}b_n n(n-1)\right)+\left(\sum_{n=0}^{\infty}b_n x^{n}\right)x}{x}=0
\end{aligned}
\tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}2C\,x^{n+4}a_n(1+n)\right)+\left(\sum_{n=0}^{\infty}4C\,x^{n+2}a_n(1+n)\right)\\
&+\left(\sum_{n=0}^{\infty}2C\,x^{n}a_n(1+n)\right)+\sum_{n=0}^{\infty}\left(-C\,x^{n+4}a_n\right)+\sum_{n=0}^{\infty}\left(-2C\,x^{n+2}a_n\right)\\
&+\sum_{n=0}^{\infty}\left(-Ca_n x^n\right)+\left(\sum_{n=0}^{\infty}n\,x^{n+3}b_n(n-1)\right)+\left(\sum_{n=0}^{\infty}2n\,x^{1+n}b_n(n-1)\right)\\
&+\left(\sum_{n=0}^{\infty}n\,x^{n-1}b_n(n-1)\right)+\left(\sum_{n=0}^{\infty}b_n x^n\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\,x^{n+4} a_n (1+n) &= \sum_{n=5}^{\infty} 2C a_{n-5} (n-4)\, x^{n-1} \\
\sum_{n=0}^{\infty} 4C\,x^{n+2} a_n (1+n) &= \sum_{n=3}^{\infty} 4C a_{-3+n} (n-2)\, x^{n-1} \\
\sum_{n=0}^{\infty} 2C\,x^{n} a_n (1+n) &= \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-C\,x^{n+4} a_n\right) &= \sum_{n=5}^{\infty} \left(-C a_{n-5} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-2C\,x^{n+2} a_n\right) &= \sum_{n=3}^{\infty} \left(-2C a_{-3+n} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-C a_n x^{n}\right) &= \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} n\,x^{n+3} b_n (n-1) &= \sum_{n=4}^{\infty} (n-4)\, b_{n-4} (n-5)\, x^{n-1} \\
\sum_{n=0}^{\infty} 2n\,x^{1+n} b_n (n-1) &= \sum_{n=2}^{\infty} 2(n-2)\, b_{n-2} (-3+n)\, x^{n-1} \\
\sum_{n=0}^{\infty} b_n x^{n} &= \sum_{n=1}^{\infty} b_{n-1} x^{n-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$
\begin{aligned}
&\left(\sum_{n=5}^{\infty} 2Ca_{n-5}(n-4)\,x^{n-1}\right)+\left(\sum_{n=3}^{\infty} 4Ca_{-3+n}(n-2)\,x^{n-1}\right)\\
&+\left(\sum_{n=1}^{\infty} 2Ca_{n-1}n\,x^{n-1}\right)+\sum_{n\,=5}^{\infty}\left(-Ca_{n-5}x^{n-1}\right)+\sum_{n\,=3}^{\infty}\left(-2Ca_{-3+n}x^{n-1}\right)\\
&+\sum_{n\,=1}^{\infty}\left(-Ca_{n-1}x^{n-1}\right)+\left(\sum_{n=4}^{\infty}(n-4)\,b_{n-4}(n-5)\,x^{n-1}\right)\\
&+\left(\sum_{n=2}^{\infty} 2(n-2)\,b_{n-2}(-3+n)\,x^{n-1}\right)\\
&+\left(\sum_{n=0}^{\infty} n\,x^{n-1}b_n(n-1)\right)+\left(\sum_{n=1}^{\infty} b_{n-1}x^{n-1}\right)=0
\end{aligned}
\tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C+1=0$$

Which is solved for C. Solving for C gives

$$C=-1$$

For $n=2$, Eq (2B) gives

$$3Ca_1+b_1+2b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2+\frac{3}{2}=0$$

Solving the above for b_2 gives

$$b_2=-\frac{3}{4}$$

For $n=3$, Eq (2B) gives

$$(2a_0+5a_2)\,C+b_2+6b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{19}{6}+6b_3=0$$

Solving the above for b_3 gives

$$b_3 = \frac{19}{36}$$

For $n = 4$, Eq (2B) gives

$$(6a_1 + 7a_3) C + 4b_2 + b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{85}{144} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{85}{1728}$$

For $n = 5$, Eq (2B) gives

$$(a_0 + 10a_2 + 9a_4) C + 12b_3 + b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{21907}{4320} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{21907}{86400}$$

For $n = 6$, Eq (2B) gives

$$(3a_1 + 14a_3 + 11a_5) C + 2b_2 + 24b_4 + b_5 + 30b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{787}{2700} + 30b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{787}{81000}$$

For $n = 7$, Eq (2B) gives

$$(5a_2 + 18a_4 + 13a_6) C + 6b_3 + 40b_5 + b_6 + 42b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{5987917}{864000} + 42b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = \frac{5987917}{36288000}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = (-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{23x^3}{144} - \frac{167x^4}{2880} - \frac{7993x^5}{86400} + \frac{23599x^6}{518400} + \frac{1860281x^7}{29030400}\right.\right. \\ \left.\left. + O\left(x^8\right)\right)\right) \ln(x) \\ + 1 - \frac{3x^2}{4} + \frac{19x^3}{36} + \frac{85x^4}{1728} - \frac{21907x^5}{86400} + \frac{787x^6}{81000} + \frac{5987917x^7}{36288000} + O\left(x^8\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{23x^3}{144} - \frac{167x^4}{2880} - \frac{7993x^5}{86400} + \frac{23599x^6}{518400} + \frac{1860281x^7}{29030400} + O\left(x^8\right)\right) \\ &\quad + c_2\left((-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{23x^3}{144} - \frac{167x^4}{2880} - \frac{7993x^5}{86400} + \frac{23599x^6}{518400} + \frac{1860281x^7}{29030400} + O\left(x^8\right)\right)\right) \ln(x) \right. \\ &\quad \left. + 1 - \frac{3x^2}{4} + \frac{19x^3}{36} + \frac{85x^4}{1728} - \frac{21907x^5}{86400} + \frac{787x^6}{81000} + \frac{5987917x^7}{36288000} + O\left(x^8\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{23x^3}{144} - \frac{167x^4}{2880} - \frac{7993x^5}{86400} + \frac{23599x^6}{518400} + \frac{1860281x^7}{29030400} + O\left(x^8\right)\right) \\ &\quad + c_2\left(-x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{23x^3}{144} - \frac{167x^4}{2880} - \frac{7993x^5}{86400} + \frac{23599x^6}{518400} + \frac{1860281x^7}{29030400} \right.\right. \\ &\quad \left.\left. + O\left(x^8\right)\right) \ln(x) + 1 - \frac{3x^2}{4} + \frac{19x^3}{36} + \frac{85x^4}{1728} - \frac{21907x^5}{86400} + \frac{787x^6}{81000} + \frac{5987917x^7}{36288000} + O\left(x^8\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+\frac{23599x^6}{518400}+\frac{1860281x^7}{29030400}+O\left(x^8\right)\right) \\
& + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+\frac{23599x^6}{518400}+\frac{1860281x^7}{29030400}\right.\right. \\
& \left.+O\left(x^8\right)\right)\ln\left(x\right)+1-\frac{3x^2}{4}+\frac{19x^3}{36}+\frac{85x^4}{1728}-\frac{21907x^5}{86400}+\frac{787x^6}{81000}+\frac{5987917x^7}{36288000} \\
& \left.+O\left(x^8\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+\frac{23599x^6}{518400}+\frac{1860281x^7}{29030400}+O\left(x^8\right)\right) \\
& + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+\frac{23599x^6}{518400}+\frac{1860281x^7}{29030400}\right.\right. \\
& \left.\left.+O\left(x^8\right)\right)\ln\left(x\right)+1-\frac{3x^2}{4}+\frac{19x^3}{36}+\frac{85x^4}{1728}-\frac{21907x^5}{86400}+\frac{787x^6}{81000}+\frac{5987917x^7}{36288000}+O\left(x^8\right)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 70

```
Order:=8;
dsolve(x*(x^2+1)^2*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = c_1 x\left(1-\frac{1}{2}x+\frac{1}{12}x^2+\frac{23}{144}x^3-\frac{167}{2880}x^4-\frac{7993}{86400}x^5+\frac{23599}{518400}x^6+\frac{1860281}{29030400}x^7\right. \\
\left.+\operatorname{O}\left(x^8\right)\right)+c_2\left(\ln\left(x\right)\left(-x+\frac{1}{2}x^2-\frac{1}{12}x^3-\frac{23}{144}x^4+\frac{167}{2880}x^5+\frac{7993}{86400}x^6\right.\right. \\
\left.-\frac{23599}{518400}x^7+\operatorname{O}\left(x^8\right)\right) \\
\left.+\left(1-\frac{3}{4}x^2+\frac{19}{36}x^3+\frac{85}{1728}x^4-\frac{21907}{86400}x^5+\frac{787}{81000}x^6+\frac{5987917}{36288000}x^7+\operatorname{O}\left(x^8\right)\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 121

```
AsymptoticDSolveValue[x*(x^2+1)^2*y''[x]+y[x]==0,y[x],{x,0,7}]
```

$$
\begin{aligned}
y(x) \to\ & c_1\left(\frac{x(7993x^5+5010x^4-13800x^3-7200x^2+43200x-86400)\log(x)}{86400}\right. \\
&\left.+\frac{-107303x^6-403755x^5+270750x^4+792000x^3-1620000x^2+1296000x+1296000}{1296000}\right) \\
&+c_2\left(\frac{23599x^7}{518400}-\frac{7993x^6}{86400}-\frac{167x^5}{2880}+\frac{23x^4}{144}+\frac{x^3}{12}-\frac{x^2}{2}+x\right)
\end{aligned}
$$

2.9 problem 9

Internal problem ID [6592]
Internal file name [`OUTPUT/5840_Sunday_June_05_2022_03_57_06_PM_62684521/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^3(x^2-25)(-2+x)^2 y''+3x(-2+x)y'+7(5+x)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^7-4x^6-21x^5+100x^4-100x^3)y''+(3x^2-6x)y'+(7x+35)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{x^2(-2+x)(x-5)(5+x)}$$
$$q(x)=\frac{7}{(x-5)(-2+x)^2x^3}$$

Table 38: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x^2(-2+x)(x-5)(5+x)}$	
singularity	type
$x = -5$	"regular"
$x = 0$	"irregular"
$x = 2$	"regular"
$x = 5$	"regular"

$q(x) = \frac{7}{(x-5)(-2+x)^2x^3}$	
singularity	type
$x = 0$	"irregular"
$x = 2$	"regular"
$x = 5$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-5, 2, 5, \infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

2.9.1 Maple step by step solution

Let's solve

$$y''x^3(x-5)(5+x)(-2+x)^2 + (3x^2-6x)y' + (7x+35)y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = -\frac{3y'}{x^2(-2+x)(x-5)(5+x)} - \frac{7y}{x^3(x-5)(-2+x)^2}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{3y'}{x^2(-2+x)(x-5)(5+x)} + \frac{7y}{x^3(x-5)(-2+x)^2} = 0$$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

$$\left[P_2(x) = \frac{3}{x^2(-2+x)(x-5)(5+x)}, P_3(x) = \frac{7}{(x-5)(-2+x)^2x^3}\right]$$

  ○ $(5+x) \cdot P_2(x)$ is analytic at $x = -5$

  $\left((5+x)\cdot P_2(x)\right)\Big|_{x=-5} = \frac{3}{1750}$

  - $(5+x)^2\cdot P_3(x)$ is analytic at $x=-5$

    $\left((5+x)^2\cdot P_3(x)\right)\Big|_{x=-5} = 0$

  - $x=-5$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-5$

- Multiply by denominators

  $y''x^3(x-5)(5+x)(-2+x)^2 + 3x(-2+x)y' + (7x+35)y = 0$

- Change variables using $x=u-5$ so that the regular singular point is at $u=0$

  $\left(u^7-39u^6+624u^5-5250u^4+24525u^3-60375u^2+61250u\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(3u^2-36u+105\right)$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $u\cdot y(u)$ to series expansion

    $u\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+1}$

  - Shift index using $k->k-1$

    $u\cdot y(u) = \sum_{k=1}^{\infty} a_{k-1} u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..7$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$35a_0r(-1747+1750r)\,u^{-1+r} + (35a_1(1+r)\,(3+1750r) - 3a_0r(-20113+20125r))\,u^r + (35a_2($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$35r(-1747+1750r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1747}{1750}\right\}$$

- The coefficients of each power of u must be 0

  $$[35a_1(1+r)\,(3+1750r) - 3a_0r(-20113+20125r) = 0, 35a_2(2+r)\,(1753+1750r) - 3a_1(1+r$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{3a_0r(-20113+20125r)}{35(1750r^2+1753r+3)}, a_2 = \frac{a_0(2142984375r^3-2141396250r^2-26144r-735)}{1225(3062500r^3+9198000r^2+6151259r+10518)}, a_3 = \frac{3a_0(19461900390625r^5+194685}{42875(5359375000r^5+3754}\right.$$

- Each term in the series must be 0, giving the recursion relation

  $$(-60375a_k + a_{k-5} - 39a_{k-4} + 624a_{k-3} - 5250a_{k-2} + 24525a_{k-1} + 61250a_{k+1})\,k^2 + (2(-60375a_k$$

- Shift index using $k->k+5$

  $$(-60375a_{k+5} + a_k - 39a_{k+1} + 624a_{k+2} - 5250a_{k+3} + 24525a_{k+4} + 61250a_{k+6})\,(k+5)^2 + (2(-60$$

- Recursion relation that defines series solution to ODE

  $$a_{k+6} = -\frac{k^2a_k-39k^2a_{k+1}+624k^2a_{k+2}-5250k^2a_{k+3}+24525k^2a_{k+4}-60375k^2a_{k+5}+2kra_k-78kra_{k+1}+1248kra_{k+2}-10500kra}{}$$

- Recursion relation for $r=0$

  $$a_{k+6} = -\frac{k^2a_k-39k^2a_{k+1}+624k^2a_{k+2}-5250k^2a_{k+3}+24525k^2a_{k+4}-60375k^2a_{k+5}-ka_k-39ka_{k+1}+1872ka_{k+2}-26250ka_{k+3}+}{35(1750k^2+19253k+52518)}$$

- Solution for $r=0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+6} = -\frac{k^2a_k-39k^2a_{k+1}+624k^2a_{k+2}-5250k^2a_{k+3}+24525k^2a_{k+4}-60375k^2a_{k+5}-ka_k-39ka_{k+1}+18}{35(1750}\right.$$

- Revert the change of variables $u=5+x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(5+x)^k, a_{k+6} = -\frac{k^2a_k-39k^2a_{k+1}+624k^2a_{k+2}-5250k^2a_{k+3}+24525k^2a_{k+4}-60375k^2a_{k+5}-ka_k-39ka_{k+1}}{35(}\right.$$

- Recursion relation for $r=\frac{1747}{1750}$

  $$a_{k+6} = -\frac{k^2a_k-39k^2a_{k+1}+624k^2a_{k+2}-5250k^2a_{k+3}+24525k^2a_{k+4}-60375k^2a_{k+5}+\frac{872}{875}ka_k-\frac{102258}{875}ka_{k+1}+\frac{2728128}{875}ka_{k+2}-367}{35(}$$

- Solution for $r=\frac{1747}{1750}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1747}{1750}}, a_{k+6} = -\frac{k^2 a_k - 39k^2 a_{k+1} + 624k^2 a_{k+2} - 5250k^2 a_{k+3} + 24525k^2 a_{k+4} - 60375k^2 a_{k+5} + \frac{872}{875} k a_k - \frac{102}{87}}{}\right.$$

- Revert the change of variables $u = 5 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (5+x)^{k+\frac{1747}{1750}}, a_{k+6} = -\frac{k^2 a_k - 39k^2 a_{k+1} + 624k^2 a_{k+2} - 5250k^2 a_{k+3} + 24525k^2 a_{k+4} - 60375k^2 a_{k+5} + \frac{872}{875} k a_k -}{}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (5+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (5+x)^{k+\frac{1747}{1750}}\right), a_{k+6} = -\frac{k^2 a_k - 39k^2 a_{k+1} + 624k^2 a_{k+2} - 5250k^2 a_{k+3} + 24525}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
Order:=8;
dsolve(x^3*(x^2-25)*(x-2)^2*diff(y(x),x$2)+3*x*(x-2)*diff(y(x),x)+7*(x+5)*y(x)=0,y(x),type='
```

No solution found

✓ Solution by Mathematica

Time used: 0.149 (sec). Leaf size: 127

```
AsymptoticDSolveValue[x^3*(x^2-25)*(x-2)^2*y''[x]+3*x*(x-2)*y'[x]+7*(x+5)*y[x]==0,y[x],{x,0,
```

$$y(x) \to c_2\left(-\frac{8443721376476278698619699192242145x^7}{24679069470425088} + \frac{2562764390339723899972072769\,99x^6}{228509902503936} - \frac{13376987201697821906188 81x^5}{352638738432} + \frac{4284030153765326450 5x^4}{3265173504} - \frac{344729362309955x^3}{7558272} + \frac{3590248795x^2}{23328} - \frac{50309x}{108} + 1\right)x^{35/6}$$

$$+ \frac{c_1 e^{\frac{3}{50}/x}\left(\frac{27670480145177700385838149741665715823829792301x^7}{4118764073681250000000000000000000000000} + \frac{3104172516869718247583976968553108060901x^6}{4358480501250000000000000000000000} - \frac{379071980}{5380}\right.}{}$$

2.10 problem 10

Internal problem ID [6593]
Internal file name [`OUTPUT/5841_Sunday_June_05_2022_03_57_09_PM_30924215/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{\left(x^{3}-2x^{2}+3x\right)^{2} y''+x(x-3)^{2} y'-(1+x)\, y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^{6}-4x^{5}+10x^{4}-12x^{3}+9x^{2}\right) y''+\left(x^{3}-6x^{2}+9x\right) y'+(-1-x)\, y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{(x-3)^{2}}{x\left(x^{2}-2x+3\right)^{2}}$$
$$q(x)=-\frac{1+x}{x^{2}\left(x^{2}-2x+3\right)^{2}}$$

Table 40: Table $p(x), q(x)$ singularites.

$p(x) = \frac{(x-3)^2}{x(x^2-2x+3)^2}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}+1$	"irregular"
$x = i\sqrt{2}+1$	"irregular"

$q(x) = -\frac{1+x}{x^2(x^2-2x+3)^2}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}+1$	"regular"
$x = i\sqrt{2}+1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $\left[-i\sqrt{2}+1, i\sqrt{2}+1\right]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2\left(x^4-4x^3+10x^2-12x+9\right)y''+\left(x^3-6x^2+9x\right)y'+\left(-1-x\right)y=0$$

Let the solution be represented as Frobenius power series of the form

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

Then

$$y'=\sum_{n=0}^{\infty}\left(n+r\right)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}\left(n+r\right)\left(n+r-1\right)a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned}&x^2\left(x^4-4x^3+10x^2-12x+9\right)\left(\sum_{n=0}^{\infty}\left(n+r\right)\left(n+r-1\right)a_n x^{n+r-2}\right)\\&+\left(x^3-6x^2+9x\right)\left(\sum_{n=0}^{\infty}\left(n+r\right)a_n x^{n+r-1}\right)+\left(-1-x\right)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0\end{aligned}\tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r+4} a_n (n+r)\,(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-4 x^{n+r+3} a_n (n+r)\,(n+r-1)\right)\\
&+\left(\sum_{n=0}^{\infty} 10 x^{n+r+2} a_n (n+r)\,(n+r-1)\right)\\
&+\sum_{n=0}^{\infty}\left(-12 x^{1+n+r} a_n (n+r)\,(n+r-1)\right)\\
&+\left(\sum_{n=0}^{\infty} 9 x^{n+r} a_n (n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right)\\
&+\sum_{n=0}^{\infty}\left(-6 x^{1+n+r} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} 9 x^{n+r} a_n (n+r)\right)\\
&+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+4} a_n (n+r)\,(n+r-1)=\sum_{n=4}^{\infty} a_{n-4} (n-4+r)\,(n-5+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-4 x^{n+r+3} a_n (n+r)\,(n+r-1)\right)=\sum_{n=3}^{\infty}\left(-4 a_{n-3} (-3+n+r)\,(n-4+r)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 10 x^{n+r+2} a_n (n+r)\,(n+r-1)=\sum_{n=2}^{\infty} 10 a_{n-2} (n+r-2)\,(-3+n+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-12 x^{1+n+r} a_n (n+r)\,(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-12 a_{n-1} (n+r-1)\,(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)=\sum_{n=2}^{\infty} a_{n-2} (n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-6 x^{1+n+r} a_n (n+r)\right)=\sum_{n=1}^{\infty}\left(-6 a_{n-1} (n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=4}^{\infty} a_{n-4}(n-4+r)(n-5+r)\, x^{n+r}\right) \\
&+\sum_{n=3}^{\infty} \left(-4a_{n-3}(-3+n+r)(n-4+r)\, x^{n+r}\right) \\
&\quad+\left(\sum_{n=2}^{\infty} 10a_{n-2}(n+r-2)(-3+n+r)\, x^{n+r}\right) \\
&\quad+\sum_{n=1}^{\infty} \left(-12a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) \\
&\quad+\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&\quad+\sum_{n=1}^{\infty} \left(-6a_{n-1}(n+r-1)\, x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n(n+r)\right) \\
&\quad+\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right)+\sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)=0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n(n+r)(n+r-1)+9x^{n+r} a_n(n+r)-a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r)+9x^r a_0 r-a_0 x^r=0$$

Or

$$\left(9x^r r(-1+r)+9x^r r-x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2-1\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2-1=0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{1}{3} \\
r_2 &= -\frac{1}{3}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(9r^2 - 1)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = \frac{12r^2 - 6r + 1}{9r^2 + 18r + 8}
$$

Substituting $n = 2$ in Eq. (2B) gives

$$
a_2 = \frac{54r^4 + 45r^3 + 70r^2 + 48r + 7}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)}
$$

Substituting $n = 3$ in Eq. (2B) gives

$$
a_3 = \frac{-108r^6 - 540r^5 + 528r^4 + 3732r^3 + 3489r^2 + 739r + 224}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)}
$$

For $4 \leq n$ the recursive equation is

$$
\begin{aligned}
&a_{n-4}(n-4+r)(n-5+r) - 4a_{n-3}(-3+n+r)(n-4+r) \\
&\quad + 10a_{n-2}(n+r-2)(-3+n+r) - 12a_{n-1}(n+r-1)(n+r-2) \\
&\quad + 9a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) \\
&\quad - 6a_{n-1}(n+r-1) + 9a_n(n+r) - a_n - a_{n-1} = 0
\end{aligned}
\tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{n^2a_{n-4} - 4n^2a_{n-3} + 10n^2a_{n-2} - 12n^2a_{n-1} + 2nra_{n-4} - 8nra_{n-3} + 20nra_{n-2} - 24nra_{n-1} + r^2a_{n-}}{}
\tag{4}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
a_n = \frac{(-9a_{n-4} + 36a_{n-3} - 90a_{n-2} + 108a_{n-1})\,n^2 + (75a_{n-4} - 228a_{n-3} + 381a_{n-2} - 198a_{n-1})\,n - 154a_{n-4}}{81n^2 + 54n}
\tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	$\frac{1}{45}$
a_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{149}{3240}$
a_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{2701}{192456}$

For $n = 4$, using the above recursive equation gives

$$
a_4 = \frac{-2997r^8 - 29079r^7 - 99054r^6 - 125883r^5 + 9826r^4 + 138853r^3 + 80233r^2 + 5477r + 8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
a_4 = \frac{236933}{121247280}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	$\frac{1}{45}$
a_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{149}{3240}$
a_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{2701}{192456}$
a_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{236933}{121247280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-17496r^{10} - 292572r^9 - 2032830r^8 - 7628094r^7 - 16915518r^6 - 23128560r^5 - 20473273r^4 - 1293}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)(9r^2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{67092967}{92754169200}$$

And the table now becomes

n	$a_{n,r}$	a
a_0	1	1
a_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	1 4
a_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	1 3
a_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	1
a_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	1
a_5	$\frac{-17496r^{10}-292572r^9-2032830r^8-7628094r^7-16915518r^6-23128560r^5-20473273r^4-12932200r^3-6220667r^2-1947462r-14560}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$	$-$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-14580r^{12} - 351378r^{11} - 4256712r^{10} - 35012979r^9 - 208451367r^8 - 881028765r^7 - 2557230254r^6}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 +}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_6 = -\frac{30839263691}{50087251368000}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$
a_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$
a_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$
a_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$
a_5	$\frac{-17496r^{10}-292572r^9-2032830r^8-7628094r^7-16915518r^6-23128560r^5-20473273r^4-12932200r^3-6220667r^2-1947462r-14560}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$
a_6	$\frac{-14580r^{12}-351378r^{11}-4256712r^{10}-35012979r^9-208451367r^8-881028765r^7-2557230254r^6-4930900584r^5-6050506046r^4-4415685}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)(9r^2+108r+323)}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{507384r^{14} + 18895680r^{13} + 306898308r^{12} + 2845233540r^{11} + 16479937161r^{10} + 60728997375r^9 + 1}{(9r^2 + 18r + 8)\,(9r^2 + 36r + 35}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_7 = -\frac{14846109458423}{72576427232232000}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$
a_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$
a_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$
a_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$
a_5	$\frac{-17496r^{10}-292572r^9-2032830r^8-7628094r^7-16915518r^6-23128560r^5-20473273r^4-12932200r^3-6220667r^2-1947462r-14560}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$
a_6	$\frac{-14580r^{12}-351378r^{11}-4256712r^{10}-35012979r^9-208451367r^8-881028765r^7-2557230254r^6-4930900584r^5-6050506046r^4-4415685}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)(9r^2+108r+323)}$
a_7	$\frac{507384r^{14}+18895680r^{13}+306898308r^{12}+2845233540r^{11}+16479937161r^{10}+60728997375r^9+134237365392r^8+125355164070r^7-17212}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+9}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 + \frac{x}{45} + \frac{149x^2}{3240} + \frac{2701x^3}{192456} + \frac{236933x^4}{121247280} - \frac{67092967x^5}{92754169200} - \frac{30839263691x^6}{50087251368000} - \frac{14846109458}{72576427232} \right. \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{12r^2 - 6r + 1}{9r^2 + 18r + 8}$$

Substituting $n = 2$ in Eq. (2B) gives

$$b_2 = \frac{54r^4 + 45r^3 + 70r^2 + 48r + 7}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$b_3 = \frac{-108r^6 - 540r^5 + 528r^4 + 3732r^3 + 3489r^2 + 739r + 224}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)}$$

For $4 \leq n$ the recursive equation is

$$\begin{aligned} &b_{n-4}(n-4+r)(n-5+r) - 4b_{n-3}(-3+n+r)(n-4+r) \\ &\quad + 10b_{n-2}(n+r-2)(-3+n+r) - 12b_{n-1}(n+r-1)(n+r-2) \\ &\quad + 9b_n(n+r)(n+r-1) + b_{n-2}(n+r-2) \\ &\quad - 6b_{n-1}(n+r-1) + 9b_n(n+r) - b_n - b_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{n^2 b_{n-4} - 4n^2 b_{n-3} + 10n^2 b_{n-2} - 12n^2 b_{n-1} + 2nrb_{n-4} - 8nrb_{n-3} + 20nrb_{n-2} - 24nrb_{n-1} + r^2 b_{n-4} -}{} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_n = \frac{(-9b_{n-4} + 36b_{n-3} - 90b_{n-2} + 108b_{n-1})\,n^2 + (87b_{n-4} - 276b_{n-3} + 501b_{n-2} - 342b_{n-1})\,n - 208b_{n-4} +}{81n^2 - 54n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	$\frac{13}{9}$
b_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{5}{162}$
b_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1591}{30618}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{-2997r^8 - 29079r^7 - 99054r^6 - 125883r^5 + 9826r^4 + 138853r^3 + 80233r^2 + 5477r + 8064}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = \frac{106583}{5511240}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	$\frac{13}{9}$
b_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{5}{162}$
b_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1591}{30618}$
b_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{106583}{5511240}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-17496r^{10} - 292572r^9 - 2032830r^8 - 7628094r^7 - 16915518r^6 - 23128560r^5 - 20473273r^4 - 1293}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)(9r^2}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_5 = \frac{7435523}{3224075400}$$

And the table now becomes

n	$b_{n,r}$	b
b_0	1	1
b_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	$\frac{13}{9}$
b_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	$-$
b_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1}{30}$
b_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{1}{55}$
b_5	$\frac{-17496r^{10}-292572r^9-2032830r^8-7628094r^7-16915518r^6-23128560r^5-20473273r^4-12932200r^3-6220667r^2-1947462r-14560}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$	$\frac{}{32}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{-14580r^{12} - 351378r^{11} - 4256712r^{10} - 35012979r^9 - 208451367r^8 - 881028765r^7 - 2557230254r^6}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 +}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_6 = -\frac{70024699}{43525017900}$$

And the table now becomes

n	$b_{n,r}$
b_0	1
b_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$
b_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$
b_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$
b_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$
b_5	$\frac{-17496r^{10}-292572r^9-2032830r^8-7628094r^7-16915518r^6-23128560r^5-20473273r^4-12932200r^3-6220667r^2-1947462r-14560}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$
b_6	$\frac{-14580r^{12}-351378r^{11}-4256712r^{10}-35012979r^9-208451367r^8-881028765r^7-2557230254r^6-4930900584r^5-6050506046r^4-44156854}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)(9r^2+108r+323)}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{507384r^{14} + 18895680r^{13} + 306898308r^{12} + 2845233540r^{11} + 16479937161r^{10} + 60728997375r^9 + 1}{(9r^2 + 18r + 8)(9r^2 + 36r + 35}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_7 = -\frac{2917066898}{2604972321315}$$

And the table now becomes

n	$b_{n,r}$
b_0	1
b_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$
b_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$
b_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$
b_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$
b_5	$\frac{-17496r^{10}-292572r^9-2032830r^8-7628094r^7-16915518r^6-23128560r^5-20473273r^4-12932200r^3-6220667r^2-1947462r-14560}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$
b_6	$\frac{-14580r^{12}-351378r^{11}-4256712r^{10}-35012979r^9-208451367r^8-881028765r^7-2557230254r^6-4930900584r^5-6050506046r^4-44156854}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)(9r^2+108r+323)}$
b_7	$\frac{507384r^{14}+18895680r^{13}+306898308r^{12}+2845233540r^{11}+16479937161r^{10}+60728997375r^9+134237365392r^8+125355164070r^7-172129}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{3}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6+b_7x^7+b_8x^8\ldots\right) \\
&= \frac{1+\frac{13x}{9}-\frac{5x^2}{162}+\frac{1591x^3}{30618}+\frac{106583x^4}{5511240}+\frac{7435523x^5}{3224075400}-\frac{70024699x^6}{43525017900}-\frac{2917066898x^7}{2604972321315}+O(x^8)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x) \\
&= c_1x^{\frac{1}{3}}\left(1+\frac{x}{45}+\frac{149x^2}{3240}+\frac{2701x^3}{192456}+\frac{236933x^4}{121247280}-\frac{67092967x^5}{92754169200}\right. \\
&\qquad \left.-\frac{30839263691x^6}{50087251368000}-\frac{14846109458423x^7}{72576427232232000}+O(x^8)\right) \\
&\quad+\frac{c_2\left(1+\frac{13x}{9}-\frac{5x^2}{162}+\frac{1591x^3}{30618}+\frac{106583x^4}{5511240}+\frac{7435523x^5}{3224075400}-\frac{70024699x^6}{43525017900}-\frac{2917066898x^7}{2604972321315}+O(x^8)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Hence the final solution is

$$y = y_h$$

$$
= c_1 x^{\frac{1}{3}} \left(1 + \frac{x}{45} + \frac{149x^2}{3240} + \frac{2701x^3}{192456} + \frac{236933x^4}{121247280} - \frac{67092967x^5}{92754169200} - \frac{30839263691x^6}{50087251368000} - \frac{14846109458423x^7}{72576427232232000} + O(x^8)\right) + \frac{c_2 \left(1 + \frac{13x}{9} - \frac{5x^2}{162} + \frac{1591x^3}{30618} + \frac{106583x^4}{5511240} + \frac{7435523x^5}{3224075400} - \frac{70024699x^6}{43525017900} - \frac{2917066898x^7}{2604972321315} + O(x^8)\right)}{x^{\frac{1}{3}}}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{3}} \left(1 + \frac{x}{45} + \frac{149x^2}{3240} + \frac{2701x^3}{192456} + \frac{236933x^4}{121247280} - \frac{67092967x^5}{92754169200} - \frac{30839263691x^6}{50087251368000} - \frac{14846109458423x^7}{72576427232232000} + O(x^8)\right) + \frac{c_2 \left(1 + \frac{13x}{9} - \frac{5x^2}{162} + \frac{1591x^3}{30618} + \frac{106583x^4}{5511240} + \frac{7435523x^5}{3224075400} - \frac{70024699x^6}{43525017900} - \frac{2917066898x^7}{2604972321315} + O(x^8)\right)}{x^{\frac{1}{3}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{3}} \left(1 + \frac{x}{45} + \frac{149x^2}{3240} + \frac{2701x^3}{192456} + \frac{236933x^4}{121247280} - \frac{67092967x^5}{92754169200} - \frac{30839263691x^6}{50087251368000} - \frac{14846109458423x^7}{72576427232232000} + O(x^8)\right) + \frac{c_2 \left(1 + \frac{13x}{9} - \frac{5x^2}{162} + \frac{1591x^3}{30618} + \frac{106583x^4}{5511240} + \frac{7435523x^5}{3224075400} - \frac{70024699x^6}{43525017900} - \frac{2917066898x^7}{2604972321315} + O(x^8)\right)}{x^{\frac{1}{3}}}
$$

Verified OK.

2.10.1 Maple step by step solution

Let's solve

$x^2(x^4 - 4x^3 + 10x^2 - 12x + 9)\, y'' + (x^3 - 6x^2 + 9x)\, y' + (-1 - x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(1+x)y}{x^2(x^4-4x^3+10x^2-12x+9)} - \frac{(x^2-6x+9)y'}{x(x^4-4x^3+10x^2-12x+9)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(x^2-6x+9)y'}{x(x^4-4x^3+10x^2-12x+9)} - \frac{(1+x)y}{x^2(x^4-4x^3+10x^2-12x+9)} = 0$$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $$\left[P_2(x) = \frac{x^2-6x+9}{x(x^4-4x^3+10x^2-12x+9)}, P_3(x) = -\frac{1+x}{x^2(x^4-4x^3+10x^2-12x+9)}\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = 1$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{9}$$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- ● Multiply by denominators

  $$x^2(x^4-4x^3+10x^2-12x+9)\,y'' + x(x^2-6x+9)\,y' + (-1-x)\,y = 0$$

- ● Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- ○ Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- ○ Convert $x^m \cdot y''$ to series expansion for $m = 2..6$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)(-1+3r)\, x^r + \left(a_1(4+3r)(2+3r) - a_0(12r^2-6r+1)\right) x^{1+r} + \left(a_2(7+3r)(5+3r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+3r)(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$$

- The coefficients of each power of x must be 0

$$[a_1(4+3r)(2+3r) - a_0(12r^2-6r+1) = 0, a_2(7+3r)(5+3r) - a_1(12r^2+18r+7) + a_0r(-9$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = \frac{a_0(12r^2-6r+1)}{9r^2+18r+8}, a_2 = \frac{a_0(54r^4+45r^3+70r^2+48r+7)}{81r^4+486r^3+1035r^2+918r+280}, a_3 = -\frac{a_0(108r^6+540r^5-528r^4-3732r^3-3489r^2-739r}{729r^6+8748r^5+42039r^4+103032r^3+134892r^2+885}\right.$$

- Each term in the series must be 0, giving the recursion relation

$$(9a_k + a_{k-4} - 4a_{k-3} + 10a_{k-2} - 12a_{k-1})\, k^2 + (2(9a_k + a_{k-4} - 4a_{k-3} + 10a_{k-2} - 12a_{k-1})\, r - 9a_k$$

- Shift index using $k- >k+4$

$$(9a_{k+4} + a_k - 4a_{k+1} + 10a_{k+2} - 12a_{k+3})\, (k+4)^2 + (2(9a_{k+4} + a_k - 4a_{k+1} + 10a_{k+2} - 12a_{k+3})\, r$$

- Recursion relation that defines series solution to ODE

$$a_{k+4} = -\frac{k^2a_k - 4k^2a_{k+1} + 10k^2a_{k+2} - 12k^2a_{k+3} + 2kra_k - 8kra_{k+1} + 20kra_{k+2} - 24kra_{k+3} + r^2a_k - 4r^2a_{k+1} + 10r^2a_{k+2} - 12r^2a_{k+}}{9k^2+18kr+9r^2+72k+72r+14}$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+4} = -\frac{k^2a_k - 4k^2a_{k+1} + 10k^2a_{k+2} - 12k^2a_{k+3} - \frac{5}{3}ka_k - \frac{4}{3}ka_{k+1} + \frac{73}{3}ka_{k+2} - 58ka_{k+3} + \frac{4}{9}a_k + \frac{8}{9}a_{k+1} + \frac{115}{9}a_{k+2} - \frac{211}{3}a_{k+3}}{9k^2+66k+120}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+4} = -\frac{k^2a_k - 4k^2a_{k+1} + 10k^2a_{k+2} - 12k^2a_{k+3} - \frac{5}{3}ka_k - \frac{4}{3}ka_{k+1} + \frac{73}{3}ka_{k+2} - 58ka_{k+3} + \frac{4}{9}a_k + \frac{8}{9}a_{k+1} +}{9k^2+66k+120}\right.$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+4} = -\frac{k^2a_k - 4k^2a_{k+1} + 10k^2a_{k+2} - 12k^2a_{k+3} - \frac{1}{3}ka_k - \frac{20}{3}ka_{k+1} + \frac{113}{3}ka_{k+2} - 74ka_{k+3} - \frac{2}{9}a_k - \frac{16}{9}a_{k+1} + \frac{301}{9}a_{k+2} - \frac{343}{3}a_{k+3}}{9k^2+78k+168}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+4} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 10k^2 a_{k+2} - 12k^2 a_{k+3} - \frac{1}{3} k a_k - \frac{20}{3} k a_{k+1} + \frac{113}{3} k a_{k+2} - 74 k a_{k+3} - \frac{2}{9} a_k - \frac{16}{9} a_{k+1} -}{9k^2 + 78k + 168}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+4} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 10k^2 a_{k+2} - 12k^2 a_{k+3} - \frac{5}{3} k a_k - \frac{4}{3} k a_{k+1} + \frac{73}{3} k a_{k+2}}{9k^2 + 66k + 120}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 55

```
Order:=8;
dsolve((x^3-2*x^2+3*x)^2*diff(y(x),x$2)+x*(x-3)^2*diff(y(x),x)-(x+1)*y(x)=0,y(x),type='serie
```

$$y(x) = \frac{c_2 x^{\frac{2}{3}} \left(1 + \frac{1}{45}x + \frac{149}{3240}x^2 + \frac{2701}{192456}x^3 + \frac{236933}{121247280}x^4 - \frac{67092967}{92754169200}x^5 - \frac{30839263691}{50087251368000}x^6 - \frac{14846109458423}{72576427232232000}x^7 + \right.}{}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 118

```
AsymptoticDSolveValue[(x^3-2*x^2+3*x)^2*y''[x]+x*(x-3)^2*y'[x]-(x+1)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \sqrt[3]{x} \left(-\frac{14846109458423x^7}{72576427232232000} - \frac{30839263691x^6}{50087251368000} - \frac{67092967x^5}{92754169200} + \frac{236933x^4}{121247280} + \frac{2701x^3}{192456} + \frac{149x^2}{3240} + \frac{x}{45} + 1\right) + \frac{c_2 \left(-\frac{2917066898x^7}{2604972321315} - \frac{70024699x^6}{43525017900} + \frac{7435523x^5}{3224075400} + \frac{106583x^4}{5511240} + \frac{1591x^3}{30618} - \frac{5x^2}{162} + \frac{13x}{9} + 1\right)}{\sqrt[3]{x}}$$

2.11 problem 11

Internal problem ID [6594]
Internal file name [`OUTPUT/5842_Sunday_June_05_2022_03_57_15_PM_17863028/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^{2}-1\right) y''+5(1+x) y'+\left(x^{2}-x\right) y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots \\ &= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (118)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (119)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$F_0 = -\frac{yx^2 - xy + 5xy' + 5y'}{x^2 - 1}$$

$$\begin{aligned}
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(-x^4 + x^3 + 31x^2 + 59x + 30)\, y' + 5(x-1)\, y\left(x^2 + \frac{4}{5}x + \frac{1}{5}\right)}{(x^2 - 1)^2}
\end{aligned}$$

$$\begin{aligned}
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(10x^5 - 2x^4 - 226x^3 - 630x^2 - 624x - 208)\, y' + y(x-1)\,(x^5 - x^4 - 36x^3 - 62x^2 - 39x - 3)}{(x^2 - 1)^3}
\end{aligned}$$

$$\begin{aligned}
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x-1)\,(1+x)\,\big((1+x)\,(x^7 - 3x^6 - 93x^5 + 33x^4 + 1793x^3 + 5077x^2 + 4971x + 1661)\, y' - 10\big(x^6 -}{(x^2 - 1)^5}
\end{aligned}$$

$$\begin{aligned}
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-15x^9 + 21x^8 + 972x^7 + 1452x^6 - 15804x^5 - 78540x^4 - 152496x^3 - 149952x^2 - 74577x - 14901)}{(x^2 -}
\end{aligned}$$

$$\begin{aligned}
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{-(1+x)\,(x^{11} - 4x^{10} - 191x^9 + 330x^8 + 10200x^7 + 14928x^6 - 156756x^5 - 789144x^4 - 1524105x^3 -}{}
\end{aligned}$$

$$\begin{aligned}
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{(x-1)\,\big(20\big(x^{12} - \frac{18}{5}x^{11} - 123x^{10} + \frac{572}{5}x^9 + 5976x^8 + 14232x^7 - \frac{385752}{5}x^6 - \frac{2609664}{5}x^5 - \frac{6360777}{5}x^4 -}{}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 5y'(0) \\
F_1 &= -y(0) + 30y'(0) \\
F_2 &= -3y(0) + 208y'(0) \\
F_3 &= -36y(0) + 1661y'(0) \\
F_4 &= -272y(0) + 14901y'(0) \\
F_5 &= -2985y(0) + 148810y'(0) \\
F_6 &= -31119y(0) + 1634384y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1 - \frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{3}{10}x^5 - \frac{17}{45}x^6 - \frac{199}{336}x^7 - \frac{10373}{13440}x^8\right) y(0) \\
& + \left(x + \frac{5}{2}x^2 + 5x^3 + \frac{26}{3}x^4 + \frac{1661}{120}x^5 + \frac{4967}{240}x^6 + \frac{14881}{504}x^7 + \frac{102149}{2520}x^8\right) y'(0) + O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x^2 - 1)\, y'' + (5x + 5)\, y' + (x^2 - x)\, y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x^2 - 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + (5x + 5) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + (x^2 - x) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\sum_{n=2}^{\infty}\left(-n(n-1) a_{n} x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 5 n a_{n} x^{n}\right) \\
+\left(\sum_{n=1}^{\infty} 5 n a_{n} x^{n-1}\right)+\left(\sum_{n=0}^{\infty} x^{n+2} a_{n}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n} a_{n}\right)=0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty}\left(-n(n-1) a_{n} x^{n-2}\right) &= \sum_{n=0}^{\infty}\left(-(n+2) a_{n+2}(1+n) x^{n}\right) \\
\sum_{n=1}^{\infty} 5 n a_{n} x^{n-1} &= \sum_{n=0}^{\infty} 5(1+n) a_{1+n} x^{n} \\
\sum_{n=0}^{\infty} x^{n+2} a_{n} &= \sum_{n=2}^{\infty} a_{n-2} x^{n} \\
\sum_{n=0}^{\infty}\left(-x^{1+n} a_{n}\right) &= \sum_{n=1}^{\infty}\left(-a_{n-1} x^{n}\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\sum_{n=0}^{\infty}\left(-(n+2) a_{n+2}(1+n) x^{n}\right)+\left(\sum_{n=1}^{\infty} 5 n a_{n} x^{n}\right) \\
+\left(\sum_{n=0}^{\infty} 5(1+n) a_{1+n} x^{n}\right)+\left(\sum_{n=2}^{\infty} a_{n-2} x^{n}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n}\right)=0 \tag{3}
$$

$n = 0$ gives

$$
-2a_2 + 5a_1 = 0
$$

$$
a_2 = \frac{5a_1}{2}
$$

$n = 1$ gives

$$-6a_3 + 5a_1 + 10a_2 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} + 5a_1$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) - (n+2)\,a_{n+2}(1+n) + 5na_n + 5(1+n)\,a_{1+n} + a_{n-2} - a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_n + 4na_n + 5na_{1+n} + 5a_{1+n} + a_{n-2} - a_{n-1}}{(n+2)\,(1+n)} \\ &= \frac{(n^2+4n)\,a_n}{(n+2)\,(1+n)} + \frac{(5n+5)\,a_{1+n}}{(n+2)\,(1+n)} + \frac{a_{n-2}}{(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_2 - 12a_4 + 15a_3 + a_0 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{26a_1}{3} - \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$21a_3 - 20a_5 + 20a_4 + a_1 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{3a_0}{10} + \frac{1661a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$32a_4 - 30a_6 + 25a_5 + a_2 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{4967a_1}{240} - \frac{17a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$45a_5 - 42a_7 + 30a_6 + a_3 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{199a_0}{336} + \frac{14881a_1}{504}$$

For $n = 6$ the recurrence equation gives

$$60a_6 - 56a_8 + 35a_7 + a_4 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{102149a_1}{2520} - \frac{10373a_0}{13440}$$

For $n = 7$ the recurrence equation gives

$$77a_7 - 72a_9 + 40a_8 + a_5 - a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{48131a_0}{45360} + \frac{435461a_1}{8064}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1x + \frac{5a_1x^2}{2} + \left(-\frac{a_0}{6} + 5a_1\right)x^3 + \left(\frac{26a_1}{3} - \frac{a_0}{8}\right)x^4 + \left(-\frac{3a_0}{10} + \frac{1661a_1}{120}\right)x^5 \\ + \left(\frac{4967a_1}{240} - \frac{17a_0}{45}\right)x^6 + \left(-\frac{199a_0}{336} + \frac{14881a_1}{504}\right)x^7 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y &= \left(1-\frac{1}{6}x^{3}-\frac{1}{8}x^{4}-\frac{3}{10}x^{5}-\frac{17}{45}x^{6}-\frac{199}{336}x^{7}\right)a_{0} \\
&\quad+\left(x+\frac{5}{2}x^{2}+5x^{3}+\frac{26}{3}x^{4}+\frac{1661}{120}x^{5}+\frac{4967}{240}x^{6}+\frac{14881}{504}x^{7}\right)a_{1}+O\left(x^{8}\right)
\end{aligned} \tag{3}
$$

At $x = 0$ the solution above becomes

$$
\begin{aligned}
y &= \left(1-\frac{1}{6}x^{3}-\frac{1}{8}x^{4}-\frac{3}{10}x^{5}-\frac{17}{45}x^{6}-\frac{199}{336}x^{7}\right)c_{1} \\
&\quad+\left(x+\frac{5}{2}x^{2}+5x^{3}+\frac{26}{3}x^{4}+\frac{1661}{120}x^{5}+\frac{4967}{240}x^{6}+\frac{14881}{504}x^{7}\right)c_{2}+O\left(x^{8}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \left(1-\frac{1}{6}x^{3}-\frac{1}{8}x^{4}-\frac{3}{10}x^{5}-\frac{17}{45}x^{6}-\frac{199}{336}x^{7}-\frac{10373}{13440}x^{8}\right)y(0) \\
&\quad+\left(x+\frac{5}{2}x^{2}+5x^{3}+\frac{26}{3}x^{4}+\frac{1661}{120}x^{5}+\frac{4967}{240}x^{6}+\frac{14881}{504}x^{7}+\frac{102149}{2520}x^{8}\right)y'(0)+O\left(x^{8}\right)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
y &= \left(1-\frac{1}{6}x^{3}-\frac{1}{8}x^{4}-\frac{3}{10}x^{5}-\frac{17}{45}x^{6}-\frac{199}{336}x^{7}\right)c_{1} \\
&\quad+\left(x+\frac{5}{2}x^{2}+5x^{3}+\frac{26}{3}x^{4}+\frac{1661}{120}x^{5}+\frac{4967}{240}x^{6}+\frac{14881}{504}x^{7}\right)c_{2}+O\left(x^{8}\right)
\end{aligned} \tag{2}
$$

Verification of solutions

$$
\begin{aligned}
y &= \left(1-\frac{1}{6}x^{3}-\frac{1}{8}x^{4}-\frac{3}{10}x^{5}-\frac{17}{45}x^{6}-\frac{199}{336}x^{7}-\frac{10373}{13440}x^{8}\right)y(0) \\
&\quad+\left(x+\frac{5}{2}x^{2}+5x^{3}+\frac{26}{3}x^{4}+\frac{1661}{120}x^{5}+\frac{4967}{240}x^{6}+\frac{14881}{504}x^{7}+\frac{102149}{2520}x^{8}\right)y'(0)+O\left(x^{8}\right)
\end{aligned}
$$

Verified OK.

$$
\begin{aligned}
y &= \left(1-\frac{1}{6}x^{3}-\frac{1}{8}x^{4}-\frac{3}{10}x^{5}-\frac{17}{45}x^{6}-\frac{199}{336}x^{7}\right)c_{1} \\
&\quad+\left(x+\frac{5}{2}x^{2}+5x^{3}+\frac{26}{3}x^{4}+\frac{1661}{120}x^{5}+\frac{4967}{240}x^{6}+\frac{14881}{504}x^{7}\right)c_{2}+O\left(x^{8}\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 69

```
Order:=8;
dsolve((x^2-1)*diff(y(x),x$2)+5*(x+1)*diff(y(x),x)+(x^2-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{6}x^3-\frac{1}{8}x^4-\frac{3}{10}x^5-\frac{17}{45}x^6-\frac{199}{336}x^7\right)y(0) + \left(x+\frac{5}{2}x^2+5x^3+\frac{26}{3}x^4+\frac{1661}{120}x^5+\frac{4967}{240}x^6+\frac{14881}{504}x^7\right)D(y)(0)+O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 89

```
AsymptoticDSolveValue[(x^2-1)*y''[x]+5*(x+1)*y'[x]+(x^2-x)*y[x]==0,y[x],{x,0,7}]
```

$$
\begin{aligned}
y(x) \to\ & c_1\left(-\frac{199x^7}{336}-\frac{17x^6}{45}-\frac{3x^5}{10}-\frac{x^4}{8}-\frac{x^3}{6}+1\right) \\
& + c_2\left(\frac{14881x^7}{504}+\frac{4967x^6}{240}+\frac{1661x^5}{120}+\frac{26x^4}{3}+5x^3+\frac{5x^2}{2}+x\right)
\end{aligned}
$$

2.12 problem 12

Internal problem ID [6595]
Internal file name [`OUTPUT/5843_Sunday_June_05_2022_03_57_18_PM_905368/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (x+3)\,y' + 7yx^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (x+3)\,y' + 7yx^2 = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{x+3}{x} \\ q(x) &= 7x \end{aligned}$$

Table 42: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 7x$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (x+3)\,y' + 7yx^2 = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x + (x+3)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + 7\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x^2 = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 7x^{2+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 7x^{2+n+r} a_n = \sum_{n=3}^{\infty} 7a_{n-3} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=3}^{\infty} 7a_{n-3} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r)=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1&=0\\r_2&=-2\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1-r_2=2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x)&=x^{r_1}\left(\sum_{n=0}^{\infty}a_n x^n\right)\\y_2(x)&=Cy_1(x)\ln(x)+x^{r_2}\left(\sum_{n=0}^{\infty}b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x)&=\sum_{n=0}^{\infty}a_n x^n\\y_2(x)&=Cy_1(x)\ln(x)+\frac{\sum_{n=0}^{\infty}b_n x^n}{x^2}\end{aligned}$$

Or

$$\begin{aligned}y_1(x)&=\sum_{n=0}^{\infty}a_n x^n\\y_2(x)&=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n-2}\right)\end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. Substituting $n=1$ in Eq. (2B) gives

$$a_1=-\frac{r}{r^2+4r+3}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{r}{r^3 + 9r^2 + 26r + 24}$$

For $3 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + 3a_n(n+r) + 7a_{n-3} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{na_{n-1} + ra_{n-1} + 7a_{n-3} - a_{n-1}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{-na_{n-1} - 7a_{n-3} + a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+4r+3}$	0
a_2	$\frac{r}{r^3+9r^2+26r+24}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-7r^2 - 50r - 84}{(5+r)(r+3)^2(4+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{7}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+4r+3}$	0
a_2	$\frac{r}{r^3+9r^2+26r+24}$	0
a_3	$\frac{-7r^2-50r-84}{(5+r)(r+3)^2(4+r)}$	$-\frac{7}{15}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{14r^3 + 120r^2 + 274r + 84}{(r+6)(4+r)^2(r+3)(1+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{7}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+4r+3}$	0
a_2	$\frac{r}{r^3+9r^2+26r+24}$	0
a_3	$\frac{-7r^2-50r-84}{(5+r)(r+3)^2(4+r)}$	$-\frac{7}{15}$
a_4	$\frac{14r^3+120r^2+274r+84}{(r+6)(4+r)^2(r+3)(1+r)(5+r)}$	$\frac{7}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-21r^4 - 232r^3 - 801r^2 - 842r - 168}{(7+r)(5+r)^2(r+3)(4+r)(2+r)(1+r)(r+6)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{150}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+4r+3}$	0
a_2	$\frac{r}{r^3+9r^2+26r+24}$	0
a_3	$\frac{-7r^2-50r-84}{(5+r)(r+3)^2(4+r)}$	$-\frac{7}{15}$
a_4	$\frac{14r^3+120r^2+274r+84}{(r+6)(4+r)^2(r+3)(1+r)(5+r)}$	$\frac{7}{120}$
a_5	$\frac{-21r^4-232r^3-801r^2-842r-168}{(7+r)(5+r)^2(r+3)(4+r)(2+r)(1+r)(r+6)}$	$-\frac{1}{150}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{49r^6 + 1155r^5 + 10550r^4 + 47403r^3 + 109365r^2 + 121470r + 49896}{(r+8)(r+6)^2(1+r)(2+r)(4+r)(r+3)^2(5+r)(7+r)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{11}{160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+4r+3}$	0
a_2	$\frac{r}{r^3+9r^2+26r+24}$	0
a_3	$\frac{-7r^2-50r-84}{(5+r)(r+3)^2(4+r)}$	$-\frac{7}{15}$
a_4	$\frac{14r^3+120r^2+274r+84}{(r+6)(4+r)^2(r+3)(1+r)(5+r)}$	$\frac{7}{120}$
a_5	$\frac{-21r^4-232r^3-801r^2-842r-168}{(7+r)(5+r)^2(r+3)(4+r)(2+r)(1+r)(r+6)}$	$-\frac{1}{150}$
a_6	$\frac{49r^6+1155r^5+10550r^4+47403r^3+109365r^2+121470r+49896}{(r+8)(r+6)^2(1+r)(2+r)(4+r)(r+3)^2(5+r)(7+r)}$	$\frac{11}{160}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-147r^7 - 4151r^6 - 47314r^5 - 279891r^4 - 917231r^3 - 1631386r^2 - 1397784r - 397152}{(r+9)(7+r)^2(5+r)(r+3)^2(4+r)^2(2+r)(1+r)(r+6)(r+8)}$$

Which for the root $r = 0$ becomes

$$a_7 = -\frac{197}{15120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+4r+3}$	0
a_2	$\frac{r}{r^3+9r^2+26r+24}$	0
a_3	$\frac{-7r^2-50r-84}{(5+r)(r+3)^2(4+r)}$	$-\frac{7}{15}$
a_4	$\frac{14r^3+120r^2+274r+84}{(r+6)(4+r)^2(r+3)(1+r)(5+r)}$	$\frac{7}{120}$
a_5	$\frac{-21r^4-232r^3-801r^2-842r-168}{(7+r)(5+r)^2(r+3)(4+r)(2+r)(1+r)(r+6)}$	$-\frac{1}{150}$
a_6	$\frac{49r^6+1155r^5+10550r^4+47403r^3+109365r^2+121470r+49896}{(r+8)(r+6)^2(1+r)(2+r)(4+r)(r+3)^2(5+r)(7+r)}$	$\frac{11}{160}$
a_7	$\frac{-147r^7-4151r^6-47314r^5-279891r^4-917231r^3-1631386r^2-1397784r-397152}{(r+9)(7+r)^2(5+r)(r+3)^2(4+r)^2(2+r)(1+r)(r+6)(r+8)}$	$-\frac{197}{15120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\
&= 1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + \frac{11x^6}{160} - \frac{197x^7}{15120} + O(x^8)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_2 \\
&= \frac{r}{r^3 + 9r^2 + 26r + 24}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{r}{r^3 + 9r^2 + 26r + 24} &= \lim_{r\to -2} \frac{r}{r^3 + 9r^2 + 26r + 24} \\
&= \mathit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\\ &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)\\ \frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\\ &\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\\ &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)\end{aligned}$$

Substituting these back into the given ode $xy'' + (x+3)\,y' + 7yx^2 = 0$ gives

$$\begin{aligned}&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right.\\ &\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\ &+ (x+3)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\ &+ 7\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right)x^2 = 0\end{aligned}$$

Which can be written as

$$\begin{aligned}&\left(\left(y_1''(x)\,x + (x+3)\,y_1'(x) + 7y_1(x)\,x^2\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x\right.\\ &\left.+ \frac{(x+3)\,y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\ &+ (x+3)\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 7\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x^2 = 0\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x + (x+3)\,y_1'(x) + 7y_1(x)\,x^2 = 0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + \frac{(x+3)\,y_1(x)}{x}\right)C \\ &+ \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\ &+ (x+3)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 7\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x^2 = 0\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right)x + (x+2)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C}{x} \\ &+ \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\,(-1+n+r_2)\right)x^2 + (x^2+3x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) + 7\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x^3}{x} \\ &= 0\end{aligned} \tag{9}$$

Since $r_1 = 0$ and $r_2 = -2$ then the above becomes

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{n-1} a_n n\right)x + (x+2)\left(\sum_{n=0}^{\infty} a_n x^n\right)\right)C}{x} \\ &+ \frac{\left(\sum_{n=0}^{\infty} x^{-4+n} b_n(n-2)\,(n-3)\right)x^2 + (x^2+3x)\left(\sum_{n=0}^{\infty} x^{n-3} b_n(n-2)\right) + 7\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)x^3}{x} \\ &= 0\end{aligned} \tag{10}$$

Which simplifies to

$$
\begin{gathered}
\left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n n\right)+\left(\sum_{n=0}^{\infty} C a_n x^n\right)+\left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n\right) \\
+\left(\sum_{n=0}^{\infty} x^{n-3} b_n\left(n^2-5n+6\right)\right)+\left(\sum_{n=0}^{\infty} x^{n-2} b_n(n-2)\right) \\
+\left(\sum_{n=0}^{\infty} 3x^{n-3} b_n(n-2)\right)+\left(\sum_{n=0}^{\infty} 7b_n x^n\right)=0
\end{gathered} \tag{2A}
$$

The next step is to make all powers of x be $n-3$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-3} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n n &= \sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{n-3} \\
\sum_{n=0}^{\infty} C a_n x^n &= \sum_{n=3}^{\infty} C a_{n-3} x^{n-3} \\
\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n &= \sum_{n=2}^{\infty} 2C a_{n-2} x^{n-3} \\
\sum_{n=0}^{\infty} x^{n-2} b_n(n-2) &= \sum_{n=1}^{\infty} b_{n-1}(n-3)\, x^{n-3} \\
\sum_{n=0}^{\infty} 7b_n x^n &= \sum_{n=3}^{\infty} 7b_{n-3} x^{n-3}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-3$.

$$
\begin{gathered}
\left(\sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{n-3}\right)+\left(\sum_{n=3}^{\infty} C a_{n-3} x^{n-3}\right)+\left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{n-3}\right) \\
+\left(\sum_{n=0}^{\infty} x^{n-3} b_n\left(n^2-5n+6\right)\right)+\left(\sum_{n=1}^{\infty} b_{n-1}(n-3)\, x^{n-3}\right) \\
+\left(\sum_{n=0}^{\infty} 3x^{n-3} b_n(n-2)\right)+\left(\sum_{n=3}^{\infty} 7b_{n-3} x^{n-3}\right)=0
\end{gathered} \tag{2B}
$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-b_1 - 2b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 - 2 = 0$$

Solving the above for b_1 gives

$$b_1 = -2$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$2C + 2 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 3$, Eq (2B) gives

$$(a_0 + 4a_1)\,C + 7b_0 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6 + 3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -2$$

For $n = 4$, Eq (2B) gives

$$(a_1 + 6a_2)\,C + 7b_1 + b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-16 + 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = 2$$

For $n = 5$, Eq (2B) gives

$$(a_2 + 8a_3)\,C + 7b_2 + 2b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{116}{15} + 15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{116}{225}$$

For $n = 6$, Eq (2B) gives

$$(a_3 + 10a_4)\, C + 7b_3 + 3b_5 + 24b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{4699}{300} + 24b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{4699}{7200}$$

For $n = 7$, Eq (2B) gives

$$(a_4 + 12a_5)\, C + 7b_4 + 4b_6 + 35b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{14969}{900} + 35b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = -\frac{14969}{31500}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1)\left(1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + \frac{11x^6}{160} - \frac{197x^7}{15120} + O(x^8)\right)\ln(x) + \frac{1 - 2x - 2x^3 + 2x^4 - \frac{116x^5}{225} + \frac{4699x^6}{7200} - \frac{14969x^7}{31500} + O(x^8)}{x^2}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + \frac{11x^6}{160} - \frac{197x^7}{15120} + O(x^8)\right) \\
&\quad + c_2 \left((-1)\left(1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + \frac{11x^6}{160} - \frac{197x^7}{15120} + O(x^8)\right)\ln(x)\right. \\
&\qquad \left. + \frac{1 - 2x - 2x^3 + 2x^4 - \frac{116x^5}{225} + \frac{4699x^6}{7200} - \frac{14969x^7}{31500} + O(x^8)}{x^2}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + \frac{11x^6}{160} - \frac{197x^7}{15120} + O(x^8)\right) \\
&\quad + c_2 \left(\left(-1 + \frac{7x^3}{15} - \frac{7x^4}{120} + \frac{x^5}{150} - \frac{11x^6}{160} + \frac{197x^7}{15120} - O(x^8)\right)\ln(x)\right. \\
&\qquad \left. + \frac{1 - 2x - 2x^3 + 2x^4 - \frac{116x^5}{225} + \frac{4699x^6}{7200} - \frac{14969x^7}{31500} + O(x^8)}{x^2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + \frac{11x^6}{160} - \frac{197x^7}{15120} + O(x^8)\right) \\
&\quad + c_2 \left(\left(-1 + \frac{7x^3}{15} - \frac{7x^4}{120} + \frac{x^5}{150} - \frac{11x^6}{160} + \frac{197x^7}{15120} - O(x^8)\right)\ln(x)\right. \\
&\qquad \left. + \frac{1 - 2x - 2x^3 + 2x^4 - \frac{116x^5}{225} + \frac{4699x^6}{7200} - \frac{14969x^7}{31500} + O(x^8)}{x^2}\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1 \left(1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + \frac{11x^6}{160} - \frac{197x^7}{15120} + O(x^8)\right) + c_2 \left(\left(-1 + \frac{7x^3}{15} - \frac{7x^4}{120} + \frac{x^5}{150} - \frac{11x^6}{160} + \frac{197x^7}{15120} - O(x^8)\right) \ln (x) + \frac{1 - 2x - 2x^3 + 2x^4 - \frac{116x^5}{225} + \frac{4699x^6}{7200} - \frac{14969x^7}{31500} + O(x^8)}{x^2}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 68

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(x+3)*diff(y(x),x)+7*x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1-\frac{7}{15}x^3+\frac{7}{120}x^4-\frac{1}{150}x^5+\frac{11}{160}x^6-\frac{197}{15120}x^7+\mathrm{O}\left(x^8\right)\right)$$
$$+\frac{c_2\left(\ln\left(x\right)\left(2x^2-\frac{14}{15}x^5+\frac{7}{60}x^6-\frac{1}{75}x^7+\mathrm{O}\left(x^8\right)\right)+\left(-2+4x-3x^2+4x^3-4x^4+\frac{547}{225}x^5-\frac{5329}{3600}x^6+\frac{764}{787}\right.\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 96

```
AsymptoticDSolveValue[x*y''[x]+(x+3)*y'[x]+7*x^2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(\frac{11x^6}{160}-\frac{x^5}{150}+\frac{7x^4}{120}-\frac{7x^3}{15}+1\right)$$
$$+c_1\left(\frac{5539x^6-10432x^5+14400x^4-14400x^3+14400x^2-14400x+7200}{7200x^2}\right.$$
$$\left.-\frac{1}{120}\left(7x^4-56x^3+120\right)\log(x)\right)$$

2.13 problem 13

Internal problem ID [6596]
Internal file name [`OUTPUT/5844_Sunday_June_05_2022_03_57_24_PM_73542075/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned}p(x) &= \frac{3x+5}{3x}\\ q(x) &= -\frac{1}{3x^2}\end{aligned}$$

Table 43: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x+5}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{3x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ + \left(\frac{5}{3}x + x^2\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \frac{\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)}{3} = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)\right) \\ +\left(\sum_{n=0}^{\infty} \frac{5x^{n+r} a_n(n+r)}{3}\right)+\sum_{n=0}^{\infty}\left(-\frac{a_n x^{n+r}}{3}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)=\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\,x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\,x^{n+r}\right) \\ +\left(\sum_{n=0}^{\infty} \frac{5x^{n+r} a_n(n+r)}{3}\right)+\sum_{n=0}^{\infty}\left(-\frac{a_n x^{n+r}}{3}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)\,(n+r-1)+\frac{5x^{n+r} a_n(n+r)}{3}-\frac{a_n x^{n+r}}{3}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)+\frac{5x^r a_0 r}{3}-\frac{a_0 x^r}{3}=0$$

Or

$$\left(x^r r(-1+r)+\frac{5x^r r}{3}-\frac{x^r}{3}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$\frac{(3r^2+2r-1)\,x^r}{3}=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + \frac{2}{3}r - \frac{1}{3} = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{3} \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(3r^2 + 2r - 1)\, x^r}{3} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)\,(n + r - 1) + a_{n-1}(n + r - 1) + \frac{5a_n(n + r)}{3} - \frac{a_n}{3} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}(n + r - 1)}{3n^2 + 6nr + 3r^2 + 2n + 2r - 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{a_{n-1}(2 - 3n)}{3n^2 + 4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{3r}{3r^2 + 8r + 4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = -\frac{1}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9r(1+r)}{(3r^2 + 8r + 4)(3r^2 + 14r + 15)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{1}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$
a_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{1}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27r(1+r)}{(3r^2 + 20r + 32)(3r^2 + 14r + 15)(3r + 2)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{1}{195}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$
a_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{1}{35}$
a_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	$-\frac{1}{195}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{1}{1248}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$
a_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{1}{35}$
a_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	$-\frac{1}{195}$
a_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	$\frac{1}{1248}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{1}{9120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$
a_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{1}{35}$
a_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	$-\frac{1}{195}$
a_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	$\frac{1}{1248}$
a_5	$-\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$	$-\frac{1}{9120}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{729r(1+r)}{(3r^2+38r+119)(3r+11)(3r+2)(3r+5)(3r+8)(3r^2+32r+84)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_6 = \frac{1}{75240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$
a_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{1}{35}$
a_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	$-\frac{1}{195}$
a_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	$\frac{1}{1248}$
a_5	$-\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$	$-\frac{1}{9120}$
a_6	$\frac{729r(1+r)}{(3r^2+38r+119)(3r+11)(3r+2)(3r+5)(3r+8)(3r^2+32r+84)}$	$\frac{1}{75240}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{2187r(1+r)}{(3r^2+44r+160)(3r+14)(3r+8)(3r+5)(3r+2)(3r+11)(3r^2+38r+119)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_7 = -\frac{1}{693000}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$
a_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{1}{35}$
a_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	$-\frac{1}{195}$
a_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	$\frac{1}{1248}$
a_5	$-\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$	$-\frac{1}{9120}$
a_6	$\frac{729r(1+r)}{(3r^2+38r+119)(3r+11)(3r+2)(3r+5)(3r+8)(3r^2+32r+84)}$	$\frac{1}{75240}$
a_7	$-\frac{2187r(1+r)}{(3r^2+44r+160)(3r+14)(3r+8)(3r+5)(3r+2)(3r+11)(3r^2+38r+119)}$	$-\frac{1}{693000}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{x}{7} + \frac{x^2}{35} - \frac{x^3}{195} + \frac{x^4}{1248} - \frac{x^5}{9120} + \frac{x^6}{75240} - \frac{x^7}{693000} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) + \frac{5b_n(n+r)}{3} - \frac{b_n}{3} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{3b_{n-1}(n+r-1)}{3n^2 + 6nr + 3r^2 + 2n + 2r - 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{3b_{n-1}(n-2)}{n(3n-4)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{3r}{3r^2 + 8r + 4}$$

Which for the root $r = -1$ becomes

$$b_1 = -3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9r(1 + r)}{(3r^2 + 8r + 4)(3r^2 + 14r + 15)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3
b_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{27r(1 + r)}{(3r^2 + 20r + 32)(3r^2 + 14r + 15)(3r + 2)}$$

Which for the root $r = -1$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3
b_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	0
b_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$$

Which for the root $r = -1$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3
b_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	0
b_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	0
b_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$$

Which for the root $r = -1$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3
b_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	0
b_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	0
b_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	0
b_5	$-\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{729r(1+r)}{(3r^2+38r+119)(3r+11)(3r+2)(3r+5)(3r+8)(3r^2+32r+84)}$$

Which for the root $r = -1$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3
b_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	0
b_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	0
b_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	0
b_5	$-\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$	0
b_6	$\frac{729r(1+r)}{(3r^2+38r+119)(3r+11)(3r+2)(3r+5)(3r+8)(3r^2+32r+84)}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{2187r(1+r)}{(3r^2+44r+160)(3r+14)(3r+8)(3r+5)(3r+2)(3r+11)(3r^2+38r+119)}$$

Which for the root $r = -1$ becomes

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3
b_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	0
b_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	0
b_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	0
b_5	$-\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$	0
b_6	$\frac{729r(1+r)}{(3r^2+38r+119)(3r+11)(3r+2)(3r+5)(3r+8)(3r^2+32r+84)}$	0
b_7	$-\frac{2187r(1+r)}{(3r^2+44r+160)(3r+14)(3r+8)(3r+5)(3r+2)(3r+11)(3r^2+38r+119)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 - 3x + O(x^8)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{x}{7} + \frac{x^2}{35} - \frac{x^3}{195} + \frac{x^4}{1248} - \frac{x^5}{9120} + \frac{x^6}{75240} - \frac{x^7}{693000} + O(x^8)\right) \\
&\quad + \frac{c_2(1 - 3x + O(x^8))}{x}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{x}{7} + \frac{x^2}{35} - \frac{x^3}{195} + \frac{x^4}{1248} - \frac{x^5}{9120} + \frac{x^6}{75240} - \frac{x^7}{693000} + O(x^8)\right) \\
&\quad + \frac{c_2(1 - 3x + O(x^8))}{x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{3}} \left(1-\frac{x}{7}+\frac{x^2}{35}-\frac{x^3}{195}+\frac{x^4}{1248}-\frac{x^5}{9120}+\frac{x^6}{75240}-\frac{x^7}{693000}+O(x^8)\right) \\
&\quad + \frac{c_2(1-3x+O(x^8))}{x}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{3}} \left(1-\frac{x}{7}+\frac{x^2}{35}-\frac{x^3}{195}+\frac{x^4}{1248}-\frac{x^5}{9120}+\frac{x^6}{75240}-\frac{x^7}{693000}+O(x^8)\right) \\
&\quad + \frac{c_2(1-3x+O(x^8))}{x}
\end{aligned}
$$

Verified OK.

2.13.1 Maple step by step solution

Let's solve

$x^2y''+\left(\frac{5}{3}x+x^2\right)y'-\frac{y}{3}=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{3x^2}-\frac{(3x+5)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3x+5)y'}{3x}-\frac{y}{3x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x)=\frac{3x+5}{3x}, P_3(x)=-\frac{1}{3x^2}\right]$

  ○ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{5}{3}$

  ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{3}$

  ○ $x=0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $3x^2y'' + x(3x+5)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ○ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ○ Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+3r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(3k+3r-1) + 3a_{k-1}(k+r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3\left(k+r-\frac{1}{3}\right)(k+r+1)\,a_k + 3a_{k-1}(k+r-1) = 0$

- Shift index using $k- >k+1$

  $3\left(k+\frac{2}{3}+r\right)(k+2+r)\,a_{k+1} + 3a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k(k+r)}{(3k+2+3r)(k+2+r)}$

- Recursion relation for $r = -1$; series terminates at $k = 1$

$$a_{k+1} = -\frac{3a_k(k-1)}{(3k-1)(k+1)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -3a_0$$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

$$y = a_0 \cdot (1 - 3x)$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{(3k+3)\left(k+\frac{7}{3}\right)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{(3k+3)\left(k+\frac{7}{3}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot (1 - 3x) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{k+1} = -\frac{3b_k\left(k+\frac{1}{3}\right)}{(3k+3)\left(k+\frac{7}{3}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+(5/3*x+x^2)*diff(y(x),x)-1/3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}}\left(1-\frac{1}{7}x+\frac{1}{35}x^2-\frac{1}{195}x^3+\frac{1}{1248}x^4-\frac{1}{9120}x^5+\frac{1}{75240}x^6-\frac{1}{693000}x^7+\mathrm{O}\left(x^8\right)\right)+c_1\left(1-3x+\mathrm{O}\left(x^8\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 72

```
AsymptoticDSolveValue[x^2*y''[x]+(5/3*x+x^2)*y'[x]-1/3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(-\frac{x^7}{693000}+\frac{x^6}{75240}-\frac{x^5}{9120}+\frac{x^4}{1248}-\frac{x^3}{195}+\frac{x^2}{35}-\frac{x}{7}+1\right)+\frac{c_2(1-3x)}{x}$$

2.14 problem 14

Internal problem ID [6597]
Internal file name [`OUTPUT/5845_Sunday_June_05_2022_03_57_28_PM_14834089/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' + 10y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + 10y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{10}{x}$$

Table 45: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{10}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + 10y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 10\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 10a_n x^{n+r} = \sum_{n=1}^{\infty} 10a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 10a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + 10a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{10a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{10a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{10}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -10$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{(r+1)^2}$	-10

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{100}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 25$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{(r+1)^2}$	-10
a_2	$\frac{100}{(r+1)^2(r+2)^2}$	25

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1000}{(r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{250}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{(r+1)^2}$	-10
a_2	$\frac{100}{(r+1)^2(r+2)^2}$	25
a_3	$-\frac{1000}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{250}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{10000}{\left(r+1\right)^{2} \left(r+2\right)^{2} \left(r+3\right)^{2} \left(4+r\right)^{2}}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{625}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{\left(r+1\right)^{2}}$	-10
a_2	$\frac{100}{\left(r+1\right)^{2} \left(r+2\right)^{2}}$	25
a_3	$-\frac{1000}{\left(r+1\right)^{2} \left(r+2\right)^{2} \left(r+3\right)^{2}}$	$-\frac{250}{9}$
a_4	$\frac{10000}{\left(r+1\right)^{2} \left(r+2\right)^{2} \left(r+3\right)^{2} \left(4+r\right)^{2}}$	$\frac{625}{36}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{100000}{\left(r+1\right)^{2} \left(r+2\right)^{2} \left(r+3\right)^{2} \left(4+r\right)^{2} \left(5+r\right)^{2}}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{125}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{\left(r+1\right)^{2}}$	-10
a_2	$\frac{100}{\left(r+1\right)^{2} \left(r+2\right)^{2}}$	25
a_3	$-\frac{1000}{\left(r+1\right)^{2} \left(r+2\right)^{2} \left(r+3\right)^{2}}$	$-\frac{250}{9}$
a_4	$\frac{10000}{\left(r+1\right)^{2} \left(r+2\right)^{2} \left(r+3\right)^{2} \left(4+r\right)^{2}}$	$\frac{625}{36}$
a_5	$-\frac{100000}{\left(r+1\right)^{2} \left(r+2\right)^{2} \left(r+3\right)^{2} \left(4+r\right)^{2} \left(5+r\right)^{2}}$	$-\frac{125}{18}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1000000}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2 (5+r)^2 (r+6)^2}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{625}{324}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{(r+1)^2}$	-10
a_2	$\frac{100}{(r+1)^2(r+2)^2}$	25
a_3	$-\frac{1000}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{250}{9}$
a_4	$\frac{10000}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{625}{36}$
a_5	$-\frac{100000}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{125}{18}$
a_6	$\frac{1000000}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2(r+6)^2}$	$\frac{625}{324}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{10000000}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2 (5+r)^2 (r+6)^2 (7+r)^2}$$

Which for the root $r = 0$ becomes

$$a_7 = -\frac{3125}{7938}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{(r+1)^2}$	-10
a_2	$\frac{100}{(r+1)^2(r+2)^2}$	25
a_3	$-\frac{1000}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{250}{9}$
a_4	$\frac{10000}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{625}{36}$
a_5	$-\frac{100000}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{125}{18}$
a_6	$\frac{1000000}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2(r+6)^2}$	$\frac{625}{324}$
a_7	$-\frac{10000000}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2(r+6)^2(7+r)^2}$	$-\frac{3125}{7938}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\
&= 25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + \frac{625x^6}{324} - \frac{3125x^7}{7938} + O(x^8)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$-\frac{10}{(r+1)^{2}}$	-10	$\frac{20}{(r+1)^{3}}$
b_2	$\frac{100}{(r+1)^{2}(r+2)^{2}}$	25	$\frac{-400r-600}{(r+1)^{3}(r+2)^{3}}$
b_3	$-\frac{1000}{(r+1)^{2}(r+2)^{2}(r+3)^{2}}$	$-\frac{250}{9}$	$\frac{6000r^{2}+24000r+22000}{(r+1)^{3}(r+2)^{3}(r+3)^{3}}$
b_4	$\frac{10000}{(r+1)^{2}(r+2)^{2}(r+3)^{2}(4+r)^{2}}$	$\frac{625}{36}$	$-\frac{80000\left(r+\frac{5}{2}\right)\left(r^{2}+5r+5\right)}{(r+1)^{3}(r+2)^{3}(r+3)^{3}(4+r)^{3}}$
b_5	$-\frac{100000}{(r+1)^{2}(r+2)^{2}(r+3)^{2}(4+r)^{2}(5+r)^{2}}$	$-\frac{125}{18}$	$\frac{1000000r^{4}+12000000r^{3}+51000000r^{2}+90000000r+54800000}{(r+1)^{3}(r+2)^{3}(r+3)^{3}(4+r)^{3}(5+r)^{3}}$
b_6	$\frac{1000000}{(r+1)^{2}(r+2)^{2}(r+3)^{2}(4+r)^{2}(5+r)^{2}(r+6)^{2}}$	$\frac{625}{324}$	$-\frac{12000000\left(r+\frac{7}{2}\right)\left(r^{4}+14r^{3}+\frac{203}{3}r^{2}+\frac{392}{3}r+84\right)}{(r+1)^{3}(r+2)^{3}(r+3)^{3}(4+r)^{3}(5+r)^{3}(r+6)^{3}}$
b_7	$-\frac{10000000}{(r+1)^{2}(r+2)^{2}(r+3)^{2}(4+r)^{2}(5+r)^{2}(r+6)^{2}(7+r)^{2}}$	$-\frac{3125}{7938}$	$\frac{140000000r^{6}+3360000000r^{5}+32200000000r^{4}+156800000000r^{3}+4061}{(r+1)^{3}(r+2)^{3}(r+3)^{3}(4+r)^{3}(5+r)^{3}(r}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= \left(25x^{2}-10x+1-\frac{250x^{3}}{9}+\frac{625x^{4}}{36}-\frac{125x^{5}}{18}+\frac{625x^{6}}{324}-\frac{3125x^{7}}{7938}+O\left(x^{8}\right)\right)\ln(x) \\
&\quad -75x^{2}+20x+\frac{2750x^{3}}{27}-\frac{15625x^{4}}{216}+\frac{3425x^{5}}{108}-\frac{6125x^{6}}{648}+\frac{75625x^{7}}{37044}+O\left(x^{8}\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(25x^{2}-10x+1-\frac{250x^{3}}{9}+\frac{625x^{4}}{36}-\frac{125x^{5}}{18}+\frac{625x^{6}}{324}-\frac{3125x^{7}}{7938}+O\left(x^{8}\right)\right) \\
&\quad + c_2\left(\left(25x^{2}-10x+1-\frac{250x^{3}}{9}+\frac{625x^{4}}{36}-\frac{125x^{5}}{18}+\frac{625x^{6}}{324}-\frac{3125x^{7}}{7938}+O\left(x^{8}\right)\right)\ln(x)\right. \\
&\quad \left. -75x^{2}+20x+\frac{2750x^{3}}{27}-\frac{15625x^{4}}{216}+\frac{3425x^{5}}{108}-\frac{6125x^{6}}{648}+\frac{75625x^{7}}{37044}+O\left(x^{8}\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$= c_1\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + \frac{625x^6}{324} - \frac{3125x^7}{7938} + O(x^8)\right) + c_2\left(\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + \frac{625x^6}{324} - \frac{3125x^7}{7938} + O(x^8)\right)\ln(x) - 75x^2 + 20x + \frac{2750x^3}{27} - \frac{15625x^4}{216} + \frac{3425x^5}{108} - \frac{6125x^6}{648} + \frac{75625x^7}{37044} + O(x^8)\right)$$

Summary

The solution(s) found are the following

$$y = c_1\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + \frac{625x^6}{324} - \frac{3125x^7}{7938} + O(x^8)\right) + c_2\left(\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + \frac{625x^6}{324} - \frac{3125x^7}{7938} + O(x^8)\right)\ln(x) - 75x^2 + 20x + \frac{2750x^3}{27} - \frac{15625x^4}{216} + \frac{3425x^5}{108} - \frac{6125x^6}{648} + \frac{75625x^7}{37044} + O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + \frac{625x^6}{324} - \frac{3125x^7}{7938} + O(x^8)\right) + c_2\left(\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + \frac{625x^6}{324} - \frac{3125x^7}{7938} + O(x^8)\right)\ln(x) - 75x^2 + 20x + \frac{2750x^3}{27} - \frac{15625x^4}{216} + \frac{3425x^5}{108} - \frac{6125x^6}{648} + \frac{75625x^7}{37044} + O(x^8)\right)$$

Verified OK.

2.14.1 Maple step by step solution

Let's solve

$y''x + y' + 10y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{10y}{x} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{10y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{10}{x}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- ● Multiply by denominators

  $y''x + y' + 10y = 0$

- ● Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- ○ Shift index using $k- >k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- ○ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- ○ Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + 10a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + 10a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{10a_k}{(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{10a_k}{(k+1)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{10a_k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 71

```
Order:=8;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+10*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & (c_2 \ln(x) + c_1)\left(1 - 10x + 25x^2 - \frac{250}{9}x^3 + \frac{625}{36}x^4 - \frac{125}{18}x^5 + \frac{625}{324}x^6 - \frac{3125}{7938}x^7 \right. \\
& \left. + \mathrm{O}\left(x^8\right)\right) \\
& + \left(20x - 75x^2 + \frac{2750}{27}x^3 - \frac{15625}{216}x^4 + \frac{3425}{108}x^5 - \frac{6125}{648}x^6 + \frac{75625}{37044}x^7 + \mathrm{O}\left(x^8\right)\right) c_2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 147

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+10*y[x]==0,y[x],{x,0,7}]
```

$$
\begin{aligned}
y(x) \to {} & c_1\left(-\frac{3125x^7}{7938} + \frac{625x^6}{324} - \frac{125x^5}{18} + \frac{625x^4}{36} - \frac{250x^3}{9} + 25x^2 - 10x + 1\right) \\
& + c_2\left(\frac{75625x^7}{37044} - \frac{6125x^6}{648} + \frac{3425x^5}{108} - \frac{15625x^4}{216} + \frac{2750x^3}{27} - 75x^2 \right. \\
& \left. + \left(-\frac{3125x^7}{7938} + \frac{625x^6}{324} - \frac{125x^5}{18} + \frac{625x^4}{36} - \frac{250x^3}{9} + 25x^2 - 10x + 1\right)\log(x) + 20x\right)
\end{aligned}
$$

2.15 problem 15

Internal problem ID [6598]
Internal file name [`OUTPUT/5846_Sunday_June_05_2022_03_57_31_PM_1502836/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2xy'' - y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' - y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = \frac{1}{x}$$

Table 47: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' - y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x - \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) - (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) - r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) - r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-3+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-3+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^n
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2a_n(n+r)(n+r-1) - a_n(n+r) + 2a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{2a_{n-1}}{2n^2 + 4nr + 2r^2 - 3n - 3r} \tag{4}
$$

Which for the root $r = \frac{3}{2}$ becomes

$$
a_n = -\frac{2a_{n-1}}{n(2n+3)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = -\frac{2}{2r^2 + r - 1}
$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = -\frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+r-1}$	$-\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{4r^4 + 12r^3 + 7r^2 - 3r - 2}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{2}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	$\frac{2}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{8r^6 + 60r^5 + 158r^4 + 165r^3 + 32r^2 - 45r - 18}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = -\frac{4}{945}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	$\frac{2}{35}$
a_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$-\frac{4}{945}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{16r^8 + 224r^7 + 1256r^6 + 3584r^5 + 5369r^4 + 3626r^3 + 19r^2 - 1134r - 360}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{2}{10395}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	$\frac{2}{35}$
a_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$-\frac{4}{945}$
a_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$\frac{2}{10395}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{32r^{10} + 720r^9 + 6880r^8 + 36360r^7 + 115626r^6 + 223965r^5 + 249595r^4 + 124965r^3 - 19333r^2 - 45}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = -\frac{4}{675675}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	$\frac{2}{35}$
a_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$-\frac{4}{945}$
a_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$\frac{2}{10395}$
a_5	$-\frac{32}{32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600}$	$-\frac{4}{675675}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64}{(32r^{10} + 720r^9 + 6880r^8 + 36360r^7 + 115626r^6 + 223965r^5 + 249595r^4 + 124965r^3 - 19333r^2 - 458}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_6 = \frac{4}{30405375}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	$\frac{2}{35}$
a_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$-\frac{4}{945}$
a_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$\frac{2}{10395}$
a_5	$-\frac{32}{32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600}$	$-\frac{4}{675675}$
a_6	$\frac{64}{(32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600)(2r^2+21r+54)}$	$\frac{4}{30405375}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{128}{(32r^{10} + 720r^9 + 6880r^8 + 36360r^7 + 115626r^6 + 223965r^5 + 249595r^4 + 124965r^3 - 19333r^2 - 4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_7 = -\frac{8}{3618239625}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$-\frac{2}{2r^2+r-1}$
a_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$
a_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$
a_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$
a_5	$-\frac{32}{32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600}$
a_6	$\frac{64}{(32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600)(2r^2+21r+54)}$
a_7	$-\frac{128}{(32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600)(2r^2+21r+54)(2r^2+25r+77)}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \ldots\right) \\
&= x^{\frac{3}{2}}\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + \frac{4x^6}{30405375} - \frac{8x^7}{3618239625} + O\left(x^8\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) - (n+r)\,b_n + 2b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{2b_{n-1}}{2n^2 + 4nr + 2r^2 - 3n - 3r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{2b_{n-1}}{n\,(2n-3)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{2}{2r^2 + r - 1}$$

Which for the root $r = 0$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+r-1}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{4r^4 + 12r^3 + 7r^2 - 3r - 2}$$

Which for the root $r = 0$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+r-1}$	2
b_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{8r^6 + 60r^5 + 158r^4 + 165r^3 + 32r^2 - 45r - 18}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{4}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+r-1}$	2
b_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	-2
b_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{4}{9}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{16r^8 + 224r^7 + 1256r^6 + 3584r^5 + 5369r^4 + 3626r^3 + 19r^2 - 1134r - 360}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{2}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+r-1}$	2
b_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	-2
b_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{4}{9}$
b_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$-\frac{2}{45}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{32r^{10} + 720r^9 + 6880r^8 + 36360r^7 + 115626r^6 + 223965r^5 + 249595r^4 + 124965r^3 - 19333r^2 - 45}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{4}{1575}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+r-1}$	2
b_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	-2
b_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{4}{9}$
b_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$-\frac{2}{45}$
b_5	$-\frac{32}{32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600}$	$\frac{4}{1575}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64}{(32r^{10} + 720r^9 + 6880r^8 + 36360r^7 + 115626r^6 + 223965r^5 + 249595r^4 + 124965r^3 - 19333r^2 - 45}$$

Which for the root $r = 0$ becomes

$$b_6 = -\frac{4}{42525}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+r-1}$	2
b_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	-2
b_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{4}{9}$
b_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$-\frac{2}{45}$
b_5	$-\frac{32}{32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600}$	$\frac{4}{1575}$
b_6	$\frac{64}{(32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600)(2r^2+21r+54)}$	$-\frac{4}{42525}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{128}{(32r^{10} + 720r^9 + 6880r^8 + 36360r^7 + 115626r^6 + 223965r^5 + 249595r^4 + 124965r^3 - 19333r^2 - 4}$$

Which for the root $r = 0$ becomes

$$b_7 = \frac{8}{3274425}$$

And the table now becomes

n	$b_{n,r}$
b_0	1
b_1	$-\frac{2}{2r^2+r-1}$
b_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$
b_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$
b_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$
b_5	$-\frac{32}{32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600}$
b_6	$\frac{64}{(32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600)(2r^2+21r+54)}$
b_7	$-\frac{128}{(32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600)(2r^2+21r+54)(2r^2+25r+77)}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \dots \\ &= 1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} - \frac{4x^6}{42525} + \frac{8x^7}{3274425} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1 x^{\frac{3}{2}} \left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + \frac{4x^6}{30405375} - \frac{8x^7}{3618239625} + O(x^8)\right) \\ &\quad + c_2 \left(1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} - \frac{4x^6}{42525} + \frac{8x^7}{3274425} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{3}{2}} \left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + \frac{4x^6}{30405375} - \frac{8x^7}{3618239625} + O(x^8)\right) \\ &\quad + c_2 \left(1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} - \frac{4x^6}{42525} + \frac{8x^7}{3274425} + O(x^8)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 x^{\frac{3}{2}} &\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + \frac{4x^6}{30405375} - \frac{8x^7}{3618239625} + O(x^8)\right) \\ &+ c_2 \left(1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} - \frac{4x^6}{42525} + \frac{8x^7}{3274425} + O(x^8)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = c_1 x^{\frac{3}{2}} &\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + \frac{4x^6}{30405375} - \frac{8x^7}{3618239625} + O(x^8)\right) \\ &+ c_2 \left(1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} - \frac{4x^6}{42525} + \frac{8x^7}{3274425} + O(x^8)\right) \end{aligned}$$

Verified OK.

2.15.1 Maple step by step solution

Let's solve

$2y''x - y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{2x} - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} + \frac{y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{1}{2}$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x - y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

  ○ Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- ○ Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- ○ Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2(k+1+r)\left(k-\frac{1}{2}+r\right) a_{k+1} + 2a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{2a_k}{(k+1+r)(2k-1+2r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}\right]$$

- Recursion relation for $r = \frac{3}{2}$

  $$a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$$

- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}, b_{k+1}=-\frac{2b_k}{(k+\frac{5}{2})(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 52

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)-diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1 x^{\frac{3}{2}}\left(1-\frac{2}{5}x+\frac{2}{35}x^2-\frac{4}{945}x^3+\frac{2}{10395}x^4-\frac{4}{675675}x^5+\frac{4}{30405375}x^6-\frac{8}{3618239625}x^7+\mathrm{O}\left(x^8\right)\right)+c_2\left(1+2x-2x^2+\frac{4}{9}x^3-\frac{2}{45}x^4+\frac{4}{1575}x^5-\frac{4}{42525}x^6+\frac{8}{3274425}x^7+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 109

```
AsymptoticDSolveValue[2*x*y''[x]-y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x)\to c_2\left(\frac{8x^7}{3274425}-\frac{4x^6}{42525}+\frac{4x^5}{1575}-\frac{2x^4}{45}+\frac{4x^3}{9}-2x^2+2x+1\right)+c_1\left(-\frac{8x^7}{3618239625}+\frac{4x^6}{30405375}-\frac{4x^5}{675675}+\frac{2x^4}{10395}-\frac{4x^3}{945}+\frac{2x^2}{35}-\frac{2x}{5}+1\right)x^{3/2}$$

2.16 problem 16

Internal problem ID [6599]
Internal file name [`OUTPUT/5847_Sunday_June_05_2022_03_57_37_PM_28878954/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + 5y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + 5y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{5}{2x} \\ q(x) &= \frac{1}{2} \end{aligned}$$

Table 49: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{2}$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + 5y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + 5\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + 5(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + 5r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + 5r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(3+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -\frac{3}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(3+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
2a_n(n+r)(n+r-1) + 5a_n(n+r) + a_{n-2} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-2}}{2n^2 + 4nr + 2r^2 + 3n + 3r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
a_n = -\frac{a_{n-2}}{n(2n+3)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{2r^2 + 11r + 14}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 60r^3 + 325r^2 + 750r + 616}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{616}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{616}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{616}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{1}{8r^6 + 228r^5 + 2630r^4 + 15675r^3 + 50732r^2 + 84132r + 55440}$$

Which for the root $r = 0$ becomes

$$a_6 = -\frac{1}{55440}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{616}$
a_5	0	0
a_6	$-\frac{1}{8r^6+228r^5+2630r^4+15675r^3+50732r^2+84132r+55440}$	$-\frac{1}{55440}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{616}$
a_5	0	0
a_6	$-\frac{1}{8r^6+228r^5+2630r^4+15675r^3+50732r^2+84132r+55440}$	$-\frac{1}{55440}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 - \frac{x^2}{14} + \frac{x^4}{616} - \frac{x^6}{55440} + O\left(x^8\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + 5(n+r)\, b_n + b_{n-2} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{2n^2 + 4nr + 2r^2 + 3n + 3r} \tag{4}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_n = -\frac{b_{n-2}}{n\,(2n-3)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{2r^2 + 11r + 14}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 60r^3 + 325r^2 + 750r + 616}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_4 = \frac{1}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{40}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{40}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = -\frac{1}{8r^6 + 228r^5 + 2630r^4 + 15675r^3 + 50732r^2 + 84132r + 55440}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_6 = -\frac{1}{2160}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{40}$
b_5	0	0
b_6	$-\frac{1}{8r^6+228r^5+2630r^4+15675r^3+50732r^2+84132r+55440}$	$-\frac{1}{2160}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{40}$
b_5	0	0
b_6	$-\frac{1}{8r^6+228r^5+2630r^4+15675r^3+50732r^2+84132r+55440}$	$-\frac{1}{2160}$
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= 1\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6+b_7x^7+b_8x^8\dots\right)\\ &= \frac{1-\frac{x^2}{2}+\frac{x^4}{40}-\frac{x^6}{2160}+O(x^8)}{x^{\frac{3}{2}}}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x)+c_2y_2(x)\\ &= c_1\left(1-\frac{x^2}{14}+\frac{x^4}{616}-\frac{x^6}{55440}+O\left(x^8\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{40}-\frac{x^6}{2160}+O(x^8)\right)}{x^{\frac{3}{2}}}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1\left(1-\frac{x^2}{14}+\frac{x^4}{616}-\frac{x^6}{55440}+O\left(x^8\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{40}-\frac{x^6}{2160}+O(x^8)\right)}{x^{\frac{3}{2}}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\left(1-\frac{x^2}{14}+\frac{x^4}{616}-\frac{x^6}{55440}+O\left(x^8\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{40}-\frac{x^6}{2160}+O(x^8)\right)}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = c_1\left(1-\frac{x^2}{14}+\frac{x^4}{616}-\frac{x^6}{55440}+O\left(x^8\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{40}-\frac{x^6}{2160}+O(x^8)\right)}{x^{\frac{3}{2}}}$$

Verified OK.

2.16.1 Maple step by step solution

Let's solve

$2y''x+5y'+xy=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{5y'}{2x} - \frac{y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{2x} + \frac{y}{2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{5}{2x}, P_3(x) = \frac{1}{2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{5}{2}$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + 5y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ○ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  ○ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

  ○ Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0 r(3+2r)\, x^{-1+r} + a_1(1+r)\,(5+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(2k+5+2r) + a_{k-1}\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1+r)\,(5+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r+1)\left(k+\frac{5}{2}+r\right) a_{k+1} + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $2(k+2+r)\left(k+\frac{7}{2}+r\right) a_{k+2} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(2k+7+2r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(2k+7)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(2k+7)}, 5a_1 = 0\right]$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+2} = -\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+4)}$

- Solution for $r = -\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2}=-\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+4)}, -a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2}=-\frac{a_k}{(k+2)(2k+7)}, 5a_1=0, b_{k+2}=-\frac{b_k}{\left(k+\frac{1}{2}\right)(2k+4)}, -b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 36

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)+5*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1\left(1-\frac{1}{2}x^2+\frac{1}{40}x^4-\frac{1}{2160}x^6+\mathrm{O}\left(x^8\right)\right)}{x^{\frac{3}{2}}}+c_2\left(1-\frac{1}{14}x^2+\frac{1}{616}x^4-\frac{1}{55440}x^6+\mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 61

```
AsymptoticDSolveValue[2*x*y''[x]+5*y'[x]+x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{x^6}{55440}+\frac{x^4}{616}-\frac{x^2}{14}+1\right)+\frac{c_2\left(-\frac{x^6}{2160}+\frac{x^4}{40}-\frac{x^2}{2}+1\right)}{x^{3/2}}$$

2.17 problem 17

Internal problem ID [6600]
Internal file name [`OUTPUT/5848_Sunday_June_05_2022_03_57_40_PM_68937364/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{4xy''+\frac{y'}{2}+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4xy''+\frac{y'}{2}+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{1}{8x}$$
$$q(x)=\frac{1}{4x}$$

Table 51: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{8x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{4x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' + \frac{y'}{2} + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \frac{\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right)}{2} + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} \frac{(n+r)\, a_n x^{n+r-1}}{2}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} \frac{(n+r)\, a_n x^{n+r-1}}{2}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r)(n+r-1) + \frac{(n+r)\, a_n x^{n+r-1}}{2} = 0$$

When $n=0$ the above becomes

$$4x^{-1+r} a_0 r(-1+r) + \frac{r a_0 x^{-1+r}}{2} = 0$$

Or

$$\left(4x^{-1+r} r(-1+r) + \frac{r\, x^{-1+r}}{2}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}\left(-\frac{7}{2} + 4r\right) = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - \frac{7}{2} r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{7}{8}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}\left(-\frac{7}{2}+4r\right)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{7}{8}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{7}{8}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1)+\frac{a_n(n+r)}{2}+a_{n-1}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}}{8n^2+16nr+8r^2-7n-7r} \tag{4}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_n = -\frac{2a_{n-1}}{n(8n+7)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{7}{8}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{8r^2 + 9r + 1}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_1 = -\frac{2}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{(8r^2 + 9r + 1)(8r^2 + 25r + 18)}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_2 = \frac{2}{345}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$
a_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{345}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{(8r^2 + 9r + 1)(8r^2 + 25r + 18)(8r^2 + 41r + 51)}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_3 = -\frac{4}{32085}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$
a_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{345}$
a_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{32085}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_4 = \frac{2}{1251315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$
a_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{345}$
a_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{32085}$
a_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{1251315}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_5 = -\frac{4}{294059025}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$
a_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{345}$
a_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{32085}$
a_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{1251315}$
a_5	$-\frac{32}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)}$	$-\frac{4}{294059025}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)(8r^2+89r+2}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_6 = \frac{4}{48519739125}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$
a_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{345}$
a_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{32085}$
a_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{1251315}$
a_5	$-\frac{32}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)}$	$-\frac{4}{294059025}$
a_6	$\frac{64}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)(8r^2+89r+246)}$	$\frac{4}{48519739125}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{128}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)(8r^2+89r+}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_7 = -\frac{8}{21397204954125}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$
a_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{345}$
a_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{32085}$
a_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{1251315}$
a_5	$-\frac{32}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)}$	$-\frac{4}{294059025}$
a_6	$\frac{64}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)(8r^2+89r+246)}$	$\frac{4}{48519739125}$
a_7	$-\frac{128}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)(8r^2+89r+246)(8r^2+105r+343)}$	$-\frac{8}{21397204954125}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{7}{8}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= x^{\frac{7}{8}}\left(1 - \frac{2x}{15} + \frac{2x^2}{345} - \frac{4x^3}{32085} + \frac{2x^4}{1251315} - \frac{4x^5}{294059025} + \frac{4x^6}{48519739125} - \frac{8x^7}{21397204954125} + O(x
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
4b_n(n+r)(n+r-1) + \frac{(n+r)\,b_n}{2} + b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{2b_{n-1}}{8n^2 + 16nr + 8r^2 - 7n - 7r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{2b_{n-1}}{n\,(8n-7)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{2}{8r^2 + 9r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{(8r^2 + 9r + 1)(8r^2 + 25r + 18)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{2}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2
b_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{9}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{(8r^2 + 9r + 1)(8r^2 + 25r + 18)(8r^2 + 41r + 51)}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{4}{459}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2
b_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{9}$
b_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{459}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{2}{11475}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2
b_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{9}$
b_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{459}$
b_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{11475}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{4}{1893375}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2
b_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{9}$
b_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{459}$
b_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{11475}$
b_5	$-\frac{32}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)}$	$-\frac{4}{1893375}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64}{(8r^2 + 9r + 1)\,(8r^2 + 25r + 18)\,(8r^2 + 41r + 51)\,(8r^2 + 57r + 100)\,(8r^2 + 73r + 165)\,(8r^2 + 89r + 2}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{4}{232885125}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2
b_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{9}$
b_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{459}$
b_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{11475}$
b_5	$-\frac{32}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)}$	$-\frac{4}{1893375}$
b_6	$\frac{64}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)(8r^2+89r+246)}$	$\frac{4}{232885125}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{128}{(8r^2 + 9r + 1)\,(8r^2 + 25r + 18)\,(8r^2 + 41r + 51)\,(8r^2 + 57r + 100)\,(8r^2 + 73r + 165)\,(8r^2 + 89r +}$$

Which for the root $r = 0$ becomes

$$b_7 = -\frac{8}{79879597875}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2
b_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{9}$
b_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{459}$
b_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{11475}$
b_5	$-\frac{32}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)}$	$-\frac{4}{1893375}$
b_6	$\frac{64}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)(8r^2+89r+246)}$	$\frac{4}{232885125}$
b_7	$-\frac{128}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)(8r^2+89r+246)(8r^2+105r+343)}$	$-\frac{8}{79879597875}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 - 2x + \frac{2x^2}{9} - \frac{4x^3}{459} + \frac{2x^4}{11475} - \frac{4x^5}{1893375} + \frac{4x^6}{232885125} - \frac{8x^7}{79879597875} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{7}{8}} \left(1 - \frac{2x}{15} + \frac{2x^2}{345} - \frac{4x^3}{32085} + \frac{2x^4}{1251315} - \frac{4x^5}{294059025} + \frac{4x^6}{48519739125} \right. \\
&\quad \left. - \frac{8x^7}{21397204954125} + O(x^8)\right) + c_2 \left(1 - 2x + \frac{2x^2}{9} - \frac{4x^3}{459} + \frac{2x^4}{11475} - \frac{4x^5}{1893375} \right. \\
&\quad \left. + \frac{4x^6}{232885125} - \frac{8x^7}{79879597875} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{7}{8}} \left(1 - \frac{2x}{15} + \frac{2x^2}{345} - \frac{4x^3}{32085} + \frac{2x^4}{1251315} - \frac{4x^5}{294059025} + \frac{4x^6}{48519739125} \right. \\
&\quad \left. - \frac{8x^7}{21397204954125} + O(x^8)\right) \\
&\quad + c_2 \left(1 - 2x + \frac{2x^2}{9} - \frac{4x^3}{459} + \frac{2x^4}{11475} - \frac{4x^5}{1893375} + \frac{4x^6}{232885125} - \frac{8x^7}{79879597875} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{7}{8}} \left(1-\frac{2x}{15}+\frac{2x^2}{345}-\frac{4x^3}{32085}+\frac{2x^4}{1251315}-\frac{4x^5}{294059025}+\frac{4x^6}{48519739125} \right. \\
\left. -\frac{8x^7}{21397204954125}+O\left(x^8\right)\right) + c_2\left(1-2x+\frac{2x^2}{9}-\frac{4x^3}{459}+\frac{2x^4}{11475}-\frac{4x^5}{1893375} \right. \\
\left. +\frac{4x^6}{232885125}-\frac{8x^7}{79879597875}+O\left(x^8\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^{\frac{7}{8}} \left(1-\frac{2x}{15}+\frac{2x^2}{345}-\frac{4x^3}{32085}+\frac{2x^4}{1251315}-\frac{4x^5}{294059025}+\frac{4x^6}{48519739125} \right. \\
\left. -\frac{8x^7}{21397204954125}+O\left(x^8\right)\right) \\
+ c_2\left(1-2x+\frac{2x^2}{9}-\frac{4x^3}{459}+\frac{2x^4}{11475}-\frac{4x^5}{1893375}+\frac{4x^6}{232885125}-\frac{8x^7}{79879597875}+O\left(x^8\right)\right)
\end{aligned}
$$

Verified OK.

2.17.1 Maple step by step solution

Let's solve

$4y''x + \frac{y'}{2} + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4x} - \frac{y'}{8x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{8x} + \frac{y}{4x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{1}{8x}, P_3(x) = \frac{1}{4x}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{8}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- ● Multiply by denominators

  $$8y''x + 2y + y' = 0$$

- ● Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - ○ Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$$

  - ○ Shift index using $k- >k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\, x^{k+r}$$

  - ○ Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$

  - ○ Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\,(k + r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-7 + 8r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k + 1 + r)\,(8k + 1 + 8r) + 2a_k\right) x^{k+r}\right) = 0$$

- ● $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-7 + 8r) = 0$$

- ● Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{7}{8}\right\}$$

- ● Each term in the series must be 0, giving the recursion relation

$$8(k+1+r)\left(k+\frac{1}{8}+r\right)a_{k+1}+2a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{2a_k}{(k+1+r)(8k+1+8r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=-\frac{2a_k}{(k+1)(8k+1)}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+1}=-\frac{2a_k}{(k+1)(8k+1)}\right]$$

- Recursion relation for $r=\frac{7}{8}$

$$a_{k+1}=-\frac{2a_k}{\left(k+\frac{15}{8}\right)(8k+8)}$$

- Solution for $r=\frac{7}{8}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{7}{8}}, a_{k+1}=-\frac{2a_k}{\left(k+\frac{15}{8}\right)(8k+8)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{7}{8}}\right), a_{k+1}=-\frac{2a_k}{(k+1)(8k+1)}, b_{k+1}=-\frac{2b_k}{\left(k+\frac{15}{8}\right)(8k+8)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 52

```
Order:=8;
dsolve(4*x*diff(y(x),x$2)+1/2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{7}{8}} \left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32085}x^3 + \frac{2}{1251315}x^4 - \frac{4}{294059025}x^5 + \frac{4}{48519739125}x^6 - \frac{8}{21397204954125}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2 \left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \frac{2}{11475}x^4 - \frac{4}{1893375}x^5 + \frac{4}{232885125}x^6 - \frac{8}{79879597875}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 111

```
AsymptoticDSolveValue[4*x*y''[x]+1/2*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2 \left(-\frac{8x^7}{79879597875} + \frac{4x^6}{232885125} - \frac{4x^5}{1893375} + \frac{2x^4}{11475} - \frac{4x^3}{459} + \frac{2x^2}{9} - 2x + 1\right) + c_1 x^{7/8} \left(-\frac{8x^7}{21397204954125} + \frac{4x^6}{48519739125} - \frac{4x^5}{294059025} + \frac{2x^4}{1251315} - \frac{4x^3}{32085} + \frac{2x^2}{345} - \frac{2x}{15} + 1\right)$$

2.18 problem 18

Internal problem ID [6601]
Internal file name [`OUTPUT/5849_Sunday_June_05_2022_03_57_44_PM_43943201/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - xy' + \left(x^2+1\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = \frac{x^2+1}{2x^2}$$

Table 53: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2+1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(x^2 + 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) - x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(x^2+1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{gathered}
\left(\sum_{n=0}^{\infty} 2 x^{n+r} a_{n}(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r} a_{n}(n+r)\right) \\
+\left(\sum_{n=0}^{\infty} x^{n+r+2} a_{n}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n+r}\right)=0
\end{gathered} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_{n}=\sum_{n=2}^{\infty} a_{n-2} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{gathered}
\left(\sum_{n=0}^{\infty} 2 x^{n+r} a_{n}(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r} a_{n}(n+r)\right) \\
+\left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n+r}\right)=0
\end{gathered} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2 x^{n+r} a_{n}(n+r)(n+r-1)-x^{n+r} a_{n}(n+r)+a_{n} x^{n+r}=0
$$

When $n=0$ the above becomes

$$
2 x^{r} a_{0} r(-1+r)-x^{r} a_{0} r+a_{0} x^{r}=0
$$

Or

$$
\left(2 x^{r} r(-1+r)-x^{r} r+x^{r}\right) a_{0}=0
$$

Since $a_{0} \neq 0$ then the above simplifies to

$$
\left(2 r^{2}-3 r+1\right) x^{r}=0
$$

Since the above is true for all x then the indicial equation becomes

$$
2 r^{2}-3 r+1=0
$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1 &= 1\\ r_2 &= \frac{1}{2}\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1}\\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}\end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) - a_n(n + r) + a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{1}{8r^6 + 156r^5 + 1202r^4 + 4641r^3 + 9365r^2 + 9243r + 3465}$$

Which for the root $r = 1$ becomes

$$a_6 = -\frac{1}{28080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0
a_6	$-\frac{1}{8r^6+156r^5+1202r^4+4641r^3+9365r^2+9243r+3465}$	$-\frac{1}{28080}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0
a_6	$-\frac{1}{8r^6+156r^5+1202r^4+4641r^3+9365r^2+9243r+3465}$	$-\frac{1}{28080}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{28080} + O\left(x^8\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) + b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = -\frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = -\frac{1}{8r^6 + 156r^5 + 1202r^4 + 4641r^3 + 9365r^2 + 9243r + 3465}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_6 = -\frac{1}{11088}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0
b_6	$-\frac{1}{8r^6+156r^5+1202r^4+4641r^3+9365r^2+9243r+3465}$	$-\frac{1}{11088}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0
b_6	$-\frac{1}{8r^6+156r^5+1202r^4+4641r^3+9365r^2+9243r+3465}$	$-\frac{1}{11088}$
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} - \frac{x^6}{11088} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{28080} + O(x^8)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} - \frac{x^6}{11088} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{28080} + O(x^8)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} - \frac{x^6}{11088} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{28080} + O(x^8)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} - \frac{x^6}{11088} + O(x^8)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{28080} + O(x^8)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} - \frac{x^6}{11088} + O(x^8)\right)
$$

Verified OK.

2.18.1 Maple step by step solution

Let's solve

$2x^2 y'' - x y' + (x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+1)y}{2x^2} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{y'}{2x}+\frac{(x^2+1)y}{2x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{1}{2x},P_3(x)=\frac{x^2+1}{2x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=-\frac{1}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=\frac{1}{2}$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $2x^2y''-xy'+(x^2+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$

  - Convert $x^2\cdot y''$ to series expansion

    $x^2\cdot y''=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+2r)(-1+r)x^r + a_1(1+2r)\,r\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)(k+r-1) + a_{k-2}\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{1, \frac{1}{2}\right\}$$

- Each term must be 0

  $$a_1(1+2r)\,r = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$2(k+r-1)\left(k-\frac{1}{2}+r\right)a_k + a_{k-2} = 0$$

- Shift index using $k->k+2$

  $$2(k+1+r)\left(k+\frac{3}{2}+r\right)a_{k+2} + a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k}{(k+1+r)(2k+3+2r)}$$

- Recursion relation for $r = 1$

  $$a_{k+2} = -\frac{a_k}{(k+2)(2k+5)}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(2k+5)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+4)}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+4)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k}{(k+2)(2k+5)}, a_1 = 0, b_{k+2} = -\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+4)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 37

```
Order:=8;
dsolve(2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+(x^2+1)*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = c_1\sqrt{x}\left(1-\frac{1}{6}x^2+\frac{1}{168}x^4-\frac{1}{11088}x^6+\mathrm{O}\left(x^8\right)\right) \\
+ c_2x\left(1-\frac{1}{10}x^2+\frac{1}{360}x^4-\frac{1}{28080}x^6+\mathrm{O}\left(x^8\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 62

```
AsymptoticDSolveValue[2*x^2*y''[x]-x*y'[x]+(x^2+1)*y[x]==0,y[x],{x,0,7}]
```

$$
y(x) \to c_1x\left(-\frac{x^6}{28080}+\frac{x^4}{360}-\frac{x^2}{10}+1\right)+c_2\sqrt{x}\left(-\frac{x^6}{11088}+\frac{x^4}{168}-\frac{x^2}{6}+1\right)
$$

2.19 problem 19

Internal problem ID [6602]
Internal file name [`OUTPUT/5850_Sunday_June_05_2022_03_57_48_PM_45936299/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{3xy'' + (2-x)\,y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3xy'' + (2-x)\,y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{-2+x}{3x} \\ q(x) &= -\frac{1}{3x} \end{aligned}$$

Table 55: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{-2+x}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{3x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3xy'' + (2 - x)\, y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &3\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x \\ &+ (2-x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 3x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$3x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(3x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+3r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{3} \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1+3r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$3a_n(n+r)\,(n+r-1) - a_{n-1}(n+r-1) + 2a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{3n-1+3r} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{a_{n-1}}{3n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2 + 3r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{9r^2 + 21r + 10}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{1}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$
a_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{18}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = \frac{1}{162}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$
a_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{18}$
a_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{162}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{1}{1944}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$
a_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{18}$
a_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{162}$
a_4	$\frac{1}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1}{1944}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = \frac{1}{29160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$
a_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{18}$
a_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{162}$
a_4	$\frac{1}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1}{1944}$
a_5	$\frac{1}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$\frac{1}{29160}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{729r^6 + 13851r^5 + 103275r^4 + 382185r^3 + 727596r^2 + 657324r + 209440}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_6 = \frac{1}{524880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$
a_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{18}$
a_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{162}$
a_4	$\frac{1}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1}{1944}$
a_5	$\frac{1}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$\frac{1}{29160}$
a_6	$\frac{1}{729r^6+13851r^5+103275r^4+382185r^3+727596r^2+657324r+209440}$	$\frac{1}{524880}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{2187r^7 + 56133r^6 + 586845r^5 + 3212055r^4 + 9826488r^3 + 16523892r^2 + 13774800r + 4188800}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_7 = \frac{1}{11022480}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$
a_2	$\frac{1}{9r^{2}+21r+10}$	$\frac{1}{18}$
a_3	$\frac{1}{27r^{3}+135r^{2}+198r+80}$	$\frac{1}{162}$
a_4	$\frac{1}{81r^{4}+702r^{3}+2079r^{2}+2418r+880}$	$\frac{1}{1944}$
a_5	$\frac{1}{243r^{5}+3240r^{4}+16065r^{3}+36360r^{2}+36492r+12320}$	$\frac{1}{29160}$
a_6	$\frac{1}{729r^{6}+13851r^{5}+103275r^{4}+382185r^{3}+727596r^{2}+657324r+209440}$	$\frac{1}{524880}$
a_7	$\frac{1}{2187r^{7}+56133r^{6}+586845r^{5}+3212055r^{4}+9826488r^{3}+16523892r^{2}+13774800r+4188800}$	$\frac{1}{11022480}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 + \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{162} + \frac{x^4}{1944} + \frac{x^5}{29160} + \frac{x^6}{524880} + \frac{x^7}{11022480} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$3b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + 2(n+r) b_n - b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{3n - 1 + 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}}{3n - 1} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{2 + 3r}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{9r^2 + 21r + 10}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{10}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$
b_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{80}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$
b_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{10}$
b_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{80}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{880}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$
b_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{10}$
b_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{80}$
b_4	$\frac{1}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1}{880}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{12320}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$
b_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{10}$
b_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{80}$
b_4	$\frac{1}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1}{880}$
b_5	$\frac{1}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$\frac{1}{12320}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{729r^6 + 13851r^5 + 103275r^4 + 382185r^3 + 727596r^2 + 657324r + 209440}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{1}{209440}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$
b_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{10}$
b_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{80}$
b_4	$\frac{1}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1}{880}$
b_5	$\frac{1}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$\frac{1}{12320}$
b_6	$\frac{1}{729r^6+13851r^5+103275r^4+382185r^3+727596r^2+657324r+209440}$	$\frac{1}{209440}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{1}{2187r^7 + 56133r^6 + 586845r^5 + 3212055r^4 + 9826488r^3 + 16523892r^2 + 13774800r + 4188800}$$

Which for the root $r = 0$ becomes

$$b_7 = \frac{1}{4188800}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$
b_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{10}$
b_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{80}$
b_4	$\frac{1}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1}{880}$
b_5	$\frac{1}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$\frac{1}{12320}$
b_6	$\frac{1}{729r^6+13851r^5+103275r^4+382185r^3+727596r^2+657324r+209440}$	$\frac{1}{209440}$
b_7	$\frac{1}{2187r^7+56133r^6+586845r^5+3212055r^4+9826488r^3+16523892r^2+13774800r+4188800}$	$\frac{1}{4188800}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \dots \\ &= 1 + \frac{x}{2} + \frac{x^2}{10} + \frac{x^3}{80} + \frac{x^4}{880} + \frac{x^5}{12320} + \frac{x^6}{209440} + \frac{x^7}{4188800} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x^{\frac{1}{3}}\left(1 + \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{162} + \frac{x^4}{1944} + \frac{x^5}{29160} + \frac{x^6}{524880} + \frac{x^7}{11022480} + O(x^8)\right) \\ &\quad + c_2\left(1 + \frac{x}{2} + \frac{x^2}{10} + \frac{x^3}{80} + \frac{x^4}{880} + \frac{x^5}{12320} + \frac{x^6}{209440} + \frac{x^7}{4188800} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x^{\frac{1}{3}}\left(1 + \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{162} + \frac{x^4}{1944} + \frac{x^5}{29160} + \frac{x^6}{524880} + \frac{x^7}{11022480} + O(x^8)\right) \\ &\quad + c_2\left(1 + \frac{x}{2} + \frac{x^2}{10} + \frac{x^3}{80} + \frac{x^4}{880} + \frac{x^5}{12320} + \frac{x^6}{209440} + \frac{x^7}{4188800} + O(x^8)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1x^{\frac{1}{3}}&\left(1 + \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{162} + \frac{x^4}{1944} + \frac{x^5}{29160} + \frac{x^6}{524880} + \frac{x^7}{11022480} + O(x^8)\right) \\ &+ c_2\left(1 + \frac{x}{2} + \frac{x^2}{10} + \frac{x^3}{80} + \frac{x^4}{880} + \frac{x^5}{12320} + \frac{x^6}{209440} + \frac{x^7}{4188800} + O(x^8)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = c_1x^{\frac{1}{3}}&\left(1 + \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{162} + \frac{x^4}{1944} + \frac{x^5}{29160} + \frac{x^6}{524880} + \frac{x^7}{11022480} + O(x^8)\right) \\ &+ c_2\left(1 + \frac{x}{2} + \frac{x^2}{10} + \frac{x^3}{80} + \frac{x^4}{880} + \frac{x^5}{12320} + \frac{x^6}{209440} + \frac{x^7}{4188800} + O(x^8)\right) \end{aligned}$$

Verified OK.

2.19.1 Maple step by step solution

Let's solve

$3y''x+(2-x)\,y'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{3x}+\frac{(-2+x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(-2+x)y'}{3x}-\frac{y}{3x}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x)=-\frac{-2+x}{3x},P_3(x)=-\frac{1}{3x}\right]$

  ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{2}{3}$

  ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3y''x+(2-x)\,y'-y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ○ Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(3k+2+3r) - a_k(k+1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3(k+1+r)\left(\left(k+r+\frac{2}{3}\right)a_{k+1} - \frac{a_k}{3}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{3k+2+3r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{3k+2}$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{3k+2}\right]$$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = \frac{a_k}{3k+3}$

- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{a_k}{3k+3}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = \frac{a_k}{3k+2}, b_{k+1} = \frac{b_k}{3k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form is not straightforward to achieve - returning hypergeometric solution
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 52

```
Order:=8;
dsolve(3*x*diff(y(x),x$2)+(2-x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4 + \frac{1}{29160}x^5 + \frac{1}{524880}x^6 + \frac{1}{11022480}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2 \left(1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 + \frac{1}{880}x^4 + \frac{1}{12320}x^5 + \frac{1}{209440}x^6 + \frac{1}{4188800}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 113

```
AsymptoticDSolveValue[3*x*y''[x]+(2-x)*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \sqrt[3]{x} \left(\frac{x^7}{11022480} + \frac{x^6}{524880} + \frac{x^5}{29160} + \frac{x^4}{1944} + \frac{x^3}{162} + \frac{x^2}{18} + \frac{x}{3} + 1\right) + c_2 \left(\frac{x^7}{4188800} + \frac{x^6}{209440} + \frac{x^5}{12320} + \frac{x^4}{880} + \frac{x^3}{80} + \frac{x^2}{10} + \frac{x}{2} + 1\right)$$

2.20 problem 20

Internal problem ID [6603]
Internal file name [`OUTPUT/5851_Sunday_June_05_2022_03_57_53_PM_21949440/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(x - \frac{2}{9}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x + \frac{2}{9}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{9x - 2}{9x^2} \end{aligned}$$

Table 57: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{9x-2}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + \left(-x + \frac{2}{9}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + \left(-x + \frac{2}{9}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{2 a_n x^{n+r}}{9}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} \frac{2a_n x^{n+r}}{9}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + \frac{2a_n x^{n+r}}{9} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + \frac{2a_0 x^r}{9} = 0$$

Or

$$\left(x^r r(-1+r) + \frac{2x^r}{9}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(9r^2 - 9r + 2)\, x^r}{9} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r + \frac{2}{9} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{2}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(9r^2 - 9r + 2)\, x^r}{9} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1} + \frac{2a_n}{9} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{9a_{n-1}}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_n = \frac{3a_{n-1}}{3n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{9}{9r^2 + 9r + 2}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_1 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{81}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_2 = \frac{9}{56}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$
a_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{56}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{729}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_3 = \frac{9}{560}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$
a_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{56}$
a_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{560}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{6561}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)(9r^2 + 63r + 110)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_4 = \frac{27}{29120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$
a_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{56}$
a_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{560}$
a_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{29120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{59049}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)(9r^2 + 63r + 110)(9r^2 + 81r + 182)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_5 = \frac{81}{2329600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$
a_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{56}$
a_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{560}$
a_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{29120}$
a_5	$\frac{59049}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$\frac{81}{2329600}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{531441}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)(9r^2 + 63r + 110)(9r^2 + 81r + 182)(9r^2 + 99r + 2}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_6 = \frac{81}{88524800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$
a_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{56}$
a_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{560}$
a_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{29120}$
a_5	$\frac{59049}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$\frac{81}{2329600}$
a_6	$\frac{531441}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)}$	$\frac{81}{88524800}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{4782969}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)(9r^2 + 63r + 110)(9r^2 + 81r + 182)(9r^2 + 99r + 2}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_7 = \frac{243}{13632819200}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$
a_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{56}$
a_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{560}$
a_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{29120}$
a_5	$\frac{59049}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$\frac{81}{2329600}$
a_6	$\frac{531441}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)}$	$\frac{81}{88524800}$
a_7	$\frac{4782969}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)(9r^2+117r+380)}$	$\frac{243}{13632819200}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\frac{2}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= x^{\frac{2}{3}}\left(1 + \frac{3x}{4} + \frac{9x^2}{56} + \frac{9x^3}{560} + \frac{27x^4}{29120} + \frac{81x^5}{2329600} + \frac{81x^6}{88524800} + \frac{243x^7}{13632819200} + O\left(x^8\right)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_{n-1} + \frac{2b_n}{9} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{9b_{n-1}}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_n = \frac{3b_{n-1}}{n(3n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{9}{9r^2 + 9r + 2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_1 = \frac{3}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{81}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = \frac{9}{20}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$
b_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{20}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{729}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_3 = \frac{9}{160}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$
b_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{20}$
b_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{160}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = \frac{27}{7040}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$
b_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{20}$
b_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{160}$
b_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{7040}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{59049}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_5 = \frac{81}{492800}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$
b_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{20}$
b_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{160}$
b_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{7040}$
b_5	$\frac{59049}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$\frac{81}{492800}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{531441}{(9r^2+9r+2)\,(9r^2+27r+20)\,(9r^2+45r+56)\,(9r^2+63r+110)\,(9r^2+81r+182)\,(9r^2+99r+2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_6 = \frac{81}{16755200}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$
b_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{20}$
b_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{160}$
b_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{7040}$
b_5	$\frac{59049}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$\frac{81}{492800}$
b_6	$\frac{531441}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)}$	$\frac{81}{16755200}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{4782969}{(9r^2+9r+2)\,(9r^2+27r+20)\,(9r^2+45r+56)\,(9r^2+63r+110)\,(9r^2+81r+182)\,(9r^2+99r+2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_7 = \frac{243}{2345728000}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$
b_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{20}$
b_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{160}$
b_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{7040}$
b_5	$\frac{59049}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$\frac{81}{492800}$
b_6	$\frac{531441}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)}$	$\frac{81}{16755200}$
b_7	$\frac{4782969}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)(9r^2+117r+380)}$	$\frac{243}{2345728000}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{2}{3}}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 + \frac{3x}{2} + \frac{9x^2}{20} + \frac{9x^3}{160} + \frac{27x^4}{7040} + \frac{81x^5}{492800} + \frac{81x^6}{16755200} + \frac{243x^7}{2345728000} + O\left(x^8\right)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x^{\frac{2}{3}}\left(1+\frac{3x}{4}+\frac{9x^2}{56}+\frac{9x^3}{560}+\frac{27x^4}{29120}+\frac{81x^5}{2329600}+\frac{81x^6}{88524800}+\frac{243x^7}{13632819200}+O\left(x^8\right)\right) \\
&\quad + c_2x^{\frac{1}{3}}\left(1+\frac{3x}{2}+\frac{9x^2}{20}+\frac{9x^3}{160}+\frac{27x^4}{7040}+\frac{81x^5}{492800}+\frac{81x^6}{16755200}+\frac{243x^7}{2345728000}+O\left(x^8\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1x^{\frac{2}{3}}\left(1+\frac{3x}{4}+\frac{9x^2}{56}+\frac{9x^3}{560}+\frac{27x^4}{29120}+\frac{81x^5}{2329600}+\frac{81x^6}{88524800}+\frac{243x^7}{13632819200}+O\left(x^8\right)\right) \\
&\quad + c_2x^{\frac{1}{3}}\left(1+\frac{3x}{2}+\frac{9x^2}{20}+\frac{9x^3}{160}+\frac{27x^4}{7040}+\frac{81x^5}{492800}+\frac{81x^6}{16755200}+\frac{243x^7}{2345728000}+O\left(x^8\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{2}{3}} \left(1+\frac{3 x}{4}+\frac{9 x^{2}}{56}+\frac{9 x^{3}}{560}+\frac{27 x^{4}}{29120}+\frac{81 x^{5}}{2329600}+\frac{81 x^{6}}{88524800}+\frac{243 x^{7}}{13632819200}\right. \\
\left. + O\left(x^{8}\right)\right) \\
+ c_2 x^{\frac{1}{3}} \left(1+\frac{3 x}{2}+\frac{9 x^{2}}{20}+\frac{9 x^{3}}{160}+\frac{27 x^{4}}{7040}+\frac{81 x^{5}}{492800}+\frac{81 x^{6}}{16755200}+\frac{243 x^{7}}{2345728000}+O\left(x^{8}\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^{\frac{2}{3}} \left(1+\frac{3 x}{4}+\frac{9 x^{2}}{56}+\frac{9 x^{3}}{560}+\frac{27 x^{4}}{29120}+\frac{81 x^{5}}{2329600}+\frac{81 x^{6}}{88524800}+\frac{243 x^{7}}{13632819200}+O\left(x^{8}\right)\right) \\
+ c_2 x^{\frac{1}{3}} \left(1+\frac{3 x}{2}+\frac{9 x^{2}}{20}+\frac{9 x^{3}}{160}+\frac{27 x^{4}}{7040}+\frac{81 x^{5}}{492800}+\frac{81 x^{6}}{16755200}+\frac{243 x^{7}}{2345728000}+O\left(x^{8}\right)\right)
\end{aligned}
$$

Verified OK.

2.20.1 Maple step by step solution

Let's solve

$x^{2} y''+\left(-x+\frac{2}{9}\right) y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(9 x-2) y}{9 x^{2}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(9 x-2) y}{9 x^{2}}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x)=0, P_3(x)=-\frac{9 x-2}{9 x^{2}}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left.\left(x \cdot P_2(x)\right)\right|_{x=0}=0$

  ○ $x^{2} \cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{2}{9}$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2 y'' + (-9x + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(-1 + 3r)\,(-2 + 3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (3k + 3r - 1)\,(3k + 3r - 2) - 9a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 3r)\,(-2 + 3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $9\left(k + r - \frac{1}{3}\right)\left(k + r - \frac{2}{3}\right) a_k - 9a_{k-1} = 0$

- Shift index using $k- >k + 1$

  $9\left(k + \frac{2}{3} + r\right)\left(k + \frac{1}{3} + r\right) a_{k+1} - 9a_k = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{9a_k}{(3k+2+3r)(3k+1+3r)}$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = \frac{9a_k}{(3k+3)(3k+2)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{9a_k}{(3k+3)(3k+2)}\right]$$

- Recursion relation for $r = \frac{2}{3}$

$$a_{k+1} = \frac{9a_k}{(3k+4)(3k+3)}$$

- Solution for $r = \frac{2}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+1} = \frac{9a_k}{(3k+4)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{k+1} = \frac{9a_k}{(3k+3)(3k+2)}, b_{k+1} = \frac{9b_k}{(3k+4)(3k+3)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 55

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)-(x-2/9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \frac{27}{7040}x^4 + \frac{81}{492800}x^5 + \frac{81}{16755200}x^6 + \frac{243}{2345728000}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2 x^{\frac{2}{3}} \left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \frac{27}{29120}x^4 + \frac{81}{2329600}x^5 + \frac{81}{88524800}x^6 + \frac{243}{13632819200}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 118

```
AsymptoticDSolveValue[x^2*y''[x]-(x-2/9)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2 \sqrt[3]{x}\left(\frac{243x^7}{2345728000} + \frac{81x^6}{16755200} + \frac{81x^5}{492800} + \frac{27x^4}{7040} + \frac{9x^3}{160} + \frac{9x^2}{20} + \frac{3x}{2} + 1\right) + c_1 x^{2/3}\left(\frac{243x^7}{13632819200} + \frac{81x^6}{88524800} + \frac{81x^5}{2329600} + \frac{27x^4}{29120} + \frac{9x^3}{560} + \frac{9x^2}{56} + \frac{3x}{4} + 1\right)$$

2.21 problem 21

Internal problem ID [6604]
Internal file name [`OUTPUT/5852_Sunday_June_05_2022_03_57_57_PM_53287355/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{2xy'' - (2x+3)\,y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + (-2x-3)\,y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x+3}{2x}$$
$$q(x) = \frac{1}{2x}$$

Table 59: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x+3}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (-2x - 3)\, y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned}&2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x \\ &+ (-2x-3)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0\end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) - 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) - 3r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) - 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-5+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 5r = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1 &= \frac{5}{2}\\ r_2 &= 0\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-5+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}}\\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^n\end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) - 3a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(2n+2r-3)}{2n^2+4nr+2r^2-5n-5r} \tag{4}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_n = \frac{2a_{n-1}(n+1)}{n\,(2n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1 + 2r}{2r^2 - r - 3}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_1 = \frac{4}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1 + 2r}{2r^3 + 3r^2 - 5r - 6}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_2 = \frac{4}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$
a_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$\frac{4}{21}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{3 + 2r}{2r^4 + 9r^3 + 4r^2 - 21r - 18}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_3 = \frac{32}{693}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$
a_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$\frac{4}{21}$
a_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$\frac{32}{693}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_4 = \frac{80}{9009}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$
a_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$\frac{4}{21}$
a_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$\frac{32}{693}$
a_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$\frac{80}{9009}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{7+2r}{(r+5)\left(2r^5+17r^4+40r^3-5r^2-102r-72\right)}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_5 = \frac{64}{45045}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$
a_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$\frac{4}{21}$
a_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$\frac{32}{693}$
a_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$\frac{80}{9009}$
a_5	$\frac{7+2r}{(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$\frac{64}{45045}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{9+2r}{(r+6)(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_6 = \frac{64}{328185}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$
a_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$\frac{4}{21}$
a_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$\frac{32}{693}$
a_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$\frac{80}{9009}$
a_5	$\frac{7+2r}{(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$\frac{64}{45045}$
a_6	$\frac{9+2r}{(r+6)(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$\frac{64}{328185}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{11+2r}{(r+7)(r+6)(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_7 = \frac{1024}{43648605}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$
a_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$\frac{4}{21}$
a_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$\frac{32}{693}$
a_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$\frac{80}{9009}$
a_5	$\frac{7+2r}{(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$\frac{64}{45045}$
a_6	$\frac{9+2r}{(r+6)(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$\frac{64}{328185}$
a_7	$\frac{11+2r}{(r+7)(r+6)(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$\frac{1024}{43648605}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{5}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^{\frac{5}{2}}\left(1 + \frac{4x}{7} + \frac{4x^2}{21} + \frac{32x^3}{693} + \frac{80x^4}{9009} + \frac{64x^5}{45045} + \frac{64x^6}{328185} + \frac{1024x^7}{43648605} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - 2b_{n-1}(n+r-1) - 3(n+r)b_n + b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}(2n+2r-3)}{2n^2+4nr+2r^2-5n-5r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}(2n-3)}{n(2n-5)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-1 + 2r}{2r^2 - r - 3}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1 + 2r}{2r^3 + 3r^2 - 5r - 6}$$

Which for the root $r = 0$ becomes

$$b_2 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$
b_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{3 + 2r}{2r^4 + 9r^3 + 4r^2 - 21r - 18}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$
b_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$-\frac{1}{6}$
b_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{5}{72}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$
b_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$-\frac{1}{6}$
b_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$-\frac{1}{6}$
b_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$-\frac{5}{72}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{7+2r}{(r+5)\left(2r^5+17r^4+40r^3-5r^2-102r-72\right)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{7}{360}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$
b_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$-\frac{1}{6}$
b_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$-\frac{1}{6}$
b_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$-\frac{5}{72}$
b_5	$\frac{7+2r}{(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$-\frac{7}{360}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{9+2r}{(r+6)(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$$

Which for the root $r = 0$ becomes

$$b_6 = -\frac{1}{240}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$
b_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$-\frac{1}{6}$
b_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$-\frac{1}{6}$
b_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$-\frac{5}{72}$
b_5	$\frac{7+2r}{(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$-\frac{7}{360}$
b_6	$\frac{9+2r}{(r+6)(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$-\frac{1}{240}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{11+2r}{(r+7)(r+6)(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$$

Which for the root $r = 0$ becomes

$$b_7 = -\frac{11}{15120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$
b_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$-\frac{1}{6}$
b_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$-\frac{1}{6}$
b_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$-\frac{5}{72}$
b_5	$\frac{7+2r}{(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$-\frac{7}{360}$
b_6	$\frac{9+2r}{(r+6)(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$-\frac{1}{240}$
b_7	$\frac{11+2r}{(r+7)(r+6)(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$-\frac{11}{15120}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 + \frac{x}{3} - \frac{x^2}{6} - \frac{x^3}{6} - \frac{5x^4}{72} - \frac{7x^5}{360} - \frac{x^6}{240} - \frac{11x^7}{15120} + O(x^8)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{5}{2}} \left(1 + \frac{4x}{7} + \frac{4x^2}{21} + \frac{32x^3}{693} + \frac{80x^4}{9009} + \frac{64x^5}{45045} + \frac{64x^6}{328185} + \frac{1024x^7}{43648605} + O(x^8)\right) \\
&+ c_2 \left(1 + \frac{x}{3} - \frac{x^2}{6} - \frac{x^3}{6} - \frac{5x^4}{72} - \frac{7x^5}{360} - \frac{x^6}{240} - \frac{11x^7}{15120} + O(x^8)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{5}{2}} \left(1 + \frac{4x}{7} + \frac{4x^2}{21} + \frac{32x^3}{693} + \frac{80x^4}{9009} + \frac{64x^5}{45045} + \frac{64x^6}{328185} + \frac{1024x^7}{43648605} + O(x^8)\right) \\
&+ c_2 \left(1 + \frac{x}{3} - \frac{x^2}{6} - \frac{x^3}{6} - \frac{5x^4}{72} - \frac{7x^5}{360} - \frac{x^6}{240} - \frac{11x^7}{15120} + O(x^8)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^{\frac{5}{2}} \left(1+\frac{4x}{7}+\frac{4x^2}{21}+\frac{32x^3}{693}+\frac{80x^4}{9009}+\frac{64x^5}{45045}+\frac{64x^6}{328185}+\frac{1024x^7}{43648605}+O(x^8)\right) \\
& + c_2 \left(1+\frac{x}{3}-\frac{x^2}{6}-\frac{x^3}{6}-\frac{5x^4}{72}-\frac{7x^5}{360}-\frac{x^6}{240}-\frac{11x^7}{15120}+O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^{\frac{5}{2}} \left(1+\frac{4x}{7}+\frac{4x^2}{21}+\frac{32x^3}{693}+\frac{80x^4}{9009}+\frac{64x^5}{45045}+\frac{64x^6}{328185}+\frac{1024x^7}{43648605}+O(x^8)\right) \\
& + c_2 \left(1+\frac{x}{3}-\frac{x^2}{6}-\frac{x^3}{6}-\frac{5x^4}{72}-\frac{7x^5}{360}-\frac{x^6}{240}-\frac{11x^7}{15120}+O(x^8)\right)
\end{aligned}
$$

Verified OK.

2.21.1 Maple step by step solution

Let's solve

$2y''x+(-2x-3)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{2x}+\frac{(2x+3)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(2x+3)y'}{2x}+\frac{y}{2x}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x)=-\frac{2x+3}{2x}, P_3(x)=\frac{1}{2x}\right]$

  ○ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-\frac{3}{2}$

  ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ○ $x=0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $2y''x + (-2x-3)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ○ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ○ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  ○ Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-5+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-3+2r) - a_k(2k+2r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-5+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{5}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k-\frac{3}{2}+r\right)a_{k+1} - 2a_k\left(k-\frac{1}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(2k+2r-1)}{(k+1+r)(2k-3+2r)}$

- Recursion relation for $r = 0$

$a_{k+1} = \frac{a_k(2k-1)}{(k+1)(2k-3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(2k-1)}{(k+1)(2k-3)}\right]$

- Recursion relation for $r = \frac{5}{2}$

  $a_{k+1} = \frac{a_k(2k+4)}{\left(k+\frac{7}{2}\right)(2k+2)}$

- Solution for $r = \frac{5}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+1} = \frac{a_k(2k+4)}{\left(k+\frac{7}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+1} = \frac{a_k(2k-1)}{(k+1)(2k-3)}, b_{k+1} = \frac{b_k(2k+4)}{\left(k+\frac{7}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 52

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)-(3+2*x)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{5}{2}} \left(1 + \frac{4}{7}x + \frac{4}{21}x^2 + \frac{32}{693}x^3 + \frac{80}{9009}x^4 + \frac{64}{45045}x^5 + \frac{64}{328185}x^6 + \frac{1024}{43648605}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 - \frac{5}{72}x^4 - \frac{7}{360}x^5 - \frac{1}{240}x^6 - \frac{11}{15120}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 113

```
AsymptoticDSolveValue[2*x*y''[x]-(3+2*x)*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$
\begin{aligned}
y(x) \to\; & c_2\left(-\frac{11x^7}{15120}-\frac{x^6}{240}-\frac{7x^5}{360}-\frac{5x^4}{72}-\frac{x^3}{6}-\frac{x^2}{6}+\frac{x}{3}+1\right) \\
& + c_1\left(\frac{1024x^7}{43648605}+\frac{64x^6}{328185}+\frac{64x^5}{45045}+\frac{80x^4}{9009}+\frac{32x^3}{693}+\frac{4x^2}{21}+\frac{4x}{7}+1\right)x^{5/2}
\end{aligned}
$$

2.22 problem 22

Internal problem ID [6605]
Internal file name [`OUTPUT/5853_Sunday_June_05_2022_03_58_03_PM_17453768/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{9x^2 - 4}{9x^2}$$

Table 61: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{9x^2-4}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(x^2 - \frac{4}{9}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-\frac{4 a_n x^{n+r}}{9}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n=\sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-\frac{4 a_n x^{n+r}}{9}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1)+x^{n+r} a_n (n+r)-\frac{4 a_n x^{n+r}}{9}=0$$

When $n=0$ the above becomes

$$x^{r} a_0 r(-1+r)+x^{r} a_0 r-\frac{4 a_0 x^{r}}{9}=0$$

Or

$$\left(x^{r} r(-1+r)+x^{r} r-\frac{4 x^{r}}{9}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{\left(9 r^2-4\right) x^{r}}{9}=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{4}{9} = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1 &= \frac{2}{3}\\ r_2 &= -\frac{2}{3}\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(9r^2 - 4)\, x^r}{9} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}}\\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\frac{2}{3}}\end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)\,(n + r - 1) + a_n(n + r) + a_{n-2} - \frac{4a_n}{9} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{9a_{n-2}}{9n^2 + 18nr + 9r^2 - 4} \tag{4}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_n = -\frac{3a_{n-2}}{n(3n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{9}{9r^2 + 36r + 32}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_2 = -\frac{3}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{20}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_4 = \frac{9}{1280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{20}$
a_3	0	0
a_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{1280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{20}$
a_3	0	0
a_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{1280}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{729}{(9r^2+36r+32)(9r^2+72r+140)(9r^2+108r+320)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_6 = -\frac{9}{56320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{20}$
a_3	0	0
a_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{1280}$
a_5	0	0
a_6	$-\frac{729}{(9r^2+36r+32)(9r^2+72r+140)(9r^2+108r+320)}$	$-\frac{9}{56320}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{20}$
a_3	0	0
a_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{1280}$
a_5	0	0
a_6	$-\frac{729}{(9r^2+36r+32)(9r^2+72r+140)(9r^2+108r+320)}$	$-\frac{9}{56320}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{2}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^{\frac{2}{3}}\left(1 - \frac{3x^2}{20} + \frac{9x^4}{1280} - \frac{9x^6}{56320} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - \frac{4b_n}{9} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{9b_{n-2}}{9n^2 + 18nr + 9r^2 - 4} \tag{4}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_n = -\frac{3b_{n-2}}{n(3n-4)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{9}{9r^2 + 36r + 32}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_2 = -\frac{3}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{4}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(9r^2 + 36r + 32)(9r^2 + 72r + 140)}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_4 = \frac{9}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{4}$
b_3	0	0
b_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{128}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{4}$
b_3	0	0
b_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{128}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = -\frac{729}{(9r^2+36r+32)(9r^2+72r+140)(9r^2+108r+320)}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_6 = -\frac{9}{3584}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{4}$
b_3	0	0
b_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{128}$
b_5	0	0
b_6	$-\frac{729}{(9r^2+36r+32)(9r^2+72r+140)(9r^2+108r+320)}$	$-\frac{9}{3584}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{4}$
b_3	0	0
b_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{128}$
b_5	0	0
b_6	$-\frac{729}{(9r^2+36r+32)(9r^2+72r+140)(9r^2+108r+320)}$	$-\frac{9}{3584}$
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{2}{3}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6+b_7x^7+b_8x^8\dots\right) \\ &= \frac{1-\frac{3x^2}{4}+\frac{9x^4}{128}-\frac{9x^6}{3584}+O(x^8)}{x^{\frac{2}{3}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x)+c_2y_2(x) \\ &= c_1x^{\frac{2}{3}}\left(1-\frac{3x^2}{20}+\frac{9x^4}{1280}-\frac{9x^6}{56320}+O(x^8)\right)+\frac{c_2\left(1-\frac{3x^2}{4}+\frac{9x^4}{128}-\frac{9x^6}{3584}+O(x^8)\right)}{x^{\frac{2}{3}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x^{\frac{2}{3}}\left(1-\frac{3x^2}{20}+\frac{9x^4}{1280}-\frac{9x^6}{56320}+O(x^8)\right)+\frac{c_2\left(1-\frac{3x^2}{4}+\frac{9x^4}{128}-\frac{9x^6}{3584}+O(x^8)\right)}{x^{\frac{2}{3}}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x^{\frac{2}{3}}\left(1-\frac{3x^2}{20}+\frac{9x^4}{1280}-\frac{9x^6}{56320}+O(x^8)\right)+\frac{c_2\left(1-\frac{3x^2}{4}+\frac{9x^4}{128}-\frac{9x^6}{3584}+O(x^8)\right)}{x^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{2}{3}} \left(1-\frac{3x^2}{20}+\frac{9x^4}{1280}-\frac{9x^6}{56320}+O\left(x^8\right)\right)+\frac{c_2 \left(1-\frac{3x^2}{4}+\frac{9x^4}{128}-\frac{9x^6}{3584}+O(x^8)\right)}{x^{\frac{2}{3}}}$$

Verified OK.

2.22.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2-\frac{4}{9}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2-4) y}{9x^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}+\frac{(9x^2-4) y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{9x^2-4}{9x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{4}{9}$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2 y'' + 9x y' + (9x^2-4) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ○ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ○ Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+3r)\,(-2+3r)\,x^r + a_1(5+3r)\,(1+3r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r+2)\,(3k+3r-2) + 9a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2+3r)\,(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{2}{3}, \frac{2}{3}\right\}$$

- Each term must be 0

  $$a_1(5+3r)\,(1+3r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(3k+3r+2)\,(3k+3r-2) + 9a_{k-2} = 0$$

- Shift index using $k- >k+2$

  $$a_{k+2}(3k+8+3r)\,(3k+4+3r) + 9a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{9a_k}{(3k+8+3r)(3k+4+3r)}$$

- Recursion relation for $r=-\frac{2}{3}$

  $a_{k+2}=-\frac{9a_k}{(3k+6)(3k+2)}$
- Solution for $r=-\frac{2}{3}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{2}{3}}, a_{k+2}=-\frac{9a_k}{(3k+6)(3k+2)}, a_1=0\right]$
- Recursion relation for $r=\frac{2}{3}$

  $a_{k+2}=-\frac{9a_k}{(3k+10)(3k+6)}$
- Solution for $r=\frac{2}{3}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+2}=-\frac{9a_k}{(3k+10)(3k+6)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{2}{3}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{k+2}=-\frac{9a_k}{(3k+6)(3k+2)}, a_1=0, b_{k+2}=-\frac{9b_k}{(3k+10)(3k+6)}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 39

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-4/9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}} \left(1-\frac{3}{20}x^2+\frac{9}{1280}x^4-\frac{9}{56320}x^6+\mathrm{O}\left(x^8\right)\right)+c_1\left(1-\frac{3}{4}x^2+\frac{9}{128}x^4-\frac{9}{3584}x^6+\mathrm{O}\left(x^8\right)\right)}{x^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 66

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-4/9)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 x^{2/3}\left(-\frac{9x^6}{56320}+\frac{9x^4}{1280}-\frac{3x^2}{20}+1\right)+\frac{c_2\left(-\frac{9x^6}{3584}+\frac{9x^4}{128}-\frac{3x^2}{4}+1\right)}{x^{2/3}}$$

2.23 problem 23

Internal problem ID [6606]
Internal file name [`OUTPUT/5854_Sunday_June_05_2022_03_58_06_PM_77810363/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2y'' + 9x^2y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + 9x^2y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \frac{2}{9x^2} \end{aligned}$$

Table 63: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = \frac{2}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + 9x^2y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned}
9x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\
+ 9x^2\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 9x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n(n+r)\,(n+r-1) + 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 2a_0 x^r = 0$$

Or

$$(9x^r r(-1+r) + 2x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 9r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 9r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{2}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2 - 9r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) + 9a_{n-1}(n+r-1) + 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{9a_{n-1}(n+r-1)}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_n = \frac{a_{n-1}(1-3n)}{3n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{9r}{9r^2 + 9r + 2}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_2 = \frac{5}{28}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
a_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{5}{28}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_3 = -\frac{1}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
a_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{5}{28}$
a_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{21}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_4 = \frac{11}{1092}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
a_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{5}{28}$
a_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{21}$
a_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{11}{1092}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_5 = -\frac{11}{6240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
a_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{5}{28}$
a_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{21}$
a_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{11}{1092}$
a_5	$-\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{11}{6240}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{531441r(1+r)(2+r)(3+r)(4+r)(5+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+2}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_6 = \frac{187}{711360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
a_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{5}{28}$
a_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{21}$
a_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{11}{1092}$
a_5	$-\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{11}{6240}$
a_6	$\frac{531441r(1+r)(2+r)(3+r)(4+r)(5+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)}$	$\frac{187}{711360}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{4782969r(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_7 = -\frac{17}{497952}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
a_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{5}{28}$
a_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{21}$
a_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{11}{1092}$
a_5	$-\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{11}{6240}$
a_6	$\frac{531441r(1+r)(2+r)(3+r)(4+r)(5+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)}$	$\frac{187}{711360}$
a_7	$-\frac{4782969r(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)(9r^2+117r+380)}$	$-\frac{17}{497952}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\frac{2}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots\right) \\
&= x^{\frac{2}{3}}\left(1 - \frac{x}{2} + \frac{5x^2}{28} - \frac{x^3}{21} + \frac{11x^4}{1092} - \frac{11x^5}{6240} + \frac{187x^6}{711360} - \frac{17x^7}{497952} + O(x^8)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$9b_n(n+r)(n+r-1) + 9b_{n-1}(n+r-1) + 2b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{9b_{n-1}(n+r-1)}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_n = \frac{b_{n-1}(2-3n)}{3n^2 - n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{9r}{9r^2 + 9r + 2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{81r(1 + r)}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = \frac{1}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{5}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{729r(1 + r)(2 + r)}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_3 = -\frac{7}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{5}$
b_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{7}{120}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = \frac{7}{528}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{5}$
b_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{7}{120}$
b_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{7}{528}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_5 = -\frac{13}{5280}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{5}$
b_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{7}{120}$
b_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{7}{528}$
b_5	$-\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{13}{5280}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{531441r(1+r)(2+r)(3+r)(4+r)(5+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_6 = \frac{13}{33660}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{5}$
b_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{7}{120}$
b_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{7}{528}$
b_5	$-\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{13}{5280}$
b_6	$\frac{531441r(1+r)(2+r)(3+r)(4+r)(5+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)}$	$\frac{13}{33660}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{4782969r(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_7 = -\frac{247}{4712400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{5}$
b_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{7}{120}$
b_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{7}{528}$
b_5	$-\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{13}{5280}$
b_6	$\frac{531441r(1+r)(2+r)(3+r)(4+r)(5+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)}$	$\frac{13}{33660}$
b_7	$-\frac{4782969r(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)(9r^2+99r+272)(9r^2+117r+380)}$	$-\frac{247}{4712400}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^{\frac{2}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right)\\
&= x^{\frac{1}{3}}\left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{7x^3}{120} + \frac{7x^4}{528} - \frac{13x^5}{5280} + \frac{13x^6}{33660} - \frac{247x^7}{4712400} + O(x^8)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\
&= c_1 x^{\frac{2}{3}}\left(1 - \frac{x}{2} + \frac{5x^2}{28} - \frac{x^3}{21} + \frac{11x^4}{1092} - \frac{11x^5}{6240} + \frac{187x^6}{711360} - \frac{17x^7}{497952} + O(x^8)\right)\\
&\quad + c_2 x^{\frac{1}{3}}\left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{7x^3}{120} + \frac{7x^4}{528} - \frac{13x^5}{5280} + \frac{13x^6}{33660} - \frac{247x^7}{4712400} + O(x^8)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h\\
&= c_1 x^{\frac{2}{3}}\left(1 - \frac{x}{2} + \frac{5x^2}{28} - \frac{x^3}{21} + \frac{11x^4}{1092} - \frac{11x^5}{6240} + \frac{187x^6}{711360} - \frac{17x^7}{497952} + O(x^8)\right)\\
&\quad + c_2 x^{\frac{1}{3}}\left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{7x^3}{120} + \frac{7x^4}{528} - \frac{13x^5}{5280} + \frac{13x^6}{33660} - \frac{247x^7}{4712400} + O(x^8)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{2}{3}} \left(1 - \frac{x}{2} + \frac{5x^2}{28} - \frac{x^3}{21} + \frac{11x^4}{1092} - \frac{11x^5}{6240} + \frac{187x^6}{711360} - \frac{17x^7}{497952} + O(x^8)\right) + c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{7x^3}{120} + \frac{7x^4}{528} - \frac{13x^5}{5280} + \frac{13x^6}{33660} - \frac{247x^7}{4712400} + O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{2}{3}} \left(1 - \frac{x}{2} + \frac{5x^2}{28} - \frac{x^3}{21} + \frac{11x^4}{1092} - \frac{11x^5}{6240} + \frac{187x^6}{711360} - \frac{17x^7}{497952} + O(x^8)\right) + c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{7x^3}{120} + \frac{7x^4}{528} - \frac{13x^5}{5280} + \frac{13x^6}{33660} - \frac{247x^7}{4712400} + O(x^8)\right)$$

Verified OK.

2.23.1 Maple step by step solution

Let's solve

$9x^2y'' + 9x^2y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{2y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{2y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 1, P_3(x) = \frac{2}{9x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{2}{9}$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $9x^2y'' + 9x^2y' + 2y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$
  - Shift index using $k->k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)\,(-2+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-1)\,(3k+3r-2) + 9a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)\,(-2+3r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $9\left(k+r-\frac{1}{3}\right)\left(k+r-\frac{2}{3}\right) a_k + 9a_{k-1}(k-1+r) = 0$
- Shift index using $k->k+1$

  $9\left(k+\frac{2}{3}+r\right)\left(k+\frac{1}{3}+r\right) a_{k+1} + 9a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{9a_k(k+r)}{(3k+2+3r)(3k+1+3r)}$
- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = -\frac{9a_k\left(k+\frac{1}{3}\right)}{(3k+3)(3k+2)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{9a_k\left(k+\frac{1}{3}\right)}{(3k+3)(3k+2)}\right]$$

- Recursion relation for $r = \frac{2}{3}$

$$a_{k+1} = -\frac{9a_k\left(k+\frac{2}{3}\right)}{(3k+4)(3k+3)}$$

- Solution for $r = \frac{2}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+1} = -\frac{9a_k\left(k+\frac{2}{3}\right)}{(3k+4)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{k+1} = -\frac{9a_k\left(k+\frac{1}{3}\right)}{(3k+3)(3k+2)}, b_{k+1} = -\frac{9b_k\left(k+\frac{2}{3}\right)}{(3k+4)(3k+3)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 55

```
Order:=8;
dsolve(9*x^2*diff(y(x),x$2)+9*x^2*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 - \frac{1}{2}x + \frac{1}{5}x^2 - \frac{7}{120}x^3 + \frac{7}{528}x^4 - \frac{13}{5280}x^5 + \frac{13}{33660}x^6 - \frac{247}{4712400}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2 x^{\frac{2}{3}} \left(1 - \frac{1}{2}x + \frac{5}{28}x^2 - \frac{1}{21}x^3 + \frac{11}{1092}x^4 - \frac{11}{6240}x^5 + \frac{187}{711360}x^6 - \frac{17}{497952}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 118

```
AsymptoticDSolveValue[9*x^2*y''[x]+9*x^2*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2 \sqrt[3]{x} \left(-\frac{247x^7}{4712400} + \frac{13x^6}{33660} - \frac{13x^5}{5280} + \frac{7x^4}{528} - \frac{7x^3}{120} + \frac{x^2}{5} - \frac{x}{2} + 1\right) + c_1 x^{2/3} \left(-\frac{17x^7}{497952} + \frac{187x^6}{711360} - \frac{11x^5}{6240} + \frac{11x^4}{1092} - \frac{x^3}{21} + \frac{5x^2}{28} - \frac{x}{2} + 1\right)$$

2.24 problem 24

Internal problem ID [6607]
Internal file name [`OUTPUT/5855_Sunday_June_05_2022_03_58_11_PM_8048776/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + 3xy' + (2x-1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 3xy' + (2x-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{2x}$$
$$q(x) = \frac{2x-1}{2x^2}$$

Table 65: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2x-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 3xy' + (2x - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) \\ &+ 3x\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (2x - 1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\left(n+r-1\right)\right)+\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)\right) \\ &+\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n=\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\left(n+r-1\right)\right)+\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)\right) \\ &+\left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)\left(n+r-1\right)+3x^{n+r} a_n(n+r)-a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r)+3x^r a_0 r-a_0 x^r=0$$

Or

$$\left(2x^r r(-1+r)+3x^r r-x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2+r-1\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2+r-1=0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{1}{2} \\
r_2 &= -1
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(2r^2 + r - 1\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2a_n(n+r)(n+r-1) + 3a_n(n+r) + 2a_{n-1} - a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{2a_{n-1}}{2n^2 + 4nr + 2r^2 + n + r - 1} \tag{4}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
a_n = -\frac{2a_{n-1}}{n(2n+3)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{2r^2 + 5r + 2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{2}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$
a_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	$\frac{2}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)(2r^2 + 13r + 20)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{4}{945}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$
a_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	$\frac{2}{35}$
a_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$-\frac{4}{945}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)(2r^2 + 13r + 20)(2r^2 + 17r + 35)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{2}{10395}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$
a_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	$\frac{2}{35}$
a_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$-\frac{4}{945}$
a_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$\frac{2}{10395}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)(2r^2 + 13r + 20)(2r^2 + 17r + 35)(2r^2 + 21r + 54)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{4}{675675}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$
a_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	$\frac{2}{35}$
a_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$-\frac{4}{945}$
a_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$\frac{2}{10395}$
a_5	$-\frac{32}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)}$	$-\frac{4}{675675}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)(2r^2+25r+77)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{4}{30405375}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$
a_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	$\frac{2}{35}$
a_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$-\frac{4}{945}$
a_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$\frac{2}{10395}$
a_5	$-\frac{32}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)}$	$-\frac{4}{675675}$
a_6	$\frac{64}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)(2r^2+25r+77)}$	$\frac{4}{30405375}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{128}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)(2r^2+25r+77)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = -\frac{8}{3618239625}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$
a_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	$\frac{2}{35}$
a_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$-\frac{4}{945}$
a_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$\frac{2}{10395}$
a_5	$-\frac{32}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)}$	$-\frac{4}{675675}$
a_6	$\frac{64}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)(2r^2+25r+77)}$	$\frac{4}{30405375}$
a_7	$-\frac{128}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)(2r^2+25r+77)(2r^2+29r+104)}$	$-\frac{8}{3618239625}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + \frac{4x^6}{30405375} - \frac{8x^7}{3618239625} + O\left(x^8\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + 3b_n(n+r) + 2b_{n-1} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{2b_{n-1}}{2n^2 + 4nr + 2r^2 + n + r - 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{2b_{n-1}}{n(2n-3)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{2}{2r^2 + 5r + 2}$$

Which for the root $r = -1$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)}$$

Which for the root $r = -1$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2
b_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)(2r^2 + 13r + 20)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{4}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2
b_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	-2
b_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$\frac{4}{9}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{2}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2
b_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	-2
b_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$\frac{4}{9}$
b_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$-\frac{2}{45}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{4}{1575}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2
b_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	-2
b_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$\frac{4}{9}$
b_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$-\frac{2}{45}$
b_5	$-\frac{32}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)}$	$\frac{4}{1575}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)(2r^2+25r+77)}$$

Which for the root $r = -1$ becomes

$$b_6 = -\frac{4}{42525}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2
b_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	-2
b_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$\frac{4}{9}$
b_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$-\frac{2}{45}$
b_5	$-\frac{32}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)}$	$\frac{4}{1575}$
b_6	$\frac{64}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)(2r^2+25r+77)}$	$-\frac{4}{42525}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{128}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)(2r^2+25r+77)}$$

Which for the root $r = -1$ becomes

$$b_7 = \frac{8}{3274425}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2
b_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	-2
b_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$\frac{4}{9}$
b_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$-\frac{2}{45}$
b_5	$-\frac{32}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)}$	$\frac{4}{1575}$
b_6	$\frac{64}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)(2r^2+25r+77)}$	$-\frac{4}{42525}$
b_7	$-\frac{128}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)(2r^2+25r+77)(2r^2+29r+104)}$	$\frac{8}{3274425}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} - \frac{4x^6}{42525} + \frac{8x^7}{3274425} + O(x^8)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + \frac{4x^6}{30405375} - \frac{8x^7}{3618239625} + O(x^8)\right) \\
&\quad + \frac{c_2\left(1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} - \frac{4x^6}{42525} + \frac{8x^7}{3274425} + O(x^8)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + \frac{4x^6}{30405375} - \frac{8x^7}{3618239625} + O(x^8)\right) \\
&\quad + \frac{c_2\left(1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} - \frac{4x^6}{42525} + \frac{8x^7}{3274425} + O(x^8)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1-\frac{2x}{5}+\frac{2x^2}{35}-\frac{4x^3}{945}+\frac{2x^4}{10395}-\frac{4x^5}{675675}+\frac{4x^6}{30405375}-\frac{8x^7}{3618239625}+O(x^8)\right) + \frac{c_2\left(1+2x-2x^2+\frac{4x^3}{9}-\frac{2x^4}{45}+\frac{4x^5}{1575}-\frac{4x^6}{42525}+\frac{8x^7}{3274425}+O(x^8)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{2x}{5}+\frac{2x^2}{35}-\frac{4x^3}{945}+\frac{2x^4}{10395}-\frac{4x^5}{675675}+\frac{4x^6}{30405375}-\frac{8x^7}{3618239625}+O(x^8)\right) + \frac{c_2\left(1+2x-2x^2+\frac{4x^3}{9}-\frac{2x^4}{45}+\frac{4x^5}{1575}-\frac{4x^6}{42525}+\frac{8x^7}{3274425}+O(x^8)\right)}{x}$$

Verified OK.

2.24.1 Maple step by step solution

Let's solve

$2x^2y'' + 3xy' + (2x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{2x} - \frac{(2x-1)y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2x} + \frac{(2x-1)y}{2x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{3}{2x}, P_3(x) = \frac{2x-1}{2x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{3}{2}$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{2}$

  - ○ $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $2x^2y''+3xy'+(2x-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - ○ Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  - ○ Convert $x^2\cdot y''$ to series expansion

    $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(1+r)\,(-1+2r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)\,(2k+2r-1)+2a_{k-1}\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-1,\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r+1)\left(k-\frac{1}{2}+r\right)a_k+2a_{k-1}=0$

- Shift index using $k->k+1$

  $2(k+2+r)\left(k+\frac{1}{2}+r\right)a_{k+1}+2a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k}{(k+2+r)(2k+1+2r)}$

- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}, b_{k+1} = -\frac{2b_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 55

```
Order:=8;
dsolve(2*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+(2*x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{3}{2}} (1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{945}x^3 + \frac{2}{10395}x^4 - \frac{4}{675675}x^5 + \frac{4}{30405375}x^6 - \frac{8}{3618239625}x^7 + O(x^8)) + c_1 (1 + 2x - 2}{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 112

```
AsymptoticDSolveValue[2*x^2*y''[x]+3*x*y'[x]+(2*x-1)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \sqrt{x} \left(-\frac{8x^7}{3618239625} + \frac{4x^6}{30405375} - \frac{4x^5}{675675} + \frac{2x^4}{10395} - \frac{4x^3}{945} + \frac{2x^2}{35} - \frac{2x}{5} + 1 \right) + \frac{c_2 \left(\frac{8x^7}{3274425} - \frac{4x^6}{42525} + \frac{4x^5}{1575} - \frac{2x^4}{45} + \frac{4x^3}{9} - 2x^2 + 2x + 1 \right)}{x}$$

2.25 problem 25

Internal problem ID [6608]
Internal file name [`OUTPUT/5856_Sunday_June_05_2022_03_58_17_PM_17907405/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + 2y' - xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = -1$$

Table 67: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y' - xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)\,(n+r-1) + 2(n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_n(n+r) - a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-2}}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r^2 + 5r + 6}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 14r^3 + 71r^2 + 154r + 120}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{r^6 + 27r^5 + 295r^4 + 1665r^3 + 5104r^2 + 8028r + 5040}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$
a_5	0	0
a_6	$\frac{1}{r^6+27r^5+295r^4+1665r^3+5104r^2+8028r+5040}$	$\frac{1}{5040}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$
a_5	0	0
a_6	$\frac{1}{r^6+27r^5+295r^4+1665r^3+5104r^2+8028r+5040}$	$\frac{1}{5040}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 + \frac{x^2}{6} + \frac{x^4}{120} + \frac{x^6}{5040} + O(x^8)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r \rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \rightarrow r_2} 0 &= \lim_{r \rightarrow -1} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 2(n+r) b_n - b_{n-2} = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + 2(n-1) b_n - b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{b_{n-2}}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{(r^2+5r+6)(r^2+9r+20)(r^2+13r+42)}$$

Which for the root $r = -1$ becomes

$$b_6 = \frac{1}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$
b_5	0	0
b_6	$\frac{1}{(r^2+5r+6)(r^2+9r+20)(r^2+13r+42)}$	$\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$
b_5	0	0
b_6	$\frac{1}{(r^2+5r+6)(r^2+9r+20)(r^2+13r+42)}$	$\frac{1}{720}$
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\
&= \frac{1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + O(x^8)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 + \frac{x^2}{6} + \frac{x^4}{120} + \frac{x^6}{5040} + O(x^8)\right) + \frac{c_2 \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + O(x^8)\right)}{x}
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1+\frac{x^2}{6}+\frac{x^4}{120}+\frac{x^6}{5040}+O\left(x^8\right)\right)+\frac{c_2\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+O(x^8)\right)}{x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\left(1+\frac{x^2}{6}+\frac{x^4}{120}+\frac{x^6}{5040}+O\left(x^8\right)\right)+\frac{c_2\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+O(x^8)\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1+\frac{x^2}{6}+\frac{x^4}{120}+\frac{x^6}{5040}+O\left(x^8\right)\right)+\frac{c_2\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+O(x^8)\right)}{x}
$$

Verified OK.

2.25.1 Maple step by step solution

Let's solve

$y''x+2y'-xy=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y'}{x}+y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2y'}{x}-y=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x)=\frac{2}{x}, P_3(x)=-1\right]$

  ○ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=2$

  ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = 0$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- ● Multiply by denominators

  $y''x + 2y' - xy = 0$

- ● Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - ○ Shift index using $k- >k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - ○ Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - ○ Shift index using $k- >k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

  - ○ Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - ○ Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

    Rewrite ODE with series expansions

    $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1}\right) x^{k+r}\right) = 0$$

- ● $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r)-a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+3+r)-a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=\frac{a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=\frac{a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{a_k}{(k+2)(k+3)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2}=\frac{a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=\frac{b_k}{(k+2)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.063 (sec). Leaf size: 36

```
Order:=8;
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 \left(1 + \frac{1}{6}x^2 + \frac{1}{120}x^4 + \frac{1}{5040}x^6 + \mathrm{O}\left(x^8\right)\right) + \frac{c_2 \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \mathrm{O}\left(x^8\right)\right)}{x}$$

✓ Solution by Mathematica
Time used: 0.014 (sec). Leaf size: 56

```
AsymptoticDSolveValue[x*y''[x]+2*y'[x]-x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(\frac{x^5}{720} + \frac{x^3}{24} + \frac{x}{2} + \frac{1}{x}\right) + c_2 \left(\frac{x^6}{5040} + \frac{x^4}{120} + \frac{x^2}{6} + 1\right)$$

2.26 problem 26

Internal problem ID [6609]
Internal file name [`OUTPUT/5857_Sunday_June_05_2022_03_58_20_PM_88084394/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{4x^2 - 1}{4x^2} \end{aligned}$$

Table 69: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2-1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ + x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(x^2 - \frac{1}{4}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)\,(n+r-1)+x^{n+r} a_n (n+r)-\frac{a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^{r} a_0 r(-1+r)+x^{r} a_0 r-\frac{a_0 x^{r}}{4} = 0$$

Or

$$\left(x^{r} r(-1+r)+x^{r} r-\frac{x^{r}}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2-1)\,x^{r}}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1 &= \frac{1}{2}\\ r_2 &= -\frac{1}{2}\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 1)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= \sqrt{x}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}\\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)\end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)\,(n+r-1)+a_n(n+r)+a_{n-2}-\frac{a_n}{4}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{4n^2+8nr+4r^2-1} \tag{4}$$

Which for the root $r=\frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{n\,(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$a_2 = -\frac{4}{4r^2+16r+15}$$

Which for the root $r=\frac{1}{2}$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$

For $n=3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{64}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)(4r^2 + 48r + 143)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = -\frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0
a_6	$-\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$-\frac{1}{5040}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0
a_6	$-\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$-\frac{1}{5040}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \ldots\right)\\ &= \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + O(x^8)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_1\\ &= 0\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\rightarrow r_2} 0 &= \lim_{r\rightarrow -\frac{1}{2}} 0\\ &= 0\end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r}\\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)\,(n+r-1) + b_n(n+r) + b_{n-2} - \frac{b_n}{4} = 0 \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$b_n \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) + b_n \left(n - \frac{1}{2}\right) + b_{n-2} - \frac{b_n}{4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{4b_{n-2}}{4n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = -\frac{64}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)(4r^2 + 48r + 143)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_6 = -\frac{1}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0
b_6	$-\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$-\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0
b_6	$-\frac{64}{(4r^2+16r+15)(4r^2+32r+63)(4r^2+48r+143)}$	$-\frac{1}{720}$
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= \sqrt{x}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \dots\right)\\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)}{\sqrt{x}}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x)\\ &= c_1\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + O(x^8)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)\right)}{\sqrt{x}}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + O(x^8)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)\right)}{\sqrt{x}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + O(x^8)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + O(x^8)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)\right)}{\sqrt{x}}$$

Verified OK.

2.26.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - ○ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$

  - ○ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

    Rewrite ODE with series expansions

    $$a_0(1+2r)(-1+2r)\, x^r + a_1(3+2r)(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2r-1) + 4a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

  $$a_1(3+2r)(1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$$

- Shift index using $k- > k+2$

  $$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{4a_k}{4k^2 + 8kr + 4r^2 + 16k + 16r + 15}$$

- Recursion relation for $r = -\frac{1}{2}$

  $$a_{k+2} = -\frac{4a_k}{4k^2 + 12k + 8}$$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2 + 12k + 8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = -\frac{4a_k}{4k^2 + 20k + 24}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2 + 20k + 24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 39

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1x\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4-\frac{1}{5040}x^6+\mathrm{O}\left(x^8\right)\right)+c_2\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\mathrm{O}\left(x^8\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 76

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],{x,0,7}]
```

$$y(x)\rightarrow c_1\left(-\frac{x^{11/2}}{720}+\frac{x^{7/2}}{24}-\frac{x^{3/2}}{2}+\frac{1}{\sqrt{x}}\right)+c_2\left(-\frac{x^{13/2}}{5040}+\frac{x^{9/2}}{120}-\frac{x^{5/2}}{6}+\sqrt{x}\right)$$

2.27 problem 27

Internal problem ID [6610]
Internal file name [`OUTPUT/5858_Sunday_June_05_2022_03_58_23_PM_39785138/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Laguerre, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{xy'' - xy' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - xy' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -1 \\ q(x) &= \frac{1}{x} \end{aligned}$$

Table 71: Table $p(x), q(x)$ singularites.

$p(x) = -1$	
singularity	type

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - xy' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x - x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}a_n x^{n+r}=\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)\,(n+r-1)\right)\\&+\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}\right)=0\end{aligned}\tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)\,(n+r-1)=0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0r(-1+r)=0$$

Or

$$x^{-1+r}a_0r(-1+r)=0$$

Since $a_0\neq 0$ then the above simplifies to

$$x^{-1+r}r(-1+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r)=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1&=1\\r_2&=0\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}$$

Or

$$\begin{aligned}
y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}$$

Or

$$\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-2)}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}(n-1)}{(n+1)\, n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1+r}{(1+r)\, r}$$

Which for the root $r = 1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1+r}{(1+r)^2\, (2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-1+r}{(1+r)\,(2+r)^2\,(3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0
a_4	$\frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1+r}{(1+r)(2+r)(3+r)(4+r)^2(5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0
a_4	$\frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$	0
a_5	$\frac{-1+r}{(1+r)(2+r)(3+r)(4+r)^2(5+r)}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-1+r}{(1+r)(2+r)(3+r)(4+r)(5+r)^2(6+r)}$$

Which for the root $r = 1$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0
a_4	$\frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$	0
a_5	$\frac{-1+r}{(1+r)(2+r)(3+r)(4+r)^2(5+r)}$	0
a_6	$\frac{-1+r}{(1+r)(2+r)(3+r)(4+r)(5+r)^2(6+r)}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-1+r}{(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)^2(7+r)}$$

Which for the root $r = 1$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0
a_4	$\frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$	0
a_5	$\frac{-1+r}{(1+r)(2+r)(3+r)(4+r)^2(5+r)}$	0
a_6	$\frac{-1+r}{(1+r)(2+r)(3+r)(4+r)(5+r)^2(6+r)}$	0
a_7	$\frac{-1+r}{(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)^2(7+r)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots\big) \\
&= x\big(1 + O(x^8)\big)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_1 \\
&= \frac{-1+r}{(1+r)\, r}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{-1+r}{(1+r)\, r} &= \lim_{r\to 0} \frac{-1+r}{(1+r)\, r} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $xy'' - xy' + y = 0$ gives

$$
\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x \\
&- x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(y_1''(x)\,x - y_1'(x)\,x + y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - y_1(x)\right)C \\
&+ \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&- x\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x - y_1'(x)\,x + y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - y_1(x)\right)C\\ &+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\ &- x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right)x - (1+x)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C}{x}\\ &+\frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)x^2 - \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x}{x}\\ &= 0\end{aligned} \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{n} a_n(n+1)\right)x - (1+x)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right)C}{x}\\ &+\frac{\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right)x^2 - \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right)x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n}\right)x}{x} = 0\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} 2C\,x^n a_n(n+1)\right) + \sum_{n=0}^{\infty}(-C a_n x^n) + \sum_{n=0}^{\infty}\left(-C\,x^{n+1} a_n\right)\\ &+\left(\sum_{n=0}^{\infty} n\,x^{n-1} b_n(n-1)\right) + \sum_{n=0}^{\infty}(-x^n b_n n) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n} a_{n} (n+1) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-C a_{n} x^{n}\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{n+1} a_{n}\right) = \sum_{n=2}^{\infty} \left(-C a_{-2+n} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{n} b_{n} n\right) = \sum_{n=1}^{\infty} \left(-(n-1)\, b_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} b_{n} x^{n} = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned} \left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) + \sum_{n=2}^{\infty} \left(-C a_{-2+n} x^{n-1}\right) \\ + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_{n} (n-1)\right) + \sum_{n=1}^{\infty} \left(-(n-1)\, b_{n-1} x^{n-1}\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0 \end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 2$, Eq (2B) gives

$$(-a_0 + 3a_1)\, C + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 + 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$(-a_1 + 5a_2)\, C - b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{2} + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{12}$$

For $n = 4$, Eq (2B) gives

$$(-a_2 + 7a_3)\, C - 2b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{6} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1}{72}$$

For $n = 5$, Eq (2B) gives

$$(-a_3 + 9a_4)\, C - 3b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{24} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{1}{480}$$

For $n = 6$, Eq (2B) gives

$$(-a_4 + 11a_5)\, C - 4b_5 + 30b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{120}+30b_6=0$$

Solving the above for b_6 gives

$$b_6=-\frac{1}{3600}$$

For $n=7$, Eq (2B) gives

$$(-a_5+13a_6)\,C-5b_6+42b_7=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{720}+42b_7=0$$

Solving the above for b_7 gives

$$b_7=-\frac{1}{30240}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_nx^{n+r_2}\right)$$

Using the above value found for $C=-1$ and all b_n, then the second solution becomes

$$y_2(x)=(-1)\left(x\left(1+O\left(x^8\right)\right)\right)\ln(x)+1-\frac{x^2}{2}-\frac{x^3}{12}-\frac{x^4}{72}-\frac{x^5}{480}-\frac{x^6}{3600}-\frac{x^7}{30240}+O\left(x^8\right)$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x)&=c_1y_1(x)+c_2y_2(x)\\&=c_1x\left(1+O\left(x^8\right)\right)+c_2\Bigg((-1)\left(x\left(1+O\left(x^8\right)\right)\right)\ln(x)+1-\frac{x^2}{2}-\frac{x^3}{12}-\frac{x^4}{72}-\frac{x^5}{480}\\&\qquad-\frac{x^6}{3600}-\frac{x^7}{30240}+O\left(x^8\right)\Bigg)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y&=y_h\\&=c_1x\left(1+O\left(x^8\right)\right)\\&\quad+c_2\left(-x\left(1+O\left(x^8\right)\right)\ln(x)+1-\frac{x^2}{2}-\frac{x^3}{12}-\frac{x^4}{72}-\frac{x^5}{480}-\frac{x^6}{3600}-\frac{x^7}{30240}+O\left(x^8\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\left(1+O\left(x^8\right)\right) + c_2\left(-x\left(1+O\left(x^8\right)\right)\ln\left(x\right)+1-\frac{x^2}{2}-\frac{x^3}{12}-\frac{x^4}{72}-\frac{x^5}{480}-\frac{x^6}{3600}-\frac{x^7}{30240}+O\left(x^8\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x\left(1+O\left(x^8\right)\right) + c_2\left(-x\left(1+O\left(x^8\right)\right)\ln\left(x\right)+1-\frac{x^2}{2}-\frac{x^3}{12}-\frac{x^4}{72}-\frac{x^5}{480}-\frac{x^6}{3600}-\frac{x^7}{30240}+O\left(x^8\right)\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 46

```
Order:=8;
dsolve(x*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x\left(1+\mathrm{O}\left(x^8\right)\right)+\left(-x+\mathrm{O}\left(x^8\right)\right)\ln\left(x\right)c_2 + \left(1+x-\frac{1}{2}x^2-\frac{1}{12}x^3-\frac{1}{72}x^4-\frac{1}{480}x^5-\frac{1}{3600}x^6-\frac{1}{30240}x^7+\mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 51

```
AsymptoticDSolveValue[x*y''[x]-x*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{-2x^6-15x^5-100x^4-600x^3-3600x^2+14400x+7200}{7200}-x\log(x)\right)+c_2x$$

2.28 problem 28

Internal problem ID [6611]
Internal file name [`OUTPUT/5859_Sunday_June_05_2022_03_58_29_PM_58851504/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \frac{3y'}{x} - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + \frac{3y'}{x} - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = -2$$

Table 72: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -2$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-2xy + xy'' + 3y' = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &-2x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x \\ &+ 3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_n(n+r) - 2a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-2}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{2a_{n-2}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2}{r^2 + 6r + 8}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r^2+6r+8}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r^2+6r+8}$	$\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4}{(4 + r)^2 (2 + r) (r + 6)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r^2+6r+8}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{4}{(4+r)^2(2+r)(r+6)}$	$\frac{1}{48}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r^2+6r+8}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{4}{(4+r)^2(2+r)(r+6)}$	$\frac{1}{48}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{8}{(4+r)^2 (2+r) (r+6)^2 (r+8)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{1}{1152}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r^2+6r+8}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{4}{(4+r)^2(2+r)(r+6)}$	$\frac{1}{48}$
a_5	0	0
a_6	$\frac{8}{(4+r)^2(2+r)(r+6)^2(r+8)}$	$\frac{1}{1152}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r^2+6r+8}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{4}{(4+r)^2(2+r)(r+6)}$	$\frac{1}{48}$
a_5	0	0
a_6	$\frac{8}{(4+r)^2(2+r)(r+6)^2(r+8)}$	$\frac{1}{1152}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots \\ &= 1 + \frac{x^2}{4} + \frac{x^4}{48} + \frac{x^6}{1152} + O\left(x^8\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{2}{r^2+6r+8} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{2}{r^2+6r+8} &= \lim_{r\to -2} \frac{2}{r^2+6r+8} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $-2xy + xy'' + 3y' = 0$ gives

$$
\begin{aligned}
&-2x\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) + \left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x}\right. \\
&\left. - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&+ 3Cy_1'(x)\ln(x) + \frac{3Cy_1(x)}{x} + 3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left((-2y_1(x)\,x + y_1''(x)\,x + 3y_1'(x))\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + \frac{3y_1(x)}{x}\right)C \\
&- 2x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&+ 3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
-2y_1(x)\,x + y_1''(x)\,x + 3y_1'(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + \frac{3y_1(x)}{x}\right)C - 2x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) \\
&+ \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&+ 3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) = 0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 - 2x^2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) + 3\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x}{x} = 0 \tag{9}$$

Since $r_1 = 0$ and $r_2 = -2$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n} a_n n\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^{n}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 - 2x^2\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) + 3\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n\right) + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n^2-5n+6)\right) \\ &+ \sum_{n=0}^{\infty} \left(-2x^{-1+n} b_n\right) + \left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n (n-2)\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $-3+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-3+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n n = \sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n = \sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} \left(-2x^{-1+n} b_n\right) = \sum_{n=2}^{\infty} \left(-2b_{n-2} x^{-3+n}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-3+n$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}\right)+\left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{-3+n} b_n\left(n^2-5n+6\right)\right)+\sum_{n=2}^{\infty}\left(-2 b_{n-2} x^{-3+n}\right) \\
&+\left(\sum_{n=0}^{\infty} 3 x^{-3+n} b_n(n-2)\right)=0
\end{aligned}
\tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1=0$$

Solving the above for b_1 gives

$$b_1=0$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C-2=0$$

Which is solved for C. Solving for C gives

$$C=1$$

For $n=3$, Eq (2B) gives

$$4Ca_1-2b_1+3b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3=0$$

Solving the above for b_3 gives

$$b_3=0$$

For $n=4$, Eq (2B) gives

$$6Ca_2-2b_2+8b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 + \frac{3}{2} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{3}{16}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 - 2b_3 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

For $n = 6$, Eq (2B) gives

$$10Ca_4 - 2b_4 + 24b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$24b_6 + \frac{7}{12} = 0$$

Solving the above for b_6 gives

$$b_6 = -\frac{7}{288}$$

For $n = 7$, Eq (2B) gives

$$12Ca_5 - 2b_5 + 35b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$35b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$y_2(x) = 1\left(1 + \frac{x^2}{4} + \frac{x^4}{48} + \frac{x^6}{1152} + O(x^8)\right)\ln(x) + \frac{1 - \frac{3x^4}{16} - \frac{7x^6}{288} + O(x^8)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1+\frac{x^2}{4}+\frac{x^4}{48}+\frac{x^6}{1152}+O\left(x^8\right)\right) \\
&\quad + c_2\left(1\left(1+\frac{x^2}{4}+\frac{x^4}{48}+\frac{x^6}{1152}+O\left(x^8\right)\right)\ln\left(x\right)+\frac{1-\frac{3x^4}{16}-\frac{7x^6}{288}+O(x^8)}{x^2}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\left(1+\frac{x^2}{4}+\frac{x^4}{48}+\frac{x^6}{1152}+O\left(x^8\right)\right) \\
&\quad + c_2\left(\left(1+\frac{x^2}{4}+\frac{x^4}{48}+\frac{x^6}{1152}+O\left(x^8\right)\right)\ln\left(x\right)+\frac{1-\frac{3x^4}{16}-\frac{7x^6}{288}+O(x^8)}{x^2}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1\left(1+\frac{x^2}{4}+\frac{x^4}{48}+\frac{x^6}{1152}+O\left(x^8\right)\right) \\
&\quad + c_2\left(\left(1+\frac{x^2}{4}+\frac{x^4}{48}+\frac{x^6}{1152}+O\left(x^8\right)\right)\ln\left(x\right)+\frac{1-\frac{3x^4}{16}-\frac{7x^6}{288}+O(x^8)}{x^2}\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1\left(1+\frac{x^2}{4}+\frac{x^4}{48}+\frac{x^6}{1152}+O\left(x^8\right)\right) \\
&\quad + c_2\left(\left(1+\frac{x^2}{4}+\frac{x^4}{48}+\frac{x^6}{1152}+O\left(x^8\right)\right)\ln\left(x\right)+\frac{1-\frac{3x^4}{16}-\frac{7x^6}{288}+O(x^8)}{x^2}\right)
\end{aligned}$$

Verified OK.

2.28.1 Maple step by step solution

Let's solve

$-2xy+y''x+3y'=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{x}+2y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{x}-2y=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{3}{x},P_3(x)=-2\right]$

  ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=3$

  ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

  ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $-2xy+y''x+3y'=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x\cdot y$ to series expansion

  $x\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+1}$

  ○ Shift index using $k->k-1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(2+r)\, x^{-1+r} + a_1 (1+r)\,(3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+3+r) - 2a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 0\}$$

- Each term must be 0

  $$a_1 (1+r)\,(3+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+r+1)\,(k+3+r) - 2a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$a_{k+2}(k+r+2)\,(k+4+r) - 2a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{2a_k}{(k+r+2)(k+4+r)}$$

- Recursion relation for $r = -2$

  $$a_{k+2} = \frac{2a_k}{k(k+2)}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = \frac{2a_k}{k(k+2)}, -a_1 = 0\right]$$

- Recursion relation for $r = 0$

  $$a_{k+2} = \frac{2a_k}{(k+2)(k+4)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+2} = \frac{2a_k}{(k+2)(k+4)}, 3a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2} = \frac{2a_k}{k(k+2)}, -a_1 = 0, b_{k+2} = \frac{2b_k}{(k+2)(k+4)}, 3b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 52

```
Order:=8;
dsolve(diff(y(x),x$2)+3/x*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1+\frac{1}{4}x^2+\frac{1}{48}x^4+\frac{1}{1152}x^6+\mathrm{O}\left(x^8\right)\right)x^2+c_2\left(\ln\left(x\right)\left(\left(-2\right)x^2-\frac{1}{2}x^4-\frac{1}{24}x^6+\mathrm{O}\left(x^8\right)\right)+\left(-2+\frac{3}{8}x^4+\right.\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 74

```
AsymptoticDSolveValue[y''[x]+3/x*y'[x]-2*y[x]==0,y[x],{x,0,7}]
```

$$
\begin{aligned}
y(x) \to\ & c_2\left(\frac{x^6}{1152}+\frac{x^4}{48}+\frac{x^2}{4}+1\right) \\
& + c_1\left(\frac{1}{48}\left(x^4+12x^2+48\right)\log(x)-\frac{5x^6+45x^4+72x^2-144}{144x^2}\right)
\end{aligned}
$$

2.29 problem 29

Internal problem ID [6612]
Internal file name [`OUTPUT/5860_Sunday_June_05_2022_03_58_34_PM_10511330/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{xy'' + (1 - x)\, y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (1 - x)\, y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x - 1}{x}$$
$$q(x) = -\frac{1}{x}$$

Table 74: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (1 - x)\, y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{multline}\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) x \\ + (1 - x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}\end{multline}$$

Which simplifies to

$$\begin{gathered}\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)\,(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right) \\ +\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0\end{gathered} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{gathered}\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)\,(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r-1}\right)=0\end{gathered} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)\,(n+r-1)+(n+r)\,a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r)+r a_0 x^{-1+r}=0$$

Or

$$\left(x^{-1+r} r(-1+r)+r\,x^{-1+r}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{1+r}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(2+r)(1+r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(3+r)(2+r)(1+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(2+r)(1+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(4+r)(3+r)(2+r)(1+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(2+r)(1+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(4+r)(3+r)(2+r)(1+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(4+r)(3+r)(2+r)(1+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(2+r)(1+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(4+r)(3+r)(2+r)(1+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(4+r)(3+r)(2+r)(1+r)(5+r)}$	$\frac{1}{120}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(6 + r)(4 + r)(3 + r)(2 + r)(1 + r)(5 + r)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(2+r)(1+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(4+r)(3+r)(2+r)(1+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(4+r)(3+r)(2+r)(1+r)(5+r)}$	$\frac{1}{120}$
a_6	$\frac{1}{(6+r)(4+r)(3+r)(2+r)(1+r)(5+r)}$	$\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{(6 + r)(4 + r)(3 + r)(2 + r)(1 + r)(5 + r)(7 + r)}$$

Which for the root $r = 0$ becomes

$$a_7 = \frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(2+r)(1+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(4+r)(3+r)(2+r)(1+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(4+r)(3+r)(2+r)(1+r)(5+r)}$	$\frac{1}{120}$
a_6	$\frac{1}{(6+r)(4+r)(3+r)(2+r)(1+r)(5+r)}$	$\frac{1}{720}$
a_7	$\frac{1}{(6+r)(4+r)(3+r)(2+r)(1+r)(5+r)(7+r)}$	$\frac{1}{5040}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{1+r}$	1	$-\frac{1}{(1+r)^2}$	-1
b_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$	$\frac{-3-2r}{(2+r)^2(1+r)^2}$	$-\frac{3}{4}$
b_3	$\frac{1}{(3+r)(2+r)(1+r)}$	$\frac{1}{6}$	$\frac{-3r^2-12r-11}{(3+r)^2(2+r)^2(1+r)^2}$	$-\frac{11}{36}$
b_4	$\frac{1}{(4+r)(3+r)(2+r)(1+r)}$	$\frac{1}{24}$	$\frac{-4r^3-30r^2-70r-50}{(4+r)^2(3+r)^2(2+r)^2(1+r)^2}$	$-\frac{25}{288}$
b_5	$\frac{1}{(4+r)(3+r)(2+r)(1+r)(5+r)}$	$\frac{1}{120}$	$\frac{-5r^4-60r^3-255r^2-450r-274}{(4+r)^2(3+r)^2(2+r)^2(1+r)^2(5+r)^2}$	$-\frac{137}{7200}$
b_6	$\frac{1}{(6+r)(4+r)(3+r)(2+r)(1+r)(5+r)}$	$\frac{1}{720}$	$\frac{-6r^5-105r^4-700r^3-2205r^2-3248r-1764}{(6+r)^2(4+r)^2(3+r)^2(2+r)^2(1+r)^2(5+r)^2}$	$-\frac{49}{14400}$
b_7	$\frac{1}{(6+r)(4+r)(3+r)(2+r)(1+r)(5+r)(7+r)}$	$\frac{1}{5040}$	$\frac{-7r^6-168r^5-1610r^4-7840r^3-20307r^2-26264r-13068}{(6+r)^2(4+r)^2(3+r)^2(2+r)^2(1+r)^2(5+r)^2(7+r)^2}$	$-\frac{121}{235200}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)\ln(x) \\
&\quad - x - \frac{3x^2}{4} - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} - \frac{49x^6}{14400} - \frac{121x^7}{235200} + O(x^8)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)\ln(x) - x - \frac{3x^2}{4}\right. \\
&\quad \left. - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} - \frac{49x^6}{14400} - \frac{121x^7}{235200} + O(x^8)\right)
\end{aligned}$$

Hence the final solution is

$$y = y_h$$

$$
\begin{aligned}
&= c_1\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right)\\
&\quad + c_2\left(\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right)\ln\left(x\right)-x-\frac{3x^2}{4}\right.\\
&\qquad \left.-\frac{11x^3}{36}-\frac{25x^4}{288}-\frac{137x^5}{7200}-\frac{49x^6}{14400}-\frac{121x^7}{235200}+O\left(x^8\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right)\\
&\quad + c_2\left(\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right)\ln\left(x\right)-x-\frac{3x^2}{4}\right.\\
&\qquad \left.-\frac{11x^3}{36}-\frac{25x^4}{288}-\frac{137x^5}{7200}-\frac{49x^6}{14400}-\frac{121x^7}{235200}+O\left(x^8\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right)\\
&\quad + c_2\left(\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+O\left(x^8\right)\right)\ln\left(x\right)-x-\frac{3x^2}{4}\right.\\
&\qquad \left.-\frac{11x^3}{36}-\frac{25x^4}{288}-\frac{137x^5}{7200}-\frac{49x^6}{14400}-\frac{121x^7}{235200}+O\left(x^8\right)\right)
\end{aligned}
$$

Verified OK.

2.29.1 Maple step by step solution

Let's solve

$y''x + (1-x)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-1)y'}{x} + \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} - \frac{y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{1}{x}\right]$
  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$
  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - ○ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + (1 - x)\, y' - y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$
  - ○ Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$
  - ○ Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-1}$
  - ○ Shift index using $k- >k + 1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\, (k + r)\, x^{k+r}$

    Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(k+1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\left(a_{k+1}(k+1) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 71

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(1-x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(-x - \frac{3}{4}x^2 - \frac{11}{36}x^3 - \frac{25}{288}x^4 - \frac{137}{7200}x^5 - \frac{49}{14400}x^6 - \frac{121}{235200}x^7 + \mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 149

```
AsymptoticDSolveValue[x*y''[x]+(1-x)*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^7}{5040} + \frac{x^6}{720} + \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right) + c_2\left(-\frac{121x^7}{235200} - \frac{49x^6}{14400} - \frac{137x^5}{7200} - \frac{25x^4}{288} - \frac{11x^3}{36} - \frac{3x^2}{4} + \left(\frac{x^7}{5040} + \frac{x^6}{720} + \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right)\log(x) - x\right)$$

2.30 problem 30

Internal problem ID [6613]
Internal file name [`OUTPUT/5861_Sunday_June_05_2022_03_58_38_PM_92457733/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{1}{x}$$

Table 76: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r) (n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad (2A)$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)\,(n+r-1) + (n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{\left(r+1\right)^{2} \left(r+2\right)^{2} \left(r+3\right)^{2} \left(4+r\right)^{2}}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{\left(r+1\right)^{2} \left(r+2\right)^{2} \left(r+3\right)^{2} \left(4+r\right)^{2} \left(5+r\right)^{2}}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{14400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{576}$
a_5	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{1}{14400}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2 (5+r)^2 (r+6)^2}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{1}{518400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{576}$
a_5	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{1}{14400}$
a_6	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2(r+6)^2}$	$\frac{1}{518400}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2 (5+r)^2 (r+6)^2 (7+r)^2}$$

Which for the root $r = 0$ becomes

$$a_7 = -\frac{1}{25401600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{576}$
a_5	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{1}{14400}$
a_6	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2(r+6)^2}$	$\frac{1}{518400}$
a_7	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2(r+6)^2(7+r)^2}$	$-\frac{1}{25401600}$

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\ &= 1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + \frac{x^6}{518400} - \frac{x^7}{25401600} + O\left(x^8\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n($
b_0	1	1	N/A since b_n starts from 1	N/
b_1	$-\frac{1}{(r+1)^2}$	-1	$\frac{2}{(r+1)^3}$	2
b_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$	$\frac{-4r-6}{(r+1)^3(r+2)^3}$	$-\frac{3}{4}$
b_3	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$	$\frac{6r^2+24r+22}{(r+1)^3(r+2)^3(r+3)^3}$	$\frac{11}{108}$
b_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{576}$	$\frac{-8r^3-60r^2-140r-100}{(r+1)^3(r+2)^3(r+3)^3(4+r)^3}$	$-\frac{}{3}$
b_5	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{1}{14400}$	$\frac{10r^4+120r^3+510r^2+900r+548}{(r+1)^3(r+2)^3(r+3)^3(4+r)^3(5+r)^3}$	$\frac{13}{432}$
b_6	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2(r+6)^2}$	$\frac{1}{518400}$	$\frac{-12r^5-210r^4-1400r^3-4410r^2-6496r-3528}{(r+1)^3(r+2)^3(r+3)^3(4+r)^3(5+r)^3(r+6)^3}$	$-\frac{}{5}$
b_7	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2(r+6)^2(7+r)^2}$	$-\frac{1}{25401600}$	$\frac{14r^6+336r^5+3220r^4+15680r^3+40614r^2+52528r+26136}{(r+1)^3(r+2)^3(r+3)^3(4+r)^3(5+r)^3(r+6)^3(7+r)^3}$	$\frac{}{592}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= \left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + \frac{x^6}{518400} - \frac{x^7}{25401600} + O(x^8)\right)\ln(x) \\
&\quad + 2x - \frac{3x^2}{4} + \frac{11x^3}{108} - \frac{25x^4}{3456} + \frac{137x^5}{432000} - \frac{49x^6}{5184000} + \frac{121x^7}{592704000} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + \frac{x^6}{518400} - \frac{x^7}{25401600} + O(x^8)\right) \\
&\quad + c_2\left(\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + \frac{x^6}{518400} - \frac{x^7}{25401600} + O(x^8)\right)\ln(x)\right. \\
&\quad\left. + 2x - \frac{3x^2}{4} + \frac{11x^3}{108} - \frac{25x^4}{3456} + \frac{137x^5}{432000} - \frac{49x^6}{5184000} + \frac{121x^7}{592704000} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$y = y_h$$

$$
\begin{aligned}
&= c_1\left(1-x+\frac{x^2}{4}-\frac{x^3}{36}+\frac{x^4}{576}-\frac{x^5}{14400}+\frac{x^6}{518400}-\frac{x^7}{25401600}+O\left(x^8\right)\right)\\
&\quad + c_2\left(\left(1-x+\frac{x^2}{4}-\frac{x^3}{36}+\frac{x^4}{576}-\frac{x^5}{14400}+\frac{x^6}{518400}-\frac{x^7}{25401600}+O\left(x^8\right)\right)\ln\left(x\right)\right.\\
&\qquad \left. + 2x-\frac{3x^2}{4}+\frac{11x^3}{108}-\frac{25x^4}{3456}+\frac{137x^5}{432000}-\frac{49x^6}{5184000}+\frac{121x^7}{592704000}+O\left(x^8\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(1-x+\frac{x^2}{4}-\frac{x^3}{36}+\frac{x^4}{576}-\frac{x^5}{14400}+\frac{x^6}{518400}-\frac{x^7}{25401600}+O\left(x^8\right)\right)\\
&\quad + c_2\left(\left(1-x+\frac{x^2}{4}-\frac{x^3}{36}+\frac{x^4}{576}-\frac{x^5}{14400}+\frac{x^6}{518400}-\frac{x^7}{25401600}+O\left(x^8\right)\right)\ln\left(x\right)\right.\\
&\qquad \left. + 2x-\frac{3x^2}{4}+\frac{11x^3}{108}-\frac{25x^4}{3456}+\frac{137x^5}{432000}-\frac{49x^6}{5184000}+\frac{121x^7}{592704000}+O\left(x^8\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(1-x+\frac{x^2}{4}-\frac{x^3}{36}+\frac{x^4}{576}-\frac{x^5}{14400}+\frac{x^6}{518400}-\frac{x^7}{25401600}+O\left(x^8\right)\right)\\
&\quad + c_2\left(\left(1-x+\frac{x^2}{4}-\frac{x^3}{36}+\frac{x^4}{576}-\frac{x^5}{14400}+\frac{x^6}{518400}-\frac{x^7}{25401600}+O\left(x^8\right)\right)\ln\left(x\right)\right.\\
&\qquad \left. + 2x-\frac{3x^2}{4}+\frac{11x^3}{108}-\frac{25x^4}{3456}+\frac{137x^5}{432000}-\frac{49x^6}{5184000}+\frac{121x^7}{592704000}+O\left(x^8\right)\right)
\end{aligned}
$$

Verified OK.

2.30.1 Maple step by step solution

Let's solve

$y''x+y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}+\frac{y}{x}=0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{1}{x}\right]$
  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$$
  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$
  - ○ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- ● Multiply by denominators

  $y''x + y' + y = 0$
- ● Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - ○ Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$
  - ○ Shift index using $k- >k+1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$$
  - ○ Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$
  - ○ Shift index using $k- >k+1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

    Rewrite ODE with series expansions

$$a_0r^2x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)^2+a_k\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2=0$
- Values of r that satisfy the indicial equation

  $r=0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k}{(k+1)^2}$
- Recursion relation for $r=0$

  $a_{k+1}=-\frac{a_k}{(k+1)^2}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+1}=-\frac{a_k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 71

```
Order:=8;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x)=(c_2\ln(x)+c_1)\left(1-x+\frac{1}{4}x^2-\frac{1}{36}x^3+\frac{1}{576}x^4-\frac{1}{14400}x^5+\frac{1}{518400}x^6-\frac{1}{25401600}x^7+\mathrm{O}\left(x^8\right)\right)+\left(2x-\frac{3}{4}x^2+\frac{11}{108}x^3-\frac{25}{3456}x^4+\frac{137}{432000}x^5-\frac{49}{5184000}x^6+\frac{121}{592704000}x^7+\mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 153

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x)\to c_1\left(-\frac{x^7}{25401600}+\frac{x^6}{518400}-\frac{x^5}{14400}+\frac{x^4}{576}-\frac{x^3}{36}+\frac{x^2}{4}-x+1\right)+c_2\left(\frac{121x^7}{592704000}-\frac{49x^6}{5184000}+\frac{137x^5}{432000}-\frac{25x^4}{3456}+\frac{11x^3}{108}-\frac{3x^2}{4}+\left(-\frac{x^7}{25401600}+\frac{x^6}{518400}-\frac{x^5}{14400}+\frac{x^4}{576}-\frac{x^3}{36}+\frac{x^2}{4}-x+1\right)\log(x)+2x\right)$$

2.31 problem 31

Internal problem ID [6614]
Internal file name [`OUTPUT/5862_Sunday_June_05_2022_03_58_41_PM_76390262/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (x-6)\,y' - 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (x-6)\,y' - 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x-6}{x}$$
$$q(x) = -\frac{3}{x}$$

Table 78: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-6}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{3}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (x - 6)\, y' - 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + (x-6) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-6(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-6(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 6(n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 6r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 6r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-7+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-7+r)=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1&=7\\ r_2&=0\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-7+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 7$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x)&=x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x)&=Cy_1(x)\ln(x)+x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x)&=x^{7}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x)&=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x)&=\sum_{n=0}^{\infty} a_n x^{n+7}\\ y_2(x)&=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1)+a_{n-1}(n+r-1)-6a_n(n+r)-3a_{n-1}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-4)}{n^2+2nr+r^2-7n-7r} \tag{4}$$

Which for the root $r=7$ becomes

$$a_n = -\frac{a_{n-1}(n+3)}{n(n+7)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=7$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n=1$, using the above recursive equation gives

$$a_1 = \frac{-r+3}{r^2-5r-6}$$

Which for the root $r=7$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$

For $n=2$, using the above recursive equation gives

$$a_2 = \frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$$

Which for the root $r=7$ becomes

$$a_2 = \frac{5}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(-2+r)(r-3)(-1+r)}{r^6 - 9r^5 - 5r^4 + 165r^3 + 4r^2 - 876r - 720}$$

Which for the root $r = 7$ becomes

$$a_3 = -\frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$
a_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-2+r)\,r(-1+r)}{(r+4)\,(r^6 - 9r^5 - 5r^4 + 165r^3 + 4r^2 - 876r - 720)}$$

Which for the root $r = 7$ becomes

$$a_4 = \frac{7}{1584}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$
a_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{36}$
a_4	$\frac{(-2+r)r(-1+r)}{(r+4)(r^6-9r^5-5r^4+165r^3+4r^2-876r-720)}$	$\frac{7}{1584}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{r(-1+r)}{(r+5)\,(r^5 - 10r^4 + 5r^3 + 160r^2 - 156r - 720)\,(r+4)}$$

Which for the root $r = 7$ becomes

$$a_5 = -\frac{7}{11880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$
a_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{36}$
a_4	$\frac{(-2+r)r(-1+r)}{(r+4)(r^6-9r^5-5r^4+165r^3+4r^2-876r-720)}$	$\frac{7}{1584}$
a_5	$-\frac{r(-1+r)}{(r+5)(r^5-10r^4+5r^3+160r^2-156r-720)(r+4)}$	$-\frac{7}{11880}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r}{(r+6)(r+4)(r-5)(r-6)(r+3)(r-4)(r+5)}$$

Which for the root $r = 7$ becomes

$$a_6 = \frac{7}{102960}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$
a_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{36}$
a_4	$\frac{(-2+r)r(-1+r)}{(r+4)(r^6-9r^5-5r^4+165r^3+4r^2-876r-720)}$	$\frac{7}{1584}$
a_5	$-\frac{r(-1+r)}{(r+5)(r^5-10r^4+5r^3+160r^2-156r-720)(r+4)}$	$-\frac{7}{11880}$
a_6	$\frac{r}{(r+6)(r+4)(r-5)(r-6)(r+3)(r-4)(r+5)}$	$\frac{7}{102960}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)}$$

Which for the root $r = 7$ becomes

$$a_7 = -\frac{1}{144144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$
a_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{36}$
a_4	$\frac{(-2+r)r(-1+r)}{(r+4)(r^6-9r^5-5r^4+165r^3+4r^2-876r-720)}$	$\frac{7}{1584}$
a_5	$-\frac{r(-1+r)}{(r+5)(r^5-10r^4+5r^3+160r^2-156r-720)(r+4)}$	$-\frac{7}{11880}$
a_6	$\frac{r}{(r+6)(r+4)(r-5)(r-6)(r+3)(r-4)(r+5)}$	$\frac{7}{102960}$
a_7	$-\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)}$	$-\frac{1}{144144}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^7\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= x^7\left(1 - \frac{x}{2} + \frac{5x^2}{36} - \frac{x^3}{36} + \frac{7x^4}{1584} - \frac{7x^5}{11880} + \frac{7x^6}{102960} - \frac{x^7}{144144} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 7$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_7(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_7 \\ &= -\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)} &= \lim_{r \to 0} -\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)} \\ &= \frac{1}{100800} \end{aligned}$$

The limit is $\frac{1}{100800}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) - 6(n+r)b_n - 3b_{n-1} = 0 \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_n n(n-1) + b_{n-1}(n-1) - 6nb_n - 3b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n+r-4)}{n^2 + 2nr + r^2 - 7n - 7r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{b_{n-1}(n-4)}{n^2 - 7n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r-3}{r^2 - 5r - 6}$$

Which for the root $r = 0$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(r-3)(-2+r)}{(r^2-5r-6)(r^2-3r-10)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{10}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(-2+r)(r-3)(-1+r)}{(r^2-5r-6)(r^2-3r-10)(r^2-r-12)}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$
b_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{120}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(-2+r)\,r(-1+r)}{(r+4)(r^2-5r-6)(r^2-3r-10)(r^2-r-12)}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$
b_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{120}$
b_4	$\frac{(-2+r)r(-1+r)}{r^7-5r^6-41r^5+145r^4+664r^3-860r^2-4224r-2880}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{r(-1+r)}{(r+5)(r^2-r-12)(r^2-3r-10)(r-6)(r+4)}$$

Which for the root $r = 0$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$
b_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{120}$
b_4	$\frac{(-2+r)r(-1+r)}{r^7-5r^6-41r^5+145r^4+664r^3-860r^2-4224r-2880}$	0
b_5	$-\frac{r(-1+r)}{(r+5)(r-5)(r-6)(r+3)(2+r)(r-4)(r+4)}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{r}{(r+6)(r+4)(r-6)(r-5)(r^2-r-12)(r+5)}$$

Which for the root $r = 0$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$
b_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{120}$
b_4	$\frac{(-2+r)r(-1+r)}{r^7-5r^6-41r^5+145r^4+664r^3-860r^2-4224r-2880}$	0
b_5	$-\frac{r(-1+r)}{(r+5)(r-5)(r-6)(r+3)(2+r)(r-4)(r+4)}$	0
b_6	$\frac{r}{(r+4)(r^2-r-12)(r^2-36)(r^2-25)}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{1}{(r+6)(r+4)(r-5)(r-6)(r-4)(r+5)(r+7)}$$

Which for the root $r = 0$ becomes

$$b_7 = \frac{1}{100800}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$
b_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{120}$
b_4	$\frac{(-2+r)r(-1+r)}{r^7-5r^6-41r^5+145r^4+664r^3-860r^2-4224r-2880}$	0
b_5	$-\frac{r(-1+r)}{(r+5)(r-5)(r-6)(r+3)(2+r)(r-4)(r+4)}$	0
b_6	$\frac{r}{(r+4)(r^2-r-12)(r^2-36)(r^2-25)}$	0
b_7	$-\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)}$	$\frac{1}{100800}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\ &= 1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \frac{x^7}{100800} + O(x^8)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^7 \left(1 - \frac{x}{2} + \frac{5x^2}{36} - \frac{x^3}{36} + \frac{7x^4}{1584} - \frac{7x^5}{11880} + \frac{7x^6}{102960} - \frac{x^7}{144144} + O(x^8)\right) \\ &\quad + c_2 \left(1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \frac{x^7}{100800} + O(x^8)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1 x^7 \left(1 - \frac{x}{2} + \frac{5x^2}{36} - \frac{x^3}{36} + \frac{7x^4}{1584} - \frac{7x^5}{11880} + \frac{7x^6}{102960} - \frac{x^7}{144144} + O(x^8)\right) \\ &\quad + c_2 \left(1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \frac{x^7}{100800} + O(x^8)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y = c_1 x^7 &\left(1 - \frac{x}{2} + \frac{5x^2}{36} - \frac{x^3}{36} + \frac{7x^4}{1584} - \frac{7x^5}{11880} + \frac{7x^6}{102960} - \frac{x^7}{144144} + O(x^8)\right) \\ &+ c_2 \left(1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \frac{x^7}{100800} + O(x^8)\right)\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}y = c_1 x^7 &\left(1 - \frac{x}{2} + \frac{5x^2}{36} - \frac{x^3}{36} + \frac{7x^4}{1584} - \frac{7x^5}{11880} + \frac{7x^6}{102960} - \frac{x^7}{144144} + O(x^8)\right) \\ &+ c_2 \left(1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \frac{x^7}{100800} + O(x^8)\right)\end{aligned}$$

Verified OK.

2.31.1 Maple step by step solution

Let's solve

$y''x + (x-6)\,y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y}{x} - \frac{(x-6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-6)y'}{x} - \frac{3y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{x-6}{x}, P_3(x) = -\frac{3}{x}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -6$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ○ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + (x-6)\,y' - 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ○ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ○ Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- ○ Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-7+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-6+r) + a_k(k+r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-7+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 7\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-6+r) + a_k(k+r-3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-3)}{(k+1+r)(k-6+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = -\frac{a_k(k-3)}{(k+1)(k-6)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0}{2}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{5}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{10}$

- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{12}$

- Express in terms of a_0

  $a_3 = -\frac{a_0}{120}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right)$
- Recursion relation for $r = 7$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}$
- Solution for $r = 7$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), b_{k+1} = -\frac{b_k(k+4)}{(k+8)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 46

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(x-6)*diff(y(x),x)-3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^7 \left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1584}x^4 - \frac{7}{11880}x^5 + \frac{7}{102960}x^6 - \frac{1}{144144}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(3628800 - 1814400x + 362880x^2 - 30240x^3 + 36x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.107 (sec). Leaf size: 77

```
AsymptoticDSolveValue[x*y''[x]+(x-6)*y'[x]-3*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{x^3}{120}+\frac{x^2}{10}-\frac{x}{2}+1\right)+c_2\left(\frac{7x^{13}}{102960}-\frac{7x^{12}}{11880}+\frac{7x^{11}}{1584}-\frac{x^{10}}{36}+\frac{5x^9}{36}-\frac{x^8}{2}+x^7\right)$$

2.32 problem 32

Internal problem ID [6615]
Internal file name [`OUTPUT/5863_Sunday_June_05_2022_03_58_46_PM_61854185/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _exact, _linear, _homogeneous]]`

$$\boxed{x(x-1)\,y''+3y'-2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2-x\right)y''-2y+3y'=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{3}{x\,(x-1)}\\ q(x)&=-\frac{2}{x\,(x-1)}\end{aligned}$$

Table 80: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{2}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x-1)\, y'' + 3y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{gathered} x(x-1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ +3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{gathered} \tag{1}$$

Which simplifies to

$$\begin{aligned}\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r-1} a_n(n+r)(n+r-1)\right) \\ +\left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)=0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)=\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)=\sum_{n=1}^{\infty}\left(-2 a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned}\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-x^{n+r-1} a_n(n+r)(n+r-1)\right) \\ +\left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-2 a_{n-1} x^{n+r-1}\right)=0\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-x^{n+r-1} a_n(n+r)(n+r-1)+3(n+r) a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$-x^{-1+r} a_0 r(-1+r)+3 r a_0 x^{-1+r}=0$$

Or

$$\left(-x^{-1+r} r(-1+r)+3 r\, x^{-1+r}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(4-r)=0$$

Since the above is true for all x then the indicial equation becomes

$$-r(-4+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 4 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(4-r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^4\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+4} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) - a_n(n+r)(n+r-1) + 3a_n(n+r) - 2a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r-3)\, a_{n-1}}{n-4+r} \tag{4}$$

Which for the root $r = 4$ becomes

$$a_n = \frac{(n+1)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_3 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_4 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{2+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_5 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5
a_5	$\frac{2+r}{r-3}$	6

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{3+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_6 = 7$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5
a_5	$\frac{2+r}{r-3}$	6
a_6	$\frac{3+r}{r-3}$	7

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{4+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_7 = 8$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5
a_5	$\frac{2+r}{r-3}$	6
a_6	$\frac{3+r}{r-3}$	7
a_7	$\frac{4+r}{r-3}$	8

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^4\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots\right) \\ &= x^4\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{1+r}{r-3} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{1+r}{r-3} &= \lim_{r\to 0} \frac{1+r}{r-3} \\ &= -\frac{1}{3} \end{aligned}$$

The limit is $-\frac{1}{3}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_{n-1}(n+r-1)(n+r-2) - b_n(n+r)(n+r-1) + 3(n+r)b_n - 2b_{n-1} = 0 \quad (4)$$

Which for for the root $r = 0$ becomes

$$b_{n-1}(n-1)(n-2) - b_n n(n-1) + 3nb_n - 2b_{n-1} = 0 \quad (4A)$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{(n+r-3)\, b_{n-1}}{n-4+r} \quad (5)$$

Which for the root $r = 0$ becomes

$$b_n = \frac{(n-3)\, b_{n-1}}{n-4} \quad (6)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-2+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-1+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{2+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$
b_5	$\frac{2+r}{r-3}$	$-\frac{2}{3}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{3+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_6 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$
b_5	$\frac{2+r}{r-3}$	$-\frac{2}{3}$
b_6	$\frac{3+r}{r-3}$	-1

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{4+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_7 = -\frac{4}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$
b_5	$\frac{2+r}{r-3}$	$-\frac{2}{3}$
b_6	$\frac{3+r}{r-3}$	-1
b_7	$\frac{4+r}{r-3}$	$-\frac{4}{3}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\ &= 1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} - x^6 - \frac{4x^7}{3} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + O(x^8)\right) \\ &\quad + c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} - x^6 - \frac{4x^7}{3} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + O(x^8)\right) \\ &\quad + c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} - x^6 - \frac{4x^7}{3} + O(x^8)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + O(x^8)\right) \\ &\quad + c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} - x^6 - \frac{4x^7}{3} + O(x^8)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + O(x^8)\right) \\ &\quad + c_2 \left(1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} - x^6 - \frac{4x^7}{3} + O(x^8)\right) \end{aligned}$$

Verified OK.

2.32.1 Maple step by step solution

Let's solve

$x(x-1)\, y'' + 3y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x(x-1)} - \frac{3y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{3y'}{x(x-1)} - \frac{2y}{x(x-1)} = 0$$

- ☐ Check to see if x_0 is a regular singular point
  - ○ Define functions

    $$\left[P_2(x) = \frac{3}{x(x-1)}, P_3(x) = -\frac{2}{x(x-1)}\right]$$

  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -3$$

  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

  - ○ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $$x_0 = 0$$

- ● Multiply by denominators

  $$x(x-1)\, y'' + 3y' - 2y = 0$$

- ● Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ○ Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - ○ Shift index using $k- >k+1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

  - ○ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - ○ Shift index using $k- >k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

    Rewrite ODE with series expansions

$$-a_0 r(-4+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-3+r) + a_k(k+1+r)\,(k+r-2)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 4\}$

- Each term in the series must be 0, giving the recursion relation

  $((-k-r+3)\, a_{k+1} + a_k(k+r-2))\,(k+1+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)}{k-3+r}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{k-3}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{2a_0}{3}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{a_1}{2}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{3}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + \frac{2}{3}x + \frac{1}{3}x^2\right)$

- Recursion relation for $r = 4$

  $a_{k+1} = \frac{a_k(k+2)}{k+1}$

- Solution for $r = 4$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1} = \frac{a_k(k+2)}{k+1}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 + \frac{2}{3}x + \frac{1}{3}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+4}\right), b_{k+1} = \frac{b_k(k+2)}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 50

```
Order:=8;
dsolve(x*(x-1)*diff(y(x),x$2)+3*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$\begin{aligned}y(x) = {} & c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + \mathrm{O}\left(x^8\right)\right) \\ & + c_2\left(-144 - 96x - 48x^2 + 48x^4 + 96x^5 + 144x^6 + 192x^7 + \mathrm{O}\left(x^8\right)\right)\end{aligned}$$

✓ Solution by Mathematica

Time used: 0.376 (sec). Leaf size: 77

```
AsymptoticDSolveValue[x*(x-1)*y''[x]+3*y'[x]-2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(-x^6 - \frac{2x^5}{3} - \frac{x^4}{3} + \frac{x^2}{3} + \frac{2x}{3} + 1\right) + c_2\left(7x^{10} + 6x^9 + 5x^8 + 4x^7 + 3x^6 + 2x^5 + x^4\right)$$

2.33 problem 33(b)

Internal problem ID [6616]
Internal file name [`OUTPUT/5864_Sunday_June_05_2022_03_58_50_PM_73551297/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 33(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \frac{2y'}{t} + \lambda y = 0}$$

With the expansion point for the power series method at $t = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + \frac{2y'}{t} + \lambda y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(t)y' + q(t)y = 0$$

Where

$$p(t) = \frac{2}{t}$$
$$q(t) = \lambda$$

Table 82: Table $p(t), q(t)$ singularites.

$p(t) = \frac{2}{t}$	
singularity	type
$t = 0$	"regular"

$q(t) = \lambda$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $t = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$\lambda y t + y'' t + 2y' = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n t^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}$$

Substituting the above back into the ode gives

$$\lambda \left(\sum_{n=0}^{\infty} a_n t^{n+r}\right) t + \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}\right) t + 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} t^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n t^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} t^{1+n+r} \lambda a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of t be $n+r-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} t^{1+n+r}\lambda a_n = \sum_{n=2}^{\infty} \lambda a_{n-2} t^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} t^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n t^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} \lambda a_{n-2} t^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$t^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r)\, a_n t^{n+r-1} = 0$$

When $n=0$ the above becomes

$$t^{-1+r} a_0 r(-1+r) + 2r a_0 t^{-1+r} = 0$$

Or

$$\left(t^{-1+r} r(-1+r) + 2r\, t^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, t^{-1+r}(1+r) = 0$$

Since the above is true for all t then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, t^{-1+r}(1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(t) &= t^{r_1}\left(\sum_{n=0}^{\infty} a_n t^n\right) \\
y_2(t) &= C y_1(t) \ln(t) + t^{r_2}\left(\sum_{n=0}^{\infty} b_n t^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(t) &= \sum_{n=0}^{\infty} a_n t^n \\
y_2(t) &= C y_1(t) \ln(t) + \frac{\sum_{n=0}^{\infty} b_n t^n}{t}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(t) &= \sum_{n=0}^{\infty} a_n t^n \\
y_2(t) &= C y_1(t) \ln(t) + \left(\sum_{n=0}^{\infty} b_n t^{n-1}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_n(n+r) + \lambda a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{\lambda a_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{\lambda a_{n-2}}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{\lambda}{r^2 + 5r + 6}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{\lambda}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{\lambda^2}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{\lambda^2}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{6}$
a_3	0	0
a_4	$\frac{\lambda^2}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{\lambda^2}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{6}$
a_3	0	0
a_4	$\frac{\lambda^2}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{\lambda^2}{120}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{\lambda^3}{(r^2+5r+6)(r^2+9r+20)(r^2+13r+42)}$$

Which for the root $r = 0$ becomes

$$a_6 = -\frac{\lambda^3}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{6}$
a_3	0	0
a_4	$\frac{\lambda^2}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{\lambda^2}{120}$
a_5	0	0
a_6	$-\frac{\lambda^3}{(r^2+5r+6)(r^2+9r+20)(r^2+13r+42)}$	$-\frac{\lambda^3}{5040}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{6}$
a_3	0	0
a_4	$\frac{\lambda^2}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{\lambda^2}{120}$
a_5	0	0
a_6	$-\frac{\lambda^3}{(r^2+5r+6)(r^2+9r+20)(r^2+13r+42)}$	$-\frac{\lambda^3}{5040}$
a_7	0	0

Using the above table, then the solution $y_1(t)$ is

$$\begin{aligned} y_1(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_8 t^8 \dots \\ &= 1 - \frac{\lambda t^2}{6} + \frac{\lambda^2 t^4}{120} - \frac{\lambda^3 t^6}{5040} + O(t^8) \end{aligned}$$

Now the second solution $y_2(t)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} 0 &= \lim_{r\rightarrow -1} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(t) &= \sum_{n=0}^{\infty} b_n t^{n+r} \\ &= \sum_{n=0}^{\infty} b_n t^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 2(n+r)\, b_n + \lambda b_{n-2} = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + 2(n-1)\, b_n + \lambda b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{\lambda b_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{\lambda b_{n-2}}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{\lambda}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{\lambda}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{\lambda^2}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{\lambda^2}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{2}$
b_3	0	0
b_4	$\frac{\lambda^2}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{\lambda^2}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{2}$
b_3	0	0
b_4	$\frac{\lambda^2}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{\lambda^2}{24}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = -\frac{\lambda^3}{(r^2+5r+6)(r^2+9r+20)(r^2+13r+42)}$$

Which for the root $r = -1$ becomes

$$b_6 = -\frac{\lambda^3}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{2}$
b_3	0	0
b_4	$\frac{\lambda^2}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{\lambda^2}{24}$
b_5	0	0
b_6	$-\frac{\lambda^3}{(r^2+5r+6)(r^2+9r+20)(r^2+13r+42)}$	$-\frac{\lambda^3}{720}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{\lambda}{r^2+5r+6}$	$-\frac{\lambda}{2}$
b_3	0	0
b_4	$\frac{\lambda^2}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{\lambda^2}{24}$
b_5	0	0
b_6	$-\frac{\lambda^3}{(r^2+5r+6)(r^2+9r+20)(r^2+13r+42)}$	$-\frac{\lambda^3}{720}$
b_7	0	0

Using the above table, then the solution $y_2(t)$ is

$$\begin{aligned}
y_2(t) &= 1\left(b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 + b_7 t^7 + b_8 t^8 \dots\right) \\
&= \frac{1 - \frac{\lambda t^2}{2} + \frac{\lambda^2 t^4}{24} - \frac{\lambda^3 t^6}{720} + O(t^8)}{t}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(t) &= c_1 y_1(t) + c_2 y_2(t) \\
&= c_1\left(1 - \frac{\lambda t^2}{6} + \frac{\lambda^2 t^4}{120} - \frac{\lambda^3 t^6}{5040} + O(t^8)\right) + \frac{c_2\left(1 - \frac{\lambda t^2}{2} + \frac{\lambda^2 t^4}{24} - \frac{\lambda^3 t^6}{720} + O(t^8)\right)}{t}
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-\frac{\lambda\, t^2}{6}+\frac{\lambda^2 t^4}{120}-\frac{\lambda^3 t^6}{5040}+O\left(t^8\right)\right)+\frac{c_2\left(1-\frac{\lambda\, t^2}{2}+\frac{\lambda^2 t^4}{24}-\frac{\lambda^3 t^6}{720}+O(t^8)\right)}{t}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1-\frac{\lambda\, t^2}{6}+\frac{\lambda^2 t^4}{120}-\frac{\lambda^3 t^6}{5040}+O\left(t^8\right)\right)+\frac{c_2\left(1-\frac{\lambda\, t^2}{2}+\frac{\lambda^2 t^4}{24}-\frac{\lambda^3 t^6}{720}+O(t^8)\right)}{t} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1-\frac{\lambda\, t^2}{6}+\frac{\lambda^2 t^4}{120}-\frac{\lambda^3 t^6}{5040}+O\left(t^8\right)\right)+\frac{c_2\left(1-\frac{\lambda\, t^2}{2}+\frac{\lambda^2 t^4}{24}-\frac{\lambda^3 t^6}{720}+O(t^8)\right)}{t}
$$

Verified OK.

2.33.1 Maple step by step solution

Let's solve

$\lambda y t + y'' t + 2y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{t} - \lambda y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{t} + \lambda y = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(t) = \frac{2}{t}, P_3(t) = \lambda\right]$

  ○ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = 2$

  ○ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

$$\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 0$$

- ○ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- ● Multiply by denominators

  $\lambda y t + y'' t + 2y' = 0$

- ● Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $t \cdot y$ to series expansion

    $t \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+1}$
  - ○ Shift index using $k- >k-1$

    $t \cdot y = \sum_{k=1}^{\infty} a_{k-1} t^{k+r}$
  - ○ Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1}$
  - ○ Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, t^{k+r}$
  - ○ Convert $t \cdot y''$ to series expansion

    $t \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r-1}$
  - ○ Shift index using $k- >k+1$

    $t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, t^{-1+r} + a_1 (1+r)\,(2+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + \lambda a_{k-1}\right) t^{k+r}\right) = 0$$

- ● $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+r+1)(k+2+r) + \lambda a_{k-1} = 0$

- Shift index using $k- >k+1$

$a_{k+2}(k+2+r)(k+3+r) + \lambda a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{\lambda a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

$a_{k+2} = -\frac{\lambda a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

$\left[y = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+2} = -\frac{\lambda a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

$a_{k+2} = -\frac{\lambda a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

$\left[y = \sum_{k=0}^{\infty} a_k t^{k}, a_{k+2} = -\frac{\lambda a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k}\right), a_{k+2} = -\frac{\lambda a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{\lambda b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 56

```
Order:=8;
dsolve(diff(y(t),t$2)+2/t*diff(y(t),t)+lambda*y(t)=0,y(t),type='series',t=0);
```

$$y(t) = c_1\left(1-\frac{1}{6}\lambda\, t^2+\frac{1}{120}\lambda^2 t^4-\frac{1}{5040}\lambda^3 t^6+\mathrm{O}\left(t^8\right)\right) + \frac{c_2\left(1-\frac{1}{2}\lambda\, t^2+\frac{1}{24}\lambda^2 t^4-\frac{1}{720}\lambda^3 t^6+\mathrm{O}\left(t^8\right)\right)}{t}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 70

```
AsymptoticDSolveValue[y''[t]+2/t*y'[t]+\[Lambda]*y[t]==0,y[t],{t,0,7}]
```

$$y(t) \to c_1\left(-\frac{1}{720}\lambda^3 t^5+\frac{\lambda^2 t^3}{24}-\frac{\lambda t}{2}+\frac{1}{t}\right)+c_2\left(-\frac{\lambda^3 t^6}{5040}+\frac{\lambda^2 t^4}{120}-\frac{\lambda t^2}{6}+1\right)$$

2.34 problem 36 (a)

Internal problem ID [6617]
Internal file name [`OUTPUT/5865_Sunday_June_05_2022_03_58_53_PM_60445809/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 36 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^3y'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{1}{x^3} \end{aligned}$$

Table 84: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x^3}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✗ Solution by Maple

```
Order:=8;
dsolve(x^3*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 294

```
AsymptoticDSolveValue[x^3*y''[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 e^{-\frac{2i}{\sqrt{x}}} x^{3/4} \left(-\frac{11100458801337530625 i x^{13/2}}{4611686018427387904} + \frac{1327867167401775 i x^{11/2}}{4503599627370496} - \frac{468131288625 i x^{9/2}}{8796093022208} + \frac{66891825 i x^{7/2}}{4294967296} - \frac{72765 i x^{5/2}}{8388608} + \frac{105 i x^{3/2}}{8192} + \frac{1149690375852815671875 x^7}{147573952589676412928} - \frac{2323767542953106 25 x^6}{288230376151711744} + \frac{33424574007825 x^5}{281474976710656} - \frac{14783093325 x^4}{549755813888} + \frac{2837835 x^3}{268435456} - \frac{4725 x^2}{524288} + \frac{15 x}{512} - \frac{3 i \sqrt{x}}{16} + 1 \right) + c_2 e^{\frac{2i}{\sqrt{x}}} x^{3/4} \left(\frac{11100458801337530625 i x^{13/2}}{4611686018427387904} - \frac{1327867167401775 i x^{11/2}}{4503599627370496} + \frac{468131288625 i x^{9/2}}{8796093022208} - \frac{668}{42}$$

2.35 problem 36(b)

Internal problem ID [6618]
Internal file name [`OUTPUT/5866_Sunday_June_05_2022_03_58_55_PM_35748752/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.3 SOLUTIONS ABOUT SINGULAR POINTS. EXERCISES 6.3. Page 255
Problem number: 36(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' + (3x-1)\,y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (3x-1)\,y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3x-1}{x^2}$$
$$q(x) = \frac{1}{x^2}$$

Table 85: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x-1}{x^2}$	
singularity	type
$x = 0$	"irregular"

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✗ Solution by Maple

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+(3*x-1)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 53

```
AsymptoticDSolveValue[x^2*y''[x]+(3*x-1)*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(5040x^7 + 720x^6 + 120x^5 + 24x^4 + 6x^3 + 2x^2 + x + 1\right) + \frac{c_2 e^{-1/x}}{x}$$

3 CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267

3.1 problem 1

Internal problem ID [6619]
Internal file name [`OUTPUT/5867_Sunday_June_05_2022_03_58_57_PM_34084305/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0}$$

3.1.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= 1\\ n &= -\frac{1}{3}\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, x\right) + c_2 \operatorname{BesselY}\left(-\frac{1}{3}, x\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, x\right) + c_2 \operatorname{BesselY}\left(-\frac{1}{3}, x\right) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, x\right) + c_2 \operatorname{BesselY}\left(-\frac{1}{3}, x\right)$$

Verified OK.

3.1.2 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2-1)y}{9x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(9x^2-1)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{9x^2-1}{9x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{9}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- ● Multiply by denominators

  $9x^2y'' + 9xy' + (9x^2 - 1)\, y = 0$

- ● Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $$a_0(1 + 3r)\,(-1 + 3r)\, x^r + a_1(4 + 3r)\,(2 + 3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k + 3r + 1)\,(3k + 3r - 1) + 9a_{k-}\right.\right.$$

- ● $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+3r)(-1+3r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{-\frac{1}{3},\frac{1}{3}\right\}$$

- Each term must be 0

$$a_1(4+3r)(2+3r)=0$$

- Solve for the dependent coefficient(s)

$$a_1=0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(3k+3r+1)(3k+3r-1)+9a_{k-2}=0$$

- Shift index using $k->k+2$

$$a_{k+2}(3k+7+3r)(3k+5+3r)+9a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{9a_k}{(3k+7+3r)(3k+5+3r)}$$

- Recursion relation for $r=-\frac{1}{3}$

$$a_{k+2}=-\frac{9a_k}{(3k+6)(3k+4)}$$

- Solution for $r=-\frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{3}},a_{k+2}=-\frac{9a_k}{(3k+6)(3k+4)},a_1=0\right]$$

- Recursion relation for $r=\frac{1}{3}$

$$a_{k+2}=-\frac{9a_k}{(3k+8)(3k+6)}$$

- Solution for $r=\frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{3}},a_{k+2}=-\frac{9a_k}{(3k+8)(3k+6)},a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{3}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right),a_{k+2}=-\frac{9a_k}{(3k+6)(3k+4)},a_1=0,b_{k+2}=-\frac{9b_k}{(3k+8)(3k+6)},b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/9)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}\left(\frac{1}{3}, x\right) + c_2 \operatorname{BesselY}\left(\frac{1}{3}, x\right)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/9)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(\frac{1}{3}, x\right) + c_2 \operatorname{BesselY}\left(\frac{1}{3}, x\right)$$

3.2 problem 2

Internal problem ID [6620]
Internal file name [`OUTPUT/5868_Sunday_June_05_2022_03_59_00_PM_89193964/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Bessel]`

$$\boxed{x^2y'' + xy' + (x^2 - 1)\, y = 0}$$

3.2.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + (x^2 - 1)\, y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= 1\\ n &= -1\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = -c_1 \operatorname{BesselJ}(1, x) - c_2 \operatorname{BesselY}(1, x)$$

<u>Summary</u>
The solution(s) found are the following

$$y = -c_1 \operatorname{BesselJ}(1, x) - c_2 \operatorname{BesselY}(1, x) \tag{1}$$

<u>Verification of solutions</u>

$$y = -c_1 \operatorname{BesselJ}(1, x) - c_2 \operatorname{BesselY}(1, x)$$

Verified OK.

3.2.2 Maple step by step solution

Let's solve

$x^2y'' + xy' + (x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-1)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-1)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -1$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x y' + (x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- ○ Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+r)\, x^r + a_1(2+r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)\,(k+r-1) + a_{k-2})\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term must be 0

  $a_1(2+r)\,r=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-1)+a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+3+r)\,(k+r+1)+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+3+r)(k+r+1)}$
- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_k}{(k+2)k}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+2)k}, a_1=0\right]$
- Recursion relation for $r=1$

  $a_{k+2}=-\frac{a_k}{(k+4)(k+2)}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{a_k}{(k+4)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2}=-\frac{a_k}{(k+2)k}, a_1=0, b_{k+2}=-\frac{b_k}{(k+4)(k+2)}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}(1, x) + c_2 \operatorname{BesselY}(1, x)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}(1, x) + c_2 \operatorname{BesselY}(1, x)$$

3.3 problem 3

Internal problem ID [6621]
Internal file name [`OUTPUT/5869_Sunday_June_05_2022_03_59_04_PM_5400465/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4xy' + (4x^2 - 25)\, y = 0}$$

3.3.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= 1\\ n &= -\frac{5}{2}\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = -\frac{c_1\sqrt{2}\left(\cos\left(x\right)x^2 - 3\sin\left(x\right)x - 3\cos\left(x\right)\right)}{\sqrt{\pi}\, x^{\frac{5}{2}}} - \frac{c_2\sqrt{2}\left(x^2\sin\left(x\right) + 3\cos\left(x\right)x - 3\sin\left(x\right)\right)}{\sqrt{\pi}\, x^{\frac{5}{2}}}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &-\frac{c_1\sqrt{2}\left(\cos\left(x\right)x^2 - 3\sin\left(x\right)x - 3\cos\left(x\right)\right)}{\sqrt{\pi}\, x^{\frac{5}{2}}}\\ &- \frac{c_2\sqrt{2}\left(x^2\sin\left(x\right) + 3\cos\left(x\right)x - 3\sin\left(x\right)\right)}{\sqrt{\pi}\, x^{\frac{5}{2}}}\end{aligned} \tag{1}$$

Verification of solutions

$$y = -\frac{c_1\sqrt{2}\left(\cos\left(x\right)x^2 - 3\sin\left(x\right)x - 3\cos\left(x\right)\right)}{\sqrt{\pi}\, x^{\frac{5}{2}}} - \frac{c_2\sqrt{2}\left(x^2\sin\left(x\right) + 3\cos\left(x\right)x - 3\sin\left(x\right)\right)}{\sqrt{\pi}\, x^{\frac{5}{2}}}$$

Verified OK.

3.3.2 Solving using Kovacic algorithm

Writing the ode as

$$4x^2y'' + 4xy' + \left(4x^2 - 25\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2\\ B &= 4x\\ C &= 4x^2 - 25\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 88: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty}\\ &= -\frac{2}{x} + (-)\,(i)\\ &= -\frac{2}{x} - i\\ &= -\frac{2}{x} - i\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2+6}{x^2}\right)\right) &= 0\\ \frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3, a_1 = -3i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 3ix - 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx}\\ &= \left(x^2 - 3ix - 3\right) \mathrm{e}^{\int \left(-\frac{2}{x} - i\right) dx}\\ &= \left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix - 2\ln(x)}\\ &= \frac{\left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix}}{x^2}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{4x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}\right) + c_2\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2\mathrm{e}^{ix}(ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2\mathrm{e}^{ix}(ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}}$$

Verified OK.

3.3.3 Maple step by step solution

Let's solve

$4x^2y'' + 4xy' + (4x^2 - 25)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-25)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-25)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-25}{4x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{25}{4}$

  ○ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 25)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(5+2r)\,(-5+2r)\,x^r + a_1(7+2r)\,(-3+2r)\,x^{1+r} + \Bigg(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-5) + 4a$$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(5+2r)\,(-5+2r) = 0$$

- • Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$$

- • Each term must be 0

  $$a_1(7+2r)\,(-3+2r) = 0$$

- • Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- • Each term in the series must be 0, giving the recursion relation

  $$a_k(2k+2r+5)\,(2k+2r-5) + 4a_{k-2} = 0$$

- • Shift index using $k- >k+2$

  $$a_{k+2}(2k+9+2r)\,(2k-1+2r) + 4a_k = 0$$

- • Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$$

- Recursion relation for $r=-\frac{5}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}$

- Solution for $r=-\frac{5}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0\right]$

- Recursion relation for $r=\frac{5}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}$

- Solution for $r=\frac{5}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0, b_{k+2}=-\frac{4b_k}{(2k+14)(2k+4)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 47

```
dsolve(4*x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2-25)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{-3c_2\left(ix-\frac{1}{3}x^2+1\right)\mathrm{e}^{-ix}+3\,\mathrm{e}^{ix}\left(ix+\frac{1}{3}x^2-1\right)c_1}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 59

```
DSolve[4*x^2*y''[x]+4*x*y'[x]+(4*x^2-25)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}((-c_2x^2+3c_1x+3c_2)\cos(x)+(c_1(x^2-3)+3c_2x)\sin(x))}{x^{5/2}}$$

3.4 problem 4

Internal problem ID [6622]
Internal file name [`OUTPUT/5870_Sunday_June_05_2022_03_59_07_PM_68484540/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{16x^2y'' + 16xy' + \left(16x^2 - 1\right) y = 0}$$

3.4.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{16}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= 1\\ n &= -\frac{1}{4}\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(-\frac{1}{4}, x\right) + c_2 \operatorname{BesselY}\left(-\frac{1}{4}, x\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}\left(-\frac{1}{4}, x\right) + c_2 \operatorname{BesselY}\left(-\frac{1}{4}, x\right) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}\left(-\frac{1}{4}, x\right) + c_2 \operatorname{BesselY}\left(-\frac{1}{4}, x\right)$$

Verified OK.

3.4.2 Maple step by step solution

Let's solve

$16x^2y'' + 16xy' + (16x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(16x^2-1)y}{16x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(16x^2-1)y}{16x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{16x^2-1}{16x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$
- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{1}{16}$
- ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- ● Multiply by denominators

  $16x^2y''+16xy'+\left(16x^2-1\right)y=0$
- ● Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
  - ○ Convert $x^2\cdot y''$ to series expansion

    $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+4r)\,(-1+4r)\,x^r+a_1(5+4r)\,(3+4r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(4k+4r+1)\,(4k+4r-1)+16a_k\right.\right.$
- ● $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+4r)(-1+4r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{4}, \frac{1}{4}\right\}$
- Each term must be 0

  $a_1(5+4r)(3+4r)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k+4r+1)(4k+4r-1)+16a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(4k+9+4r)(4k+7+4r)+16a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{16a_k}{(4k+9+4r)(4k+7+4r)}$
- Recursion relation for $r=-\frac{1}{4}$

  $a_{k+2}=-\frac{16a_k}{(4k+8)(4k+6)}$
- Solution for $r=-\frac{1}{4}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2}=-\frac{16a_k}{(4k+8)(4k+6)}, a_1=0\right]$
- Recursion relation for $r=\frac{1}{4}$

  $a_{k+2}=-\frac{16a_k}{(4k+10)(4k+8)}$
- Solution for $r=\frac{1}{4}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2}=-\frac{16a_k}{(4k+10)(4k+8)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2}=-\frac{16a_k}{(4k+8)(4k+6)}, a_1=0, b_{k+2}=-\frac{16b_k}{(4k+10)(4k+8)}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(16*x^2*diff(y(x),x$2)+16*x*diff(y(x),x)+(16*x^2-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}\left(\frac{1}{4}, x\right) + c_2 \operatorname{BesselY}\left(\frac{1}{4}, x\right)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 22

```
DSolve[16*x^2*y''[x]+16*x*y'[x]+(16*x^2-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(\frac{1}{4}, x\right) + c_2 \operatorname{BesselY}\left(\frac{1}{4}, x\right)$$

3.5 problem 5

Internal problem ID [6623]
Internal file name [`OUTPUT/5871_Sunday_June_05_2022_03_59_11_PM_93638397/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + y' + xy = 0}$$

3.5.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + yx^2 = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= 1\\ n &= 0\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}(0, x) + c_2 \operatorname{BesselY}(0, x)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}(0, x) + c_2 \operatorname{BesselY}(0, x) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}(0, x) + c_2 \operatorname{BesselY}(0, x)$$

Verified OK.

3.5.2 Maple step by step solution

Let's solve

$y''x + y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = 1\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- ● Multiply by denominators

  $$y''x + y' + xy = 0$$

- ● Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - ○ Shift index using $k- >k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - ○ Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - ○ Shift index using $k- >k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

  - ○ Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - ○ Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + a_1 (1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1} (k+r+1)^2 + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1(1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}(0, x) + c_2 \operatorname{BesselY}(0, x)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
DSolve[x*y''[x]+y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}(0, x) + c_2 \operatorname{BesselY}(0, x)$$

3.6 problem 6

Internal problem ID [6624]
Internal file name [`OUTPUT/5872_Sunday_June_05_2022_03_59_14_PM_42879518/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Bessel]`

$$\boxed{xy'' + y' + \left(x - \frac{4}{x}\right) y = 0}$$

3.6.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + (x^2 - 4) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + (-n^2 + x^2) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha) xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= 1\\ n &= -2\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}(2, x) + c_2 \operatorname{BesselY}(2, x)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}(2, x) + c_2 \operatorname{BesselY}(2, x) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}(2, x) + c_2 \operatorname{BesselY}(2, x)$$

Verified OK.

3.6.2 Maple step by step solution

Let's solve

$x^2y'' + xy' + (x^2-4)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-4)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-4)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-4}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \bigg|_{x=0} = 1$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x)) \bigg|_{x=0} = -4$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- ● Multiply by denominators

  $$x^2y'' + xy' + (x^2 - 4)\, y = 0$$

- ● Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2 + r)\,(-2 + r)\, x^r + a_1(3 + r)\,(-1 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k + r + 2)\,(k + r - 2) + a_{k-2})\, x^{k+r}\right)$$

- ● $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2 + r)\,(-2 + r) = 0$$

- ● Values of r that satisfy the indicial equation

  $$r \in \{-2, 2\}$$

- Each term must be 0

  $a_1(3+r)(-1+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-2)+a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)(k+r)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+4+r)(k+r)}$

- Recursion relation for $r=-2$

  $a_{k+2}=-\frac{a_k}{(k+2)(k-2)}$

- Series not valid for $r=-2$, division by 0 in the recursion relation at $k=2$

  $a_{k+2}=-\frac{a_k}{(k+2)(k-2)}$

- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+6)(k+2)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+6)(k+2)}, a_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(x*diff(y(x),x),x)+(x-4/x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\operatorname{BesselJ}(0, x)\, c_1 x - \operatorname{BesselY}(0, x)\, c_2 x + 2c_1 \operatorname{BesselJ}(1, x) + 2c_2 \operatorname{BesselY}(1, x)}{x}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 18

```
DSolve[D[x*y'[x],x]+(x-4/x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}(2, x) + c_2 \operatorname{BesselY}(2, x)$$

3.7 problem 7

Internal problem ID [6625]
Internal file name [`OUTPUT/5873_Sunday_June_05_2022_03_59_18_PM_75210480/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(9x^2 - 4\right) y = 0}$$

3.7.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + \left(9x^2 - 4\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= 3\\ n &= -2\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}(2, 3x) + c_2 \operatorname{BesselY}(2, 3x)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}(2, 3x) + c_2 \operatorname{BesselY}(2, 3x) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}(2, 3x) + c_2 \operatorname{BesselY}(2, 3x)$$

Verified OK.

3.7.2 Maple step by step solution

Let's solve

$x^2y'' + xy' + (9x^2 - 4)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2-4)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(9x^2-4)y}{x^2} = 0$

- Simplify ODE

  $9yx^2 + x^2y'' + xy' - 4y = 0$

- Make a change of variables

  $t = 3x$

- Compute y'

$y' = 3\frac{d}{dt}y(t)$

- Compute second derivative

  $y'' = 9\frac{d^2}{dt^2}y(t)$

- Apply change of variables to the ODE

  $y(t)\, t^2 + t^2\left(\frac{d^2}{dt^2}y(t)\right) + t\left(\frac{d}{dt}y(t)\right) - 4y(t) = 0$

- ODE is now of the Bessel form
- Solution to Bessel ODE

  $y(t) = c_1 \mathit{BesselJ}(2, t) + c_2 \mathit{BesselY}(2, t)$

- Make the change from t back to x

  $y = c_1 \mathit{BesselJ}(2, 3x) + c_2 \mathit{BesselY}(2, 3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(9*x^2-4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-3\,\mathrm{BesselY}\left(0, 3x\right) c_2 x - 3\,\mathrm{BesselJ}\left(0, 3x\right) c_1 x + 2\,\mathrm{BesselY}\left(1, 3x\right) c_2 + 2\,\mathrm{BesselJ}\left(1, 3x\right) c_1}{3x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]+x*y'[x]+(9*x^2-4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\,\mathrm{BesselJ}(2, 3x) + c_2\,\mathrm{BesselY}(2, 3x)$$

3.8 problem 8

Internal problem ID [6626]
Internal file name [`OUTPUT/5874_Sunday_June_05_2022_03_59_21_PM_97338649/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0}$$

3.8.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{36x^2 - \frac{1}{4}}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{36x^2 - \frac{1}{4}}{x^2} - \frac{\left(\frac{1}{x}\right)'}{2} - \frac{\left(\frac{1}{x}\right)^2}{4} \\
&= \frac{36x^2 - \frac{1}{4}}{x^2} - \frac{\left(-\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\
&= \frac{36x^2 - \frac{1}{4}}{x^2} - \left(-\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\
&= 36
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{1}{x}}{2}} \\
&= \frac{1}{\sqrt{x}} \qquad (5)
\end{aligned}$$

Hence (3) becomes

$$y = \frac{v(x)}{\sqrt{x}} \tag{4}$$

Applying this change of variable to the original ode results in

$$x^{\frac{3}{2}}(v''(x) + 36v(x)) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 36$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 36\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 36 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 36$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(36)}\\ &= \pm 6i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +6i\\ \lambda_2 &= -6i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 6i\\ \lambda_2 &= -6i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 6$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^{0}(c_1 \cos(6x) + c_2 \sin(6x))$$

Or

$$v(x) = c_1 \cos(6x) + c_2 \sin(6x)$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x)\\ &= (c_1 \cos(6x) + c_2 \sin(6x))\,(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{\sqrt{x}}$$

Hence (7) becomes

$$y = \frac{c_1 \cos(6x) + c_2 \sin(6x)}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(6x) + c_2 \sin(6x)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos(6x) + c_2 \sin(6x)}{\sqrt{x}}$$

Verified OK.

3.8.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = 6$$
$$n = -\frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y=\frac{c_1\sqrt{2}\,\sqrt{6}\,\cos(6x)}{6\sqrt{\pi}\,\sqrt{x}}+\frac{c_2\sqrt{2}\,\sqrt{6}\,\sin(6x)}{6\sqrt{\pi}\,\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y=\frac{c_1\sqrt{2}\,\sqrt{6}\,\cos(6x)}{6\sqrt{\pi}\,\sqrt{x}}+\frac{c_2\sqrt{2}\,\sqrt{6}\,\sin(6x)}{6\sqrt{\pi}\,\sqrt{x}} \tag{1}$$

Verification of solutions

$$y=\frac{c_1\sqrt{2}\,\sqrt{6}\,\cos(6x)}{6\sqrt{\pi}\,\sqrt{x}}+\frac{c_2\sqrt{2}\,\sqrt{6}\,\sin(6x)}{6\sqrt{\pi}\,\sqrt{x}}$$

Verified OK.

3.8.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2y''+xy'+\left(36x^2-\frac{1}{4}\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2\\B&=x\\C&=36x^2-\frac{1}{4}\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int\frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-36}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -36 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -36z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 94: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -36$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(6x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(6x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(6x)}{6}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{\cos(6x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(6x)}{\sqrt{x}}\left(\frac{\tan(6x)}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\cos(6x)}{\sqrt{x}} + \frac{c_2\sin(6x)}{6\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\cos(6x)}{\sqrt{x}} + \frac{c_2\sin(6x)}{6\sqrt{x}}
$$

Verified OK.

3.8.4 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(144x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(144x^2-1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{144x^2-1}{4x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - ○ $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $4x^2y''+4xy'+\left(144x^2-1\right)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  - ○ Convert $x^2\cdot y''$ to series expansion

    $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(1+2r)\,(-1+2r)\,x^r+a_1(3+2r)\,(1+2r)\,x^{1+r}+\Bigg(\sum_{k=2}^{\infty}\left(a_k(2k+2r+1)\,(2k+2r-1)+144a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{1}{2},\frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1)+144a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2+8(k+2)r+4r^2-1\right)+144a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{144a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=-\frac{144a_k}{4k^2+12k+8}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}},a_{k+2}=-\frac{144a_k}{4k^2+12k+8},a_1=0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{144a_k}{4k^2+20k+24}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}},a_{k+2}=-\frac{144a_k}{4k^2+20k+24},a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}}\right),a_{k+2}=-\frac{144a_k}{4k^2+12k+8},a_1=0,b_{k+2}=-\frac{144b_k}{4k^2+20k+24},b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(36*x^2-1/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(6x) + c_2 \cos(6x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(36*x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-6ix}(12c_1 - ic_2e^{12ix})}{12\sqrt{x}}$$

3.9 problem 9

Internal problem ID [6627]
Internal file name [`OUTPUT/5875_Sunday_June_05_2022_03_59_24_PM_1534104/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0}$$

3.9.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= 5\\ n &= -\frac{2}{3}\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(-\frac{2}{3}, 5x\right) + c_2 \operatorname{BesselY}\left(-\frac{2}{3}, 5x\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}\left(-\frac{2}{3}, 5x\right) + c_2 \operatorname{BesselY}\left(-\frac{2}{3}, 5x\right) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}\left(-\frac{2}{3}, 5x\right) + c_2 \operatorname{BesselY}\left(-\frac{2}{3}, 5x\right)$$

Verified OK.

3.9.2 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(225x^2-4)y}{9x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(225x^2-4)y}{9x^2} = 0$

- Simplify ODE

  $x^2y'' + xy' + 25yx^2 - \frac{4y}{9} = 0$

- Make a change of variables

$t = 5x$

- Compute y'

  $y' = 5\frac{d}{dt}y(t)$

- Compute second derivative

  $y'' = 25\frac{d^2}{dt^2}y(t)$

- Apply change of variables to the ODE

  $t^2\left(\frac{d^2}{dt^2}y(t)\right) + t\left(\frac{d}{dt}y(t)\right) + y(t)\, t^2 - \frac{4y(t)}{9} = 0$

- ODE is now of the Bessel form
- Solution to Bessel ODE

  $y(t) = c_1 BesselJ\left(\frac{2}{3}, t\right) + c_2 BesselY\left(\frac{2}{3}, t\right)$

- Make the change from t back to x

  $y = c_1 BesselJ\left(\frac{2}{3}, 5x\right) + c_2 BesselY\left(\frac{2}{3}, 5x\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(25*x^2-4/9)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \,\mathrm{BesselJ}\left(\frac{2}{3}, 5x\right) + c_2 \,\mathrm{BesselY}\left(\frac{2}{3}, 5x\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]+x*y'[x]+(25*x^2-4/9)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \,\mathrm{BesselJ}\left(\frac{2}{3}, 5x\right) + c_2 \,\mathrm{BesselY}\left(\frac{2}{3}, 5x\right)$$

3.10 problem 10

Internal problem ID [6628]
Internal file name [`OUTPUT/5876_Sunday_June_05_2022_03_59_27_PM_23916409/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^{2} y'' + x y' + \left(2 x^{2}-64\right) y = 0}$$

3.10.1 Solving as second order bessel ode ode

Writing the ode as

$$x^{2} y'' + x y' + \left(2 x^{2}-64\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^{2} y'' + x y' + \left(-n^{2}+x^{2}\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^{2} y'' + \left(1-2 \alpha \right) x y' + \left(\beta^{2} \gamma^{2} x^{2 \gamma}-n^{2} \gamma^{2}+\alpha^{2}\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_{1} \operatorname{BesselJ}\left(n , \beta \,x^{\gamma}\right)+c_{2} \operatorname{BesselY}\left(n , \beta \,x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= \sqrt{2}\\ n &= -8\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(8, x\sqrt{2}\right) + c_2 \operatorname{BesselY}\left(8, x\sqrt{2}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}\left(8, x\sqrt{2}\right) + c_2 \operatorname{BesselY}\left(8, x\sqrt{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}\left(8, x\sqrt{2}\right) + c_2 \operatorname{BesselY}\left(8, x\sqrt{2}\right)$$

Verified OK.

3.10.2 Maple step by step solution

Let's solve

$x^2y'' + xy' + (2x^2 - 64)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(x^2-32)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{2(x^2-32)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{2(x^2-32)}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\Big|_{x=0} = 1$$

  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$(x^2 \cdot P_3(x))\Big|_{x=0} = -64$$

  - ○ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $$x^2 y'' + x y' + (2x^2 - 64)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ○ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ○ Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(8+r)\,(-8+r)\, x^r + a_1(9+r)\,(-7+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+8)\,(k+r-8) + 2a_{k-2}\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(8+r)\,(-8+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-8, 8\}$$

- Each term must be 0

  $a_1(9+r)(-7+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+8)(k+r-8) + 2a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+10+r)(k-6+r) + 2a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k}{(k+10+r)(k-6+r)}$

- Recursion relation for $r = -8$

  $a_{k+2} = -\frac{2a_k}{(k+2)(k-14)}$

- Series not valid for $r = -8$, division by 0 in the recursion relation at $k = 14$

  $a_{k+2} = -\frac{2a_k}{(k+2)(k-14)}$

- Recursion relation for $r = 8$

  $a_{k+2} = -\frac{2a_k}{(k+18)(k+2)}$

- Solution for $r = 8$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+8}, a_{k+2} = -\frac{2a_k}{(k+18)(k+2)}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 101

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(2*x^2-64)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-16\sqrt{2}\,c_1(x^6-75x^4+1080x^2-2520)\,\text{BesselJ}\left(1,\sqrt{2}\,x\right)-16\sqrt{2}\,c_2(x^6-75x^4+1080x^2-2520)\,\text{Bessel}}{x^7}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 30

```
DSolve[x^2*y''[x]+x*y'[x]+(2*x^2-64)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1\,\text{BesselJ}\left(8,\sqrt{2}x\right)+c_2\,\text{BesselY}\left(8,\sqrt{2}x\right)$$

3.11 problem 13

Internal problem ID [6629]
Internal file name [`OUTPUT/5877_Sunday_June_05_2022_03_59_31_PM_72457781/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{xy'' + 2y' + 4y = 0}$$

3.11.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 2xy' + 4xy = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha) xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -\frac{1}{2}\\
\beta &= 4\\
n &= 1\\
\gamma &= \frac{1}{2}
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \operatorname{BesselJ}\left(1, 4\sqrt{x}\right)}{\sqrt{x}} + \frac{c_2 \operatorname{BesselY}\left(1, 4\sqrt{x}\right)}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \operatorname{BesselJ}\left(1, 4\sqrt{x}\right)}{\sqrt{x}} + \frac{c_2 \operatorname{BesselY}\left(1, 4\sqrt{x}\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \operatorname{BesselJ}\left(1, 4\sqrt{x}\right)}{\sqrt{x}} + \frac{c_2 \operatorname{BesselY}\left(1, 4\sqrt{x}\right)}{\sqrt{x}}$$

Verified OK.

3.11.2 Maple step by step solution

Let's solve

$y''x + 2y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x} - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + \frac{4y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x) = \frac{2}{x}, P_3(x) = \frac{4}{x}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- ● Multiply by denominators

  $$y''x + 2y' + 4y = 0$$

- ● Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- ○ Shift index using $k- >k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- ○ Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- ○ Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+2+r) + 4a_k\right) x^{k+r}\right) = 0$$

- ● $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+1+r)\,(k+2+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{4a_k}{(k+1+r)(k+2+r)}$

- Recursion relation for $r = -1$

$a_{k+1} = -\frac{4a_k}{k(k+1)}$

- Solution for $r = -1$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{4a_k}{k(k+1)}\right]$

- Recursion relation for $r = 0$

$a_{k+1} = -\frac{4a_k}{(k+1)(k+2)}$

- Solution for $r = 0$

$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{4a_k}{(k+1)(k+2)}\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+1} = -\frac{4a_k}{k(k+1)}, b_{k+1} = -\frac{4b_k}{(k+1)(k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 27

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 \operatorname{BesselY}\left(1, 4\sqrt{x}\right) + c_1 \operatorname{BesselJ}\left(1, 4\sqrt{x}\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 42

```
DSolve[x*y''[x]+2*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 \operatorname{BesselJ}\left(1, 4\sqrt{x}\right) - 2ic_2 \operatorname{BesselY}\left(1, 4\sqrt{x}\right)}{2\sqrt{x}}$$

3.12 problem 14

Internal problem ID [6630]
Internal file name [`OUTPUT/5878_Sunday_June_05_2022_03_59_33_PM_65251362/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + 3y' + xy = 0}$$

3.12.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 3xy' + yx^2 = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = -1$$
$$\beta = 1$$
$$n = 1$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \operatorname{BesselJ}(1, x)}{x} + \frac{c_2 \operatorname{BesselY}(1, x)}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \operatorname{BesselJ}(1, x)}{x} + \frac{c_2 \operatorname{BesselY}(1, x)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \operatorname{BesselJ}(1, x)}{x} + \frac{c_2 \operatorname{BesselY}(1, x)}{x}$$

Verified OK.

3.12.2 Maple step by step solution

Let's solve

$y''x + 3y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} + y = 0$

- Simplify ODE

  $x^2 y'' + 3xy' + yx^2 = 0$

- Make a change of variables

  $y = \frac{u(x)}{x}$

- Compute y'

  $y' = -\frac{u(x)}{x^2} + \frac{u'(x)}{x}$
- Compute y''

  $y'' = \frac{2u(x)}{x^3} - \frac{2u'(x)}{x^2} + \frac{u''(x)}{x}$
- Apply change of variables to the ODE

  $x^2u(x) + u''(x)\,x^2 + u'(x)\,x - u(x) = 0$
- ODE is now of the Bessel form
- Solution to Bessel ODE

  $u(x) = c_1 \mathit{BesselJ}(1, x) + c_2 \mathit{BesselY}(1, x)$
- Make the change from y back to y

  $y = \frac{c_1 \mathit{BesselJ}(1,x) + c_2 \mathit{BesselY}(1,x)}{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \operatorname{BesselJ}(1, x) + c_2 \operatorname{BesselY}(1, x)}{x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[x*y''[x]+3*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 \operatorname{BesselJ}(1, x) + c_2 \operatorname{BesselY}(1, x)}{x}$$

3.13 problem 15

Internal problem ID [6631]
Internal file name [`OUTPUT/5879_Sunday_June_05_2022_03_59_36_PM_19963015/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' - y' + xy = 0}$$

3.13.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - xy' + yx^2 = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha) xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 1\\ \beta &= 1\\ n &= 1\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 x \operatorname{BesselJ}(1, x) + c_2 x \operatorname{BesselY}(1, x)$$

Summary

The solution(s) found are the following

$$y = c_1 x \operatorname{BesselJ}(1, x) + c_2 x \operatorname{BesselY}(1, x) \tag{1}$$

Verification of solutions

$$y = c_1 x \operatorname{BesselJ}(1, x) + c_2 x \operatorname{BesselY}(1, x)$$

Verified OK.

3.13.2 Maple step by step solution

Let's solve

$y''x - y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + y = 0$

- Simplify ODE

  $x^2 y'' - xy' + yx^2 = 0$

- Make a change of variables

  $y = xu(x)$

- Compute y'

$y' = u(x) + xu'(x)$

- Compute y''

  $y'' = 2u'(x) + xu''(x)$
- Apply change of variables to the ODE

  $u''(x)\,x^2 + x^2u(x) + xu'(x) - u(x) = 0$
- ODE is now of the Bessel form
- Solution to Bessel ODE

  $u(x) = c_1 \mathit{BesselJ}(1, x) + c_2 \mathit{BesselY}(1, x)$
- Make the change from y back to y

  $y = x(c_1 \mathit{BesselJ}(1, x) + c_2 \mathit{BesselY}(1, x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x$2)-diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = x(c_1 \operatorname{BesselJ}(1, x) + c_2 \operatorname{BesselY}(1, x))$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 20

```
DSolve[x*y''[x]-y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_1 \operatorname{BesselJ}(1, x) + c_2 \operatorname{BesselY}(1, x))$$

3.14 problem 16

Internal problem ID [6632]
Internal file name [`OUTPUT/5880_Sunday_June_05_2022_03_59_40_PM_757059/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' - 5y' + xy = 0}$$

3.14.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - 5xy' + yx^2 = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha) xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= 3 \\
\beta &= 1 \\
n &= 3 \\
\gamma &= 1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$y = c_1 x^3 \operatorname{BesselJ}(3, x) + c_2 x^3 \operatorname{BesselY}(3, x)$$

Summary
The solution(s) found are the following

$$y = c_1 x^3 \operatorname{BesselJ}(3, x) + c_2 x^3 \operatorname{BesselY}(3, x) \tag{1}$$

Verification of solutions

$$y = c_1 x^3 \operatorname{BesselJ}(3, x) + c_2 x^3 \operatorname{BesselY}(3, x)$$

Verified OK.

3.14.2 Maple step by step solution

Let's solve

$y''x - 5y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{x} + y = 0$

- Simplify ODE

  $x^2 y'' - 5xy' + yx^2 = 0$

- Make a change of variables

  $y = x^3 u(x)$

- Compute y'

$y' = 3x^2 u(x) + x^3 u'(x)$

- Compute y''

  $y'' = 6xu(x) + 6x^2 u'(x) + x^3 u''(x)$
- Apply change of variables to the ODE

  $x^2 u(x) + u''(x)\, x^2 + u'(x)\, x - 9u(x) = 0$
- ODE is now of the Bessel form
- Solution to Bessel ODE

  $u(x) = c_1 \mathit{BesselJ}(3, x) + c_2 \mathit{BesselY}(3, x)$
- Make the change from y back to y

  $y = (c_1 \mathit{BesselJ}(3, x) + c_2 \mathit{BesselY}(3, x))\, x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
dsolve(x*diff(y(x),x$2)-5*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$\begin{aligned}y(x) = -\left(c_1\left(x^2-8\right)\operatorname{BesselJ}(1,x)+c_2\left(x^2-8\right)\operatorname{BesselY}(1,x)\right.\\ \left.+4x(c_1\operatorname{BesselJ}(0,x)+c_2\operatorname{BesselY}(0,x))\right)x\end{aligned}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[x*y''[x]-5*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3(c_1 \operatorname{BesselJ}(3,x) + c_2 \operatorname{BesselY}(3,x))$$

3.15 problem 17

Internal problem ID [6633]
Internal file name [`OUTPUT/5881_Sunday_June_05_2022_03_59_43_PM_84206040/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(x^2-2\right)y = 0}$$

3.15.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(x^2-2\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2+x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= 1 \\
n &= -\frac{3}{2} \\
\gamma &= 1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = -\frac{c_1 \sqrt{2}\,(\sin(x)\, x + \cos(x))}{x\sqrt{\pi}} - \frac{c_2 \sqrt{2}\,(-\cos(x)\, x + \sin(x))}{x\sqrt{\pi}}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{c_1 \sqrt{2}\,(\sin(x)\, x + \cos(x))}{x\sqrt{\pi}} - \frac{c_2 \sqrt{2}\,(-\cos(x)\, x + \sin(x))}{x\sqrt{\pi}} \tag{1}
$$

Verification of solutions

$$
y = -\frac{c_1 \sqrt{2}\,(\sin(x)\, x + \cos(x))}{x\sqrt{\pi}} - \frac{c_2 \sqrt{2}\,(-\cos(x)\, x + \sin(x))}{x\sqrt{\pi}}
$$

Verified OK.

3.15.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
x^2 y'' + \left(x^2 - 2\right) y &= 0 && (1) \\
A y'' + B y' + C y &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= 0 \\
C &= x^2 - 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 102: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2-2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(i) \\
&= -\frac{1}{x} - i \\
&= -\frac{1}{x} - i
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2+2}{x^2}\right)\right) = 0$$

$$\frac{2ia_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -i + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (-i + x)\, \mathrm{e}^{\int \left(-\frac{1}{x} - i\right) dx} \\
&= (-i + x)\, \mathrm{e}^{-ix - \ln(x)} \\
&= \frac{(-i + x)\, \mathrm{e}^{-ix}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(-i+x)\,\mathrm{e}^{-ix}}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-i+x)\,\mathrm{e}^{-ix}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{(-i+x)\,\mathrm{e}^{-ix}}{x} \int \frac{1}{\frac{(-i+x)^2\mathrm{e}^{-2ix}}{x^2}}\,dx \\ &= \frac{(-i+x)\,\mathrm{e}^{-ix}}{x} \left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{2i-2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(-i+x)\,\mathrm{e}^{-ix}}{x}\right) + c_2 \left(\frac{(-i+x)\,\mathrm{e}^{-ix}}{x} \left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{2i-2x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(-i+x)\,\mathrm{e}^{-ix}}{x} - \frac{c_2(ix-1)\,\mathrm{e}^{ix}}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(-i+x)\,\mathrm{e}^{-ix}}{x} - \frac{c_2(ix-1)\,\mathrm{e}^{ix}}{2x}$$

Verified OK.

3.15.3 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 - 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-2) y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-2) y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{x^2-2}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -2$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + (x^2 - 2) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+r)\,(-2+r)\,x^{r}+a_{1}(2+r)\,(-1+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_{k}(k+r+1)\,(k+r-2)+a_{k-2}\right) x^{k+r}\right)$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-1,2\}$

- Each term must be 0

  $a_{1}(2+r)\,(-1+r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+1)\,(k+r-2)+a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+3+r)\,(k+r)+a_{k}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_{k}}{(k+3+r)(k+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_{k}}{(k+2)(k-1)}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_{k}x^{k-1}, a_{k+2}=-\frac{a_{k}}{(k+2)(k-1)}, a_{1}=0\right]$

- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_{k}}{(k+5)(k+2)}$

- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+5)(k+2)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+5)(k+2)}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+(x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{(c_1x+c_2)\cos(x)+\sin(x)(c_2x-c_1)}{x}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 21

```
DSolve[x^2*y''[x]+(x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to x(c_1j_1(x)-c_2y_1(x))$$

3.16 problem 18

Internal problem ID [6634]
Internal file name [`OUTPUT/5882_Sunday_June_05_2022_03_59_46_PM_90080849/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + \left(16x^2+1\right)y = 0}$$

3.16.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(4x^2+\frac{1}{4}\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2+x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = 0$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}(0, 2x) + c_2\sqrt{x}\ \mathrm{BesselY}(0, 2x)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}(0, 2x) + c_2\sqrt{x}\ \mathrm{BesselY}(0, 2x) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}(0, 2x) + c_2\sqrt{x}\ \mathrm{BesselY}(0, 2x)$$

Verified OK.

3.16.2 Maple step by step solution

Let's solve

$4x^2y'' + (16x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(16x^2+1)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(16x^2+1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{16x^2+1}{4x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- • Multiply by denominators

  $$4x^2y'' + (16x^2 + 1)\, y = 0$$

- • Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ○ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1 + 2r)^2\, x^r + a_1(1 + 2r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k + 2r - 1)^2 + 16a_{k-2}\right) x^{k+r}\right) = 0$$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1 + 2r)^2 = 0$$

- • Values of r that satisfy the indicial equation

  $$r = \frac{1}{2}$$

- • Each term must be 0

  $$a_1(1 + 2r)^2 = 0$$

- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)^2 a_k + 16a_{k-2} = 0$
- Shift index using $k- >k+2$

  $4\left(k+\frac{3}{2}+r\right)^2 a_{k+2} + 16a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{16a_k}{(2k+3+2r)^2}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{16a_k}{(2k+4)^2}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{16a_k}{(2k+4)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(4*x^2*diff(y(x),x$2)+(16*x^2+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = (\mathrm{BesselJ}\,(0, 2x)\, c_1 + \mathrm{BesselY}\,(0, 2x)\, c_2)\, \sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 28

```
DSolve[4*x^2*y''[x]+(16*x^2+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}(c_1\, \mathrm{BesselJ}(0, 2x) + c_2\, \mathrm{BesselY}(0, 2x))$$

3.17 problem 19

Internal problem ID [6635]
Internal file name [`OUTPUT/5883_Sunday_June_05_2022_03_59_49_PM_89770355/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{xy'' + 3y' + yx^3 = 0}$$

3.17.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 3xy' + yx^4 = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha) xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= -1 \\
\beta &= \frac{1}{2} \\
n &= \frac{1}{2} \\
\gamma &= 2
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = \frac{2c_1 \sin\left(\frac{x^2}{2}\right)}{x\sqrt{\pi}\,\sqrt{x^2}} - \frac{2c_2 \cos\left(\frac{x^2}{2}\right)}{x\sqrt{\pi}\,\sqrt{x^2}}
$$

Summary
The solution(s) found are the following

$$
y = \frac{2c_1 \sin\left(\frac{x^2}{2}\right)}{x\sqrt{\pi}\,\sqrt{x^2}} - \frac{2c_2 \cos\left(\frac{x^2}{2}\right)}{x\sqrt{\pi}\,\sqrt{x^2}} \tag{1}
$$

Verification of solutions

$$
y = \frac{2c_1 \sin\left(\frac{x^2}{2}\right)}{x\sqrt{\pi}\,\sqrt{x^2}} - \frac{2c_2 \cos\left(\frac{x^2}{2}\right)}{x\sqrt{\pi}\,\sqrt{x^2}}
$$

Verified OK.

3.17.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
xy'' + 3y' + yx^3 &= 0 && (1) \\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x \\
B &= 3 \\
C &= x^3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 4\\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ix - \frac{3i}{8x^3} - \frac{9i}{128x^7} - \frac{27i}{1024x^{11}} - \frac{405i}{32768x^{15}} - \frac{1701i}{262144x^{19}} - \frac{15309i}{4194304x^{23}} - \frac{72171i}{33554432x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-4x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(-x^2\right) + \left(\frac{3}{4x^2}\right) \\ &= -x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= ix \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) \quad = \frac{1}{2}\left(\frac{0}{i} - 1\right) \quad = -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{i} - 1\right) = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-4x^4 + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	ix	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty}\\ &= -\frac{1}{2x} + (-)\left(ix\right)\\ &= -\frac{1}{2x} - ix\\ &= -\frac{1}{2x} - ix\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{2x} - ix\right)(0) + \left(\left(\frac{1}{2x^2} - i\right) + \left(-\frac{1}{2x} - ix\right)^2 - \left(\frac{-4x^4+3}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int \left(-\frac{1}{2x} - ix\right)dx}\\ &= \frac{\mathrm{e}^{-\frac{ix^2}{2}}}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{x}\,dx}\\ &= z_1 e^{-\frac{3\ln(x)}{2}}\\ &= z_1\left(\frac{1}{x^{\frac{3}{2}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{ix^2}{2}}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{i\mathrm{e}^{ix^2}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\mathrm{e}^{-\frac{ix^2}{2}}}{x^2}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{ix^2}{2}}}{x^2} \left(-\frac{i\mathrm{e}^{ix^2}}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-\frac{ix^2}{2}}}{x^2} - \frac{ic_2 \mathrm{e}^{\frac{ix^2}{2}}}{2x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{ix^2}{2}}}{x^2} - \frac{ic_2 \mathrm{e}^{\frac{ix^2}{2}}}{2x^2}$$

Verified OK.

3.17.3 Maple step by step solution

Let's solve

$y''x+3y'+yx^3=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{x}-yx^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{x}+yx^2=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{3}{x},P_3(x)=x^2\right]$

  ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=3$

  ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x+3y'+yx^3=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^3\cdot y$ to series expansion

  $x^3\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+3}$

  ○ Shift index using $k->k-3$

$$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

  - ○ Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - ○ Shift index using $k- >k+1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$$

  - ○ Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - ○ Shift index using $k- >k+1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^{r} + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \Bigg(\sum_{k=3}^{\infty} ($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 0\}$$

- The coefficients of each power of x must be 0

  $$[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_1 = 0, a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(k+r+3) + a_{k-3} = 0$$

- Shift index using $k- >k+3$

  $$a_{k+4}(k+4+r)\,(k+6+r) + a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+4} = -\frac{a_k}{(k+4+r)(k+6+r)}$$

- Recursion relation for $r = -2$

$$a_{k+4} = -\frac{a_k}{(k+2)(k+4)}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+4} = -\frac{a_k}{(k+4)(k+6)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+4} = -\frac{a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+4} = -\frac{a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{b_k}{(k+4)(k+6)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+x^3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin\left(\frac{x^2}{2}\right) + c_2 \cos\left(\frac{x^2}{2}\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 43

```
DSolve[x*y''[x]+3*y'[x]+x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{ix^2}{2}} \left(2c_1 - ic_2 e^{ix^2}\right)}{2x^2}$$

3.18 problem 20

Internal problem ID [6636]
Internal file name [`OUTPUT/5884_Sunday_June_05_2022_03_59_53_PM_88974195/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' + 9xy' + \left(x^6 - 36\right) y = 0}$$

3.18.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + \left(\frac{x^6}{9} - 4\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= \frac{1}{9}\\ n &= -\frac{2}{3}\\ \gamma &= 3\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{x^3}{9}\right) + c_2 \operatorname{BesselY}\left(-\frac{2}{3}, \frac{x^3}{9}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{x^3}{9}\right) + c_2 \operatorname{BesselY}\left(-\frac{2}{3}, \frac{x^3}{9}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{x^3}{9}\right) + c_2 \operatorname{BesselY}\left(-\frac{2}{3}, \frac{x^3}{9}\right)$$

Verified OK.

3.18.2 Maple step by step solution

Let's solve

$9x^2y'' + 9xy' + (x^6 - 36)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{(x^6-36)y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^6-36)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{x^6-36}{9x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=-4$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $9x^2y''+9xy'+\left(x^6-36\right)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..6$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$

  - Convert $x^2\cdot y''$ to series expansion

    $x^2\cdot y''=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $9a_0(2+r)\,(-2+r)\,x^r+9a_1(3+r)\,(-1+r)\,x^{1+r}+9a_2(4+r)\,r\,x^{2+r}+9a_3(5+r)\,(1+r)\,x^{3+r}+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$9(2+r)(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-2,2\}$

- The coefficients of each power of x must be 0

  $[9a_1(3+r)(-1+r)=0, 9a_2(4+r)r=0, 9a_3(5+r)(1+r)=0, 9a_4(6+r)(2+r)=0, 9a_5(7+$

- Solve for the dependent coefficient(s)

  $\{a_1=0, a_2=0, a_3=0, a_4=0, a_5=0\}$

- Each term in the series must be 0, giving the recursion relation

  $9a_k(k+r+2)(k+r-2)+a_{k-6}=0$

- Shift index using $k->k+6$

  $9a_{k+6}(k+8+r)(k+4+r)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+6}=-\frac{a_k}{9(k+8+r)(k+4+r)}$

- Recursion relation for $r=-2$

  $a_{k+6}=-\frac{a_k}{9(k+6)(k+2)}$

- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-2}, a_{k+6}=-\frac{a_k}{9(k+6)(k+2)}, a_1=0, a_2=0, a_3=0, a_4=0, a_5=0\right]$

- Recursion relation for $r=2$

  $a_{k+6}=-\frac{a_k}{9(k+10)(k+6)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+6}=-\frac{a_k}{9(k+10)(k+6)}, a_1=0, a_2=0, a_3=0, a_4=0, a_5=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-2}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+2}\right), a_{k+6}=-\frac{a_k}{9(k+6)(k+2)}, a_1=0, a_2=0, a_3=0, a_4=0, a_5=0,\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(9*x^2*diff(y(x),x$2)+9*x*diff(y(x),x)+(x^6-36)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}\left(\frac{2}{3}, \frac{x^3}{9}\right) + c_2 \operatorname{BesselY}\left(\frac{2}{3}, \frac{x^3}{9}\right)$$

✓ Solution by Mathematica

Time used: 0.151 (sec). Leaf size: 43

```
DSolve[9*x^2*y''[x]+9*x*y'[x]+(x^6-36)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3c_1 \operatorname{Gamma}\left(\frac{4}{3}\right) \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{x^3}{9}\right) + c_2 \operatorname{Gamma}\left(\frac{5}{3}\right) \operatorname{BesselJ}\left(\frac{2}{3}, \frac{x^3}{9}\right)$$

3.19 problem 22(a)

Internal problem ID [6637]
Internal file name [`OUTPUT/5885_Sunday_June_05_2022_03_59_56_PM_77102067/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 22(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - y x^2 = 0}$$

3.19.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - y x^4 = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + x y' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}\left(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)\right) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= \frac{1}{2}\\ \beta &= \frac{i}{2}\\ n &= \frac{1}{4}\\ \gamma &= 2\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)$$

Verified OK.

3.19.2 Maple step by step solution

Let's solve

$y'' - y x^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ○ Convert $x^2 \cdot y$ to series expansion

  $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

  - Shift index using $k- > k-2$

    $$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

  - Convert y'' to series expansion

    $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

  - Shift index using $k- > k+2$

    $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 = 0, 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_{k-2} = 0$$

- Shift index using $k- > k+2$

  $$\left((k+2)^2 + 3k + 8\right) a_{k+4} - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_2 + \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1\right) \sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 37

```
DSolve[y''[x]-x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \operatorname{ParabolicCylinderD}\left(-\frac{1}{2}, i\sqrt{2}x\right) + c_1 \operatorname{ParabolicCylinderD}\left(-\frac{1}{2}, \sqrt{2}x\right)$$

3.20 problem 22 (b)

Internal problem ID [6638]
Internal file name [`OUTPUT/5886_Sunday_June_05_2022_04_00_00_PM_35605720/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 22 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' - 7yx^3 = 0}$$

3.20.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' - 7yx^4 = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = \frac{i\sqrt{7}}{2}$$
$$n = 0$$
$$\gamma = 2$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselI}\left(0, \frac{\sqrt{7}\,x^2}{2}\right) + c_2 \operatorname{BesselY}\left(0, \frac{i\sqrt{7}\,x^2}{2}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselI}\left(0, \frac{\sqrt{7}\,x^2}{2}\right) + c_2 \operatorname{BesselY}\left(0, \frac{i\sqrt{7}\,x^2}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselI}\left(0, \frac{\sqrt{7}\,x^2}{2}\right) + c_2 \operatorname{BesselY}\left(0, \frac{i\sqrt{7}\,x^2}{2}\right)$$

Verified OK.

3.20.2 Maple step by step solution

Let's solve

$y''x + y' - 7yx^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + 7yx^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - 7yx^2 = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x) = \frac{1}{x}, P_3(x) = -7x^2\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x + y' - 7yx^3 = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^3 \cdot y$ to series expansion

  $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

- ○ Shift index using $k- >k-3$

  $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

- ○ Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- ○ Shift index using $k- >k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$$

- ○ Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- ○ Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1(1+r)^2 x^r + a_2(2+r)^2 x^{1+r} + a_3(3+r)^2 x^{2+r} + \Bigg(\sum_{k=3}^{\infty} \big(a_{k+1}(k+1+r)^2 - 7a_{k-3}\big)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- The coefficients of each power of x must be 0

  $\left[a_1(1+r)^2 = 0, a_2(2+r)^2 = 0, a_3(3+r)^2 = 0\right]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - 7a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4)^2 - 7a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = \frac{7a_k}{(k+4)^2}$

- Recursion relation for $r = 0$

  $a_{k+4} = \frac{7a_k}{(k+4)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{7a_k}{(k+4)^2}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(x*diff(y(x),x$2)+diff(y(x),x)-7*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselI}\left(0, \frac{\sqrt{7}\, x^2}{2}\right) + c_2 \operatorname{BesselK}\left(0, \frac{\sqrt{7}\, x^2}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.089 (sec). Leaf size: 41

```
DSolve[x*y''[x]+y'[x]-7*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselI}\left(0, \frac{\sqrt{7}x^2}{2}\right) + 2c_2 K_0\left(\frac{\sqrt{7}x^2}{2}\right)$$

3.21 problem 23

Internal problem ID [6639]
Internal file name [`OUTPUT/5887_Sunday_June_05_2022_04_00_03_PM_80547132/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

3.21.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) \tag{1}$$

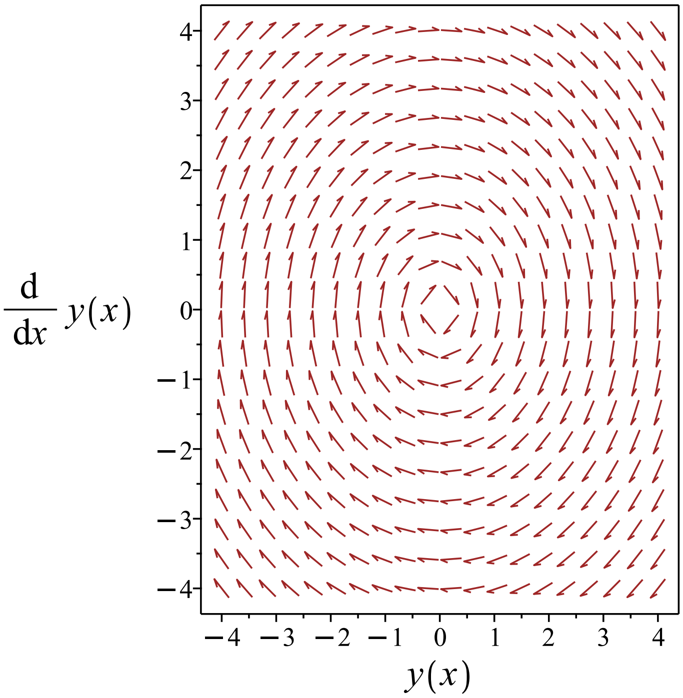

Figure 23: Slope field plot

Verification of solutions

$$y = \cos(x)\, c_1 + c_2 \sin(x)$$

Verified OK.

3.21.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + y'y)\, dx = 0$$

$$\frac{y'^2}{2} + \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{-y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2+2c_1}}dy = \int dx$$

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_2 + x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2+2c_1}}dy = \int dx$$

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_3$$

Summary

The solution(s) found are the following

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_2 + x \tag{1}$$

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_3 \tag{2}$$

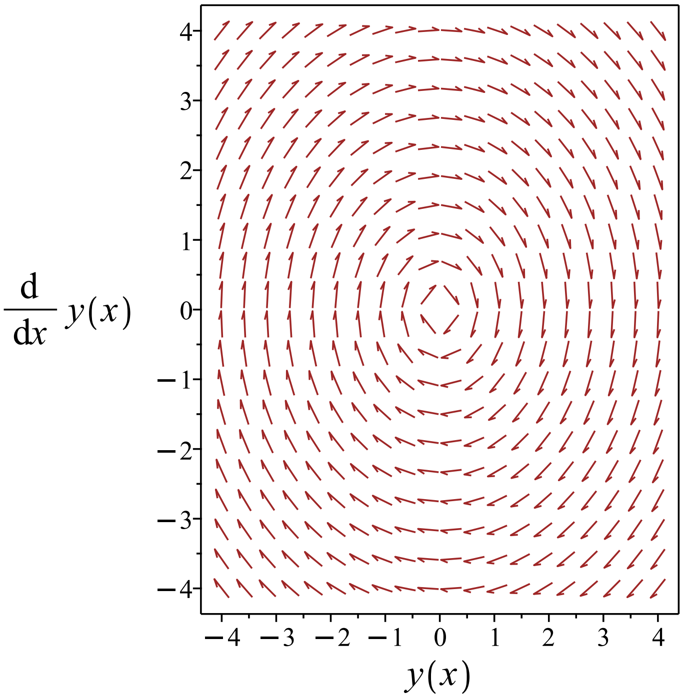

Figure 24: Slope field plot

Verification of solutions

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right)=c_2+x$$

Verified OK.

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right)=x+c_3$$

Verified OK.

3.21.3 Solving using Kovacic algorithm

Writing the ode as

$$y''+y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 110: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\&= 0 - 0\\&= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\&= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\, c_1 + c_2 \sin(x) \tag{1}$$

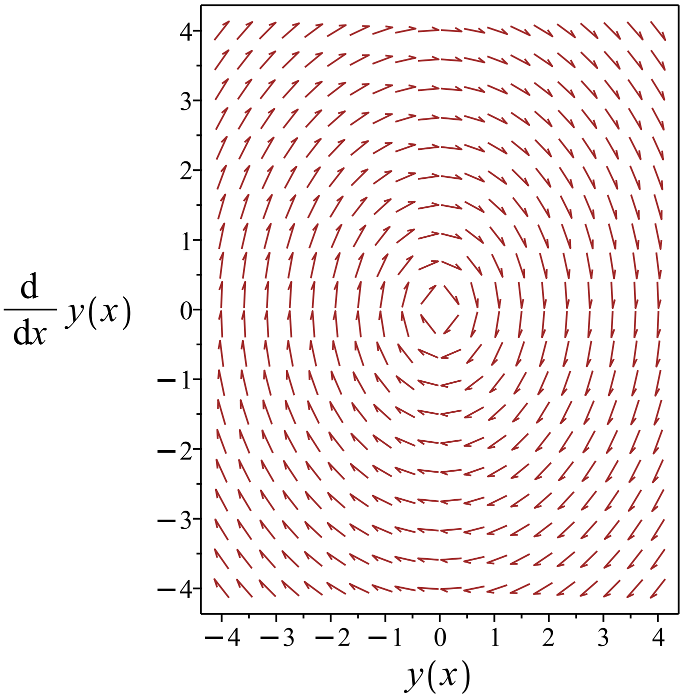

Figure 25: Slope field plot

<u>Verification of solutions</u>

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Verified OK.

3.21.4 Maple step by step solution

Let's solve

$y'' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = \cos(x)\, c_1 + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x$2)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(x) + \cos(x)\, c_2$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[y''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(x) + c_2 \sin(x)$$

3.22 problem 24

Internal problem ID [6640]
Internal file name [`OUTPUT/5888_Sunday_June_05_2022_04_00_05_PM_69853832/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 4xy' + y(x^2 + 2) = 0}$$

3.22.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{x^2 + 2}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x^2+2}{x^2} - \frac{\left(\frac{4}{x}\right)'}{2} - \frac{\left(\frac{4}{x}\right)^2}{4} \\
&= \frac{x^2+2}{x^2} - \frac{\left(-\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\
&= \frac{x^2+2}{x^2} - \left(-\frac{2}{x^2}\right) - \frac{4}{x^2} \\
&= 1
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{4}{x}}{2}} \\
&= \frac{1}{x^2}
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) + v(x) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(\cos(x)\, c_1 + c_2 \sin(x))$$

Or

$$v(x) = \cos(x)\, c_1 + c_2 \sin(x)$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x)\\ &= (\cos(x)\, c_1 + c_2 \sin(x))\,(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x^2}$$

Hence (7) becomes

$$y = \frac{\cos(x) c_1 + c_2 \sin(x)}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{\cos(x) c_1 + c_2 \sin(x)}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{\cos(x) c_1 + c_2 \sin(x)}{x^2}$$

Verified OK.

3.22.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + 4xy' + y(x^2 + 2) = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + xy' + (-n^2 + x^2) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= -\frac{3}{2} \\ \beta &= 1 \\ n &= -\frac{1}{2} \\ \gamma &= 1 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\,\cos(x)}{x^2\sqrt{\pi}} + \frac{c_2\sqrt{2}\,\sin(x)}{x^2\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,\cos(x)}{x^2\sqrt{\pi}} + \frac{c_2\sqrt{2}\,\sin(x)}{x^2\sqrt{\pi}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\cos(x)}{x^2\sqrt{\pi}} + \frac{c_2\sqrt{2}\,\sin(x)}{x^2\sqrt{\pi}}$$

Verified OK.

3.22.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 4xy' + y(x^2+2) = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 112: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1\left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x^2}\right) + c_2 \left(\frac{\cos(x)}{x^2} (\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{x^2} + \frac{c_2 \sin(x)}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{x^2} + \frac{c_2 \sin(x)}{x^2}
$$

Verified OK.

3.22.4 Maple step by step solution

Let's solve

$x^2 y'' + 4xy' + y(x^2 + 2) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} - \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

● Multiply by denominators

  $x^2 y'' + 4xy' + y(x^2 + 2) = 0$

● Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  ○ Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (2 + r)\,(1 + r)\, x^r + a_1 (3 + r)\,(2 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k + r + 2)\,(k + r + 1) + a_{k-2}\right) x^{k+r}\right) =$

● $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2 + r)\,(1 + r) = 0$

● Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$

● Each term must be 0

  $a_1 (3 + r)\,(2 + r) = 0$

● Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r+1)+a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)(k+3+r)+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+4+r)(k+3+r)}$
- Recursion relation for $r=-2$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$
- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$
- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x) + \cos(x) c_2}{x^2}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]+4*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x^2}$$

3.23 problem 25

Internal problem ID [6641]
Internal file name [`OUTPUT/5889_Sunday_June_05_2022_04_00_08_PM_87415694/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{16x^2y'' + 32xy' + \left(x^4 - 12\right) y = 0}$$

3.23.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 2xy' + \left(\frac{x^4}{16} - \frac{3}{4}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= -\frac{1}{2}\\ \beta &= \frac{1}{8}\\ n &= -\frac{1}{2}\\ \gamma &= 2\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{4c_1 \cos\left(\frac{x^2}{8}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{x^2}} + \frac{4c_2 \sin\left(\frac{x^2}{8}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{x^2}}$$

Summary

The solution(s) found are the following

$$y = \frac{4c_1 \cos\left(\frac{x^2}{8}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{x^2}} + \frac{4c_2 \sin\left(\frac{x^2}{8}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{x^2}} \tag{1}$$

Verification of solutions

$$y = \frac{4c_1 \cos\left(\frac{x^2}{8}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{x^2}} + \frac{4c_2 \sin\left(\frac{x^2}{8}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{x^2}}$$

Verified OK.

3.23.2 Solving using Kovacic algorithm

Writing the ode as

$$16x^2y'' + 32xy' + (x^4 - 12)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 16x^2\\ B &= 32x\\ C &= x^4 - 12\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^4 + 12 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 114: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{x^2}{16} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ix}{4} - \frac{3i}{2x^3} - \frac{9i}{2x^7} - \frac{27i}{x^{11}} - \frac{405i}{2x^{15}} - \frac{1701i}{x^{19}} - \frac{15309i}{x^{23}} - \frac{144342i}{x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{ix}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{-x^4 + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(-\frac{x^2}{16}\right) + \left(\frac{3}{4x^2}\right) \\ &= -\frac{x^2}{16} + \frac{3}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{ix}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{i}{4}} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{i}{4}} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^4 + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{ix}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)\left(\frac{ix}{4}\right) \\
&= -\frac{1}{2x} - \frac{ix}{4} \\
&= -\frac{1}{2x} - \frac{ix}{4}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - \frac{ix}{4}\right)(0) + \left(\left(\frac{1}{2x^2} - \frac{i}{4}\right) + \left(-\frac{1}{2x} - \frac{ix}{4}\right)^2 - \left(\frac{-x^4+12}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{ix}{4}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{ix^2}{8}}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{32x}{16x^2} \, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{i x^2}{8}}}{x^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{32x}{16x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-2i\mathrm{e}^{\frac{i x^2}{4}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{i x^2}{8}}}{x^{\frac{3}{2}}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{i x^2}{8}}}{x^{\frac{3}{2}}} \left(-2i\mathrm{e}^{\frac{i x^2}{4}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-\frac{i x^2}{8}}}{x^{\frac{3}{2}}} - \frac{2 i c_2 \mathrm{e}^{\frac{i x^2}{8}}}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{i x^2}{8}}}{x^{\frac{3}{2}}} - \frac{2 i c_2 \mathrm{e}^{\frac{i x^2}{8}}}{x^{\frac{3}{2}}}$$

Verified OK.

3.23.3 Maple step by step solution

Let's solve

$16x^2y'' + 32xy' + (x^4 - 12)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - \frac{(x^4-12)y}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + \frac{(x^4-12)y}{16x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = \frac{x^4-12}{16x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 2$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{3}{4}$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2y'' + 32xy' + (x^4 - 12)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^{m} \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r}$

  - Convert $x^{2} \cdot y''$ to series expansion

    $x^{2} \cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

    Rewrite ODE with series expansions

    $4a_{0}(3+2r)\,(-1+2r)\, x^{r} + 4a_{1}(5+2r)\,(1+2r)\, x^{1+r} + 4a_{2}(7+2r)\,(3+2r)\, x^{2+r} + 4a_{3}(9+2r)\,($

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $4(3+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{1}{2}\right\}$

- The coefficients of each power of x must be 0

  $[4a_{1}(5+2r)\,(1+2r) = 0, 4a_{2}(7+2r)\,(3+2r) = 0, 4a_{3}(9+2r)\,(5+2r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_{1} = 0, a_{2} = 0, a_{3} = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $16\left(k+r+\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right) a_{k} + a_{k-4} = 0$

- Shift index using $k- >k+4$

  $16\left(k+\frac{11}{2}+r\right)\left(k+\frac{7}{2}+r\right) a_{k+4} + a_{k} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{a_{k}}{4(2k+11+2r)(2k+7+2r)}$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+4} = -\frac{a_{k}}{4(2k+8)(2k+4)}$

- Solution for $r = -\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_{k} x^{k-\frac{3}{2}}, a_{k+4} = -\frac{a_{k}}{4(2k+8)(2k+4)}, a_{1} = 0, a_{2} = 0, a_{3} = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+4} = -\frac{a_k}{4(2k+12)(2k+8)}$
- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+4} = -\frac{a_k}{4(2k+12)(2k+8)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+4} = -\frac{a_k}{4(2k+8)(2k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{}{4(2k-}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 25

```
dsolve(16*x^2*diff(y(x),x$2)+32*x*diff(y(x),x)+(x^4-12)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin\left(\frac{x^2}{8}\right) + c_2 \cos\left(\frac{x^2}{8}\right)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 42

```
DSolve[16*x^2*y''[x]+32*x*y'[x]+(x^4-12)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{ix^2}{8}}\left(c_1 - 2ic_2 e^{\frac{ix^2}{4}}\right)}{x^{3/2}}$$

3.24 problem 26

Internal problem ID [6642]
Internal file name [`OUTPUT/5890_Sunday_June_05_2022_04_00_11_PM_10941972/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. 6.4 SPECIAL FUNCTIONS. EXERCISES 6.4. Page 267
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4xy' + (16x^2 + 3)\, y = 0}$$

3.24.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{16x^2 + 3}{4x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{16x^2+3}{4x^2} - \frac{\left(-\frac{1}{x}\right)'}{2} - \frac{\left(-\frac{1}{x}\right)^2}{4} \\
&= \frac{16x^2+3}{4x^2} - \frac{\left(\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\
&= \frac{16x^2+3}{4x^2} - \left(\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\
&= 4
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)} \\
&= e^{-\int \frac{-\frac{1}{x}}{2}} \\
&= \sqrt{x}
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\sqrt{x} \tag{4}$$

Applying this change of variable to the original ode results in

$$4x^{\frac{5}{2}}\left(v''(x) + 4v(x)\right) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$v(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x)\\ &= (c_1 \cos(2x) + c_2 \sin(2x))\,(z(x)) \qquad (7)\end{aligned}$$

But from (5)

$$z(x) = \sqrt{x}$$

Hence (7) becomes

$$y = (c_1 \cos(2x) + c_2 \sin(2x))\sqrt{x}$$

Summary

The solution(s) found are the following

$$y = (c_1 \cos(2x) + c_2 \sin(2x))\sqrt{x} \tag{1}$$

Verification of solutions

$$y = (c_1 \cos(2x) + c_2 \sin(2x))\sqrt{x}$$

Verified OK.

3.24.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - x y' + \left(4x^2 + \frac{3}{4}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + x y' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= 1 \\
\beta &= 2 \\
n &= -\frac{1}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{x}\,\cos(2x)}{\sqrt{\pi}} + \frac{c_2\sqrt{x}\,\sin(2x)}{\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{x}\,\cos(2x)}{\sqrt{\pi}} + \frac{c_2\sqrt{x}\,\sin(2x)}{\sqrt{\pi}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{x}\,\cos(2x)}{\sqrt{\pi}} + \frac{c_2\sqrt{x}\,\sin(2x)}{\sqrt{\pi}}$$

Verified OK.

3.24.3 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align} 4x^2y'' - 4xy' + (16x^2+3)\,y &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align}$$

Comparing (1) and (2) shows that

$$\begin{align} A &= 4x^2 \\ B &= -4x \tag{3} \\ C &= 16x^2+3 \end{align}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 116: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos (2x) \sqrt{x}\right) + c_2 \left(\cos (2x) \sqrt{x} \left(\frac{\tan (2x)}{2}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 \sqrt{x}\, \cos (2x) + \frac{c_2 \sqrt{x}\, \sin (2x)}{2} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1 \sqrt{x}\, \cos (2x) + \frac{c_2 \sqrt{x}\, \sin (2x)}{2}
$$

Verified OK.

3.24.4 Maple step by step solution

Let's solve

$4x^2 y'' - 4xy' + (16x^2 + 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(16x^2+3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{(16x^2+3)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{16x^2+3}{4x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{3}{4}$

    - ○ $x = 0$is a regular singular point

      Check to see if $x_0 = 0$ is a regular singular point

      $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4xy' + (16x^2 + 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + 2r)\,(-3 + 2r)\, x^r + a_1(1 + 2r)\,(-1 + 2r)\, x^{1+r} + \Bigg(\sum_{k=2}^{\infty} (a_k(2k + 2r - 1)\,(2k + 2r - 3) + 1$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 2r)\,(-3 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1 + 2r)\,(-1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right)a_k+16a_{k-2}=0$$

- Shift index using $k->k+2$

  $$4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right)a_{k+2}+16a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{16a_k}{(2k+1+2r)(2k+3+2r)}$$

- Recursion relation for $r=\frac{1}{2}$

  $$a_{k+2}=-\frac{16a_k}{(2k+2)(2k+4)}$$

- Solution for $r=\frac{1}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}},a_{k+2}=-\frac{16a_k}{(2k+2)(2k+4)},a_1=0\right]$$

- Recursion relation for $r=\frac{3}{2}$

  $$a_{k+2}=-\frac{16a_k}{(2k+4)(2k+6)}$$

- Solution for $r=\frac{3}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}},a_{k+2}=-\frac{16a_k}{(2k+4)(2k+6)},a_1=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right),a_{k+2}=-\frac{16a_k}{(2k+2)(2k+4)},a_1=0,b_{k+2}=-\frac{16b_k}{(2k+4)(2k+6)},b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(16*x^2+3)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\,(c_1 \sin(2x) + c_2 \cos(2x))$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 39

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(16*x^2+3)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-2ix}\sqrt{x}\left(4c_1 - ic_2e^{4ix}\right)$$

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4.1 problem 9

Internal problem ID [6643]
Internal file name [`OUTPUT/5891_Sunday_June_05_2022_04_00_14_PM_73135955/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{2xy'' + y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{1}{2x}$$

Table 118: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n + r) (n + r - 1)\right) + \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{1}{1247400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$
a_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{1247400}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{64 \left(r+2\right) \left(r+3\right) \left(r+\frac{9}{2}\right) \left(r+\frac{5}{2}\right) \left(r+\frac{7}{2}\right) \left(r+1\right) \left(r+\frac{3}{2}\right) \left(r+\frac{1}{2}\right) \left(5+r\right) \left(r+4\right) \left(r+6\right) \left(r+\frac{11}{2}\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = \frac{1}{97297200}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$
a_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{1247400}$
a_6	$\frac{1}{64(r+2)(r+3)\left(r+\frac{9}{2}\right)\left(r+\frac{5}{2}\right)\left(r+\frac{7}{2}\right)(r+1)\left(r+\frac{3}{2}\right)\left(r+\frac{1}{2}\right)(5+r)(r+4)(r+6)\left(r+\frac{11}{2}\right)}$	$\frac{1}{97297200}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{1}{\left(r+2\right) \left(r+3\right) \left(2r+9\right) \left(2r+5\right) \left(2r+7\right) \left(r+1\right) \left(2r+3\right) \left(2r+1\right) \left(5+r\right) \left(r+4\right) \left(r+6\right) \left(2r+11\right) (}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = -\frac{1}{10216206000}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$
a_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{1247400}$
a_6	$\frac{1}{64(r+2)(r+3)\left(r+\frac{9}{2}\right)\left(r+\frac{5}{2}\right)\left(r+\frac{7}{2}\right)(r+1)\left(r+\frac{3}{2}\right)\left(r+\frac{1}{2}\right)(5+r)(r+4)(r+6)\left(r+\frac{11}{2}\right)}$	$\frac{1}{97297200}$
a_7	$-\frac{1}{(r+2)(r+3)(2r+9)(2r+5)(2r+7)(r+1)(2r+3)(2r+1)(5+r)(r+4)(r+6)(2r+11)(2r^2+27r+91)}$	$-\frac{1}{10216206000}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots) \\
&= \sqrt{x}\left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + \frac{x^6}{97297200} - \frac{x^7}{10216206000} + O(x^8)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + (n+r)\,b_n + b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{b_{n-1}}{n\,(2n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{1}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$
b_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{113400}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{64\,(r+2)\,(r+3)\left(r+\frac{9}{2}\right)\left(r+\frac{5}{2}\right)\left(r+\frac{7}{2}\right)(r+1)\left(r+\frac{3}{2}\right)\left(r+\frac{1}{2}\right)(5+r)\,(r+4)\,(r+6)\left(r+\frac{11}{2}\right)}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{1}{7484400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$
b_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{113400}$
b_6	$\frac{1}{64(r+2)(r+3)\left(r+\frac{9}{2}\right)\left(r+\frac{5}{2}\right)\left(r+\frac{7}{2}\right)(r+1)\left(r+\frac{3}{2}\right)\left(r+\frac{1}{2}\right)(5+r)(r+4)(r+6)\left(r+\frac{11}{2}\right)}$	$\frac{1}{7484400}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{1}{(r+2)\,(r+3)\,(2r+9)\,(2r+5)\,(2r+7)\,(r+1)\,(2r+3)\,(2r+1)\,(5+r)\,(r+4)\,(r+6)\,(2r+11)\,(}$$

Which for the root $r = 0$ becomes

$$b_7 = -\frac{1}{681080400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$
b_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{113400}$
b_6	$\frac{1}{64(r+2)(r+3)\left(r+\frac{9}{2}\right)\left(r+\frac{5}{2}\right)\left(r+\frac{7}{2}\right)(r+1)\left(r+\frac{3}{2}\right)\left(r+\frac{1}{2}\right)(5+r)(r+4)(r+6)\left(r+\frac{11}{2}\right)}$	$\frac{1}{7484400}$
b_7	$-\frac{1}{(r+2)(r+3)(2r+9)(2r+5)(2r+7)(r+1)(2r+3)(2r+1)(5+r)(r+4)(r+6)(2r+11)(2r^2+27r+91)}$	$-\frac{1}{681080400}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\ &= 1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + \frac{x^6}{7484400} - \frac{x^7}{681080400} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x} \left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + \frac{x^6}{97297200} - \frac{x^7}{10216206000} + O(x^8)\right) \\ &\quad + c_2 \left(1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + \frac{x^6}{7484400} - \frac{x^7}{681080400} + O(x^8)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x} \left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + \frac{x^6}{97297200} - \frac{x^7}{10216206000} + O(x^8)\right) \\ &\quad + c_2 \left(1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + \frac{x^6}{7484400} - \frac{x^7}{681080400} + O(x^8)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+\frac{x^6}{97297200}-\frac{x^7}{10216206000}+O(x^8)\right) + c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+\frac{x^6}{7484400}-\frac{x^7}{681080400}+O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+\frac{x^6}{97297200}-\frac{x^7}{10216206000}+O(x^8)\right) + c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+\frac{x^6}{7484400}-\frac{x^7}{681080400}+O(x^8)\right)$$

Verified OK.

4.1.1 Maple step by step solution

Let's solve

$2y''x + y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x} - \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} + \frac{y}{2x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{1}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $2y''x + y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$

  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-1+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) + a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k+\frac{1}{2}+r\right)a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+1+r)(2k+1+2r)}$

- Recursion relation for $r = 0$

$a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}, b_{k+1} = -\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 52

```
Order:=8;
dsolve(2*x*diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \frac{1}{22680}x^4 - \frac{1}{1247400}x^5 + \frac{1}{97297200}x^6 - \frac{1}{10216206000}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \frac{1}{2520}x^4 - \frac{1}{113400}x^5 + \frac{1}{7484400}x^6 - \frac{1}{681080400}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 111

```
AsymptoticDSolveValue[2*x*y''[x]+y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{x^7}{10216206000}+\frac{x^6}{97297200}-\frac{x^5}{1247400}+\frac{x^4}{22680}-\frac{x^3}{630}+\frac{x^2}{30}-\frac{x}{3}+1\right) + c_2\left(-\frac{x^7}{681080400}+\frac{x^6}{7484400}-\frac{x^5}{113400}+\frac{x^4}{2520}-\frac{x^3}{90}+\frac{x^2}{6}-x+1\right)$$

4.2 problem 10

Internal problem ID [6644]
Internal file name [`OUTPUT/5892_Sunday_June_05_2022_04_00_18_PM_43670742/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - xy' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (180) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (181) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= xy' + y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x^2 y' + xy + 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 + 5x\right) y' + y\left(x^2 + 3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 + 9x^2 + 8\right) y' + yx\left(x^2 + 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^5 + 14x^3 + 33x\right) y' + y\left(x^4 + 12x^2 + 15\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(x^6 + 20x^4 + 87x^2 + 48\right) y' + yx\left(x^4 + 18x^2 + 57\right) \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(x^7 + 27x^5 + 185x^3 + 279x\right) y' + y\left(x^2 + 7\right)\left(x^4 + 18x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0)\\
F_1 &= 2y'(0)\\
F_2 &= 3y(0)\\
F_3 &= 8y'(0)\\
F_4 &= 15y(0)\\
F_5 &= 48y'(0)\\
F_6 &= 105y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \frac{1}{384}x^8\right) y(0) + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7\right) y'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}\\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{105}$$

For $n = 6$ the recurrence equation gives

$$56a_8 - 7a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{384}$$

For $n = 7$ the recurrence equation gives

$$72a_9 - 8a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{945}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \frac{1}{48} a_0 x^6 + \frac{1}{105} a_1 x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6\right) a_0 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7\right) a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7\right) c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \frac{1}{384}x^8\right) y(0) + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7\right) y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \frac{1}{384}x^8\right) y(0) + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7\right) c_2 + O(x^8)$$

Verified OK.

4.2.1 Maple step by step solution

Let's solve

$y'' = xy' + y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - xy' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\left(a_{k+2}(k+2) - a_k\right) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 44

```
Order:=8;
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{48}x^6\right)y(0)+\left(x+\frac{1}{3}x^3+\frac{1}{15}x^5+\frac{1}{105}x^7\right)D(y)\left(0\right)+O\left(x^8\right)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]-x*y'[x]-y[x]==0,y[x],{x,0,7}]
```

$$y(x)\to c_2\left(\frac{x^7}{105}+\frac{x^5}{15}+\frac{x^3}{3}+x\right)+c_1\left(\frac{x^6}{48}+\frac{x^4}{8}+\frac{x^2}{2}+1\right)$$

4.3 problem 11

Internal problem ID [6645]
Internal file name [`OUTPUT/5893_Sunday_June_05_2022_04_00_20_PM_75879269/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(x-1)\,y''+3y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (183)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (184)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3y}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(3-3x)\, y' + 3y}{(x-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(6x-6)\, y' + (9x-15)\, y}{(x-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(9x^2-36x+27)\, y' + (-36x+54)\, y}{(x-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-54x^2+180x-126)\, y' - 27\left(x^2-8x+\frac{29}{3}\right) y}{(x-1)^5} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{(-27x^3+405x^2-1089x+711)\, y' + 243y\left(x^2-\frac{50}{9}x+\frac{163}{27}\right)}{(x-1)^6} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{(324x^3-3132x^2+7452x-4644)\, y' + 81\left(x^3-27x^2+\frac{353}{3}x-\frac{355}{3}\right) y}{(x-1)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 3y(0)\\
F_1 &= 3y(0) + 3y'(0)\\
F_2 &= 15y(0) + 6y'(0)\\
F_3 &= 54y(0) + 27y'(0)\\
F_4 &= 261y(0) + 126y'(0)\\
F_5 &= 1467y(0) + 711y'(0)\\
F_6 &= 9585y(0) + 4644y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \frac{9}{20}x^5 + \frac{29}{80}x^6 + \frac{163}{560}x^7 + \frac{213}{896}x^8\right) y(0)\\
&+ \left(x + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{9}{40}x^5 + \frac{7}{40}x^6 + \frac{79}{560}x^7 + \frac{129}{1120}x^8\right) y'(0) + O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x - 1)\, y'' + 3y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}\\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$(x - 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\,x^{n-1} a_n (n-1)\right) + \sum_{n\,=2}^{\infty} \left(-n(n-1)\,a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n\,=2}^{\infty} n\,x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\,a_{n+1} n\,x^n$$

$$\sum_{n\,=2}^{\infty} \left(-n(n-1)\,a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\,a_{n+2}(n+1)\,x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\,a_{n+1} n\,x^n\right) + \sum_{n\,=0}^{\infty} \left(-(n+2)\,a_{n+2}(n+1)\,x^n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 + 3a_0 = 0$$

$$a_2 = \frac{3a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\,a_{n+1} n - (n+2)\,a_{n+2}(n+1) + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_{n+1} + n a_{n+1} + 3a_n}{(n+2)\,(n+1)} \\ &= \frac{3a_n}{(n+2)\,(n+1)} + \frac{(n^2+n)\,a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 - 6a_3 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{2} + \frac{a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$6a_3 - 12a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{8} + \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$12a_4 - 20a_5 + 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{9a_0}{20} + \frac{9a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$20a_5 - 30a_6 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{29a_0}{80} + \frac{7a_1}{40}$$

For $n = 5$ the recurrence equation gives

$$30a_6 - 42a_7 + 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{163a_0}{560} + \frac{79a_1}{560}$$

For $n = 6$ the recurrence equation gives

$$42a_7 - 56a_8 + 3a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{213a_0}{896} + \frac{129a_1}{1120}$$

For $n = 7$ the recurrence equation gives

$$56a_8 - 72a_9 + 3a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{331a_0}{1680} + \frac{1283a_1}{13440}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x + \frac{3a_0 x^2}{2} + \left(\frac{a_0}{2} + \frac{a_1}{2}\right) x^3 + \left(\frac{5a_0}{8} + \frac{a_1}{4}\right) x^4 \\ + \left(\frac{9a_0}{20} + \frac{9a_1}{40}\right) x^5 + \left(\frac{29a_0}{80} + \frac{7a_1}{40}\right) x^6 + \left(\frac{163a_0}{560} + \frac{79a_1}{560}\right) x^7 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y = \left(1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \frac{9}{20}x^5 + \frac{29}{80}x^6 + \frac{163}{560}x^7\right) a_0 \\ + \left(x + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{9}{40}x^5 + \frac{7}{40}x^6 + \frac{79}{560}x^7\right) a_1 + O(x^8) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned} y = \left(1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \frac{9}{20}x^5 + \frac{29}{80}x^6 + \frac{163}{560}x^7\right) c_1 \\ + \left(x + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{9}{40}x^5 + \frac{7}{40}x^6 + \frac{79}{560}x^7\right) c_2 + O(x^8) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1+\frac{3}{2}x^{2}+\frac{1}{2}x^{3}+\frac{5}{8}x^{4}+\frac{9}{20}x^{5}+\frac{29}{80}x^{6}+\frac{163}{560}x^{7}+\frac{213}{896}x^{8}\right) y(0) \\
& + \left(x+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{9}{40}x^{5}+\frac{7}{40}x^{6}+\frac{79}{560}x^{7}+\frac{129}{1120}x^{8}\right) y'(0)+O\left(x^{8}\right)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
y = {} & \left(1+\frac{3}{2}x^{2}+\frac{1}{2}x^{3}+\frac{5}{8}x^{4}+\frac{9}{20}x^{5}+\frac{29}{80}x^{6}+\frac{163}{560}x^{7}\right) c_{1} \\
& + \left(x+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{9}{40}x^{5}+\frac{7}{40}x^{6}+\frac{79}{560}x^{7}\right) c_{2}+O\left(x^{8}\right)
\end{aligned} \tag{2}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1+\frac{3}{2}x^{2}+\frac{1}{2}x^{3}+\frac{5}{8}x^{4}+\frac{9}{20}x^{5}+\frac{29}{80}x^{6}+\frac{163}{560}x^{7}+\frac{213}{896}x^{8}\right) y(0) \\
& + \left(x+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{9}{40}x^{5}+\frac{7}{40}x^{6}+\frac{79}{560}x^{7}+\frac{129}{1120}x^{8}\right) y'(0)+O\left(x^{8}\right)
\end{aligned}
$$

Verified OK.

$$
\begin{aligned}
y = {} & \left(1+\frac{3}{2}x^{2}+\frac{1}{2}x^{3}+\frac{5}{8}x^{4}+\frac{9}{20}x^{5}+\frac{29}{80}x^{6}+\frac{163}{560}x^{7}\right) c_{1} \\
& + \left(x+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{9}{40}x^{5}+\frac{7}{40}x^{6}+\frac{79}{560}x^{7}\right) c_{2}+O\left(x^{8}\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=8;
dsolve((x-1)*diff(y(x),x$2)+3*y(x)=0,y(x),type='series',x=0);
```

$$\begin{aligned}y(x) = {} & \left(1+\frac{3}{2}x^2+\frac{1}{2}x^3+\frac{5}{8}x^4+\frac{9}{20}x^5+\frac{29}{80}x^6+\frac{163}{560}x^7\right)y(0) \\ & +\left(x+\frac{1}{2}x^3+\frac{1}{4}x^4+\frac{9}{40}x^5+\frac{7}{40}x^6+\frac{79}{560}x^7\right)D(y)(0)+O\left(x^8\right)\end{aligned}$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 91

```
AsymptoticDSolveValue[(x-1)*y''[x]+3*y[x]==0,y[x],{x,0,7}]
```

$$
\begin{aligned}
y(x) \to\ & c_2\left(\frac{79x^7}{560}+\frac{7x^6}{40}+\frac{9x^5}{40}+\frac{x^4}{4}+\frac{x^3}{2}+x\right) \\
& + c_1\left(\frac{163x^7}{560}+\frac{29x^6}{80}+\frac{9x^5}{20}+\frac{5x^4}{8}+\frac{x^3}{2}+\frac{3x^2}{2}+1\right)
\end{aligned}
$$

4.4 problem 12

Internal problem ID [6646]
Internal file name [`OUTPUT/5894_Sunday_June_05_2022_04_00_23_PM_35193772/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - x^2y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f\big|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'\big|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (186) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (187) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= x^2 y' - xy \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^4 + x\right) y' + \left(-x^3 - 1\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= x^2 \left(x^3 + 4\right) (-y + xy') \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^8 + 9x^5 + 8x^2\right) y' + \left(-x^7 - 9x^4 - 8x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^9 + 16x^6 + 44x^3 + 8\right) (-y + xy') \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= x^2 \left(x^9 + 25x^6 + 140x^3 + 140\right) (-y + xy') \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= x \left(x^{12} + 36x^9 + 340x^6 + 840x^3 + 280\right) (-y + xy')
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= -8y(0) \\
F_5 &= 0 \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6\right) y(0) + xy'(0) + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = x^2 \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty} \left(-n\, x^{1+n} a_n\right) = \sum_{n=2}^{\infty} \left(-(n-1)\, a_{n-1} x^n\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \sum_{n=2}^{\infty} \left(-(n-1)\, a_{n-1} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0 \quad (3)$$

$n = 1$ gives

$$6a_3 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - (n-1)\, a_{n-1} + a_{n-1} = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-1}(n-2)}{(n+2)\,(1+n)} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{90}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

For $n = 6$ the recurrence equation gives

$$56a_8 - 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$72a_9 - 5a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{a_0}{1296}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{90} a_0 x^6 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6\right) a_0 + a_1 x + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6\right) c_1 + c_2 x + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6\right) y(0) + x y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6\right) c_1 + c_2 x + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6\right) y(0) + x y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6\right) c_1 + c_2 x + O(x^8)$$

Verified OK.

4.4.1 Maple step by step solution

Let's solve

$y'' = x^2 y' - xy$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - x^2 y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ○ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ○ Shift index using $k- > k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ○ Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  ○ Shift index using $k- > k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k- > k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k-2)\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_{k-1}(k-2) = 0$

- Shift index using $k- > k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_k(k-1) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k(k-1)}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=8;
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6\right) y(0) + D(y)(0)\, x + O(x^8)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 27

```
AsymptoticDSolveValue[y''[x]-x^2*y'[x]+x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(-\frac{x^6}{90}-\frac{x^3}{6}+1\right)+c_2x$$

4.5 problem 13

Internal problem ID [6647]
Internal file name [`OUTPUT/5895_Sunday_June_05_2022_04_00_25_PM_60904034/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{xy'' - (x+2)\,y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (-x-2)\,y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x+2}{x}$$
$$q(x) = \frac{2}{x}$$

Table 122: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-x - 2)\, y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) x \\ + (-x - 2) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_{n} (n+r) (n+r-1)\right)+\sum_{n =0}^{\infty} \left(-x^{n+r} a_{n} (n+r)\right) + \sum_{n =0}^{\infty} \left(-2 (n+r) a_{n} x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} 2 a_{n} x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n =0}^{\infty} \left(-x^{n+r} a_{n} (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) x^{n+r-1}\right)$$

$$\sum_{n =0}^{\infty} 2 a_{n} x^{n+r} = \sum_{n=1}^{\infty} 2 a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_{n} (n+r) (n+r-1)\right)+\sum_{n =1}^{\infty} \left(-a_{n-1} (n+r-1) x^{n+r-1}\right) + \sum_{n =0}^{\infty} \left(-2 (n+r) a_{n} x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty} 2 a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_{n} (n+r) (n+r-1)-2 (n+r) a_{n} x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_{0} r (-1+r)-2 r a_{0} x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r (-1+r)-2 r \,x^{-1+r}\right) a_{0} = 0$$

Since $a_{0}\neq 0$ then the above simplifies to

$$r \,x^{-1+r} (-3+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 3 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^3\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 2a_n(n+r) + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n+r} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = \frac{a_{n-1}}{n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{1+r}$$

Which for the root $r = 3$ becomes

$$a_1 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)(2+r)}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{1}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(1+r)(2+r)(3+r)}$$

Which for the root $r = 3$ becomes

$$a_3 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{20}$
a_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{120}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(2+r)(3+r)(4+r)(1+r)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{20}$
a_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{120}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{840}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$$

Which for the root $r = 3$ becomes

$$a_5 = \frac{1}{6720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{20}$
a_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{120}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{840}$
a_5	$\frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$	$\frac{1}{6720}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(4+r)(1+r)(2+r)(5+r)(3+r)(6+r)}$$

Which for the root $r = 3$ becomes

$$a_6 = \frac{1}{60480}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{20}$
a_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{120}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{840}$
a_5	$\frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$	$\frac{1}{6720}$
a_6	$\frac{1}{(4+r)(1+r)(2+r)(5+r)(3+r)(6+r)}$	$\frac{1}{60480}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{1}{(1+r)(2+r)(5+r)(3+r)(6+r)(7+r)(4+r)}$$

Which for the root $r = 3$ becomes

$$a_7 = \frac{1}{604800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{20}$
a_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{120}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{840}$
a_5	$\frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$	$\frac{1}{6720}$
a_6	$\frac{1}{(4+r)(1+r)(2+r)(5+r)(3+r)(6+r)}$	$\frac{1}{60480}$
a_7	$\frac{1}{(1+r)(2+r)(5+r)(3+r)(6+r)(7+r)(4+r)}$	$\frac{1}{604800}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^3\left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + \frac{x^6}{60480} + \frac{x^7}{604800} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= \frac{1}{(1+r)(2+r)(3+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1}{(1+r)(2+r)(3+r)} &= \lim_{r \to 0} \frac{1}{(1+r)(2+r)(3+r)} \\ &= \frac{1}{6} \end{aligned}$$

The limit is $\frac{1}{6}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^n
\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) - 2(n+r)b_n + 2b_{n-1} = 0 \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_n n(n-1) - b_{n-1}(n-1) - 2nb_n + 2b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{n+r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{1+r}$$

Which for the root $r = 0$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{(1+r)(2+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(2+r)(3+r)(4+r)(1+r)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{24}$
b_5	$\frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$	$\frac{1}{120}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{(4+r)(1+r)(2+r)(5+r)(3+r)(6+r)}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{1}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{24}$
b_5	$\frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$	$\frac{1}{120}$
b_6	$\frac{1}{(4+r)(1+r)(2+r)(5+r)(3+r)(6+r)}$	$\frac{1}{720}$

For $n = 7$, using the above recursive equation gives

$$b_7 = \frac{1}{(1+r)(2+r)(5+r)(3+r)(6+r)(7+r)(4+r)}$$

Which for the root $r = 0$ becomes

$$b_7 = \frac{1}{5040}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{24}$
b_5	$\frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$	$\frac{1}{120}$
b_6	$\frac{1}{(4+r)(1+r)(2+r)(5+r)(3+r)(6+r)}$	$\frac{1}{720}$
b_7	$\frac{1}{(1+r)(2+r)(5+r)(3+r)(6+r)(7+r)(4+r)}$	$\frac{1}{5040}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + \frac{x^6}{60480} + \frac{x^7}{604800} + O(x^8)\right) \\
&\quad + c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + \frac{x^6}{60480} + \frac{x^7}{604800} + O(x^8)\right) \\
&\quad + c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x^3 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + \frac{x^6}{60480} + \frac{x^7}{604800} + O(x^8)\right) \\
&\quad + c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 x^3 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + \frac{x^6}{60480} + \frac{x^7}{604800} + O(x^8)\right) \\
&\quad + c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8)\right)
\end{aligned}$$

Verified OK.

4.5.1 Maple step by step solution

Let's solve

$y''x+(-x-2)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{x}+\frac{(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x+2)y'}{x}+\frac{2y}{x}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x)=-\frac{x+2}{x},P_3(x)=\frac{2}{x}\right]$

  ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-2$

  ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x+(-x-2)\,y'+2y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ○ Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-2) - a_k(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 3\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-2)\,(a_{k+1}(k+1+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 3$

  $$a_{k+1} = \frac{a_k}{k+4}$$

- Solution for $r = 3$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k}{k+4}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 52

```
Order:=8;
dsolve(x*diff(y(x),x$2)-(x+2)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 \left(1 + \frac{1}{4}x + \frac{1}{20}x^2 + \frac{1}{120}x^3 + \frac{1}{840}x^4 + \frac{1}{6720}x^5 + \frac{1}{60480}x^6 + \frac{1}{604800}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2\left(12 + 12x + 6x^2 + 2x^3 + \frac{1}{2}x^4 + \frac{1}{10}x^5 + \frac{1}{60}x^6 + \frac{1}{420}x^7 + \mathrm{O}\left(x^8\right)\right)$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 94

```
AsymptoticDSolveValue[x*y''[x]-(x+2)*y'[x]+2*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^6}{720} + \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right) + c_2\left(\frac{x^9}{60480} + \frac{x^8}{6720} + \frac{x^7}{840} + \frac{x^6}{120} + \frac{x^5}{20} + \frac{x^4}{4} + x^3\right)$$

4.6 problem 14

Internal problem ID [6648]
Internal file name [`OUTPUT/5896_Sunday_June_05_2022_04_00_29_PM_1035037/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\cos(x)\,y''+y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{190}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{191}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y}{\cos(x)} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -\sec(x)\,(y\tan(x) + y') \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2\sec(x)\left(\tan(x)\,y' + (\sec(x) - 1)\,y\left(\sec(x) + \frac{1}{2}\right)\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \sec(x)\left(\left(-6\sec(x)^2 + \sec(x) + 3\right)y' + \tan(x)\sec(x)^2\,y\left(\cos(x)^2 + 4\cos(x) - 6\right)\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -24\left(-\frac{\sec(x)^2\left(\cos(x)^2 + \frac{3\cos(x)}{2} - 6\right)\tan(x)\,y'}{6} + y\left(\sec(x)^4 - \frac{3\sec(x)^3}{4} - \frac{19\sec(x)^2}{24} + \frac{11\sec(}{24}\right.\right. \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= -\left(\left(120\sec(x)^4 - 36\sec(x)^3 - 99\sec(x)^2 + 23\sec(x) + 5\right)y' + y\tan(x)\sec(x)^4(\cos(x) - 1)\left(\cos(\right.\right. \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= -720\left(\frac{\tan(x)\sec(x)^4\left(120 + \cos(x)^4 + 12\cos(x)^3 - 58\cos(x)^2 - 40\cos(x)\right)y'}{120} + y\left(\sec(x)^6 - \frac{5\,\mathrm{s}}{}\right.\right.
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= 0 \\
F_3 &= -2y'(0) \\
F_4 &= y(0) \\
F_5 &= -13y'(0) \\
F_6 &= 13y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6 + \frac{13}{40320}x^8\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5 - \frac{13}{5040}x^7\right) y'(0) + O(x^8)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\frac{\sum_{n=0}^{\infty} a_n x^n}{\cos(x)} \tag{1}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$\begin{aligned}
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \dots \\
&= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8
\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(1-\frac{1}{2}x^{2}+\frac{1}{24}x^{4}-\frac{1}{720}x^{6}+\frac{1}{40320}x^{8}\right)\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)=0$$

Expanding the first term in (1) gives

$$1\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)-\frac{x^{2}}{2}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\frac{x^{4}}{24}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)$$
$$-\frac{x^{6}}{720}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\frac{x^{8}}{40320}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)=0$$

Which simplifies to

$$\begin{aligned}\left(\sum_{n=2}^{\infty}\frac{n\,x^{n+6}a_{n}(n-1)}{40320}\right)+\sum_{n\,=2}^{\infty}\left(-\frac{n\,x^{n+4}a_{n}(n-1)}{720}\right)+\left(\sum_{n=2}^{\infty}\frac{n\,x^{n+2}a_{n}(n-1)}{24}\right)\\+\sum_{n\,=2}^{\infty}\left(-\frac{na_{n}x^{n}(n-1)}{2}\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)=0\end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n\,=2}^{\infty}\frac{n\,x^{n+6}a_{n}(n-1)}{40320}=\sum_{n=8}^{\infty}\frac{(n-6)\,a_{n-6}(n-7)\,x^{n}}{40320}$$

$$\sum_{n\,=2}^{\infty}\left(-\frac{n\,x^{n+4}a_{n}(n-1)}{720}\right)=\sum_{n=6}^{\infty}\left(-\frac{(n-4)\,a_{n-4}(n-5)\,x^{n}}{720}\right)$$

$$\sum_{n\,=2}^{\infty}\frac{n\,x^{n+2}a_{n}(n-1)}{24}=\sum_{n=4}^{\infty}\frac{(n-2)\,a_{n-2}(n-3)\,x^{n}}{24}$$

$$\sum_{n\,=2}^{\infty}n(n-1)\,a_{n}x^{n-2}=\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=8}^{\infty} \frac{(n-6)\, a_{n-6} (n-7)\, x^{n}}{40320}\right)+\sum_{n=6}^{\infty}\left(-\frac{(n-4)\, a_{n-4} (n-5)\, x^{n}}{720}\right) \\ +\left(\sum_{n=4}^{\infty} \frac{(n-2)\, a_{n-2} (n-3)\, x^{n}}{24}\right)+\sum_{n=2}^{\infty}\left(-\frac{n a_{n} x^{n} (n-1)}{2}\right) \\ +\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^{n}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)=0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

$n = 3$ gives

$$-2a_3 + 20a_5 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{a_1}{3} + 20a_5 = 0$$

Or

$$a_5 = -\frac{a_1}{60}$$

$n = 4$ gives

$$\frac{a_2}{12} - 5a_4 + 30a_6 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{a_0}{24}+30a_6=0$$

Or

$$a_6=\frac{a_0}{720}$$

$n=5$ gives

$$\frac{a_3}{4}-9a_5+42a_7=0$$

Which after substituting earlier equations, simplifies to

$$\frac{13a_1}{120}+42a_7=0$$

Or

$$a_7=-\frac{13a_1}{5040}$$

$n=6$ gives

$$-\frac{a_2}{360}+\frac{a_4}{2}-14a_6+56a_8=0$$

Which after substituting earlier equations, simplifies to

$$-\frac{13a_0}{720}+56a_8=0$$

Or

$$a_8=\frac{13a_0}{40320}$$

$n=7$ gives

$$-\frac{a_3}{120}+\frac{5a_5}{6}-20a_7+72a_9=0$$

Which after substituting earlier equations, simplifies to

$$\frac{197a_1}{5040}+72a_9=0$$

Or

$$a_9 = -\frac{197a_1}{362880}$$

For $8 \leq n$, the recurrence equation is

$$\begin{aligned}&\frac{(n-6)\,a_{n-6}(n-7)}{40320} - \frac{(n-4)\,a_{n-4}(n-5)}{720} + \frac{(n-2)\,a_{n-2}(n-3)}{24}\\&-\frac{na_n(n-1)}{2} + (n+2)\,a_{n+2}(n+1) + a_n = 0\end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned}a_{n+2}\\&= \frac{20160n^2a_n - n^2a_{n-6} + 56n^2a_{n-4} - 1680n^2a_{n-2} - 20160na_n + 13na_{n-6} - 504na_{n-4} + 8400na_{n-2} - 403}{40320\,(n+2)\,(n+1)}\end{aligned}$$

$$\begin{aligned}(5)\quad &= \frac{(20160n^2 - 20160n - 40320)\,a_n}{40320\,(n+2)\,(n+1)} + \frac{(-n^2+13n-42)\,a_{n-6}}{40320\,(n+2)\,(n+1)}\\&+\frac{(56n^2 - 504n + 1120)\,a_{n-4}}{40320\,(n+2)\,(n+1)} + \frac{(-1680n^2 + 8400n - 10080)\,a_{n-2}}{40320\,(n+2)\,(n+1)}\end{aligned}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n\\&= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{6}a_1x^3 - \frac{1}{60}a_1x^5 + \frac{1}{720}a_0x^6 - \frac{13}{5040}a_1x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6\right)a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5 - \frac{13}{5040}x^7\right)a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6\right)c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5 - \frac{13}{5040}x^7\right)c_2 + O(x^8)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6 + \frac{13}{40320}x^8\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5 - \frac{13}{5040}x^7\right) y'(0) + O(x^8) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5 - \frac{13}{5040}x^7\right) c_2 + O(x^8) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6 + \frac{13}{40320}x^8\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5 - \frac{13}{5040}x^7\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5 - \frac{13}{5040}x^7\right) c_2 + O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0,
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      u(t)-t^2*diff(u(t),t)+(-t^3+t)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
Order:=8;
dsolve(cos(x)*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5 - \frac{13}{5040}x^7\right) D(y)\,(0) + O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 49

```
AsymptoticDSolveValue[Cos[x]*y''[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{x^6}{720} - \frac{x^2}{2} + 1\right) + c_2\left(-\frac{13x^7}{5040} - \frac{x^5}{60} - \frac{x^3}{6} + x\right)$$

4.7 problem 15

Internal problem ID [6649]
Internal file name [`OUTPUT/5897_Sunday_June_05_2022_04_00_31_PM_69610406/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + xy' + 2y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = -2]$$

With the expansion point for the power series method at $x = 0$.

4.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + xy' + 2y = 0$$

The domain of $p(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{193}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{194}\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -xy' - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x^2 y' + 2xy - 3y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -y'x^3 - 2yx^2 + 7xy' + 8y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 12x^2 + 15\right) y' + 2\left(x^3 - 9x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 18x^3 - 57x\right) y' - 2y\left(x^4 - 15x^2 + 24\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \left(x^6 - 25x^4 + 141x^2 - 105\right) y' + 2yx\left(x^4 - 22x^2 + 87\right) \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(-x^7 + 33x^5 - 285x^3 + 561x\right) y' - 2y\left(x^6 - 30x^4 + 207x^2 - 192\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 3$ and

$y'(0) = -2$ gives

$$\begin{aligned}
F_0 &= -6\\
F_1 &= 6\\
F_2 &= 24\\
F_3 &= -30\\
F_4 &= -144\\
F_5 &= 210\\
F_6 &= 1152
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = x^4 + x^3 - 3x^2 - 2x + 3 - \frac{x^5}{4} - \frac{x^6}{5} + \frac{x^7}{24} + \frac{x^8}{35} + O(x^8)$$

$$y = x^4 + x^3 - 3x^2 - 2x + 3 - \frac{x^5}{4} - \frac{x^6}{5} + \frac{x^7}{24} + \frac{x^8}{35} + O(x^8)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}\\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+1} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 7a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{48}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + 8a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{105}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + 9a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{384}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - a_0x^2 - \frac{1}{2}a_1x^3 + \frac{1}{3}a_0x^4 + \frac{1}{8}a_1x^5 - \frac{1}{15}a_0x^6 - \frac{1}{48}a_1x^7 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6\right)a_0 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7\right)a_1 + O(x^8) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6\right)c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7\right)c_2 + O(x^8)$$

$$y = 3 - 3x^2 + x^4 - \frac{x^6}{5} - 2x + x^3 - \frac{x^5}{4} + \frac{x^7}{24} + O(x^8)$$

Summary

The solution(s) found are the following

$$y = x^4 + x^3 - 3x^2 - 2x + 3 - \frac{x^5}{4} - \frac{x^6}{5} + \frac{x^7}{24} + \frac{x^8}{35} + O(x^8) \tag{1}$$

$$y = 3 - 3x^2 + x^4 - \frac{x^6}{5} - 2x + x^3 - \frac{x^5}{4} + \frac{x^7}{24} + O(x^8) \tag{2}$$

Verification of solutions

$$y = x^4 + x^3 - 3x^2 - 2x + 3 - \frac{x^5}{4} - \frac{x^6}{5} + \frac{x^7}{24} + \frac{x^8}{35} + O(x^8)$$

Verified OK.

$$y = 3 - 3x^2 + x^4 - \frac{x^6}{5} - 2x + x^3 - \frac{x^5}{4} + \frac{x^7}{24} + O(x^8)$$

Verified OK.

4.7.2 Maple step by step solution

Let's solve

$$\left[y'' = -xy' - 2y, y(0) = 3, y'\Big|_{\{x=0\}} = -2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + xy' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=8;
dsolve([diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(0) = 3, D(y)(0) = -2],y(x),type='series',x=
```

$$y(x)=3-2x-3x^{2}+x^{3}+x^{4}-\frac{1}{4}x^{5}-\frac{1}{5}x^{6}+\frac{1}{24}x^{7}+\mathrm{O}\left(x^{8}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[{y''[x]+x*y'[x]+2*y[x]==0,{y[0]==3,y'[0]==-2}},y[x],{x,0,7}]
```

$$y(x)\to\frac{x^{7}}{24}-\frac{x^{6}}{5}-\frac{x^{5}}{4}+x^{4}+x^{3}-3x^{2}-2x+3$$

4.8 problem 16

Internal problem ID [6650]
Internal file name [`OUTPUT/5898_Sunday_June_05_2022_04_00_35_PM_80550766/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(x+2)\,y''+3y=0}$$

With initial conditions

$$[y(0)=0, y'(0)=1]$$

With the expansion point for the power series method at $x=0$.

4.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x)&=0\\ q(x)&=\frac{3}{x+2}\\ F&=0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{3y}{x + 2} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{3}{x+2}$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (196)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (197)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4) \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5) \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n |_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3y}{x+2}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= \frac{(-3x-6)\,y'+3y}{(x+2)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= \frac{(6x+12)\,y'+(9x+12)\,y}{(x+2)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \frac{9x^2y'+18xy'-36xy-54y}{(x+2)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= \frac{(-54x^2-144x-72)\,y'-27\left(x^2-2x-\frac{16}{3}\right)y}{(x+2)^5}\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x}+\frac{\partial F_4}{\partial y}y'+\frac{\partial F_4}{\partial y'}F_4\\
&= \frac{(-27x^3+162x^2+612x+360)\,y'+243\left(x^2+\frac{4}{9}x-\frac{44}{27}\right)y}{(x+2)^6}\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x}+\frac{\partial F_5}{\partial y}y'+\frac{\partial F_5}{\partial y'}F_5\\
&= \frac{(324x^3-216x^2-2592x-1728)\,y'+81y\left(x^3-18x^2-\frac{52}{3}x+\frac{56}{3}\right)}{(x+2)^7}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -\frac{3}{2} \\
F_2 &= \frac{3}{2} \\
F_3 &= 0 \\
F_4 &= -\frac{9}{4} \\
F_5 &= \frac{45}{8} \\
F_6 &= -\frac{27}{2}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x - \frac{x^3}{4} + \frac{x^4}{16} - \frac{x^6}{320} + \frac{x^7}{896} - \frac{3x^8}{8960} + O\left(x^8\right)
$$

$$
y = x - \frac{x^3}{4} + \frac{x^4}{16} - \frac{x^6}{320} + \frac{x^7}{896} - \frac{3x^8}{8960} + O\left(x^8\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x + 2)\, y'' + 3y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$(x+2)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+3\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\,x^{n-1}a_n(n-1)\right)+\left(\sum_{n=2}^{\infty} 2n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} 3a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\,x^{n-1}a_n(n-1)=\sum_{n=1}^{\infty}(n+1)\,a_{n+1}n\,x^n$$

$$\sum_{n=2}^{\infty} 2n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty} 2(n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty}(n+1)\,a_{n+1}n\,x^n\right)+\left(\sum_{n=0}^{\infty} 2(n+2)\,a_{n+2}(n+1)\,x^n\right)+\left(\sum_{n=0}^{\infty} 3a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$4a_2+3a_0=0$$

$$a_2=-\frac{3a_0}{4}$$

For $1\leq n$, the recurrence equation is

$$(n+1)\,a_{n+1}n+2(n+2)\,a_{n+2}(n+1)+3a_n=0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n^2 a_{n+1} + n a_{n+1} + 3a_n}{2\left(n+2\right)\left(n+1\right)} \\
&= -\frac{3a_n}{2\left(n+2\right)\left(n+1\right)} - \frac{\left(n^2+n\right) a_{n+1}}{2\left(n+2\right)\left(n+1\right)}
\end{aligned}
\tag{5}
$$

For $n = 1$ the recurrence equation gives

$$2a_2 + 12a_3 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{8} - \frac{a_1}{4}$$

For $n = 2$ the recurrence equation gives

$$6a_3 + 24a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{16} + \frac{a_1}{16}$$

For $n = 3$ the recurrence equation gives

$$12a_4 + 40a_5 + 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{9a_0}{320}$$

For $n = 4$ the recurrence equation gives

$$20a_5 + 60a_6 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{160} - \frac{a_1}{320}$$

For $n = 5$ the recurrence equation gives

$$30a_6 + 84a_7 + 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{11a_0}{8960} + \frac{a_1}{896}$$

For $n = 6$ the recurrence equation gives

$$42a_7 + 112a_8 + 3a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{3a_0}{10240} - \frac{3a_1}{8960}$$

For $n = 7$ the recurrence equation gives

$$56a_8 + 144a_9 + 3a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{19a_0}{215040} + \frac{23a_1}{215040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1x - \frac{3a_0x^2}{4} + \left(\frac{a_0}{8} - \frac{a_1}{4}\right)x^3 + \left(\frac{a_0}{16} + \frac{a_1}{16}\right)x^4 \\ - \frac{9a_0x^5}{320} + \left(\frac{a_0}{160} - \frac{a_1}{320}\right)x^6 + \left(-\frac{11a_0}{8960} + \frac{a_1}{896}\right)x^7 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = &\left(1-\frac{3}{4}x^2+\frac{1}{8}x^3+\frac{1}{16}x^4-\frac{9}{320}x^5+\frac{1}{160}x^6-\frac{11}{8960}x^7\right)a_0 \\
&+\left(x-\frac{1}{4}x^3+\frac{1}{16}x^4-\frac{1}{320}x^6+\frac{1}{896}x^7\right)a_1+O(x^8)
\end{aligned}
\tag{3}
$$

At $x = 0$ the solution above becomes

$$
\begin{aligned}
y = &\left(1-\frac{3}{4}x^2+\frac{1}{8}x^3+\frac{1}{16}x^4-\frac{9}{320}x^5+\frac{1}{160}x^6-\frac{11}{8960}x^7\right)c_1 \\
&+\left(x-\frac{1}{4}x^3+\frac{1}{16}x^4-\frac{1}{320}x^6+\frac{1}{896}x^7\right)c_2+O(x^8)
\end{aligned}
$$

$$
y = x-\frac{x^3}{4}+\frac{x^4}{16}-\frac{x^6}{320}+\frac{x^7}{896}+O(x^8)
$$

Summary

The solution(s) found are the following

$$
y = x-\frac{x^3}{4}+\frac{x^4}{16}-\frac{x^6}{320}+\frac{x^7}{896}-\frac{3x^8}{8960}+O(x^8) \tag{1}
$$

$$
y = x-\frac{x^3}{4}+\frac{x^4}{16}-\frac{x^6}{320}+\frac{x^7}{896}+O(x^8) \tag{2}
$$

Verification of solutions

$$
y = x-\frac{x^3}{4}+\frac{x^4}{16}-\frac{x^6}{320}+\frac{x^7}{896}-\frac{3x^8}{8960}+O(x^8)
$$

Verified OK.

$$
y = x-\frac{x^3}{4}+\frac{x^4}{16}-\frac{x^6}{320}+\frac{x^7}{896}+O(x^8)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=8;
dsolve([(x+2)*diff(y(x),x$2)+3*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='series',x=0);
```

$$y(x) = x - \frac{1}{4}x^3 + \frac{1}{16}x^4 - \frac{1}{320}x^6 + \frac{1}{896}x^7 + \mathrm{O}\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 33

```
AsymptoticDSolveValue[{(x+2)*y''[x]+3*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,7}]
```

$$y(x) \to \frac{x^7}{896} - \frac{x^6}{320} + \frac{x^4}{16} - \frac{x^3}{4} + x$$

4.9 problem 17

Internal problem ID [6651]
Internal file name [`OUTPUT/5899_Sunday_June_05_2022_04_00_38_PM_11393271/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-2\sin(x))\,y''+xy=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (199)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (200)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$F_0 = \frac{xy}{-1 + 2\sin(x)}$$

$$\begin{aligned}F_1 &= \frac{dF_0}{dx} \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\ &= \frac{x(-1 + 2\sin(x))\, y' - 2\left(\cos(x)\, x - \sin(x) + \frac{1}{2}\right) y}{(-1 + 2\sin(x))^2}\end{aligned}$$

$$\begin{aligned}F_2 &= \frac{dF_1}{dx} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\ &= \frac{\big(8\cos(x)^2 + (8\sin(x)\, x - 4x)\cos(x) + 8\sin(x) - 10\big)\, y' - 2y\big(2\cos(x)^2 x + (-4\sin(x) + 2)\cos(x)}{(8\sin(x) - 12)\cos(x)^2 - 14\sin(x) + 13}\end{aligned}$$

$$\begin{aligned}F_3 &= \frac{dF_2}{dx} \\ &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\ &= \frac{\big(48\cos(x)^3 - 4x(x - 6\sin(x))\cos(x)^2 + (48\sin(x) - 60)\cos(x) + (-4x^2 + 30x)\sin(x) + 5x^2 - 2}{16\cos(x)^4 + (32\sin(}\end{aligned}$$

$$\begin{aligned}F_4 &= \frac{dF_3}{dx} \\ &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\ &= \frac{\Big(-192\cos(x)^4 + 48\Big(x - \frac{4\sin(x)}{3} - 2\Big)\, x\cos(x)^3 + ((-48x - 96)\sin(x) + 72x - 48)\cos(x)^2 + ((48x}{}\end{aligned}$$

$$\begin{aligned}F_5 &= \frac{dF_4}{dx} \\ &= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\ &= \frac{\big(-640\cos(x)^5 + (368x^2 - 160\sin(x)\, x - 800x - 160)\cos(x)^4 + ((-736x + 640)\sin(x) + 1104x -}{}\end{aligned}$$

$$\begin{aligned}F_6 &= \frac{dF_5}{dx} \\ &= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\ &= \frac{\big(1920\cos(x)^6 + (-2304x^2 + 384\sin(x)\, x + 4800x + 3072)\cos(x)^5 + ((6912x - 8640)\sin(x) - 192}{}\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -2y'(0) - 4y(0) \\
F_3 &= -12y'(0) - 24y(0) \\
F_4 &= -96y'(0) - 180y(0) \\
F_5 &= -910y'(0) - 1700y(0) \\
F_6 &= -10368y'(0) - 19308y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$\begin{aligned}
y = &\left(1 - \frac{1}{6}x^3 - \frac{1}{6}x^4 - \frac{1}{5}x^5 - \frac{1}{4}x^6 - \frac{85}{252}x^7 - \frac{1609}{3360}x^8\right) y(0) \\
&+ \left(x - \frac{1}{12}x^4 - \frac{1}{10}x^5 - \frac{2}{15}x^6 - \frac{13}{72}x^7 - \frac{9}{35}x^8\right) y'(0) + O(x^8)
\end{aligned}$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \frac{x\left(\sum_{n=0}^{\infty} a_n x^n\right)}{-1 + 2\sin(x)} \tag{1}$$

Expanding $1 - 2\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$\begin{aligned}
1 - 2\sin(x) &= 1 - 2x + \frac{1}{3}x^3 - \frac{1}{60}x^5 + \frac{1}{2520}x^7 - \frac{1}{181440}x^9 + \dots \\
&= 1 - 2x + \frac{1}{3}x^3 - \frac{1}{60}x^5 + \frac{1}{2520}x^7 - \frac{1}{181440}x^9
\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(1-2x+\frac{1}{3}x^{3}-\frac{1}{60}x^{5}+\frac{1}{2520}x^{7}-\frac{1}{181440}x^{9}\right)\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)$$
$$+x\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)=0$$

Expanding the first term in (1) gives

$$1\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)-2x\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\frac{x^{3}}{3}$$
$$\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)-\frac{x^{5}}{60}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\frac{x^{7}}{2520}$$
$$\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)-\frac{x^{9}}{181440}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+x\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)=0$$

Which simplifies to

$$\begin{aligned}&\sum_{n=2}^{\infty}\left(-\frac{n\,x^{n+7}a_{n}(n-1)}{181440}\right)+\left(\sum_{n=2}^{\infty}\frac{n\,x^{n+5}a_{n}(n-1)}{2520}\right)\\&+\sum_{n=2}^{\infty}\left(-\frac{n\,x^{n+3}a_{n}(n-1)}{60}\right)+\left(\sum_{n=2}^{\infty}\frac{n\,x^{1+n}a_{n}(n-1)}{3}\right)\\&+\sum_{n=2}^{\infty}\left(-2n\,x^{n-1}a_{n}(n-1)\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\left(\sum_{n=0}^{\infty}x^{1+n}a_{n}\right)=0\end{aligned}\tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-\frac{n\,x^{n+7}a_{n}(n-1)}{181440}\right)=\sum_{n=9}^{\infty}\left(-\frac{(n-7)\,a_{n-7}(n-8)\,x^{n}}{181440}\right)$$
$$\sum_{n=2}^{\infty}\frac{n\,x^{n+5}a_{n}(n-1)}{2520}=\sum_{n=7}^{\infty}\frac{(n-5)\,a_{n-5}(n-6)\,x^{n}}{2520}$$

$$\sum_{n=2}^{\infty}\left(-\frac{n\,x^{n+3}a_n(n-1)}{60}\right)=\sum_{n=5}^{\infty}\left(-\frac{(n-3)\,a_{n-3}(n-4)\,x^{n}}{60}\right)$$

$$\sum_{n=2}^{\infty}\frac{n\,x^{1+n}a_n(n-1)}{3}=\sum_{n=3}^{\infty}\frac{(n-1)\,a_{n-1}(n-2)\,x^{n}}{3}$$

$$\sum_{n=2}^{\infty}\left(-2n\,x^{n-1}a_n(n-1)\right)=\sum_{n=1}^{\infty}\left(-2(1+n)\,a_{1+n}n\,x^{n}\right)$$

$$\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(1+n)\,x^{n}$$

$$\sum_{n=0}^{\infty}x^{1+n}a_n=\sum_{n=1}^{\infty}a_{n-1}x^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}
&\sum_{n=9}^{\infty}\left(-\frac{(n-7)\,a_{n-7}(n-8)\,x^{n}}{181440}\right)+\left(\sum_{n=7}^{\infty}\frac{(n-5)\,a_{n-5}(n-6)\,x^{n}}{2520}\right)\\
&+\sum_{n=5}^{\infty}\left(-\frac{(n-3)\,a_{n-3}(n-4)\,x^{n}}{60}\right)\\
&+\left(\sum_{n=3}^{\infty}\frac{(n-1)\,a_{n-1}(n-2)\,x^{n}}{3}\right)+\sum_{n=1}^{\infty}\left(-2(1+n)\,a_{1+n}n\,x^{n}\right)\\
&+\left(\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(1+n)\,x^{n}\right)+\left(\sum_{n=1}^{\infty}a_{n-1}x^{n}\right)=0
\end{aligned}\tag{3}$$

$n=1$ gives

$$-4a_2+6a_3+a_0=0$$

Which after substituting earlier equations, simplifies to

$$a_3=-\frac{a_0}{6}$$

$n=2$ gives

$$-12a_3+12a_4+a_1=0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{a_0}{6} - \frac{a_1}{12}$$

$n = 3$ gives

$$\frac{5a_2}{3} - 24a_4 + 20a_5 = 0$$

Which after substituting earlier equations, simplifies to

$$4a_0 + 2a_1 + 20a_5 = 0$$

Or

$$a_5 = -\frac{a_0}{5} - \frac{a_1}{10}$$

$n = 4$ gives

$$3a_3 - 40a_5 + 30a_6 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{15a_0}{2} + 4a_1 + 30a_6 = 0$$

Or

$$a_6 = -\frac{a_0}{4} - \frac{2a_1}{15}$$

$n = 5$ gives

$$-\frac{a_2}{30} + 5a_4 - 60a_6 + 42a_7 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{85a_0}{6} + \frac{91a_1}{12} + 42a_7 = 0$$

Or

$$a_7 = -\frac{85a_0}{252} - \frac{13a_1}{72}$$

$n = 6$ gives

$$-\frac{a_3}{10} + \frac{23a_5}{3} - 84a_7 + 56a_8 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{1609a_0}{60} + \frac{72a_1}{5} + 56a_8 = 0$$

Or

$$a_8 = -\frac{1609a_0}{3360} - \frac{9a_1}{35}$$

$n = 7$ gives

$$\frac{a_2}{1260} - \frac{a_4}{5} + 11a_6 - 112a_8 + 72a_9 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{611a_0}{12} + \frac{547a_1}{20} + 72a_9 = 0$$

Or

$$a_9 = -\frac{611a_0}{864} - \frac{547a_1}{1440}$$

For $9 \leq n$, the recurrence equation is

$$\begin{aligned} &-\frac{(n-7)\, a_{n-7}(n-8)}{181440} + \frac{(n-5)\, a_{n-5}(n-6)}{2520} - \frac{(n-3)\, a_{n-3}(n-4)}{60} \\ &+ \frac{(n-1)\, a_{n-1}(n-2)}{3} - 2(1+n)\, a_{1+n} n + (n+2)\, a_{n+2}(1+n) + a_{n-1} = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

(5)

$$\begin{aligned} &a_{n+2} \\ &= \frac{362880n^2 a_{1+n} + n^2 a_{n-7} - 72n^2 a_{n-5} + 3024n^2 a_{n-3} - 60480n^2 a_{n-1} + 362880n a_{1+n} - 15n a_{n-7} + 792n a_n}{181440\,(n+2)\,(1+n} \\ &= \frac{(362880n^2 + 362880n)\, a_{1+n}}{181440\,(n+2)\,(1+n)} + \frac{(n^2 - 15n + 56)\, a_{n-7}}{181440\,(n+2)\,(1+n)} + \frac{(-72n^2 + 792n - 2160)\, a_{n-5}}{181440\,(n+2)\,(1+n)} \\ &+ \frac{(3024n^2 - 21168n + 36288)\, a_{n-3}}{181440\,(n+2)\,(1+n)} + \frac{(-60480n^2 + 181440n - 302400)\, a_{n-1}}{181440\,(n+2)\,(1+n)} \end{aligned}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned}y = a_0 + a_1 x - \frac{a_0 x^3}{6} + \left(-\frac{a_0}{6} - \frac{a_1}{12}\right) x^4 + \left(-\frac{a_0}{5} - \frac{a_1}{10}\right) x^5 \\ + \left(-\frac{a_0}{4} - \frac{2a_1}{15}\right) x^6 + \left(-\frac{85a_0}{252} - \frac{13a_1}{72}\right) x^7 + \dots\end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned}y = \left(1 - \frac{1}{6}x^3 - \frac{1}{6}x^4 - \frac{1}{5}x^5 - \frac{1}{4}x^6 - \frac{85}{252}x^7\right) a_0 \\ + \left(x - \frac{1}{12}x^4 - \frac{1}{10}x^5 - \frac{2}{15}x^6 - \frac{13}{72}x^7\right) a_1 + O(x^8)\end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned}y = \left(1 - \frac{1}{6}x^3 - \frac{1}{6}x^4 - \frac{1}{5}x^5 - \frac{1}{4}x^6 - \frac{85}{252}x^7\right) c_1 \\ + \left(x - \frac{1}{12}x^4 - \frac{1}{10}x^5 - \frac{2}{15}x^6 - \frac{13}{72}x^7\right) c_2 + O(x^8)\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y = \left(1 - \frac{1}{6}x^3 - \frac{1}{6}x^4 - \frac{1}{5}x^5 - \frac{1}{4}x^6 - \frac{85}{252}x^7 - \frac{1609}{3360}x^8\right) y(0) \\ + \left(x - \frac{1}{12}x^4 - \frac{1}{10}x^5 - \frac{2}{15}x^6 - \frac{13}{72}x^7 - \frac{9}{35}x^8\right) y'(0) + O(x^8)\end{aligned} \tag{1}$$

$$\begin{aligned}y = \left(1 - \frac{1}{6}x^3 - \frac{1}{6}x^4 - \frac{1}{5}x^5 - \frac{1}{4}x^6 - \frac{85}{252}x^7\right) c_1 \\ + \left(x - \frac{1}{12}x^4 - \frac{1}{10}x^5 - \frac{2}{15}x^6 - \frac{13}{72}x^7\right) c_2 + O(x^8)\end{aligned} \tag{2}$$

Verification of solutions

$$y=\left(1-\frac{1}{6}x^{3}-\frac{1}{6}x^{4}-\frac{1}{5}x^{5}-\frac{1}{4}x^{6}-\frac{85}{252}x^{7}-\frac{1609}{3360}x^{8}\right)y(0)+\left(x-\frac{1}{12}x^{4}-\frac{1}{10}x^{5}-\frac{2}{15}x^{6}-\frac{13}{72}x^{7}-\frac{9}{35}x^{8}\right)y'(0)+O\left(x^{8}\right)$$

Verified OK.

$$y=\left(1-\frac{1}{6}x^{3}-\frac{1}{6}x^{4}-\frac{1}{5}x^{5}-\frac{1}{4}x^{6}-\frac{85}{252}x^{7}\right)c_{1}+\left(x-\frac{1}{12}x^{4}-\frac{1}{10}x^{5}-\frac{2}{15}x^{6}-\frac{13}{72}x^{7}\right)c_{2}+O\left(x^{8}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
   -> trying with_periodic_functions in the coefficients
      --- Trying Lie symmetry methods, 2nd order ---
      `, `-> Computing symmetries using: way = 5`[0, y]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
Order:=8;
dsolve((1-2*sin(x))*diff(y(x),x$2)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{6}x^3-\frac{1}{6}x^4-\frac{1}{5}x^5-\frac{1}{4}x^6-\frac{85}{252}x^7\right)y(0) + \left(x-\frac{1}{12}x^4-\frac{1}{10}x^5-\frac{2}{15}x^6-\frac{13}{72}x^7\right)D(y)(0)+O(x^8)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 77

```
AsymptoticDSolveValue[(1-2*Sin[x])*y''[x]+x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_2\left(-\frac{13x^7}{72}-\frac{2x^6}{15}-\frac{x^5}{10}-\frac{x^4}{12}+x\right)+c_1\left(-\frac{85x^7}{252}-\frac{x^6}{4}-\frac{x^5}{5}-\frac{x^4}{6}-\frac{x^3}{6}+1\right)$$

4.10 problem 18

Internal problem ID [6652]
Internal file name [`OUTPUT/5900_Sunday_June_05_2022_04_00_42_PM_10570403/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + xy' + y = 0}$$

With initial conditions

$$[y(1) = -6, y'(1) = 3]$$

With the expansion point for the power series method at $x = 1$.

4.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x \\ q(x) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + xy' + y = 0$$

The domain of $p(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) + (t+1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$\begin{aligned} y(0) &= -6 \\ y'(0) &= 3 \end{aligned}$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (202) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (203) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -t\left(\frac{d}{dt}y(t)\right) - \frac{d}{dt}y(t) - y(t)\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0\\
&= \left(t^2 + 2t - 1\right)\left(\frac{d}{dt}y(t)\right) + \left(t + 1\right)y(t)\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1\\
&= \left(-t^3 - 3t^2 + 2t + 4\right)\left(\frac{d}{dt}y(t)\right) - y(t)\left(t^2 + 2t - 2\right)\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2\\
&= \left(t^4 + 4t^3 - 3t^2 - 14t\right)\left(\frac{d}{dt}y(t)\right) + y(t)\left(t + 1\right)\left(t^2 + 2t - 6\right)\\
F_4 &= \frac{dF_3}{dt}\\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3\\
&= \left(-t^5 - 5t^4 + 4t^3 + 32t^2 + 4t - 20\right)\left(\frac{d}{dt}y(t)\right) - y(t)\left(t^4 + 4t^3 - 6t^2 - 20t + 4\right)\\
F_5 &= \frac{dF_4}{dt}\\
&= \frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_4}{\partial \frac{d}{dt}y\left(t\right)}F_4\\
&= \left(t^6 + 6t^5 - 5t^4 - 60t^3 - 18t^2 + 100t + 20\right)\left(\frac{d}{dt}y(t)\right) + y(t)\left(t + 1\right)\left(t^4 + 4t^3 - 12t^2 - 32t + 40\right)\\
F_6 &= \frac{dF_5}{dt}\\
&= \frac{\partial F_5}{\partial t} + \frac{\partial F_5}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_5}{\partial \frac{d}{dt}y\left(t\right)}F_5\\
&= \left(-t^7 - 7t^6 + 6t^5 + 100t^4 + 50t^3 - 306t^2 - 148t + 120\right)\left(\frac{d}{dt}y(t)\right) - y(t)\left(t^2 + 2t - 6\right)\left(t^4 + 4t^3 - 12
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = -6$ and $y'(0) = 3$ gives

$$
\begin{aligned}
F_0 &= 3 \\
F_1 &= -9 \\
F_2 &= 0 \\
F_3 &= 36 \\
F_4 &= -36 \\
F_5 &= -180 \\
F_6 &= 432
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = -6 + 3t + \frac{3t^2}{2} - \frac{3t^3}{2} + \frac{3t^5}{10} - \frac{t^6}{20} - \frac{t^7}{28} + \frac{3t^8}{280} + O\left(t^8\right)
$$

$$
y(t) = -6 + 3t + \frac{3t^2}{2} - \frac{3t^3}{2} + \frac{3t^5}{10} - \frac{t^6}{20} - \frac{t^7}{28} + \frac{3t^8}{280} + O\left(t^8\right)
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = -t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n t^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n \\ \sum_{n=1}^{\infty} n a_n t^{n-1} &= \sum_{n=0}^{\infty} (n+1)\, a_{n+1} t^n\end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2} - \frac{a_1}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + (n+1)\, a_{n+1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{a_n + a_{n+1}}{n+2} \\ &= -\frac{a_n}{n+2} - \frac{a_{n+1}}{n+2} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_1 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6} + \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_2 + 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{12} + \frac{a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 + 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_4 + 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{180} - \frac{a_1}{36}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_5 + 6a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{126} + \frac{a_1}{252}$$

For $n = 6$ the recurrence equation gives

$$56a_8 + 7a_6 + 7a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_8 = -\frac{a_0}{3360} + \frac{a_1}{336}$$

For $n = 7$ the recurrence equation gives

$$72a_9 + 8a_7 + 8a_8 = 0$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{11a_0}{12960} - \frac{a_1}{1296}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t + \left(-\frac{a_0}{2} - \frac{a_1}{2}\right) t^2 + \left(-\frac{a_1}{6} + \frac{a_0}{6}\right) t^3 + \left(\frac{a_0}{12} + \frac{a_1}{6}\right) t^4 \\ - \frac{a_0 t^5}{20} + \left(-\frac{a_0}{180} - \frac{a_1}{36}\right) t^6 + \left(\frac{a_0}{126} + \frac{a_1}{252}\right) t^7 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y(t) = &\left(1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{20}t^5 - \frac{1}{180}t^6 + \frac{1}{126}t^7\right) a_0 \\ &+ \left(t - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{6}t^4 - \frac{1}{36}t^6 + \frac{1}{252}t^7\right) a_1 + O(t^8) \end{aligned} \tag{3}$$

At $t = 0$ the solution above becomes

$$\begin{aligned} y(t) = &\left(1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{20}t^5 - \frac{1}{180}t^6 + \frac{1}{126}t^7\right) c_1 \\ &+ \left(t - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{6}t^4 - \frac{1}{36}t^6 + \frac{1}{252}t^7\right) c_2 + O(t^8) \end{aligned}$$

$$y(t) = -6 + \frac{3t^2}{2} - \frac{3t^3}{2} + \frac{3t^5}{10} - \frac{t^6}{20} - \frac{t^7}{28} + 3t + O(t^8)$$

Replacing t in the above with the original independent variable x using $t = x - 1$ results in

$$\begin{aligned} y = -9 + 3x &+ \frac{3(x-1)^2}{2} - \frac{3(x-1)^3}{2} + \frac{3(x-1)^5}{10} \\ &- \frac{(x-1)^6}{20} - \frac{(x-1)^7}{28} + \frac{3(x-1)^8}{280} + O\left((x-1)^8\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = -9 + 3x &+ \frac{3(x-1)^2}{2} - \frac{3(x-1)^3}{2} + \frac{3(x-1)^5}{10} \\ &- \frac{(x-1)^6}{20} - \frac{(x-1)^7}{28} + \frac{3(x-1)^8}{280} + O\left((x-1)^8\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = -9 + 3x &+ \frac{3(x-1)^2}{2} - \frac{3(x-1)^3}{2} + \frac{3(x-1)^5}{10} \\ &- \frac{(x-1)^6}{20} - \frac{(x-1)^7}{28} + \frac{3(x-1)^8}{280} + O\left((x-1)^8\right) \end{aligned}$$

Verified OK.

4.10.2 Maple step by step solution

Let's solve

$$\left[y'' + xy' + y = 0, y(1) = -6, y'\Big|_{\{x=1\}} = 3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

  - Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1) \, x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+1)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+1)\,(a_{k+2}(k+2) + a_k) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
Order:=8;
dsolve([diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(1) = -6, D(y)(1) = 3],y(x),type='series',x=1)
```

$$y(x) = -6 + 3(x-1) + \frac{3}{2}(x-1)^2 - \frac{3}{2}(x-1)^3 + \frac{3}{10}(x-1)^5 - \frac{1}{20}(x-1)^6 - \frac{1}{28}(x-1)^7 + \mathrm{O}\left((x-1)^8\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 55

```
AsymptoticDSolveValue[{y''[x]+x*y'[x]+y[x]==0,{y[1]==-6,y'[1]==3}},y[x],{x,1,7}]
```

$$y(x) \to -\frac{1}{28}(x-1)^7 - \frac{1}{20}(x-1)^6 + \frac{3}{10}(x-1)^5 - \frac{3}{2}(x-1)^3 + \frac{3}{2}(x-1)^2 + 3(x-1) - 6$$

4.11 problem 19

Internal problem ID [6653]
Internal file name [`OUTPUT/5901_Sunday_June_05_2022_04_00_46_PM_64195058/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{xy'' + (-\cos(x) + 1)\, y' + y x^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (205)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (206)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{-yx^2 + \cos(x)\, y' - y'}{x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(-x^3 - \sin(x)\, x + \cos(x)^2 - 3\cos(x) + 2\right) y' - \cos(x)\, yx^2}{x^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\big(\cos(x)^3 - 6\cos(x)^2 + (-2x^3 - x^2 - 3\sin(x)\, x + 11)\cos(x) + 5\sin(x)\, x - 6\big)\, y' + yx^2 \big(x^3 + 2\sin(}{x^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\big(\cos(x)^4 - 10\cos(x)^3 + (-3x^3 - 7x^2 - 6\sin(x)\, x + 35)\cos(x)^2 + (5x^3 + 7x^2 + 26\sin(x)\, x - 50)\, c}{} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\big(\cos(x)^5 - 15\cos(x)^4 + (-4x^3 - 25x^2 - 10\sin(x)\, x + 85)\cos(x)^3 + (18x^3 + 80x^2 + 80\sin(x)\, x -}{} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= \frac{\big(\cos(x)^6 - 21\cos(x)^5 + (-5x^3 - 65x^2 - 15\sin(x)\, x + 175)\cos(x)^4 + (42x^3 + 420x^2 + 190\sin(x)\, x}{} \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \frac{\big(\cos(x)^7 - 28\cos(x)^6 + (-6x^3 - 140x^2 - 21\sin(x)\, x + 322)\cos(x)^5 + (80x^3 + 1505x^2 + 385\sin(x}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) - \frac{y'(0)}{2} \\
F_2 &= -2y'(0) \\
F_3 &= \frac{3y(0)}{2} + y'(0) \\
F_4 &= 6y'(0) + 4y(0) \\
F_5 &= \frac{73y'(0)}{12} - \frac{25y(0)}{4} \\
F_6 &= -34y'(0) - 21y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{1}{6}x^3 + \frac{1}{80}x^5 + \frac{1}{180}x^6 - \frac{5}{4032}x^7 - \frac{1}{1920}x^8\right) y(0) \\
&+ \left(x - \frac{1}{12}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{1}{120}x^6 + \frac{73}{60480}x^7 - \frac{17}{20160}x^8\right) y'(0) + O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
xy'' + (-\cos(x) + 1) y' + yx^2 = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) x + (-\cos(x) + 1) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 = 0 \quad (1)
$$

Expanding $-\cos(x)+1$ as Taylor series around $x=0$ and keeping only the first 8 terms gives

$$\begin{aligned}-\cos(x)+1 &= \frac{1}{2}x^2-\frac{1}{24}x^4+\frac{1}{720}x^6-\frac{1}{40320}x^8+\dots\\ &= \frac{1}{2}x^2-\frac{1}{24}x^4+\frac{1}{720}x^6-\frac{1}{40320}x^8\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)x\\ &+\left(\frac{1}{2}x^2-\frac{1}{24}x^4+\frac{1}{720}x^6-\frac{1}{40320}x^8\right)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)x^2=0\end{aligned}$$

Expanding the second term in (1) gives

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)x+\frac{x^2}{2}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-\frac{x^4}{24}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)\\ &+\frac{x^6}{720}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-\frac{x^8}{40320}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)x^2=0\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} n\,x^{n-1}a_n(n-1)\right)+\left(\sum_{n=1}^{\infty}\frac{n\,x^{1+n}a_n}{2}\right)+\sum_{n=1}^{\infty}\left(-\frac{n\,x^{n+3}a_n}{24}\right)\\ &+\left(\sum_{n=1}^{\infty}\frac{n\,x^{n+5}a_n}{720}\right)+\sum_{n=1}^{\infty}\left(-\frac{n\,x^{n+7}a_n}{40320}\right)+\left(\sum_{n=0}^{\infty} x^{n+2}a_n\right)=0\end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\,x^{n-1}a_n(n-1)=\sum_{n=1}^{\infty}(1+n)\,a_{1+n}n\,x^n$$

$$\sum_{n=1}^{\infty}\frac{n\,x^{1+n}a_n}{2}=\sum_{n=2}^{\infty}\frac{(n-1)\,a_{n-1}x^n}{2}$$

$$\sum_{n=1}^{\infty} \left(-\frac{n\, x^{n+3} a_n}{24}\right) = \sum_{n=4}^{\infty} \left(-\frac{(n-3)\, a_{n-3} x^n}{24}\right)$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+5} a_n}{720} = \sum_{n=6}^{\infty} \frac{(n-5)\, a_{n-5} x^n}{720}$$

$$\sum_{n=1}^{\infty} \left(-\frac{n\, x^{n+7} a_n}{40320}\right) = \sum_{n=8}^{\infty} \left(-\frac{(n-7)\, a_{n-7} x^n}{40320}\right)$$

$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{multline}\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n\right) + \left(\sum_{n=2}^{\infty} \frac{(n-1)\, a_{n-1} x^n}{2}\right) + \sum_{n=4}^{\infty} \left(-\frac{(n-3)\, a_{n-3} x^n}{24}\right) \\ + \left(\sum_{n=6}^{\infty} \frac{(n-5)\, a_{n-5} x^n}{720}\right) + \sum_{n=8}^{\infty} \left(-\frac{(n-7)\, a_{n-7} x^n}{40320}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) = 0 \end{multline} \tag{3}$$

$n = 1$ gives

$$6a_3 + \frac{a_1}{2} + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} - \frac{a_1}{12}$$

$n = 2$ gives

$$12a_4 + a_2 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{a_1}{12}$$

$n = 3$ gives

$$20a_5 + \frac{3a_3}{2} - \frac{a_1}{24} + a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_0}{80} + \frac{a_1}{120}$$

$n = 4$ gives

$$30a_6 + 2a_4 - \frac{a_2}{12} + a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$30a_6 - \frac{a_1}{4} - \frac{a_0}{6} = 0$$

Or

$$a_6 = \frac{a_0}{180} + \frac{a_1}{120}$$

$n = 5$ gives

$$42a_7 + \frac{5a_5}{2} - \frac{a_3}{8} + \frac{a_1}{720} + a_4 = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{5a_0}{4032} + \frac{73a_1}{60480}$$

$n = 6$ gives

$$56a_8 + 3a_6 - \frac{a_4}{6} + \frac{a_2}{360} + a_5 = 0$$

Which after substituting earlier equations, simplifies to

$$56a_8 + \frac{7a_0}{240} + \frac{17a_1}{360} = 0$$

Or

$$a_8 = -\frac{a_0}{1920} - \frac{17a_1}{20160}$$

For $8 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} n + \frac{(n-1)\, a_{n-1}}{2} - \frac{(n-3)\, a_{n-3}}{24} + \frac{(n-5)\, a_{n-5}}{720} - \frac{(n-7)\, a_{n-7}}{40320} + a_{n-2} = 0 \tag{4}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y = a_0 + a_1 x + \left(-\frac{a_0}{6} - \frac{a_1}{12}\right) x^3 - \frac{a_1 x^4}{12} + \left(\frac{a_0}{80} + \frac{a_1}{120}\right) x^5 \\
+ \left(\frac{a_0}{180} + \frac{a_1}{120}\right) x^6 + \left(-\frac{5a_0}{4032} + \frac{73a_1}{60480}\right) x^7 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{80}x^5 + \frac{1}{180}x^6 - \frac{5}{4032}x^7\right) a_0 \\
+ \left(x - \frac{1}{12}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{1}{120}x^6 + \frac{73}{60480}x^7\right) a_1 + O(x^8)
\end{aligned}
\tag{3}
$$

At $x = 0$ the solution above becomes

$$
\begin{aligned}
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{80}x^5 + \frac{1}{180}x^6 - \frac{5}{4032}x^7\right) c_1 \\
+ \left(x - \frac{1}{12}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{1}{120}x^6 + \frac{73}{60480}x^7\right) c_2 + O(x^8)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{80}x^5 + \frac{1}{180}x^6 - \frac{5}{4032}x^7 - \frac{1}{1920}x^8\right) y(0) \\
+ \left(x - \frac{1}{12}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{1}{120}x^6 + \frac{73}{60480}x^7 - \frac{17}{20160}x^8\right) y'(0) + O(x^8)
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{80}x^5 + \frac{1}{180}x^6 - \frac{5}{4032}x^7\right) c_1 \\
+ \left(x - \frac{1}{12}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{1}{120}x^6 + \frac{73}{60480}x^7\right) c_2 + O(x^8)
\end{aligned}
\tag{2}
$$

Verification of solutions

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{80}x^5 + \frac{1}{180}x^6 - \frac{5}{4032}x^7 - \frac{1}{1920}x^8\right) y(0) \\ + \left(x - \frac{1}{12}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{1}{120}x^6 + \frac{73}{60480}x^7 - \frac{17}{20160}x^8\right) y'(0) + O(x^8)$$

Verified OK.

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{80}x^5 + \frac{1}{180}x^6 - \frac{5}{4032}x^7\right) c_1 \\ + \left(x - \frac{1}{12}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{1}{120}x^6 + \frac{73}{60480}x^7\right) c_2 + O(x^8)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 59

```
Order:=8;
dsolve(x*diff(y(x),x$2)+(1-cos(x))*diff(y(x),x)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & \left(1-\frac{1}{6}x^{3}+\frac{1}{80}x^{5}+\frac{1}{180}x^{6}-\frac{5}{4032}x^{7}\right)y(0) \\
& +\left(x-\frac{1}{12}x^{3}-\frac{1}{12}x^{4}+\frac{1}{120}x^{5}+\frac{1}{120}x^{6}+\frac{73}{60480}x^{7}\right)D(y)\left(0\right)+O\left(x^{8}\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 36

```
AsymptoticDSolveValue[x*y''[x]+(1-Cos[x])*y'[x]+x^2*y[x]==0,y[x],{x,0,7}]
```

$$
y(x) \rightarrow -\frac{53x^{7}}{8640}+\frac{x^{5}}{48}+\frac{x^{4}}{6}-\frac{x^{3}}{3}-2x+3
$$

4.12 problem 20

Internal problem ID [6654]
Internal file name [`OUTPUT/5902_Sunday_June_05_2022_04_00_51_PM_15720401/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(\mathrm{e}^{x} - 1 - x)\, y'' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(\mathrm{e}^{x} - 1 - x)\, y'' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{x}{\mathrm{e}^{x} - 1 - x} \end{aligned}$$

Table 126: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{x}{e^x - 1 - x}$	
singularity	type
$x = -\operatorname{LambertW}(Z, -e^{-1}) - 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-\operatorname{LambertW}(Z, -e^{-1}) - 1]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$(e^x - 1 - x)\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Expanding $e^x - 1 - x$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$\begin{aligned} e^x - 1 - x &= \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \dots \\ &= \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 \end{aligned}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}\frac{x^{n+r+6}a_n(n+r)(n+r-1)}{40320}\right)\\
&+\left(\sum_{n=0}^{\infty}\frac{x^{n+r+5}a_n(n+r)(n+r-1)}{5040}\right)\\
&+\left(\sum_{n=0}^{\infty}\frac{x^{n+r+4}a_n(n+r)(n+r-1)}{720}\right)\\
&+\left(\sum_{n=0}^{\infty}\frac{x^{n+r+3}a_n(n+r)(n+r-1)}{120}\right)\\
&+\left(\sum_{n=0}^{\infty}\frac{x^{n+r+2}a_n(n+r)(n+r-1)}{24}\right)\\
&+\left(\sum_{n=0}^{\infty}\frac{x^{1+n+r}a_n(n+r)(n+r-1)}{6}\right)\\
&+\left(\sum_{n=0}^{\infty}\frac{x^{n+r}a_n(n+r)(n+r-1)}{2}\right)+\left(\sum_{n=0}^{\infty}x^{1+n+r}a_n\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\frac{x^{n+r+6}a_n(n+r)(n+r-1)}{40320}=\sum_{n=6}^{\infty}\frac{a_{n-6}(n-6+r)(n-7+r)x^{n+r}}{40320}$$

$$\sum_{n=0}^{\infty}\frac{x^{n+r+5}a_n(n+r)(n+r-1)}{5040}=\sum_{n=5}^{\infty}\frac{a_{n-5}(-5+n+r)(n-6+r)x^{n+r}}{5040}$$

$$\sum_{n=0}^{\infty}\frac{x^{n+r+4}a_n(n+r)(n+r-1)}{720}=\sum_{n=4}^{\infty}\frac{a_{n-4}(n+r-4)(-5+n+r)x^{n+r}}{720}$$

$$\sum_{n=0}^{\infty}\frac{x^{n+r+3}a_n(n+r)(n+r-1)}{120}=\sum_{n=3}^{\infty}\frac{a_{n-3}(n+r-3)(n+r-4)x^{n+r}}{120}$$

$$\sum_{n=0}^{\infty}\frac{x^{n+r+2}a_n(n+r)(n+r-1)}{24}=\sum_{n=2}^{\infty}\frac{a_{n-2}(n+r-2)(n+r-3)x^{n+r}}{24}$$

$$\sum_{n=0}^{\infty} \frac{x^{1+n+r} a_n (n+r)(n+r-1)}{6} = \sum_{n=1}^{\infty} \frac{a_{n-1} (n+r-1)(n+r-2) x^{n+r}}{6}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}&\left(\sum_{n=6}^{\infty} \frac{a_{n-6} (n-6+r)(n-7+r) x^{n+r}}{40320}\right)\\ &+\left(\sum_{n=5}^{\infty} \frac{a_{n-5} (-5+n+r)(n-6+r) x^{n+r}}{5040}\right)\\ &+\left(\sum_{n=4}^{\infty} \frac{a_{n-4} (n+r-4)(-5+n+r) x^{n+r}}{720}\right)\\ &+\left(\sum_{n=3}^{\infty} \frac{a_{n-3} (n+r-3)(n+r-4) x^{n+r}}{120}\right)\\ &+\left(\sum_{n=2}^{\infty} \frac{a_{n-2} (n+r-2)(n+r-3) x^{n+r}}{24}\right)\\ &+\left(\sum_{n=1}^{\infty} \frac{a_{n-1} (n+r-1)(n+r-2) x^{n+r}}{6}\right)\\ &+\left(\sum_{n=0}^{\infty} \frac{x^{n+r} a_n (n+r)(n+r-1)}{2}\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right)=0\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$\frac{x^{n+r} a_n (n+r)(n+r-1)}{2} = 0$$

When $n=0$ the above becomes

$$\frac{x^r a_0 r(-1+r)}{2} = 0$$

Or

$$\frac{x^r a_0 r(-1+r)}{2} = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{x^r r(-1+r)}{2} = 0$$

Since the above is true for all x then the indicial equation becomes

$$\frac{r(-1+r)}{2} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{x^r r(-1+r)}{2} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-r^2 + r - 6}{3r\,(1 + r)}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{r^4 + 47r^2 + 144}{36r\,(1 + r)^2\,(2 + r)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = \frac{r^6 + 3r^5 - 179r^4 - 363r^3 - 2162r^2 - 1980r - 5760}{540r\,(1 + r)^2\,(2 + r)^2\,(3 + r)}$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = \frac{-r^8 - 8r^7 + 170r^6 + 1132r^5 + 12443r^4 + 42076r^3 + 116700r^2 + 152352r + 250560}{6480r\,(1 + r)^2\,(2 + r)^2\,(3 + r)^2\,(4 + r)}$$

Substituting $n = 5$ in Eq. (2B) gives

$$a_5 = \frac{-5r^{10} - 75r^9 - 240r^8 + 1170r^7 - 36405r^6 - 390015r^5 - 2082886r^4 - 6427176r^3 - 14403600r^2 - 19}{136080r\,(1 + r)^2\,(2 + r)^2\,(3 + r)^2\,(4 + r)^2\,(5 + r)}$$

For $6 \leq n$ the recursive equation is

$$\begin{aligned} &\frac{a_{n-6}(n - 6 + r)\,(n - 7 + r)}{40320} + \frac{a_{n-5}(-5 + n + r)\,(n - 6 + r)}{5040} \\ &+ \frac{a_{n-4}(n + r - 4)\,(-5 + n + r)}{720} + \frac{a_{n-3}(n + r - 3)\,(n + r - 4)}{120} \\ &+ \frac{a_{n-2}(n + r - 2)\,(n + r - 3)}{24} + \frac{a_{n-1}(n + r - 1)\,(n + r - 2)}{6} \\ &+ \frac{a_n(n + r)\,(n + r - 1)}{2} + a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-6} + 8n^2 a_{n-5} + 56n^2 a_{n-4} + 336n^2 a_{n-3} + 1680n^2 a_{n-2} + 6720n^2 a_{n-1} + 2nra_{n-6} + 16nra_{n-5} +}{} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(-a_{n-6} - 8a_{n-5} - 56a_{n-4} - 336a_{n-3} - 1680a_{n-2} - 6720a_{n-1})\,n^2 + (11a_{n-6} + 72a_{n-5} + 392a_{n-4} + 1}{2016} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r-6}{3r(1+r)}$	-1
a_2	$\frac{r^4+47r^2+144}{36r(1+r)^2(2+r)}$	$\frac{4}{9}$
a_3	$\frac{r^6+3r^5-179r^4-363r^3-2162r^2-1980r-5760}{540r(1+r)^2(2+r)^2(3+r)}$	$-\frac{29}{216}$
a_4	$\frac{-r^8-8r^7+170r^6+1132r^5+12443r^4+42076r^3+116700r^2+152352r+250560}{6480r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$\frac{37}{1200}$
a_5	$\frac{-5r^{10}-75r^9-240r^8+1170r^7-36405r^6-390015r^5-2082886r^4-6427176r^3-14403600r^2-19843488r-24167808}{136080r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{58}{10125}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-r^{12} - 24r^{11} - 1325r^{10} - 22980r^9 - 52863r^8 + 1292688r^7 + 18497065r^6 + 125947500r^5 + 5370299}{4082400r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)^2(5}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{14209}{15876000}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{-r^2+r-6}{3r(1+r)}$
a_2	$\frac{r^4+47r^2+144}{36r(1+r)^2(2+r)}$
a_3	$\frac{r^6+3r^5-179r^4-363r^3-2162r^2-1980r-5760}{540r(1+r)^2(2+r)^2(3+r)}$
a_4	$\frac{-r^8-8r^7+170r^6+1132r^5+12443r^4+42076r^3+116700r^2+152352r+250560}{6480r(1+r)^2(2+r)^2(3+r)^2(4+r)}$
a_5	$\frac{-5r^{10}-75r^9-240r^8+1170r^7-36405r^6-390015r^5-2082886r^4-6427176r^3-14403600r^2-19843488r-24167808}{136080r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$
a_6	$\frac{-r^{12}-24r^{11}-1325r^{10}-22980r^9-52863r^8+1292688r^7+18497065r^6+125947500r^5+537029924r^4+1503323616r^3+2999558880r^2+40320}{4082400r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{149r^{14} + 5215r^{13} + 79387r^{12} + 686735r^{11} + 3713973r^{10} + 13154325r^9 + 23818345r^8 - 104697475r^7}{24494400r\left(1+r\right)^2(2}$$

Which for the root $r = 1$ becomes

$$a_7 = -\frac{107329}{889056000}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{-r^2+r-6}{3r(1+r)}$
a_2	$\frac{r^4+47r^2+144}{36r(1+r)^2(2+r)}$
a_3	$\frac{r^6+3r^5-179r^4-363r^3-2162r^2-1980r-5760}{540r(1+r)^2(2+r)^2(3+r)}$
a_4	$\frac{-r^8-8r^7+170r^6+1132r^5+12443r^4+42076r^3+116700r^2+152352r+250560}{6480r(1+r)^2(2+r)^2(3+r)^2(4+r)}$
a_5	$\frac{-5r^{10}-75r^9-240r^8+1170r^7-36405r^6-390015r^5-2082886r^4-6427176r^3-14403600r^2-19843488r-24167808}{136080r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$
a_6	$\frac{-r^{12}-24r^{11}-1325r^{10}-22980r^9-52863r^8+1292688r^7+18497065r^6+125947500r^5+537029924r^4+1503323616r^3+2999558880r^2+40320}{4082400r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$
a_7	$\frac{149r^{14}+5215r^{13}+79387r^{12}+686735r^{11}+3713973r^{10}+13154325r^9+23818345r^8-104697475r^7-1414855462r^6-8172285680r^5-3021876}{24494400r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)^2(7+r)}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x\left(1 - x + \frac{4x^2}{9} - \frac{29x^3}{216} + \frac{37x^4}{1200} - \frac{58x^5}{10125} + \frac{14209x^6}{15876000} - \frac{107329x^7}{889056000} + O\left(x^8\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-r^2 + r - 6}{3r\left(1 + r\right)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{-r^2 + r - 6}{3r\left(1 + r\right)} &= \lim_{r \to 0} \frac{-r^2 + r - 6}{3r\left(1 + r\right)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $(\mathrm{e}^x - 1 - x)\,y'' + xy = 0$ gives

$$\begin{aligned}
&(\mathrm{e}^x - 1 - x)\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&\quad + x\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}
&\left(\left((\mathrm{e}^x - 1 - x)\,y_1''(x) + y_1(x)\,x\right)\ln(x) + (\mathrm{e}^x - 1 - x)\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right)C \\
&\quad + (\mathrm{e}^x - 1 - x)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \qquad (7) \\
&\quad + x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

But since $y_1(x)$ is a solution to the ode, then

$$(\mathrm{e}^x - 1 - x)\, y_1''(x) + y_1(x)\, x = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&(\mathrm{e}^x - 1 - x)\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) C \\
&+ (\mathrm{e}^x - 1 - x)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\frac{\left(-2x(-\mathrm{e}^x + 1 + x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) + (-\mathrm{e}^x + 1 + x)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x^2} \\
&+ \frac{-x^2(-\mathrm{e}^x + 1 + x)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)\,(-1+n+r_2)\right) + x^3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{x^2} = 0
\end{aligned} \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\begin{aligned}
&\frac{\left(-2x(-\mathrm{e}^x + 1 + x)\left(\sum_{n=0}^{\infty} x^n a_n (1+n)\right) + (-\mathrm{e}^x + 1 + x)\left(\sum_{n=0}^{\infty} a_n x^{1+n}\right)\right) C}{x^2} \\
&+ \frac{-x^2(-\mathrm{e}^x + 1 + x)\left(\sum_{n=0}^{\infty} x^{n-2} b_n n(n-1)\right) + x^3\left(\sum_{n=0}^{\infty} b_n x^n\right)}{x^2} = 0
\end{aligned} \tag{10}$$

Expanding $\frac{2C\,\mathrm{e}^x}{x}$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$\begin{aligned}
\frac{2C\,\mathrm{e}^x}{x} &= \frac{2C}{x} + 2C + Cx + \frac{C\,x^2}{3} + \frac{C\,x^3}{12} + \frac{C\,x^4}{60} + \frac{C\,x^5}{360} + \frac{C\,x^6}{2520} + \frac{C\,x^7}{20160} + \frac{C\,x^8}{181440} + \dots \\
&= \frac{2C}{x} + 2C + Cx + \frac{C\,x^2}{3} + \frac{C\,x^3}{12} + \frac{C\,x^4}{60} + \frac{C\,x^5}{360} + \frac{C\,x^6}{2520} + \frac{C\,x^7}{20160} + \frac{C\,x^8}{181440}
\end{aligned}$$

Expanding $-\frac{C\,\mathrm{e}^{x}}{x}$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
-\frac{C\,\mathrm{e}^{x}}{x} &= -\frac{C}{x}-C-\frac{Cx}{2}-\frac{C\,x^{2}}{6}-\frac{C\,x^{3}}{24}-\frac{C\,x^{4}}{120}-\frac{C\,x^{5}}{720}-\frac{C\,x^{6}}{5040}-\frac{C\,x^{7}}{40320}-\frac{C\,x^{8}}{362880}+\dots \\
&= -\frac{C}{x}-C-\frac{Cx}{2}-\frac{C\,x^{2}}{6}-\frac{C\,x^{3}}{24}-\frac{C\,x^{4}}{120}-\frac{C\,x^{5}}{720}-\frac{C\,x^{6}}{5040}-\frac{C\,x^{7}}{40320}-\frac{C\,x^{8}}{362880}
\end{aligned}
$$

Expanding $\frac{\mathrm{e}^{x}}{x^{2}}$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
\frac{\mathrm{e}^{x}}{x^{2}} &= \frac{1}{x^{2}}+\frac{1}{x}+\frac{1}{2}+\frac{x}{6}+\frac{x^{2}}{24}+\frac{x^{3}}{120}+\frac{x^{4}}{720}+\frac{x^{5}}{5040}+\frac{x^{6}}{40320}+\frac{x^{7}}{362880}+\frac{x^{8}}{3628800}+\dots \\
&= \frac{1}{x^{2}}+\frac{1}{x}+\frac{1}{2}+\frac{x}{6}+\frac{x^{2}}{24}+\frac{x^{3}}{120}+\frac{x^{4}}{720}+\frac{x^{5}}{5040}+\frac{x^{6}}{40320}+\frac{x^{7}}{362880}+\frac{x^{8}}{3628800}
\end{aligned}
$$

Which simplifies to

$$\text{Expression too large to display} \tag{2A}$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \frac{C\,x^{n+8} a_{n} (1+n)}{181440} = \sum_{n=10}^{\infty} \frac{C a_{n-10} (n-9)\,x^{n-2}}{181440}$$

$$\sum_{n=0}^{\infty} \frac{C\,x^{n+7} a_{n} (1+n)}{20160} = \sum_{n=9}^{\infty} \frac{C a_{n-9} (n-8)\,x^{n-2}}{20160}$$

$$\sum_{n=0}^{\infty} \frac{C\,x^{n+6} a_{n} (1+n)}{2520} = \sum_{n=8}^{\infty} \frac{C a_{n-8} (-7+n)\,x^{n-2}}{2520}$$

$$\sum_{n=0}^{\infty} \frac{C\,x^{n+5} a_{n} (1+n)}{360} = \sum_{n=7}^{\infty} \frac{C a_{-7+n} (n-6)\,x^{n-2}}{360}$$

$$\sum_{n=0}^{\infty} \frac{C\,x^{n+4} a_{n} (1+n)}{60} = \sum_{n=6}^{\infty} \frac{C a_{n-6} (n-5)\,x^{n-2}}{60}$$

$$\sum_{n=0}^{\infty} \frac{C\,x^{n+3} a_{n} (1+n)}{12} = \sum_{n=5}^{\infty} \frac{C a_{n-5} (n-4)\,x^{n-2}}{12}$$

$$\sum_{n=0}^{\infty} \frac{C\,x^{n+2} a_{n} (1+n)}{3} = \sum_{n=4}^{\infty} \frac{C a_{n-4} (n-3)\,x^{n-2}}{3}$$

$$\sum_{n=0}^{\infty} C\,x^{1+n} a_n (1+n) = \sum_{n=3}^{\infty} C a_{n-3} (n-2)\, x^{n-2}$$

$$\sum_{n=0}^{\infty} 2 a_n x^n C (1+n) = \sum_{n=2}^{\infty} 2 C a_{n-2} (n-1)\, x^{n-2}$$

$$\sum_{n=0}^{\infty} 2C\,x^{n-1} a_n (1+n) = \sum_{n=1}^{\infty} 2 C a_{n-1} n\, x^{n-2}$$

$$\sum_{n=0}^{\infty} \left(-2C\,x^{n-1} a_n (1+n)\right) = \sum_{n=1}^{\infty} \left(-2 C a_{n-1} n\, x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} \left(-2 a_n x^n C (1+n)\right) = \sum_{n=2}^{\infty} \left(-2 C a_{n-2} (n-1)\, x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} \left(-C\,x^{n-1} a_n\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} \left(-a_n x^n C\right) = \sum_{n=2}^{\infty} \left(-C a_{n-2} x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{C\,x^{1+n} a_n}{2}\right) = \sum_{n=3}^{\infty} \left(-\frac{C a_{n-3} x^{n-2}}{2}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{C\,x^{n+2} a_n}{6}\right) = \sum_{n=4}^{\infty} \left(-\frac{C a_{n-4} x^{n-2}}{6}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{C\,x^{n+3} a_n}{24}\right) = \sum_{n=5}^{\infty} \left(-\frac{C a_{n-5} x^{n-2}}{24}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{C\,x^{n+4} a_n}{120}\right) = \sum_{n=6}^{\infty} \left(-\frac{C a_{n-6} x^{n-2}}{120}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{C\,x^{n+5} a_n}{720}\right) = \sum_{n=7}^{\infty} \left(-\frac{C a_{-7+n} x^{n-2}}{720}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{C\,x^{n+6} a_n}{5040}\right) = \sum_{n=8}^{\infty} \left(-\frac{C a_{n-8} x^{n-2}}{5040}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{C\,x^{n+7} a_n}{40320}\right) = \sum_{n=9}^{\infty} \left(-\frac{C a_{n-9} x^{n-2}}{40320}\right)$$

$$\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+8}a_n}{362880}\right)=\sum_{n=10}^{\infty}\left(-\frac{Ca_{n-10}x^{n-2}}{362880}\right)$$

$$\sum_{n=0}^{\infty}C\,x^{n-1}a_n=\sum_{n=1}^{\infty}Ca_{n-1}x^{n-2}$$

$$\sum_{n=0}^{\infty}a_nx^nC=\sum_{n=2}^{\infty}Ca_{n-2}x^{n-2}$$

$$\sum_{n=0}^{\infty}\frac{n\,x^{n+8}b_n(n-1)}{3628800}=\sum_{n=10}^{\infty}\frac{(n-10)\,b_{n-10}(n-11)\,x^{n-2}}{3628800}$$

$$\sum_{n=0}^{\infty}\frac{n\,x^{n+7}b_n(n-1)}{362880}=\sum_{n=9}^{\infty}\frac{(n-9)\,b_{n-9}(n-10)\,x^{n-2}}{362880}$$

$$\sum_{n=0}^{\infty}\frac{n\,x^{n+6}b_n(n-1)}{40320}=\sum_{n=8}^{\infty}\frac{(n-8)\,b_{n-8}(n-9)\,x^{n-2}}{40320}$$

$$\sum_{n=0}^{\infty}\frac{n\,x^{n+5}b_n(n-1)}{5040}=\sum_{n=7}^{\infty}\frac{(-7+n)\,b_{-7+n}(n-8)\,x^{n-2}}{5040}$$

$$\sum_{n=0}^{\infty}\frac{n\,x^{n+4}b_n(n-1)}{720}=\sum_{n=6}^{\infty}\frac{(n-6)\,b_{n-6}(-7+n)\,x^{n-2}}{720}$$

$$\sum_{n=0}^{\infty}\frac{n\,x^{n+3}b_n(n-1)}{120}=\sum_{n=5}^{\infty}\frac{(n-5)\,b_{n-5}(n-6)\,x^{n-2}}{120}$$

$$\sum_{n=0}^{\infty}\frac{n\,x^{n+2}b_n(n-1)}{24}=\sum_{n=4}^{\infty}\frac{(n-4)\,b_{n-4}(n-5)\,x^{n-2}}{24}$$

$$\sum_{n=0}^{\infty}\frac{n\,x^{1+n}b_n(n-1)}{6}=\sum_{n=3}^{\infty}\frac{(n-3)\,b_{n-3}(n-4)\,x^{n-2}}{6}$$

$$\sum_{n=0}^{\infty}\frac{x^nb_nn(n-1)}{2}=\sum_{n=2}^{\infty}\frac{(n-2)\,b_{n-2}(n-3)\,x^{n-2}}{2}$$

$$\sum_{n=0}^{\infty}n\,x^{n-1}b_n(n-1)=\sum_{n=1}^{\infty}(n-1)\,b_{n-1}(n-2)\,x^{n-2}$$

$$\sum_{n=0}^{\infty}\left(-n\,x^{n-1}b_n(n-1)\right)=\sum_{n=1}^{\infty}\left(-(n-1)\,b_{n-1}(n-2)\,x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} b_n = \sum_{n=3}^{\infty} b_{n-3} x^{n-2}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n - 2$.

$$\text{Expression too large to display} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$0 = 0$$

Which is solved for C. Solving for C gives

$$C = 1$$

For $n = 3$, Eq (2B) gives

$$\frac{Ca_0}{2} + b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{3}{2} = 0$$

For $n = 4$, Eq (2B) gives

$$\frac{(a_0 + 9a_1) C}{6} + b_1 + b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{4}{3} + b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{4}{3}$$

For $n = 5$, Eq (2B) gives

$$\frac{(a_0 + 12a_1 + 60a_2) C}{24} + \frac{4b_2}{3} + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{175}{72} + 3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{175}{216}$$

For $n = 6$, Eq (2B) gives

$$\frac{(a_0 + 15a_1 + 100a_2 + 420a_3)\, C}{120} + \frac{b_2}{12} + 2b_3 + 6b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{3727}{2160} + 6b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{3727}{12960}$$

For $n = 7$, Eq (2B) gives

$$\frac{(a_0 + 18a_1 + 150a_2 + 840a_3 + 3240a_4)\, C}{720} + 10b_5 + \frac{b_2}{60} + \frac{b_3}{4} + 3b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{47531}{64800} + 10b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{47531}{648000}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$y_2(x) = 1\left(x\left(1 - x + \frac{4x^2}{9} - \frac{29x^3}{216} + \frac{37x^4}{1200} - \frac{58x^5}{10125} + \frac{14209x^6}{15876000} - \frac{107329x^7}{889056000} + O(x^8)\right)\right) \ln(x) + 1 + \frac{4x^2}{3} - \frac{175x^3}{216} + \frac{3727x^4}{12960} - \frac{47531x^5}{648000} + O(x^8)$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1-x+\frac{4x^2}{9}-\frac{29x^3}{216}+\frac{37x^4}{1200}-\frac{58x^5}{10125}+\frac{14209x^6}{15876000}-\frac{107329x^7}{889056000}+O\left(x^8\right)\right) \\
&\quad + c_2\left(1\left(x\left(1-x+\frac{4x^2}{9}-\frac{29x^3}{216}+\frac{37x^4}{1200}-\frac{58x^5}{10125}+\frac{14209x^6}{15876000}-\frac{107329x^7}{889056000}+O\left(x^8\right)\right)\right)\ln\left(x\right)\right. \\
&\qquad \left. + 1+\frac{4x^2}{3}-\frac{175x^3}{216}+\frac{3727x^4}{12960}-\frac{47531x^5}{648000}+O\left(x^8\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x\left(1-x+\frac{4x^2}{9}-\frac{29x^3}{216}+\frac{37x^4}{1200}-\frac{58x^5}{10125}+\frac{14209x^6}{15876000}-\frac{107329x^7}{889056000}+O\left(x^8\right)\right) \\
&\quad + c_2\left(x\left(1-x+\frac{4x^2}{9}-\frac{29x^3}{216}+\frac{37x^4}{1200}-\frac{58x^5}{10125}+\frac{14209x^6}{15876000}-\frac{107329x^7}{889056000}+O\left(x^8\right)\right)\ln\left(x\right)\right. \\
&\qquad \left. + 1+\frac{4x^2}{3}-\frac{175x^3}{216}+\frac{3727x^4}{12960}-\frac{47531x^5}{648000}+O\left(x^8\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x\left(1-x+\frac{4x^2}{9}-\frac{29x^3}{216}+\frac{37x^4}{1200}-\frac{58x^5}{10125}+\frac{14209x^6}{15876000}-\frac{107329x^7}{889056000}+O\left(x^8\right)\right) \\
&\quad + c_2\left(x\left(1-x+\frac{4x^2}{9}-\frac{29x^3}{216}+\frac{37x^4}{1200}-\frac{58x^5}{10125}+\frac{14209x^6}{15876000}-\frac{107329x^7}{889056000}\right.\right. \\
&\qquad \left.\left. +O\left(x^8\right)\right)\ln\left(x\right) + 1+\frac{4x^2}{3}-\frac{175x^3}{216}+\frac{3727x^4}{12960}-\frac{47531x^5}{648000}+O\left(x^8\right)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 x\left(1-x+\frac{4x^2}{9}-\frac{29x^3}{216}+\frac{37x^4}{1200}-\frac{58x^5}{10125}+\frac{14209x^6}{15876000}-\frac{107329x^7}{889056000}+O\left(x^8\right)\right) \\
&\quad + c_2\left(x\left(1-x+\frac{4x^2}{9}-\frac{29x^3}{216}+\frac{37x^4}{1200}-\frac{58x^5}{10125}+\frac{14209x^6}{15876000}-\frac{107329x^7}{889056000}\right.\right. \\
&\qquad \left.\left. +O\left(x^8\right)\right)\ln\left(x\right) + 1+\frac{4x^2}{3}-\frac{175x^3}{216}+\frac{3727x^4}{12960}-\frac{47531x^5}{648000}+O\left(x^8\right)\right)
\end{aligned}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✓ Solution by Maple

Time used: 0.312 (sec). Leaf size: 70

```
Order:=8;
dsolve((exp(x)-1-x)*diff(y(x),x$2)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - x + \frac{4}{9}x^2 - \frac{29}{216}x^3 + \frac{37}{1200}x^4 - \frac{58}{10125}x^5 + \frac{14209}{15876000}x^6 - \frac{107329}{889056000}x^7 + \mathrm{O}\left(x^8\right)\right) + c_2 \left(\ln\left(x\right)\left((-2)\,x + 2x^2 - \frac{8}{9}x^3 + \frac{29}{108}x^4 - \frac{37}{600}x^5 + \frac{116}{10125}x^6 - \frac{14209}{7938000}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(1 - \frac{8}{3}x^2 + \frac{175}{108}x^3 - \frac{3727}{6480}x^4 + \frac{47531}{324000}x^5 - \frac{3003737}{102060000}x^6 + \frac{48833381}{10001880000}x^7 + \mathrm{O}\left(x^8\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.366 (sec). Leaf size: 133

```
AsymptoticDSolveValue[(Exp[x]-1-x)*y''[x]+x*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 \left(x^6 \left(\frac{116 \log(x)}{10125} - \frac{3003737}{102060000}\right) + x^5 \left(\frac{47531}{324000} - \frac{37 \log(x)}{600}\right) + x^4 \left(\frac{29 \log(x)}{108} - \frac{3727}{6480}\right) + x^3 \left(\frac{175}{108} - \frac{8 \log(x)}{9}\right) + x^2 \left(2 \log(x) - \frac{8}{3}\right) - 2x \log(x) + 1\right) + c_2 x \left(-\frac{107329x^7}{889056000} + \frac{14209x^6}{15876000} - \frac{58x^5}{10125} + \frac{37x^4}{1200} - \frac{29x^3}{216} + \frac{4x^2}{9} - x + 1\right)$$

4.13 problem 21

Internal problem ID [6655]
Internal file name [`OUTPUT/5903_Sunday_June_05_2022_04_00_57_PM_50472387/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 6 SERIES SOLUTIONS OF LINEAR EQUATIONS. CHAPTER 6 IN REVIEW. Page 271
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + x^2 y' + 2xy = 10x^3 - 2x + 5}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (209) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (210) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -x^2y' - 2xy + 10x^3 - 2x + 5\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= \left(x^4 - 4x\right)y' - 10x^5 + 2yx^3 + 2x^3 + 25x^2 - 2y - 2\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= \left(-x^6 + 10x^3 - 6\right)y' + \left(-2x^5 + 14x^2\right)y + 10x^7 - 2x^5 - 85x^4 + 14x^2 + 30x\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= -10\left(\left(-\frac{1}{10}x^7 + \frac{9}{5}x^4 - 5x\right)y' + \left(-\frac{1}{5}x^6 + 3x^3 - 4\right)y + x^8 - \frac{x^6}{5} - \frac{33x^5}{2} + 3x^3 + 35x^2 - 4\right)x\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= \left(-x^{10} + 28x^7 - 170x^4 + 140x\right)y' + \left(-2x^9 + 50x^6 - 220x^3 + 40\right)y + 10x^{11} - 2x^9 - 265x^8 + 50x^6 +\\
F_5 &= \frac{dF_4}{dx}\\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4\\
&= \left(x^{12} - 40x^9 + 416x^6 - 1040x^3 + 180\right)y' - 10x\left(\left(-\frac{1}{5}x^{10} + \frac{37}{5}x^7 - 64x^4 + 94x\right)y + x^{12} - \frac{x^{10}}{5} - \right.\\
F_6 &= \frac{dF_5}{dx}\\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}y' + \frac{\partial F_5}{\partial y'}F_5\\
&= 10x\left(-\frac{x\left(x^{12} - 54x^9 + 850x^6 - 4176x^3 + 4240\right)y'}{10} + \left(-\frac{1}{5}x^{12} + \frac{51}{5}x^9 - \frac{712}{5}x^6 + 528x^3 - 224\right)y + \right.
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 5 \\
F_1 &= -2y(0) - 2 \\
F_2 &= -6y'(0) \\
F_3 &= 0 \\
F_4 &= 40 + 40y(0) \\
F_5 &= 180y'(0) \\
F_6 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{3}x^3 + \frac{1}{18}x^6\right) y(0) + \left(x - \frac{1}{4}x^4 + \frac{1}{28}x^7\right) y'(0) + \frac{5x^2}{2} - \frac{x^3}{3} + \frac{x^6}{18} + O(x^8)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -x^2 \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2x \left(\sum_{n=0}^{\infty} a_n x^n\right) + 10x^3 - 2x + 5 \quad (1)
$$

Expanding $10x^3 - 2x + 5$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
10x^3 - 2x + 5 &= 10x^3 - 2x + 5 + \ldots \\
&= 10x^3 - 2x + 5
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$
\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + x^2 \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2x \left(\sum_{n=0}^{\infty} a_n x^n\right) = 10x^3 - 2x + 5
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n\, x^{1+n} a_n\right)+\left(\sum_{n=0}^{\infty} 2x^{1+n} a_n\right) = 10x^3-2x+5 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n \\
\sum_{n=1}^{\infty} n\, x^{1+n} a_n &= \sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^n \\
\sum_{n=0}^{\infty} 2x^{1+n} a_n &= \sum_{n=1}^{\infty} 2a_{n-1} x^n
\end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right)+\left(\sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^n\right)+\left(\sum_{n=1}^{\infty} 2a_{n-1} x^n\right) = 10x^3-2x+5 \quad (3)$$

$n=0$ gives

$$\begin{aligned}
(2a_2)\, 1 &= 5 \\
2a_2 &= 5
\end{aligned}$$

Or

$$a_2 = \frac{5}{2}$$

$n=1$ gives

$$\begin{aligned}
(6a_3+2a_0)\, x &= -2x \\
6a_3+2a_0 &= -2
\end{aligned}$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{3} - \frac{1}{3}$$

For $2 \leq n$, the recurrence equation is

$$\left(\left(n+2\right) a_{n+2}(1+n) + \left(n-1\right) a_{n-1} + 2a_{n-1}\right) x^n = 10x^3 - 2x + 5 \tag{4}$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}\left(12a_4 + 3a_1\right) x^2 &= 0\\ 12a_4 + 3a_1 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}\left(20a_5 + 4a_2\right) x^3 &= 10x^3\\ 20a_5 + 4a_2 &= 10\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}\left(30a_6 + 5a_3\right) x^4 &= 0\\ 30a_6 + 5a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1}{18} + \frac{a_0}{18}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(42a_7 + 6a_4)\, x^5 &= 0\\ 42a_7 + 6a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{28}$$

For $n = 6$ the recurrence equation gives

$$\begin{aligned}(56a_8 + 7a_5)\, x^6 &= 0\\ 56a_8 + 7a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_8 = 0$$

For $n = 7$ the recurrence equation gives

$$\begin{aligned}(72a_9 + 8a_6)\, x^7 &= 0\\ 72a_9 + 8a_6 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_9 = -\frac{1}{162} - \frac{a_0}{162}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n\\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{5x^2}{2} + \left(-\frac{a_0}{3} - \frac{1}{3}\right) x^3 - \frac{a_1 x^4}{4} + \left(\frac{1}{18} + \frac{a_0}{18}\right) x^6 + \frac{a_1 x^7}{28} + \dots$$

Collecting terms, the solution becomes

$$y = \left(1-\frac{1}{3}x^{3}+\frac{1}{18}x^{6}\right)a_{0}+\left(x-\frac{1}{4}x^{4}+\frac{1}{28}x^{7}\right)a_{1}+\frac{5x^{2}}{2}-\frac{x^{3}}{3}+\frac{x^{6}}{18}+O\left(x^{8}\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1-\frac{1}{3}x^{3}+\frac{1}{18}x^{6}\right)c_{1}+\left(x-\frac{1}{4}x^{4}+\frac{1}{28}x^{7}\right)c_{2}+\frac{5x^{2}}{2}-\frac{x^{3}}{3}+\frac{x^{6}}{18}+O\left(x^{8}\right)$$

Summary

The solution(s) found are the following

$$y = \left(1-\frac{1}{3}x^{3}+\frac{1}{18}x^{6}\right)y(0)+\left(x-\frac{1}{4}x^{4}+\frac{1}{28}x^{7}\right)y'(0)+\frac{5x^{2}}{2}-\frac{x^{3}}{3}+\frac{x^{6}}{18}+O\left(x^{8}\right) \tag{1}$$

$$y = \left(1-\frac{1}{3}x^{3}+\frac{1}{18}x^{6}\right)c_{1}+\left(x-\frac{1}{4}x^{4}+\frac{1}{28}x^{7}\right)c_{2}+\frac{5x^{2}}{2}-\frac{x^{3}}{3}+\frac{x^{6}}{18}+O\left(x^{8}\right) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{3}x^{3}+\frac{1}{18}x^{6}\right)y(0)+\left(x-\frac{1}{4}x^{4}+\frac{1}{28}x^{7}\right)y'(0)+\frac{5x^{2}}{2}-\frac{x^{3}}{3}+\frac{x^{6}}{18}+O\left(x^{8}\right)$$

Verified OK.

$$y = \left(1-\frac{1}{3}x^{3}+\frac{1}{18}x^{6}\right)c_{1}+\left(x-\frac{1}{4}x^{4}+\frac{1}{28}x^{7}\right)c_{2}+\frac{5x^{2}}{2}-\frac{x^{3}}{3}+\frac{x^{6}}{18}+O\left(x^{8}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
Order:=8;
dsolve(diff(y(x),x$2)+x^2*diff(y(x),x)+2*x*y(x)=5-2*x+10*x^3,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{3}x^3+\frac{1}{18}x^6\right)y(0)+\left(x-\frac{1}{4}x^4+\frac{1}{28}x^7\right)D(y)(0)+\frac{5x^2}{2}-\frac{x^3}{3}+\frac{x^6}{18}+O(x^8)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 63

```
AsymptoticDSolveValue[y''[x]+x^2*y'[x]+2*x*y[x]==5-2*x+10*x^3,y[x],{x,0,7}]
```

$$y(x) \to \frac{x^6}{18}-\frac{x^3}{3}+\frac{5x^2}{2}+c_2\left(\frac{x^7}{28}-\frac{x^4}{4}+x\right)+c_1\left(\frac{x^6}{18}-\frac{x^3}{3}+1\right)$$

5 CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289

5.1 problem 31

Internal problem ID [6656]
Internal file name [`OUTPUT/5904_Sunday_June_05_2022_04_00_59_PM_36707830/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y = 1}$$

With initial conditions

$$[y(0) = 0]$$

5.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -1$$
$$q(t) = 1$$

Hence the ode is

$$y' - y = 1$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - Y(s) = \frac{1}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - Y(s) = \frac{1}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{1}{s\,(s-1)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s} + \frac{1}{s-1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{s}\right) = -1$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = \mathrm{e}^{t}$$

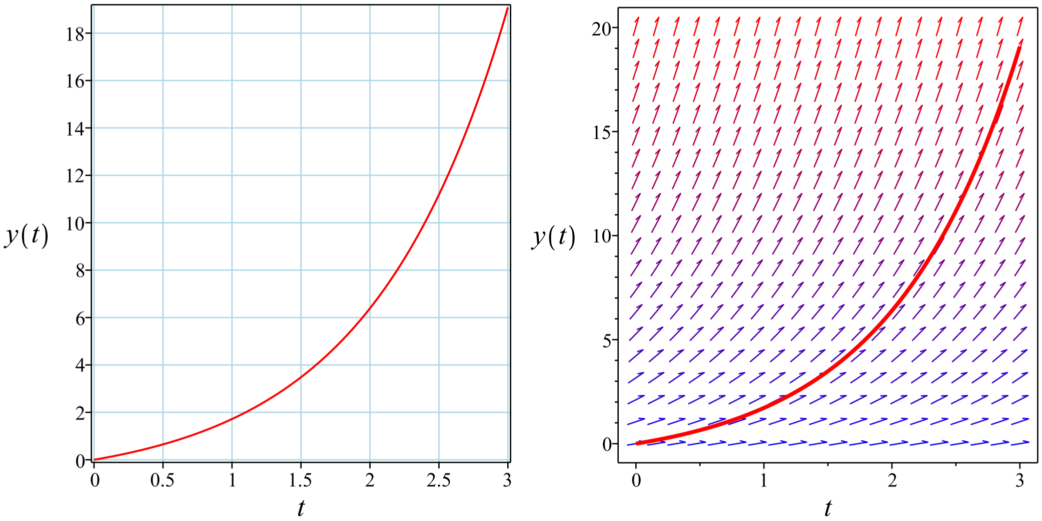
Adding the above results and simplifying gives

$$y = e^{t} - 1$$

Summary

The solution(s) found are the following

$$y = e^{t} - 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{t} - 1$$

Verified OK.

5.1.3 Maple step by step solution

Let's solve

$[y' - y = 1, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y+1} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y+1}dt = \int 1dt + c_1$
- Evaluate integral

  $\ln(y+1) = t + c_1$
- Solve for y

  $y = \mathrm{e}^{t+c_1} - 1$
- Use initial condition $y(0) = 0$

  $0 = \mathrm{e}^{c_1} - 1$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \mathrm{e}^{t} - 1$
- Solution to the IVP

  $y = \mathrm{e}^{t} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.672 (sec). Leaf size: 8

```
dsolve([diff(y(t),t)-y(t)=1,y(0) = 0],y(t), singsol=all)
```

$$y(t) = -1 + \mathrm{e}^{t}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 10

```
DSolve[{y'[t]-y[t]==1,{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{t} - 1$$

5.2 problem 32

Internal problem ID [6657]
Internal file name [`OUTPUT/5905_Sunday_June_05_2022_04_01_02_PM_89408436/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{2y' + y = 0}$$

With initial conditions

$$[y(0) = -3]$$

5.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{2}$$
$$q(t) = 0$$

Hence the ode is

$$y' + \frac{y}{2} = 0$$

The domain of $p(t) = \frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

5.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$2sY(s) - 2y(0) + Y(s) = 0 \tag{1}$$

Replacing initial condition gives

$$2sY(s) + 6 + Y(s) = 0$$

Solving for $Y(s)$ gives

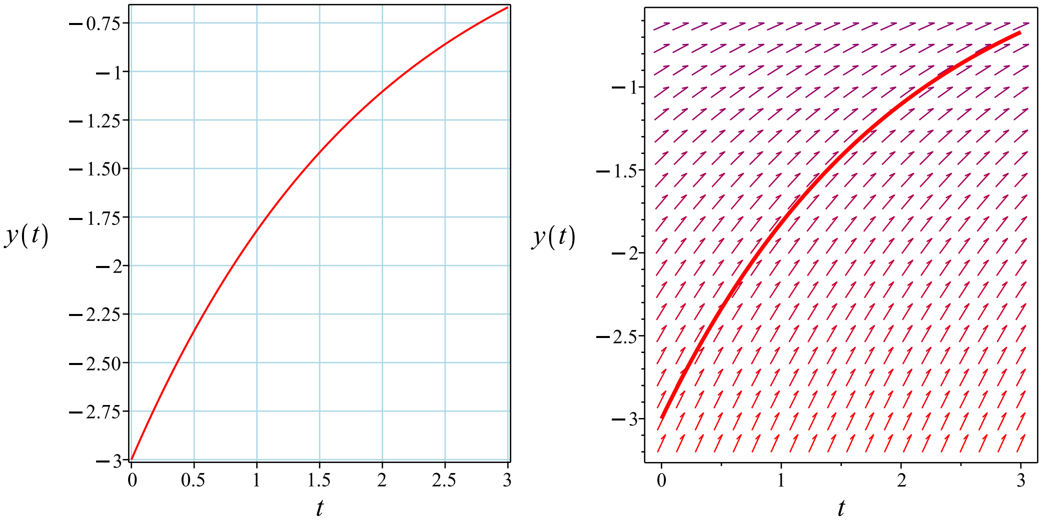
$$Y(s) = -\frac{6}{2s + 1}$$

Taking inverse Laplace transform gives

$$\mathcal{L}^{-1}\left(-\frac{6}{2s + 1}\right) = -3\,\mathrm{e}^{-\frac{t}{2}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -3\,\mathrm{e}^{-\frac{t}{2}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -3\,\mathrm{e}^{-\frac{t}{2}}$$

Verified OK.

5.2.3 Maple step by step solution

Let's solve

$[2y' + y = 0, y(0) = -3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{1}{2}$

- Integrate both sides with respect to t

  $\int \frac{y'}{y} dt = \int -\frac{1}{2} dt + c_1$

- Evaluate integral

  $\ln(y) = -\frac{t}{2} + c_1$

- Solve for y

  $y = \mathrm{e}^{-\frac{t}{2} + c_1}$

- Use initial condition $y(0) = -3$

  $-3 = \mathrm{e}^{c_1}$
- Solve for c_1

  $c_1 = \ln(3) + \mathrm{I}\pi$
- Substitute $c_1 = \ln(3) + \mathrm{I}\pi$ into general solution and simplify

  $y = -3\,\mathrm{e}^{-\frac{t}{2}}$
- Solution to the IVP

  $y = -3\,\mathrm{e}^{-\frac{t}{2}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.438 (sec). Leaf size: 10

```
dsolve([2*diff(y(t),t)+y(t)=0,y(0) = -3],y(t), singsol=all)
```

$$y(t) = -3\,\mathrm{e}^{-\frac{t}{2}}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 14

```
DSolve[{2*y'[t]+y[t]==0,{y[0]==-3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -3e^{-t/2}$$

5.3 problem 33

Internal problem ID [6658]
Internal file name [`OUTPUT/5906_Sunday_June_05_2022_04_01_04_PM_30233262/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 6y = \mathrm{e}^{4t}}$$

With initial conditions

$$[y(0) = 2]$$

5.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 6$$
$$q(t) = \mathrm{e}^{4t}$$

Hence the ode is

$$y' + 6y = e^{4t}$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = e^{4t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.3.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + 6Y(s) = \frac{1}{-4 + s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 2 + 6Y(s) = \frac{1}{-4 + s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{-7 + 2s}{(-4 + s)(s + 6)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{-40 + 10s} + \frac{19}{10(s + 6)}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{-40 + 10s}\right) = \frac{e^{4t}}{10}$$
$$\mathcal{L}^{-1}\left(\frac{19}{10(s + 6)}\right) = \frac{19\,e^{-6t}}{10}$$

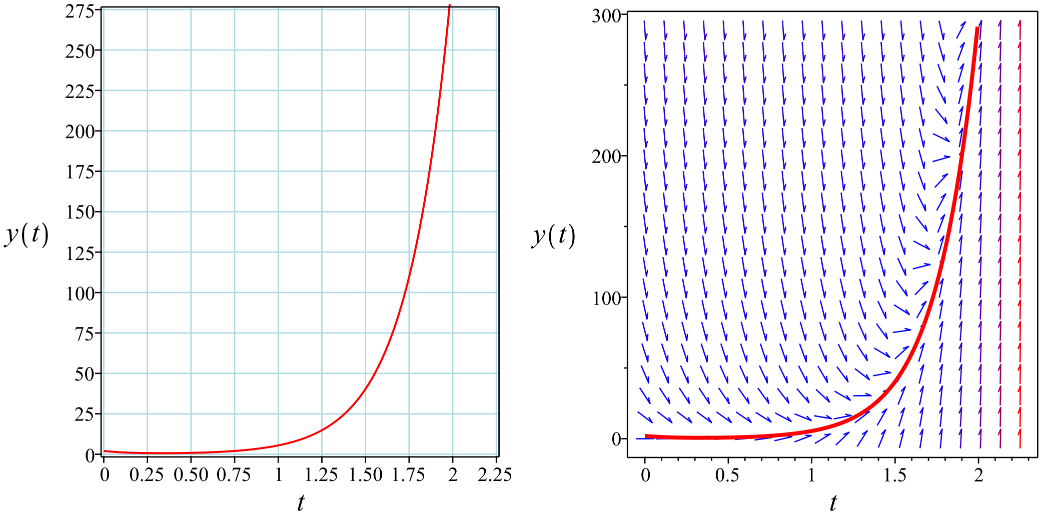
Adding the above results and simplifying gives

$$y = \frac{19\,\mathrm{e}^{-6t}}{10} + \frac{\mathrm{e}^{4t}}{10}$$

Summary

The solution(s) found are the following

$$y = \frac{19\,\mathrm{e}^{-6t}}{10} + \frac{\mathrm{e}^{4t}}{10} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{19\,\mathrm{e}^{-6t}}{10} + \frac{\mathrm{e}^{4t}}{10}$$

Verified OK.

5.3.3 Maple step by step solution

Let's solve

$[y' + 6y = \mathrm{e}^{4t}, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -6y + \mathrm{e}^{4t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 6y = \mathrm{e}^{4t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + 6y\right) = \mu(t)\,\mathrm{e}^{4t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y' + 6y\right) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = 6\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{6t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,\mathrm{e}^{4t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\,\mathrm{e}^{4t} dt + c_1$

- Solve for y

  $y = \frac{\int \mu(t)\mathrm{e}^{4t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{6t}$

  $y = \frac{\int \mathrm{e}^{6t}\mathrm{e}^{4t} dt + c_1}{\mathrm{e}^{6t}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{\mathrm{e}^{10t}}{10} + c_1}{\mathrm{e}^{6t}}$

- Simplify

  $y = \frac{\left(\mathrm{e}^{10t} + 10c_1\right)\mathrm{e}^{-6t}}{10}$

- Use initial condition $y(0) = 2$

  $2 = \frac{1}{10} + c_1$

- Solve for c_1

  $c_1 = \frac{19}{10}$

- Substitute $c_1 = \frac{19}{10}$ into general solution and simplify

  $y = \frac{\left(\mathrm{e}^{10t} + 19\right)\mathrm{e}^{-6t}}{10}$

- Solution to the IVP

  $y = \frac{(\mathrm{e}^{10t}+19)\,\mathrm{e}^{-6t}}{10}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.719 (sec). Leaf size: 17

```
dsolve([diff(y(t),t)+6*y(t)=exp(4*t),y(0) = 2],y(t), singsol=all)
```

$$y(t) = \frac{(\mathrm{e}^{10t}+19)\,\mathrm{e}^{-6t}}{10}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 21

```
DSolve[{y'[t]+6*y[t]==Exp[4*t],{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{10}e^{-6t}\left(e^{10t}+19\right)$$

5.4 problem 34

Internal problem ID [6659]
Internal file name [`OUTPUT/5907_Sunday_June_05_2022_04_01_06_PM_46553927/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = 2\cos(5t)}$$

With initial conditions

$$[y(0) = 0]$$

5.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= 2\cos(5t) \end{aligned}$$

Hence the ode is

$$y' - y = 2\cos(5t)$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2\cos(5t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.4.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - Y(s) = \frac{2s}{s^2 + 25} \tag{1}$$

Replacing initial condition gives

$$sY(s) - Y(s) = \frac{2s}{s^2 + 25}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2s}{(s^2 + 25)(s - 1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{13s - 13} + \frac{-\frac{1}{26} - \frac{5i}{26}}{s - 5i} + \frac{-\frac{1}{26} + \frac{5i}{26}}{s + 5i}$$

The inverse Laplace of each term above is now found, which gives

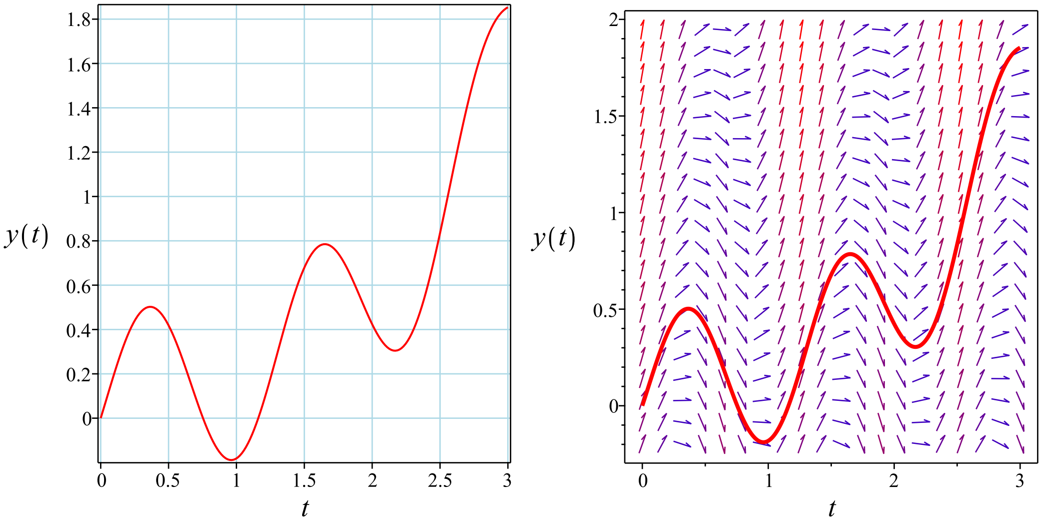
$$\mathcal{L}^{-1}\left(\frac{1}{13s-13}\right) = \frac{e^{t}}{13}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{26}-\frac{5i}{26}}{s-5i}\right) = \left(-\frac{1}{26}-\frac{5i}{26}\right)e^{5it}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{26}+\frac{5i}{26}}{s+5i}\right) = \left(-\frac{1}{26}+\frac{5i}{26}\right)e^{-5it}$$

Adding the above results and simplifying gives

$$y = \frac{e^{t}}{13}-\frac{\cos(5t)}{13}+\frac{5\sin(5t)}{13}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{t}}{13}-\frac{\cos(5t)}{13}+\frac{5\sin(5t)}{13} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{t}}{13}-\frac{\cos(5t)}{13}+\frac{5\sin(5t)}{13}$$

Verified OK.

5.4.3 Maple step by step solution

Let's solve

$[y' - y = 2\cos(5t), y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + 2\cos(5t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = 2\cos(5t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(y' - y) = 2\mu(t)\cos(5t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)(y' - y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 2\mu(t)\cos(5t)\,dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 2\mu(t)\cos(5t)\,dt + c_1$

- Solve for y

  $y = \frac{\int 2\mu(t)\cos(5t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-t}$

  $y = \frac{\int 2\,\mathrm{e}^{-t}\cos(5t)dt + c_1}{\mathrm{e}^{-t}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{\mathrm{e}^{-t}\cos(5t)}{13} + \frac{5\,\mathrm{e}^{-t}\sin(5t)}{13} + c_1}{\mathrm{e}^{-t}}$

- Simplify

  $y = c_1\mathrm{e}^{t} + \frac{5\sin(5t)}{13} - \frac{\cos(5t)}{13}$

- Use initial condition $y(0) = 0$

  $0 = c_1 - \frac{1}{13}$
- Solve for c_1

  $c_1 = \frac{1}{13}$
- Substitute $c_1 = \frac{1}{13}$ into general solution and simplify

  $y = \frac{e^t}{13} - \frac{\cos(5t)}{13} + \frac{5\sin(5t)}{13}$
- Solution to the IVP

  $y = \frac{e^t}{13} - \frac{\cos(5t)}{13} + \frac{5\sin(5t)}{13}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.703 (sec). Leaf size: 21

```
dsolve([diff(y(t),t)-y(t)=2*cos(5*t),y(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{e^t}{13} - \frac{\cos(5t)}{13} + \frac{5\sin(5t)}{13}$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 25

```
DSolve[{y'[t]-y[t]==2*Cos[5*t],{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{13}\left(e^t + 5\sin(5t) - \cos(5t)\right)$$

5.5 problem 35

Internal problem ID [6660]
Internal file name [`OUTPUT/5908_Sunday_June_05_2022_04_01_09_PM_9545444/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 5y' + 4y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

5.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 4y = 0$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 5sY(s) - 5y(0) + 4Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 5 - s + 5sY(s) + 4Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s+5}{s^2+5s+4}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{4}{3(s+1)} - \frac{1}{3(s+4)}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{4}{3(s+1)}\right) = \frac{4\,\mathrm{e}^{-t}}{3}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{3(s+4)}\right) = -\frac{\mathrm{e}^{-4t}}{3}$$

Adding the above results and simplifying gives

$$y = -\frac{\mathrm{e}^{-4t}}{3} + \frac{4\,\mathrm{e}^{-t}}{3}$$

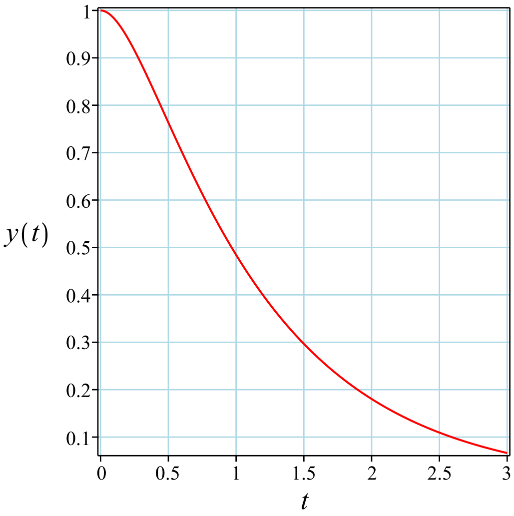
Simplifying the solution gives

$$y = -\frac{\mathrm{e}^{-4t}}{3} + \frac{4\,\mathrm{e}^{-t}}{3}$$

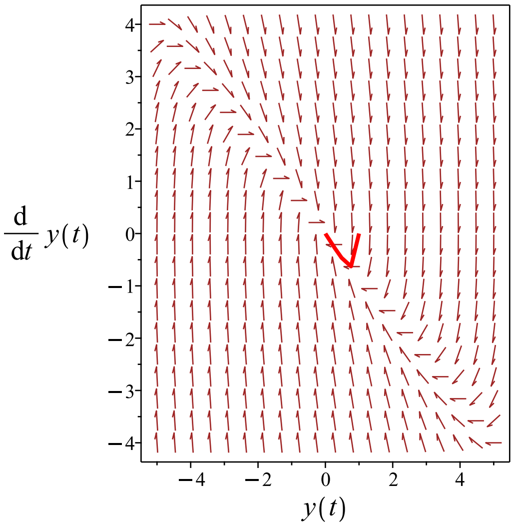
Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-4t}}{3} + \frac{4\,\mathrm{e}^{-t}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\mathrm{e}^{-4t}}{3} + \frac{4\,\mathrm{e}^{-t}}{3}$$

Verified OK.

5.5.2 Maple step by step solution

Let's solve

$$\left[y'' + 5y' + 4y = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 5r + 4 = 0$

- Factor the characteristic polynomial

  $(r+4)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -1)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-4t}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-t}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-t}$

  ○ Use initial condition $y(0) = 1$

  $1 = c_1 + c_2$

  ○ Compute derivative of the solution

  $y' = -4c_1 \mathrm{e}^{-4t} - c_2 \mathrm{e}^{-t}$

  ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -4c_1 - c_2$

  ○ Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{3}, c_2 = \frac{4}{3}\right\}$

  - Substitute constant values into general solution and simplify

    $y = -\frac{e^{-4t}}{3} + \frac{4\,e^{-t}}{3}$

- Solution to the IVP

  $y = -\frac{e^{-4t}}{3} + \frac{4\,e^{-t}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.688 (sec). Leaf size: 17

```
dsolve([diff(y(t),t$2)+5*diff(y(t),t)+4*y(t)=0,y(0) = 1, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{e^{-4t}}{3} + \frac{4\,e^{-t}}{3}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 23

```
DSolve[{y''[t]+5*y'[t]+4*y[t]==0,{y[0]==1,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to \frac{1}{3}e^{-4t}\left(4e^{3t} - 1\right)$$

5.6 problem 36

Internal problem ID [6661]
Internal file name [`OUTPUT/5909_Sunday_June_05_2022_04_01_11_PM_39893734/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 4y' = 6\,\mathrm{e}^{3t} - 3\,\mathrm{e}^{-t}}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

5.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 0 \\ F &= 6\,\mathrm{e}^{3t} - 3\,\mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$y'' - 4y' = 6\,\mathrm{e}^{3t} - 3\,\mathrm{e}^{-t}$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = 6\,\mathrm{e}^{3t} - 3\,\mathrm{e}^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4sY(s) + 4y(0) = \frac{3s+15}{(s-3)(s+1)} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 5 - s - 4sY(s) = \frac{3s+15}{(s-3)(s+1)}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^3 - 7s^2 + 10s + 30}{(s-3)(s+1)\,s\,(s-4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{5}{2s} - \frac{3}{5(s+1)} + \frac{11}{10(s-4)} - \frac{2}{s-3}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{5}{2s}\right) &= \frac{5}{2}\\ \mathcal{L}^{-1}\left(-\frac{3}{5\left(s+1\right)}\right) &= -\frac{3\,\mathrm{e}^{-t}}{5}\\ \mathcal{L}^{-1}\left(\frac{11}{10\left(s-4\right)}\right) &= \frac{11\,\mathrm{e}^{4t}}{10}\\ \mathcal{L}^{-1}\left(-\frac{2}{s-3}\right) &= -2\,\mathrm{e}^{3t}\end{aligned}$$

Adding the above results and simplifying gives

$$y = \frac{5}{2}-\frac{3\,\mathrm{e}^{-t}}{5}+\frac{11\,\mathrm{e}^{4t}}{10}-2\,\mathrm{e}^{3t}$$

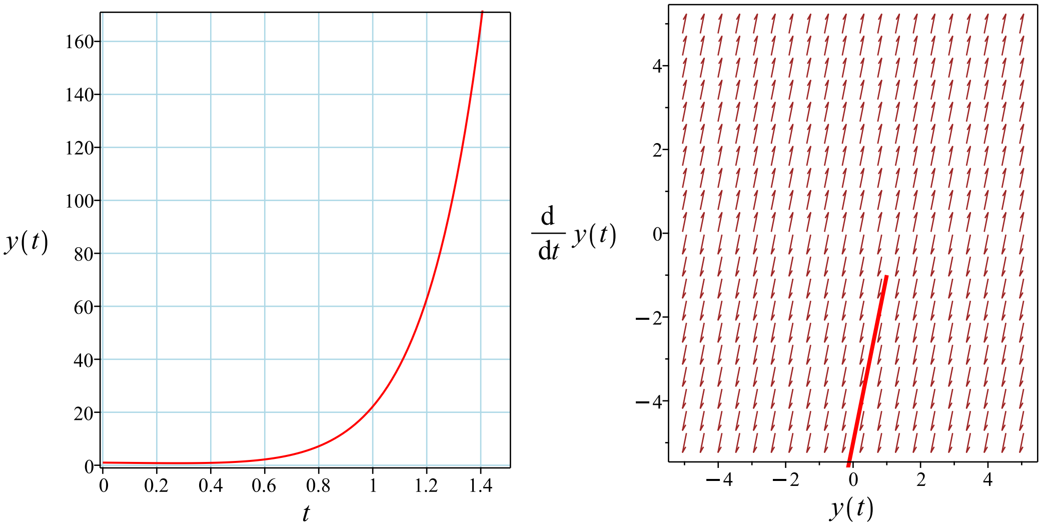
Simplifying the solution gives

$$y = \frac{5}{2}-\frac{3\,\mathrm{e}^{-t}}{5}+\frac{11\,\mathrm{e}^{4t}}{10}-2\,\mathrm{e}^{3t}$$

Summary

The solution(s) found are the following

$$y = \frac{5}{2}-\frac{3\,\mathrm{e}^{-t}}{5}+\frac{11\,\mathrm{e}^{4t}}{10}-2\,\mathrm{e}^{3t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{5}{2} - \frac{3\,\mathrm{e}^{-t}}{5} + \frac{11\,\mathrm{e}^{4t}}{10} - 2\,\mathrm{e}^{3t}$$

Verified OK.

5.6.2 Maple step by step solution

Let's solve

$\left[y'' - 4y' = 6\,\mathrm{e}^{3t} - 3\,\mathrm{e}^{-t}, y(0) = 1, y'\Big|_{\{t=0\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r = 0$
- Factor the characteristic polynomial

  $r(r-4) = 0$
- Roots of the characteristic polynomial

  $r = (0, 4)$
- 1st solution of the homogeneous ODE

  $y_1(t) = 1$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{4t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 \mathrm{e}^{4t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 6\,\mathrm{e}^{3t} - 3\,\mathrm{e}^{-t}\right]$

  ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t)\,,y_2(t)) = \begin{bmatrix} 1 & e^{4t} \\ 0 & 4\,e^{4t} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t)\,,y_2(t)) = 4\,e^{4t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{3\left(\int (2\,e^{3t}-e^{-t})dt\right)}{4} + \frac{3\,e^{4t}\left(\int (2\,e^{4t}-1)e^{-5t}dt\right)}{4}$

- ○ Compute integrals

  $y_p(t) = -\frac{3\,e^{-t}}{5} - 2\,e^{3t}$

- ● Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{4t} - \frac{3\,e^{-t}}{5} - 2\,e^{3t}$

☐ Check validity of solution $y = c_1 + c_2 e^{4t} - \frac{3e^{-t}}{5} - 2e^{3t}$

- ○ Use initial condition $y(0) = 1$

  $1 = c_1 + c_2 - \frac{13}{5}$

- ○ Compute derivative of the solution

  $y' = 4c_2 e^{4t} + \frac{3\,e^{-t}}{5} - 6\,e^{3t}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = 4c_2 - \frac{27}{5}$

- ○ Solve for c_1 and c_2

  $\left\{c_1 = \frac{5}{2}, c_2 = \frac{11}{10}\right\}$

- ○ Substitute constant values into general solution and simplify

  $y = \frac{5}{2} - \frac{3\,e^{-t}}{5} + \frac{11\,e^{4t}}{10} - 2\,e^{3t}$

- ● Solution to the IVP

  $y = \frac{5}{2} - \frac{3\,e^{-t}}{5} + \frac{11\,e^{4t}}{10} - 2\,e^{3t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 4*_b(_a)+6*exp(3*_a)-3*exp(-_a), _b(_a)
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 1.812 (sec). Leaf size: 24

```
dsolve([diff(y(t),t$2)-4*diff(y(t),t)=6*exp(3*t)-3*exp(-t),y(0) = 1, D(y)(0) = -1],y(t), sin
```

$$y(t) = \frac{5}{2} + \frac{11\,\mathrm{e}^{4t}}{10} - 2\,\mathrm{e}^{3t} - \frac{3\,\mathrm{e}^{-t}}{5}$$

✓ Solution by Mathematica

Time used: 0.139 (sec). Leaf size: 34

```
DSolve[{y''[t]-4*y'[t]==6*Exp[3*t]-3*Exp[-t],{y[0]==1,y'[0]==-1}},y[t],t,IncludeSingularSolu
```

$$y(t) \to -\frac{3e^{-t}}{5} - 2e^{3t} + \frac{11e^{4t}}{10} + \frac{5}{2}$$

5.7 problem 37

Internal problem ID [6662]
Internal file name [`OUTPUT/5910_Sunday_June_05_2022_04_01_13_PM_51186879/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sqrt{2}\, \sin\left(t\sqrt{2}\right)}$$

With initial conditions

$$[y(0) = 10, y'(0) = 0]$$

5.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \sqrt{2}\, \sin\left(t\sqrt{2}\right) \end{aligned}$$

Hence the ode is

$$y'' + y = \sqrt{2}\,\sin\left(t\sqrt{2}\right)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sqrt{2}\,\sin\left(t\sqrt{2}\right)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \frac{2}{s^2+2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 10\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 10s + Y(s) = \frac{2}{s^2+2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{10s^3 + 20s + 2}{(s^2+2)\,(s^2+1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{5-i}{s-i}+\frac{5+i}{s+i}+\frac{i\sqrt{2}}{2s-2i\sqrt{2}}-\frac{i\sqrt{2}}{2\left(s+i\sqrt{2}\right)}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{5-i}{s-i}\right) &= (5-i)\,\mathrm{e}^{it}\\
\mathcal{L}^{-1}\left(\frac{5+i}{s+i}\right) &= (5+i)\,\mathrm{e}^{-it}\\
\mathcal{L}^{-1}\left(\frac{i\sqrt{2}}{2s-2i\sqrt{2}}\right) &= \frac{i\sqrt{2}\,\mathrm{e}^{i\sqrt{2}\,t}}{2}\\
\mathcal{L}^{-1}\left(-\frac{i\sqrt{2}}{2\left(s+i\sqrt{2}\right)}\right) &= -\frac{i\sqrt{2}\,\mathrm{e}^{-i\sqrt{2}\,t}}{2}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = 10\cos(t)+2\sin(t)-\sqrt{2}\,\sin\left(t\sqrt{2}\right)$$

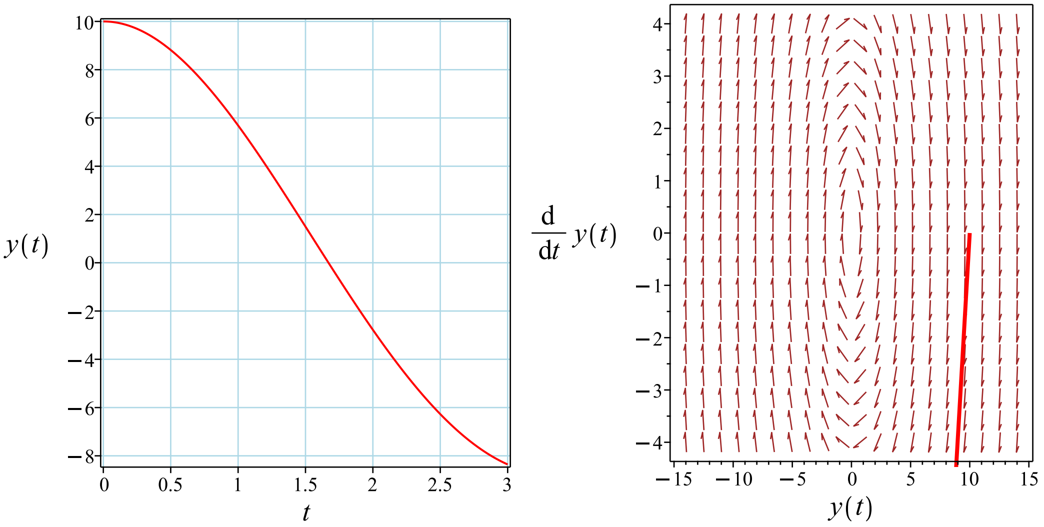
Simplifying the solution gives

$$y = 10\cos(t)+2\sin(t)-\sqrt{2}\,\sin\left(t\sqrt{2}\right)$$

Summary

The solution(s) found are the following

$$y = 10\cos(t)+2\sin(t)-\sqrt{2}\,\sin\left(t\sqrt{2}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 10\cos(t) + 2\sin(t) - \sqrt{2}\,\sin\left(t\sqrt{2}\right)$$

Verified OK.

5.7.2 Maple step by step solution

Let's solve

$\left[y'' + y = \sqrt{2}\,\sin\left(t\sqrt{2}\right), y(0) = 10, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \sqrt{2}\, \sin\left(t\sqrt{2}\right)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 1$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \sqrt{2}\left(-\cos(t)\left(\int \sin(t) \sin\left(t\sqrt{2}\right) dt\right) + \sin(t)\left(\int \cos(t) \sin\left(t\sqrt{2}\right) dt\right)\right)$

  ○ Compute integrals

  $y_p(t) = -\sqrt{2}\, \sin\left(t\sqrt{2}\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(t) + c_2 \sin(t) - \sqrt{2}\, \sin\left(t\sqrt{2}\right)$

☐ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) - \sqrt{2}\, \sin\left(t\sqrt{2}\right)$

  ○ Use initial condition $y(0) = 10$

  $10 = c_1$

  ○ Compute derivative of the solution

  $y' = -c_1 \sin(t) + c_2 \cos(t) - 2\cos\left(t\sqrt{2}\right)$

  ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_2 - 2$

  - Solve for c_1 and c_2

    $$\{c_1 = 10, c_2 = 2\}$$

  - Substitute constant values into general solution and simplify

    $$y = 10\cos(t) + 2\sin(t) - \sqrt{2}\,\sin\left(t\sqrt{2}\right)$$

- Solution to the IVP

  $$y = 10\cos(t) + 2\sin(t) - \sqrt{2}\,\sin\left(t\sqrt{2}\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.765 (sec). Leaf size: 24

```
dsolve([diff(y(t),t$2)+y(t)=sqrt(2)*sin(sqrt(2)*t),y(0) = 10, D(y)(0) = 0],y(t), singsol=all
```

$$y(t) = -\sqrt{2}\,\sin\left(\sqrt{2}\,t\right) + 10\cos(t) + 2\sin(t)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 29

```
DSolve[{y''[t]+y[t]==Sqrt[2]*Sin[Sqrt[2]*t],{y[0]==10,y'[0]==0}},y[t],t,IncludeSingularSolut
```

$$y(t) \to 2\sin(t) - \sqrt{2}\sin\left(\sqrt{2}t\right) + 10\cos(t)$$

5.8 problem 38

Internal problem ID [6663]
Internal file name [`OUTPUT/5911_Sunday_June_05_2022_04_01_15_PM_68935858/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 9y = \mathrm{e}^{t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

5.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}p(t) &= 0\\ q(t) &= 9\\ F &= \mathrm{e}^{t}\end{aligned}$$

Hence the ode is

$$y'' + 9y = \mathrm{e}^{t}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \mathrm{e}^{t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{1}{s-1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 9Y(s) = \frac{1}{s-1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{(s-1)(s^2+9)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{10s-10} + \frac{-\frac{1}{20}+\frac{i}{60}}{s-3i} + \frac{-\frac{1}{20}-\frac{i}{60}}{s+3i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{10s-10}\right) = \frac{e^{t}}{10}$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{20}+\frac{i}{60}}{s-3i}\right) = \left(-\frac{1}{20}+\frac{i}{60}\right) e^{3it}$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{20}-\frac{i}{60}}{s+3i}\right) = \left(-\frac{1}{20}-\frac{i}{60}\right) e^{-3it}$$

Adding the above results and simplifying gives

$$y = -\frac{\cos(3t)}{10} - \frac{\sin(3t)}{30} + \frac{e^{t}}{10}$$

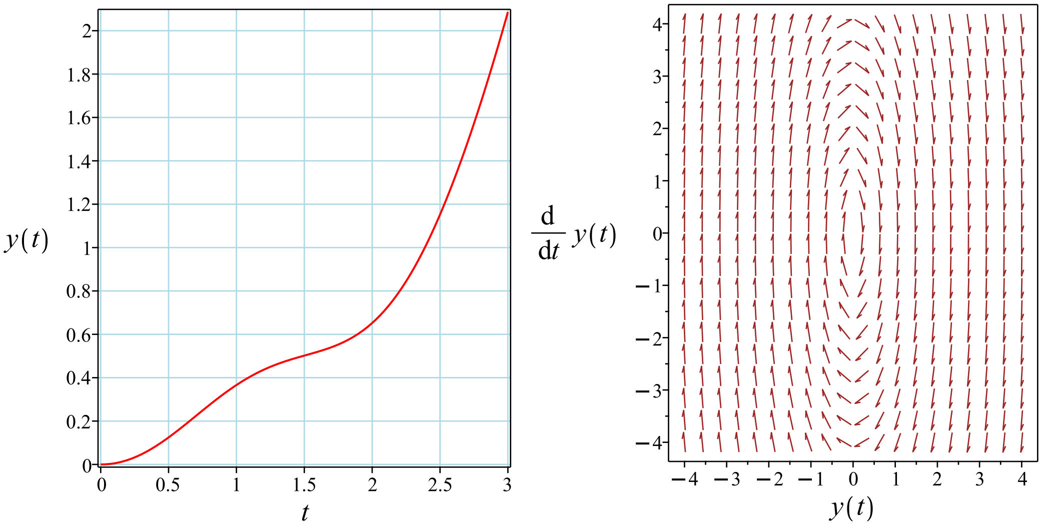
Simplifying the solution gives

$$y = -\frac{\cos(3t)}{10} - \frac{\sin(3t)}{30} + \frac{e^{t}}{10}$$

Summary

The solution(s) found are the following

$$y = -\frac{\cos(3t)}{10} - \frac{\sin(3t)}{30} + \frac{e^{t}}{10} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y=-\frac{\cos(3t)}{10}-\frac{\sin(3t)}{30}+\frac{\mathrm{e}^{t}}{10}$$

Verified OK.

5.8.2 Maple step by step solution

Let's solve

$\left[y''+9y=\mathrm{e}^{t}, y(0)=0, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2+9=0$
- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-36})}{2}$
- Roots of the characteristic polynomial

  $r=(-3\,\mathrm{I}, 3\,\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(3t)$
- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(3t)$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(3t)+c_2\sin(3t)+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\mathrm{e}^{t}\right]$

  ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3$
- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(3t)\left(\int \sin(3t)\mathrm{e}^t dt\right)}{3} + \frac{\sin(3t)\left(\int \cos(3t)\mathrm{e}^t dt\right)}{3}$
- ○ Compute integrals

  $y_p(t) = \frac{\mathrm{e}^t}{10}$

● Substitute particular solution into general solution to ODE

$y = c_1\cos(3t) + c_2\sin(3t) + \frac{\mathrm{e}^t}{10}$

☐ Check validity of solution $y = c_1\cos(3t) + c_2\sin(3t) + \frac{\mathrm{e}^t}{10}$

- ○ Use initial condition $y(0) = 0$

  $0 = \frac{1}{10} + c_1$
- ○ Compute derivative of the solution

  $y' = -3c_1\sin(3t) + 3c_2\cos(3t) + \frac{\mathrm{e}^t}{10}$
- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = \frac{1}{10} + 3c_2$
- ○ Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{10}, c_2 = -\frac{1}{30}\right\}$
- ○ Substitute constant values into general solution and simplify

  $y = -\frac{\cos(3t)}{10} - \frac{\sin(3t)}{30} + \frac{\mathrm{e}^t}{10}$

● Solution to the IVP

$y = -\frac{\cos(3t)}{10} - \frac{\sin(3t)}{30} + \frac{\mathrm{e}^t}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.828 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)+9*y(t)=exp(t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{\cos(3t)}{10} - \frac{\sin(3t)}{30} + \frac{e^{t}}{10}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 27

```
DSolve[{y''[t]+9*y[t]==Exp[t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{30}\left(3e^{t} - \sin(3t) - 3\cos(3t)\right)$$

5.9 problem 39

Internal problem ID [6664]
Internal file name [`OUTPUT/5912_Sunday_June_05_2022_04_01_18_PM_63338079/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 39.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

`[[_3rd_order, _with_linear_symmetries]]`

$$\boxed{2y''' + 3y'' - 3y' - 2y = \mathrm{e}^{-t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0, y''(0) = 1]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$2s^3Y(s) - 2y''(0) - 2sy'(0) - 2s^2y(0) + 3s^2Y(s) - 3y'(0) - 3sy(0) - 3sY(s) + 3y(0) - 2Y(s) = \frac{1}{s+1} \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= 0 \\
y''(0) &= 1
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$2s^3Y(s) - 2 + 3s^2Y(s) - 3sY(s) - 2Y(s) = \frac{1}{s+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3+2s}{(s+1)\,(2s^3+3s^2-3s-2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{9s+18} + \frac{1}{2s+2} + \frac{5}{18\,(s-1)} - \frac{8}{9\left(s+\frac{1}{2}\right)}$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{9s+18}\right) &= \frac{\mathrm{e}^{-2t}}{9} \\
\mathcal{L}^{-1}\left(\frac{1}{2s+2}\right) &= \frac{\mathrm{e}^{-t}}{2} \\
\mathcal{L}^{-1}\left(\frac{5}{18\,(s-1)}\right) &= \frac{5\,\mathrm{e}^{t}}{18} \\
\mathcal{L}^{-1}\left(-\frac{8}{9\left(s+\frac{1}{2}\right)}\right) &= -\frac{8\,\mathrm{e}^{-\frac{t}{2}}}{9}
\end{aligned}
$$

Adding the above results and simplifying gives

$$y = -\frac{8\,\mathrm{e}^{-\frac{t}{2}}}{9} + \frac{7\cosh(t)}{9} - \frac{2\sinh(t)}{9} + \frac{\mathrm{e}^{-2t}}{9}$$

Summary

The solution(s) found are the following

$$y = -\frac{8\,\mathrm{e}^{-\frac{t}{2}}}{9} + \frac{7\cosh(t)}{9} - \frac{2\sinh(t)}{9} + \frac{\mathrm{e}^{-2t}}{9} \tag{1}$$

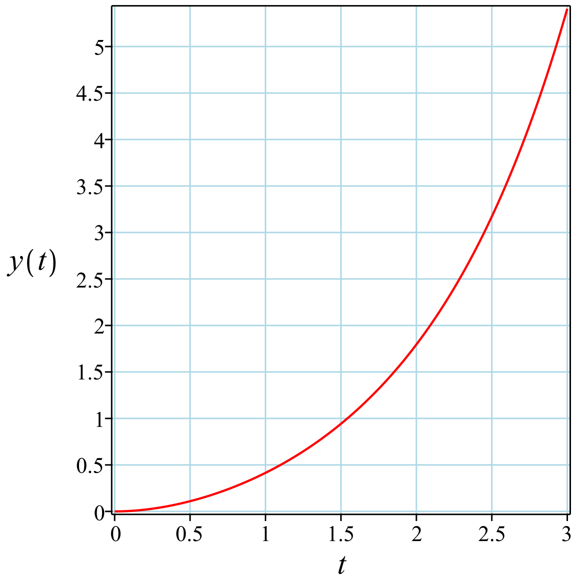


Figure 34: Solution plot

Verification of solutions

$$y = -\frac{8\,\mathrm{e}^{-\frac{t}{2}}}{9} + \frac{7\cosh(t)}{9} - \frac{2\sinh(t)}{9} + \frac{\mathrm{e}^{-2t}}{9}$$

Verified OK.

5.9.1 Maple step by step solution

Let's solve

$\left[2y''' + 3y'' - 3y' - 2y = \mathrm{e}^{-t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0, y''\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{3y''}{2} + \frac{3y'}{2} + y + \frac{\mathrm{e}^{-t}}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{3y''}{2} - \frac{3y'}{2} - y = \frac{\mathrm{e}^{-t}}{2}$

☐ Convert linear ODE into a system of first order ODEs

  ○ Define new variable $y_1(t)$

$y_1(t) = y$

- Define new variable $y_2(t)$

  $y_2(t) = y'$

- Define new variable $y_3(t)$

  $y_3(t) = y''$

- Isolate for $y_3'(t)$ using original ODE

  $y_3'(t) = \frac{e^{-t}}{2} - \frac{3y_3(t)}{2} + \frac{3y_2(t)}{2} + y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = \frac{e^{-t}}{2} - \frac{3y_3(t)}{2} + \frac{3y_2(t)}{2} + y_1(t)\right]$

- Define vector

  $$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

  $$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & \frac{3}{2} & -\frac{3}{2} \end{bmatrix} \cdot \vec{y}(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{e^{-t}}{2} \end{bmatrix}$$

- Define the forcing function

  $$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ \frac{e^{-t}}{2} \end{bmatrix}$$

- Define the coefficient matrix

  $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & \frac{3}{2} & -\frac{3}{2} \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(t) = A \cdot \vec{y}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}, \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-2t} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{-\frac{t}{2}} \cdot \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(t)$

☐ Fundamental matrix

- ○ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} \frac{\mathrm{e}^{-2t}}{4} & 4\,\mathrm{e}^{-\frac{t}{2}} & \mathrm{e}^{t} \\ -\frac{\mathrm{e}^{-2t}}{2} & -2\,\mathrm{e}^{-\frac{t}{2}} & \mathrm{e}^{t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{-\frac{t}{2}} & \mathrm{e}^{t} \end{bmatrix}$$

- ○ The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- ○ Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} \frac{\mathrm{e}^{-2t}}{4} & 4\,\mathrm{e}^{-\frac{t}{2}} & \mathrm{e}^{t} \\ -\frac{\mathrm{e}^{-2t}}{2} & -2\,\mathrm{e}^{-\frac{t}{2}} & \mathrm{e}^{t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{-\frac{t}{2}} & \mathrm{e}^{t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 4 & 1 \\ -\frac{1}{2} & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

- ○ Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{\left(2\,\mathrm{e}^{3t}+8\,\mathrm{e}^{\frac{3t}{2}}-1\right)\mathrm{e}^{-2t}}{9} & \frac{\left(5\,\mathrm{e}^{3t}-4\,\mathrm{e}^{\frac{3t}{2}}-1\right)\mathrm{e}^{-2t}}{9} & \frac{2\left(\mathrm{e}^{3t}-2\,\mathrm{e}^{\frac{3t}{2}}+1\right)\mathrm{e}^{-2t}}{9} \\ \frac{2\left(\mathrm{e}^{3t}-2\,\mathrm{e}^{\frac{3t}{2}}+1\right)\mathrm{e}^{-2t}}{9} & \frac{\left(5\,\mathrm{e}^{3t}+2\,\mathrm{e}^{\frac{3t}{2}}+2\right)\mathrm{e}^{-2t}}{9} & \frac{2\left(\mathrm{e}^{3t}+\mathrm{e}^{\frac{3t}{2}}-2\right)\mathrm{e}^{-2t}}{9} \\ \frac{2\left(\mathrm{e}^{3t}+\mathrm{e}^{\frac{3t}{2}}-2\right)\mathrm{e}^{-2t}}{9} & \frac{\left(5\,\mathrm{e}^{3t}-\mathrm{e}^{\frac{3t}{2}}-4\right)\mathrm{e}^{-2t}}{9} & \frac{\left(2\,\mathrm{e}^{3t}-\mathrm{e}^{\frac{3t}{2}}+8\right)\mathrm{e}^{-2t}}{9} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- ○ Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{y}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- ○ Take the derivative of the particular solution

  $\vec{y}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A\cdot\Phi(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$
  - Cancel like terms

    $\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$
  - Multiply by the inverse of the fundamental matrix

    $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot\vec{f}(t)$
  - Integrate to solve for $\vec{v}(t)$

    $\vec{v}(t)=\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$
  - Plug $\vec{v}(t)$ into the equation for the particular solution

    $\vec{y}_p(t)=\Phi(t)\cdot\left(\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

    $$\vec{y}_p(t)=\left[\begin{array}{c}\frac{\left(\mathrm{e}^{3t}-8\,\mathrm{e}^{\frac{3t}{2}}+9\,\mathrm{e}^{t}-2\right)\mathrm{e}^{-2t}}{18}\\ -\frac{\left(-\mathrm{e}^{3t}-4\,\mathrm{e}^{\frac{3t}{2}}+9\,\mathrm{e}^{t}-4\right)\mathrm{e}^{-2t}}{18}\\ \frac{\left(\mathrm{e}^{3t}-2\,\mathrm{e}^{\frac{3t}{2}}+9\,\mathrm{e}^{t}-8\right)\mathrm{e}^{-2t}}{18}\end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{y}(t)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3+\left[\begin{array}{c}\frac{\left(\mathrm{e}^{3t}-8\,\mathrm{e}^{\frac{3t}{2}}+9\,\mathrm{e}^{t}-2\right)\mathrm{e}^{-2t}}{18}\\ -\frac{\left(-\mathrm{e}^{3t}-4\,\mathrm{e}^{\frac{3t}{2}}+9\,\mathrm{e}^{t}-4\right)\mathrm{e}^{-2t}}{18}\\ \frac{\left(\mathrm{e}^{3t}-2\,\mathrm{e}^{\frac{3t}{2}}+9\,\mathrm{e}^{t}-8\right)\mathrm{e}^{-2t}}{18}\end{array}\right]$$
- First component of the vector is the solution to the ODE

  $$y=\frac{\left(36c_3\mathrm{e}^{3t}+2\,\mathrm{e}^{3t}+144c_2\mathrm{e}^{\frac{3t}{2}}-16\,\mathrm{e}^{\frac{3t}{2}}+18\,\mathrm{e}^{t}+9c_1-4\right)\mathrm{e}^{-2t}}{36}$$
- Use the initial condition $y(0)=0$

  $0=c_3+4c_2+\frac{c_1}{4}$
- Calculate the 1st derivative of the solution

$$y' = \frac{\left(108 c_3 e^{3t}+6\,e^{3t}+216 c_2 e^{\frac{3t}{2}}-24\,e^{\frac{3t}{2}}+18\,e^{t}\right) e^{-2t}}{36} - \frac{\left(36 c_3 e^{3t}+2\,e^{3t}+144 c_2 e^{\frac{3t}{2}}-16\,e^{\frac{3t}{2}}+18\,e^{t}+9 c_1-4\right) e^{-2t}}{18}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = c_3 - 2c_2 - \frac{c_1}{2}$$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{\left(324 c_3 e^{3t}+18\,e^{3t}+324 c_2 e^{\frac{3t}{2}}-36\,e^{\frac{3t}{2}}+18\,e^{t}\right) e^{-2t}}{36} - \frac{\left(108 c_3 e^{3t}+6\,e^{3t}+216 c_2 e^{\frac{3t}{2}}-24\,e^{\frac{3t}{2}}+18\,e^{t}\right) e^{-2t}}{9} + \left(36 c_3 e^{3t}+2\,e^{3t}+1$$

- Use the initial condition $y''\Big|_{\{t=0\}} = 1$

$$1 = c_1 + c_2 + c_3$$

- Solve for the unknown coefficients

$$\left\{c_1 = \frac{8}{9}, c_2 = -\frac{1}{9}, c_3 = \frac{2}{9}\right\}$$

- Solution to the IVP

$$y = \frac{\left(5\,e^{3t}-16\,e^{\frac{3t}{2}}+9\,e^{t}+2\right) e^{-2t}}{18}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.859 (sec). Leaf size: 25

```
dsolve([2*diff(y(t),t$3)+3*diff(y(t),t$2)-3*diff(y(t),t)-2*y(t)=exp(-t),y(0) = 0, D(y)(0) =
```

$$y(t) = \frac{7\cosh(t)}{9} - \frac{2\sinh(t)}{9} + \frac{e^{-2t}}{9} - \frac{8\,e^{-\frac{t}{2}}}{9}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 37

```
DSolve[{2*y'''[t]+3*y''[t]-3*y'[t]-2*y[t]==Exp[-t],{y[0]==0,y'[0]==0,y''[0]==1}},y[t],t,Incl
```

$$y(t) \to \frac{1}{18}e^{-2t}\left(9e^{t} - 16e^{3t/2} + 5e^{3t} + 2\right)$$

5.10 problem 40

Internal problem ID [6665]
Internal file name [`OUTPUT/5913_Sunday_June_05_2022_04_01_20_PM_70541589/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 40.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + 2y'' - y' - 2y = \sin(3t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0, y''(0) = 1]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\\ \mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) + 2s^2Y(s) - 2y'(0) - 2sy(0) - sY(s) + y(0) - 2Y(s) = \frac{3}{s^2+9} \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= 0 \\
y''(0) &= 1
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^3 Y(s) - 1 + 2s^2 Y(s) - sY(s) - 2Y(s) = \frac{3}{s^2+9}
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = \frac{s^2+12}{(s^2+9)\,(s^3+2s^2-s-2)}
$$

Applying partial fractions decomposition results in

$$
Y(s) = \frac{13}{60\,(s-1)} + \frac{\frac{3}{260}+\frac{i}{130}}{s-3i} + \frac{\frac{3}{260}-\frac{i}{130}}{s+3i} + \frac{16}{39\,(s+2)} - \frac{13}{20\,(s+1)}
$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{13}{60\,(s-1)}\right) &= \frac{13\,\mathrm{e}^{t}}{60} \\
\mathcal{L}^{-1}\left(\frac{\frac{3}{260}+\frac{i}{130}}{s-3i}\right) &= \left(\frac{3}{260}+\frac{i}{130}\right)\mathrm{e}^{3it} \\
\mathcal{L}^{-1}\left(\frac{\frac{3}{260}-\frac{i}{130}}{s+3i}\right) &= \left(\frac{3}{260}-\frac{i}{130}\right)\mathrm{e}^{-3it} \\
\mathcal{L}^{-1}\left(\frac{16}{39\,(s+2)}\right) &= \frac{16\,\mathrm{e}^{-2t}}{39} \\
\mathcal{L}^{-1}\left(-\frac{13}{20\,(s+1)}\right) &= -\frac{13\,\mathrm{e}^{-t}}{20}
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = \frac{3\cos(3t)}{130} - \frac{\sin(3t)}{65} - \frac{13\cosh(t)}{30} + \frac{13\sinh(t)}{15} + \frac{16\,\mathrm{e}^{-2t}}{39}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{3\cos(3t)}{130} - \frac{\sin(3t)}{65} - \frac{13\cosh(t)}{30} + \frac{13\sinh(t)}{15} + \frac{16\,\mathrm{e}^{-2t}}{39} \tag{1}
$$

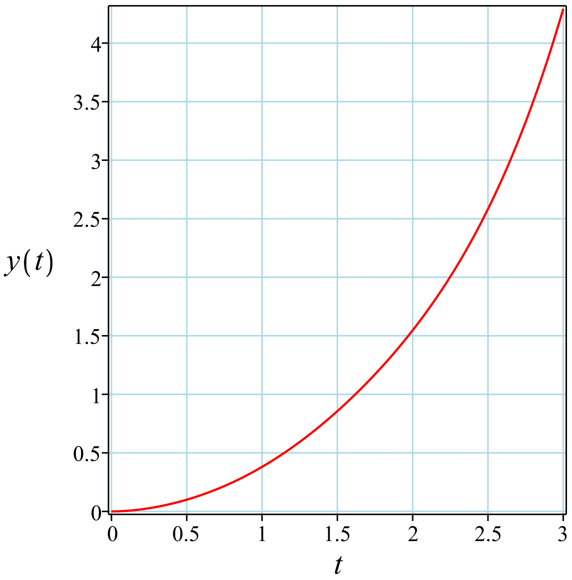


Figure 35: Solution plot

<u>Verification of solutions</u>

$$y = \frac{3\cos(3t)}{130} - \frac{\sin(3t)}{65} - \frac{13\cosh(t)}{30} + \frac{13\sinh(t)}{15} + \frac{16\,\mathrm{e}^{-2t}}{39}$$

Verified OK.

5.10.1 Maple step by step solution

Let's solve

$$\left[y''' + 2y'' - y' - 2y = \sin(3t), y(0) = 0, y'\Big|_{\{t=0\}} = 0, y''\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

☐ Convert linear ODE into a system of first order ODEs

  - ○ Define new variable $y_1(t)$

    $y_1(t) = y$

  - ○ Define new variable $y_2(t)$

    $y_2(t) = y'$

  - ○ Define new variable $y_3(t)$

$y_3(t) = y''$

- Isolate for $y_3'(t)$ using original ODE

  $y_3'(t) = \sin(3t) - 2y_3(t) + y_2(t) + 2y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = \sin(3t) - 2y_3(t) + y_2(t) + 2y_1(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \cdot \vec{y}(t) + \begin{bmatrix} 0 \\ 0 \\ \sin(3t) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ \sin(3t) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(t) = A \cdot \vec{y}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_1 = \mathrm{e}^{-2t} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

  $$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_2 = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

  $$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_3 = \mathrm{e}^{t} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(t)$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t)=\left[\begin{array}{ccc}\frac{{\mathrm e}^{-2t}}{4} & {\mathrm e}^{-t} & {\mathrm e}^{t} \\ -\frac{{\mathrm e}^{-2t}}{2} & -{\mathrm e}^{-t} & {\mathrm e}^{t} \\ {\mathrm e}^{-2t} & {\mathrm e}^{-t} & {\mathrm e}^{t} \end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where I is th

  $\Phi(t)=\phi(t)\cdot\frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t)=\left[\begin{array}{ccc}\frac{{\mathrm e}^{-2t}}{4} & {\mathrm e}^{-t} & {\mathrm e}^{t} \\ -\frac{{\mathrm e}^{-2t}}{2} & -{\mathrm e}^{-t} & {\mathrm e}^{t} \\ {\mathrm e}^{-2t} & {\mathrm e}^{-t} & {\mathrm e}^{t} \end{array}\right]\cdot\frac{1}{\left[\begin{array}{ccc}\frac{1}{4} & 1 & 1 \\ -\frac{1}{2} & -1 & 1 \\ 1 & 1 & 1 \end{array}\right]}$$
- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t)=\left[\begin{array}{ccc}\frac{\left({\mathrm e}^{3t}+3\,{\mathrm e}^{t}-1\right){\mathrm e}^{-2t}}{3} & \frac{{\mathrm e}^{t}}{2}-\frac{{\mathrm e}^{-t}}{2} & \frac{\left({\mathrm e}^{3t}-3\,{\mathrm e}^{t}+2\right){\mathrm e}^{-2t}}{6} \\ \frac{\left({\mathrm e}^{3t}-3\,{\mathrm e}^{t}+2\right){\mathrm e}^{-2t}}{3} & \frac{{\mathrm e}^{-t}}{2}+\frac{{\mathrm e}^{t}}{2} & \frac{\left({\mathrm e}^{3t}+3\,{\mathrm e}^{t}-4\right){\mathrm e}^{-2t}}{6} \\ \frac{\left({\mathrm e}^{3t}+3\,{\mathrm e}^{t}-4\right){\mathrm e}^{-2t}}{3} & \frac{{\mathrm e}^{t}}{2}-\frac{{\mathrm e}^{-t}}{2} & \frac{\left({\mathrm e}^{3t}-3\,{\mathrm e}^{t}+8\right){\mathrm e}^{-2t}}{6} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{y}_{p}(t)=\Phi(t)\cdot\overrightarrow{v}(t)$
- Take the derivative of the particular solution

  $\overrightarrow{y}_{p}^{\prime}(t)=\Phi^{\prime}(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}^{\prime}(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi^{\prime}(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}^{\prime}(t)=A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\overrightarrow{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}^{\prime}(t)=A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\overrightarrow{f}(t)$
- Cancel like terms

  $\Phi(t)\cdot\overrightarrow{v}^{\prime}(t)=\overrightarrow{f}(t)$
- Multiply by the inverse of the fundamental matrix

$\overrightarrow{v}^{\prime}(t)=\frac{1}{\Phi(t)}\cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t)=\int_{0}^{t}\frac{1}{\Phi(s)}\cdot \overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{y}_{p}(t)=\Phi(t)\cdot \left(\int_{0}^{t}\frac{1}{\Phi(s)}\cdot \overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_{p}(t)=\left[\begin{array}{c} -\frac{\left(4\,\mathrm{e}^{2t}\sin(3t)-13\,\mathrm{e}^{3t}-6\,\mathrm{e}^{2t}\cos(3t)+39\,\mathrm{e}^{t}-20\right)\mathrm{e}^{-2t}}{260} \\ -\frac{\left(18\,\mathrm{e}^{2t}\sin(3t)-13\,\mathrm{e}^{3t}+12\,\mathrm{e}^{2t}\cos(3t)-39\,\mathrm{e}^{t}+40\right)\mathrm{e}^{-2t}}{260} \\ \frac{\left(36\,\mathrm{e}^{2t}\sin(3t)+13\,\mathrm{e}^{3t}-54\,\mathrm{e}^{2t}\cos(3t)-39\,\mathrm{e}^{t}+80\right)\mathrm{e}^{-2t}}{260} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(t)=c_{1}\overrightarrow{y}_{1}+c_{2}\overrightarrow{y}_{2}+c_{3}\overrightarrow{y}_{3}+\left[\begin{array}{c} -\frac{\left(4\,\mathrm{e}^{2t}\sin(3t)-13\,\mathrm{e}^{3t}-6\,\mathrm{e}^{2t}\cos(3t)+39\,\mathrm{e}^{t}-20\right)\mathrm{e}^{-2t}}{260} \\ -\frac{\left(18\,\mathrm{e}^{2t}\sin(3t)-13\,\mathrm{e}^{3t}+12\,\mathrm{e}^{2t}\cos(3t)-39\,\mathrm{e}^{t}+40\right)\mathrm{e}^{-2t}}{260} \\ \frac{\left(36\,\mathrm{e}^{2t}\sin(3t)+13\,\mathrm{e}^{3t}-54\,\mathrm{e}^{2t}\cos(3t)-39\,\mathrm{e}^{t}+80\right)\mathrm{e}^{-2t}}{260} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y=-\frac{\left(-260c_{3}\mathrm{e}^{3t}+4\,\mathrm{e}^{2t}\sin(3t)-13\,\mathrm{e}^{3t}-6\,\mathrm{e}^{2t}\cos(3t)-260c_{2}\mathrm{e}^{t}+39\,\mathrm{e}^{t}-65c_{1}-20\right)\mathrm{e}^{-2t}}{260}$

- Use the initial condition $y(0)=0$

  $0=c_{3}+c_{2}+\frac{c_{1}}{4}$

- Calculate the 1st derivative of the solution

  $y^{\prime}=-\frac{\left(-780c_{3}\mathrm{e}^{3t}+26\,\mathrm{e}^{2t}\sin(3t)-39\,\mathrm{e}^{3t}-260c_{2}\mathrm{e}^{t}+39\,\mathrm{e}^{t}\right)\mathrm{e}^{-2t}}{260}+\frac{\left(-260c_{3}\mathrm{e}^{3t}+4\,\mathrm{e}^{2t}\sin(3t)-13\,\mathrm{e}^{3t}-6\,\mathrm{e}^{2t}\cos(3t)-260c_{2}\mathrm{e}^{t}+39\,\mathrm{e}^{t}-}{130}$

- Use the initial condition $y^{\prime}\Big|_{\{t=0\}}=0$

  $0=c_{3}-c_{2}-\frac{c_{1}}{2}$

- Calculate the 2nd derivative of the solution

  $y^{\prime\prime}=-\frac{\left(-2340c_{3}\mathrm{e}^{3t}+52\,\mathrm{e}^{2t}\sin(3t)+78\,\mathrm{e}^{2t}\cos(3t)-117\,\mathrm{e}^{3t}-260c_{2}\mathrm{e}^{t}+39\,\mathrm{e}^{t}\right)\mathrm{e}^{-2t}}{260}+\frac{\left(-780c_{3}\mathrm{e}^{3t}+26\,\mathrm{e}^{2t}\sin(3t)-39\,\mathrm{e}^{3t}-260c_{2}\mathrm{e}^{t}+}{65}$

- Use the initial condition $y^{\prime\prime}\Big|_{\{t=0\}}=1$

  $1=c_{1}+c_{2}+c_{3}$

- Solve for the unknown coefficients

  $\left\{c_1 = \frac{4}{3}, c_2 = -\frac{1}{2}, c_3 = \frac{1}{6}\right\}$

- Solution to the IVP

  $y = -\frac{(12\,\mathrm{e}^{2t}\sin(3t) - 169\,\mathrm{e}^{3t} - 18\,\mathrm{e}^{2t}\cos(3t) + 507\,\mathrm{e}^{t} - 320)\,\mathrm{e}^{-2t}}{780}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.875 (sec). Leaf size: 31

```
dsolve([diff(y(t),t$3)+2*diff(y(t),t$2)-diff(y(t),t)-2*y(t)=sin(3*t),y(0) = 0, D(y)(0) = 0,
```

$$y(t) = \frac{3\cos(3t)}{130} - \frac{\sin(3t)}{65} - \frac{13\cosh(t)}{30} + \frac{13\sinh(t)}{15} + \frac{16\,\mathrm{e}^{-2t}}{39}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 42

```
DSolve[{y'''[t]+2*y''[t]-y'[t]-2*y[t]==Sin[3*t],{y[0]==0,y'[0]==0,y''[0]==1}},y[t],t,Include
```

$$y(t) \to \frac{1}{780}\left(e^{-2t}\left(-507e^{t} + 169e^{3t} + 320\right) - 12\sin(3t) + 18\cos(3t)\right)$$

5.11 problem 41

Internal problem ID [6666]
Internal file name [`OUTPUT/5914_Sunday_June_05_2022_04_01_23_PM_90062451/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \mathrm{e}^{-3t} \cos(2t)}$$

With initial conditions

$$[y(0) = 0]$$

5.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= \mathrm{e}^{-3t} \cos(2t) \end{aligned}$$

Hence the ode is

$$y' + y = e^{-3t} \cos(2t)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = e^{-3t} \cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.11.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + Y(s) = \frac{s+3}{(s+3)^2 + 4} \tag{1}$$

Replacing initial condition gives

$$sY(s) + Y(s) = \frac{s+3}{(s+3)^2 + 4}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{s+3}{(s^2 + 6s + 13)(s+1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{4s+4} + \frac{-\frac{1}{8} - \frac{i}{8}}{s+3-2i} + \frac{-\frac{1}{8} + \frac{i}{8}}{s+3+2i}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{4s+4}\right) &= \frac{e^{-t}}{4} \\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{8}-\frac{i}{8}}{s+3-2i}\right) &= \left(-\frac{1}{8}-\frac{i}{8}\right) e^{(-3+2i)t} \\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{8}+\frac{i}{8}}{s+3+2i}\right) &= \left(-\frac{1}{8}+\frac{i}{8}\right) e^{(-3-2i)t}
\end{aligned}$$

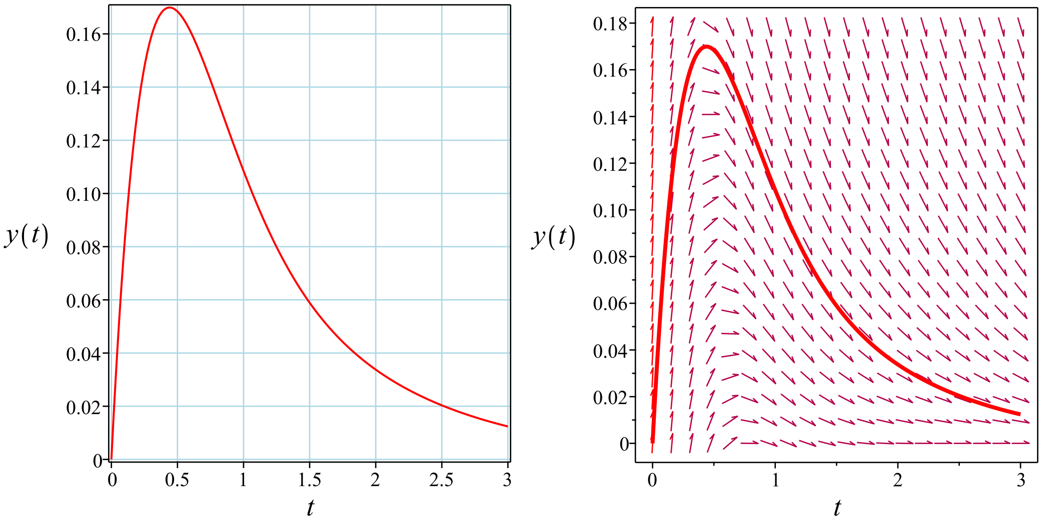
Adding the above results and simplifying gives

$$y = \frac{e^{-t}}{4} + \frac{e^{-3t}(-\cos(2t)+\sin(2t))}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-t}}{4} + \frac{e^{-3t}(-\cos(2t)+\sin(2t))}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-t}}{4} + \frac{e^{-3t}(-\cos(2t)+\sin(2t))}{4}$$

Verified OK.

5.11.3 Maple step by step solution

Let's solve

$[y' + y = \mathrm{e}^{-3t} \cos(2t), y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + \mathrm{e}^{-3t} \cos(2t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = \mathrm{e}^{-3t} \cos(2t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(y' + y) = \mu(t) \mathrm{e}^{-3t} \cos(2t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) y)$

  $\mu(t)(y' + y) = \mu'(t) y + \mu(t) y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t) y)\right) dt = \int \mu(t) \mathrm{e}^{-3t} \cos(2t) dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t) y = \int \mu(t) \mathrm{e}^{-3t} \cos(2t) dt + c_1$

- Solve for y

  $y = \frac{\int \mu(t)\mathrm{e}^{-3t} \cos(2t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{t}$

  $y = \frac{\int \mathrm{e}^{t}\mathrm{e}^{-3t} \cos(2t)dt + c_1}{\mathrm{e}^{t}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{(-2\cos(t) + 2\sin(t))\mathrm{e}^{-2t} \cos(t)}{4} + \frac{1}{4(\mathrm{e}^{t})^2} + c_1}{\mathrm{e}^{t}}$

- Simplify

$y = \frac{(-\cos(2t)+\sin(2t))\mathrm{e}^{-t}\mathrm{e}^{-2t}}{4} + \mathrm{e}^{-t}c_1$

- Use initial condition $y(0) = 0$

  $0 = -\frac{1}{4} + c_1$

- Solve for c_1

  $c_1 = \frac{1}{4}$

- Substitute $c_1 = \frac{1}{4}$ into general solution and simplify

  $y = -\frac{(-1+(\cos(2t)-\sin(2t))\mathrm{e}^{-2t})\mathrm{e}^{-t}}{4}$

- Solution to the IVP

  $y = -\frac{(-1+(\cos(2t)-\sin(2t))\mathrm{e}^{-2t})\mathrm{e}^{-t}}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.0 (sec). Leaf size: 28

```
dsolve([diff(y(t),t)+y(t)=exp(-3*t)*cos(2*t),y(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{\mathrm{e}^{-t}}{4} + \frac{\mathrm{e}^{-3t}(-\cos(2t) + \sin(2t))}{4}$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 30

```
DSolve[{y'[t]+y[t]==Exp[-3*t]*Cos[2*t],{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{4}e^{-3t}\left(e^{2t} + \sin(2t) - \cos(2t)\right)$$

5.12 problem 42

Internal problem ID [6667]
Internal file name [`OUTPUT/5915_Sunday_June_05_2022_04_01_26_PM_36930774/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.2.2 TRANSFORMS OF DERIVATIVES Page 289
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 5y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 3]$$

5.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 5y = 0$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + 5Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 3\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - s - 2sY(s) + 5Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s+1}{s^2 - 2s + 5}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{1}{2} - \frac{i}{2}}{s - 1 - 2i} + \frac{\frac{1}{2} + \frac{i}{2}}{s - 1 + 2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{1}{2}-\frac{i}{2}}{s-1-2i}\right)=\left(\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{(1+2i)t}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{1}{2}+\frac{i}{2}}{s-1+2i}\right)=\left(\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{(1-2i)t}$$

Adding the above results and simplifying gives

$$y=\mathrm{e}^{t}(\cos(2t)+\sin(2t))$$

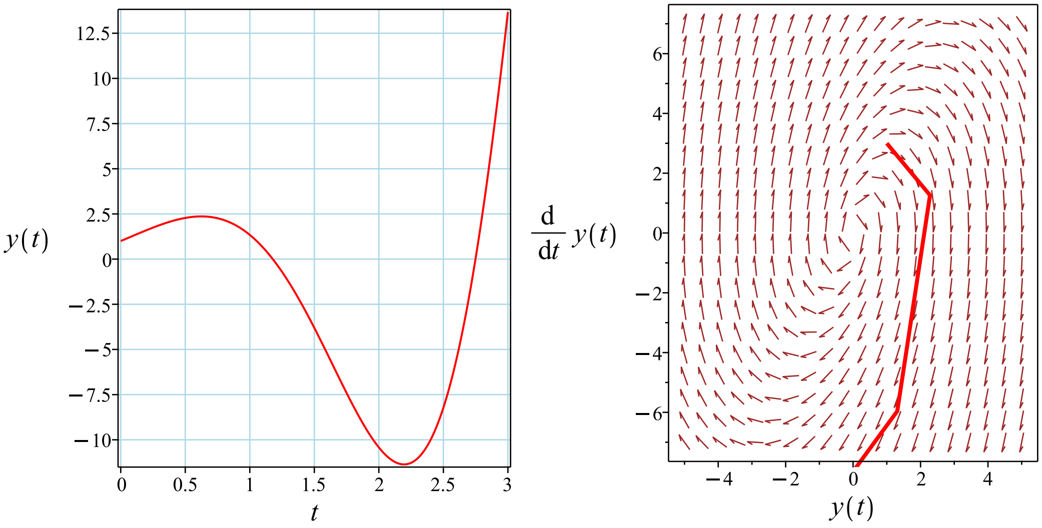
Simplifying the solution gives

$$y=\mathrm{e}^{t}(\cos(2t)+\sin(2t))$$

Summary

The solution(s) found are the following

$$y=\mathrm{e}^{t}(\cos(2t)+\sin(2t)) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y=\mathrm{e}^{t}(\cos(2t)+\sin(2t))$$

Verified OK.

5.12.2 Maple step by step solution

Let's solve

$\left[y''-2y'+5y=0, y(0)=1, y'\Big|_{\{t=0\}}=3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2-2r+5=0$

- Use quadratic formula to solve for r

  $r=\frac{2\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r=(1-2\,\mathrm{I}, 1+2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^t\cos(2t)$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^t\sin(2t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\mathrm{e}^t\cos(2t)+c_2\mathrm{e}^t\sin(2t)$

☐ Check validity of solution $y=c_1\mathrm{e}^t\cos(2t)+c_2\mathrm{e}^t\sin(2t)$

  ○ Use initial condition $y(0)=1$

  $1=c_1$

  ○ Compute derivative of the solution

  $y'=c_1\mathrm{e}^t\cos(2t)-2c_1\mathrm{e}^t\sin(2t)+c_2\mathrm{e}^t\sin(2t)+2c_2\mathrm{e}^t\cos(2t)$

  ○ Use the initial condition $y'\Big|_{\{t=0\}}=3$

  $3=c_1+2c_2$

  ○ Solve for c_1 and c_2

  $\{c_1=1, c_2=1\}$

- Solution to the IVP

  - Substitute constant values into general solution and simplify

    $y = \mathrm{e}^{t}(\cos(2t) + \sin(2t))$

  $y = \mathrm{e}^{t}(\cos(2t) + \sin(2t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.765 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)+5*y(t)=0,y(0) = 1, D(y)(0) = 3],y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{t}(\cos(2t) + \sin(2t))$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 18

```
DSolve[{y''[t]-2*y'[t]+5*y[t]==0,{y[0]==1,y'[0]==3}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to e^{t}(\sin(2t) + \cos(2t))$$

6 CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297

6.1 problem 21

Internal problem ID [6668]
Internal file name [`OUTPUT/5916_Sunday_June_05_2022_04_01_29_PM_85455668/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 4y = \mathrm{e}^{-4t}}$$

With initial conditions

$$[y(0) = 2]$$

6.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 4$$
$$q(t) = \mathrm{e}^{-4t}$$

Hence the ode is

$$y' + 4y = e^{-4t}$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = e^{-4t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + 4Y(s) = \frac{1}{s+4} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 2 + 4Y(s) = \frac{1}{s+4}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{9+2s}{(s+4)^2}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{(s+4)^2} + \frac{2}{s+4}$$

The inverse Laplace of each term above is now found, which gives

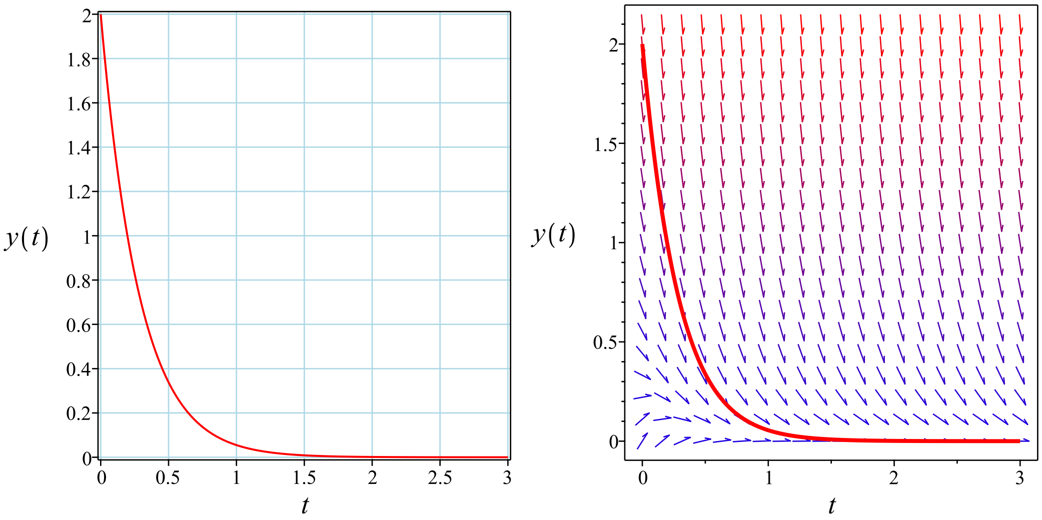
$$\mathcal{L}^{-1}\left(\frac{1}{(s+4)^2}\right) = e^{-4t}t$$
$$\mathcal{L}^{-1}\left(\frac{2}{s+4}\right) = 2\,e^{-4t}$$

Adding the above results and simplifying gives

$$y = (t + 2)\, e^{-4t}$$

Summary
The solution(s) found are the following

$$y = (t + 2)\, e^{-4t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = (t + 2)\, e^{-4t}$$

Verified OK.

6.1.3 Maple step by step solution

Let's solve

$[y' + 4y = e^{-4t}, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -4y + e^{-4t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 4y = \mathrm{e}^{-4t}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + 4y) = \mu(t)\,\mathrm{e}^{-4t}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + 4y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = 4\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{4t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,\mathrm{e}^{-4t} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\,\mathrm{e}^{-4t} dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t)\mathrm{e}^{-4t} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{4t}$

  $y = \frac{\int \mathrm{e}^{4t}\mathrm{e}^{-4t} dt + c_1}{\mathrm{e}^{4t}}$
- Evaluate the integrals on the rhs

  $y = \frac{t + c_1}{\mathrm{e}^{4t}}$
- Simplify

  $y = \mathrm{e}^{-4t}(t + c_1)$
- Use initial condition $y(0) = 2$

  $2 = c_1$
- Solve for c_1

  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

  $y = (t + 2)\,\mathrm{e}^{-4t}$
- Solution to the IVP

$$y = (t + 2)\,\mathrm{e}^{-4t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.672 (sec). Leaf size: 12

```
dsolve([diff(y(t),t)+4*y(t)=exp(-4*t),y(0) = 2],y(t), singsol=all)
```

$$y(t) = (t + 2)\,\mathrm{e}^{-4t}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 14

```
DSolve[{y'[t]+4*y[t]==Exp[-4*t],{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-4t}(t + 2)$$

6.2 problem 22

Internal problem ID [6669]
Internal file name [`OUTPUT/5917_Sunday_June_05_2022_04_01_32_PM_59360507/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = 1 + t\,\mathrm{e}^{t}}$$

With initial conditions

$$[y(0) = 0]$$

6.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -1$$
$$q(t) = 1 + t\,\mathrm{e}^{t}$$

Hence the ode is

$$y' - y = 1 + t\,\mathrm{e}^{t}$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1 + t\,\mathrm{e}^{t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - Y(s) = \frac{1}{s} + \frac{1}{(s-1)^2} \tag{1}$$

Replacing initial condition gives

$$sY(s) - Y(s) = \frac{1}{s} + \frac{1}{(s-1)^2}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{s^2 - s + 1}{s\,(s-1)^3}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s-1} - \frac{1}{s} + \frac{1}{(s-1)^3}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = \mathrm{e}^{t}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s}\right) = -1$$
$$\mathcal{L}^{-1}\left(\frac{1}{\left(s-1\right)^{3}}\right) = \frac{\mathrm{e}^{t} t^{2}}{2}$$

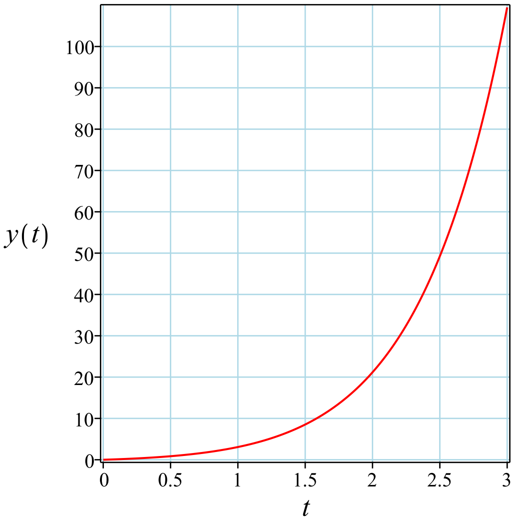
Adding the above results and simplifying gives

$$y = -1+\frac{\mathrm{e}^{t} \left(t^{2}+2\right)}{2}$$

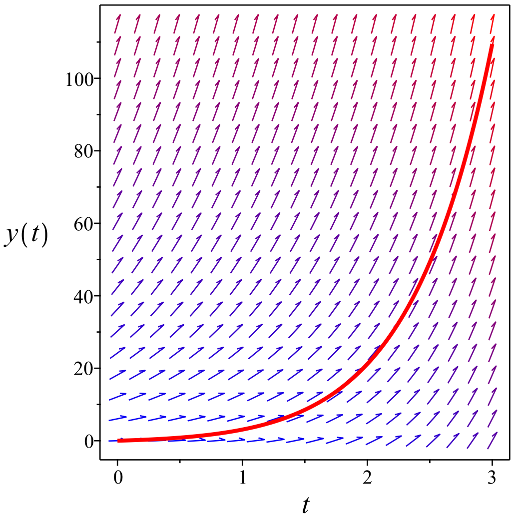
Summary

The solution(s) found are the following

$$y = -1+\frac{\mathrm{e}^{t} \left(t^{2}+2\right)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -1+\frac{\mathrm{e}^{t} \left(t^{2}+2\right)}{2}$$

Verified OK.

6.2.3 Maple step by step solution

Let's solve

$[y' - y = 1 + t\,\mathrm{e}^{t}, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'
- Isolate the derivative

  $y' = y + 1 + t\,\mathrm{e}^{t}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = 1 + t\,\mathrm{e}^{t}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - y) = \mu(t)\,(1 + t\,\mathrm{e}^{t})$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' - y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,(1 + t\,\mathrm{e}^{t})\,dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\,(1 + t\,\mathrm{e}^{t})\,dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t)(1+t\,\mathrm{e}^{t})dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-t}$

  $y = \frac{\int \mathrm{e}^{-t}(1+t\,\mathrm{e}^{t})dt + c_1}{\mathrm{e}^{-t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\frac{t^2}{2} - \frac{1}{\mathrm{e}^{t}} + c_1}{\mathrm{e}^{-t}}$
- Simplify

$y = -1 + \frac{(t^2+2c_1)\,\mathrm{e}^t}{2}$

- Use initial condition $y(0) = 0$

  $0 = -1 + c_1$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = -1 + \frac{\mathrm{e}^t t^2}{2} + \mathrm{e}^t$

- Solution to the IVP

  $y = -1 + \frac{\mathrm{e}^t t^2}{2} + \mathrm{e}^t$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.719 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)-y(t)=1+t*exp(t),y(0) = 0],y(t), singsol=all)
```

$$y(t) = -1 + \frac{\mathrm{e}^t t^2}{2} + \mathrm{e}^t$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 19

```
DSolve[{y'[t]-y[t]==1+t*Exp[t],{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow \frac{1}{2}e^t\left(t^2+2\right)-1$$

6.3 problem 23

Internal problem ID [6670]
Internal file name [`OUTPUT/5918_Sunday_June_05_2022_04_01_34_PM_3748528/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

6.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}p(t) &= 2\\ q(t) &= 1\\ F &= 0\end{aligned}$$

Hence the ode is

$$y'' + 2y' + y = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 - s + 2sY(s) + Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s+3}{s^2 + 2s + 1}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s+1} + \frac{2}{(s+1)^2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = \mathrm{e}^{-t}$$
$$\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2}\right) = 2t\,\mathrm{e}^{-t}$$

Adding the above results and simplifying gives

$$y = (2t+1)\,\mathrm{e}^{-t}$$

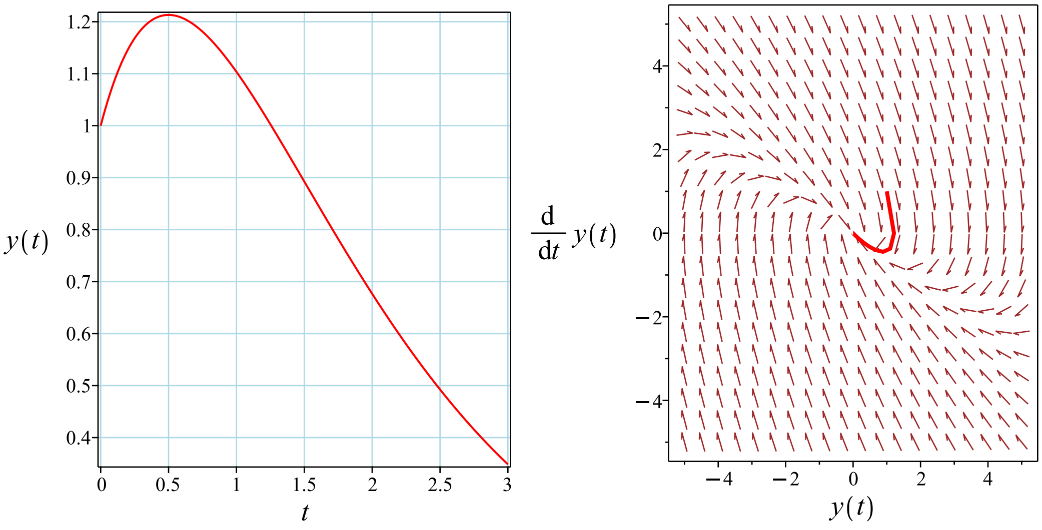
Simplifying the solution gives

$$y = (2t+1)\,\mathrm{e}^{-t}$$

Summary

The solution(s) found are the following

$$y = (2t+1)\,\mathrm{e}^{-t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = (2t+1)\,\mathrm{e}^{-t}$$

Verified OK.

6.3.2 Maple step by step solution

Let's solve

$$\left[y''+2y'+y=0, y(0)=1, y'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2+2r+1=0$
- Factor the characteristic polynomial

  $(r+1)^2=0$
- Root of the characteristic polynomial

  $r=-1$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t)=t\,\mathrm{e}^{-t}$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y=\mathrm{e}^{-t}c_1+t\,\mathrm{e}^{-t}c_2$

☐ Check validity of solution $y=\mathrm{e}^{-t}c_1+t\mathrm{e}^{-t}c_2$

  ○ Use initial condition $y(0)=1$

  $1=c_1$

  ○ Compute derivative of the solution

  $y'=-\mathrm{e}^{-t}c_1+c_2\mathrm{e}^{-t}-t\,\mathrm{e}^{-t}c_2$

  ○ Use the initial condition $y'\Big|_{\{t=0\}}=1$

  $1=-c_1+c_2$

  ○ Solve for c_1 and c_2

  $\{c_1=1, c_2=2\}$

- Substitute constant values into general solution and simplify

  $y = (2t + 1) \mathrm{e}^{-t}$

- Solution to the IVP

  $y = (2t + 1) \mathrm{e}^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.687 (sec). Leaf size: 14

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t)=0,y(0) = 1, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = (2t + 1) \mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 16

```
DSolve[{y''[t]+2*y'[t]+y[t]==0,{y[0]==1,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t}(2t + 1)$$

6.4 problem 24

Internal problem ID [6671]
Internal file name [`OUTPUT/5919_Sunday_June_05_2022_04_01_36_PM_63478124/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 4y = t^3 \mathrm{e}^{2t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

6.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 4 \\ F &= t^3 \mathrm{e}^{2t} \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 4y = t^3 e^{2t}$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t^3 e^{2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2 Y(s) - y'(0) - sy(0) - 4sY(s) + 4y(0) + 4Y(s) = \frac{6}{(s-2)^4} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) - 4sY(s) + 4Y(s) = \frac{6}{(s-2)^4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{6}{(s-2)^4 (s^2 - 4s + 4)}$$

Taking inverse Laplace transform gives

$$\mathcal{L}^{-1}\left(\frac{6}{\left(s-2\right)^{4}\left(s^{2}-4s+4\right)}\right) = \frac{t^{5}\mathrm{e}^{2t}}{20}$$

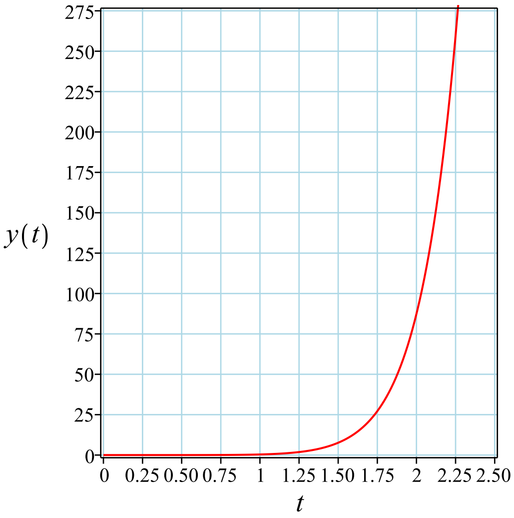
Simplifying the solution gives

$$y = \frac{t^{5}\mathrm{e}^{2t}}{20}$$

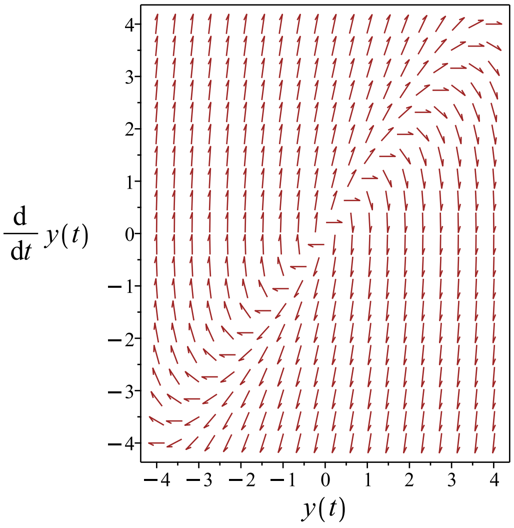
Summary

The solution(s) found are the following

$$y = \frac{t^{5}\mathrm{e}^{2t}}{20} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{t^{5}\mathrm{e}^{2t}}{20}$$

Verified OK.

6.4.2 Maple step by step solution

Let's solve

$$\left[y'' - 4y' + 4y = t^3 \mathrm{e}^{2t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{2t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{2t} + c_2 t\,\mathrm{e}^{2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = t^3 \mathrm{e}^{2t}\right]$$

  ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{2t} & t\,\mathrm{e}^{2t} \\ 2\,\mathrm{e}^{2t} & \mathrm{e}^{2t} + 2t\,\mathrm{e}^{2t} \end{bmatrix}$$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{4t}$

  ○ Substitute functions into equation for $y_p(t)$

$y_p(t) = \mathrm{e}^{2t}\left(-\left(\int t^4 dt\right) + \left(\int t^3 dt\right) t\right)$

- ○ Compute integrals

  $y_p(t) = \frac{t^5 \mathrm{e}^{2t}}{20}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{2t} + c_2 t\, \mathrm{e}^{2t} + \frac{t^5 \mathrm{e}^{2t}}{20}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{2t} + c_2 t \mathrm{e}^{2t} + \frac{t^5 \mathrm{e}^{2t}}{20}$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1$

- ○ Compute derivative of the solution

  $y' = 2c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{2t} + 2c_2 t\, \mathrm{e}^{2t} + \frac{t^4 \mathrm{e}^{2t}}{4} + \frac{t^5 \mathrm{e}^{2t}}{10}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 2c_1 + c_2$

- ○ Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- ○ Substitute constant values into general solution and simplify

  $y = \frac{t^5 \mathrm{e}^{2t}}{20}$

- Solution to the IVP

  $y = \frac{t^5 \mathrm{e}^{2t}}{20}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.829 (sec). Leaf size: 13

```
dsolve([diff(y(t),t$2)-4*diff(y(t),t)+4*y(t)=t^3*exp(2*t),y(0) = 0, D(y)(0) = 0],y(t), sings
```

$$y(t) = \frac{t^5 \mathrm{e}^{2t}}{20}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 17

```
DSolve[{y''[t]-4*y'[t]+4*y[t]==t^3*Exp[2*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to \frac{1}{20} e^{2t} t^5$$

6.5 problem 25

Internal problem ID [6672]
Internal file name [`OUTPUT/5920_Sunday_June_05_2022_04_01_38_PM_35627410/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 6y' + 9y = t}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

6.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -6 \\ q(t) &= 9 \\ F &= t \end{aligned}$$

Hence the ode is

$$y'' - 6y' + 9y = t$$

The domain of $p(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 6sY(s) + 6y(0) + 9Y(s) = \frac{1}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 6sY(s) + 9Y(s) = \frac{1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 + 1}{s^2\,(s^2 - 6s + 9)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{9s^2} + \frac{2}{27s} + \frac{10}{9\left(s-3\right)^2} - \frac{2}{27\left(s-3\right)}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{9s^2}\right) &= \frac{t}{9} \\
\mathcal{L}^{-1}\left(\frac{2}{27s}\right) &= \frac{2}{27} \\
\mathcal{L}^{-1}\left(\frac{10}{9\left(s-3\right)^2}\right) &= \frac{10t\,\mathrm{e}^{3t}}{9} \\
\mathcal{L}^{-1}\left(-\frac{2}{27\left(s-3\right)}\right) &= -\frac{2\,\mathrm{e}^{3t}}{27}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = \frac{2}{27} + \frac{t}{9} + \frac{2\,\mathrm{e}^{3t}(-1+15t)}{27}$$

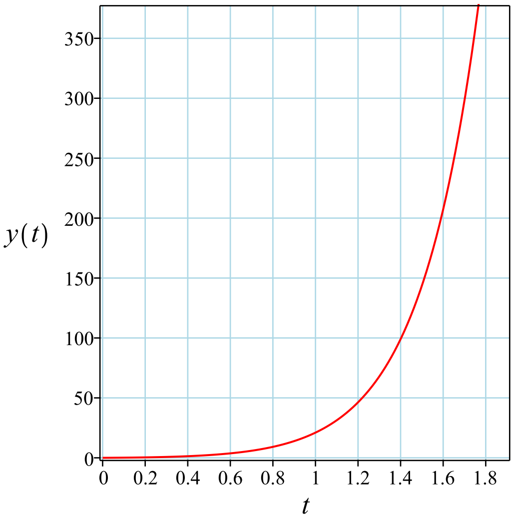
Simplifying the solution gives

$$y = \frac{\left(30t-2\right)\mathrm{e}^{3t}}{27} + \frac{t}{9} + \frac{2}{27}$$

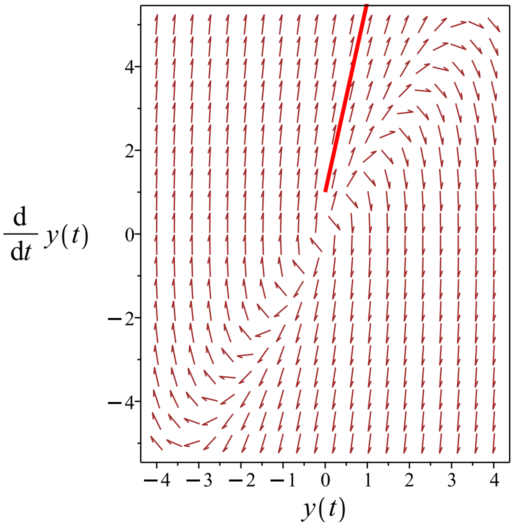
Summary

The solution(s) found are the following

$$y = \frac{\left(30t-2\right)\mathrm{e}^{3t}}{27} + \frac{t}{9} + \frac{2}{27} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(30t - 2)\, e^{3t}}{27} + \frac{t}{9} + \frac{2}{27}$$

Verified OK.

6.5.2 Maple step by step solution

Let's solve

$$\left[y'' - 6y' + 9y = t, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r - 3)^2 = 0$

- Root of the characteristic polynomial

  $r = 3$

- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{3t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{3t} + c_2 t\,\mathrm{e}^{3t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = t\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t)\,, y_2(t)) = \left[\begin{array}{cc} \mathrm{e}^{3t} & t\,\mathrm{e}^{3t} \\ 3\,\mathrm{e}^{3t} & \mathrm{e}^{3t} + 3t\,\mathrm{e}^{3t} \end{array}\right]$

  - Compute Wronskian

    $W(y_1(t)\,, y_2(t)) = \mathrm{e}^{6t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \mathrm{e}^{3t}\left(-\left(\int t^2 \mathrm{e}^{-3t} dt\right) + \left(\int t\,\mathrm{e}^{-3t} dt\right) t\right)$

  - Compute integrals

    $y_p(t) = \frac{t}{9} + \frac{2}{27}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{3t} + c_2 t\,\mathrm{e}^{3t} + \frac{t}{9} + \frac{2}{27}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{3t} + c_2 t \mathrm{e}^{3t} + \frac{t}{9} + \frac{2}{27}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + \frac{2}{27}$

  - Compute derivative of the solution

    $y' = 3c_1 \mathrm{e}^{3t} + \mathrm{e}^{3t} c_2 + 3c_2 t\,\mathrm{e}^{3t} + \frac{1}{9}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = 3c_1 + c_2 + \frac{1}{9}$

  - Solve for c_1 and c_2

    $\left\{c_1 = -\frac{2}{27}, c_2 = \frac{10}{9}\right\}$
  - Substitute constant values into general solution and simplify

    $y = \frac{(30t-2)\,\mathrm{e}^{3t}}{27} + \frac{t}{9} + \frac{2}{27}$
- Solution to the IVP

  $y = \frac{(30t-2)\,\mathrm{e}^{3t}}{27} + \frac{t}{9} + \frac{2}{27}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.641 (sec). Leaf size: 20

```
dsolve([diff(y(t),t$2)-6*diff(y(t),t)+9*y(t)=t,y(0) = 0, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{(30t-2)\,\mathrm{e}^{3t}}{27} + \frac{t}{9} + \frac{2}{27}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 25

```
DSolve[{y''[t]-6*y'[t]+9*y[t]==t,{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to \frac{1}{27}\left(3t + e^{3t}(30t-2) + 2\right)$$

6.6 problem 26

Internal problem ID [6673]
Internal file name [`OUTPUT/5921_Sunday_June_05_2022_04_01_41_PM_87756984/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 4y = t^3}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

6.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 4 \\ F &= t^3 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 4y = t^3$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t^3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4sY(s) + 4y(0) + 4Y(s) = \frac{6}{s^4} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4 - s - 4sY(s) + 4Y(s) = \frac{6}{s^4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^5 - 4s^4 + 6}{s^4\,(s^2 - 4s + 4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{4s-8} + \frac{9}{8s^2} + \frac{3}{2s^4} + \frac{3}{2s^3} + \frac{3}{4s} - \frac{13}{8\left(s-2\right)^2}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{4s-8}\right) &= \frac{e^{2t}}{4} \\
\mathcal{L}^{-1}\left(\frac{9}{8s^2}\right) &= \frac{9t}{8} \\
\mathcal{L}^{-1}\left(\frac{3}{2s^4}\right) &= \frac{t^3}{4} \\
\mathcal{L}^{-1}\left(\frac{3}{2s^3}\right) &= \frac{3t^2}{4} \\
\mathcal{L}^{-1}\left(\frac{3}{4s}\right) &= \frac{3}{4} \\
\mathcal{L}^{-1}\left(-\frac{13}{8\left(s-2\right)^2}\right) &= -\frac{13t\,e^{2t}}{8}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = \frac{3}{4} + \frac{t^3}{4} + \frac{3t^2}{4} + \frac{9t}{8} - \frac{e^{2t}(-2+13t)}{8}$$

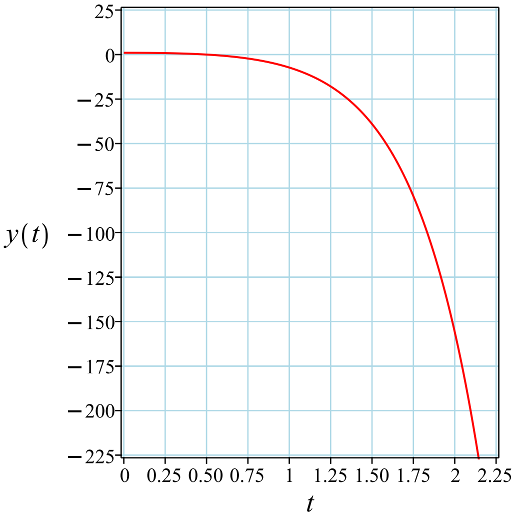
Simplifying the solution gives

$$y = \frac{\left(2-13t\right)e^{2t}}{8} + \frac{t^3}{4} + \frac{3t^2}{4} + \frac{9t}{8} + \frac{3}{4}$$

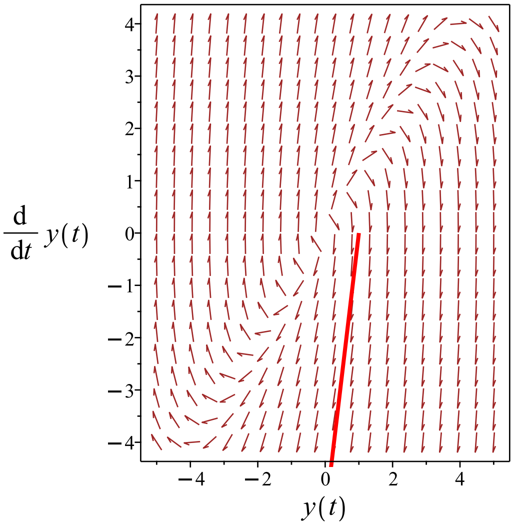
Summary

The solution(s) found are the following

$$y = \frac{\left(2-13t\right)e^{2t}}{8} + \frac{t^3}{4} + \frac{3t^2}{4} + \frac{9t}{8} + \frac{3}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(2-13t)\,e^{2t}}{8} + \frac{t^3}{4} + \frac{3t^2}{4} + \frac{9t}{8} + \frac{3}{4}$$

Verified OK.

6.6.2 Maple step by step solution

Let's solve

$$\left[y'' - 4y' + 4y = t^3, y(0) = 1, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{2t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{2t} + c_2 t\,\mathrm{e}^{2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = t^3\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \left[\begin{array}{cc} \mathrm{e}^{2t} & t\,\mathrm{e}^{2t} \\ 2\,\mathrm{e}^{2t} & \mathrm{e}^{2t} + 2t\,\mathrm{e}^{2t} \end{array}\right]$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{4t}$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \mathrm{e}^{2t}\left(-\left(\int t^4 \mathrm{e}^{-2t} dt\right) + \left(\int t^3 \mathrm{e}^{-2t} dt\right) t\right)$

  ○ Compute integrals

  $y_p(t) = \frac{1}{4}t^3 + \frac{3}{4}t^2 + \frac{9}{8}t + \frac{3}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{2t} + c_2 t\,\mathrm{e}^{2t} + \frac{t^3}{4} + \frac{3t^2}{4} + \frac{9t}{8} + \frac{3}{4}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{2t} + c_2 t \mathrm{e}^{2t} + \frac{t^3}{4} + \frac{3t^2}{4} + \frac{9t}{8} + \frac{3}{4}$

  ○ Use initial condition $y(0) = 1$

  $1 = c_1 + \frac{3}{4}$

  ○ Compute derivative of the solution

  $y' = 2c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{2t} + 2c_2 t\,\mathrm{e}^{2t} + \frac{3t^2}{4} + \frac{3t}{2} + \frac{9}{8}$

  ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 2c_1 + c_2 + \frac{9}{8}$

  - Solve for c_1 and c_2

    $\left\{c_1 = \frac{1}{4}, c_2 = -\frac{13}{8}\right\}$

  - Substitute constant values into general solution and simplify

    $y = \frac{(2-13t)\mathrm{e}^{2t}}{8} + \frac{t^3}{4} + \frac{3t^2}{4} + \frac{9t}{8} + \frac{3}{4}$

- Solution to the IVP

  $y = \frac{(2-13t)\mathrm{e}^{2t}}{8} + \frac{t^3}{4} + \frac{3t^2}{4} + \frac{9t}{8} + \frac{3}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.688 (sec). Leaf size: 30

```
dsolve([diff(y(t),t$2)-4*diff(y(t),t)+4*y(t)=t^3,y(0) = 1, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{(2-13t)\,\mathrm{e}^{2t}}{8} + \frac{t^3}{4} + \frac{3t^2}{4} + \frac{9t}{8} + \frac{3}{4}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 35

```
DSolve[{y''[t]-4*y'[t]+4*y[t]==t^3,{y[0]==1,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to \frac{1}{8}\left(2t^3 + 6t^2 + 9t + e^{2t}(2-13t) + 6\right)$$

6.7 problem 27

Internal problem ID [6674]
Internal file name [`OUTPUT/5922_Sunday_June_05_2022_04_01_43_PM_33068858/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 6y' + 13y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = -3]$$

6.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -6 \\ q(t) &= 13 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 6y' + 13y = 0$$

The domain of $p(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 13$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 6sY(s) + 6y(0) + 13Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= -3\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3 - 6sY(s) + 13Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{3}{s^2 - 6s + 13}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{3i}{4(s - 3 - 2i)} - \frac{3i}{4(s - 3 + 2i)}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{3i}{4(s-3-2i)}\right)=\frac{3i\mathrm{e}^{(3+2i)t}}{4}$$
$$\mathcal{L}^{-1}\left(-\frac{3i}{4(s-3+2i)}\right)=-\frac{3i\mathrm{e}^{(3-2i)t}}{4}$$

Adding the above results and simplifying gives

$$y=-\frac{3\,\mathrm{e}^{3t}\sin(2t)}{2}$$

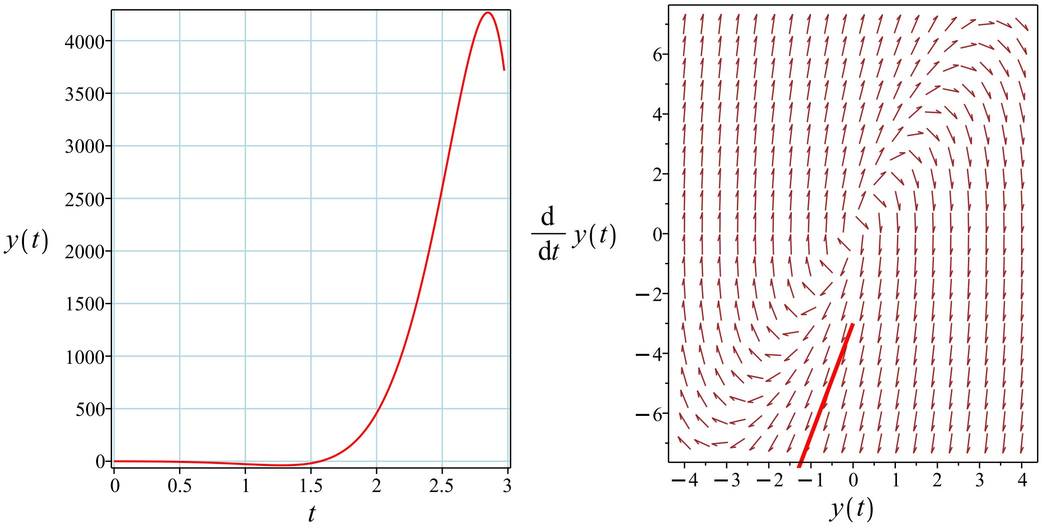
Simplifying the solution gives

$$y=-\frac{3\,\mathrm{e}^{3t}\sin(2t)}{2}$$

Summary

The solution(s) found are the following

$$y=-\frac{3\,\mathrm{e}^{3t}\sin(2t)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y=-\frac{3\,\mathrm{e}^{3t}\sin(2t)}{2}$$

Verified OK.

6.7.2 Maple step by step solution

Let's solve

$$\left[y''-6y'+13y=0, y(0)=0, y'\Big|_{\{t=0\}}=-3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2-6r+13=0$

- Use quadratic formula to solve for r

  $r=\frac{6\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r=(3-2\,\mathrm{I}, 3+2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{3t}\cos(2t)$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{3t}\sin(2t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{3t}\cos(2t)+c_2\mathrm{e}^{3t}\sin(2t)$

☐ Check validity of solution $y=c_1\mathrm{e}^{3t}\cos(2t)+c_2\mathrm{e}^{3t}\sin(2t)$

  ○ Use initial condition $y(0)=0$

  $0=c_1$

  ○ Compute derivative of the solution

  $y'=3c_1\mathrm{e}^{3t}\cos(2t)-2c_1\mathrm{e}^{3t}\sin(2t)+3c_2\mathrm{e}^{3t}\sin(2t)+2c_2\mathrm{e}^{3t}\cos(2t)$

  ○ Use the initial condition $y'\Big|_{\{t=0\}}=-3$

  $-3=3c_1+2c_2$

  ○ Solve for c_1 and c_2

  $\left\{c_1=0, c_2=-\frac{3}{2}\right\}$

  - Substitute constant values into general solution and simplify

    $y = -\frac{3\,\mathrm{e}^{3t} \sin(2t)}{2}$

- Solution to the IVP

  $y = -\frac{3\,\mathrm{e}^{3t} \sin(2t)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.735 (sec). Leaf size: 14

```
dsolve([diff(y(t),t$2)-6*diff(y(t),t)+13*y(t)=0,y(0) = 0, D(y)(0) = -3],y(t), singsol=all)
```

$$y(t) = -\frac{3\,\mathrm{e}^{3t} \sin(2t)}{2}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 16

```
DSolve[{y''[t]-6*y'[t]+13*y[t]==0,{y[0]==0,y'[0]==-3}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to -3e^{3t} \sin(t) \cos(t)$$

6.8 problem 28

Internal problem ID [6675]
Internal file name [`OUTPUT/5923_Sunday_June_05_2022_04_01_45_PM_85455363/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' + 20y' + 51y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0]$$

6.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 10 \\ q(t) &= \frac{51}{2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 10y' + \frac{51y}{2} = 0$$

The domain of $p(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{51}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$2s^2Y(s) - 2y'(0) - 2sy(0) + 20sY(s) - 20y(0) + 51Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$2s^2Y(s) - 40 - 4s + 20sY(s) + 51Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4s + 40}{2s^2 + 20s + 51}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1 - 5i\sqrt{2}}{s + 5 - \frac{i\sqrt{2}}{2}} + \frac{1 + 5i\sqrt{2}}{s + 5 + \frac{i\sqrt{2}}{2}}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1-5i\sqrt{2}}{s+5-\frac{i\sqrt{2}}{2}}\right) = \left(1-5i\sqrt{2}\right)\mathrm{e}^{-\frac{\left(-i\sqrt{2}+10\right)t}{2}}$$
$$\mathcal{L}^{-1}\left(\frac{1+5i\sqrt{2}}{s+5+\frac{i\sqrt{2}}{2}}\right) = \left(1+5i\sqrt{2}\right)\mathrm{e}^{-\frac{\left(i\sqrt{2}+10\right)t}{2}}$$

Adding the above results and simplifying gives

$$y = 2\,\mathrm{e}^{-5t}\left(\cos\left(\frac{t\sqrt{2}}{2}\right) + 5\sqrt{2}\,\sin\left(\frac{t\sqrt{2}}{2}\right)\right)$$

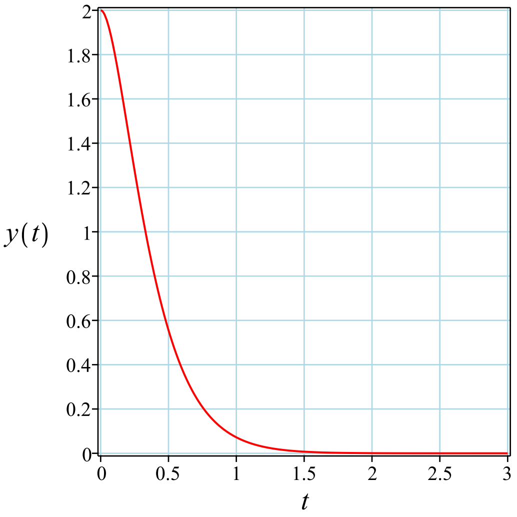
Simplifying the solution gives

$$y = 2\,\mathrm{e}^{-5t}\left(\cos\left(\frac{t\sqrt{2}}{2}\right) + 5\sqrt{2}\,\sin\left(\frac{t\sqrt{2}}{2}\right)\right)$$

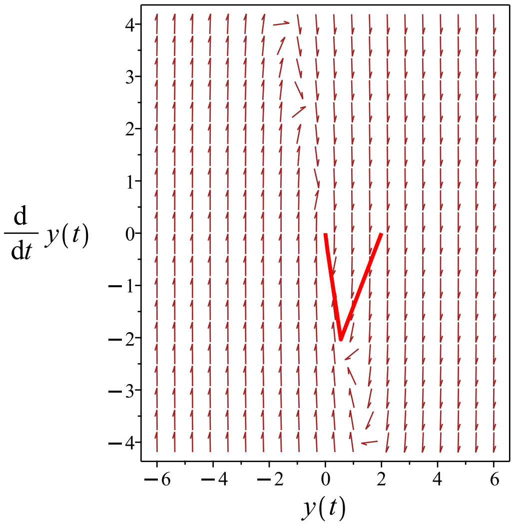
Summary

The solution(s) found are the following

$$y = 2\,\mathrm{e}^{-5t}\left(\cos\left(\frac{t\sqrt{2}}{2}\right) + 5\sqrt{2}\,\sin\left(\frac{t\sqrt{2}}{2}\right)\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{-5t}\left(\cos\left(\frac{t\sqrt{2}}{2}\right) + 5\sqrt{2}\,\sin\left(\frac{t\sqrt{2}}{2}\right)\right)$$

Verified OK.

6.8.2 Maple step by step solution

Let's solve

$\left[2y'' + 20y' + 51y = 0, y(0) = 2, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -10y' - \frac{51y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 10y' + \frac{51y}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 + 10r + \frac{51}{2} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-10)\pm(\sqrt{-2})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-5 - \frac{\mathrm{I}\sqrt{2}}{2}, -5 + \frac{\mathrm{I}\sqrt{2}}{2}\right)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-5t}\cos\left(\frac{t\sqrt{2}}{2}\right)$

- 2nd solution of the ODE

  $y_2(t) = \sin\left(\frac{t\sqrt{2}}{2}\right)\mathrm{e}^{-5t}$

- General solution of the ODE

  $y = c_1y_1(t) + c_2y_2(t)$

- Substitute in solutions

$$y = c_1 \mathrm{e}^{-5t} \cos\left(\frac{t\sqrt{2}}{2}\right) + c_2 \sin\left(\frac{t\sqrt{2}}{2}\right) \mathrm{e}^{-5t}$$

- ☐ Check validity of solution $y = c_1 \mathrm{e}^{-5t} \cos\left(\frac{t\sqrt{2}}{2}\right) + c_2 \sin\left(\frac{t\sqrt{2}}{2}\right) \mathrm{e}^{-5t}$
  - Use initial condition $y(0) = 2$

    $$2 = c_1$$
  - Compute derivative of the solution

    $$y' = -5c_1 \mathrm{e}^{-5t} \cos\left(\frac{t\sqrt{2}}{2}\right) - \frac{c_1 \mathrm{e}^{-5t} \sqrt{2}\, \sin\left(\frac{t\sqrt{2}}{2}\right)}{2} + \frac{c_2 \sqrt{2}\, \cos\left(\frac{t\sqrt{2}}{2}\right) \mathrm{e}^{-5t}}{2} - 5c_2 \sin\left(\frac{t\sqrt{2}}{2}\right) \mathrm{e}^{-5t}$$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $$0 = -5c_1 + \frac{c_2\sqrt{2}}{2}$$
  - Solve for c_1 and c_2

    $$\left\{c_1 = 2, c_2 = 10\sqrt{2}\right\}$$
  - Substitute constant values into general solution and simplify

    $$y = 2\,\mathrm{e}^{-5t}\left(\cos\left(\frac{t\sqrt{2}}{2}\right) + 5\sqrt{2}\, \sin\left(\frac{t\sqrt{2}}{2}\right)\right)$$
- Solution to the IVP

  $$y = 2\,\mathrm{e}^{-5t}\left(\cos\left(\frac{t\sqrt{2}}{2}\right) + 5\sqrt{2}\, \sin\left(\frac{t\sqrt{2}}{2}\right)\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.938 (sec). Leaf size: 30

```
dsolve([2*diff(y(t),t$2)+20*diff(y(t),t)+51*y(t)=0,y(0) = 2, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = 2\,\mathrm{e}^{-5t}\left(\cos\left(\frac{\sqrt{2}\,t}{2}\right) + 5\sqrt{2}\, \sin\left(\frac{\sqrt{2}\,t}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 36

```
DSolve[{2*y''[t]+20*y'[t]+51*y[t]==0,{y[0]==2,y'[0]==0}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to 2e^{-5t}\left(5\sqrt{2}\sin\left(\frac{t}{\sqrt{2}}\right)+\cos\left(\frac{t}{\sqrt{2}}\right)\right)$$

6.9 problem 29

Internal problem ID [6676]
Internal file name [`OUTPUT/5924_Sunday_June_05_2022_04_01_47_PM_75428172/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = \mathrm{e}^{t} \cos(t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

6.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -1 \\ F &= \mathrm{e}^{t} \cos(t) \end{aligned}$$

Hence the ode is

$$y'' - y = e^t \cos(t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^t \cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - Y(s) = \frac{s-1}{(s-1)^2 + 1} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - Y(s) = \frac{s-1}{(s-1)^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{(s+1)(s^2 - 2s + 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-\frac{1}{10}-\frac{i}{5}}{s-1-i}+\frac{-\frac{1}{10}+\frac{i}{5}}{s-1+i}+\frac{1}{5s+5}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{10}-\frac{i}{5}}{s-1-i}\right) = \left(-\frac{1}{10}-\frac{i}{5}\right)\mathrm{e}^{(1+i)t}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{10}+\frac{i}{5}}{s-1+i}\right) = \left(-\frac{1}{10}+\frac{i}{5}\right)\mathrm{e}^{(1-i)t}$$
$$\mathcal{L}^{-1}\left(\frac{1}{5s+5}\right) = \frac{\mathrm{e}^{-t}}{5}$$

Adding the above results and simplifying gives

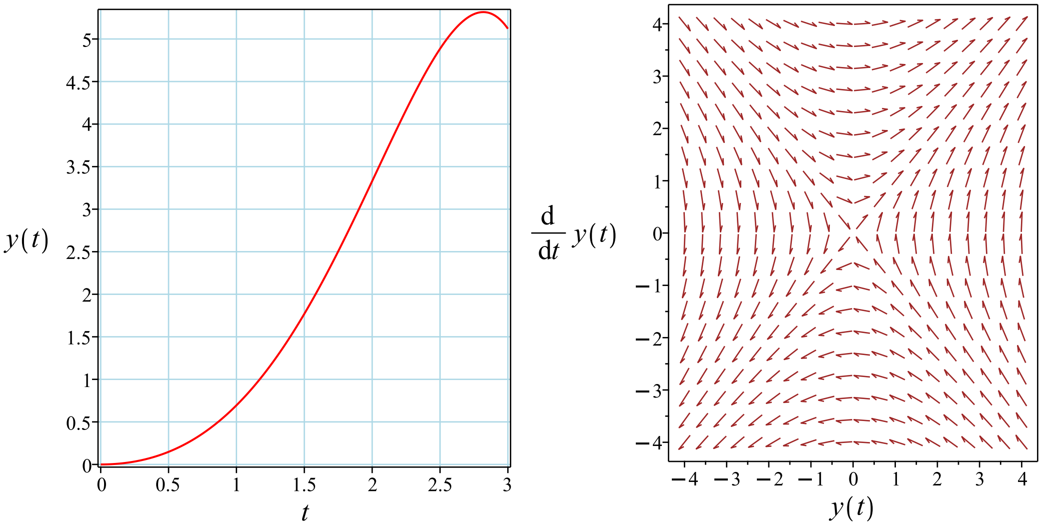
$$y = \frac{\mathrm{e}^{-t}}{5}+\frac{(-\cos(t)+2\sin(t))\,\mathrm{e}^{t}}{5}$$

Simplifying the solution gives

$$y = \frac{\mathrm{e}^{-t}}{5}+\frac{(-\cos(t)+2\sin(t))\,\mathrm{e}^{t}}{5}$$

Summary
The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-t}}{5}+\frac{(-\cos(t)+2\sin(t))\,\mathrm{e}^{t}}{5} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\mathrm{e}^{-t}}{5} + \frac{(-\cos(t) + 2\sin(t))\,\mathrm{e}^{t}}{5}$$

Verified OK.

6.9.2 Maple step by step solution

Let's solve

$\left[y'' - y = \mathrm{e}^{t}\cos(t)\,, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-t}c_1 + c_2\mathrm{e}^{t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathrm{e}^{t}\cos(t)\right]$

  ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t)\,, y_2(t)) = \left[\begin{array}{cc} \mathrm{e}^{-t} & \mathrm{e}^{t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{t} \end{array}\right]$$

- ○ Compute Wronskian

  $W(y_1(t)\,, y_2(t)) = 2$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\mathrm{e}^{-t}(\int \mathrm{e}^{2t}\cos(t)dt)}{2} + \frac{\mathrm{e}^{t}(\int \cos(t)dt)}{2}$

- ○ Compute integrals

  $y_p(t) = -\frac{(\cos(t)-2\sin(t))\mathrm{e}^{t}}{5}$

- ● Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-t}c_1 + c_2\mathrm{e}^{t} - \frac{(\cos(t)-2\sin(t))\mathrm{e}^{t}}{5}$

- ☐ Check validity of solution $y = \mathrm{e}^{-t}c_1 + c_2\mathrm{e}^{t} - \frac{(\cos(t)-2\sin(t))\mathrm{e}^{t}}{5}$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - \frac{1}{5}$

- ○ Compute derivative of the solution

  $y' = -\mathrm{e}^{-t}c_1 + c_2\mathrm{e}^{t} - \frac{(-\sin(t)-2\cos(t))\mathrm{e}^{t}}{5} - \frac{(\cos(t)-2\sin(t))\mathrm{e}^{t}}{5}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -c_1 + c_2 + \frac{1}{5}$

- ○ Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{5}, c_2 = 0\right\}$

- ○ Substitute constant values into general solution and simplify

  $y = \frac{\mathrm{e}^{-t}}{5} + \frac{(-\cos(t)+2\sin(t))\mathrm{e}^{t}}{5}$

- ● Solution to the IVP

  $y = \frac{\mathrm{e}^{-t}}{5} + \frac{(-\cos(t)+2\sin(t))\mathrm{e}^{t}}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.0 (sec). Leaf size: 24

```
dsolve([diff(y(t),t$2)-y(t)=exp(t)*cos(t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{e^{-t}}{5} + \frac{e^{t}(-\cos(t) + 2\sin(t))}{5}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 29

```
DSolve[{y''[t]-y[t]==Exp[t]*Cos[t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to \frac{1}{5}\left(e^{-t} + 2e^{t}\sin(t) - e^{t}\cos(t)\right)$$

6.10 problem 30

Internal problem ID [6677]
Internal file name [`OUTPUT/5925_Sunday_June_05_2022_04_01_50_PM_31220714/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y' + 5y = t + 1}$$

With initial conditions

$$[y(0) = 0, y'(0) = 4]$$

6.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 5 \\ F &= t + 1 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 5y = t + 1$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t + 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + 5Y(s) = \frac{s+1}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 4\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4 - 2sY(s) + 5Y(s) = \frac{s+1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4s^2 + s + 1}{s^2\,(s^2 - 2s + 5)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-\frac{7}{50}-\frac{51i}{50}}{s-1-2i}+\frac{-\frac{7}{50}+\frac{51i}{50}}{s-1+2i}+\frac{1}{5s^{2}}+\frac{7}{25s}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{-\frac{7}{50}-\frac{51i}{50}}{s-1-2i}\right) &= \left(-\frac{7}{50}-\frac{51i}{50}\right) \mathrm{e}^{(1+2i)t} \\
\mathcal{L}^{-1}\left(\frac{-\frac{7}{50}+\frac{51i}{50}}{s-1+2i}\right) &= \left(-\frac{7}{50}+\frac{51i}{50}\right) \mathrm{e}^{(1-2i)t} \\
\mathcal{L}^{-1}\left(\frac{1}{5s^{2}}\right) &= \frac{t}{5} \\
\mathcal{L}^{-1}\left(\frac{7}{25s}\right) &= \frac{7}{25}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = \frac{7}{25}+\frac{t}{5}+\frac{\left(-7\cos\left(2t\right)+51\sin\left(2t\right)\right)\mathrm{e}^{t}}{25}$$

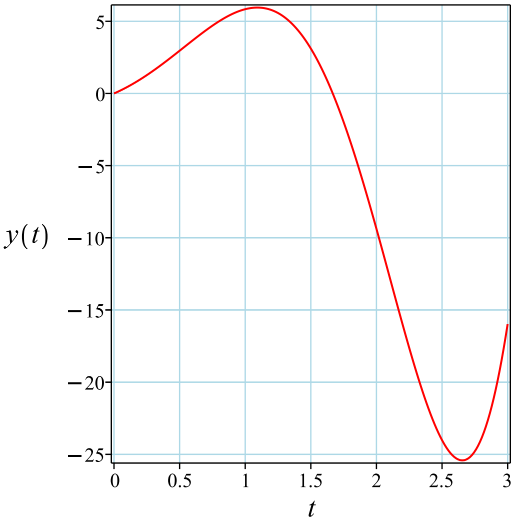
Simplifying the solution gives

$$y = -\frac{7\,\mathrm{e}^{t}\cos\left(2t\right)}{25}+\frac{51\,\mathrm{e}^{t}\sin\left(2t\right)}{25}+\frac{t}{5}+\frac{7}{25}$$

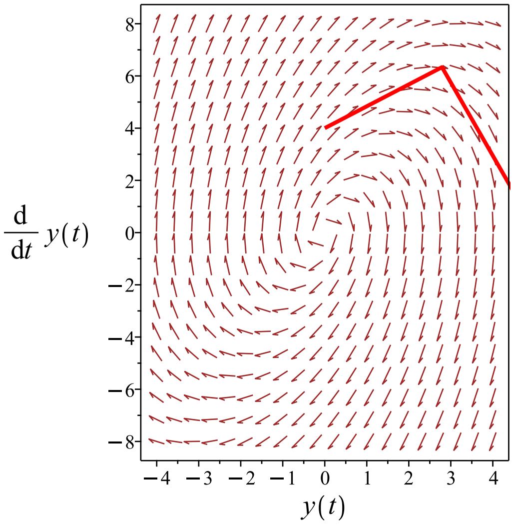
Summary

The solution(s) found are the following

$$y = -\frac{7\,\mathrm{e}^{t}\cos\left(2t\right)}{25}+\frac{51\,\mathrm{e}^{t}\sin\left(2t\right)}{25}+\frac{t}{5}+\frac{7}{25} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{7\,e^{t}\cos(2t)}{25} + \frac{51\,e^{t}\sin(2t)}{25} + \frac{t}{5} + \frac{7}{25}$$

Verified OK.

6.10.2 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + 5y = t + 1, y(0) = 0, y'\Big|_{\{t=0\}} = 4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = e^t \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^t \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = t+1\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^t \cos(2t) & e^t \sin(2t) \\ e^t \cos(2t) - 2e^t \sin(2t) & e^t \sin(2t) + 2e^t \cos(2t) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2e^{2t}$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{e^t\left(\cos(2t)\left(\int (t+1)\sin(2t)e^{-t}dt\right) - \sin(2t)\left(\int (t+1)\cos(2t)e^{-t}dt\right)\right)}{2}$

  ○ Compute integrals

  $y_p(t) = \frac{7}{25} + \frac{t}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \frac{7}{25} + \frac{t}{5}$

☐ Check validity of solution $y = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \frac{7}{25} + \frac{t}{5}$

  ○ Use initial condition $y(0) = 0$

  $0 = c_1 + \frac{7}{25}$

  ○ Compute derivative of the solution

  $y' = c_1 e^t \cos(2t) - 2c_1 e^t \sin(2t) + c_2 e^t \sin(2t) + 2c_2 e^t \cos(2t) + \frac{1}{5}$

  ○ Use the initial condition $y'\Big|_{\{t=0\}} = 4$

  $4 = c_1 + \frac{1}{5} + 2c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = -\frac{7}{25}, c_2 = \frac{51}{25}\right\}$

  - Substitute constant values into general solution and simplify

    $y = -\frac{7\,\mathrm{e}^t \cos(2t)}{25} + \frac{51\,\mathrm{e}^t \sin(2t)}{25} + \frac{t}{5} + \frac{7}{25}$

- Solution to the IVP

  $y = -\frac{7\,\mathrm{e}^t \cos(2t)}{25} + \frac{51\,\mathrm{e}^t \sin(2t)}{25} + \frac{t}{5} + \frac{7}{25}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.797 (sec). Leaf size: 26

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)+5*y(t)=1+t,y(0) = 0, D(y)(0) = 4],y(t), singsol=all)
```

$$y(t) = -\frac{7\,\mathrm{e}^t \cos(2t)}{25} + \frac{51\,\mathrm{e}^t \sin(2t)}{25} + \frac{t}{5} + \frac{7}{25}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 32

```
DSolve[{y''[t]-2*y'[t]+5*y[t]==1+t,{y[0]==0,y'[0]==4}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to \frac{1}{25}\left(5t + 51e^t \sin(2t) - 7e^t \cos(2t) + 7\right)$$

6.11 problem 31

Internal problem ID [6678]
Internal file name [`OUTPUT/5926_Sunday_June_05_2022_04_01_52_PM_95041200/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + y = 0}$$

With initial conditions

$$[y(1) = 2, y'(0) = 2]$$

Since initial condition $y(0)$ is not at zero, then let

$$y(0) = c_1$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= 2 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 2 - sc_1 + 2sY(s) - 2c_1 + Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{sc_1 + 2c_1 + 2}{s^2 + 2s + 1}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{c_1}{s+1} + \frac{2 + c_1}{(s+1)^2}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{c_1}{s+1}\right) &= \mathrm{e}^{-t} c_1 \\ \mathcal{L}^{-1}\left(\frac{2 + c_1}{(s+1)^2}\right) &= (2 + c_1)\, t\, \mathrm{e}^{-t} \end{aligned}$$

Adding the above results and simplifying gives

$$y = \mathrm{e}^{-t}(2t + c_1(t+1))$$

Since one initial condition $y(1)$ is not at zero then we need to setup one equation to solve for c_1. Using the above solution then

$$2 = \mathrm{e}^{-1}(2 + 2c_1)$$

Solving the above for c_1 gives Solving gives

$$c_1 = -\left(\mathrm{e}^{-1} - 1\right) \mathrm{e}$$

Subtituting this in the solution obtained above gives

$$\begin{aligned} y &= \mathrm{e}^{-t}\left(2t - \left(\mathrm{e}^{-1} - 1\right) \mathrm{e}(t+1)\right) \\ &= \mathrm{e}^{-t}(t - 1 + \mathrm{e}t + \mathrm{e}) \end{aligned}$$

Simplifying the solution gives

$$y = \mathrm{e}^{-t}(t - 1 + \mathrm{e}t + \mathrm{e})$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-t}(t - 1 + \mathrm{e}t + \mathrm{e}) \tag{1}$$

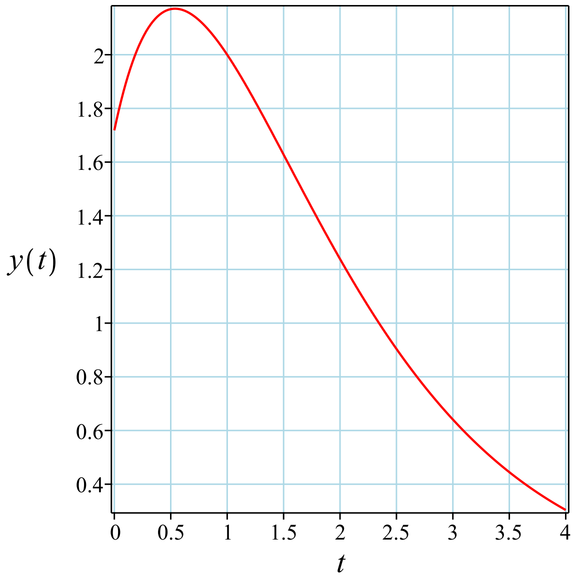


Figure 48: Solution plot

Verification of solutions

$$y = \mathrm{e}^{-t}(t - 1 + \mathrm{e}t + \mathrm{e})$$

Verified OK.

6.11.1 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + y = 0, y(1) = 2, y'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = \mathrm{e}^{-t} c_1 + t\,\mathrm{e}^{-t} c_2$

☐ Check validity of solution $y = \mathrm{e}^{-t} c_1 + t\mathrm{e}^{-t} c_2$

  - Use initial condition $y(1) = 2$

    $2 = \mathrm{e}^{-1} c_1 + \mathrm{e}^{-1} c_2$

  - Compute derivative of the solution

    $y' = -\mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{-t} - t\,\mathrm{e}^{-t} c_2$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 2$

    $2 = -c_1 + c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = -\frac{\mathrm{e}^{-1}-1}{\mathrm{e}^{-1}}, c_2 = \frac{1+\mathrm{e}^{-1}}{\mathrm{e}^{-1}}\right\}$

  - Substitute constant values into general solution and simplify

    $y = \mathrm{e}^{-t+1}(t+1) + \mathrm{e}^{-t}(t-1)$

- Solution to the IVP

  $y = \mathrm{e}^{-t+1}(t+1) + \mathrm{e}^{-t}(t-1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.547 (sec). Leaf size: 26

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t)=0,y(1) = 2, D(y)(0) = 2],y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{-t}(t - 1 + \mathrm{e}t + \mathrm{e})$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[{y''[t]+2*y'[t]+y[t]==0,{y[1]==2,y'[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t}(et + t + e - 1)$$

6.12 problem 32

Internal problem ID [6679]
Internal file name [`OUTPUT/5927_Sunday_June_05_2022_04_01_54_PM_48314026/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 8y' + 20y = 0}$$

With initial conditions

$$[y(0) = 0, y'(\pi) = 0]$$

Since initial condition $y'(0)$ is not at zero, then let

$$y'(0) = c_2$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 8sY(s) - 8y(0) + 20Y(s) = 0 \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= c_2
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^2Y(s) - c_2 + 8sY(s) + 20Y(s) = 0
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = \frac{c_2}{s^2 + 8s + 20}
$$

Applying partial fractions decomposition results in

$$
Y(s) = \frac{\left(\frac{1}{2} - \frac{i}{4}\right) c_2 - \frac{c_2}{2}}{s + 4 - 2i} + \frac{\left(\frac{1}{2} + \frac{i}{4}\right) c_2 - \frac{c_2}{2}}{s + 4 + 2i}
$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{\left(\frac{1}{2} - \frac{i}{4}\right) c_2 - \frac{c_2}{2}}{s + 4 - 2i}\right) &= -\frac{i c_2 \mathrm{e}^{(-4+2i)t}}{4} \\
\mathcal{L}^{-1}\left(\frac{\left(\frac{1}{2} + \frac{i}{4}\right) c_2 - \frac{c_2}{2}}{s + 4 + 2i}\right) &= \frac{i c_2 \mathrm{e}^{(-4-2i)t}}{4}
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = \frac{c_2 \mathrm{e}^{-4t} \sin(2t)}{2}
$$

Since one initial condition $y'(\pi)$ is not at zero, then we need to setup an equation to solve for c_2. At $t = \pi$. Taking derivative of the solution and evaluating at $t = \pi$ gives

$$
0 = c_2 \mathrm{e}^{-4\pi}
$$

Solving gives

$$
c_2 = 0
$$

Subtituting this in the solution obtained above gives

$$
\begin{aligned}
y &= 0 \\
&= 0
\end{aligned}
$$

Simplifying the solution gives

$$y = 0$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

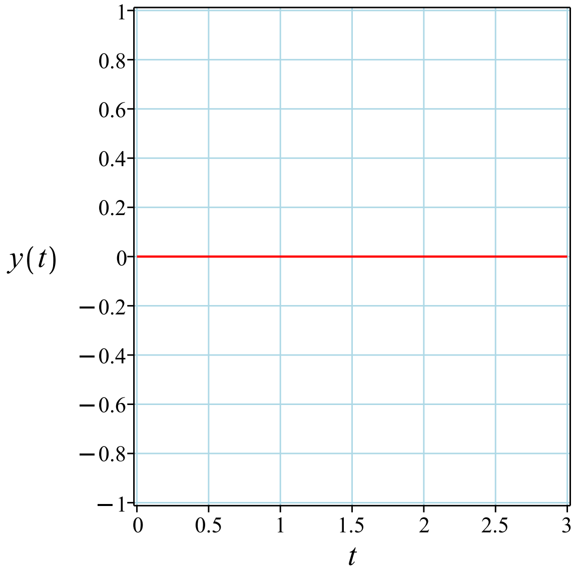


Figure 49: Solution plot

Verification of solutions

$$y = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.782 (sec). Leaf size: 5

```
dsolve([diff(y(t),t$2)+8*diff(y(t),t)+20*y(t)=0,y(0) = 0, D(y)(Pi) = 0],y(t), singsol=all)
```

$$y(t) = 0$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 6

```
DSolve[{y''[t]+8*y'[t]+20*y[t]==0,{y[0]==0,y'[Pi]==0}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to 0$$

6.13 problem 63

Internal problem ID [6680]
Internal file name [`OUTPUT/5928_Sunday_June_05_2022_04_01_56_PM_12801013/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 63.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \begin{cases} 0 & 0 \leq t < 1 \\ 5 & 1 \leq t \end{cases}}$$

With initial conditions

$$[y(0) = 0]$$

6.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 1$$
$$q(t) = \begin{cases} 0 & t < 1 \\ 5 & 1 \leq t \end{cases}$$

Hence the ode is

$$y' + y = \begin{cases} 0 & t < 1 \\ 5 & 1 \leq t \end{cases}$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \begin{cases} 0 & t < 1 \\ 5 & 1 \leq t \end{cases}$ is

$$\{t < 1 \vee 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.13.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + Y(s) = \frac{5\,\mathrm{e}^{-s}}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) + Y(s) = \frac{5\,\mathrm{e}^{-s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{5\,\mathrm{e}^{-s}}{s\,(s+1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{5\,\mathrm{e}^{-s}}{s\,(s+1)}\right) \\ &= 5\,\mathrm{Heaviside}\,(t-1)\left(1-\mathrm{e}^{-t+1}\right) \end{aligned}$$

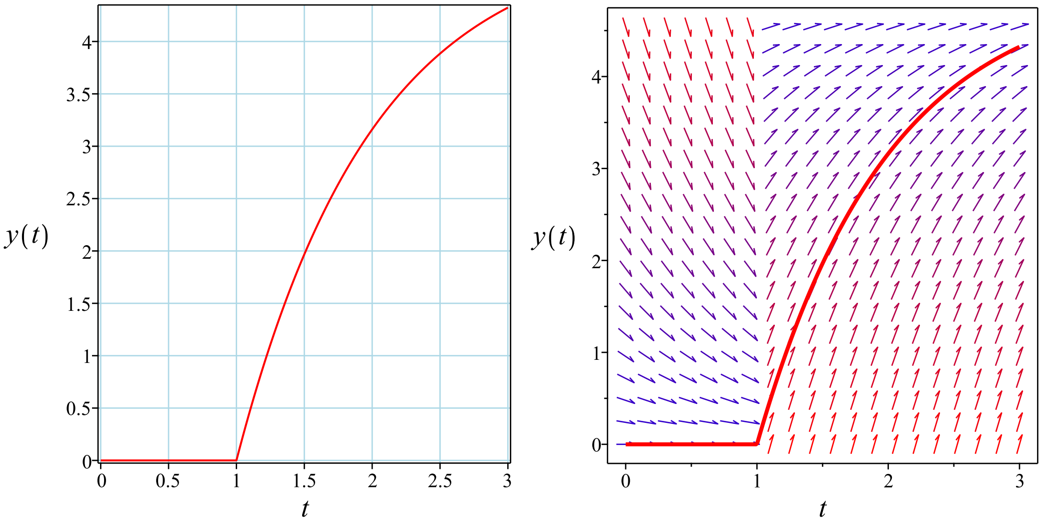
Hence the final solution is

$$y = 5\,\mathrm{Heaviside}\,(t-1)\left(1-\mathrm{e}^{-t+1}\right)$$

Summary

The solution(s) found are the following

$$y = 5\,\mathrm{Heaviside}\,(t-1)\left(1-\mathrm{e}^{-t+1}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 5\,\mathrm{Heaviside}\,(t-1)\left(1-\mathrm{e}^{-t+1}\right)$$

Verified OK.

6.13.3 Maple step by step solution

Let's solve

$$\left[y' + y = \left\{\begin{array}{cc} 0 & 0 \le t < 1 \\ 5 & 1 \le t \end{array}\right., y(0) = 0\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $$y' = -y + \left\{\begin{array}{cc} 0 & 0 \le t < 1 \\ 5 & 1 \le t \end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $$y' + y = \left\{\begin{array}{cc} 0 & 0 \le t < 1 \\ 5 & 1 \le t \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $$\mu(t)\,(y' + y) = \mu(t)\left(\left\{\begin{array}{cc} 0 & 0 \le t < 1 \\ 5 & 1 \le t \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$

- Integrate both sides with respect to t

  $$\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\left(\left\{\begin{array}{cc} 0 & 0 \le t < 1 \\ 5 & 1 \le t \end{array}\right.\right) dt + c_1$$

- Evaluate the integral on the lhs

  $$\mu(t)\,y = \int \mu(t)\left(\left\{\begin{array}{cc} 0 & 0 \le t < 1 \\ 5 & 1 \le t \end{array}\right.\right) dt + c_1$$

- Solve for y

$$y = \frac{\int \mu(t) \left(\left\{\begin{array}{cc} 0 & 0 \le t < 1 \\ 5 & 1 \le t \end{array}\right.\right) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{t}$

$$y = \frac{\int \mathrm{e}^{t} \left(\left\{\begin{array}{cc} 0 & 0 \le t < 1 \\ 5 & 1 \le t \end{array}\right.\right) dt + c_1}{\mathrm{e}^{t}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} 0 & t \le 1 \\ 5\,\mathrm{e}^{t} - 5\,\mathrm{e} & 1 < t \end{array}\right. + c_1}{\mathrm{e}^{t}}$$

- Simplify

$$y = \mathrm{e}^{-t} \left(\left\{\begin{array}{cc} 0 & t \le 1 \\ 5\,\mathrm{e}^{t} - 5\,\mathrm{e} & 1 < t \end{array}\right. + c_1\right)$$

- Use initial condition $y(0) = 0$

$$0 = c_1$$

- Solve for c_1

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = \left\{\begin{array}{cc} 0 & t \le 1 \\ 5 - 5\,\mathrm{e}^{-t+1} & 1 < t \end{array}\right.$$

- Solution to the IVP

$$y = \left\{\begin{array}{cc} 0 & t \le 1 \\ 5 - 5\,\mathrm{e}^{-t+1} & 1 < t \end{array}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.625 (sec). Leaf size: 22

```
dsolve([diff(y(t),t)+y(t)=piecewise(0<=t and t<1,0,t>=1,5),y(0) = 0],y(t), singsol=all)
```

$$y(t) = \begin{cases} 0 & t < 1 \\ 5 - 5\,\mathrm{e}^{1-t} & 1 \leq t \end{cases}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 23

```
DSolve[{y'[t]+y[t]==Piecewise[{{0,0<=t<1},{5,t>=1}}],{y[0]==0}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to \left\{ \begin{array}{cc} 0 & t \leq 1 \\ 5 - 5e^{1-t} & \text{True} \end{array} \right.$$

6.14 problem 64

Internal problem ID [6681]
Internal file name [`OUTPUT/5929_Sunday_June_05_2022_04_01_59_PM_39823723/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 64.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[[_linear, `class A`]]`

$$\boxed{y' + y = \begin{cases} 1 & 0 \leq t < 1 \\ -1 & 1 \leq t \end{cases}}$$

With initial conditions

$$[y(0) = 0]$$

6.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= 1 \\
q(t) &= \begin{cases} 0 & t < 0 \\ 1 & t < 1 \\ -1 & 1 \leq t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y' + y = \begin{cases} 0 & t < 0 \\ 1 & t < 1 \\ -1 & 1 \leq t \end{cases}
$$

The domain of $p(t) = 1$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \begin{cases} 0 & t < 0 \\ 1 & t < 1 \\ -1 & 1 \leq t \end{cases}$ is

$$
\{0 \leq t \leq 1, 1 \leq t \leq \infty, -\infty \leq t \leq 0\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.14.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\mathcal{L}(y') = sY(s) - y(0)
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$
sY(s) - y(0) + Y(s) = \frac{1 - 2\,\mathrm{e}^{-s}}{s} \tag{1}
$$

Replacing initial condition gives

$$sY(s) + Y(s) = \frac{1 - 2\,\mathrm{e}^{-s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{-1 + 2\,\mathrm{e}^{-s}}{s\,(s+1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-1 + 2\,\mathrm{e}^{-s}}{s\,(s+1)}\right) \\ &= 1 - \mathrm{e}^{-t} - 2\,\mathrm{Heaviside}\,(t-1)\left(1 - \mathrm{e}^{-t+1}\right) \end{aligned}$$

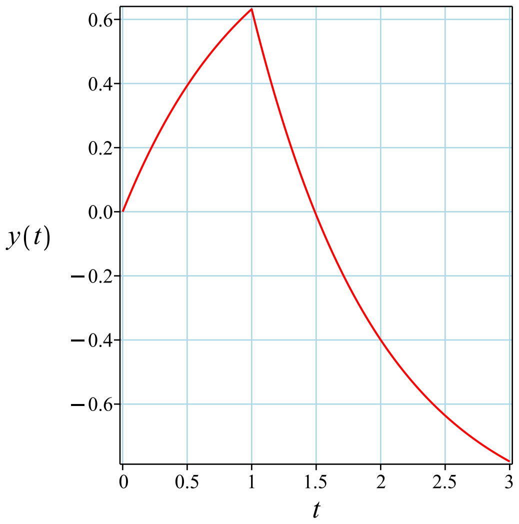
Hence the final solution is

$$y = 1 - \mathrm{e}^{-t} - 2\,\mathrm{Heaviside}\,(t-1)\left(1 - \mathrm{e}^{-t+1}\right)$$

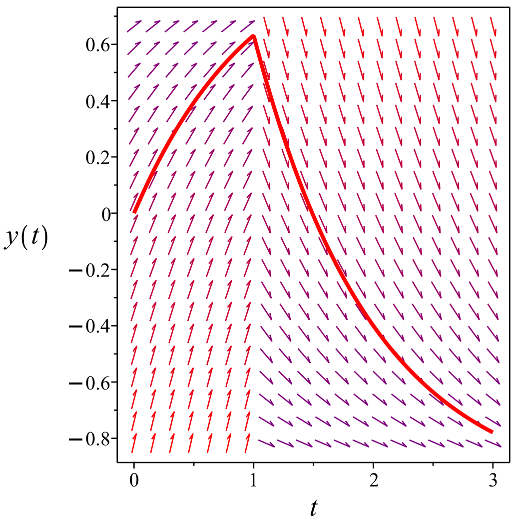
Summary

The solution(s) found are the following

$$y = 1 - \mathrm{e}^{-t} - 2\,\mathrm{Heaviside}\,(t-1)\left(1 - \mathrm{e}^{-t+1}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 - \mathrm{e}^{-t} - 2\,\mathrm{Heaviside}\,(t-1)\left(1 - \mathrm{e}^{-t+1}\right)$$

Verified OK.

6.14.3 Maple step by step solution

Let's solve

$$\left[y' + y = \left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right., y(0) = 0\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $$y' = -y + \left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $$y' + y = \left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $$\mu(t)\,(y' + y) = \mu(t)\left(\left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$

- Integrate both sides with respect to t

  $$\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\left(\left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right) dt + c_1$$

- Evaluate the integral on the lhs

  $$\mu(t)\,y = \int \mu(t)\left(\left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right) dt + c_1$$

- Solve for y

$$y = \frac{\int \mu(t) \left(\left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{t}$

$$y = \frac{\int \mathrm{e}^{t} \left(\left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right) dt + c_1}{\mathrm{e}^{t}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} 0 & t \le 0 \\ \mathrm{e}^{t} - 1 & t \le 1 \\ -\mathrm{e}^{t} + 2\,\mathrm{e} - 1 & 1 < t \end{array}\right. + c_1}{\mathrm{e}^{t}}$$

- Simplify

$$y = \mathrm{e}^{-t} \left(-\left(\left\{\begin{array}{cc} 0 & t \le 0 \\ 1 - \mathrm{e}^{t} & t \le 1 \\ \mathrm{e}^{t} - 2\,\mathrm{e} + 1 & 1 < t \end{array}\right.\right) + c_1 \right)$$

- Use initial condition $y(0) = 0$

$0 = c_1$

- Solve for c_1

$c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = \left\{\begin{array}{cc} 0 & t \le 0 \\ 1 - \mathrm{e}^{-t} & t \le 1 \\ -1 - \mathrm{e}^{-t} + 2\,\mathrm{e}^{-t+1} & 1 < t \end{array}\right.$$

- Solution to the IVP

$$y = \left\{\begin{array}{cc} 0 & t \le 0 \\ 1 - \mathrm{e}^{-t} & t \le 1 \\ -1 - \mathrm{e}^{-t} + 2\,\mathrm{e}^{-t+1} & 1 < t \end{array}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.782 (sec). Leaf size: 35

```
dsolve([diff(y(t),t)+y(t)=piecewise(0<=t and t<1,1,t>=1,-1),y(0) = 0],y(t), singsol=all)
```

$$y(t) = -\mathrm{e}^{-t} - \left(\begin{cases} -1 & t<1 \\ 1-2\,\mathrm{e}^{1-t} & 1\leq t \end{cases}\right)$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 43

```
DSolve[{y'[t]+y[t]==Piecewise[{{1,0<=t<1},{-1,t>=1}}],{y[0]==0}},y[t],t,IncludeSingularSolut
```

$$y(t) \to \begin{cases} 0 & t\leq 0 \\ 1-e^{-t} & 0<t\leq 1 \\ -e^{-t}(1-2e+e^{t}) & \text{True} \end{cases}$$

6.15 problem 65

Internal problem ID [6682]
Internal file name [`OUTPUT/5930_Sunday_June_05_2022_04_02_04_PM_51701053/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 65.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \begin{cases} t & 0 \leq t < 1 \\ 0 & 1 \leq t \end{cases}}$$

With initial conditions

$$[y(0) = 0]$$

6.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= 1 \\
q(t) &= \begin{cases} 0 & t < 0 \\ t & t < 1 \\ 0 & 1 \le t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y' + y = \begin{cases} 0 & t < 0 \\ t & t < 1 \\ 0 & 1 \le t \end{cases}
$$

The domain of $p(t) = 1$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \begin{cases} 0 & t < 0 \\ t & t < 1 \\ 0 & 1 \le t \end{cases}$ is

$$
\{t < 1 \lor 1 < t\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.15.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\mathcal{L}(y') = sY(s) - y(0)
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$
sY(s) - y(0) + Y(s) = \frac{-(s+1)\,\mathrm{e}^{-s} + 1}{s^2} \tag{1}
$$

Replacing initial condition gives

$$sY(s) + Y(s) = \frac{-(s+1)\,\mathrm{e}^{-s} + 1}{s^2}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{\mathrm{e}^{-s} s + \mathrm{e}^{-s} - 1}{s^2\,(s+1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{\mathrm{e}^{-s} s + \mathrm{e}^{-s} - 1}{s^2\,(s+1)}\right) \\
&= \operatorname{Heaviside}(-t+1)\,t + \left(\mathrm{e}^{-t+1} + 2\,\mathrm{e}^{\frac{1}{2}-\frac{t}{2}} \sinh\left(\frac{t}{2} - \frac{1}{2}\right)\right) \operatorname{Heaviside}(t-1) - 1 + \mathrm{e}^{-t}
\end{aligned}$$

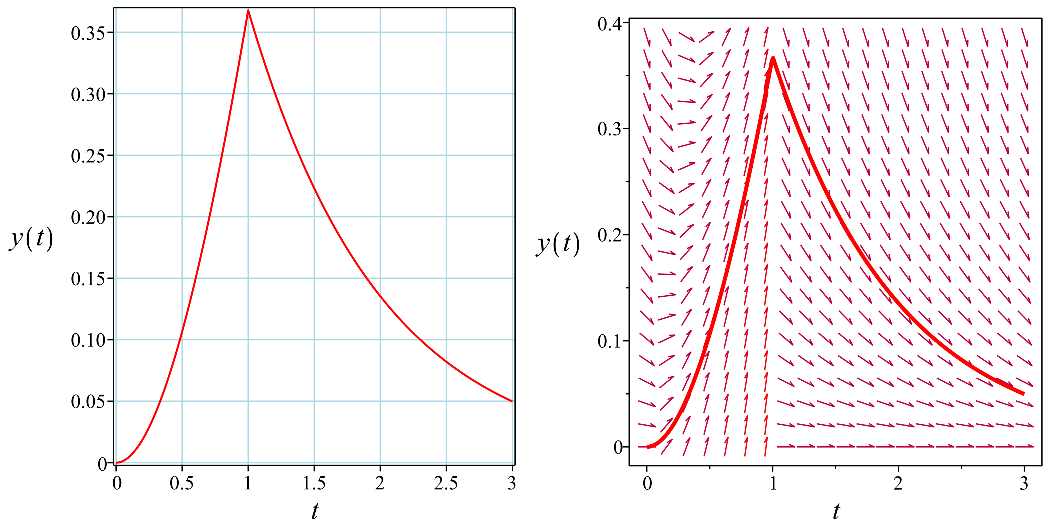
Hence the final solution is

$$y = \operatorname{Heaviside}(-t+1)\,t + \left(\mathrm{e}^{-t+1} + 2\,\mathrm{e}^{\frac{1}{2}-\frac{t}{2}} \sinh\left(\frac{t}{2} - \frac{1}{2}\right)\right) \operatorname{Heaviside}(t-1) - 1 + \mathrm{e}^{-t}$$

Summary

The solution(s) found are the following

$$y = \operatorname{Heaviside}(-t+1)\,t + \left(\mathrm{e}^{-t+1} + 2\,\mathrm{e}^{\frac{1}{2}-\frac{t}{2}} \sinh\left(\frac{t}{2} - \frac{1}{2}\right)\right) \operatorname{Heaviside}(t-1) - 1 + \mathrm{e}^{-t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \text{Heaviside}\left(-t+1\right) t + \left(\mathrm{e}^{-t+1} + 2\,\mathrm{e}^{\frac{1}{2}-\frac{t}{2}} \sinh\left(\frac{t}{2}-\frac{1}{2}\right)\right) \text{Heaviside}\left(t-1\right) - 1 + \mathrm{e}^{-t}$$

Verified OK.

6.15.3 Maple step by step solution

Let's solve

$$\left[y' + y = \left\{\begin{array}{cc} t & 0 \le t < 1 \\ 0 & 1 \le t \end{array}\right., y(0) = 0\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $$y' = -y + \left\{\begin{array}{cc} t & 0 \le t < 1 \\ 0 & 1 \le t \end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $$y' + y = \left\{\begin{array}{cc} t & 0 \le t < 1 \\ 0 & 1 \le t \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $$\mu(t)\left(y' + y\right) = \mu(t)\left(\left\{\begin{array}{cc} t & 0 \le t < 1 \\ 0 & 1 \le t \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}\left(\mu(t)\,y\right)$

  $\mu(t)\left(y' + y\right) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\,y)\right)dt = \int \mu(t)\left(\left\{\begin{array}{cc} t & 0\le t<1 \\ 0 & 1\le t \end{array}\right.\right)dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\,y = \int \mu(t)\left(\left\{\begin{array}{cc} t & 0\le t<1 \\ 0 & 1\le t \end{array}\right.\right)dt + c_1$$

- Solve for y

$$y = \frac{\int \mu(t)\left(\left\{\begin{array}{cc} t & 0\le t<1 \\ 0 & 1\le t \end{array}\right.\right)dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{t}$

$$y = \frac{\int \left(\left\{\begin{array}{cc} t & 0\le t<1 \\ 0 & 1\le t \end{array}\right.\right)\mathrm{e}^{t}dt + c_1}{\mathrm{e}^{t}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} 0 & t\le 0 \\ (t-1)\,\mathrm{e}^{t}+1 & t\le 1 \\ 1 & 1<t \end{array}\right. + c_1}{\mathrm{e}^{t}}$$

- Simplify

$$y = \left(\left\{\begin{array}{cc} 0 & t\le 0 \\ (t-1)\,\mathrm{e}^{t}+1 & t\le 1 \\ 1 & 1<t \end{array}\right. + c_1\right)\mathrm{e}^{-t}$$

- Use initial condition $y(0) = 0$

$0 = c_1$

- Solve for c_1

$c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = \begin{cases} 0 & t \leq 0 \\ -1 + \mathrm{e}^{-t} + t & t \leq 1 \\ \mathrm{e}^{-t} & 1 < t \end{cases}$$

- Solution to the IVP

$$y = \begin{cases} 0 & t \leq 0 \\ -1 + \mathrm{e}^{-t} + t & t \leq 1 \\ \mathrm{e}^{-t} & 1 < t \end{cases}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 3.313 (sec). Leaf size: 51

```
dsolve([diff(y(t),t)+y(t)=piecewise(0<=t and t<1,t,t>=1,0),y(0) = 0],y(t), singsol=all)
```

$$y(t) = \begin{cases} -1 + \mathrm{e}^{-t} + t & t < 1 \\ 1 + \mathrm{e}^{-1} & t = 1 \\ \mathrm{e}^{-t} & 1 < t \end{cases}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 32

```
DSolve[{y'[t]+y[t]==Piecewise[{{t,0<=t<1},{0,t>=1}}],{y[0]==0}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to \{ \begin{array}{cc} 0 & t \leq 0 \\ t + e^{-t} - 1 & 0 < t \leq 1 \\ e^{-t} & \text{True} \end{array}$$

6.16 problem 66

Internal problem ID [6683]
Internal file name [`OUTPUT/5931_Sunday_June_05_2022_04_02_09_PM_90726123/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 66.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & 1 \leq t \end{cases}}$$

With initial conditions

$$[y(0) = 0, y'(0) = -1]$$

6.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$
\begin{aligned}
p(t) &= 0 \\
q(t) &= 4 \\
F &= \begin{cases} 0 & t<0 \\ 1 & t<1 \\ 0 & 1\leq t \end{cases}
\end{aligned}
$$

Hence the ode is

$$y''+4y=\begin{cases} 0 & t<0 \\ 1 & t<1 \\ 0 & 1\leq t \end{cases}$$

The domain of $p(t)=0$ is

$$\{-\infty<t<\infty\}$$

And the point $t_0=0$ is inside this domain. The domain of $q(t)=4$ is

$$\{-\infty<t<\infty\}$$

And the point $t_0=0$ is also inside this domain. The domain of $F=\begin{cases} 0 & t<0 \\ 1 & t<1 \\ 0 & 1\leq t \end{cases}$ is

$$\{0\leq t\leq 1, 1\leq t\leq \infty, -\infty\leq t\leq 0\}$$

And the point $t_0=0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y)=Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$
\begin{aligned}
\mathcal{L}(y') &= sY(s)-y(0) \\
\mathcal{L}(y'') &= s^2Y(s)-y'(0)-sy(0)
\end{aligned}
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{1 - e^{-s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= -1 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 1 + 4Y(s) = \frac{1 - e^{-s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-1 + e^{-s} + s}{s\,(s^2 + 4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-1 + e^{-s} + s}{s\,(s^2 + 4)}\right) \\ &= \frac{1}{4} - \frac{\cos(2t)}{4} - \frac{\text{Heaviside}(t-1)\sin(t-1)^2}{2} - \frac{\sin(2t)}{2} \end{aligned}$$

Hence the final solution is

$$y = \frac{1}{4} - \frac{\cos(2t)}{4} - \frac{\text{Heaviside}(t-1)\sin(t-1)^2}{2} - \frac{\sin(2t)}{2}$$

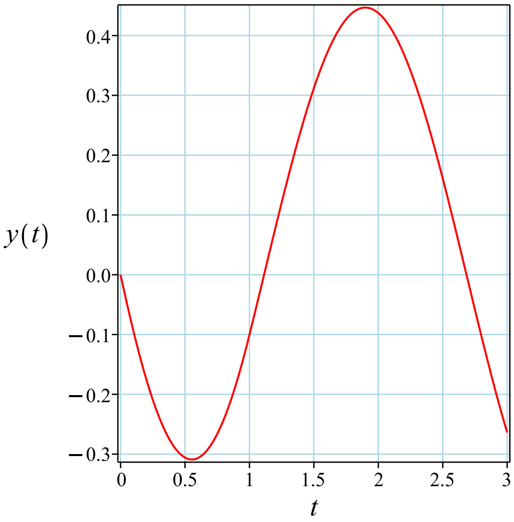
Simplifying the solution gives

$$y = \frac{1}{4} - \frac{\cos(2t)}{4} - \frac{\text{Heaviside}(t-1)\sin(t-1)^2}{2} - \frac{\sin(2t)}{2}$$

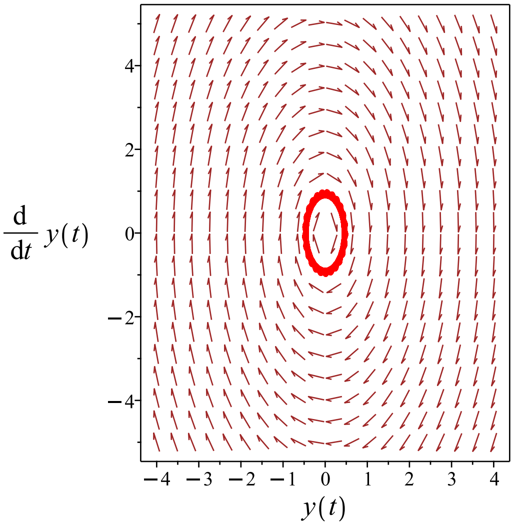
Summary

The solution(s) found are the following

$$y = \frac{1}{4} - \frac{\cos(2t)}{4} - \frac{\text{Heaviside}(t-1)\sin(t-1)^2}{2} - \frac{\sin(2t)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{4} - \frac{\cos(2t)}{4} - \frac{\text{Heaviside}(t-1)\sin(t-1)^2}{2} - \frac{\sin(2t)}{2}$$

Verified OK.

6.16.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y = \left\{\begin{array}{cc} 0 & t < 0 \\ 1 & t < 1 \\ 0 & 1 \leq t \end{array}\right., y(0) = 0, y'\Big|_{\{t=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \left\{\begin{array}{cc} 0 & t < 0 \\ 1 & t < 1 \\ 0 & 1 \le t \end{array}\right.\right]$$

  ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \left[\begin{array}{cc} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{array}\right]$$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

  ○ Substitute functions into equation for $y_p(t)$

  $$y_p(t) = -\cos(2t)\left(\int \left(\left\{\begin{array}{cc} 0 & t < 0 \\ \frac{\sin(2t)}{2} & t < 1 \\ 0 & 1 \le t \end{array}\right.\right) dt\right) + \sin(2t)\left(\int \left(\left\{\begin{array}{cc} 0 & t < 0 \\ \frac{\cos(2t)}{2} & t < 1 \\ 0 & 1 \le t \end{array}\right.\right) dt\right)$$

  ○ Compute integrals

  $$y_p(t) = -\frac{\left(\left\{\begin{array}{cc} 0 & t \le 0 \\ -1 + \cos(2t) & t \le 1 \\ 2\sin(1)^2 \cos(2t) - \sin(2)\sin(2t) & 1 < t \end{array}\right.\right)}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{\left(\begin{cases} 0 & t \leq 0 \\ -1+\cos(2t) & t \leq 1 \\ 2\sin(1)^2 \cos(2t) - \sin(2)\sin(2t) & 1 < t \end{cases}\right)}{4}$$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{\left(\begin{cases} 0 \\ -1+\cos(2t) \\ 2\sin(1)^2 \cos(2t) - \sin(2)\sin(2t) \end{cases}\right.}{4}$

- Use initial condition $y(0) = 0$

  $0 = c_1$
- Compute derivative of the solution

  $$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{\left(\begin{cases} 0 & t \leq 0 \\ -2\sin(2t) & t \leq 1 \\ -4\sin(1)^2 \sin(2t) - 2\sin(2)\cos(2t) & 1 < t \end{cases}\right)}{4}$$
- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = 2c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = 0, c_2 = -\frac{1}{2}\right\}$
- Substitute constant values into general solution and simplify

  $$y = -\frac{\sin(2t)}{2} - \frac{\left(\begin{cases} 0 & t \leq 0 \\ -1+\cos(2t) & t \leq 1 \\ 2\sin(1)^2 \cos(2t) - \sin(2)\sin(2t) & 1 < t \end{cases}\right)}{4}$$

● Solution to the IVP

$$y = -\frac{\sin(2t)}{2} - \frac{\left(\begin{cases} 0 & t \leq 0 \\ -1+\cos(2t) & t \leq 1 \\ 2\sin(1)^2 \cos(2t) - \sin(2)\sin(2t) & 1 < t \end{cases}\right)}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.734 (sec). Leaf size: 47

```
dsolve([diff(y(t),t$2)+4*y(t)=piecewise(0<=t and t<1,1,t>=1,0),y(0) = 0, D(y)(0) = -1],y(t),
```

$$y(t) = -\frac{\sin(2t)}{2} - \frac{\cos(2t)}{4} + \frac{\left(\begin{cases} 1 & t < 1 \\ \cos(2t-2) & 1 \leq t \end{cases}\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 65

```
DSolve[{y''[t]+4*y[t]==Piecewise[{{1,0<=t<1},{0,t>=1}}],{y[0]==0,y'[0]==-1}},y[t],t,IncludeS
```

$$y(t) \to \begin{cases} -\cos(t)\sin(t) & t \leq 0 \\ \frac{1}{4}(-\cos(2t) - 2\sin(2t) + 1) & 0 < t \leq 1 \\ \frac{1}{4}(\cos(2-2t) - \cos(2t) - 2\sin(2t)) & \text{True} \end{cases}$$

6.17 problem 67

Internal problem ID [6684]
Internal file name [`OUTPUT/5932_Sunday_June_05_2022_04_02_15_PM_53286991/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 67.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \sin(t)\,\mathrm{Heaviside}(t - 2\pi)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

6.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= \sin(t)\,\mathrm{Heaviside}(t - 2\pi) \end{aligned}$$

Hence the ode is

$$y'' + 4y = \sin(t) \operatorname{Heaviside}(t - 2\pi)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(t) \operatorname{Heaviside}(t - 2\pi)$ is

$$\{t < 2\pi \vee 2\pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{e^{-2\pi s}}{s^2 + 1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - s + 4Y(s) = \frac{e^{-2\pi s}}{s^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^3 + e^{-2\pi s} + s}{(s^2 + 1)(s^2 + 4)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{s^3 + \mathrm{e}^{-2\pi s} + s}{(s^2+1)\,(s^2+4)}\right) \\
&= \cos(2t) + \frac{(2\sin(t) - \sin(2t))\,\mathrm{Heaviside}(t-2\pi)}{6}
\end{aligned}
$$

Hence the final solution is

$$
y = \cos(2t) + \frac{(2\sin(t) - \sin(2t))\,\mathrm{Heaviside}(t-2\pi)}{6}
$$

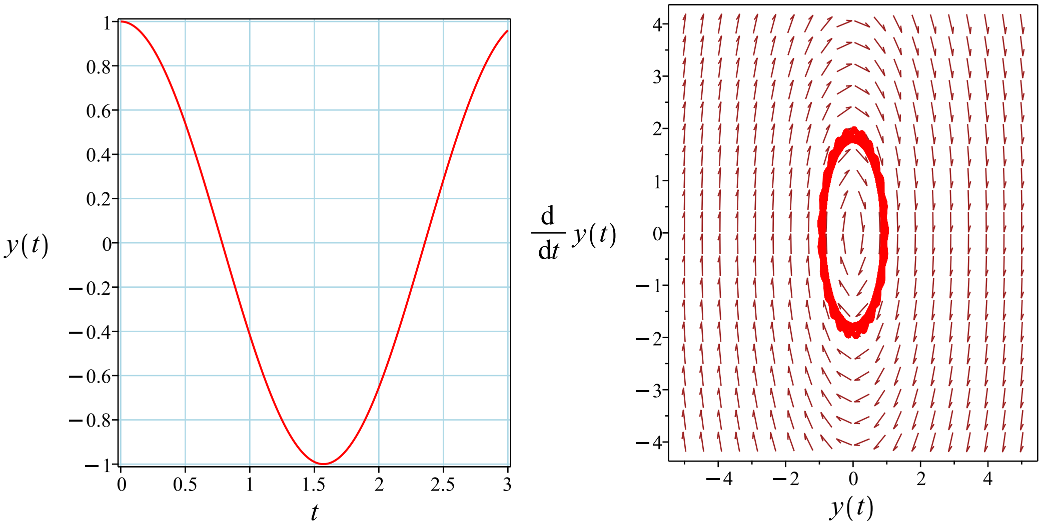
Simplifying the solution gives

$$
y = -\frac{\sin(t)\,(\cos(t)-1)\,\mathrm{Heaviside}(t-2\pi)}{3} + 2\cos(t)^2 - 1
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\sin(t)\,(\cos(t)-1)\,\mathrm{Heaviside}(t-2\pi)}{3} + 2\cos(t)^2 - 1 \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$
y = -\frac{\sin(t)\,(\cos(t)-1)\,\mathrm{Heaviside}(t-2\pi)}{3} + 2\cos(t)^2 - 1
$$

Verified OK.

6.17.2 Maple step by step solution

Let's solve

$\left[y'' + 4y = \sin(t)\, \mathit{Heaviside}(t - 2\pi), y(0) = 1, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \sin(t)\, \mathit{Heaviside}(t - 2\pi)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(2t)\left(\int \sin(2t)\sin(t)\mathit{Heaviside}(t-2\pi)dt\right)}{2} + \frac{\sin(2t)\left(\int \cos(2t)\sin(t)\mathit{Heaviside}(t-2\pi)dt\right)}{2}$

  - Compute integrals

    $y_p(t) = -\frac{\sin(t)(\cos(t)-1)\mathit{Heaviside}(t-2\pi)}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(2t) + c_2\sin(2t) - \frac{\sin(t)(\cos(t)-1)\mathit{Heaviside}(t-2\pi)}{3}$

☐ Check validity of solution $y = c_1\cos(2t) + c_2\sin(2t) - \frac{\sin(t)(\cos(t)-1)\mathit{Heaviside}(t-2\pi)}{3}$

  - Use initial condition $y(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $y' = -2c_1\sin(2t) + 2c_2\cos(2t) - \frac{\cos(t)(\cos(t)-1)\mathit{Heaviside}(t-2\pi)}{3} + \frac{\sin(t)^2\mathit{Heaviside}(t-2\pi)}{3} - \frac{\sin(t)(\cos(t)-1)\mathit{Di}}{3}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

    $y = -\frac{\sin(t)(\cos(t)-1)\mathit{Heaviside}(t-2\pi)}{3} + 2\cos(t)^2 - 1$

- Solution to the IVP

  $y = -\frac{\sin(t)(\cos(t)-1)\mathit{Heaviside}(t-2\pi)}{3} + 2\cos(t)^2 - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.031 (sec). Leaf size: 28

```
dsolve([diff(y(t),t$2)+4*y(t)=sin(t)*Heaviside(t-2*Pi),y(0) = 1, D(y)(0) = 0],y(t), singsol=
```

$$y(t) = -\frac{(\cos(t)-1)\sin(t)\,\mathrm{Heaviside}(t-2\pi)}{3} + 2\cos(t)^2 - 1$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 36

```
DSolve[{y''[t]+4*y[t]==Sin[t]*UnitStep[t-2*Pi],{y[0]==1,y'[0]==0}},y[t],t,IncludeSingularSol
```

$$y(t) \to \left\{ \begin{array}{cc} \cos(2t) & t \leq 2\pi \\ \frac{1}{3}(3\cos(2t) - \cos(t)\sin(t) + \sin(t)) & \text{True} \end{array} \right.$$

6.18 problem 68

Internal problem ID [6685]
Internal file name [`OUTPUT/5933_Sunday_June_05_2022_04_02_20_PM_64677786/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 68.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 5y' + 6y = \text{Heaviside}\,(t-1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

6.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -5 \\ q(t) &= 6 \\ F &= \text{Heaviside}\,(t-1) \end{aligned}$$

Hence the ode is

$$y'' - 5y' + 6y = \text{Heaviside}\,(t-1)$$

The domain of $p(t) = -5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \text{Heaviside}\,(t-1)$ is

$$\{t < 1 \vee 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 5sY(s) + 5y(0) + 6Y(s) = \frac{\mathrm{e}^{-s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 5sY(s) + 6Y(s) = \frac{\mathrm{e}^{-s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-s} + s}{s\,(s^2 - 5s + 6)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s))\\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-s}+s}{s\,(s^2-5s+6)}\right)\\
&= \frac{\mathrm{Heaviside}\,(t-1)}{6}-\mathrm{e}^{2t}+\mathrm{e}^{3t}+\frac{\mathrm{e}^{3t-3}}{3}-\frac{\mathrm{e}^{2t-2}}{2}+\frac{(-2\,\mathrm{e}^{3t-3}+3\,\mathrm{e}^{2t-2})\,\mathrm{Heaviside}\,(-t+1)}{6}
\end{aligned}
$$

Hence the final solution is

$$
y=\frac{\mathrm{Heaviside}\,(t-1)}{6}-\mathrm{e}^{2t}+\mathrm{e}^{3t}+\frac{\mathrm{e}^{3t-3}}{3}-\frac{\mathrm{e}^{2t-2}}{2}+\frac{(-2\,\mathrm{e}^{3t-3}+3\,\mathrm{e}^{2t-2})\,\mathrm{Heaviside}\,(-t+1)}{6}
$$

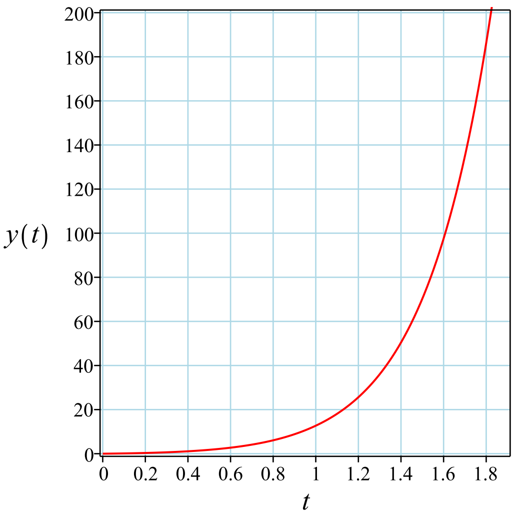
Simplifying the solution gives

$$
y=\frac{\mathrm{Heaviside}\,(t-1)}{6}-\mathrm{e}^{2t}+\mathrm{e}^{3t}+\frac{\mathrm{e}^{3t-3}\,\mathrm{Heaviside}\,(t-1)}{3}-\frac{\mathrm{e}^{2t-2}\,\mathrm{Heaviside}\,(t-1)}{2}
$$

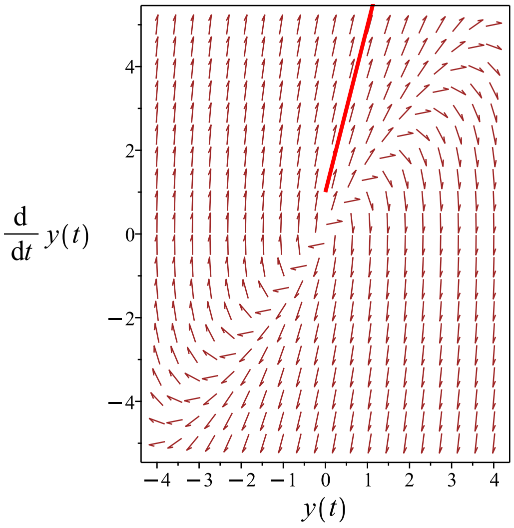
<u>Summary</u>

The solution(s) found are the following

$$
y=\frac{\mathrm{Heaviside}\,(t-1)}{6}-\mathrm{e}^{2t}+\mathrm{e}^{3t}+\frac{\mathrm{e}^{3t-3}\,\mathrm{Heaviside}\,(t-1)}{3}-\frac{\mathrm{e}^{2t-2}\,\mathrm{Heaviside}\,(t-1)}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\text{Heaviside}(t-1)}{6} - e^{2t} + e^{3t} + \frac{e^{3t-3}\,\text{Heaviside}(t-1)}{3} - \frac{e^{2t-2}\,\text{Heaviside}(t-1)}{2}$$

Verified OK.

6.18.2 Maple step by step solution

Let's solve

$$\left[y'' - 5y' + 6y = \mathit{Heaviside}(t-1)\,, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2t} + e^{3t} c_2 + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \mathit{Heaviside}(t-1)\right]$$

  ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{2t} & e^{3t} \\ 2\,e^{2t} & 3\,e^{3t} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{5t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -e^{2t}\left(\int \mathit{Heaviside}(t-1)\,e^{-2t}dt\right) + e^{3t}\left(\int \mathit{Heaviside}(t-1)\,e^{-3t}dt\right)$

- ○ Compute integrals

  $y_p(t) = \frac{\mathit{Heaviside}(t-1)\left(1+2\,e^{3t-3}-3\,e^{2t-2}\right)}{6}$

- ● Substitute particular solution into general solution to ODE

  $y = c_1 e^{2t} + e^{3t}c_2 + \frac{\mathit{Heaviside}(t-1)\left(1+2\,e^{3t-3}-3\,e^{2t-2}\right)}{6}$

☐ Check validity of solution $y = c_1 e^{2t} + e^{3t}c_2 + \frac{\mathit{Heaviside}(t-1)\left(1+2e^{3t-3}-3e^{2t-2}\right)}{6}$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

- ○ Compute derivative of the solution

  $y' = 2c_1 e^{2t} + 3\,e^{3t}c_2 + \frac{\mathit{Dirac}(t-1)\left(1+2\,e^{3t-3}-3\,e^{2t-2}\right)}{6} + \frac{\mathit{Heaviside}(t-1)\left(6\,e^{3t-3}-6\,e^{2t-2}\right)}{6}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = 2c_1 + 3c_2$

- ○ Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = 1\}$

- ○ Substitute constant values into general solution and simplify

  $y = \frac{\mathit{Heaviside}(t-1)}{6} - e^{2t} + e^{3t} + \frac{e^{3t-3}\mathit{Heaviside}(t-1)}{3} - \frac{e^{2t-2}\mathit{Heaviside}(t-1)}{2}$

- ● Solution to the IVP

  $y = \frac{\mathit{Heaviside}(t-1)}{6} - e^{2t} + e^{3t} + \frac{e^{3t-3}\mathit{Heaviside}(t-1)}{3} - \frac{e^{2t-2}\mathit{Heaviside}(t-1)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.906 (sec). Leaf size: 62

```
dsolve([diff(y(t),t$2)-5*diff(y(t),t)+6*y(t)=Heaviside(t-1),y(0) = 0, D(y)(0) = 1],y(t), sin
```

$$y(t) = \frac{\text{Heaviside}\,(t-1)}{6} - \mathrm{e}^{2t} + \mathrm{e}^{3t} - \frac{\text{Heaviside}\,(t-1)\,\mathrm{e}^{2t-2}}{2} + \frac{\text{Heaviside}\,(t-1)\,\mathrm{e}^{3t-3}}{3}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 60

```
DSolve[{y''[t]-5*y'[t]+6*y[t]==UnitStep[t-1],{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolut
```

$$y(t) \to \begin{cases} e^{2t}(-1+e^t) & t \leq 1 \\ \frac{1}{6} - e^{2t} + e^{3t} - \frac{1}{2}e^{2t-2} + \frac{1}{3}e^{3t-3} & \text{True} \end{cases}$$

6.19 problem 69

Internal problem ID [6686]
Internal file name [`OUTPUT/5934_Sunday_June_05_2022_04_02_24_PM_76411984/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 69.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \begin{cases} 0 & 0 \leq t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & 2\pi \leq t \end{cases}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

6.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \begin{cases} 0 & t < \pi \\ 1 & t < 2\pi \\ 0 & 2\pi \leq t \end{cases} \end{aligned}$$

Hence the ode is

$$y'' + y = \begin{cases} 0 & t < \pi \\ 1 & t < 2\pi \\ 0 & 2\pi \leq t \end{cases}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < \pi \\ 1 & t < 2\pi \\ 0 & 2\pi \leq t \end{cases}$ is

$$\{\pi \leq t \leq 2\pi, 2\pi \leq t \leq \infty, -\infty \leq t \leq \pi\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$
\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$
s^2Y(s) - y'(0) - sy(0) + Y(s) = \frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}}{s} \tag{1}
$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= 1
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^2Y(s) - 1 + Y(s) = \frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}}{s}
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = \frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s} + s}{s\,(s^2 + 1)}
$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s} + s}{s\,(s^2 + 1)}\right) \\
&= 2\,\mathrm{Heaviside}\,(t - \pi)\cos\left(\frac{t}{2}\right)^2 - 2\,\mathrm{Heaviside}\,(t - 2\pi)\sin\left(\frac{t}{2}\right)^2 + \sin\,(t)
\end{aligned}
$$

Hence the final solution is

$$
y = 2\,\mathrm{Heaviside}\,(t - \pi)\cos\left(\frac{t}{2}\right)^2 - 2\,\mathrm{Heaviside}\,(t - 2\pi)\sin\left(\frac{t}{2}\right)^2 + \sin\,(t)
$$

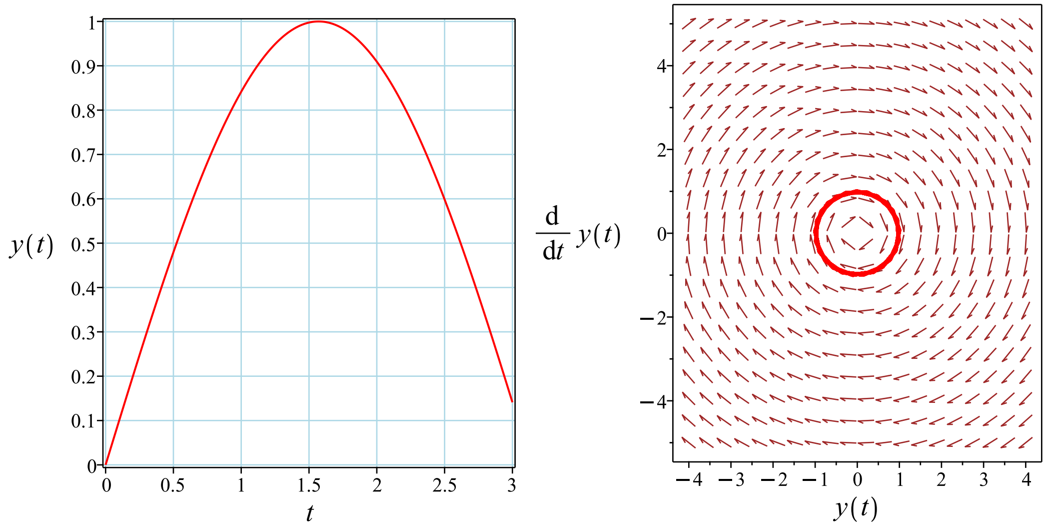
Simplifying the solution gives

$$
y = \mathrm{Heaviside}\,(t - \pi)\,(1 + \cos\,(t)) + \mathrm{Heaviside}\,(t - 2\pi)\,(\cos\,(t) - 1) + \sin\,(t)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{Heaviside}\,(t - \pi)\,(1 + \cos\,(t)) + \mathrm{Heaviside}\,(t - 2\pi)\,(\cos\,(t) - 1) + \sin\,(t) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \text{Heaviside}\,(t-\pi)\,(1+\cos(t)) + \text{Heaviside}\,(t-2\pi)\,(\cos(t)-1) + \sin(t)$$

Verified OK.

6.19.2 Maple step by step solution

Let's solve

$$\left[y''+y=\left\{\begin{array}{cc} 0 & t<\pi \\ 1 & t<2\pi \\ 0 & 2\pi \leq t \end{array}\right., y(0)=0, y'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+1=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=(-\mathrm{I},\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))}dt\right), f(t) = \left\{\begin{array}{cc} 0 & t < \pi \\ 1 & t < 2\pi \\ 0 & 2\pi \leq t \end{array}\right.\right]$$
  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(t), y_2(t)) = \left[\begin{array}{cc} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{array}\right]$$
  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 1$
  - Substitute functions into equation for $y_p(t)$

    $$y_p(t) = -\cos(t)\left(\int \left(\left\{\begin{array}{cc} 0 & t < \pi \\ \sin(t) & t < 2\pi \\ 0 & 2\pi \leq t \end{array}\right.\right)dt\right) + \sin(t)\left(\int \left(\left\{\begin{array}{cc} 0 & t < \pi \\ \cos(t) & t < 2\pi \\ 0 & 2\pi \leq t \end{array}\right.\right)dt\right)$$
  - Compute integrals

    $$y_p(t) = \left\{\begin{array}{cc} 0 & t \leq \pi \\ 1 + \cos(t) & t \leq 2\pi \\ 2\cos(t) & 2\pi < t \end{array}\right.$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(t) + c_2 \sin(t) + \left\{\begin{array}{cc} 0 & t \leq \pi \\ 1+\cos(t) & t \leq 2\pi \\ 2\cos(t) & 2\pi < t \end{array}\right.$$

☐ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) + \left\{\begin{array}{cc} 0 & t \leq \pi \\ 1+\cos(t) & t \leq 2\pi \\ 2\cos(t) & 2\pi < t \end{array}\right.$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1$

- ○ Compute derivative of the solution

  $$y' = -c_1 \sin(t) + c_2 \cos(t) + \left\{\begin{array}{cc} 0 & t \leq \pi \\ -\sin(t) & t \leq 2\pi \\ -2\sin(t) & 2\pi < t \end{array}\right.$$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = c_2$

- ○ Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 1\}$

- ○ Substitute constant values into general solution and simplify

  $$y = \sin(t) + \left\{\begin{array}{cc} 0 & t \leq \pi \\ 1+\cos(t) & t \leq 2\pi \\ 2\cos(t) & 2\pi < t \end{array}\right.$$

- Solution to the IVP

$$y = \sin(t) + \left\{\begin{array}{cc} 0 & t \leq \pi \\ 1+\cos(t) & t \leq 2\pi \\ 2\cos(t) & 2\pi < t \end{array}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.968 (sec). Leaf size: 50

```
dsolve([diff(y(t),t$2)+y(t)=piecewise(0<=t and t<Pi,0,Pi<=t and t<2*Pi,1,t>=2*Pi,0),y(0) = 0
```

$$y(t) = \sin(t) + \left(\begin{cases} 0 & t < \pi \\ \cos(t) + 1 & t < 2\pi \\ 2\cos(t) & 2\pi \leq t \end{cases}\right)$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 35

```
DSolve[{y''[t]+y[t]==Piecewise[{{0,0<=t<Pi},{1,Pi<=t<2*Pi},{0,t>=2*Pi}}],{y[0]==0,y'[0]==1}}
```

$$y(t) \to \begin{cases} \sin(t) & t \leq \pi \\ \cos(t) + \sin(t) + 1 & \pi < t \leq 2\pi \\ 2\cos(t) + \sin(t) & \text{True} \end{cases}$$

6.20 problem 70

Internal problem ID [6687]
Internal file name [`OUTPUT/5935_Sunday_June_05_2022_04_02_31_PM_23667262/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.3.1 TRANSLATION ON THE s-AXIS. Page 297
Problem number: 70.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 3y = 1 - \text{Heaviside}\left(-2 + t\right) - \text{Heaviside}\left(t - 4\right) + \text{Heaviside}\left(t - 6\right)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

6.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 3 \\ F &= 1 - \text{Heaviside}\left(-2 + t\right) - \text{Heaviside}\left(t - 4\right) + \text{Heaviside}\left(t - 6\right) \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 3y = 1 - \text{Heaviside}\left(-2 + t\right) - \text{Heaviside}\left(t - 4\right) + \text{Heaviside}\left(t - 6\right)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 1 - \text{Heaviside}\left(-2 + t\right) - \text{Heaviside}\left(t - 4\right) + \text{Heaviside}\left(t - 6\right)$ is

$$\{2 \le t \le 4, 4 \le t \le 6, 6 \le t \le \infty, -\infty \le t \le 2\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2 Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 3Y(s) = \frac{\mathrm{e}^{-6s} - \mathrm{e}^{-4s} - \mathrm{e}^{-2s} + 1}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) + 4sY(s) + 3Y(s) = \frac{\mathrm{e}^{-6s} - \mathrm{e}^{-4s} - \mathrm{e}^{-2s} + 1}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-6s} - \mathrm{e}^{-4s} - \mathrm{e}^{-2s} + 1}{s\,(s^2 + 4s + 3)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-6s} - \mathrm{e}^{-4s} - \mathrm{e}^{-2s} + 1}{s\,(s^2 + 4s + 3)}\right) \\
&= \frac{\mathrm{Heaviside}\,(-t+4)}{3} + \frac{\mathrm{e}^{-3t}}{6} - \frac{\mathrm{e}^{-t}}{2} + \frac{(-\mathrm{e}^{12-3t} + 3\,\mathrm{e}^{-t+4})\,\mathrm{Heaviside}\,(t-4)}{6} + \frac{(-2 - \mathrm{e}^{6-3t} + 3\,\mathrm{e}^{2-t})\,\mathrm{Hea}}{6}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y = {} & \frac{\mathrm{Heaviside}\,(-t+4)}{3} + \frac{\mathrm{e}^{-3t}}{6} - \frac{\mathrm{e}^{-t}}{2} + \frac{(-\mathrm{e}^{12-3t} + 3\,\mathrm{e}^{-t+4})\,\mathrm{Heaviside}\,(t-4)}{6} \\
& + \frac{(-2 - \mathrm{e}^{6-3t} + 3\,\mathrm{e}^{2-t})\,\mathrm{Heaviside}\,(-2+t)}{6} \\
& + \frac{(2 - 3\,\mathrm{e}^{-t+6} + \mathrm{e}^{-3t+18})\,\mathrm{Heaviside}\,(t-6)}{6}
\end{aligned}$$

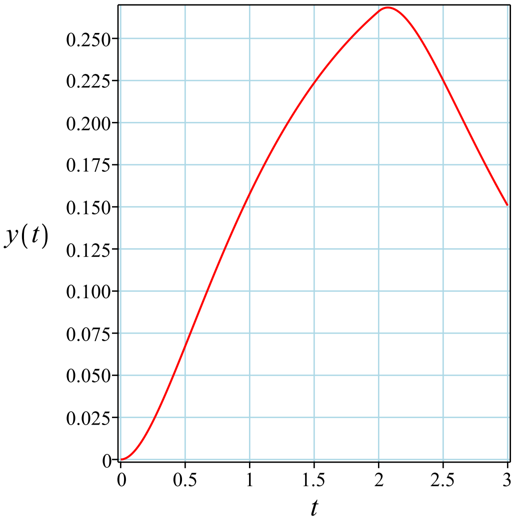
Simplifying the solution gives

$$\begin{aligned}
y = {} & \frac{1}{3} - \frac{\mathrm{Heaviside}\,(t-4)}{3} + \frac{\mathrm{e}^{-3t}}{6} - \frac{\mathrm{e}^{-t}}{2} - \frac{\mathrm{Heaviside}\,(t-4)\,\mathrm{e}^{12-3t}}{6} + \frac{\mathrm{Heaviside}\,(t-4)\,\mathrm{e}^{-t+4}}{2} \\
& - \frac{\mathrm{Heaviside}\,(-2+t)\,\mathrm{e}^{6-3t}}{6} + \frac{\mathrm{Heaviside}\,(-2+t)\,\mathrm{e}^{2-t}}{2} - \frac{\mathrm{Heaviside}\,(-2+t)}{3} \\
& - \frac{\mathrm{Heaviside}\,(t-6)\,\mathrm{e}^{-t+6}}{2} + \frac{\mathrm{Heaviside}\,(t-6)\,\mathrm{e}^{-3t+18}}{6} + \frac{\mathrm{Heaviside}\,(t-6)}{3}
\end{aligned}$$

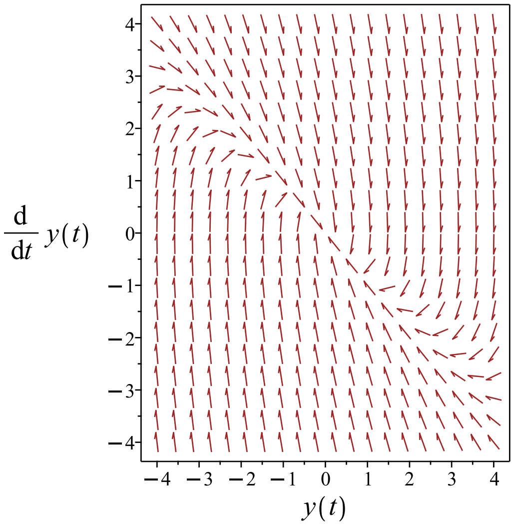
Summary

The solution(s) found are the following

$$\begin{aligned}
y = {} & \frac{1}{3} - \frac{\mathrm{Heaviside}\,(t-4)}{3} + \frac{\mathrm{e}^{-3t}}{6} - \frac{\mathrm{e}^{-t}}{2} - \frac{\mathrm{Heaviside}\,(t-4)\,\mathrm{e}^{12-3t}}{6} \\
& + \frac{\mathrm{Heaviside}\,(t-4)\,\mathrm{e}^{-t+4}}{2} - \frac{\mathrm{Heaviside}\,(-2+t)\,\mathrm{e}^{6-3t}}{6} \\
& + \frac{\mathrm{Heaviside}\,(-2+t)\,\mathrm{e}^{2-t}}{2} - \frac{\mathrm{Heaviside}\,(-2+t)}{3} \\
& - \frac{\mathrm{Heaviside}\,(t-6)\,\mathrm{e}^{-t+6}}{2} + \frac{\mathrm{Heaviside}\,(t-6)\,\mathrm{e}^{-3t+18}}{6} + \frac{\mathrm{Heaviside}\,(t-6)}{3}
\end{aligned} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y=\frac{1}{3}-\frac{\text{Heaviside}\left(t-4\right)}{3}+\frac{\mathrm{e}^{-3t}}{6}-\frac{\mathrm{e}^{-t}}{2}-\frac{\text{Heaviside}\left(t-4\right)\mathrm{e}^{12-3t}}{6}+\frac{\text{Heaviside}\left(t-4\right)\mathrm{e}^{-t+4}}{2}$$
$$-\frac{\text{Heaviside}\left(-2+t\right)\mathrm{e}^{6-3t}}{6}+\frac{\text{Heaviside}\left(-2+t\right)\mathrm{e}^{2-t}}{2}-\frac{\text{Heaviside}\left(-2+t\right)}{3}$$
$$-\frac{\text{Heaviside}\left(t-6\right)\mathrm{e}^{-t+6}}{2}+\frac{\text{Heaviside}\left(t-6\right)\mathrm{e}^{-3t+18}}{6}+\frac{\text{Heaviside}\left(t-6\right)}{3}$$

Verified OK.

6.20.2 Maple step by step solution

Let's solve

$$\left[y''+4y'+3y=1-\mathit{Heaviside}(-2+t)-\mathit{Heaviside}(t-4)+\mathit{Heaviside}(t-6)\,,y(0)=0,y'\Big|_{\{t=0}\right.$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+4r+3=0$

- Factor the characteristic polynomial

  $(r+3)\,(r+1)=0$

● Roots of the characteristic polynomial

$r = (-3, -1)$

● 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{-3t}$

● 2nd solution of the homogeneous ODE

$y_2(t) = \mathrm{e}^{-t}$

● General solution of the ODE

$y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

● Substitute in solutions of the homogeneous ODE

$y = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 1 - \mathit{Heaviside}(-2+t) - \mathit{H}\right.$

○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t} & \mathrm{e}^{-t} \\ -3\,\mathrm{e}^{-3t} & -\mathrm{e}^{-t} \end{bmatrix}$$

○ Compute Wronskian

$W(y_1(t), y_2(t)) = 2\,\mathrm{e}^{-4t}$

○ Substitute functions into equation for $y_p(t)$

$y_p(t) = \frac{\mathrm{e}^{-3t}\left(\int(-1+\mathit{Heaviside}(-2+t)+\mathit{Heaviside}(t-4)-\mathit{Heaviside}(t-6))\mathrm{e}^{3t}dt\right)}{2} - \frac{\mathrm{e}^{-t}\left(\int(-1+\mathit{Heaviside}(-2+t)+\mathit{Heaviside}(t-4\right.}{2}$

○ Compute integrals

$y_p(t) = -\frac{\mathit{Heaviside}(-2+t)}{3} - \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{6-3t}}{6} - \frac{\mathit{Heaviside}(t-4)}{3} - \frac{\mathit{Heaviside}(t-4)\mathrm{e}^{12-3t}}{6} + \frac{\mathit{Heaviside}(t-6)}{3} + \mathit{He}$

● Substitute particular solution into general solution to ODE

$y = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-t} - \frac{\mathit{Heaviside}(-2+t)}{3} - \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{6-3t}}{6} - \frac{\mathit{Heaviside}(t-4)}{3} - \frac{\mathit{Heaviside}(t-4)\mathrm{e}^{12-3t}}{6} + \mathit{Heavis}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-t} - \frac{\mathit{Heaviside}(-2+t)}{3} - \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{6-3t}}{6} - \frac{\mathit{Heaviside}(t-4)}{3}$

○ Use initial condition $y(0) = 0$

$0 = c_1 + c_2 + \frac{1}{3}$

○ Compute derivative of the solution

$y' = -3c_1 \mathrm{e}^{-3t} - c_2 \mathrm{e}^{-t} - \frac{\mathit{Dirac}(-2+t)}{3} - \frac{\mathit{Dirac}(-2+t)\mathrm{e}^{6-3t}}{6} + \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{6-3t}}{2} - \frac{\mathit{Dirac}(t-4)}{3} - \frac{\mathit{Dirac}(t-4)\mathrm{e}^{1}}{6}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -3c_1 - c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{6}, c_2 = -\frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{1}{3} - \frac{\mathit{Heaviside}(t-4)}{3} + \frac{\mathrm{e}^{-3t}}{6} - \frac{\mathrm{e}^{-t}}{2} - \frac{\mathit{Heaviside}(t-4)\mathrm{e}^{12-3t}}{6} + \frac{\mathit{Heaviside}(t-4)\mathrm{e}^{-t+4}}{2} - \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{6-3t}}{6} + \mathit{He}$

- Solution to the IVP

  $y = \frac{1}{3} - \frac{\mathit{Heaviside}(t-4)}{3} + \frac{\mathrm{e}^{-3t}}{6} - \frac{\mathrm{e}^{-t}}{2} - \frac{\mathit{Heaviside}(t-4)\mathrm{e}^{12-3t}}{6} + \frac{\mathit{Heaviside}(t-4)\mathrm{e}^{-t+4}}{2} - \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{6-3t}}{6} + \mathit{He}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.109 (sec). Leaf size: 94

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+3*y(t)=1-Heaviside(t-2)-Heaviside(t-4)+Heaviside(t-6),
```

$$
\begin{aligned}
y(t) = {} & \frac{1}{3} - \frac{\mathrm{Heaviside}\,(t-4)}{3} + \frac{\mathrm{e}^{-3t}}{6} - \frac{\mathrm{e}^{-t}}{2} - \frac{\mathrm{Heaviside}\,(t-6)\,\mathrm{e}^{-t+6}}{2} \\
& + \frac{\mathrm{Heaviside}\,(t-6)\,\mathrm{e}^{-3t+18}}{6} + \frac{\mathrm{Heaviside}\,(t-6)}{3} \\
& + \frac{\mathrm{Heaviside}\,(t-4)\,\mathrm{e}^{-t+4}}{2} - \frac{\mathrm{Heaviside}\,(t-4)\,\mathrm{e}^{-3t+12}}{6} \\
& + \frac{\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{-t+2}}{2} - \frac{\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{-3t+6}}{6} - \frac{\mathrm{Heaviside}\,(t-2)}{3}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 175

```
DSolve[{y''[t]+4*y'[t]+3*y[t]==1-UnitStep[t-2]-UnitStep[t-4]+UnitStep[t-6],{y[0]==0,y'[0]==0
```

$y(t)$

$$\rightarrow \begin{cases} \frac{1}{6}e^{-3t}(-1+e^t)^2(1+2e^t) & t\leq 2 \\ -\frac{1}{6}e^{-3t}(-1+e^2)(1+e^2+e^4-3e^{2t}) & 2<t\leq 4 \\ \frac{1}{6}e^{-3t}(-1+e^2)^2(1+e^2)(1+e^2+2e^4+e^6+2e^8+e^{10}+e^{12}-3e^{2t}) & t>6 \\ -\frac{1}{6}e^{-3t}(-1+e^6+e^{12}+3e^{2t}+2e^{3t}-3e^{2t+2}-3e^{2t+4}) & \text{True} \end{cases}$$

7 CHAPTER 7 THE LAPLACE TRANSFORM. 7.4.1 DERIVATIVES OF A TRANSFORM. Page 309

7.1 problem 9

Internal problem ID [6688]
Internal file name [`OUTPUT/5936_Sunday_June_05_2022_04_02_46_PM_9679260/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.4.1 DERIVATIVES OF A TRANSFORM. Page 309
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \sin(t)\, t}$$

With initial conditions

$$[y(0) = 0]$$

7.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 1$$
$$q(t) = \sin(t)\, t$$

Hence the ode is

$$y' + y = \sin(t)\, t$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \sin(t)\, t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + Y(s) = \frac{2s}{\left(s^2+1\right)^2} \tag{1}$$

Replacing initial condition gives

$$sY(s) + Y(s) = \frac{2s}{\left(s^2+1\right)^2}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2s}{\left(s^2+1\right)^2 (s+1)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{2(s+1)} + \frac{-\frac{1}{4} - \frac{i}{4}}{(s-i)^2} + \frac{-\frac{1}{4} + \frac{i}{4}}{(s+i)^2} + \frac{1}{4s-4i} + \frac{1}{4s+4i}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{1}{2\left(s+1\right)}\right) &= -\frac{\mathrm{e}^{-t}}{2}\\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{4}-\frac{i}{4}}{\left(s-i\right)^{2}}\right) &= \left(-\frac{1}{4}-\frac{i}{4}\right) t\,\mathrm{e}^{it}\\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{4}+\frac{i}{4}}{\left(s+i\right)^{2}}\right) &= \left(-\frac{1}{4}+\frac{i}{4}\right) t\,\mathrm{e}^{-it}\\
\mathcal{L}^{-1}\left(\frac{1}{4s-4i}\right) &= \frac{\mathrm{e}^{it}}{4}\\
\mathcal{L}^{-1}\left(\frac{1}{4s+4i}\right) &= \frac{\mathrm{e}^{-it}}{4}
\end{aligned}$$

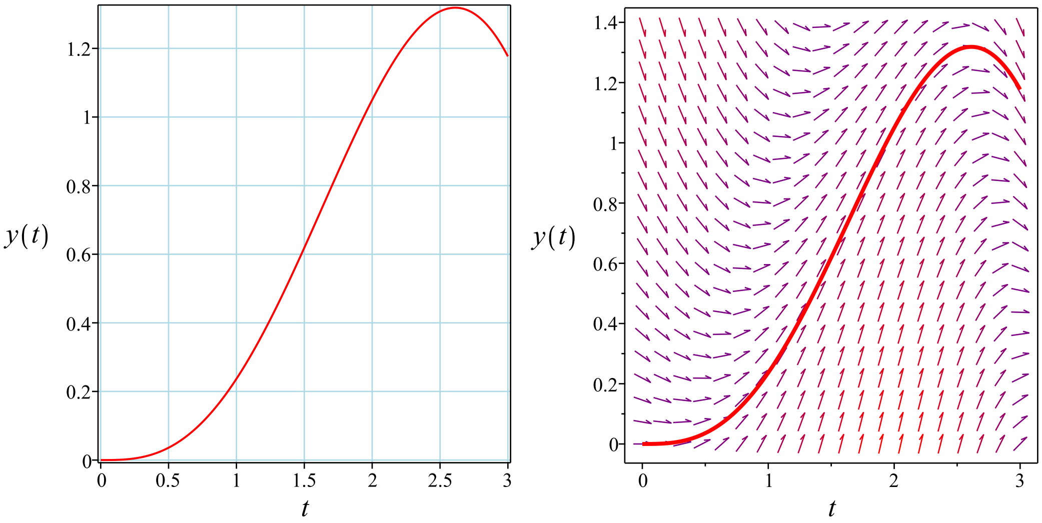
Adding the above results and simplifying gives

$$y = \frac{\sin\left(t\right) t}{2}-\frac{\mathrm{e}^{-t}}{2}-\frac{\cos\left(t\right) \left(t-1\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin\left(t\right) t}{2}-\frac{\mathrm{e}^{-t}}{2}-\frac{\cos\left(t\right) \left(t-1\right)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\sin(t)\,t}{2} - \frac{e^{-t}}{2} - \frac{\cos(t)\,(t-1)}{2}$$

Verified OK.

7.1.3 Maple step by step solution

Let's solve

$[y' + y = \sin(t)\,t, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + \sin(t)\,t$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = \sin(t)\,t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + y) = \mu(t)\sin(t)\,t$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$

- Solve to find the integrating factor

  $\mu(t) = e^{t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\sin(t)\,t dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\sin(t)\,t dt + c_1$

- Solve for y

  $y = \frac{\int \mu(t)\sin(t) t dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{t}$

  $y = \frac{\int e^{t}\sin(t) t dt + c_1}{e^{t}}$

- Evaluate the integrals on the rhs

  $y = \frac{\left(\frac{1}{2}-\frac{t}{2}\right) \mathrm{e}^{t} \cos(t)+\frac{\mathrm{e}^{t} \sin(t) t}{2}+c_1}{\mathrm{e}^{t}}$

- Simplify

  $y = \mathrm{e}^{-t} c_1 + \frac{(-t+1) \cos(t)}{2} + \frac{\sin(t) t}{2}$

- Use initial condition $y(0) = 0$

  $0 = \frac{1}{2} + c_1$

- Solve for c_1

  $c_1 = -\frac{1}{2}$

- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

  $y = -\frac{\mathrm{e}^{-t}}{2} + \frac{(-t+1) \cos(t)}{2} + \frac{\sin(t) t}{2}$

- Solution to the IVP

  $y = -\frac{\mathrm{e}^{-t}}{2} + \frac{(-t+1) \cos(t)}{2} + \frac{\sin(t) t}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.703 (sec). Leaf size: 23

```
dsolve([diff(y(t),t)+y(t)=t*sin(t),y(0) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{e^{-t}}{2} + \frac{(1-t)\cos(t)}{2} + \frac{t\sin(t)}{2}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 28

```
DSolve[{y'[t]+y[t]==t*Sin[t],{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}\left(-e^{-t} + t\sin(t) - t\cos(t) + \cos(t)\right)$$

7.2 problem 10

Internal problem ID [6689]
Internal file name [`OUTPUT/5937_Sunday_June_05_2022_04_02_49_PM_7573469/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.4.1 DERIVATIVES OF A TRANSFORM. Page 309
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = \mathrm{e}^{t} \sin(t)\, t}$$

With initial conditions

$$[y(0) = 0]$$

7.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= \mathrm{e}^{t} \sin(t)\, t \end{aligned}$$

Hence the ode is

$$y' - y = e^t \sin(t)\, t$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = e^t \sin(t)\, t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - Y(s) = \frac{2s-2}{\left((s-1)^2+1\right)^2} \tag{1}$$

Replacing initial condition gives

$$sY(s) - Y(s) = \frac{2s-2}{\left((s-1)^2+1\right)^2}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2}{(s^2-2s+2)^2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{2(s-1-i)^2} - \frac{1}{2(s-1+i)^2} - \frac{i}{2(s-1-i)} + \frac{i}{2s-2+2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{2\left(s-1-i\right)^{2}}\right)=-\frac{t\,\mathrm{e}^{\left(1+i\right)t}}{2}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{2\left(s-1+i\right)^{2}}\right)=-\frac{t\,\mathrm{e}^{\left(1-i\right)t}}{2}$$
$$\mathcal{L}^{-1}\left(-\frac{i}{2\left(s-1-i\right)}\right)=-\frac{i\mathrm{e}^{\left(1+i\right)t}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{i}{2s-2+2i}\right)=\frac{i\mathrm{e}^{\left(1-i\right)t}}{2}$$

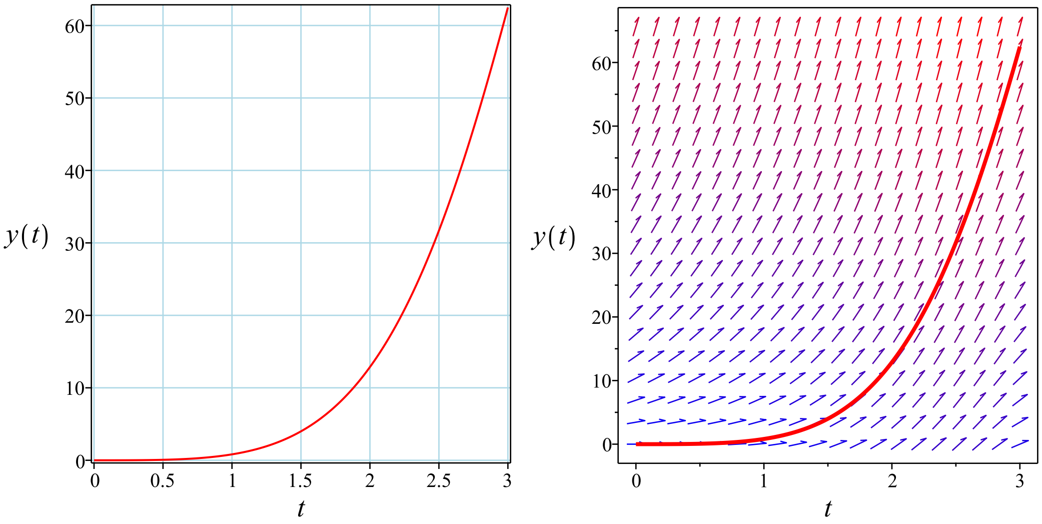
Adding the above results and simplifying gives

$$y=\mathrm{e}^{t}\left(\sin\left(t\right)-\cos\left(t\right)t\right)$$

Summary

The solution(s) found are the following

$$y=\mathrm{e}^{t}\left(\sin\left(t\right)-\cos\left(t\right)t\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y=\mathrm{e}^{t}\left(\sin\left(t\right)-\cos\left(t\right)t\right)$$

Verified OK.

7.2.3 Maple step by step solution

Let's solve

$[y' - y = \mathrm{e}^{t} \sin(t)\, t, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + \mathrm{e}^{t} \sin(t)\, t$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = \mathrm{e}^{t} \sin(t)\, t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - y) = \mu(t)\, \mathrm{e}^{t} \sin(t)\, t$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\,(y' - y) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\, \mathrm{e}^{t} \sin(t)\, t dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)\, \mathrm{e}^{t} \sin(t)\, t dt + c_1$

- Solve for y

  $y = \frac{\int \mu(t) \mathrm{e}^{t} \sin(t) t dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-t}$

  $y = \frac{\int \mathrm{e}^{-t} \mathrm{e}^{t} \sin(t) t dt + c_1}{\mathrm{e}^{-t}}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(t) - \cos(t) t + c_1}{\mathrm{e}^{-t}}$

- Simplify

  $y = \mathrm{e}^{t}(\sin(t) - \cos(t)\, t + c_1)$

- Use initial condition $y(0) = 0$

  $0 = c_1$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \mathrm{e}^t(\sin(t) - \cos(t)\, t)$
- Solution to the IVP

  $y = \mathrm{e}^t(\sin(t) - \cos(t)\, t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.922 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)-y(t)=t*exp(t)*sin(t),y(0) = 0],y(t), singsol=all)
```

$$y(t) = -\mathrm{e}^t(-\sin(t) + \cos(t)\, t)$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 17

```
DSolve[{y'[t]-y[t]==t*Exp[t]*Sin[t],{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^t(\sin(t) - t\cos(t))$$

7.3 problem 11

Internal problem ID [6690]
Internal file name [`OUTPUT/5938_Sunday_June_05_2022_04_02_52_PM_6088064/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.4.1 DERIVATIVES OF A TRANSFORM. Page 309
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = \cos(3t)}$$

With initial conditions

$$[y(0) = 2, y'(0) = 5]$$

7.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 9 \\ F &= \cos(3t) \end{aligned}$$

Hence the ode is

$$y'' + 9y = \cos(3t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(3t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{s}{s^2+9} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= 5\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 5 - 2s + 9Y(s) = \frac{s}{s^2+9}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s^3 + 5s^2 + 19s + 45}{(s^2+9)^2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{i}{12\left(s-3i\right)^{2}}+\frac{i}{12\left(s+3i\right)^{2}}+\frac{1-\frac{5i}{6}}{s-3i}+\frac{1+\frac{5i}{6}}{s+3i}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{i}{12\left(s-3i\right)^{2}}\right) &= -\frac{it\,\mathrm{e}^{3it}}{12}\\
\mathcal{L}^{-1}\left(\frac{i}{12\left(s+3i\right)^{2}}\right) &= \frac{it\,\mathrm{e}^{-3it}}{12}\\
\mathcal{L}^{-1}\left(\frac{1-\frac{5i}{6}}{s-3i}\right) &= \left(1-\frac{5i}{6}\right)\mathrm{e}^{3it}\\
\mathcal{L}^{-1}\left(\frac{1+\frac{5i}{6}}{s+3i}\right) &= \left(1+\frac{5i}{6}\right)\mathrm{e}^{-3it}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = 2\cos\left(3t\right)+\frac{\sin\left(3t\right)\left(t+10\right)}{6}$$

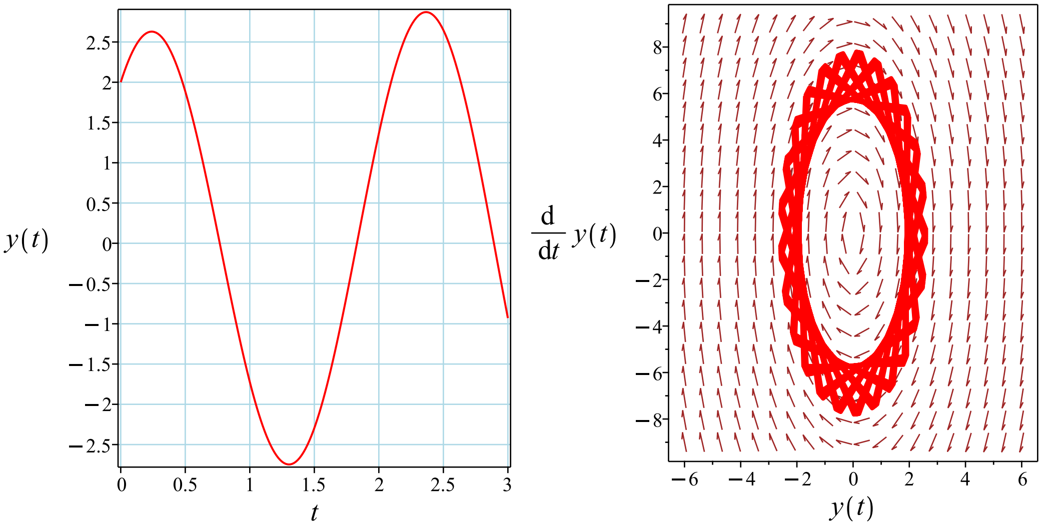
Simplifying the solution gives

$$y = 2\cos\left(3t\right)+\frac{\sin\left(3t\right)\left(t+10\right)}{6}$$

Summary

The solution(s) found are the following

$$y = 2\cos\left(3t\right)+\frac{\sin\left(3t\right)\left(t+10\right)}{6} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = 2\cos(3t) + \frac{\sin(3t)(t+10)}{6}$$

Verified OK.

7.3.2 Maple step by step solution

Let's solve

$$\left[y'' + 9y = \cos(3t), y(0) = 2, y'\Big|_{\{t=0\}} = 5\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \cos(3t)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(3t)\left(\int \sin(6t)dt\right)}{6} + \frac{\sin(3t)\left(\int \cos(3t)^2 dt\right)}{3}$

  ○ Compute integrals

  $y_p(t) = \frac{\cos(3t)}{36} + \frac{\sin(3t)t}{6}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{\cos(3t)}{36} + \frac{\sin(3t)t}{6}$

☐ Check validity of solution $y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{\cos(3t)}{36} + \frac{\sin(3t)t}{6}$

  ○ Use initial condition $y(0) = 2$

  $2 = c_1 + \frac{1}{36}$

  ○ Compute derivative of the solution

  $y' = -3c_1 \sin(3t) + 3c_2 \cos(3t) + \frac{\sin(3t)}{12} + \frac{\cos(3t)t}{2}$

  ○ Use the initial condition $y'\Big|_{\{t=0\}} = 5$

  $5 = 3c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = \frac{71}{36}, c_2 = \frac{5}{3}\right\}$

  - Substitute constant values into general solution and simplify

    $y = 2\cos(3t) + \frac{\sin(3t)(t+10)}{6}$

- Solution to the IVP

  $y = 2\cos(3t) + \frac{\sin(3t)(t+10)}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.718 (sec). Leaf size: 20

```
dsolve([diff(y(t),t$2)+9*y(t)=cos(3*t),y(0) = 2, D(y)(0) = 5],y(t), singsol=all)
```

$$y(t) = 2\cos(3t) + \frac{\sin(3t)(10+t)}{6}$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 23

```
DSolve[{y''[t]+9*y[t]==Cos[3*t],{y[0]==2,y'[0]==5}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{6}(t+10)\sin(3t) + 2\cos(3t)$$

7.4 problem 12

Internal problem ID [6691]
Internal file name [`OUTPUT/5939_Sunday_June_05_2022_04_02_54_PM_11383534/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.4.1 DERIVATIVES OF A TRANSFORM. Page 309
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sin(t)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

7.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \sin(t) \end{aligned}$$

Hence the ode is

$$y'' + y = \sin(t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \frac{1}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 1 - s + Y(s) = \frac{1}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s(s^2 - s + 1)}{(s^2+1)^2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{4 \left(s-i\right)^{2}}-\frac{1}{4 \left(s+i\right)^{2}}+\frac{\frac{1}{2}+\frac{i}{4}}{s-i}+\frac{\frac{1}{2}-\frac{i}{4}}{s+i}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{1}{4 \left(s-i\right)^{2}}\right) &= -\frac{t\,\mathrm{e}^{i t}}{4} \\
\mathcal{L}^{-1}\left(-\frac{1}{4 \left(s+i\right)^{2}}\right) &= -\frac{t\,\mathrm{e}^{-i t}}{4} \\
\mathcal{L}^{-1}\left(\frac{\frac{1}{2}+\frac{i}{4}}{s-i}\right) &= \left(\frac{1}{2}+\frac{i}{4}\right) \mathrm{e}^{i t} \\
\mathcal{L}^{-1}\left(\frac{\frac{1}{2}-\frac{i}{4}}{s+i}\right) &= \left(\frac{1}{2}-\frac{i}{4}\right) \mathrm{e}^{-i t}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = -\frac{\sin \left(t \right)}{2}-\frac{\cos \left(t \right) \left(-2+t \right)}{2}$$

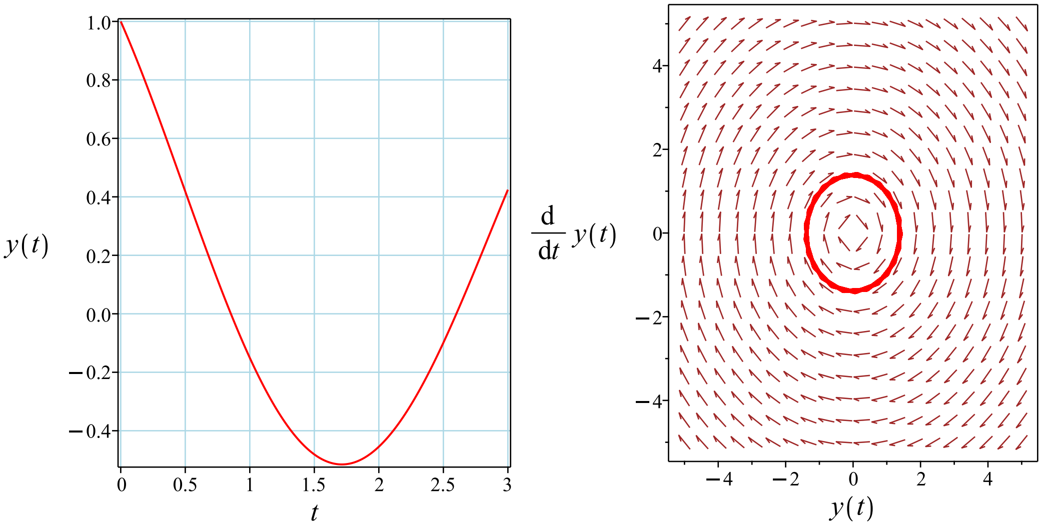
Simplifying the solution gives

$$y = -\frac{\sin \left(t \right)}{2}-\frac{\cos \left(t \right) \left(-2+t \right)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sin \left(t \right)}{2}-\frac{\cos \left(t \right) \left(-2+t \right)}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\sin(t)}{2} - \frac{\cos(t)(-2+t)}{2}$$

Verified OK.

7.4.2 Maple step by step solution

Let's solve

$$\left[y'' + y = \sin(t), y(0) = 1, y'\Big|_{\{t=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \sin(t)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 1$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\cos(t) \left(\int \sin(t)^2 \, dt\right) + \frac{\sin(t)\left(\int \sin(2t) dt\right)}{2}$

  ○ Compute integrals

  $y_p(t) = \frac{\sin(t)}{4} - \frac{\cos(t) t}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + \frac{\sin(t)}{4} - \frac{\cos(t) t}{2}$

☐ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) + \frac{\sin(t)}{4} - \frac{\cos(t) t}{2}$

  ○ Use initial condition $y(0) = 1$

  $1 = c_1$

  ○ Compute derivative of the solution

  $y' = -c_1 \sin(t) + c_2 \cos(t) - \frac{\cos(t)}{4} + \frac{\sin(t) t}{2}$

  ○ Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = -\frac{1}{4} + c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = 1, c_2 = -\frac{3}{4}\right\}$
  - Substitute constant values into general solution and simplify

    $y = -\frac{\cos(t)t}{2} - \frac{\sin(t)}{2} + \cos(t)$
- Solution to the IVP

  $y = -\frac{\cos(t)t}{2} - \frac{\sin(t)}{2} + \cos(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.735 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$2)+y(t)=sin(t),y(0) = 1, D(y)(0) = -1],y(t), singsol=all)
```

$$y(t) = -\frac{\sin(t)}{2} - \frac{\cos(t)(t-2)}{2}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 21

```
DSolve[{y''[t]+y[t]==Sin[t],{y[0]==1,y'[0]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{\sin(t)}{2} - \frac{1}{2}t\cos(t) + \cos(t)$$

7.5 problem 13

Internal problem ID [6692]
Internal file name [`OUTPUT/5940_Sunday_June_05_2022_04_02_56_PM_13728986/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.4.1 DERIVATIVES OF A TRANSFORM. Page 309
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 16y = \left\{\begin{array}{cc} \cos(4t) & 0 \leq t < \pi \\ 0 & \pi \leq t \end{array}\right.}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

7.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$
\begin{aligned}
p(t) &= 0 \\
q(t) &= 16 \\
F &= \begin{cases} 0 & t < 0 \\ \cos(4t) & t < \pi \\ 0 & \pi \le t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y'' + 16y = \begin{cases} 0 & t < 0 \\ \cos(4t) & t < \pi \\ 0 & \pi \le t \end{cases}
$$

The domain of $p(t) = 0$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 16$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < 0 \\ \cos(4t) & t < \pi \\ 0 & \pi \le t \end{cases}$ is

$$
\{0 \le t \le \pi, \pi \le t \le \infty, -\infty \le t \le 0\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0)
\end{aligned}
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^{2}Y(s)-y'(0)-sy(0)+16Y(s)=\frac{\left(-\mathrm{e}^{-\pi s}+1\right)s}{s^{2}+16} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0)&=0\\ y'(0)&=1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^{2}Y(s)-1+16Y(s)=\frac{\left(-\mathrm{e}^{-\pi s}+1\right)s}{s^{2}+16}$$

Solving the above equation for $Y(s)$ results in

$$Y(s)=-\frac{s\,\mathrm{e}^{-\pi s}-s^{2}-s-16}{\left(s^{2}+16\right)^{2}}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}y&=\mathcal{L}^{-1}(Y(s))\\ &=\mathcal{L}^{-1}\left(-\frac{s\,\mathrm{e}^{-\pi s}-s^{2}-s-16}{\left(s^{2}+16\right)^{2}}\right)\\ &=\frac{\sin\left(4t\right)\left(2+t-\operatorname{Heaviside}\left(t-\pi\right)\left(t-\pi\right)\right)}{8}\end{aligned}$$

Converting the above solution to piecewise it becomes

$$y=\begin{cases}\frac{\sin(4t)(t+2)}{8} & t\le\pi\\ \frac{\sin(4t)(2+\pi)}{8} & \pi<t\end{cases}$$

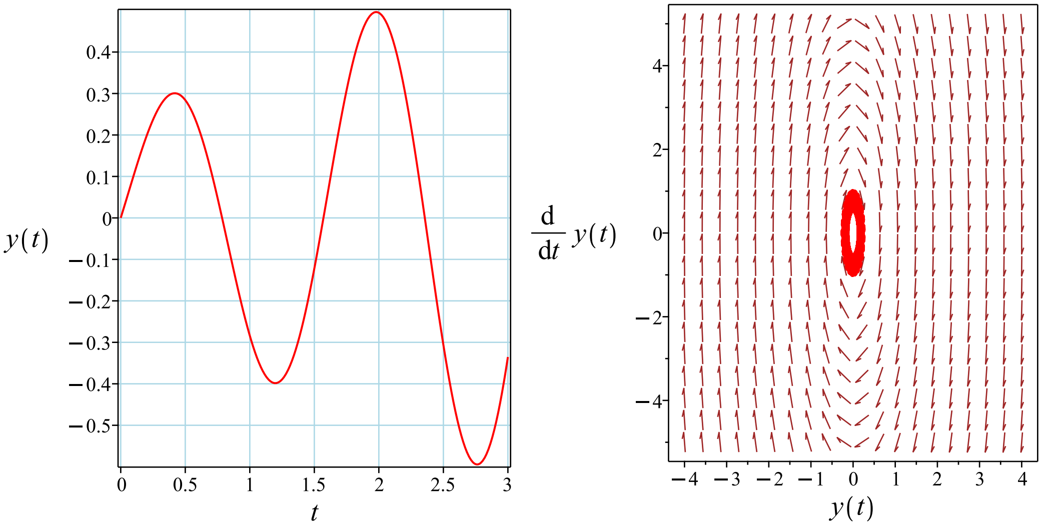
Simplifying the solution gives

$$y=\frac{\sin\left(4t\right)\left(2+\left(\left\{\begin{array}{cc}t & t\le\pi\\ \pi & \pi<t\end{array}\right.\right)\right)}{8}$$

Summary

The solution(s) found are the following

$$y=\frac{\sin\left(4t\right)\left(2+\left(\left\{\begin{array}{cc}t & t\le\pi\\ \pi & \pi<t\end{array}\right.\right)\right)}{8} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\sin(4t)\left(2+\left(\left\{\begin{array}{cc} t & t \leq \pi \\ \pi & \pi < t \end{array}\right.\right)\right)}{8}$$

Verified OK.

7.5.2 Maple step by step solution

Let's solve

$$\left[y''+16y = \left\{\begin{array}{cc} 0 & t<0 \\ \cos(4t) & t<\pi \\ 0 & \pi \leq t \end{array}\right., y(0)=0, y'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+16=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-4\,\mathrm{I}, 4\,\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(4t)$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(4t)$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(4t) + c_2 \sin(4t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \left\{\begin{array}{cc} 0 & t < 0 \\ \cos(4t) & t < \pi \\ 0 & \pi \leq t \end{array}\right.\right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \left[\begin{array}{cc} \cos(4t) & \sin(4t) \\ -4\sin(4t) & 4\cos(4t) \end{array}\right]$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 4$
- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\cos(4t)\left(\int \left(\left\{\begin{array}{cc} 0 & t < 0 \\ \frac{\sin(8t)}{8} & t < \pi \\ 0 & \pi \leq t \end{array}\right.\right) dt\right) + \sin(4t)\left(\int \left(\left\{\begin{array}{cc} 0 & t < 0 \\ \frac{\cos(4t)^2}{4} & t < \pi \\ 0 & \pi \leq t \end{array}\right.\right) dt\right)$$

- ○ Compute integrals

$$y_p(t) = \frac{\sin(4t)\left(\left\{\begin{array}{cc} 0 & t \leq 0 \\ t & t \leq \pi \\ \pi & \pi < t \end{array}\right.\right)}{8}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(4t) + c_2 \sin(4t) + \frac{\sin(4t)\left(\begin{cases} 0 & t \leq 0 \\ t & t \leq \pi \\ \pi & \pi < t \end{cases}\right)}{8}$$

☐ Check validity of solution $y = c_1 \cos(4t) + c_2 \sin(4t) + \frac{\sin(4t)\left(\begin{cases} 0 & t \leq 0 \\ t & t \leq \pi \\ \pi & \pi < t \end{cases}\right)}{8}$

  - ○ Use initial condition $y(0) = 0$

    $0 = c_1$

  - ○ Compute derivative of the solution

    $$y' = -4c_1 \sin(4t) + 4c_2 \cos(4t) + \frac{\cos(4t)\left(\begin{cases} 0 & t \leq 0 \\ t & t \leq \pi \\ \pi & \pi < t \end{cases}\right)}{2} + \frac{\sin(4t)\left(\begin{cases} 0 & t < 0 \\ \mathit{undefined} & t = 0 \\ 1 & t < \pi \\ \mathit{undefined} & t = \pi \\ 0 & \pi < t \end{cases}\right)}{8}$$

  - ○ Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = \mathit{undefined} + 4c_2$

  - ○ Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = \mathit{undefined}\}$

  - ○ Substitute constant values into general solution and simplify

    $y = \mathit{undefined}$

- Solution to the IVP

  $y = \mathit{undefined}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.703 (sec). Leaf size: 29

```
dsolve([diff(y(t),t$2)+16*y(t)=piecewise(0<=t and t<Pi,cos(4*t),t>= Pi,0),y(0) = 0, D(y)(0)
```

$$y(t) = \frac{\sin(4t)\left(2+\left(\begin{cases} t & t<\pi \\ \pi & \pi \le t \end{cases}\right)\right)}{8}$$

✓ Solution by Mathematica

Time used: 0.135 (sec). Leaf size: 60

```
DSolve[{y''[t]+16*y[t]==Piecewise[{{Cos[4*t],0<=t<Pi},{0,t>=Pi}}],{y[0]==1,y'[0]==1}},y[t],t
```

$$y(t) \to \left\{ \begin{array}{cc} \cos(4t)+\frac{1}{4}\sin(4t) & t \le 0 \\ \cos(4t)+\frac{1}{8}(2+\pi)\sin(4t) & t>\pi \\ \cos(4t)+\frac{1}{8}(t+2)\sin(4t) & \text{True} \end{array} \right.$$

7.6 problem 14

Internal problem ID [6693]
Internal file name [`OUTPUT/5941_Sunday_June_05_2022_04_03_03_PM_50687365/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.4.1 DERIVATIVES OF A TRANSFORM. Page 309
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \begin{cases} 1 & 0 \leq t < \frac{\pi}{2} \\ \sin(t) & \frac{\pi}{2} \leq t \end{cases}}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

7.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \begin{cases} 0 & t < 0 \\ 1 & t < \frac{\pi}{2} \\ \sin(t) & \frac{\pi}{2} \le t \end{cases} \end{aligned}$$

Hence the ode is

$$y'' + y = \begin{cases} 0 & t < 0 \\ 1 & t < \frac{\pi}{2} \\ \sin(t) & \frac{\pi}{2} \le t \end{cases}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < 0 \\ 1 & t < \frac{\pi}{2} \\ \sin(t) & \frac{\pi}{2} \le t \end{cases}$ is

$$\left\{0 \le t \le \frac{\pi}{2}, \frac{\pi}{2} \le t \le \infty, -\infty \le t \le 0\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2 Y(s) - y'(0) - s y(0) + Y(s) = \frac{-\mathrm{e}^{-\frac{\pi s}{2}} + 1 + s^2}{s\,(s^2+1)} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) - s + Y(s) = \frac{-\mathrm{e}^{-\frac{\pi s}{2}} + 1 + s^2}{s\,(s^2+1)}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-s^4 - 2s^2 + \mathrm{e}^{-\frac{\pi s}{2}} - 1}{s\,(s^2+1)^2}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-s^4 - 2s^2 + \mathrm{e}^{-\frac{\pi s}{2}} - 1}{s\,(s^2+1)^2}\right) \\ &= 1 + \frac{(-4 + 4\sin(t) - \cos(t)\,(2t-\pi))\,\mathrm{Heaviside}\left(t - \frac{\pi}{2}\right)}{4} \end{aligned}$$

Hence the final solution is

$$y = 1 + \frac{(-4 + 4\sin(t) - \cos(t)\,(2t-\pi))\,\mathrm{Heaviside}\left(t - \frac{\pi}{2}\right)}{4}$$

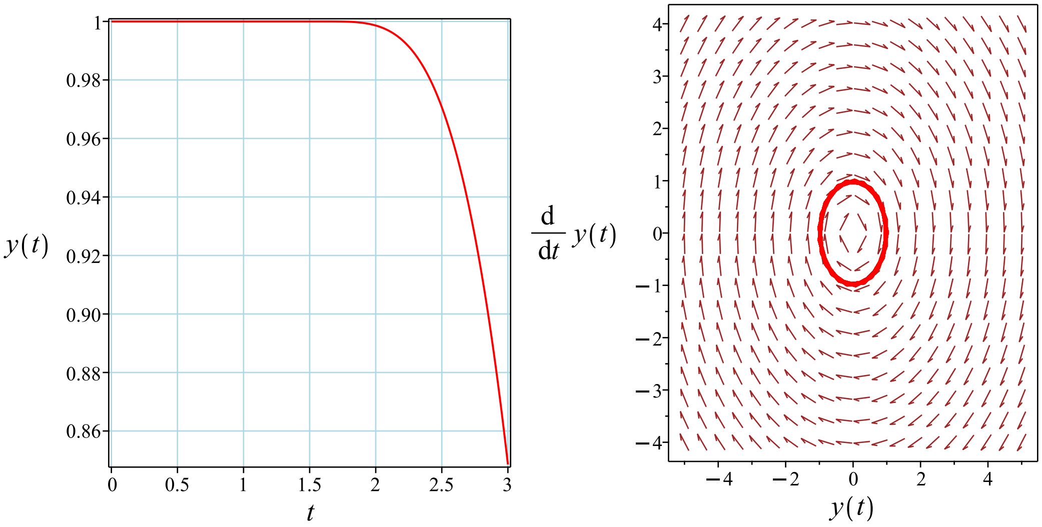
Simplifying the solution gives

$$y = 1 + \frac{((-2t+\pi)\cos(t) + 4\sin(t) - 4)\,\mathrm{Heaviside}\left(t - \frac{\pi}{2}\right)}{4}$$

Summary

The solution(s) found are the following

$$y = 1 + \frac{((-2t+\pi)\cos(t) + 4\sin(t) - 4)\,\mathrm{Heaviside}\left(t - \frac{\pi}{2}\right)}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 1 + \frac{\left(\left(-2t + \pi\right)\cos\left(t\right) + 4\sin\left(t\right) - 4\right)\mathrm{Heaviside}\left(t - \frac{\pi}{2}\right)}{4}$$

Verified OK.

7.6.2 Maple step by step solution

Let's solve

$$\left[y'' + y = \begin{cases} 0 & t < 0 \\ 1 & t < \frac{\pi}{2} \\ \sin\left(t\right) & \frac{\pi}{2} \le t \end{cases}, y(0) = 1, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4}\right)}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))}dt\right), f(t) = \left\{\begin{array}{cc} 0 & t < 0 \\ 1 & t < \frac{\pi}{2} \\ \sin(t) & \frac{\pi}{2} \leq t \end{array}\right.\right]$$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(t), y_2(t)) = \left[\begin{array}{cc} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{array}\right]$$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 1$

  - Substitute functions into equation for $y_p(t)$

    $$y_p(t) = -\cos(t)\left(\int \left(\left\{\begin{array}{cc} 0 & t < 0 \\ \sin(t) & t < \frac{\pi}{2} \\ \sin(t)^2 & \frac{\pi}{2} \leq t \end{array}\right.\right) dt\right) + \sin(t)\left(\int \cos(t)\left(\left\{\begin{array}{cc} 0 & t < 0 \\ 1 & t < \frac{\pi}{2} \\ \sin(t) & \frac{\pi}{2} \leq t \end{array}\right.\right) dt\right)$$

  - Compute integrals

    $$y_p(t) = \left\{\begin{array}{cc} 0 & t \leq 0 \\ -\cos(t) + 1 & t \leq \frac{\pi}{2} \\ \frac{(\pi - 2t - 4)\cos(t)}{4} + \sin(t) & \frac{\pi}{2} < t \end{array}\right.$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(t) + c_2 \sin(t) + \left\{\begin{array}{cc} 0 & t \le 0 \\ -\cos(t) + 1 & t \le \frac{\pi}{2} \\ \frac{(\pi - 2t - 4)\cos(t)}{4} + \sin(t) & \frac{\pi}{2} < t \end{array}\right.$$

☐ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) + \left\{\begin{array}{cc} 0 & t \le 0 \\ -\cos(t) + 1 & t \le \frac{\pi}{2} \\ \frac{(\pi - 2t - 4)\cos(t)}{4} + \sin(t) & \frac{\pi}{2} < t \end{array}\right.$

- Use initial condition $y(0) = 1$

  $1 = c_1$
- Compute derivative of the solution

  $$y' = -c_1 \sin(t) + c_2 \cos(t) + \left\{\begin{array}{cc} 0 & t \le 0 \\ \sin(t) & t \le \frac{\pi}{2} \\ \frac{\cos(t)}{2} - \frac{(\pi - 2t - 4)\sin(t)}{4} & \frac{\pi}{2} < t \end{array}\right.$$
- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_2$
- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 0\}$
- Substitute constant values into general solution and simplify

  $$y = \left\{\begin{array}{cc} \cos(t) & t \le 0 \\ 1 & t \le \frac{\pi}{2} \\ \frac{(-2t + \pi)\cos(t)}{4} + \sin(t) & \frac{\pi}{2} < t \end{array}\right.$$

- Solution to the IVP

  $$y = \left\{\begin{array}{cc} \cos(t) & t \le 0 \\ 1 & t \le \frac{\pi}{2} \\ \frac{(-2t + \pi)\cos(t)}{4} + \sin(t) & \frac{\pi}{2} < t \end{array}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.734 (sec). Leaf size: 30

```
dsolve([diff(y(t),t$2)+y(t)=piecewise(0<=t and t<Pi/2,1,t>= Pi/2,sin(t)),y(0) = 1, D(y)(0) =
```

$$y(t)=\left\{\begin{array}{cc}1 & t<\frac{\pi}{2}\\ \frac{(-2t+\pi)\cos(t)}{4}+\sin(t) & \frac{\pi}{2}\le t\end{array}\right.$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 38

```
DSolve[{y''[t]+y[t]==Piecewise[{{1,0<=t<Pi/2},{Sin[t],t>=Pi/2}}],{y[0]==1,y'[0]==0}},y[t],t,
```

$$y(t)\to\ \{\begin{array}{cc}\cos(t) & t\le 0\\ 1 & t>0\land 2t\le\pi\\ \frac{1}{4}(\pi-2t)\cos(t)+\sin(t) & \text{True}\end{array}$$

7.7 problem 17

Internal problem ID [6694]
Internal file name [`OUTPUT/5942_Sunday_June_05_2022_04_03_09_PM_59262026/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.4.1 DERIVATIVES OF A TRANSFORM. Page 309
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{ty'' - y' = 2t^2}$$

With initial conditions

$$[y(0) = 0]$$

7.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}p(t) &= -\frac{1}{t}\\ q(t) &= 0\\ F &= 2t\end{aligned}$$

Hence the ode is

$$y'' - \frac{y'}{t} = 2t$$

The domain of $p(t) = -\frac{1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

But the point $t_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

7.7.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\begin{aligned}\int (ty'' - y')\, dt &= \int 2t^2 dt\\ ty' - 2y &= \frac{2t^3}{3} + c_1\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned}p(t) &= -\frac{2}{t}\\ q(t) &= \frac{2t^3 + 3c_1}{3t}\end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{t} = \frac{2t^3 + 3c_1}{3t}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{t}dt} \\ &= \frac{1}{t^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{2t^3 + 3c_1}{3t}\right) \\ \frac{d}{dt}\left(\frac{y}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(\frac{2t^3 + 3c_1}{3t}\right) \\ d\left(\frac{y}{t^2}\right) &= \left(\frac{2t^3 + 3c_1}{3t^3}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{t^2} &= \int \frac{2t^3 + 3c_1}{3t^3}\, dt \\ \frac{y}{t^2} &= \frac{2t}{3} - \frac{c_1}{2t^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$y = t^2\left(\frac{2t}{3} - \frac{c_1}{2t^2}\right) + c_2 t^2$$

which simplifies to

$$y = \frac{2}{3}t^3 - \frac{1}{2}c_1 + c_2 t^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{2}{3}t^3 - \frac{1}{2}c_1 + c_2 t^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{c_1}{2} \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$

Substituting these values back in above solution results in

$$y = \frac{2}{3}t^3 + c_2 t^2$$

Summary

The solution(s) found are the following

$$y = \frac{2}{3}t^3 + c_2 t^2 \tag{1}$$

Verification of solutions

$$y = \frac{2}{3}t^3 + c_2 t^2$$

Verified OK.

7.7.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$p'(t)\,t - p(t) - 2t^2 = 0$$

Which is now solve for $p(t)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(t) + p(t)p(t) = q(t)$$

Where here

$$p(t) = -\frac{1}{t}$$
$$q(t) = 2t$$

Hence the ode is

$$p'(t) - \frac{p(t)}{t} = 2t$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{t}dt} \\ &= \frac{1}{t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu p) &= (\mu)(2t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{t}\right) &= \left(\frac{1}{t}\right)(2t) \\ \mathrm{d}\left(\frac{p}{t}\right) &= 2\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{t} &= \int 2\,\mathrm{d}t \\ \frac{p}{t} &= 2t + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t}$ results in

$$p(t) = c_1 t + 2t^2$$

which simplifies to

$$p(t) = t(c_1 + 2t)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = t(c_1 + 2t)$$

Integrating both sides gives

$$\begin{aligned}y &= \int t(c_1 + 2t)\,\mathrm{d}t \\ &= \frac{2}{3}t^3 + \frac{1}{2}t^2 c_1 + c_2\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2$$

This shows that no solution exist. Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{2}{3}t^3 + \frac{1}{2}t^2c_1 + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Equations $\{1\mathrm{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = \frac{2}{3}t^3 + \frac{1}{2}t^2c_1$$

Summary

The solution(s) found are the following

$$y = \frac{2}{3}t^3 + \frac{1}{2}t^2c_1 \tag{1}$$

Verification of solutions

$$y = \frac{2}{3}t^3 + \frac{1}{2}t^2c_1$$

Verified OK.

7.7.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(t)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}y' &= B'v + v'B\\ y'' &= B''v + B'v' + v''B + v'B'\\ &= v''B + 2v' + B' + B''v\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0\\ ABv'' + \left(2AB' + B^2\right)v' + (AB'' + BB' + CB)\,v &= 0 \qquad (1)\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}A &= t\\ B &= -1\\ C &= 0\\ F &= 2t^2\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}AB'' + BB' + CB &= (t)\,(0) + (-1)\,(0) + (0)\,(-1)\\ &= 0\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-tv'' + (1)\,v' = 0$$

Now by applying $v' = u$ the above becomes

$$-tu'(t) + u(t) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= \frac{u}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{1}{t}\,dt \\ \int \frac{1}{u}\,du &= \int \frac{1}{t}\,dt \\ \ln(u) &= c_1 + \ln(t) \\ u &= e^{c_1+\ln(t)} \\ &= c_1 t \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1 t \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(t) &= \int c_1 t \,\mathrm{d}t \\ &= \frac{t^2 c_1}{2} + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(t) &= Bv \\ &= (-1)\left(\frac{t^2 c_1}{2} + c_2\right) \\ &= -\frac{t^2 c_1}{2} - c_2 \end{aligned}$$

And now the particular solution $y_p(t)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = -1$$
$$y_2 = t^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -1 & t^2 \\ \frac{d}{dt}(-1) & \frac{d}{dt}(t^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -1 & t^2 \\ 0 & 2t \end{vmatrix}$$

Therefore

$$W = (-1)(2t) - (t^2)(0)$$

Which simplifies to

$$W = -2t$$

Which simplifies to

$$W = -2t$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2t^4}{-2t^2}\,dt$$

Which simplifies to

$$u_1 = -\int -t^2 dt$$

Hence

$$u_1 = \frac{t^3}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-2t^2}{-2t^2}\,dt$$

Which simplifies to

$$u_2 = \int 1 dt$$

Hence

$$u_2 = t$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{2t^3}{3}$$

Hence the complete solution is

$$\begin{aligned}
y(t) &= y_h + y_p \\
&= \left(-\frac{t^2 c_1}{2} - c_2\right) + \left(\frac{2t^3}{3}\right) \\
&= -\frac{1}{2}t^2 c_1 - c_2 + \frac{2}{3}t^3
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{1}{2}t^2 c_1 - c_2 + \frac{2}{3}t^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -c_2 \tag{1A}$$

Equations {1A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -\frac{1}{2}t^2c_1 + \frac{2}{3}t^3$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{2}t^2c_1 + \frac{2}{3}t^3 \tag{1}$$

Verification of solutions

$$y = -\frac{1}{2}t^2c_1 + \frac{2}{3}t^3$$

Verified OK.

7.7.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$ty'' - y' = 2t^2$$

Integrating both sides of the ODE w.r.t t gives

$$\int (ty'' - y')\,dt = \int 2t^2 dt$$
$$ty' - 2y = \frac{2t^3}{3} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= -\frac{2}{t} \\
q(t) &= \frac{2t^3 + 3c_1}{3t}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{2y}{t} = \frac{2t^3 + 3c_1}{3t}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{t}dt} \\
&= \frac{1}{t^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{2t^3 + 3c_1}{3t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(\frac{2t^3 + 3c_1}{3t}\right) \\
\mathrm{d}\left(\frac{y}{t^2}\right) &= \left(\frac{2t^3 + 3c_1}{3t^3}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{t^2} &= \int \frac{2t^3 + 3c_1}{3t^3}\,\mathrm{d}t \\
\frac{y}{t^2} &= \frac{2t}{3} - \frac{c_1}{2t^2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$
y = t^2\left(\frac{2t}{3} - \frac{c_1}{2t^2}\right) + c_2 t^2
$$

which simplifies to

$$
y = \frac{2}{3}t^3 - \frac{1}{2}c_1 + c_2 t^2
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{2}{3}t^3 - \frac{1}{2}c_1 + c_2 t^2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{c_1}{2} \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$

Substituting these values back in above solution results in

$$y = \frac{2}{3}t^3 + c_2t^2$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{2}{3}t^3 + c_2t^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{2}{3}t^3 + c_2t^2$$

Verified OK.

7.7.6 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align} ty'' - y' &= 0 \tag{1}\\ Ay'' + By' + Cy &= 0 \tag{2}\end{align}$$

Comparing (1) and (2) shows that

$$\begin{align} A &= t \\ B &= -1 \tag{3}\\ C &= 0 \end{align}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 164: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4t^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + (-)\,(0) \\
&= -\frac{1}{2t} \\
&= -\frac{1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(t)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0)+2\left(-\frac{1}{2t}\right)(0)+\left(\left(\frac{1}{2t^2}\right)+\left(-\frac{1}{2t}\right)^2-\left(\frac{3}{4t^2}\right)\right)&=0\\ 0&=0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(t)&=pe^{\int \omega\,dt}\\ &=\mathrm{e}^{\int -\frac{1}{2t}dt}\\ &=\frac{1}{\sqrt{t}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &=z_1e^{-\int \frac{1}{2}\frac{-1}{t}\,dt}\\ &=z_1e^{\frac{\ln(t)}{2}}\\ &=z_1\left(\sqrt{t}\right)\end{aligned}$$

Which simplifies to

$$y_1=1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(\frac{t^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{t^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
ty'' - y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + \frac{c_2t^2}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \frac{t^2}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} 1 & \frac{t^2}{2} \\ \frac{d}{dt}(1) & \frac{d}{dt}\left(\frac{t^2}{2}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} 1 & \frac{t^2}{2} \\ 0 & t \end{vmatrix}
$$

Therefore

$$
W = (1)\,(t) - \left(\frac{t^2}{2}\right)(0)
$$

Which simplifies to

$$
W = t
$$

Which simplifies to

$$
W = t
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{t^4}{t^2}\, dt
$$

Which simplifies to

$$u_1 = -\int t^2 dt$$

Hence

$$u_1 = -\frac{t^3}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2t^2}{t^2}\, dt$$

Which simplifies to

$$u_2 = \int 2dt$$

Hence

$$u_2 = 2t$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{2t^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 t^2}{2}\right) + \left(\frac{2t^3}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + \frac{1}{2} c_2 t^2 + \frac{2}{3} t^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$

Substituting these values back in above solution results in

$$y = \frac{1}{2}c_2t^2 + \frac{2}{3}t^3$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2}c_2t^2 + \frac{2}{3}t^3 \tag{1}$$

Verification of solutions

$$y = \frac{1}{2}c_2t^2 + \frac{2}{3}t^3$$

Verified OK.

7.7.7 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= t \\ q(x) &= -1 \\ r(x) &= 0 \\ s(x) &= 2t^2 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,y' + (q(t) - p'(t))\,y)' = s(x)$$

Integrating gives

$$p(t)\,y' + (q(t) - p'(t))\,y = \int s(t)\,\,dt$$

Substituting the above values for p, q, r, s gives

$$ty' - 2y = \int 2t^2\,dt$$

We now have a first order ode to solve which is

$$ty' - 2y = \frac{2t^3}{3} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned}p(t) &= -\frac{2}{t}\\ q(t) &= \frac{2t^3 + 3c_1}{3t}\end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{t} = \frac{2t^3 + 3c_1}{3t}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2}{t}dt}\\ &= \frac{1}{t^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{2t^3 + 3c_1}{3t}\right)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(\frac{2t^3 + 3c_1}{3t}\right)\\ \mathrm{d}\left(\frac{y}{t^2}\right) &= \left(\frac{2t^3 + 3c_1}{3t^3}\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{t^2} &= \int \frac{2t^3+3c_1}{3t^3}\,\mathrm{d}t\\ \frac{y}{t^2} &= \frac{2t}{3}-\frac{c_1}{2t^2}+c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$y = t^2\left(\frac{2t}{3}-\frac{c_1}{2t^2}\right)+c_2t^2$$

which simplifies to

$$y = \frac{2}{3}t^3-\frac{1}{2}c_1+c_2t^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{2}{3}t^3-\frac{1}{2}c_1+c_2t^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{c_1}{2} \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$

Substituting these values back in above solution results in

$$y = \frac{2}{3}t^3+c_2t^2$$

Summary

The solution(s) found are the following

$$y = \frac{2}{3}t^3+c_2t^2 \tag{1}$$

Verification of solutions

$$y = \frac{2}{3}t^3+c_2t^2$$

Verified OK.

7.7.8 Maple step by step solution

Let's solve

$[y''t - y' = 2t^2, y(0) = 0]$

- Highest derivative means the order of the ODE is 2

  y''
- Make substitution $u = y'$ to reduce order of ODE

  $u'(t)\, t - u(t) = 2t^2$
- Isolate the derivative

  $u'(t) = \frac{u(t)}{t} + 2t$
- Group terms with $u(t)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(t) - \frac{u(t)}{t} = 2t$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t) \left(u'(t) - \frac{u(t)}{t}\right) = 2\mu(t)\, t$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, u(t))$

  $\mu(t) \left(u'(t) - \frac{u(t)}{t}\right) = \mu'(t)\, u(t) + \mu(t)\, u'(t)$
- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{\mu(t)}{t}$
- Solve to find the integrating factor

  $\mu(t) = \frac{1}{t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, u(t))\right) dt = \int 2\mu(t)\, t dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, u(t) = \int 2\mu(t)\, t dt + c_1$
- Solve for $u(t)$

  $u(t) = \frac{\int 2\mu(t) t dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \frac{1}{t}$

  $u(t) = t\left(\int 2dt + c_1\right)$
- Evaluate the integrals on the rhs

$$u(t) = t(c_1 + 2t)$$

- Solve 1st ODE for $u(t)$

  $$u(t) = t(c_1 + 2t)$$

- Make substitution $u = y'$

  $$y' = t(c_1 + 2t)$$

- Integrate both sides to solve for y

  $$\int y' dt = \int t(c_1 + 2t)\, dt + c_2$$

- Compute integrals

  $$y = \frac{2}{3}t^3 + \frac{1}{2}t^2 c_1 + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (2*_a^2+_b(_a))/_a, _b(_a)`   *** Suble
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve([t*diff(y(t),t$2)-diff(y(t),t)=2*t^2,y(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{t^2(4t + 3c_1)}{6}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 29

```
DSolve[{y''[t]-y'[t]==2*t^2,{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{2t^3}{3} - 2t^2 - 4t + c_1(e^t - 1)$$

7.8 problem 18

Internal problem ID [6695]
Internal file name [`OUTPUT/5943_Sunday_June_05_2022_04_03_12_PM_67799813/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.4.1 DERIVATIVES OF A TRANSFORM. Page 309
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{2y'' + ty' - 2y = 10}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

7.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = \frac{t}{2}$$
$$q(t) = -1$$
$$F = 5$$

Hence the ode is

$$y'' + \frac{ty'}{2} - y = 5$$

The domain of $p(t) = \frac{t}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.8.2 Solving using Kovacic algorithm

Writing the ode as

$$\begin{aligned} 2y'' + ty' - 2y &= 0 \qquad (1) \\ Ay'' + By' + Cy &= 0 \qquad (2) \end{aligned}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= t \qquad (3) \\ C &= -2 \end{aligned}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \qquad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \qquad (5) \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 20}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 + 20 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2}{16} + \frac{5}{4}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 166: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{1} a_i t^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{4} + \frac{5}{2t} - \frac{25}{2t^3} + \frac{125}{t^5} - \frac{3125}{2t^7} + \frac{21875}{t^9} - \frac{328125}{t^{11}} + \frac{5156250}{t^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\
&= \frac{t}{4} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2+20}{16} \\ &= Q + \frac{R}{16} \\ &= \left(\frac{t^2}{16} + \frac{5}{4}\right) + (0) \\ &= \frac{t^2}{16} + \frac{5}{4} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{t}$ in the quotient is $\frac{5}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{4}\right) - (0) \\ &= \frac{5}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{t}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{4}}{\frac{1}{4}} - 1\right) &&= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{4}}{\frac{1}{4}} - 1\right) &&= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2}{16} + \frac{5}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{t}{4}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{t}{4}\right) \\ &= \frac{t}{4} \\ &= \frac{t}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(\frac{t}{4}\right)(2t + a_1) + \left(\left(\frac{1}{4}\right) + \left(\frac{t}{4}\right)^2 - \left(\frac{t^2}{16} + \frac{5}{4}\right)\right) &= 0 \\ 2 - \frac{a_1 t}{2} - a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2, a_1 = 0\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^2+2\right) \mathrm{e}^{\int \frac{t}{4} dt} \\
&= \left(t^2+2\right) \mathrm{e}^{\frac{t^2}{8}} \\
&= \left(t^2+2\right) \mathrm{e}^{\frac{t^2}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t}{2} \, dt} \\
&= z_1 e^{-\frac{t^2}{8}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t^2}{8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t^2 + 2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{2} \, dt}}{\left(y_1\right)^2} \, dt \\
&= y_1 \int \frac{e^{-\frac{t^2}{4}}}{\left(y_1\right)^2} \, dt \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{t^2}{4}}}{\left(t^2+2\right)^2} dt\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^2+2\right) + c_2 \left(t^2+2 \left(\int \frac{\mathrm{e}^{-\frac{t^2}{4}}}{\left(t^2+2\right)^2} dt\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$2y'' + ty' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1(t^2+2) + c_2(t^2+2)\left(\int \frac{\mathrm{e}^{-\frac{t^2}{4}}}{(t^2+2)^2}dt\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= t^2+2 \\ y_2 &= (t^2+2)\left(\int \frac{\mathrm{e}^{-\frac{t^2}{4}}}{(t^2+2)^2}dt\right) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t^2+2 & (t^2+2)\left(\int \frac{e^{-\frac{t^2}{4}}}{(t^2+2)^2}dt\right) \\ \frac{d}{dt}(t^2+2) & \frac{d}{dt}\left((t^2+2)\left(\int \frac{e^{-\frac{t^2}{4}}}{(t^2+2)^2}dt\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t^2+2 & (t^2+2)\left(\int \frac{e^{-\frac{t^2}{4}}}{(t^2+2)^2}dt\right) \\ 2t & 2t\left(\int \frac{e^{-\frac{t^2}{4}}}{(t^2+2)^2}dt\right) + \frac{e^{-\frac{t^2}{4}}}{t^2+2} \end{vmatrix}$$

Therefore

$$W = (t^2+2)\left(2t\left(\int \frac{e^{-\frac{t^2}{4}}}{(t^2+2)^2}dt\right) + \frac{e^{-\frac{t^2}{4}}}{t^2+2}\right) - \left((t^2+2)\left(\int \frac{e^{-\frac{t^2}{4}}}{(t^2+2)^2}dt\right)\right)(2t)$$

Which simplifies to

$$W = e^{-\frac{t^2}{4}}$$

Which simplifies to

$$W = e^{-\frac{t^2}{4}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{10(t^2+2)\left(\int \frac{e^{-\frac{t^2}{4}}}{(t^2+2)^2}dt\right)}{2\,e^{-\frac{t^2}{4}}}\,dt$$

Which simplifies to

$$u_1 = -\int 5(t^2+2)\left(\int \frac{e^{-\frac{t^2}{4}}}{(t^2+2)^2}dt\right)e^{\frac{t^2}{4}}\,dt$$

Hence

$$u_1 = -\left(\int_0^t 5\left(\alpha^2+2\right)\left(\int \frac{e^{-\frac{\alpha^2}{4}}}{\left(\alpha^2+2\right)^2}d\alpha\right) e^{\frac{\alpha^2}{4}} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{10t^2+20}{2\,e^{-\frac{t^2}{4}}}\,dt$$

Which simplifies to

$$u_2 = \int 5\left(t^2+2\right) e^{\frac{t^2}{4}} dt$$

Hence

$$u_2 = 10t\,e^{\frac{t^2}{4}}$$

Which simplifies to

$$u_1 = -5\left(\int_0^t \left(\alpha^2+2\right)\left(\int \frac{e^{-\frac{\alpha^2}{4}}}{\left(\alpha^2+2\right)^2}d\alpha\right) e^{\frac{\alpha^2}{4}} d\alpha\right)$$

$$u_2 = 10t\,e^{\frac{t^2}{4}}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -5\left(\int_0^t \left(\alpha^2+2\right)\left(\int \frac{e^{-\frac{\alpha^2}{4}}}{\left(\alpha^2+2\right)^2}d\alpha\right) e^{\frac{\alpha^2}{4}} d\alpha\right)\left(t^2+2\right) + 10t\,e^{\frac{t^2}{4}}\left(t^2+2\right)\left(\int \frac{e^{-\frac{t^2}{4}}}{\left(t^2+2\right)^2}dt\right)$$

Which simplifies to

$$y_p(t) = 10\left(t\,e^{\frac{t^2}{4}}\left(\int \frac{e^{-\frac{t^2}{4}}}{\left(t^2+2\right)^2}dt\right) - \frac{\left(\int_0^t \left(\alpha^2+2\right)\left(\int \frac{e^{-\frac{\alpha^2}{4}}}{\left(\alpha^2+2\right)^2}d\alpha\right) e^{\frac{\alpha^2}{4}} d\alpha\right)}{2}\right)\left(t^2+2\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 (t^2+2) + c_2 (t^2+2) \left(\int \frac{\mathrm{e}^{-\frac{t^2}{4}}}{(t^2+2)^2} dt\right)\right) \\
&\quad + \left(10 \left(t\,\mathrm{e}^{\frac{t^2}{4}} \left(\int \frac{\mathrm{e}^{-\frac{t^2}{4}}}{(t^2+2)^2} dt\right) - \frac{\left(\int_0^t (\alpha^2+2)\left(\int \frac{\mathrm{e}^{-\frac{\alpha^2}{4}}}{(\alpha^2+2)^2} d\alpha\right) \mathrm{e}^{\frac{\alpha^2}{4}} d\alpha\right)}{2}\right) (t^2+2)\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y = {} & (t^2+2)\left(c_2 \left(\int \frac{\mathrm{e}^{-\frac{t^2}{4}}}{(t^2+2)^2} dt\right) + c_1\right) \\
& + 10 \left(t\,\mathrm{e}^{\frac{t^2}{4}} \left(\int \frac{\mathrm{e}^{-\frac{t^2}{4}}}{(t^2+2)^2} dt\right) - \frac{\left(\int_0^t (\alpha^2+2)\left(\int \frac{\mathrm{e}^{-\frac{\alpha^2}{4}}}{(\alpha^2+2)^2} d\alpha\right) \mathrm{e}^{\frac{\alpha^2}{4}} d\alpha\right)}{2}\right) (t^2+2)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = (t^2+2)\left(c_2 \left(\int \frac{\mathrm{e}^{-\frac{t^2}{4}}}{(t^2+2)^2} dt\right) + c_1\right) + 10 \Bigg(t\,\mathrm{e}^{\frac{t^2}{4}} \left(\int \frac{\mathrm{e}^{-\frac{t^2}{4}}}{(t^2+2)^2} dt\right) - \frac{\Big(\int_0^t (\alpha^2+2)\left(\int \frac{\mathrm{e}^{-\frac{\alpha^2}{4}}}{(\alpha^2+2)^2} d\alpha\right) \mathrm{e}^{\frac{\alpha^2}{4}}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = \lim_{t \to 0} 10 \left(-\frac{\left(\int_0^t (\alpha^2+2)\left(\int \frac{\mathrm{e}^{-\frac{\alpha^2}{4}}}{(\alpha^2+2)^2} d\alpha\right) \mathrm{e}^{\frac{\alpha^2}{4}} d\alpha\right)}{2} + \left(t\,\mathrm{e}^{\frac{t^2}{4}} + \frac{c_2}{10}\right)\left(\int \frac{\mathrm{e}^{-\frac{t^2}{4}}}{(t^2+2)^2} dt\right) + \frac{c_1}{10}\right)(t^2+2) \tag{1A}
$$

Taking derivative of the solution gives

$$y' = 2t\left(c_2\left(\int \frac{e^{-\frac{t^2}{4}}}{\left(t^2+2\right)^2}dt\right)+c_1\right)+\frac{c_2 e^{-\frac{t^2}{4}}}{t^2+2}+10\left(e^{\frac{t^2}{4}}\left(\int \frac{e^{-\frac{t^2}{4}}}{\left(t^2+2\right)^2}dt\right)+\frac{t^2 e^{\frac{t^2}{4}}\left(\int \frac{e^{-\frac{t^2}{4}}}{\left(t^2+2\right)^2}dt\right)}{2}+\frac{t\, e^{\frac{t^2}{4}} e^{-}}{(t^2+2}\right.$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \lim_{t\to 0}\frac{\left(-10t^3-20t\right)\left(\int_0^t \left(\alpha^2+2\right)\left(\int \frac{e^{-\frac{\alpha^2}{4}}}{\left(\alpha^2+2\right)^2}d\alpha\right)e^{\frac{\alpha^2}{4}}d\alpha\right)+20\left(t\, e^{\frac{t^2}{4}}+\frac{c_2}{10}\right)t\left(t^2+2\right)\left(\int \frac{e^{-\frac{t^2}{4}}}{\left(t^2+2\right)^2}dt\right)+c_2 e}{t^2+2} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

<u>Verification of solutions</u> N/A

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
      Solution has integrals. Trying a special function solution free of integrals...
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         <- Kummer successful
      <- special function solution successful
         -> Trying to convert hypergeometric functions to elementary form...
         <- elementary form is not straightforward to achieve - returning special function s
      <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 9

```
dsolve([2*diff(y(t),t$2)+t*diff(y(t),t)-2*y(t)=10,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{5t^2}{2}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 10

```
DSolve[{y''[t]+t*y'[t]-2*y[t]==10,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \rightarrow 5t^2$$

7.9 problem 36

Internal problem ID [6696]
Internal file name [`OUTPUT/5944_Sunday_June_05_2022_04_03_18_PM_79239426/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. 7.4.1 DERIVATIVES OF A TRANSFORM. Page 309
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sin(t) + \sin(t)\, t}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

7.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \sin(t)\,(t+1) \end{aligned}$$

Hence the ode is

$$y'' + y = \sin(t)(t+1)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(t)(t+1)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \frac{s^2+2s+1}{(s^2+1)^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + Y(s) = \frac{s^2+2s+1}{(s^2+1)^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2+2s+1}{(s^2+1)^3}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{4\left(s-i\right)^{3}}-\frac{1}{4\left(s+i\right)^{3}}+\frac{-\frac{1}{4}-\frac{i}{8}}{\left(s-i\right)^{2}}+\frac{-\frac{1}{4}+\frac{i}{8}}{\left(s+i\right)^{2}}-\frac{i}{4\left(s-i\right)}+\frac{i}{4s+4i}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{1}{4\left(s-i\right)^{3}}\right) &= -\frac{\mathrm{e}^{it} t^{2}}{8}\\
\mathcal{L}^{-1}\left(-\frac{1}{4\left(s+i\right)^{3}}\right) &= -\frac{\mathrm{e}^{-it} t^{2}}{8}\\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{4}-\frac{i}{8}}{\left(s-i\right)^{2}}\right) &= \left(-\frac{1}{4}-\frac{i}{8}\right) t\,\mathrm{e}^{it}\\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{4}+\frac{i}{8}}{\left(s+i\right)^{2}}\right) &= \left(-\frac{1}{4}+\frac{i}{8}\right) t\,\mathrm{e}^{-it}\\
\mathcal{L}^{-1}\left(-\frac{i}{4\left(s-i\right)}\right) &= -\frac{i\mathrm{e}^{it}}{4}\\
\mathcal{L}^{-1}\left(\frac{i}{4s+4i}\right) &= \frac{i\mathrm{e}^{-it}}{4}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = \frac{\left(t+2\right)\left(\sin\left(t\right)-\cos\left(t\right) t\right)}{4}$$

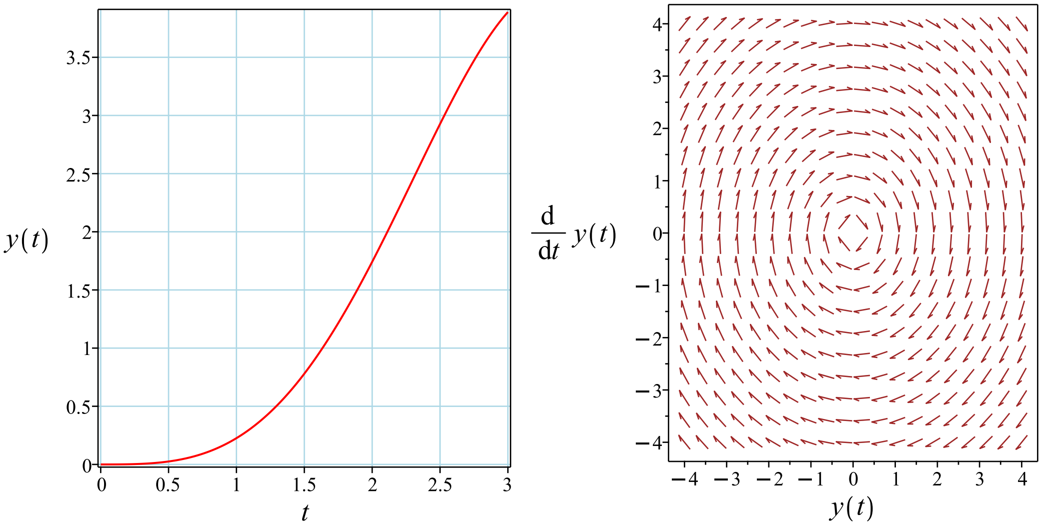
Simplifying the solution gives

$$y = \frac{\left(t+2\right)\left(\sin\left(t\right)-\cos\left(t\right) t\right)}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(t+2\right)\left(\sin\left(t\right)-\cos\left(t\right) t\right)}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{(t+2)(\sin(t) - \cos(t)\, t)}{4}$$

Verified OK.

7.9.2 Maple step by step solution

Let's solve

$$\left[y'' + y = \sin(t)(t+1), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t)=\cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(t)+c_2\sin(t)+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t)=-y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\sin(t)(t+1)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t),y_2(t))=\begin{bmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t),y_2(t))=1$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t)=-\cos(t)\left(\int \sin(t)^2(t+1)\,dt\right)+\frac{\sin(t)\left(\int \sin(2t)(t+1)dt\right)}{2}$

  - Compute integrals

    $y_p(t)=\frac{(-t^2-2t)\cos(t)}{4}+\frac{\sin(t)(t+1)}{4}$

- Substitute particular solution into general solution to ODE

  $y=c_1\cos(t)+c_2\sin(t)+\frac{(-t^2-2t)\cos(t)}{4}+\frac{\sin(t)(t+1)}{4}$

☐ Check validity of solution $y=c_1\cos(t)+c_2\sin(t)+\frac{(-t^2-2t)\cos(t)}{4}+\frac{\sin(t)(t+1)}{4}$

  - Use initial condition $y(0)=0$

    $0=c_1$

  - Compute derivative of the solution

    $y'=-c_1\sin(t)+c_2\cos(t)+\frac{(-2t-2)\cos(t)}{4}-\frac{(-t^2-2t)\sin(t)}{4}+\frac{\cos(t)(t+1)}{4}+\frac{\sin(t)}{4}$

  - Use the initial condition $y'\Big|_{\{t=0\}}=0$

    $0=-\frac{1}{4}+c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = 0, c_2 = \frac{1}{4}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{(t+2)(\sin(t)-\cos(t)t)}{4}$

- Solution to the IVP

  $y = \frac{(t+2)(\sin(t)-\cos(t)t)}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.797 (sec). Leaf size: 17

```
dsolve([diff(y(t),t$2)+y(t)=sin(t)+t*sin(t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{(-\sin(t) + \cos(t)\, t)\,(t+2)}{4}$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 21

```
DSolve[{y''[t]+y[t]==Sin[t]+t*Sin[t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to -\frac{1}{4}(t+2)(t\cos(t) - \sin(t))$$

8 CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315

8.1 problem 1

Internal problem ID [6697]
Internal file name [`OUTPUT/5945_Sunday_June_05_2022_04_03_21_PM_12489767/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 3y = \delta(-2 + t)}$$

With initial conditions

$$[y(0) = 0]$$

8.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= \delta(-2 + t) \end{aligned}$$

Hence the ode is

$$y' - 3y = \delta(-2+t)$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \delta(-2+t)$ is

$$\{t < 2 \lor 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - 3Y(s) = \mathrm{e}^{-2s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 3Y(s) = \mathrm{e}^{-2s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{\mathrm{e}^{-2s}}{s-3}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-2s}}{s-3}\right) \\ &= \mathrm{Heaviside}\,(-2+t)\,\mathrm{e}^{-6+3t} \end{aligned}$$

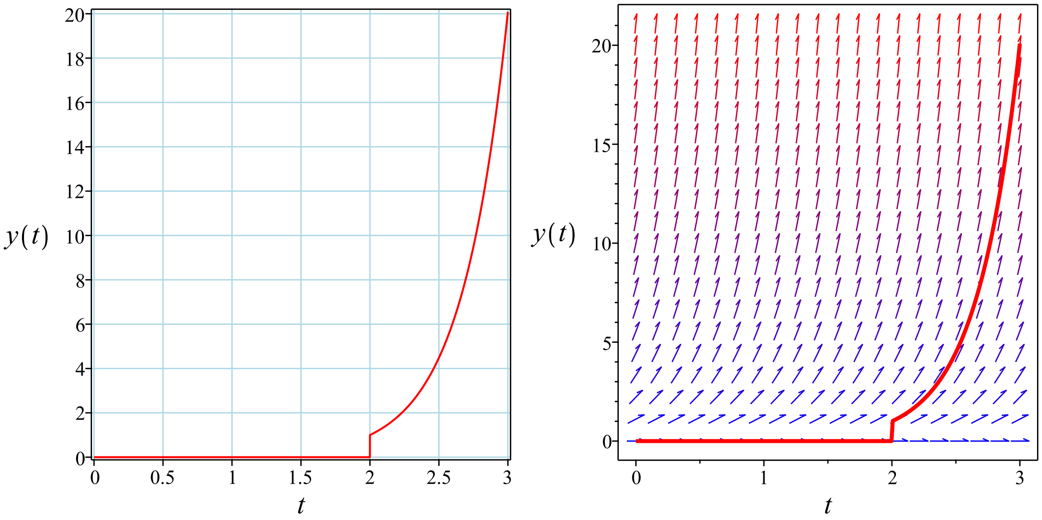
Hence the final solution is

$$y = \text{Heaviside}\,(-2+t)\,e^{-6+3t}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \text{Heaviside}\,(-2+t)\,e^{-6+3t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \text{Heaviside}\,(-2+t)\,e^{-6+3t}$$

Verified OK.

8.1.3 Maple step by step solution

Let's solve

$[y'-3y = Dirac(-2+t)\,, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 3y + Dirac(-2+t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 3y = \mathit{Dirac}(-2+t)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - 3y) = \mu(t)\,\mathit{Dirac}(-2+t)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' - 3y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -3\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-3t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,\mathit{Dirac}(-2+t)\,dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\,\mathit{Dirac}(-2+t)\,dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t)\mathit{Dirac}(-2+t)dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-3t}$

  $y = \frac{\int \mathrm{e}^{-3t}\mathit{Dirac}(-2+t)dt + c_1}{\mathrm{e}^{-3t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{-6} + c_1}{\mathrm{e}^{-3t}}$
- Simplify

  $y = \mathrm{e}^{3t}(\mathit{Heaviside}(-2+t)\,\mathrm{e}^{-6} + c_1)$
- Use initial condition $y(0) = 0$

  $0 = c_1$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \mathit{Heaviside}(-2+t)\,\mathrm{e}^{-6+3t}$
- Solution to the IVP

$$y = \mathit{Heaviside}(-2+t)\,\mathrm{e}^{-6+3t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.968 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)-3*y(t)=Dirac(t-2),y(0) = 0],y(t), singsol=all)
```

$$y(t) = \mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{3t-6}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 17

```
DSolve[{y'[t]-3*y[t]==DiracDelta[t-2],{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{3t-6}\theta(t-2)$$

8.2 problem 2

Internal problem ID [6698]
Internal file name [`OUTPUT/5946_Sunday_June_05_2022_04_03_25_PM_19810492/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \delta(t-1)}$$

With initial conditions

$$[y(0) = 2]$$

8.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= \delta(t-1) \end{aligned}$$

Hence the ode is

$$y' + y = \delta(t - 1)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \delta(t - 1)$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + Y(s) = \mathrm{e}^{-s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 2 + Y(s) = \mathrm{e}^{-s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{\mathrm{e}^{-s} + 2}{s + 1}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-s} + 2}{s + 1}\right) \\ &= \text{Heaviside}\,(t - 1)\,\mathrm{e}^{-t+1} + 2\,\mathrm{e}^{-t} \end{aligned}$$

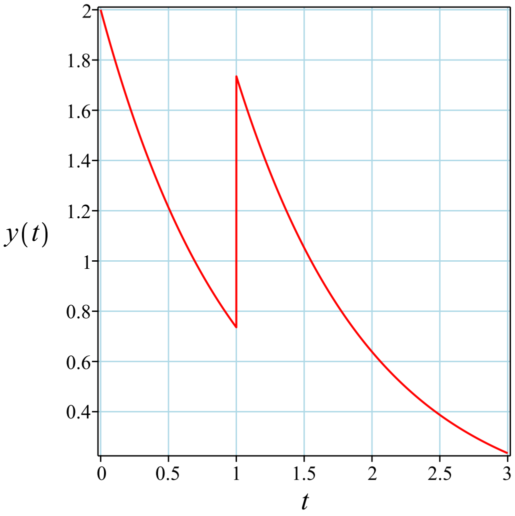
Hence the final solution is

$$y = \text{Heaviside}\left(t-1\right) \mathrm{e}^{-t+1}+2\,\mathrm{e}^{-t}$$

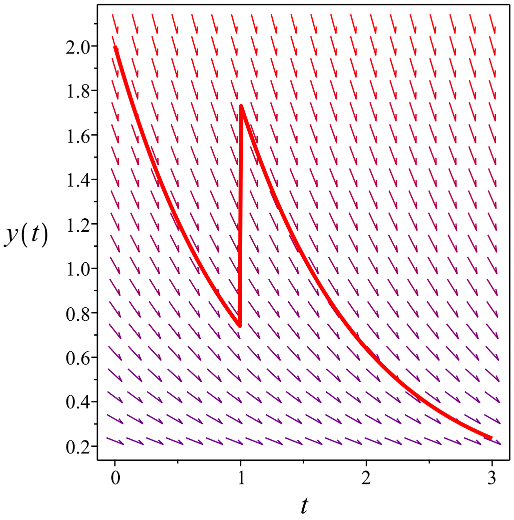
Summary

The solution(s) found are the following

$$y = \text{Heaviside}\left(t-1\right) \mathrm{e}^{-t+1}+2\,\mathrm{e}^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \text{Heaviside}\left(t-1\right) \mathrm{e}^{-t+1}+2\,\mathrm{e}^{-t}$$

Verified OK.

8.2.3 Maple step by step solution

Let's solve

$\left[y'+y=\mathit{Dirac}\left(t-1\right), y\left(0\right)=2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-y+\mathit{Dirac}\left(t-1\right)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE
  $y'+y=\mathit{Dirac}(t-1)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t)\left(y'+y\right)=\mu(t)\,\mathit{Dirac}(t-1)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$
  $\mu(t)\left(y'+y\right)=\mu'(t)\,y+\mu(t)\,y'$
- Isolate $\mu'(t)$
  $\mu'(t)=\mu(t)$
- Solve to find the integrating factor
  $\mu(t)=\mathrm{e}^{t}$
- Integrate both sides with respect to t
  $\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int\mu(t)\,\mathit{Dirac}(t-1)\,dt+c_1$
- Evaluate the integral on the lhs
  $\mu(t)\,y=\int\mu(t)\,\mathit{Dirac}(t-1)\,dt+c_1$
- Solve for y
  $y=\frac{\int\mu(t)\mathit{Dirac}(t-1)dt+c_1}{\mu(t)}$
- Substitute $\mu(t)=\mathrm{e}^{t}$
  $y=\frac{\int\mathrm{e}^{t}\mathit{Dirac}(t-1)dt+c_1}{\mathrm{e}^{t}}$
- Evaluate the integrals on the rhs
  $y=\frac{\mathit{Heaviside}(t-1)\mathrm{e}+c_1}{\mathrm{e}^{t}}$
- Simplify
  $y=\mathrm{e}^{-t}(\mathit{Heaviside}(t-1)\,\mathrm{e}+c_1)$
- Use initial condition $y(0)=2$
  $2=c_1$
- Solve for c_1
  $c_1=2$
- Substitute $c_1=2$ into general solution and simplify
  $y=\mathrm{e}^{-t}(\mathit{Heaviside}(t-1)\,\mathrm{e}+2)$
- Solution to the IVP

$$y = \mathrm{e}^{-t}(\mathit{Heaviside}(t-1)\,\mathrm{e} + 2)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.938 (sec). Leaf size: 22

```
dsolve([diff(y(t),t)+y(t)=Dirac(t-1),y(0) = 2],y(t), singsol=all)
```

$$y(t) = \mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{1-t} + 2\,\mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 19

```
DSolve[{y'[t]+y[t]==DiracDelta[t-1],{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t}(e\theta(t-1)+2)$$

8.3 problem 3

Internal problem ID [6699]
Internal file name [`OUTPUT/5947_Sunday_June_05_2022_04_03_29_PM_41960622/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \delta(t - 2\pi)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

8.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \delta(t - 2\pi) \end{aligned}$$

Hence the ode is

$$y'' + y = \delta(t - 2\pi)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - 2\pi)$ is

$$\{t < 2\pi \vee 2\pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \mathrm{e}^{-2\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + Y(s) = \mathrm{e}^{-2\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-2\pi s} + 1}{s^2 + 1}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-2\pi s}+1}{s^2+1}\right) \\ &= \sin(t)\,(\text{Heaviside}\,(t-2\pi)+1) \end{aligned}$$

Hence the final solution is

$$y = \sin(t)\,(\text{Heaviside}\,(t-2\pi)+1)$$

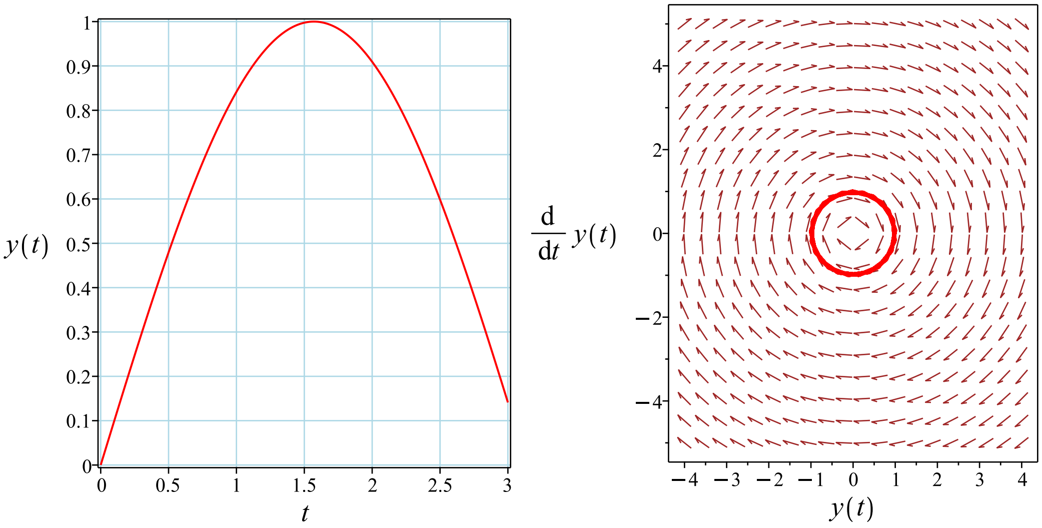
Simplifying the solution gives

$$y = \sin(t)\,(\text{Heaviside}\,(t-2\pi)+1)$$

Summary

The solution(s) found are the following

$$y = \sin(t)\,(\text{Heaviside}\,(t-2\pi)+1) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sin(t)\,(\text{Heaviside}\,(t-2\pi)+1)$$

Verified OK.

8.3.2 Maple step by step solution

Let's solve

$$\left[y''+y=\mathit{Dirac}(t-2\pi)\,,y(0)=0,y'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+1=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=(-\mathrm{I},\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(t)+c_2\sin(t)+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right),f(t)=\mathit{Dirac}(t-2\pi)\right]$$

  ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t)\,,y_2(t))=\begin{bmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{bmatrix}$$

  ○ Compute Wronskian

  $W(y_1(t)\,,y_2(t))=1$

  ○ Substitute functions into equation for $y_p(t)$

$y_p(t) = \sin(t) \left(\int \mathit{Dirac}(t-2\pi)\, dt\right)$

- ○ Compute integrals

  $y_p(t) = \sin(t)\, \mathit{Heaviside}(t-2\pi)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + \sin(t)\, \mathit{Heaviside}(t-2\pi)$

- ☐ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) + \sin(t)\, \mathit{Heaviside}(t-2\pi)$
  - ○ Use initial condition $y(0) = 0$

    $0 = c_1$
  - ○ Compute derivative of the solution

    $y' = -c_1 \sin(t) + c_2 \cos(t) + \cos(t)\, \mathit{Heaviside}(t-2\pi) + \sin(t)\, \mathit{Dirac}(t-2\pi)$
  - ○ Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = c_2$
  - ○ Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 1\}$
  - ○ Substitute constant values into general solution and simplify

    $y = \sin(t)\, (\mathit{Heaviside}(t-2\pi) + 1)$

- Solution to the IVP

  $y = \sin(t)\, (\mathit{Heaviside}(t-2\pi) + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.953 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$2)+y(t)=Dirac(t-2*Pi),y(0) = 0, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = \sin(t) \left(\mathrm{Heaviside}(t-2\pi)+1\right)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 16

```
DSolve[{y''[t]+y[t]==DiracDelta[t-2*Pi],{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to (\theta(t-2\pi)+1)\sin(t)$$

8.4 problem 4

Internal problem ID [6700]
Internal file name [`OUTPUT/5948_Sunday_June_05_2022_04_03_33_PM_56971458/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 16y = \delta(t - 2\pi)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

8.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 16 \\ F &= \delta(t - 2\pi) \end{aligned}$$

Hence the ode is

$$y'' + 16y = \delta(t - 2\pi)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 16$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - 2\pi)$ is

$$\{t < 2\pi \lor 2\pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 16Y(s) = \mathrm{e}^{-2\pi s} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 16Y(s) = \mathrm{e}^{-2\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-2\pi s}}{s^2 + 16}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{s^2+16}\right) \\
&= \frac{\text{Heaviside}\,(t-2\pi)\sin(4t)}{4}
\end{aligned}
$$

Hence the final solution is

$$y = \frac{\text{Heaviside}\,(t-2\pi)\sin(4t)}{4}$$

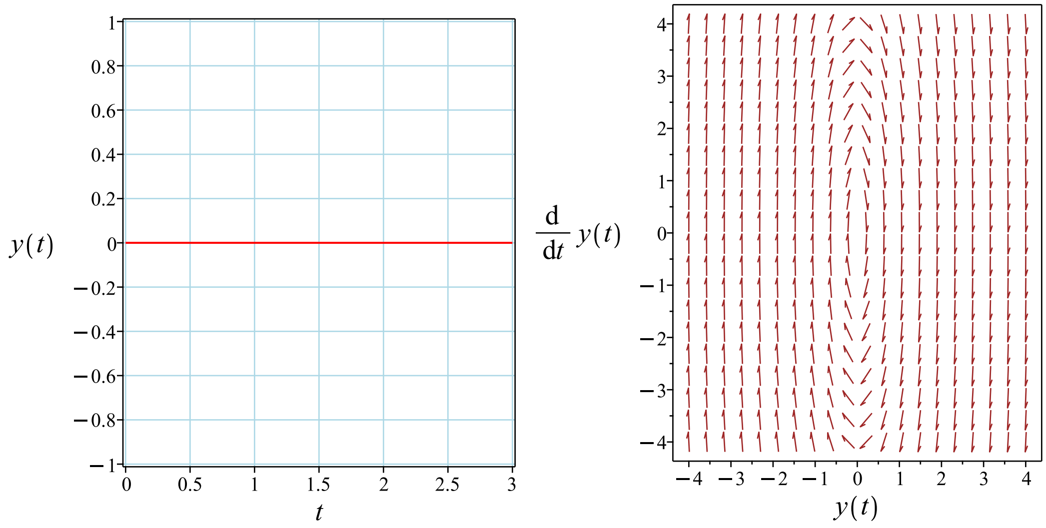
Simplifying the solution gives

$$y = \frac{\text{Heaviside}\,(t-2\pi)\sin(4t)}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{\text{Heaviside}\,(t-2\pi)\sin(4t)}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\text{Heaviside}\,(t-2\pi)\sin(4t)}{4}$$

Verified OK.

8.4.2 Maple step by step solution

Let's solve

$$\left[y''+16y=Dirac(t-2\pi)\,,y(0)=0,y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+16=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r=(-4\,\mathrm{I},4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(4t)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(4t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(4t)+c_2\sin(4t)+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right),f(t)=Dirac(t-2\pi)\right]$$

  ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t)\,,y_2(t))=\left[\begin{array}{cc}\cos(4t) & \sin(4t)\\ -4\sin(4t) & 4\cos(4t)\end{array}\right]$$

  ○ Compute Wronskian

  $W(y_1(t)\,,y_2(t))=4$

  ○ Substitute functions into equation for $y_p(t)$

$y_p(t) = \frac{\sin(4t)\left(\int Dirac(t-2\pi)dt\right)}{4}$

- ○ Compute integrals

  $y_p(t) = \frac{Heaviside(t-2\pi)\sin(4t)}{4}$

- ● Substitute particular solution into general solution to ODE

  $y = c_1 \cos(4t) + c_2 \sin(4t) + \frac{Heaviside(t-2\pi)\sin(4t)}{4}$

☐ Check validity of solution $y = c_1 \cos(4t) + c_2 \sin(4t) + \frac{Heaviside(t-2\pi)\sin(4t)}{4}$

  - ○ Use initial condition $y(0) = 0$

    $0 = c_1$

  - ○ Compute derivative of the solution

    $y' = -4c_1 \sin(4t) + 4c_2 \cos(4t) + \frac{Dirac(t-2\pi)\sin(4t)}{4} + Heaviside(t-2\pi)\cos(4t)$

  - ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = 4c_2$

  - ○ Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$

  - ○ Substitute constant values into general solution and simplify

    $y = \frac{Heaviside(t-2\pi)\sin(4t)}{4}$

- ● Solution to the IVP

  $y = \frac{Heaviside(t-2\pi)\sin(4t)}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.891 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$2)+16*y(t)=Dirac(t-2*Pi),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t)=\frac{\text{Heaviside}\,(t-2\pi)\sin\,(4t)}{4}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 19

```
DSolve[{y''[t]+16*y[t]==DiracDelta[t-2*Pi],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutio
```

$$y(t)\rightarrow\frac{1}{4}\theta(t-2\pi)\sin(4t)$$

8.5 problem 5

Internal problem ID [6701]
Internal file name [`OUTPUT/5949_Sunday_June_05_2022_04_03_37_PM_66215802/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \delta\left(t - \frac{\pi}{2}\right) + \delta\left(t - \frac{3\pi}{2}\right)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

8.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \delta\left(t - \frac{\pi}{2}\right) + \delta\left(t - \frac{3\pi}{2}\right) \end{aligned}$$

Hence the ode is

$$y'' + y = \delta\left(t - \frac{\pi}{2}\right) + \delta\left(t - \frac{3\pi}{2}\right)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta\left(t - \frac{\pi}{2}\right) + \delta\left(t - \frac{3\pi}{2}\right)$ is

$$\left\{\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}, \frac{3\pi}{2} \leq t \leq \infty, -\infty \leq t \leq \frac{\pi}{2}\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \mathrm{e}^{-\frac{\pi s}{2}} + \mathrm{e}^{-\frac{3\pi s}{2}} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + Y(s) = \mathrm{e}^{-\frac{\pi s}{2}} + \mathrm{e}^{-\frac{3\pi s}{2}}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-\frac{\pi s}{2}} + \mathrm{e}^{-\frac{3\pi s}{2}}}{s^2 + 1}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-\frac{\pi s}{2}}+\mathrm{e}^{-\frac{3\pi s}{2}}}{s^2+1}\right) \\
&= \cos(t)\left(-\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)+\mathrm{Heaviside}\left(t-\frac{3\pi}{2}\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = \cos(t)\left(-\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)+\mathrm{Heaviside}\left(t-\frac{3\pi}{2}\right)\right)
$$

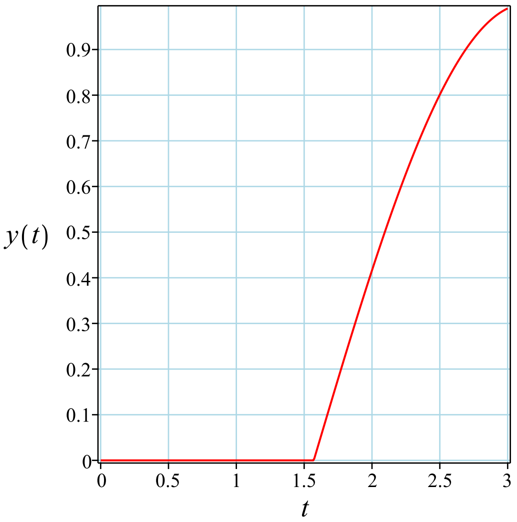
Simplifying the solution gives

$$
y = \cos(t)\left(-\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)+\mathrm{Heaviside}\left(t-\frac{3\pi}{2}\right)\right)
$$

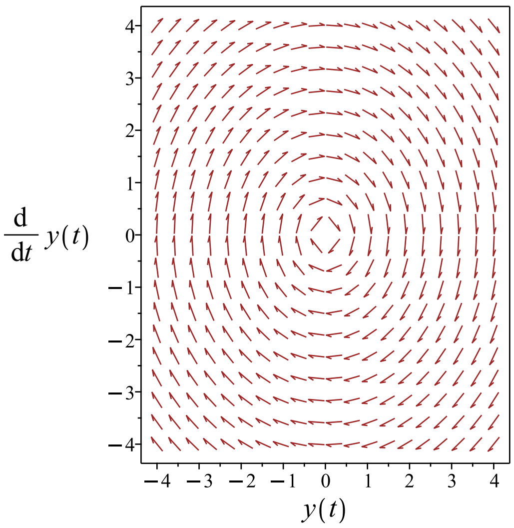
Summary

The solution(s) found are the following

$$
y = \cos(t)\left(-\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)+\mathrm{Heaviside}\left(t-\frac{3\pi}{2}\right)\right) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos(t)\left(-\text{Heaviside}\left(t-\frac{\pi}{2}\right)+\text{Heaviside}\left(t-\frac{3\pi}{2}\right)\right)$$

Verified OK.

8.5.2 Maple step by step solution

Let's solve

$\left[y'' + y = \mathit{Dirac}\left(t-\frac{\pi}{2}\right) + \mathit{Dirac}\left(t-\frac{3\pi}{2}\right), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\Big[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathit{Dirac}\left(t-\frac{\pi}{2}\right) + \mathit{Dirac}(t -$
  - ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 1$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\cos(t)\left(\int \left(\mathit{Dirac}\left(t-\frac{\pi}{2}\right) - \mathit{Dirac}\left(t-\frac{3\pi}{2}\right)\right) dt\right)$

- ○ Compute integrals

  $y_p(t) = \cos(t)\left(-\mathit{Heaviside}\left(t-\frac{\pi}{2}\right) + \mathit{Heaviside}\left(t-\frac{3\pi}{2}\right)\right)$

● Substitute particular solution into general solution to ODE

  $y = c_1\cos(t) + c_2\sin(t) + \cos(t)\left(-\mathit{Heaviside}\left(t-\frac{\pi}{2}\right) + \mathit{Heaviside}\left(t-\frac{3\pi}{2}\right)\right)$

☐ Check validity of solution $y = c_1\cos(t) + c_2\sin(t) + \cos(t)\big(-\mathit{Heaviside}\big(t-\frac{\pi}{2}\big) + \mathit{Heaviside}\big(t$ -

- ○ Use initial condition $y(0) = 0$

  $0 = c_1$

- ○ Compute derivative of the solution

  $y' = -c_1\sin(t) + c_2\cos(t) - \sin(t)\left(-\mathit{Heaviside}\left(t-\frac{\pi}{2}\right) + \mathit{Heaviside}\left(t-\frac{3\pi}{2}\right)\right) + \cos(t)\big(-\mathit{Dirac}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_2$

- ○ Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- ○ Substitute constant values into general solution and simplify

  $y = \cos(t)\left(-\mathit{Heaviside}\left(t-\frac{\pi}{2}\right) + \mathit{Heaviside}\left(t-\frac{3\pi}{2}\right)\right)$

● Solution to the IVP

  $y = \cos(t)\left(-\mathit{Heaviside}\left(t-\frac{\pi}{2}\right) + \mathit{Heaviside}\left(t-\frac{3\pi}{2}\right)\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.062 (sec). Leaf size: 22

```
dsolve([diff(y(t),t$2)+y(t)=Dirac(t-1/2*Pi)+Dirac(t-3/2*Pi),y(0) = 0, D(y)(0) = 0],y(t), sin
```

$$y(t) = \left(\mathrm{Heaviside}\left(t-\frac{3\pi}{2}\right)-\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)\right)\cos\left(t\right)$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 27

```
DSolve[{y''[t]+y[t]==DiracDelta[t-1/2*Pi]+DiracDelta[t-3/2*Pi],{y[0]==0,y'[0]==0}},y[t],t,In
```

$$y(t)\to(\theta(2t-3\pi)-\theta(2t-\pi))\cos(t)$$

8.6 problem 6

Internal problem ID [6702]
Internal file name [`OUTPUT/5950_Sunday_June_05_2022_04_03_40_PM_50583576/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \delta(t - 2\pi) + \delta(t - 4\pi)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

8.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \delta(t - 2\pi) + \delta(t - 4\pi) \end{aligned}$$

Hence the ode is

$$y'' + y = \delta(t - 2\pi) + \delta(t - 4\pi)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - 2\pi) + \delta(t - 4\pi)$ is

$$\{2\pi \leq t \leq 4\pi, 4\pi \leq t \leq \infty, -\infty \leq t \leq 2\pi\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \mathrm{e}^{-2\pi s} + \mathrm{e}^{-4\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - s + Y(s) = \mathrm{e}^{-2\pi s} + \mathrm{e}^{-4\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-2\pi s} + \mathrm{e}^{-4\pi s} + s}{s^2 + 1}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-2\pi s}+\mathrm{e}^{-4\pi s}+s}{s^2+1}\right) \\
&= \cos(t)+\sin(t)\,(\mathrm{Heaviside}(t-2\pi)+\mathrm{Heaviside}(t-4\pi))
\end{aligned}$$

Hence the final solution is

$$y = \cos(t)+\sin(t)\,(\mathrm{Heaviside}(t-2\pi)+\mathrm{Heaviside}(t-4\pi))$$

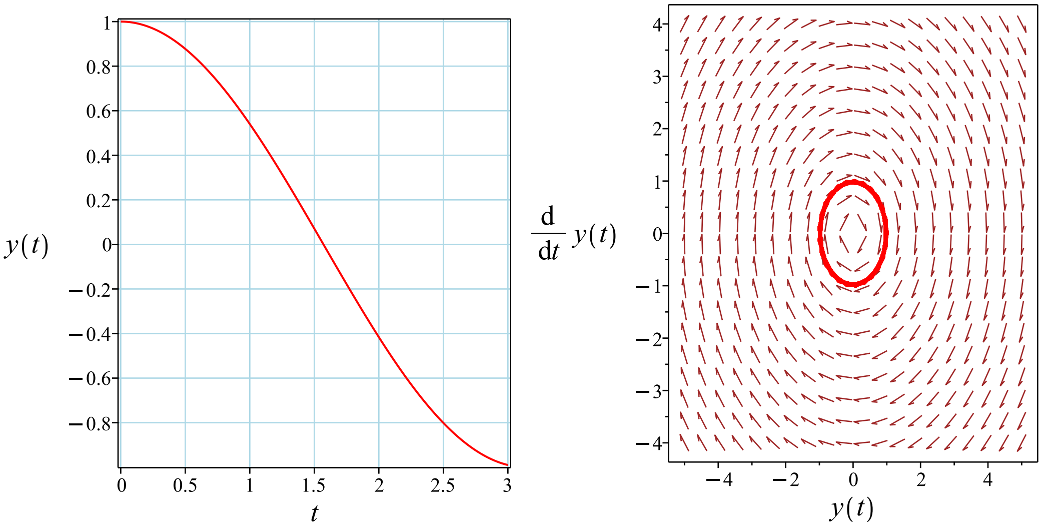
Simplifying the solution gives

$$y = \sin(t)\,\mathrm{Heaviside}(t-2\pi)+\sin(t)\,\mathrm{Heaviside}(t-4\pi)+\cos(t)$$

Summary

The solution(s) found are the following

$$y = \sin(t)\,\mathrm{Heaviside}(t-2\pi)+\sin(t)\,\mathrm{Heaviside}(t-4\pi)+\cos(t) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sin(t)\,\mathrm{Heaviside}(t-2\pi)+\sin(t)\,\mathrm{Heaviside}(t-4\pi)+\cos(t)$$

Verified OK.

8.6.2 Maple step by step solution

Let's solve

$$\left[y''+y=\mathit{Dirac}(t-2\pi)+\mathit{Dirac}(t-4\pi), y(0)=1, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2+1=0$
- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r=(-\mathrm{I},\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(t)$
- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(t)$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(t)+c_2\sin(t)+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\mathit{Dirac}(t-2\pi)+\mathit{Dirac}(t-\right.$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t),y_2(t))=\begin{bmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(t),y_2(t))=1$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \sin(t)\left(\int (\mathit{Dirac}(t-2\pi) + \mathit{Dirac}(t-4\pi))\,dt\right)$

  - ○ Compute integrals

    $y_p(t) = \sin(t)\,(\mathit{Heaviside}(t-2\pi) + \mathit{Heaviside}(t-4\pi))$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(t) + c_2\sin(t) + \sin(t)\,(\mathit{Heaviside}(t-2\pi) + \mathit{Heaviside}(t-4\pi))$

☐ Check validity of solution $y = c_1\cos(t) + c_2\sin(t) + \sin(t)\,(\mathit{Heaviside}(t-2\pi) + \mathit{Heaviside}(t -$

  - ○ Use initial condition $y(0) = 1$

    $1 = c_1$

  - ○ Compute derivative of the solution

    $y' = -c_1\sin(t) + c_2\cos(t) + \cos(t)\,(\mathit{Heaviside}(t-2\pi) + \mathit{Heaviside}(t-4\pi)) + \sin(t)\,(\mathit{Dirac}(t -$

  - ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = c_2$

  - ○ Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 0\}$

  - ○ Substitute constant values into general solution and simplify

    $y = \sin(t)\,\mathit{Heaviside}(t-2\pi) + \sin(t)\,\mathit{Heaviside}(t-4\pi) + \cos(t)$

- Solution to the IVP

  $y = \sin(t)\,\mathit{Heaviside}(t-2\pi) + \sin(t)\,\mathit{Heaviside}(t-4\pi) + \cos(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.078 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+y(t)=Dirac(t-2*Pi)+Dirac(t-4*Pi),y(0) = 1, D(y)(0) = 0],y(t), singsol
```

$$y(t) = \sin(t)\,\mathrm{Heaviside}(t-2\pi) + \sin(t)\,\mathrm{Heaviside}(t-4\pi) + \cos(t)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 26

```
DSolve[{y''[t]+y[t]==DiracDelta[t-2*Pi]+DiracDelta[t-4*Pi],{y[0]==1,y'[0]==0}},y[t],t,Includ
```

$$y(t) \to \theta(t-4\pi)\sin(t) + \theta(t-2\pi)\sin(t) + \cos(t)$$

8.7 problem 7

Internal problem ID [6703]
Internal file name [`OUTPUT/5951_Sunday_June_05_2022_04_03_45_PM_75629825/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 2y' = \delta(t-1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

8.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 0 \\ F &= \delta(t-1) \end{aligned}$$

Hence the ode is

$$y'' + 2y' = \delta(t-1)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = \delta(t-1)$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) = \mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + 2sY(s) = \mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-s} + 1}{s\,(s+2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}y &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-s}+1}{s\,(s+2)}\right)\\ &= \frac{\text{Heaviside}\,(t-1)\,(1-\mathrm{e}^{-2t+2})}{2} + \frac{1}{2} - \frac{\mathrm{e}^{-2t}}{2}\end{aligned}$$

Hence the final solution is

$$y = \frac{\text{Heaviside}\left(t-1\right)\left(1-\mathrm{e}^{-2t+2}\right)}{2}+\frac{1}{2}-\frac{\mathrm{e}^{-2t}}{2}$$

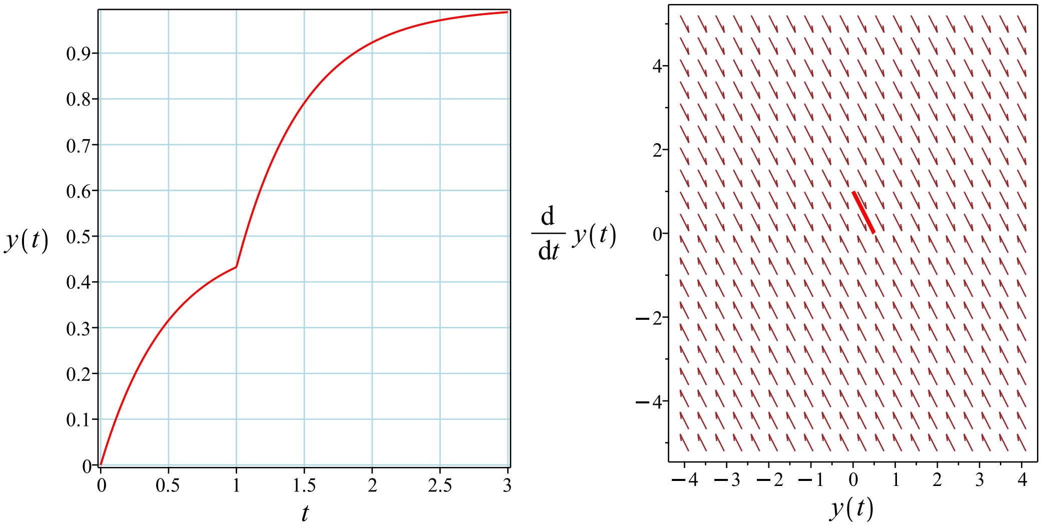
Simplifying the solution gives

$$y = -\frac{\text{Heaviside}\left(t-1\right)\mathrm{e}^{-2t+2}}{2}+\frac{\text{Heaviside}\left(t-1\right)}{2}+\frac{1}{2}-\frac{\mathrm{e}^{-2t}}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\text{Heaviside}\left(t-1\right)\mathrm{e}^{-2t+2}}{2}+\frac{\text{Heaviside}\left(t-1\right)}{2}+\frac{1}{2}-\frac{\mathrm{e}^{-2t}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{\text{Heaviside}\left(t-1\right)\mathrm{e}^{-2t+2}}{2}+\frac{\text{Heaviside}\left(t-1\right)}{2}+\frac{1}{2}-\frac{\mathrm{e}^{-2t}}{2}$$

Verified OK.

8.7.2 Maple step by step solution

Let's solve

$$\left[y''+2y'=\mathit{Dirac}(t-1)\,,y(0)=0,y'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+2r=0$

- Factor the characteristic polynomial

  $r(r+2)=0$

- Roots of the characteristic polynomial

  $r=(-2,0)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t)=1$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-2t}+c_2+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right),f(t)=\mathit{Dirac}(t-1)\right]$$

  ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t)\,,y_2(t))=\left[\begin{array}{cc}\mathrm{e}^{-2t} & 1\\ -2\,\mathrm{e}^{-2t} & 0\end{array}\right]$$

  ○ Compute Wronskian

  $W(y_1(t)\,,y_2(t))=2\,\mathrm{e}^{-2t}$

  ○ Substitute functions into equation for $y_p(t)$

$y_p(t) = -\frac{\left(\int \mathit{Dirac}(t-1)dt\right)\left(-1+\mathrm{e}^{-2t+2}\right)}{2}$

- Compute integrals

  $y_p(t) = -\frac{\mathit{Heaviside}(t-1)\left(-1+\mathrm{e}^{-2t+2}\right)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-2t} + c_2 - \frac{\mathit{Heaviside}(t-1)\left(-1+\mathrm{e}^{-2t+2}\right)}{2}$

☐ Check validity of solution $y = c_1\mathrm{e}^{-2t} + c_2 - \frac{\mathit{Heaviside}(t-1)\left(-1+\mathrm{e}^{-2t+2}\right)}{2}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

- Compute derivative of the solution

  $y' = -2c_1\mathrm{e}^{-2t} - \frac{\mathit{Dirac}(t-1)\left(-1+\mathrm{e}^{-2t+2}\right)}{2} + \mathit{Heaviside}(t-1)\,\mathrm{e}^{-2t+2}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = -2c_1$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\mathit{Heaviside}(t-1)\mathrm{e}^{-2t+2}}{2} + \frac{\mathit{Heaviside}(t-1)}{2} + \frac{1}{2} - \frac{\mathrm{e}^{-2t}}{2}$

- Solution to the IVP

  $y = -\frac{\mathit{Heaviside}(t-1)\mathrm{e}^{-2t+2}}{2} + \frac{\mathit{Heaviside}(t-1)}{2} + \frac{1}{2} - \frac{\mathrm{e}^{-2t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -2*_b(_a)+Dirac(_a-1), _b(_a)`   *** Su
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 1.984 (sec). Leaf size: 28

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)=Dirac(t-1),y(0) = 0, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = -\frac{\operatorname{Heaviside}(t-1)\,\mathrm{e}^{-2t+2}}{2} + \frac{\operatorname{Heaviside}(t-1)}{2} + \frac{1}{2} - \frac{\mathrm{e}^{-2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.144 (sec). Leaf size: 37

```
DSolve[{y''[t]+2*y'[t]==DiracDelta[t-1],{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to \frac{1}{2}e^{-2t}\left(\left(e^{2t} - e^{2}\right)\theta(t-1) + e^{2t} - 1\right)$$

8.8 problem 8

Internal problem ID [6704]
Internal file name [`OUTPUT/5952_Sunday_June_05_2022_04_03_49_PM_95698925/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 2y' = 1 + \delta(-2 + t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

8.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 0 \\ F &= 1 + \delta(-2 + t) \end{aligned}$$

Hence the ode is

$$y'' - 2y' = 1 + \delta(-2 + t)$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = 1 + \delta(-2 + t)$ is

$$\{t < 2 \lor 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) = \frac{1}{s} + \mathrm{e}^{-2s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 2sY(s) = \frac{1}{s} + \mathrm{e}^{-2s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-2s}s + s + 1}{s^2\,(s - 2)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-2s}s+s+1}{s^2\,(s-2)}\right) \\
&= -\frac{3}{4}-\frac{\operatorname{Heaviside}(-2+t)}{2}+\frac{3\,\mathrm{e}^{2t}}{4}-\frac{t}{2}+\frac{(1-\operatorname{Heaviside}(2-t))\,\mathrm{e}^{-4+2t}}{2}
\end{aligned}
$$

Hence the final solution is

$$
y = -\frac{3}{4}-\frac{\operatorname{Heaviside}(-2+t)}{2}+\frac{3\,\mathrm{e}^{2t}}{4}-\frac{t}{2}+\frac{(1-\operatorname{Heaviside}(2-t))\,\mathrm{e}^{-4+2t}}{2}
$$

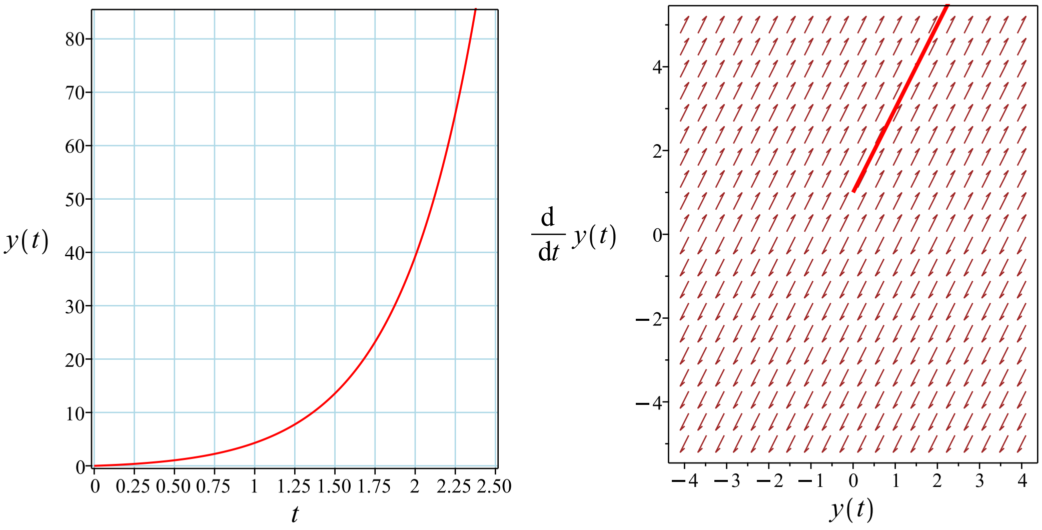
Simplifying the solution gives

$$
y = -\frac{3}{4}-\frac{\operatorname{Heaviside}(-2+t)}{2}+\frac{3\,\mathrm{e}^{2t}}{4}-\frac{t}{2}+\frac{\mathrm{e}^{-4+2t}\operatorname{Heaviside}(-2+t)}{2}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{3}{4}-\frac{\operatorname{Heaviside}(-2+t)}{2}+\frac{3\,\mathrm{e}^{2t}}{4}-\frac{t}{2}+\frac{\mathrm{e}^{-4+2t}\operatorname{Heaviside}(-2+t)}{2} \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{3}{4} - \frac{\text{Heaviside}\left(-2+t\right)}{2} + \frac{3\,\mathrm{e}^{2t}}{4} - \frac{t}{2} + \frac{\mathrm{e}^{-4+2t}\,\text{Heaviside}\left(-2+t\right)}{2}$$

Verified OK.

8.8.2 Maple step by step solution

Let's solve

$\left[y'' - 2y' = 1 + \mathit{Dirac}(-2+t)\,, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r = 0$

- Factor the characteristic polynomial

  $r(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (0, 2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 \mathrm{e}^{2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 1 + \mathit{Dirac}(-2+t)\right]$

  ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} 1 & e^{2t} \\ 0 & 2\,e^{2t} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{2t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\left(\int (1+Dirac(-2+t))dt\right)}{2} + \frac{e^{2t}\left(\int \left(e^{-4} Dirac(-2+t)+e^{-2t}\right)dt\right)}{2}$

- ○ Compute integrals

  $y_p(t) = -\frac{t}{2} - \frac{Heaviside(-2+t)}{2} + \frac{e^{-4+2t} Heaviside(-2+t)}{2} - \frac{1}{4}$

- ● Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{2t} - \frac{t}{2} - \frac{Heaviside(-2+t)}{2} + \frac{e^{-4+2t} Heaviside(-2+t)}{2} - \frac{1}{4}$

☐ Check validity of solution $y = c_1 + c_2 e^{2t} - \frac{t}{2} - \frac{Heaviside(-2+t)}{2} + \frac{e^{-4+2t} Heaviside(-2+t)}{2} - \frac{1}{4}$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - \frac{1}{4}$

- ○ Compute derivative of the solution

  $y' = 2c_2 e^{2t} - \frac{1}{2} - \frac{Dirac(-2+t)}{2} + e^{-4+2t} Heaviside(-2+t) + \frac{e^{-4+2t} Dirac(-2+t)}{2}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = 2c_2 - \frac{1}{2}$

- ○ Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{2}, c_2 = \frac{3}{4}\right\}$

- ○ Substitute constant values into general solution and simplify

  $y = -\frac{3}{4} - \frac{Heaviside(-2+t)}{2} + \frac{3\,e^{2t}}{4} - \frac{t}{2} + \frac{e^{-4+2t} Heaviside(-2+t)}{2}$

- ● Solution to the IVP

  $y = -\frac{3}{4} - \frac{Heaviside(-2+t)}{2} + \frac{3\,e^{2t}}{4} - \frac{t}{2} + \frac{e^{-4+2t} Heaviside(-2+t)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 2*_b(_a)+1+Dirac(_a-2), _b(_a)`   *** S
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 2.016 (sec). Leaf size: 39

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)=1+Dirac(t-2),y(0) = 0, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = -\frac{3}{4} - \frac{\mathrm{Heaviside}\left(t-2\right)}{2} + \frac{3\,\mathrm{e}^{2t}}{4} - \frac{t}{2} + \frac{\mathrm{Heaviside}\left(t-2\right)\mathrm{e}^{2t-4}}{2}$$

✓ Solution by Mathematica

Time used: 0.116 (sec). Leaf size: 37

```
DSolve[{y''[t]-2*y'[t]==1+DiracDelta[t-2],{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolution
```

$$y(t) \to \frac{1}{4}\left(\left(2e^{2t-4}-2\right)\theta(t-2)-2t+3e^{2t}-3\right)$$

8.9 problem 9

Internal problem ID [6705]
Internal file name [`OUTPUT/5953_Sunday_June_05_2022_04_03_53_PM_1103531/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y' + 5y = \delta(t - 2\pi)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

8.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 5 \\ F &= \delta(t - 2\pi) \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 5y = \delta(t - 2\pi)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - 2\pi)$ is

$$\{t < 2\pi \lor 2\pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 5Y(s) = \mathrm{e}^{-2\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4sY(s) + 5Y(s) = \mathrm{e}^{-2\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-2\pi s}}{s^2 + 4s + 5}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{s^2+4s+5}\right) \\ &= \text{Heaviside}\left(t-2\pi\right) e^{4\pi-2t} \sin\left(t\right) \end{aligned}$$

Hence the final solution is

$$y = \text{Heaviside}\left(t-2\pi\right) e^{4\pi-2t} \sin\left(t\right)$$

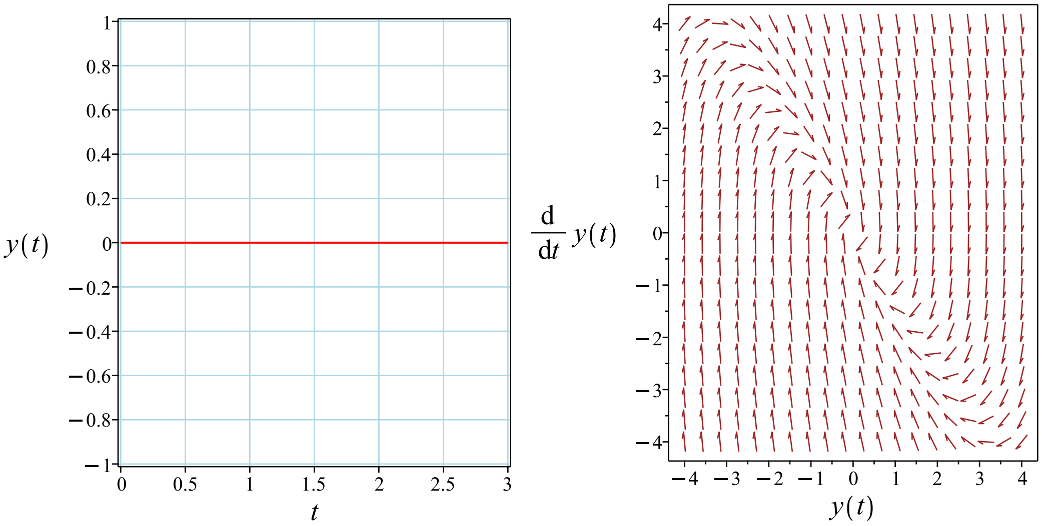
Simplifying the solution gives

$$y = \text{Heaviside}\left(t-2\pi\right) e^{4\pi-2t} \sin\left(t\right)$$

Summary

The solution(s) found are the following

$$y = \text{Heaviside}\left(t-2\pi\right) e^{4\pi-2t} \sin\left(t\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \text{Heaviside}\left(t-2\pi\right) e^{4\pi-2t} \sin\left(t\right)$$

Verified OK.

8.9.2 Maple step by step solution

Let's solve

$$\left[y''+4y'+5y=Dirac(t-2\pi)\,,y(0)=0,y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2+4r+5=0$
- Use quadratic formula to solve for r

  $r=\frac{(-4)\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r=(-2-\mathrm{I},-2+\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-2t}\cos(t)$
- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{-2t}\sin(t)$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y=\cos(t)\,\mathrm{e}^{-2t}c_1+\sin(t)\,\mathrm{e}^{-2t}c_2+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right),f(t)=Dirac(t-2\pi)\right]$$

  ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t)\,,y_2(t))=\left[\begin{array}{cc}\mathrm{e}^{-2t}\cos(t) & \mathrm{e}^{-2t}\sin(t)\\ -2\,\mathrm{e}^{-2t}\cos(t)-\mathrm{e}^{-2t}\sin(t) & -2\,\mathrm{e}^{-2t}\sin(t)+\mathrm{e}^{-2t}\cos(t)\end{array}\right]$$

  ○ Compute Wronskian

  $W(y_1(t)\,,y_2(t))=\mathrm{e}^{-4t}$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \sin(t) \left(\int \mathit{Dirac}(t-2\pi)\, dt\right) \mathrm{e}^{4\pi-2t}$

  - ○ Compute integrals

    $y_p(t) = \mathit{Heaviside}(t-2\pi)\, \mathrm{e}^{4\pi-2t} \sin(t)$

- • Substitute particular solution into general solution to ODE

  $y = \cos(t)\, \mathrm{e}^{-2t} c_1 + \sin(t)\, \mathrm{e}^{-2t} c_2 + \mathit{Heaviside}(t-2\pi)\, \mathrm{e}^{4\pi-2t} \sin(t)$

- ☐ Check validity of solution $y = \cos(t)\, \mathrm{e}^{-2t} c_1 + \sin(t)\, \mathrm{e}^{-2t} c_2 + \mathit{Heaviside}(t-2\pi)\, \mathrm{e}^{4\pi-2t} \sin(t)$
  - ○ Use initial condition $y(0) = 0$

    $0 = c_1$
  - ○ Compute derivative of the solution

    $y' = -\sin(t)\, \mathrm{e}^{-2t} c_1 - 2\cos(t)\, \mathrm{e}^{-2t} c_1 + \cos(t)\, \mathrm{e}^{-2t} c_2 - 2\sin(t)\, \mathrm{e}^{-2t} c_2 + \mathit{Dirac}(t-2\pi)\, \mathrm{e}^{4\pi-2t} \sin ($
  - ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -2c_1 + c_2$
  - ○ Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$
  - ○ Substitute constant values into general solution and simplify

    $y = \mathit{Heaviside}(t-2\pi)\, \mathrm{e}^{4\pi-2t} \sin(t)$
- • Solution to the IVP

  $y = \mathit{Heaviside}(t-2\pi)\, \mathrm{e}^{4\pi-2t} \sin(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.109 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+5*y(t)=Dirac(t-2*Pi),y(0) = 0, D(y)(0) = 0],y(t), sing
```

$$y(t) = \sin(t) \operatorname{Heaviside}(t - 2\pi) e^{4\pi - 2t}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 23

```
DSolve[{y''[t]+4*y'[t]+5*y[t]==DiracDelta[t-2*Pi],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingular
```

$$y(t) \to e^{4\pi - 2t} \theta(t - 2\pi) \sin(t)$$

8.10 problem 10

Internal problem ID [6706]
Internal file name [`OUTPUT/5954_Sunday_June_05_2022_04_03_57_PM_50975212/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + y = \delta(t-1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

8.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 1 \\ F &= \delta(t-1) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + y = \delta(t-1)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t-1)$ is

$$\{t < 1 \vee 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + Y(s) = \mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2sY(s) + Y(s) = \mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-s}}{s^2 + 2s + 1}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-s}}{s^2+2s+1}\right) \\
&= \mathrm{Heaviside}\,(t-1)\,(t-1)\,\mathrm{e}^{-t+1}
\end{aligned}
$$

Hence the final solution is

$$
y = \mathrm{Heaviside}\,(t-1)\,(t-1)\,\mathrm{e}^{-t+1}
$$

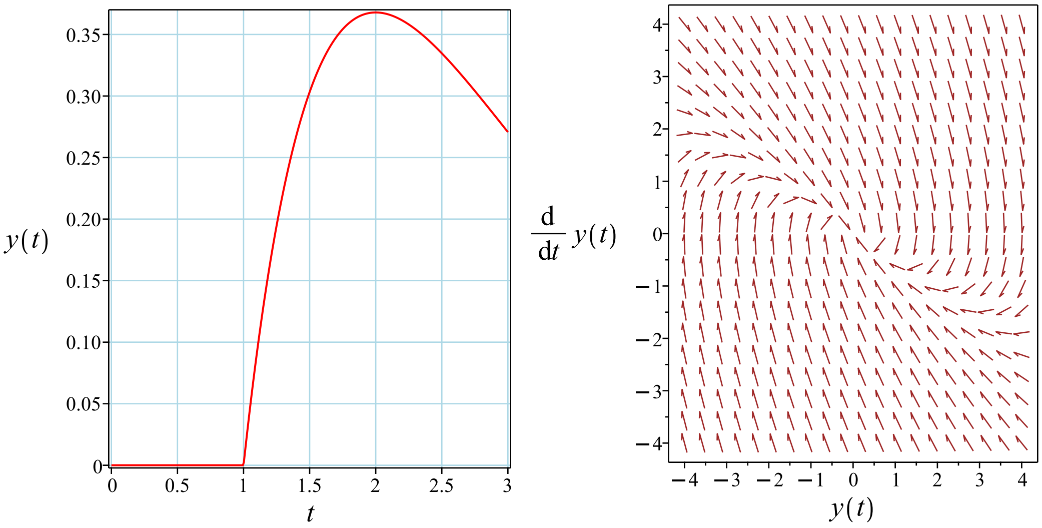
Simplifying the solution gives

$$
y = \mathrm{Heaviside}\,(t-1)\,(t-1)\,\mathrm{e}^{-t+1}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{Heaviside}\,(t-1)\,(t-1)\,\mathrm{e}^{-t+1} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = \mathrm{Heaviside}\,(t-1)\,(t-1)\,\mathrm{e}^{-t+1}
$$

Verified OK.

8.10.2 Maple step by step solution

Let's solve

$\left[y''+2y'+y=\mathit{Dirac}(t-1), y(0)=0, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+2r+1=0$

- Factor the characteristic polynomial

  $(r+1)^2=0$

- Root of the characteristic polynomial

  $r=-1$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t)=t\,\mathrm{e}^{-t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=\mathrm{e}^{-t}c_1+t\,\mathrm{e}^{-t}c_2+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\mathit{Dirac}(t-1)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t))=\begin{bmatrix}\mathrm{e}^{-t} & t\,\mathrm{e}^{-t}\\ -\mathrm{e}^{-t} & \mathrm{e}^{-t}-t\,\mathrm{e}^{-t}\end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t))=\mathrm{e}^{-2t}$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \left(\int \mathit{Dirac}(t-1)\,dt\right) \mathrm{e}^{-t+1}(t-1)$

  - Compute integrals

    $y_p(t) = \mathit{Heaviside}(t-1)\,(t-1)\,\mathrm{e}^{-t+1}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-t}c_1 + t\,\mathrm{e}^{-t}c_2 + \mathit{Heaviside}(t-1)\,(t-1)\,\mathrm{e}^{-t+1}$

- ☐ Check validity of solution $y = \mathrm{e}^{-t}c_1 + t\mathrm{e}^{-t}c_2 + \mathit{Heaviside}(t-1)\,(t-1)\,\mathrm{e}^{-t+1}$

  - Use initial condition $y(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $y' = -\mathrm{e}^{-t}c_1 + c_2\mathrm{e}^{-t} - t\,\mathrm{e}^{-t}c_2 + \mathit{Dirac}(t-1)\,(t-1)\,\mathrm{e}^{-t+1} + \mathit{Heaviside}(t-1)\,\mathrm{e}^{-t+1} - \mathit{Heaviside}(t$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

    $y = \mathit{Heaviside}(t-1)\,(t-1)\,\mathrm{e}^{-t+1}$

- Solution to the IVP

  $y = \mathit{Heaviside}(t-1)\,(t-1)\,\mathrm{e}^{-t+1}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.937 (sec). Leaf size: 18

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t)=Dirac(t-1),y(0) = 0, D(y)(0) = 0],y(t), singsol=a
```

$$y(t) = (t-1)\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{1-t}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 20

```
DSolve[{y''[t]+2*y'[t]+y[t]==DiracDelta[t-1],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolut
```

$$y(t) \to e^{1-t}(t-1)\theta(t-1)$$

8.11 problem 11

Internal problem ID [6707]
Internal file name [`OUTPUT/5955_Sunday_June_05_2022_04_04_00_PM_2173919/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

8.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 13 \\ F &= \delta(t-\pi) + \delta(t-3\pi) \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 13y = \delta(t - \pi) + \delta(t - 3\pi)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 13$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - \pi) + \delta(t - 3\pi)$ is

$$\{\pi \leq t \leq 3\pi, 3\pi \leq t \leq \infty, -\infty \leq t \leq \pi\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 13Y(s) = \mathrm{e}^{-\pi s} + \mathrm{e}^{-3\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4 - s + 4sY(s) + 13Y(s) = \mathrm{e}^{-\pi s} + \mathrm{e}^{-3\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-\pi s} + \mathrm{e}^{-3\pi s} + s + 4}{s^2 + 4s + 13}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-\pi s}+e^{-3\pi s}+s+4}{s^{2}+4s+13}\right) \\
&= e^{-2t}\cos(3t)+\frac{\sin(3t)\left(-\operatorname{Heaviside}(t-\pi)\,e^{2\pi-2t}-e^{-2t+6\pi}\operatorname{Heaviside}(t-3\pi)+2\,e^{-2t}\right)}{3}
\end{aligned}
$$

Converting the above solution to piecewise it becomes

$$
y=\left\{\begin{array}{cc}
e^{-2t}\cos(3t)+\frac{2\sin(3t)e^{-2t}}{3} & t\le\pi \\
e^{-2t}\cos(3t)+\frac{\sin(3t)\left(2\,e^{-2t}-e^{2\pi-2t}\right)}{3} & t\le 3\pi \\
e^{-2t}\cos(3t)+\frac{\sin(3t)\left(2\,e^{-2t}-e^{2\pi-2t}-e^{-2t+6\pi}\right)}{3} & 3\pi<t
\end{array}\right.
$$

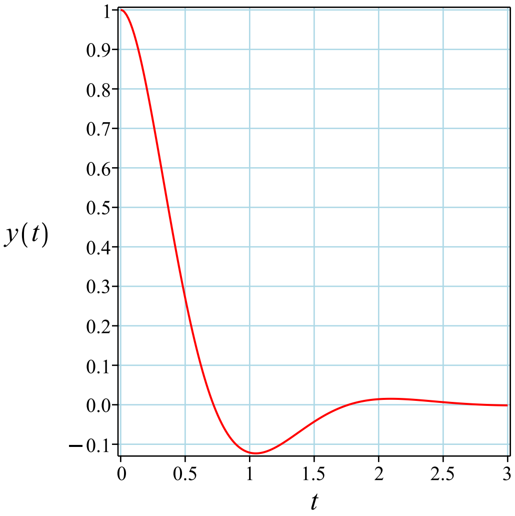
Simplifying the solution gives

$$
y=\frac{\left(\left\{\begin{array}{cc}
e^{-2t}(3\cos(3t)+2\sin(3t)) & t\le\pi \\
-\sin(3t)\,e^{2\pi-2t}+3\,e^{-2t}\left(\cos(3t)+\frac{2\sin(3t)}{3}\right) & t\le 3\pi \\
-\sin(3t)\,e^{2\pi-2t}-\sin(3t)\,e^{-2t+6\pi}+3\,e^{-2t}\left(\cos(3t)+\frac{2\sin(3t)}{3}\right) & 3\pi<t
\end{array}\right.\right)}{3}
$$

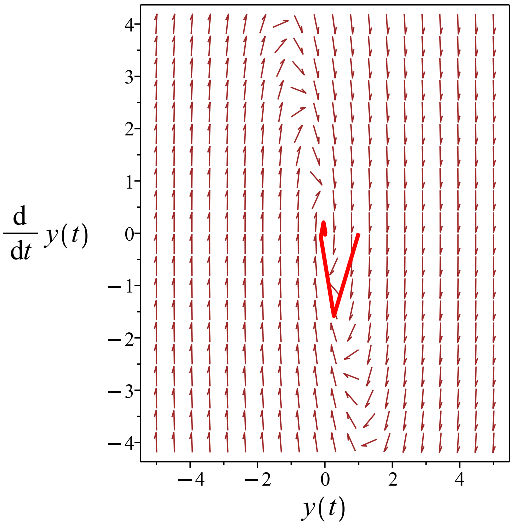
Summary

The solution(s) found are the following

$$
y=\frac{\left(\left\{\begin{array}{cc}
e^{-2t}(3\cos(3t)+2\sin(3t)) & t\le\pi \\
-\sin(3t)\,e^{2\pi-2t}+3\,e^{-2t}\left(\cos(3t)+\frac{2\sin(3t)}{3}\right) & t\le 3\pi \\
-\sin(3t)\,e^{2\pi-2t}-\sin(3t)\,e^{-2t+6\pi}+3\,e^{-2t}\left(\cos(3t)+\frac{2\sin(3t)}{3}\right) & 3\pi<t
\end{array}\right.\right)}{3} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\left(\begin{cases} \mathrm{e}^{-2t}(3\cos(3t)+2\sin(3t)) & t \le \pi \\ -\sin(3t)\,\mathrm{e}^{2\pi-2t}+3\,\mathrm{e}^{-2t}\left(\cos(3t)+\frac{2\sin(3t)}{3}\right) & t \le 3\pi \\ -\sin(3t)\,\mathrm{e}^{2\pi-2t}-\sin(3t)\,\mathrm{e}^{-2t+6\pi}+3\,\mathrm{e}^{-2t}\left(\cos(3t)+\frac{2\sin(3t)}{3}\right) & 3\pi < t \end{cases}\right)}{3}$$

Verified OK.

8.11.2 Maple step by step solution

Let's solve

$$\left[y''+4y'+13y = \mathit{Dirac}(t-\pi)+\mathit{Dirac}(t-3\pi)\,, y(0)=1, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+4r+13=0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - 3\,\mathrm{I}, -2 + 3\,\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}\cos(3t)$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)\,\mathrm{e}^{-2t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{-2t}\cos(3t) + c_2\sin(3t)\,\mathrm{e}^{-2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))}dt\right), f(t) = \mathit{Dirac}(t-\pi) + \mathit{Dirac}(t-3\right.$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t}\cos(3t) & \sin(3t)\,\mathrm{e}^{-2t} \\ -2\,\mathrm{e}^{-2t}\cos(3t) - 3\sin(3t)\,\mathrm{e}^{-2t} & 3\,\mathrm{e}^{-2t}\cos(3t) - 2\sin(3t)\,\mathrm{e}^{-2t} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 3\,\mathrm{e}^{-4t}$
  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \frac{\sin(3t)\mathrm{e}^{-2t}\left(\int\left(-\mathrm{e}^{6\pi}\mathit{Dirac}(t-3\pi) - \mathrm{e}^{2\pi}\mathit{Dirac}(t-\pi)\right)dt\right)}{3}$
  - Compute integrals

    $y_p(t) = -\frac{\sin(3t)\mathrm{e}^{-2t}\left(\mathrm{e}^{2\pi}\mathit{Heaviside}(t-\pi) + \mathrm{e}^{6\pi}\mathit{Heaviside}(t-3\pi)\right)}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-2t}\cos(3t) + c_2\sin(3t)\,\mathrm{e}^{-2t} - \frac{\sin(3t)\mathrm{e}^{-2t}\left(\mathrm{e}^{2\pi}\mathit{Heaviside}(t-\pi) + \mathrm{e}^{6\pi}\mathit{Heaviside}(t-3\pi)\right)}{3}$

☐ Check validity of solution $y = c_1\mathrm{e}^{-2t}\cos(3t) + c_2\sin(3t)\,\mathrm{e}^{-2t} - \frac{\sin(3t)\mathrm{e}^{-2t}\left(\mathrm{e}^{2\pi}\mathit{Heaviside}(t-\pi) + \mathrm{e}^{6\pi}\mathit{Heavis}\right.}{3}$

  - Use initial condition $y(0) = 1$

    $1 = c_1$
  - Compute derivative of the solution

$$y' = -2c_1 \mathrm{e}^{-2t} \cos(3t) - 3c_1 \mathrm{e}^{-2t} \sin(3t) + 3c_2 \cos(3t)\, \mathrm{e}^{-2t} - 2c_2 \sin(3t)\, \mathrm{e}^{-2t} - \cos(3t)\, \mathrm{e}^{-2t} (\mathrm{e}^{2\pi} \mathit{He}$$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $$0 = -2c_1 + 3c_2$$

  - Solve for c_1 and c_2

    $$\left\{c_1 = 1, c_2 = \frac{2}{3}\right\}$$

  - Substitute constant values into general solution and simplify

    $$y = -\frac{\mathrm{e}^{-2t} \left(\mathit{Heaviside}(t-\pi)\mathrm{e}^{2\pi} \sin(3t) + \mathit{Heaviside}(t-3\pi)\mathrm{e}^{6\pi} \sin(3t) - 3\cos(3t) - 2\sin(3t)\right)}{3}$$

- Solution to the IVP

  $$y = -\frac{\mathrm{e}^{-2t} \left(\mathit{Heaviside}(t-\pi)\mathrm{e}^{2\pi} \sin(3t) + \mathit{Heaviside}(t-3\pi)\mathrm{e}^{6\pi} \sin(3t) - 3\cos(3t) - 2\sin(3t)\right)}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.297 (sec). Leaf size: 59

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+13*y(t)=Dirac(t-Pi)+Dirac(t-3*Pi),y(0) = 1, D(y)(0) =
```

$$y(t) = \left(\cos(3t) + \frac{2\sin(3t)}{3}\right)\mathrm{e}^{-2t} - \frac{\sin(3t)\, \mathrm{e}^{-2t+6\pi}\, \mathrm{Heaviside}(t-3\pi)}{3} - \frac{\sin(3t)\, \mathrm{e}^{-2t+2\pi}\, \mathrm{Heaviside}(t-\pi)}{3}$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 59

```
DSolve[{y''[t]+4*y'[t]+13*y[t]==DiracDelta[t-Pi]+DiracDelta[t-3*Pi],{y[0]==1,y'[0]==0}},y[t]
```

$$y(t) \to -\frac{1}{3}e^{-2t}\left(e^{6\pi}\theta(t-3\pi)\sin(3t)+e^{2\pi}\theta(t-\pi)\sin(3t)-2\sin(3t)-3\cos(3t)\right)$$

8.12 problem 12

Internal problem ID [6708]
Internal file name [`OUTPUT/5956_Sunday_June_05_2022_04_04_11_PM_56284687/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 7y' + 6y = \mathrm{e}^{t} + \delta(-2+t) + \delta(t-4)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

8.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -7 \\ q(t) &= 6 \\ F &= \mathrm{e}^{t} + \delta(-2+t) + \delta(t-4) \end{aligned}$$

Hence the ode is

$$y'' - 7y' + 6y = e^t + \delta(-2+t) + \delta(t-4)$$

The domain of $p(t) = -7$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^t + \delta(-2+t) + \delta(t-4)$ is

$$\{2 \leq t \leq 4, 4 \leq t \leq \infty, -\infty \leq t \leq 2\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 7sY(s) + 7y(0) + 6Y(s) = \frac{1}{s-1} + e^{-2s} + e^{-4s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 7sY(s) + 6Y(s) = \frac{1}{s-1} + e^{-2s} + e^{-4s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-2s}s + e^{-4s}s - e^{-2s} - e^{-4s} + 1}{(s-1)(s^2 - 7s + 6)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-2s}s + \mathrm{e}^{-4s}s - \mathrm{e}^{-2s} - \mathrm{e}^{-4s} + 1}{(s-1)\left(s^{2}-7s+6\right)}\right) \\
&= \frac{\mathrm{e}^{6t}}{25} + \frac{\mathrm{e}^{-24+6t}}{5} + \frac{\mathrm{e}^{-12+6t}}{5} - \frac{\mathrm{e}^{t-4}}{5} - \frac{\mathrm{e}^{-2+t}}{5} - \frac{\mathrm{e}^{t}(1+5t)}{25} + \frac{\left(-\mathrm{e}^{-12+6t} + \mathrm{e}^{-2+t}\right)\mathrm{Heaviside}\left(2-t\right)}{5} + (-\mathrm{e}^{-2}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y = {} & \frac{\mathrm{e}^{6t}}{25} + \frac{\mathrm{e}^{-24+6t}}{5} + \frac{\mathrm{e}^{-12+6t}}{5} - \frac{\mathrm{e}^{t-4}}{5} - \frac{\mathrm{e}^{-2+t}}{5} - \frac{\mathrm{e}^{t}(1+5t)}{25} \\
& + \frac{\left(-\mathrm{e}^{-12+6t} + \mathrm{e}^{-2+t}\right)\mathrm{Heaviside}\left(2-t\right)}{5} + \frac{\left(-\mathrm{e}^{-24+6t} + \mathrm{e}^{t-4}\right)\mathrm{Heaviside}\left(-t+4\right)}{5}
\end{aligned}
$$

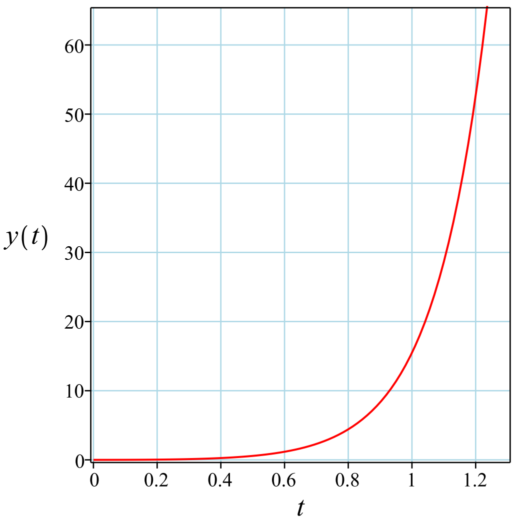
Simplifying the solution gives

$$
\begin{aligned}
y = {} & \frac{\mathrm{e}^{-24+6t}\,\mathrm{Heaviside}\left(t-4\right)}{5} + \frac{\mathrm{e}^{-12+6t}\,\mathrm{Heaviside}\left(-2+t\right)}{5} \\
& - \frac{\mathrm{e}^{-2+t}\,\mathrm{Heaviside}\left(-2+t\right)}{5} - \frac{\mathrm{e}^{t-4}\,\mathrm{Heaviside}\left(t-4\right)}{5} + \frac{\mathrm{e}^{6t}}{25} + \frac{\left(-5t-1\right)\mathrm{e}^{t}}{25}
\end{aligned}
$$

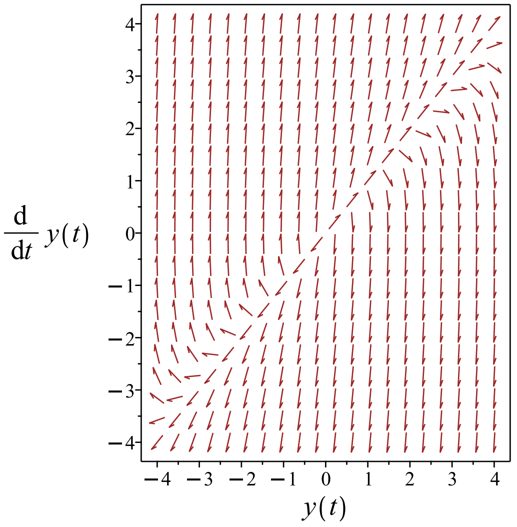
Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{\mathrm{e}^{-24+6t}\,\mathrm{Heaviside}\left(t-4\right)}{5} + \frac{\mathrm{e}^{-12+6t}\,\mathrm{Heaviside}\left(-2+t\right)}{5} \\
& - \frac{\mathrm{e}^{-2+t}\,\mathrm{Heaviside}\left(-2+t\right)}{5} - \frac{\mathrm{e}^{t-4}\,\mathrm{Heaviside}\left(t-4\right)}{5} + \frac{\mathrm{e}^{6t}}{25} + \frac{\left(-5t-1\right)\mathrm{e}^{t}}{25}
\end{aligned}
\tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{-24+6t}\,\mathrm{Heaviside}\,(t-4)}{5} + \frac{e^{-12+6t}\,\mathrm{Heaviside}\,(-2+t)}{5} - \frac{e^{-2+t}\,\mathrm{Heaviside}\,(-2+t)}{5} - \frac{e^{t-4}\,\mathrm{Heaviside}\,(t-4)}{5} + \frac{e^{6t}}{25} + \frac{(-5t-1)\,e^{t}}{25}$$

Verified OK.

8.12.2 Maple step by step solution

Let's solve

$$\left[y''-7y'+6y=e^{t}+Dirac(-2+t)+Dirac(t-4)\,,y(0)=0,y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2-7r+6=0$

- Factor the characteristic polynomial

  $(r-1)\,(r-6)=0$

- Roots of the characteristic polynomial

  $r=(1,6)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{6t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{6t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \mathrm{e}^{t} + \mathit{Dirac}(-2+t) + \mathit{Dirac}\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{t} & \mathrm{e}^{6t} \\ \mathrm{e}^{t} & 6\,\mathrm{e}^{6t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 5\,\mathrm{e}^{7t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{\mathrm{e}^{t}\left(\int \left(1+\mathrm{e}^{-2}\mathit{Dirac}(-2+t)+\mathrm{e}^{-4}\mathit{Dirac}(t-4)\right)dt\right)}{5} + \frac{\mathrm{e}^{6t}\left(\int \left(\mathrm{e}^{-12}\mathit{Dirac}(-2+t)+\mathrm{e}^{-24}\mathit{Dirac}(t-4)+\mathrm{e}^{-5t}\right)dt\right)}{5}$

  - Compute integrals

    $y_p(t) = \frac{\mathrm{e}^{-24+6t}\mathit{Heaviside}(t-4)}{5} + \frac{\mathrm{e}^{-12+6t}\mathit{Heaviside}(-2+t)}{5} - \frac{\mathrm{e}^{-2+t}\mathit{Heaviside}(-2+t)}{5} - \frac{\mathrm{e}^{t-4}\mathit{Heaviside}(t-4)}{5} + \frac{(-5t-1)\mathrm{e}^{t}}{25}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{6t} + \frac{\mathrm{e}^{-24+6t}\mathit{Heaviside}(t-4)}{5} + \frac{\mathrm{e}^{-12+6t}\mathit{Heaviside}(-2+t)}{5} - \frac{\mathrm{e}^{-2+t}\mathit{Heaviside}(-2+t)}{5} - \frac{\mathrm{e}^{t-4}\mathit{Heaviside}(t-4)}{5}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{6t} + \frac{\mathrm{e}^{-24+6t}\mathit{Heaviside}(t-4)}{5} + \frac{\mathrm{e}^{-12+6t}\mathit{Heaviside}(-2+t)}{5} - \frac{\mathrm{e}^{-2+t}\mathit{Heavi}}{5}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2 - \frac{1}{25}$

  - Compute derivative of the solution

    $y' = c_1 \mathrm{e}^{t} + 6 c_2 \mathrm{e}^{6t} + \frac{6\,\mathrm{e}^{-24+6t}\mathit{Heaviside}(t-4)}{5} + \frac{\mathrm{e}^{-24+6t}\mathit{Dirac}(t-4)}{5} + \frac{6\,\mathrm{e}^{-12+6t}\mathit{Heaviside}(-2+t)}{5} + \frac{\mathrm{e}^{-12+6t}\mathit{Dirac}(-2+}{5}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_1 + 6c_2 - \frac{6}{25}$

  - Solve for c_1 and c_2

    $\left\{c_1 = 0, c_2 = \frac{1}{25}\right\}$

  - Substitute constant values into general solution and simplify

    $y = \frac{e^{-24+6t} \mathit{Heaviside}(t-4)}{5} + \frac{e^{-12+6t} \mathit{Heaviside}(-2+t)}{5} - \frac{e^{-2+t} \mathit{Heaviside}(-2+t)}{5} - \frac{e^{t-4} \mathit{Heaviside}(t-4)}{5} + \frac{e^{6t}}{25} + \frac{(-5t-1}{25}$

- Solution to the IVP

  $y = \frac{e^{-24+6t} \mathit{Heaviside}(t-4)}{5} + \frac{e^{-12+6t} \mathit{Heaviside}(-2+t)}{5} - \frac{e^{-2+t} \mathit{Heaviside}(-2+t)}{5} - \frac{e^{t-4} \mathit{Heaviside}(t-4)}{5} + \frac{e^{6t}}{25} + \frac{(-5t-1}{25}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.266 (sec). Leaf size: 90

```
dsolve([diff(y(t),t$2)-7*diff(y(t),t)+6*y(t)=exp(t)+Dirac(t-2)+Dirac(t-4),y(0) = 0, D(y)(0)
```

$$y(t) = \frac{e^{-24+6t} \operatorname{Heaviside}(t-4)}{5} + \frac{e^{-12+6t} \operatorname{Heaviside}(t-2)}{5} - \frac{e^{t-4} \operatorname{Heaviside}(t-4)}{5} - \frac{e^{t-2} \operatorname{Heaviside}(t-2)}{5} + \frac{e^{6t}}{25} + \frac{(-5t-1)\, e^{t}}{25}$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 67

```
DSolve[{y''[t]-7*y'[t]+6*y[t]==Exp[t]+DiracDelta[t-2]+DiracDelta[t-4],{y[0]==9,y'[0]==0}},y[
```

$$y(t) \to \frac{1}{25}e^{t-24}\left(5\left(e^{5t}-e^{20}\right)\theta(t-4)+5\left(e^{5t+12}-e^{22}\right)\theta(t-2)+e^{24}\left(-5t-44e^{5t}+269\right)\right)$$

8.13 problem 15(a)

Internal problem ID [6709]
Internal file name [`OUTPUT/5957_Sunday_June_05_2022_04_04_22_PM_51342204/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 15(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 10y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

8.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 10 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 10y = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 10Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + 2sY(s) + 10Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{s^2 + 2s + 10}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{i}{6\,(s + 1 - 3i)} + \frac{i}{6s + 6 + 18i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{i}{6(s+1-3i)}\right)=-\frac{i\mathrm{e}^{(-1+3i)t}}{6}$$
$$\mathcal{L}^{-1}\left(\frac{i}{6s+6+18i}\right)=\frac{i\mathrm{e}^{(-1-3i)t}}{6}$$

Adding the above results and simplifying gives

$$y=\frac{\sin(3t)\,\mathrm{e}^{-t}}{3}$$

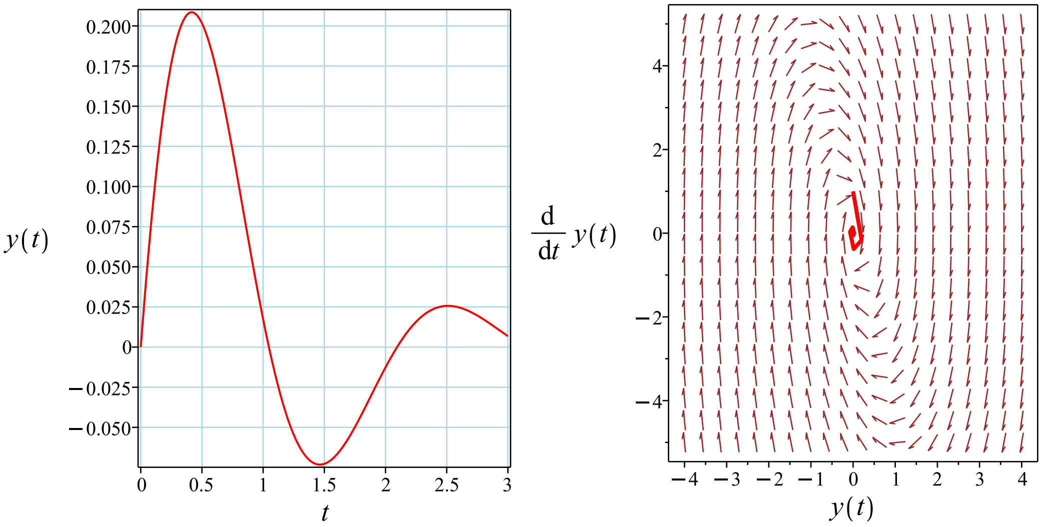
Simplifying the solution gives

$$y=\frac{\sin(3t)\,\mathrm{e}^{-t}}{3}$$

Summary

The solution(s) found are the following

$$y=\frac{\sin(3t)\,\mathrm{e}^{-t}}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y=\frac{\sin(3t)\,\mathrm{e}^{-t}}{3}$$

Verified OK.

8.13.2 Maple step by step solution

Let's solve

$\left[y''+2y'+10y=0, y(0)=0, y'\Big|_{\{t=0\}}=1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2+2r+10=0$

- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r=(-1-3\,\mathrm{I}, -1+3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t)=\cos(3t)\,\mathrm{e}^{-t}$

- 2nd solution of the ODE

  $y_2(t)=\sin(3t)\,\mathrm{e}^{-t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\cos(3t)\,\mathrm{e}^{-t}+c_2\sin(3t)\,\mathrm{e}^{-t}$

☐ Check validity of solution $y=c_1\cos(3t)\,\mathrm{e}^{-t}+c_2\sin(3t)\,\mathrm{e}^{-t}$

  ○ Use initial condition $y(0)=0$

  $0=c_1$

  ○ Compute derivative of the solution

  $y'=-3c_1\sin(3t)\,\mathrm{e}^{-t}-c_1\cos(3t)\,\mathrm{e}^{-t}+3c_2\cos(3t)\,\mathrm{e}^{-t}-c_2\sin(3t)\,\mathrm{e}^{-t}$

  ○ Use the initial condition $y'\Big|_{\{t=0\}}=1$

  $1=-c_1+3c_2$

  ○ Solve for c_1 and c_2

  $\left\{c_1=0, c_2=\frac{1}{3}\right\}$

  - Substitute constant values into general solution and simplify

    $y = \frac{\sin(3t)\,\mathrm{e}^{-t}}{3}$

- Solution to the IVP

  $y = \frac{\sin(3t)\,\mathrm{e}^{-t}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.688 (sec). Leaf size: 14

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+10*y(t)=0,y(0) = 0, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{\mathrm{e}^{-t} \sin(3t)}{3}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 18

```
DSolve[{y''[t]+2*y'[t]+10*y[t]==0,{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to \frac{1}{3} e^{-t} \sin(3t)$$

8.14 problem 15(b)

Internal problem ID [6710]
Internal file name [`OUTPUT/5958_Sunday_June_05_2022_04_04_25_PM_69753260/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 7 THE LAPLACE TRANSFORM. EXERCISES 7.5. Page 315
Problem number: 15(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 10y = \delta(t)}$$

Since no initial conditions are explicitly given, then let

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 10Y(s) = 1 \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= c_1 \\
y'(0) &= c_2
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - c_2 - sc_1 + 2sY(s) - 2c_1 + 10Y(s) = 1$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{sc_1 + 2c_1 + c_2 + 1}{s^2 + 2s + 10}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{(-1+3i)\left(-\frac{c_1}{18} - \frac{c_2}{18} - \frac{1}{18}\right) + \frac{4c_1}{9} - \frac{c_2}{18} - \frac{1}{18}}{s+1-3i} + \frac{(-1-3i)\left(-\frac{c_1}{18} - \frac{c_2}{18} - \frac{1}{18}\right) + \frac{4c_1}{9} - \frac{c_2}{18} - \frac{1}{18}}{s+1+3i}$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{(-1+3i)\left(-\frac{c_1}{18} - \frac{c_2}{18} - \frac{1}{18}\right) + \frac{4c_1}{9} - \frac{c_2}{18} - \frac{1}{18}}{s+1-3i}\right) &= \frac{\mathrm{e}^{(-1+3i)t}(-i - ic_2 + (3-i)\,c_1)}{6} \\
\mathcal{L}^{-1}\left(\frac{(-1-3i)\left(-\frac{c_1}{18} - \frac{c_2}{18} - \frac{1}{18}\right) + \frac{4c_1}{9} - \frac{c_2}{18} - \frac{1}{18}}{s+1+3i}\right) &= \frac{\mathrm{e}^{(-1-3i)t}(i + ic_2 + (3+i)\,c_1)}{6}
\end{aligned}
$$

Adding the above results and simplifying gives

$$y = \frac{\mathrm{e}^{-t}(3c_1 \cos(3t) + \sin(3t)\,(c_1 + c_2 + 1))}{3}$$

Simplifying the solution gives

$$y = \frac{\mathrm{e}^{-t}(c_1 + c_2 + 1)\sin(3t)}{3} + c_1 \cos(3t)\,\mathrm{e}^{-t}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-t}(c_1 + c_2 + 1)\sin(3t)}{3} + c_1 \cos(3t)\,\mathrm{e}^{-t} \tag{1}$$

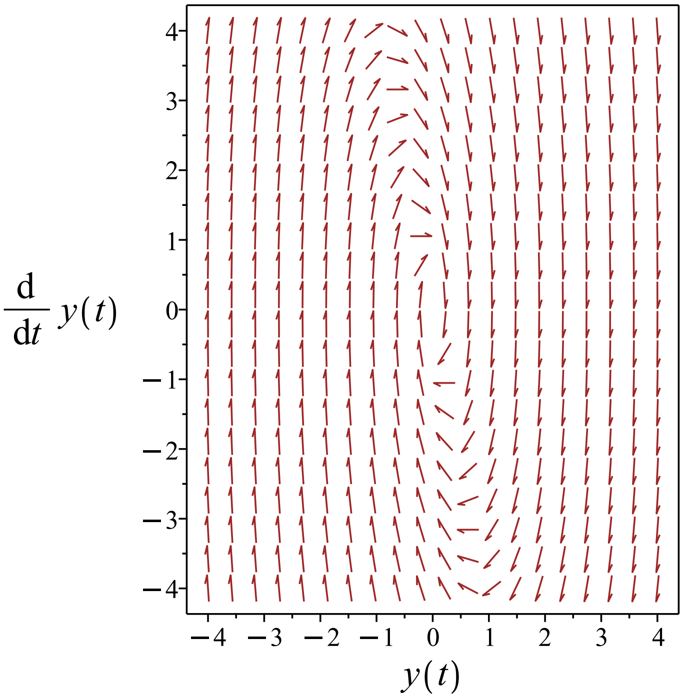


Figure 78: Slope field plot

Verification of solutions

$$y = \frac{e^{-t}(c_1 + c_2 + 1)\sin(3t)}{3} + c_1 \cos(3t)\, e^{-t}$$

Verified OK.

8.14.1 Maple step by step solution

Let's solve

$y'' + 2y' + 10y = Dirac(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-1-3\,\mathrm{I}, -1+3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(3t)\,\mathrm{e}^{-t}$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)\,\mathrm{e}^{-t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t)\,\mathrm{e}^{-t} + c_2 \sin(3t)\,\mathrm{e}^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathit{Dirac}(t)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t)\,\mathrm{e}^{-t} & \sin(3t)\,\mathrm{e}^{-t} \\ -3\sin(3t)\,\mathrm{e}^{-t} - \cos(3t)\,\mathrm{e}^{-t} & 3\cos(3t)\,\mathrm{e}^{-t} - \sin(3t)\,\mathrm{e}^{-t} \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3\,\mathrm{e}^{-2t}$
- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{\sin(3t)\mathrm{e}^{-t}\left(\int \mathit{Dirac}(t)dt\right)}{3}$
- ○ Compute integrals

  $y_p(t) = \frac{\sin(3t)\mathrm{e}^{-t}\mathit{Heaviside}(t)}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3t)\,\mathrm{e}^{-t} + c_2 \sin(3t)\,\mathrm{e}^{-t} + \frac{\sin(3t)\mathrm{e}^{-t}\mathit{Heaviside}(t)}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.86 (sec). Leaf size: 30

```
dsolve(diff(y(t),t$2)+2*diff(y(t),t)+10*y(t)=Dirac(t),y(t), singsol=all)
```

$$y(t) = \frac{\mathrm{e}^{-t}(3y(0)\cos(3t) + \sin(3t)(D(y)(0) + y(0) + 1))}{3}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 38

```
DSolve[y''[t]+2*y'[t]+10*y[t]==DiracDelta[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{3}e^{-t}(\theta(t)\sin(3t) + 3c_2\cos(3t) + 3c_1\sin(3t))$$

9 CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332

9.1 problem 1

Internal problem ID [6711]
Internal file name [`OUTPUT/5959_Sunday_June_05_2022_04_04_28_PM_35233547/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 3x(t) - 5y \\ y' &= 4x(t) + 8y \end{aligned}$$

9.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{\frac{11t}{2}} \cos\left(\frac{\sqrt{55}\,t}{2}\right) - \frac{\sqrt{55}\,\mathrm{e}^{\frac{11t}{2}} \sin\left(\frac{\sqrt{55}\,t}{2}\right)}{11} & -\frac{2\sqrt{55}\,\mathrm{e}^{\frac{11t}{2}} \sin\left(\frac{\sqrt{55}\,t}{2}\right)}{11} \\ \frac{8\sqrt{55}\,\mathrm{e}^{\frac{11t}{2}} \sin\left(\frac{\sqrt{55}\,t}{2}\right)}{55} & e^{\frac{11t}{2}} \cos\left(\frac{\sqrt{55}\,t}{2}\right) + \frac{\sqrt{55}\,\mathrm{e}^{\frac{11t}{2}} \sin\left(\frac{\sqrt{55}\,t}{2}\right)}{11} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{\frac{11t}{2}}\left(\sqrt{55}\,\sin\left(\frac{\sqrt{55}\,t}{2}\right) - 11\cos\left(\frac{\sqrt{55}\,t}{2}\right)\right)}{11} & -\frac{2\sqrt{55}\,\mathrm{e}^{\frac{11t}{2}} \sin\left(\frac{\sqrt{55}\,t}{2}\right)}{11} \\ \frac{8\sqrt{55}\,\mathrm{e}^{\frac{11t}{2}} \sin\left(\frac{\sqrt{55}\,t}{2}\right)}{55} & \frac{\mathrm{e}^{\frac{11t}{2}}\left(\sqrt{55}\,\sin\left(\frac{\sqrt{55}\,t}{2}\right) + 11\cos\left(\frac{\sqrt{55}\,t}{2}\right)\right)}{11} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{\frac{11t}{2}}\left(\sqrt{55}\,\sin\left(\frac{\sqrt{55}\,t}{2}\right) - 11\cos\left(\frac{\sqrt{55}\,t}{2}\right)\right)}{11} & -\frac{2\sqrt{55}\,\mathrm{e}^{\frac{11t}{2}} \sin\left(\frac{\sqrt{55}\,t}{2}\right)}{11} \\ \frac{8\sqrt{55}\,\mathrm{e}^{\frac{11t}{2}} \sin\left(\frac{\sqrt{55}\,t}{2}\right)}{55} & \frac{\mathrm{e}^{\frac{11t}{2}}\left(\sqrt{55}\,\sin\left(\frac{\sqrt{55}\,t}{2}\right) + 11\cos\left(\frac{\sqrt{55}\,t}{2}\right)\right)}{11} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{\frac{11t}{2}}\left(\sqrt{55}\,\sin\left(\frac{\sqrt{55}\,t}{2}\right) - 11\cos\left(\frac{\sqrt{55}\,t}{2}\right)\right)c_1}{11} - \frac{2\sqrt{55}\,\mathrm{e}^{\frac{11t}{2}} \sin\left(\frac{\sqrt{55}\,t}{2}\right)c_2}{11} \\ \frac{8\sqrt{55}\,\mathrm{e}^{\frac{11t}{2}} \sin\left(\frac{\sqrt{55}\,t}{2}\right)c_1}{55} + \frac{\mathrm{e}^{\frac{11t}{2}}\left(\sqrt{55}\,\sin\left(\frac{\sqrt{55}\,t}{2}\right) + 11\cos\left(\frac{\sqrt{55}\,t}{2}\right)\right)c_2}{11} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\left(\sqrt{55}\,(c_1+2c_2)\sin\left(\frac{\sqrt{55}\,t}{2}\right) - 11\cos\left(\frac{\sqrt{55}\,t}{2}\right)c_1\right)\mathrm{e}^{\frac{11t}{2}}}{11} \\ \frac{8\left(\sqrt{55}\,\left(c_1+\frac{5c_2}{8}\right)\sin\left(\frac{\sqrt{55}\,t}{2}\right) + \frac{55\cos\left(\frac{\sqrt{55}\,t}{2}\right)c_2}{8}\right)\mathrm{e}^{\frac{11t}{2}}}{55} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -5 \\ 4 & 8-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 11\lambda + 44 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{11}{2} + \frac{i\sqrt{55}}{2}$$
$$\lambda_2 = \frac{11}{2} - \frac{i\sqrt{55}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{11}{2} + \frac{i\sqrt{55}}{2}$	1	complex eigenvalue
$\frac{11}{2} - \frac{i\sqrt{55}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{11}{2} - \frac{i\sqrt{55}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix} - \left(\frac{11}{2} - \frac{i\sqrt{55}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -\frac{5}{2} + \frac{i\sqrt{55}}{2} & -5 \\ 4 & \frac{5}{2} + \frac{i\sqrt{55}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{5}{2}+\frac{i\sqrt{55}}{2} & -5 & 0 \\ 4 & \frac{5}{2}+\frac{i\sqrt{55}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{4R_1}{-\frac{5}{2}+\frac{i\sqrt{55}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{5}{2}+\frac{i\sqrt{55}}{2} & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{5}{2}+\frac{i\sqrt{55}}{2} & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{10t}{-5+i\sqrt{55}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{10t}{-5+\mathrm{I}\sqrt{55}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{10t}{-5+i\sqrt{55}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{10t}{-5+\mathrm{I}\sqrt{55}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{10}{-5+i\sqrt{55}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{10t}{-5+\mathrm{I}\sqrt{55}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{10}{-5+i\sqrt{55}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{10t}{-5+\mathrm{I}\sqrt{55}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{10}{-5+i\sqrt{55}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{11}{2} + \frac{i\sqrt{55}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix} - \left(\frac{11}{2} + \frac{i\sqrt{55}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{5}{2} - \frac{i\sqrt{55}}{2} & -5 \\ 4 & \frac{5}{2} - \frac{i\sqrt{55}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{5}{2} - \frac{i\sqrt{55}}{2} & -5 & 0 \\ 4 & \frac{5}{2} - \frac{i\sqrt{55}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{4R_1}{-\frac{5}{2} - \frac{i\sqrt{55}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{5}{2} - \frac{i\sqrt{55}}{2} & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{5}{2} - \frac{i\sqrt{55}}{2} & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{10t}{5+i\sqrt{55}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{10t}{5+\mathrm{I}\sqrt{55}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{10t}{5+i\sqrt{55}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{10t}{5+\mathrm{I}\sqrt{55}} \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{10}{5+i\sqrt{55}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{10t}{5+\mathrm{I}\sqrt{55}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{10}{5+i\sqrt{55}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\left[\begin{array}{c} -\frac{10t}{5+\mathrm{I}\sqrt{55}} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{10}{5+i\sqrt{55}} \\ 1 \end{array}\right]$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{11}{2}+\frac{i\sqrt{55}}{2}$	1	1	No	$\left[\begin{array}{c} -\frac{5}{\frac{5}{2}+\frac{i\sqrt{55}}{2}} \\ 1 \end{array}\right]$
$\frac{11}{2}-\frac{i\sqrt{55}}{2}$	1	1	No	$\left[\begin{array}{c} -\frac{5}{\frac{5}{2}-\frac{i\sqrt{55}}{2}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{5\,\mathrm{e}^{\left(\frac{11}{2}+\frac{i\sqrt{55}}{2}\right)t}}{\frac{5}{2}+\frac{i\sqrt{55}}{2}} \\ \mathrm{e}^{\left(\frac{11}{2}+\frac{i\sqrt{55}}{2}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{5\,\mathrm{e}^{\left(\frac{11}{2}-\frac{i\sqrt{55}}{2}\right)t}}{\frac{5}{2}-\frac{i\sqrt{55}}{2}} \\ \mathrm{e}^{\left(\frac{11}{2}-\frac{i\sqrt{55}}{2}\right)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{5ic_1\left(i+\frac{\sqrt{55}}{5}\right)\mathrm{e}^{\frac{\left(11+i\sqrt{55}\right)t}{2}}}{8} + \frac{5i\left(i-\frac{\sqrt{55}}{5}\right)c_2\mathrm{e}^{-\frac{\left(i\sqrt{55}-11\right)t}{2}}}{8} \\ c_1\mathrm{e}^{\frac{\left(11+i\sqrt{55}\right)t}{2}} + c_2\mathrm{e}^{-\frac{\left(i\sqrt{55}-11\right)t}{2}} \end{array}\right]$$

The following is the phase plot of the system.

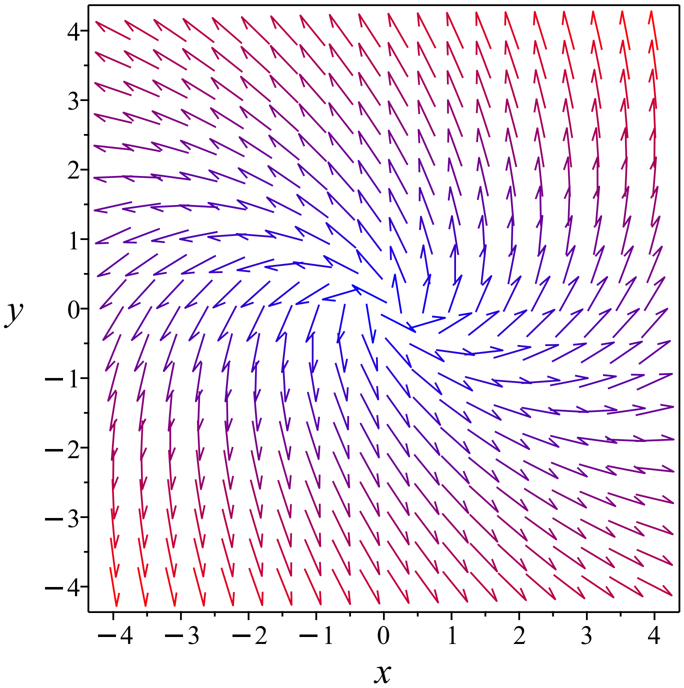


Figure 79: Phase plot

9.1.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) - 5y, y' = 4x(t) + 8y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 3 & -5 \\ 4 & 8 \end{array}\right]$$

- Rewrite the system as

  $\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[\frac{11}{2}-\frac{\mathrm{I}\sqrt{55}}{2}, \left[\begin{array}{c} -\frac{5}{\frac{5}{2}-\frac{\mathrm{I}\sqrt{55}}{2}} \\ 1 \end{array}\right]\right], \left[\frac{11}{2}+\frac{\mathrm{I}\sqrt{55}}{2}, \left[\begin{array}{c} -\frac{5}{\frac{5}{2}+\frac{\mathrm{I}\sqrt{55}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[\frac{11}{2}-\frac{\mathrm{I}\sqrt{55}}{2}, \left[\begin{array}{c} -\frac{5}{\frac{5}{2}-\frac{\mathrm{I}\sqrt{55}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{\left(\frac{11}{2}-\frac{\mathrm{I}\sqrt{55}}{2}\right)t} \cdot \left[\begin{array}{c} -\frac{5}{\frac{5}{2}-\frac{\mathrm{I}\sqrt{55}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{\frac{11t}{2}} \cdot \left(\cos\left(\frac{\sqrt{55}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{55}\,t}{2}\right)\right) \cdot \left[\begin{array}{c} -\frac{5}{\frac{5}{2}-\frac{\mathrm{I}\sqrt{55}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

  $$\mathrm{e}^{\frac{11t}{2}} \cdot \left[\begin{array}{c} -\frac{5\left(\cos\left(\frac{\sqrt{55}\,t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{55}\,t}{2}\right)\right)}{\frac{5}{2}-\frac{\mathrm{I}\sqrt{55}}{2}} \\ \cos\left(\frac{\sqrt{55}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{55}\,t}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_1(t) = \mathrm{e}^{\frac{11t}{2}} \cdot \left[\begin{array}{c} -\frac{5\cos\left(\frac{\sqrt{55}\,t}{2}\right)}{8} - \frac{\sqrt{55}\,\sin\left(\frac{\sqrt{55}\,t}{2}\right)}{8} \\ \cos\left(\frac{\sqrt{55}\,t}{2}\right) \end{array}\right], \overrightarrow{x}_2(t) = \mathrm{e}^{\frac{11t}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{55}\,t}{2}\right)\sqrt{55}}{8} + \frac{5\sin\left(\frac{\sqrt{55}\,t}{2}\right)}{8} \\ -\sin\left(\frac{\sqrt{55}\,t}{2}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{\frac{11t}{2}} \cdot \begin{bmatrix} -\frac{5\cos\left(\frac{\sqrt{55}\,t}{2}\right)}{8} - \frac{\sqrt{55}\,\sin\left(\frac{\sqrt{55}\,t}{2}\right)}{8} \\ \cos\left(\frac{\sqrt{55}\,t}{2}\right) \end{bmatrix} + c_2 \mathrm{e}^{\frac{11t}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{55}\,t}{2}\right)\sqrt{55}}{8} + \frac{5\sin\left(\frac{\sqrt{55}\,t}{2}\right)}{8} \\ -\sin\left(\frac{\sqrt{55}\,t}{2}\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{\left(\left(c_2\sqrt{55}+5c_1\right)\cos\left(\frac{\sqrt{55}\,t}{2}\right)+\sin\left(\frac{\sqrt{55}\,t}{2}\right)\left(c_1\sqrt{55}-5c_2\right)\right)\mathrm{e}^{\frac{11t}{2}}}{8} \\ \mathrm{e}^{\frac{11t}{2}}\left(\cos\left(\frac{\sqrt{55}\,t}{2}\right)c_1 - \sin\left(\frac{\sqrt{55}\,t}{2}\right)c_2\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{\left(\left(c_2\sqrt{55}+5c_1\right)\cos\left(\frac{\sqrt{55}\,t}{2}\right)+\sin\left(\frac{\sqrt{55}\,t}{2}\right)\left(c_1\sqrt{55}-5c_2\right)\right)\mathrm{e}^{\frac{11t}{2}}}{8}, y = \mathrm{e}^{\frac{11t}{2}}\left(\cos\left(\frac{\sqrt{55}\,t}{2}\right)c_1 - \sin\left(\frac{\sqrt{55}\,t}{2}\right)c_2\right)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 84

```
dsolve([diff(x(t),t)=3*x(t)-5*y(t),diff(y(t),t)=4*x(t)+8*y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{\frac{11t}{2}}\left(\sin\left(\frac{\sqrt{55}\,t}{2}\right)c_1 + \cos\left(\frac{\sqrt{55}\,t}{2}\right)c_2\right)$$

$$y(t) = \frac{\mathrm{e}^{\frac{11t}{2}}\left(\sin\left(\frac{\sqrt{55}\,t}{2}\right)\sqrt{55}\,c_2 - \cos\left(\frac{\sqrt{55}\,t}{2}\right)\sqrt{55}\,c_1 - 5\sin\left(\frac{\sqrt{55}\,t}{2}\right)c_1 - 5\cos\left(\frac{\sqrt{55}\,t}{2}\right)c_2\right)}{10}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 113

```
DSolve[{x'[t]==3*x[t]-5*y[t],y'[t]==4*x[t]+8*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{11}e^{11t/2}\left(11c_1\cos\left(\frac{\sqrt{55}t}{2}\right) - \sqrt{55}(c_1+2c_2)\sin\left(\frac{\sqrt{55}t}{2}\right)\right)$$

$$y(t) \to \frac{1}{55}e^{11t/2}\left(55c_2\cos\left(\frac{\sqrt{55}t}{2}\right) + \sqrt{55}(8c_1+5c_2)\sin\left(\frac{\sqrt{55}t}{2}\right)\right)$$

9.2 problem 2

Internal problem ID [6712]
Internal file name [`OUTPUT/5960_Sunday_June_05_2022_04_04_31_PM_78606726/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 4x(t) - 7y \\ y' &= 5x(t) \end{aligned}$$

9.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{2t}\cos\left(\sqrt{31}\,t\right)+\frac{2\,e^{2t}\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{31} & -\frac{7\,e^{2t}\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{31} \\ \frac{5\,e^{2t}\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{31} & e^{2t}\cos\left(\sqrt{31}\,t\right)-\frac{2\,e^{2t}\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{31} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{2t}\left(2\sin\left(\sqrt{31}\,t\right)\sqrt{31}+31\cos\left(\sqrt{31}\,t\right)\right)}{31} & -\frac{7\,e^{2t}\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{31} \\ \frac{5\,e^{2t}\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{31} & e^{2t}\left(\cos\left(\sqrt{31}\,t\right)-\frac{2\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{31}\right) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{2t}\left(2\sin\left(\sqrt{31}\,t\right)\sqrt{31}+31\cos\left(\sqrt{31}\,t\right)\right)}{31} & -\frac{7\,e^{2t}\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{31} \\ \frac{5\,e^{2t}\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{31} & e^{2t}\left(\cos\left(\sqrt{31}\,t\right)-\frac{2\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{31}\right) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{2t}\left(2\sin\left(\sqrt{31}\,t\right)\sqrt{31}+31\cos\left(\sqrt{31}\,t\right)\right)c_1}{31}-\frac{7\,e^{2t}\sin\left(\sqrt{31}\,t\right)\sqrt{31}\,c_2}{31} \\ \frac{5\,e^{2t}\sin\left(\sqrt{31}\,t\right)\sqrt{31}\,c_1}{31}+e^{2t}\left(\cos\left(\sqrt{31}\,t\right)-\frac{2\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{31}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2\,e^{2t}\left(\sqrt{31}\left(c_1-\frac{7c_2}{2}\right)\sin\left(\sqrt{31}\,t\right)+\frac{31\cos\left(\sqrt{31}\,t\right)c_1}{2}\right)}{31} \\ \frac{5\left(\sqrt{31}\left(c_1-\frac{2c_2}{5}\right)\sin\left(\sqrt{31}\,t\right)+\frac{31c_2\cos\left(\sqrt{31}\,t\right)}{5}\right)e^{2t}}{31} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix}4 & -7\\ 5 & 0\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix}4-\lambda & -7\\ 5 & -\lambda\end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^2-4\lambda+35=0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1&=2+i\sqrt{31}\\ \lambda_2&=2-i\sqrt{31}\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2+i\sqrt{31}$	1	complex eigenvalue
$2-i\sqrt{31}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=2-i\sqrt{31}$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\begin{aligned}\left(\begin{bmatrix}4 & -7\\ 5 & 0\end{bmatrix}-\left(2-i\sqrt{31}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}&=\begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}2+i\sqrt{31} & -7\\ 5 & i\sqrt{31}-2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}&=\begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i\sqrt{31} & -7 & 0 \\ 5 & i\sqrt{31}-2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{5R_1}{2+i\sqrt{31}} \Longrightarrow \left[\begin{array}{cc|c} 2+i\sqrt{31} & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i\sqrt{31} & -7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{7t}{2+i\sqrt{31}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{7t}{2+\mathrm{I}\sqrt{31}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{7t}{2+i\sqrt{31}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{7t}{2+\mathrm{I}\sqrt{31}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{7}{2+i\sqrt{31}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{7t}{2+\mathrm{I}\sqrt{31}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{7}{2+i\sqrt{31}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{7t}{2+\mathrm{I}\sqrt{31}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{7}{2+i\sqrt{31}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + i\sqrt{31}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix} - \left(2 + i\sqrt{31}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 - i\sqrt{31} & -7 \\ 5 & -2 - i\sqrt{31} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 - i\sqrt{31} & -7 & 0 \\ 5 & -2 - i\sqrt{31} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{5R_1}{2 - i\sqrt{31}} \Longrightarrow \left[\begin{array}{cc|c} 2 - i\sqrt{31} & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 - i\sqrt{31} & -7 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{7t}{i\sqrt{31}-2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{7t}{\mathrm{I}\sqrt{31}-2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7t}{i\sqrt{31}-2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{7t}{\mathrm{I}\sqrt{31}-2} \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{7}{i\sqrt{31}-2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{7t}{\mathrm{I}\sqrt{31}-2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7}{i\sqrt{31}-2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\left[\begin{array}{c} -\frac{7t}{\mathrm{I}\sqrt{31}-2} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{7}{i\sqrt{31}-2} \\ 1 \end{array}\right]$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2 + i\sqrt{31}$	1	1	No	$\left[\begin{array}{c} -\frac{7}{i\sqrt{31}-2} \\ 1 \end{array}\right]$
$2 - i\sqrt{31}$	1	1	No	$\left[\begin{array}{c} -\frac{7}{-2-i\sqrt{31}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{7\,\mathrm{e}^{\left(2+i\sqrt{31}\right)t}}{i\sqrt{31}-2} \\ \mathrm{e}^{\left(2+i\sqrt{31}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{7\,\mathrm{e}^{\left(2-i\sqrt{31}\right)t}}{-2-i\sqrt{31}} \\ \mathrm{e}^{\left(2-i\sqrt{31}\right)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} -\frac{2i\left(i+\frac{\sqrt{31}}{2}\right)c_2\mathrm{e}^{-\left(i\sqrt{31}-2\right)t}}{5} - \frac{2ic_1\left(i-\frac{\sqrt{31}}{2}\right)\mathrm{e}^{\left(2+i\sqrt{31}\right)t}}{5} \\ c_1\mathrm{e}^{\left(2+i\sqrt{31}\right)t} + c_2\mathrm{e}^{-\left(i\sqrt{31}-2\right)t} \end{array}\right]$$

The following is the phase plot of the system.

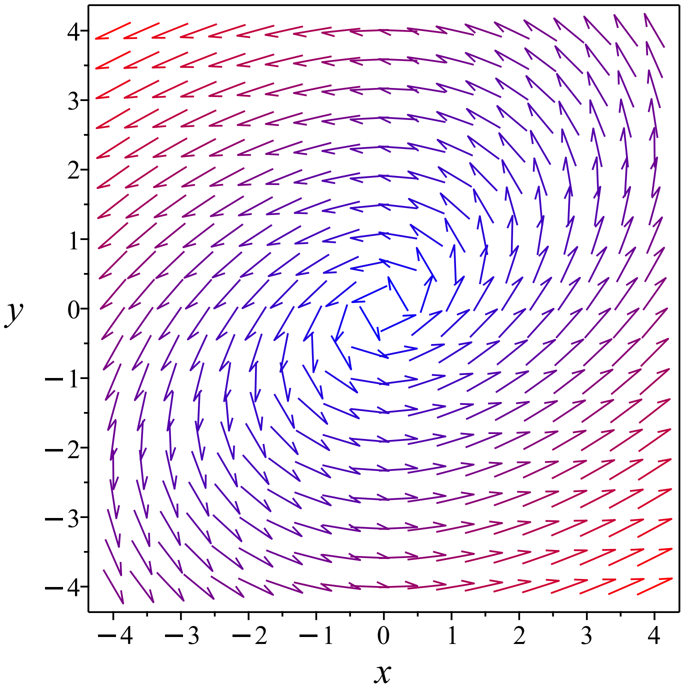


Figure 80: Phase plot

9.2.3 Maple step by step solution

Let's solve

$[x'(t) = 4x(t) - 7y, y' = 5x(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[2-\mathrm{I}\sqrt{31}, \begin{bmatrix} -\frac{7}{-2-\mathrm{I}\sqrt{31}} \\ 1 \end{bmatrix}\right], \left[2+\mathrm{I}\sqrt{31}, \begin{bmatrix} -\frac{7}{\mathrm{I}\sqrt{31}-2} \\ 1 \end{bmatrix}\right]\right]$$
- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[2-\mathrm{I}\sqrt{31}, \begin{bmatrix} -\frac{7}{-2-\mathrm{I}\sqrt{31}} \\ 1 \end{bmatrix}\right]$$
- Solution from eigenpair

  $$\mathrm{e}^{\left(2-\mathrm{I}\sqrt{31}\right)t} \cdot \begin{bmatrix} -\frac{7}{-2-\mathrm{I}\sqrt{31}} \\ 1 \end{bmatrix}$$
- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{2t} \cdot \left(\cos\left(\sqrt{31}\,t\right) - \mathrm{I}\sin\left(\sqrt{31}\,t\right)\right) \cdot \begin{bmatrix} -\frac{7}{-2-\mathrm{I}\sqrt{31}} \\ 1 \end{bmatrix}$$
- Simplify expression

  $$\mathrm{e}^{2t} \cdot \begin{bmatrix} -\frac{7\left(\cos\left(\sqrt{31}\,t\right)-\mathrm{I}\sin\left(\sqrt{31}\,t\right)\right)}{-2-\mathrm{I}\sqrt{31}} \\ \cos\left(\sqrt{31}\,t\right) - \mathrm{I}\sin\left(\sqrt{31}\,t\right) \end{bmatrix}$$
- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\vec{x}_1(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{2\cos\left(\sqrt{31}\,t\right)}{5} - \frac{\sin\left(\sqrt{31}\,t\right)\sqrt{31}}{5} \\ \cos\left(\sqrt{31}\,t\right) \end{bmatrix}, \vec{x}_2(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} -\frac{\cos\left(\sqrt{31}\,t\right)\sqrt{31}}{5} - \frac{2\sin\left(\sqrt{31}\,t\right)}{5} \\ -\sin\left(\sqrt{31}\,t\right) \end{bmatrix}\right]$$
- General solution to the system of ODEs

  $\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$
- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{2\cos\left(\sqrt{31}\, t\right)}{5} - \frac{\sin\left(\sqrt{31}\, t\right)\sqrt{31}}{5} \\ \cos\left(\sqrt{31}\, t\right) \end{bmatrix} + c_2 \mathrm{e}^{2t} \cdot \begin{bmatrix} -\frac{\cos\left(\sqrt{31}\, t\right)\sqrt{31}}{5} - \frac{2\sin\left(\sqrt{31}\, t\right)}{5} \\ -\sin\left(\sqrt{31}\, t\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{\left(\left(c_2\sqrt{31}-2c_1\right)\cos\left(\sqrt{31}\, t\right)+\sin\left(\sqrt{31}\, t\right)\left(c_1\sqrt{31}+2c_2\right)\right)\mathrm{e}^{2t}}{5} \\ \mathrm{e}^{2t}\left(\cos\left(\sqrt{31}\, t\right) c_1 - \sin\left(\sqrt{31}\, t\right) c_2\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{\left(\left(c_2\sqrt{31}-2c_1\right)\cos\left(\sqrt{31}\, t\right)+\sin\left(\sqrt{31}\, t\right)\left(c_1\sqrt{31}+2c_2\right)\right)\mathrm{e}^{2t}}{5}, y = \mathrm{e}^{2t}\left(\cos\left(\sqrt{31}\, t\right) c_1 - \sin\left(\sqrt{31}\, t\right) c_2\right)\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 78

```
dsolve([diff(x(t),t)=4*x(t)-7*y(t),diff(y(t),t)=5*x(t)],singsol=all)
```

$$x(t) = -\frac{\mathrm{e}^{2t}\left(\sqrt{31}\sin\left(\sqrt{31}\, t\right) c_2 - \sqrt{31}\cos\left(\sqrt{31}\, t\right) c_1 - 2\sin\left(\sqrt{31}\, t\right) c_1 - 2\cos\left(\sqrt{31}\, t\right) c_2\right)}{5}$$

$$y(t) = \mathrm{e}^{2t}\left(\sin\left(\sqrt{31}\, t\right) c_1 + \cos\left(\sqrt{31}\, t\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 98

```
DSolve[{x'[t]==4*x[t]-7*y[t],y'[t]==5*x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{2t}\cos\left(\sqrt{31}t\right) + \frac{(2c_1 - 7c_2)e^{2t}\sin\left(\sqrt{31}t\right)}{\sqrt{31}}$$

$$y(t) \to c_2 e^{2t}\cos\left(\sqrt{31}t\right) + \frac{(5c_1 - 2c_2)e^{2t}\sin\left(\sqrt{31}t\right)}{\sqrt{31}}$$

9.3 problem 3

Internal problem ID [6713]
Internal file name [`OUTPUT/5961_Sunday_June_05_2022_04_04_34_PM_15973523/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x'(t) &= -3x(t) + 4y - 9z(t) \\
y' &= 6x(t) - y \\
z'(t) &= 10x(t) + 4y + 3z(t)
\end{aligned}$$

9.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned}
e^{At} &= \text{Expression too large to display} \\
&= \text{Expression too large to display}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \text{Expression too large to display}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \text{Expression too large to display} \\
&= \text{Expression too large to display}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 4 & -9 \\ 6 & -1-\lambda & 0 \\ 10 & 4 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3+\lambda^2+57\lambda+369=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}$$

$$\lambda_2=\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6}-\frac{85}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}+\frac{i\sqrt{3}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}-\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_3=\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6}-\frac{85}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}-\frac{i\sqrt{3}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}-\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right)}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity
$\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6}-\frac{85}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}+\frac{i\sqrt{3}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}-\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right)}{2}$	1
$\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6}-\frac{85}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}-\frac{i\sqrt{3}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}-\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right)}{2}$	1
$-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}$	1

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{bmatrix} - \left(-\frac{\left(4726 + 306\sqrt{291}\right)^{\frac{1}{3}}}{3} + \frac{170}{3\left(4726 + 306\sqrt{291}\right)^{\frac{1}{3}}} - \frac{1}{3}\right)$$

$$\begin{bmatrix} \frac{\left(4726+306\sqrt{291}\right)^{\frac{2}{3}} - 8\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} - 170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} & 4 & -9 \\ 6 & -\frac{2\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}}{2} + \left(4726+306\sqrt{291}\right)^{\frac{1}{3}} + 85\right)}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} & 0 \\ 10 & 4 & \frac{\left(4726+306\sqrt{291}\right)^{\frac{2}{3}} + 10\left(4726+\right.}{3\left(4726+306\sqrt{291}\right.} \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} -\frac{8}{3} + \frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3} - \frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} & 4 & - \\ 6 & -\frac{2}{3} + \frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3} - \frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} & 0 \\ 10 & 4 & \frac{10}{3} + \frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3} \end{bmatrix}$$

$$R_2 = R_2 - \frac{6R_1}{-\frac{8}{3} + \frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3} - \frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}} \Longrightarrow \begin{bmatrix} \frac{\left(4726+306\sqrt{291}\right)^{\frac{2}{3}} - 8\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} - 170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} & \\ 0 & -\frac{102\left(\left(\sqrt{\right.\right.}}{\left(4726+\right.} \\ 10 & \end{bmatrix}$$

$$R_3 = R_3 - \frac{30\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} R_1}{\left(4726+306\sqrt{291}\right)^{\frac{2}{3}} - 8\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} - 170} \Longrightarrow \left[\begin{array}{c} \frac{\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-8\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-17}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} \\ 0 \\ 0 \end{array}\right.$$

$$R_3 = R_3 + \frac{\left(4\left(4726+306\sqrt{291}\right)^{\frac{2}{3}} - 152\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} - 680\right)\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}\left(-\left(4726+306\right.\right.}{102\left(\left(4726+306\sqrt{291}\right)^{\frac{2}{3}} - 8\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} - 170\right)\left(\left(\sqrt{291}+21\right)\left(4726+306\sqrt{291}\right.\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{ccc} \frac{\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-8\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} & 4 & \\ 0 & -\frac{102\left(\left(\sqrt{291}+21\right)\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-10\sqrt{291}-\frac{30\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}}{17}-60\right)}{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}\left(-\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}+8\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\right)} & \frac{}{\left(4726+\right.} \\ 0 & 0 & \end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{27\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} t\left(18\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-35\right.}{\left(30\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+102\right.} \right.$

Hence the solution is

$$\left[\begin{matrix} \frac{27\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} t\left(18\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020\right)}{\left(30\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020\right)\Bigg(\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-8\left(4726+306\sqrt{291}\right)} \\ \frac{27\left(4726+306\sqrt{291}\right)^{\frac{2}{3}} t}{30\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020} \\ t \end{matrix}\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{matrix} \frac{27\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} t\left(18\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020\right)}{\left(30\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020\right)\Bigg(\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-8\left(4726+306\sqrt{291}\right)} \\ \frac{27\left(4726+306\sqrt{291}\right)^{\frac{2}{3}} t}{30\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020} \\ t \end{matrix}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{matrix} \frac{27\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} t\left(18\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020\right)}{\left(30\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020\right)\Bigg(\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-8\left(4726+306\sqrt{291}\right)} \\ \frac{27\left(4726+306\sqrt{291}\right)^{\frac{2}{3}} t}{30\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020} \\ t \end{matrix}\right.$$

Which is normalized to

$$\left[\begin{matrix} \frac{27\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} t\left(18\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020\right)}{\left(30\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020\right)\Bigg(\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-8\left(4726+306\sqrt{291}\right)} \\ \frac{27\left(4726+306\sqrt{291}\right)^{\frac{2}{3}} t}{30\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-17\sqrt{291}\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-357\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+170\sqrt{291}+1020} \\ t \end{matrix}\right.$$

Considering the eigenvalue $\lambda_2 = \frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6} - \frac{85}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} - \frac{1}{3} - \frac{i\sqrt{3}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3} - \frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right)}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{bmatrix} - \left(\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6} - \frac{85}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right.\right.$$

$$\begin{bmatrix} -\frac{\left(1+i\sqrt{3}\right)\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}}+170i\sqrt{3}+16\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}-170}{6\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}} & 4 \\ 6 & -\frac{\left(1+i\sqrt{3}\right)\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}}+170i\sqrt{3}+4\left(4726+306\sqrt{3}\right.}{6\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}} \\ 10 & 4 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} -\frac{8}{3} - \frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6} + \frac{85}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3} - \frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right)}{2} & \\ 6 & -\frac{2}{3} - \frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6} + \\ 10 & \end{bmatrix}$$

$$R_2 = R_2 - \frac{6R_1}{-\frac{8}{3} - \frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6} + \frac{85}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3} - \frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow$$

$$R_3 = R_3 + \frac{60\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}} R_1}{\left(1+i\sqrt{3}\right)\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}} + 170i\sqrt{3} + 16\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}} - 170} \Longrightarrow$$

$$R_3 = R_3 - \frac{\left(\left(4i\sqrt{3}+4\right)\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}} + 680i\sqrt{3} + 304\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}} - 680\right)\left(170 -\right.}{2142\left(i\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}}\sqrt{3} + \left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}} + 170i\sqrt{3} + 16\left(4726+306\sqrt{3}\,\sqrt{\right.\right.}}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{cc} -\frac{\left(1+i\sqrt{3}\right)\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}}+170i\sqrt{3}+16\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}-170}{6\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}} & 4 \\ 0 & \frac{2142\left(i\sqrt{3}+\frac{i\sqrt{97}}{7}-\frac{\sqrt{291}}{21}-1\right)\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+6120i\sqrt{3}+3}{\left(170-16\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-i\left(\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}+170\right.\right.} \\ 0 & 0 \end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{54(4726+306\sqrt{291})^{\frac{1}{3}} t (-357i(4726+306\sqrt{291})^{\frac{1}{3}}\sqrt{3}-51i(}{(-357i(4726+306\sqrt{291})^{\frac{1}{3}}\sqrt{3}-51i(4726+306\sqrt{291})^{\frac{1}{3}}\sqrt{97}+17\sqrt{291}\,(4726+306\sqrt{291})^{\frac{1}{3}}+6}\right.$

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} -\frac{54(4726+306\sqrt{291})^{\frac{1}{3}} t (-357\,\mathrm{I}(4726+306\sqrt{291})^{\frac{1}{3}}\sqrt{3}-51\,\mathrm{I}(4726+306\sqrt{291})^{\frac{1}{3}}\sqrt{97}+17\sqrt{291}\,(4726+30}{(-357\,\mathrm{I}(4726+306\sqrt{291})^{\frac{1}{3}}\sqrt{3}-51\,\mathrm{I}(4726+306\sqrt{291})^{\frac{1}{3}}\sqrt{97}+17\sqrt{291}\,(4726+306\sqrt{291})^{\frac{1}{3}}+60(4726+306\sqrt{291})^{\frac{2}{3}}-1020\,\mathrm{I}\sqrt{3}-510\,\mathrm{I}\sqrt{9}} \\ \frac{54(4726+306\sqrt{}}{-357\,\mathrm{I}(4726+306\sqrt{3}\sqrt{97})^{\frac{1}{3}}\sqrt{3}-51\,\mathrm{I}(4726+306\sqrt{3}\sqrt{97})^{\frac{1}{3}}\sqrt{97}+17(4726+306\sqrt{3}\sqrt{97})^{\frac{1}{3}}\sqrt{3}\sqrt{97}+} \\ t \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

Considering the eigenvalue $\lambda_3 = \frac{(4726+306\sqrt{291})^{\frac{1}{3}}}{6} - \frac{85}{3(4726+306\sqrt{291})^{\frac{1}{3}}} - \frac{1}{3} + \frac{i\sqrt{3}\left(-\frac{(4726+306\sqrt{291})^{\frac{1}{3}}}{3} - \frac{170}{3(4726+306\sqrt{291})^{\frac{1}{3}}}\right.}{2}$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{bmatrix} - \left(\frac{\left(4726 + 306\sqrt{291}\right)^{\frac{1}{3}}}{6} - \frac{85}{3\left(4726 + 306\sqrt{291}\right)^{\frac{1}{3}}} - \frac{1}{3}\right.\right.$$

$$\left[\begin{matrix} \frac{\left(i\sqrt{3}-1\right)\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}}+170i\sqrt{3}-16\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}+170}{6\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}} & 4 \\ 6 & \frac{\left(i\sqrt{3}-1\right)\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}}+170i\sqrt{3}-4\left(4726+306\sqrt{3}\,\sqrt{97}\right)}{6\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}} \\ 10 & 4 \end{matrix}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{matrix} -\frac{8}{3} - \frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6} + \frac{85}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3} - \frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right)}{2} & \\ 6 & -\frac{2}{3} - \frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6} + \\ 10 & \end{matrix}\right.$$

$$R_2 = R_2 - \frac{6R_1}{-\frac{8}{3} - \frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6} + \frac{85}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3} - \frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow \left[\begin{matrix} \frac{\left(i\sqrt{3}-1\right.}{} \\ \\ \end{matrix}\right.$$

$$R_3 = R_3 - \frac{60\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}} R_1}{\left(i\sqrt{3}-1\right)\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}}+170i\sqrt{3}-16\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}+170} \Longrightarrow \Bigg[\ \left(i\sqrt{3}-1\right)$$

$$R_3 = R_3 - \frac{\Big(\left(4i\sqrt{3}-4\right)\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}}+680i\sqrt{3}-304\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}+680\Big)\ (4726+}{2142\ \Big(i\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}}\sqrt{3}-\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}}+170i\sqrt{3}-16\ (4726+306\sqrt{3}\,\sqrt{9}}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{cc} \frac{\left(i\sqrt{3}-1\right)\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{2}{3}}+170i\sqrt{3}-16\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}+170}{6\left(4726+306\sqrt{3}\,\sqrt{97}\right)^{\frac{1}{3}}} & 4 \\ 0 & \frac{2142\left(i\sqrt{3}+\frac{i\sqrt{97}}{7}+\frac{\sqrt{291}}{21}+1\right)\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}+6120i\sqrt{3}+306}{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}\Big(-170+16\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-i\Big(\big(4726} \\ 0 & 0 \end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{18\left(4726+306\sqrt{291}\right)^{\frac{1}{3}} t\Big(8i\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}\sqrt{3}+9i\ (4726+30}{\Big(-4i\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}\sqrt{3}+3i\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}\sqrt{97}+\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}\sqrt{291}-635i\ (4726} \right.$

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\left[\begin{array}{c} \frac{18(4726+306\sqrt{291})^{\frac{1}{3}} t (8 \mathrm{I}(4726+306\sqrt{291})^{\frac{2}{3}} \sqrt{3}+9 \mathrm{I}(4726+306\sqrt{291})^{\frac{2}{3}} \sqrt{97}+3(4726+306\sqrt{291})^{\frac{2}{3}} \sqrt{291}}{(-4 \mathrm{I}(4726+306\sqrt{291})^{\frac{2}{3}} \sqrt{3}+3 \mathrm{I} \sqrt{97}\, (4726+306\sqrt{291})^{\frac{2}{3}}+(4726+306\sqrt{291})^{\frac{2}{3}} \sqrt{291}-635 \mathrm{I}(4726+306\sqrt{291})^{\frac{1}{3}} \sqrt{3}-105 \mathrm{I}(4726+306\sqrt{291})} \\ \frac{3(-85 \mathrm{I}(4726+306\sqrt{291})^{\frac{1}{3}} \sqrt{3}+8(4726+3}{17(-4 \mathrm{I}(4726+306\sqrt{291})^{\frac{2}{3}} \sqrt{3}+3 \mathrm{I} \sqrt{97}\, (4726+306\sqrt{291})^{\frac{2}{3}}+(4726+306\sqrt{291})^{\frac{2}{3}} \sqrt{291}-635 \mathrm{I}(} \\ \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity	
	algebraic m	geometric
$-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}$	1	1
$\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{6}-\frac{85}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}+\frac{i\sqrt{3}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}-\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right)}{2}$	1	1
$\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}$ $i\sqrt{3}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}-\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right)$		

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t} \\
&= \left[\begin{array}{c}
-\frac{135\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)}{\left(-\frac{11\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{1870}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}+\frac{412}{3}+2\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)^{2}\right)\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}-\right.} \\
\frac{\frac{9\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)^{2}}{2}+3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-\frac{510}{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}+\frac{573}{2}}{-\frac{11\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{1870}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}+\frac{412}{3}+2\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)^{2}} \\
1
\end{array}\right.
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = c_1 \left[\begin{array}{c}
-\frac{135\,\mathrm{e}^{\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t}\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)}\right.}{\left(-\frac{11\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{1870}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}+\frac{412}{3}+2\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)^{2}\right)\left(-\frac{\left(4726+306\right.}{3}\right.} \\
\frac{9\,\mathrm{e}^{\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t}\left(\left(\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)^{2}+\frac{2\left(4726+306\sqrt{291}\right.}{3}\right.\right.}{2\left(-\frac{11\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{1870}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}+\frac{412}{3}+2\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}\right.\right.} \\
\mathrm{e}^{\left(-\frac{\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}{3}+\frac{170}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}-\frac{1}{3}\right)t}
\end{array}\right.
$$

Which becomes

Expression too large to display

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 2255

```
dsolve([diff(x(t),t)=-3*x(t)+4*y(t)-9*z(t),diff(y(t),t)=6*x(t)-y(t),diff(z(t),t)=10*x(t)+4*y
```

Expression too large to display

$$y(t) = \cos\left(\frac{\left(\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}+170\right) t\sqrt{3}\, 1156^{\frac{1}{3}}}{204\left(139+9\sqrt{291}\right)^{\frac{1}{3}}}\right) \mathrm{e}^{\frac{\left(-170+\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-2\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}\right)t}{6\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}} c_3 + \mathrm{e}^{\frac{\left(-170+\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}-2\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}\right)t}{6\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}} \sin\left(\frac{\left(\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}+170\right) t\sqrt{3}\, 1156^{\frac{1}{3}}}{204\left(139+9\sqrt{291}\right)^{\frac{1}{3}}}\right) c_2 + c_1 \mathrm{e}^{-\frac{\left(\left(4726+306\sqrt{291}\right)^{\frac{2}{3}}+\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}-170\right)t}{3\left(4726+306\sqrt{291}\right)^{\frac{1}{3}}}}$$

Expression too large to display

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 510

```
DSolve[{x'[t]==-3*x[t]+4*y[t]-9*z[t],y'[t]==6*x[t]-y[t],z'[t]==10*x[t]+4*y[t]+3*z[t]},{x[t],
```

$$
\begin{aligned}
x(t) \to\ & 4c_2 \text{RootSum}\left[\#1^3+\#1^2+57\#1+369\&, \frac{\#1e^{\#1t}-12e^{\#1t}}{3\#1^2+2\#1+57}\&\right] \\
& - 9c_3 \text{RootSum}\left[\#1^3+\#1^2+57\#1+369\&, \frac{\#1e^{\#1t}+e^{\#1t}}{3\#1^2+2\#1+57}\&\right] \\
& + c_1 \text{RootSum}\left[\#1^3+\#1^2+57\#1+369\&, \frac{\#1^2e^{\#1t}-2\#1e^{\#1t}-3e^{\#1t}}{3\#1^2+2\#1+57}\&\right] \\
y(t) \to\ & -54c_3 \text{RootSum}\left[\#1^3+\#1^2+57\#1+369\&, \frac{e^{\#1t}}{3\#1^2+2\#1+57}\&\right] \\
& + 6c_1 \text{RootSum}\left[\#1^3+\#1^2+57\#1+369\&, \frac{\#1e^{\#1t}-3e^{\#1t}}{3\#1^2+2\#1+57}\&\right] \\
& + c_2 \text{RootSum}\left[\#1^3+\#1^2+57\#1+369\&, \frac{\#1^2e^{\#1t}+81e^{\#1t}}{3\#1^2+2\#1+57}\&\right] \\
z(t) \to\ & 4c_2 \text{RootSum}\left[\#1^3+\#1^2+57\#1+369\&, \frac{\#1e^{\#1t}+13e^{\#1t}}{3\#1^2+2\#1+57}\&\right] \\
& + 2c_1 \text{RootSum}\left[\#1^3+\#1^2+57\#1+369\&, \frac{5\#1e^{\#1t}+17e^{\#1t}}{3\#1^2+2\#1+57}\&\right] \\
& + c_3 \text{RootSum}\left[\#1^3+\#1^2+57\#1+369\&, \frac{\#1^2e^{\#1t}+4\#1e^{\#1t}-21e^{\#1t}}{3\#1^2+2\#1+57}\&\right]
\end{aligned}
$$

9.4 problem 4

Internal problem ID [6714]
Internal file name [`OUTPUT/5962_Sunday_June_05_2022_04_04_45_PM_83937430/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= x(t) - y \\ y' &= x(t) + 2z(t) \\ z'(t) &= -x(t) + z(t) \end{aligned}$$

9.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \text{Expression too large to display} \\ &= \text{Expression too large to display} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c}\\ &= \text{Expression too large to display}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\\ &= \text{Expression too large to display}\\ &= \text{Expression too large to display}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 & 0 \\ 1 & -\lambda & 2 \\ -1 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 2\lambda^2 + 2\lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} - \frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} + \frac{2}{3}$$

$$\lambda_2 = -\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12} + \frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} + \frac{2}{3} + \frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} + \frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_3 = -\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12} + \frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} + \frac{2}{3} - \frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} + \frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type o
$-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12} + \frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} + \frac{2}{3} - \frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} + \frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}$	1	compl
$-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12} + \frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} + \frac{2}{3} + \frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} + \frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}$	1	compl
$\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} - \frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} + \frac{2}{3}$	1	real ei

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} - \frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} + \frac{2}{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} - \frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} + \frac{2}{3}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} \frac{-\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+8}{6\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} & -1 & 0 \\ 1 & \frac{-\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-4\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+8}{6\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} & 2 \\ -1 & 0 & \frac{-\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+8}{6\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} \frac{1}{3} - \frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} + \frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} & -1 & 0 \\ 1 & -\frac{2}{3} - \frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} + \frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} & 2 \\ -1 & 0 & \frac{1}{3} - \frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} + \frac{4}{3\left(244+12\right.} \end{bmatrix}$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{3} - \frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6} + \frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}} \Longrightarrow \begin{bmatrix} \frac{-\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+8}{6\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} \\ 0 & \frac{2\left(\sqrt{417}+19\right)\left(244+12\right.}{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\left(-\left(\right.\right.} \\ -1 \end{bmatrix}$$

$$R_3 = R_3 + \frac{6\left(244 + 12\sqrt{417}\right)^{\frac{1}{3}} R_1}{-\left(244 + 12\sqrt{417}\right)^{\frac{2}{3}} + 2\left(244 + 12\sqrt{417}\right)^{\frac{1}{3}} + 8} \Longrightarrow \begin{bmatrix} \frac{-\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+8}{6\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} & \\ 0 & \frac{2\left(\sqrt{41}\right.}{\left(244+\right.} \\ 0 & \end{bmatrix}$$

$$R_3 = R_3 - \frac{3\left(244 + 12\sqrt{417}\right)^{\frac{2}{3}} \left(-\left(244 + 12\sqrt{417}\right)^{\frac{2}{3}} + 2\left(244 + 12\sqrt{417}\right)^{\frac{1}{3}} + 8\right) R_2}{\left(\left(244 + 12\sqrt{417}\right)^{\frac{2}{3}} - 2\left(244 + 12\sqrt{417}\right)^{\frac{1}{3}} - 8\right) \left(\left(\sqrt{417} + 19\right)\left(244 + 12\sqrt{417}\right)^{\frac{1}{3}} + 2\sqrt{417} + \left(\right.\right.}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{-\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+8}{6\left(244+12\sqrt{417}\right)^{\frac{1}{3}}} & -1 & 0 \\ 0 & \frac{2\left(\sqrt{417}+19\right)\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+4\sqrt{417}+2\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+92}{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\left(-\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+8\right)} & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{\left(3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}-61\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-16\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+32\right)t}{96}, v_2 = -\frac{}{\sqrt{417}\left(\right.}\right.$

Hence the solution is

$$\begin{bmatrix} \frac{\left(3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}-61\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-16\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+32\right)t}{96} \\ -\frac{2t\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+4\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-6\sqrt{417}-122\right)}{\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2\sqrt{417}+19\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+46} \\ t \end{bmatrix} = \begin{bmatrix} \frac{\left(3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}-61\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\right.}{96} \\ -\frac{2t\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+4\left(244+12\sqrt{4}\right.\right.}{\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c}\frac{\left(3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}-61\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-16\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+32\right)t}{96}\\ -\frac{2t\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+4\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-6\sqrt{417}-122\right)}{\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2\sqrt{417}+19\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+46}\\ t\end{array}\right]=t\left[\begin{array}{c}\frac{\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}}{32}-\frac{61\left(244+12\sqrt{417}\right)}{96}\\ -\frac{2\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+4\left(244+12\sqrt{4\right.\right.}}{\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2}\\ 1\end{array}\right.$$

Let $t=1$ the eigenvector becomes

$$\left[\begin{array}{c}\frac{\left(3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}-61\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-16\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+32\right)t}{96}\\ -\frac{2t\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+4\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-6\sqrt{417}-122\right)}{\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2\sqrt{417}+19\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+46}\\ t\end{array}\right]=\left[\begin{array}{c}\frac{\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}}{32}-\frac{61\left(244+12\sqrt{417}\right)}{96}\\ -\frac{2\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+4\left(244+12\sqrt{4\right.\right.}}{\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2}\\ 1\end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c}\frac{\left(3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}-61\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-16\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+32\right)t}{96}\\ -\frac{2t\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+4\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-6\sqrt{417}-122\right)}{\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2\sqrt{417}+19\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+46}\\ t\end{array}\right]=\left[\begin{array}{c}\frac{\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}}{32}-\frac{61\left(244+12\sqrt{417}\right)}{96}\\ -\frac{2\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+4\left(244+12\sqrt{4\right.\right.}}{\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+2}\\ 1\end{array}\right.$$

Considering the eigenvalue $\lambda_2=-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}-\frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{array}\right] - \left(-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12} + \right.\right.$$

$$\left[\begin{array}{cc} \frac{i\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}\sqrt{3}+\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}+8i\sqrt{3}+4\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}-8}{12\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}} & -1 \\ 1 & \frac{i\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}\sqrt{3}+\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}+8i\sqrt{3}}{12\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}} \\ -1 & 0 \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc} \frac{1}{3}+\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}-\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2} & \\ 1 & -\frac{2}{3}+\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}-\frac{2}{3\left(244+12\sqrt{4\right.}} \\ -1 & \end{array}\right.$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{3}+\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}-\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow \left[\begin{array}{c} \frac{i\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}}{} \\ \\ \end{array}\right.$$

$$R_3 = R_3 + \frac{12\left(244 + 12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}} R_1}{i \left(244 + 12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}} \sqrt{3} + \left(244 + 12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}} + 8i\sqrt{3} + 4 \left(244 + 12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}} - 8} \Longrightarrow$$

$$R_3 = R_3 + \frac{6\left(244 + 12\sqrt{417}\right)^{\frac{2}{3}} \left(-8 + 4\left(244 + 12\sqrt{417}\right)^{\frac{1}{3}} + i\right.}{\left(i \left(244 + 12\sqrt{417}\right)^{\frac{2}{3}} \sqrt{3} + \left(244 + 12\sqrt{417}\right)^{\frac{2}{3}} + 8i\sqrt{3} + 4 \left(244 + 12\sqrt{417}\right)^{\frac{1}{3}} - 8\right) \left(\left(19i\sqrt{3} + 3\right.\right.}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{i\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}\sqrt{3}+\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}+8i\sqrt{3}+4\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}-8}{12\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}} & -1 \\ 0 & \frac{2\left(19i\sqrt{3}+3i\sqrt{139}-\sqrt{417}-19\right)\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-92i\sqrt{}}{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\left(-8+4\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+i\left(\left(244\right.\right.\right.} \\ 0 & 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = -\frac{\left(61i\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{3}-9i\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{139}+3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}-16i\left(244+12\right.\right.}{192} \right.$

Hence the solution is

$$\begin{bmatrix} -\frac{\left(61\,\mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{3}-9\,\mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{139}+3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}-16\,\mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\sqrt{3}-61\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-16\left(244\right.\right.}{192} \\ -\frac{4t\left(2\,\mathrm{I}\sqrt{3}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+61\,\mathrm{I}\sqrt{3}+3\sqrt{417}+9\,\mathrm{I}\sqrt{139}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-2\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+61\right)}{3\,\mathrm{I}\sqrt{139}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+19\,\mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\sqrt{3}-\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-46\,\mathrm{I}\sqrt{3}-6\,\mathrm{I}\sqrt{139}-2\sqrt{417}+2\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-19\left(244\right.} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} -\frac{\left(61 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{3}-9 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{139}+3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}-16 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\sqrt{3}-61\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-16(244\right.}{192} \\ -\frac{4t\left(2 \mathrm{I}\sqrt{3}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+61 \mathrm{I}\sqrt{3}+3\sqrt{417}+9 \mathrm{I}\sqrt{139}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-2\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+61\right)}{3 \mathrm{I}\sqrt{139}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+19 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\sqrt{3}-\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-46 \mathrm{I}\sqrt{3}-6 \mathrm{I}\sqrt{139}-2\sqrt{417}+2\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-19(244} \\ t \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} -\frac{\left(61 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{3}-9 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{139}+3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}-16 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\sqrt{3}-61\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-16(244\right.}{192} \\ -\frac{4t\left(2 \mathrm{I}\sqrt{3}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+61 \mathrm{I}\sqrt{3}+3\sqrt{417}+9 \mathrm{I}\sqrt{139}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-2\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+61\right)}{3 \mathrm{I}\sqrt{139}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+19 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\sqrt{3}-\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-46 \mathrm{I}\sqrt{3}-6 \mathrm{I}\sqrt{139}-2\sqrt{417}+2\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-19(244} \\ t \end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c} -\frac{\left(61 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{3}-9 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{139}+3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}-16 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\sqrt{3}-61\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-16(244\right.}{192} \\ -\frac{4t\left(2 \mathrm{I}\sqrt{3}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+61 \mathrm{I}\sqrt{3}+3\sqrt{417}+9 \mathrm{I}\sqrt{139}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-2\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+61\right)}{3 \mathrm{I}\sqrt{139}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+19 \mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\sqrt{3}-\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-46 \mathrm{I}\sqrt{3}-6 \mathrm{I}\sqrt{139}-2\sqrt{417}+2\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-19(244} \\ t \end{array}\right.$$

Considering the eigenvalue $\lambda_3 = -\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}+\frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{array}\right] - \left(-\frac{\left(244+12\sqrt{417}\right)}{12}\right.\right.$$

$$\left[\begin{array}{cc} \frac{-i\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}\sqrt{3}+\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}-8i\sqrt{3}+4\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}-8}{12\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}} & -1 \\ 1 & \frac{-i\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}\sqrt{3}+\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}-8i}{12\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}} \\ -1 & 0 \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc} \frac{1}{3}+\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}-\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2} & \\ 1 & -\frac{2}{3}+\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}-\frac{2}{3\left(244+12\sqrt{4\right.}} \\ -1 & \end{array}\right.$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{3}+\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}-\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow \left[\begin{array}{c} \frac{-i\left(244+12\sqrt{3}\,\sqrt{139}\right)}{} \\ \\ \end{array}\right.$$

$$R_3 = R_3 + \frac{12\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}} R_1}{-i\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}\sqrt{3} + \left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}} - 8i\sqrt{3} + 4\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}} - 8} =$$

$$R_3 = R_3 - \frac{6\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}} \left(8 - 4\left(244+12\sqrt{417}\right)^{\frac{1}{3}} + i\left(\left(\right.\right.\right.}{\left(i\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}\sqrt{3} - \left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}} + 8i\sqrt{3} - 4\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}} + 8\right)\left(\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{cc} \frac{-i\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}\sqrt{3}+\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}-8i\sqrt{3}+4\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}-8}{12\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}} & -1 \\ 0 & \frac{2\left(19i\sqrt{3}+3i\sqrt{139}+\sqrt{417}+19\right)\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-92i}{\left(8-4\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+i\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+8\right)\right.\right.} \\ 0 & 0 \end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = -\frac{\left(3i\sqrt{3}\,\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-61i\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{3}+3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}+16i\left(244\right.\right.}{192}\right.$

Hence the solution is

$$\left[\begin{array}{c} -\frac{\left(3\,\mathrm{I}\sqrt{3}\,\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-61\,\mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{3}+3\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\sqrt{417}+16\,\mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\sqrt{3}-61\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-16\left(\right.\right.}{192} \\ -\frac{4t\left(-2\,\mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\sqrt{3}+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-61\,\mathrm{I}\sqrt{3}-9\,\mathrm{I}\sqrt{139}+3\sqrt{417}-2\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+61\right)}{-19\,\mathrm{I}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\sqrt{3}-3\,\mathrm{I}\sqrt{139}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-\sqrt{417}\left(244+12\sqrt{417}\right)^{\frac{1}{3}}+2\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+46\,\mathrm{I}\sqrt{3}+6\,\mathrm{I}\sqrt{139}-2\sqrt{417}-19\left(\right.} \\ t \end{array}\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} -\frac{(3 I\sqrt{3}\,\sqrt{417}\,(244+12\sqrt{417})^{\frac{2}{3}}-61 I(244+12\sqrt{417})^{\frac{2}{3}}\sqrt{3}+3(244+12\sqrt{417})^{\frac{2}{3}}\sqrt{417}+16 I(244+12\sqrt{417})^{\frac{1}{3}}\sqrt{3}-61(244+12\sqrt{417})^{\frac{2}{3}}-16(}{192} \\ -\frac{4t\left(-2 I(244+12\sqrt{417})^{\frac{1}{3}}\sqrt{3}+(244+12\sqrt{417})^{\frac{2}{3}}-61 I\sqrt{3}-9 I\sqrt{139}+3\sqrt{417}-2(244+12\sqrt{417})^{\frac{1}{3}}+61\right)}{-19 I(244+12\sqrt{417})^{\frac{1}{3}}\sqrt{3}-3 I\sqrt{139}\,(244+12\sqrt{417})^{\frac{1}{3}}-\sqrt{417}\,(244+12\sqrt{417})^{\frac{1}{3}}+2(244+12\sqrt{417})^{\frac{2}{3}}+46 I\sqrt{3}+6 I\sqrt{139}-2\sqrt{417}-19(} \\ t \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} -\frac{(3 I\sqrt{3}\,\sqrt{417}\,(244+12\sqrt{417})^{\frac{2}{3}}-61 I(244+12\sqrt{417})^{\frac{2}{3}}\sqrt{3}+3(244+12\sqrt{417})^{\frac{2}{3}}\sqrt{417}+16 I(244+12\sqrt{417})^{\frac{1}{3}}\sqrt{3}-61(244+12\sqrt{417})^{\frac{2}{3}}-16(}{192} \\ -\frac{4t\left(-2 I(244+12\sqrt{417})^{\frac{1}{3}}\sqrt{3}+(244+12\sqrt{417})^{\frac{2}{3}}-61 I\sqrt{3}-9 I\sqrt{139}+3\sqrt{417}-2(244+12\sqrt{417})^{\frac{1}{3}}+61\right)}{-19 I(244+12\sqrt{417})^{\frac{1}{3}}\sqrt{3}-3 I\sqrt{139}\,(244+12\sqrt{417})^{\frac{1}{3}}-\sqrt{417}\,(244+12\sqrt{417})^{\frac{1}{3}}+2(244+12\sqrt{417})^{\frac{2}{3}}+46 I\sqrt{3}+6 I\sqrt{139}-2\sqrt{417}-19(} \\ t \end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c} -\frac{(3 I\sqrt{3}\,\sqrt{417}\,(244+12\sqrt{417})^{\frac{2}{3}}-61 I(244+12\sqrt{417})^{\frac{2}{3}}\sqrt{3}+3(244+12\sqrt{417})^{\frac{2}{3}}\sqrt{417}+16 I(244+12\sqrt{417})^{\frac{1}{3}}\sqrt{3}-61(244+12\sqrt{417})^{\frac{2}{3}}-16(}{192} \\ -\frac{4t\left(-2 I(244+12\sqrt{417})^{\frac{1}{3}}\sqrt{3}+(244+12\sqrt{417})^{\frac{2}{3}}-61 I\sqrt{3}-9 I\sqrt{139}+3\sqrt{417}-2(244+12\sqrt{417})^{\frac{1}{3}}+61\right)}{-19 I(244+12\sqrt{417})^{\frac{1}{3}}\sqrt{3}-3 I\sqrt{139}\,(244+12\sqrt{417})^{\frac{1}{3}}-\sqrt{417}\,(244+12\sqrt{417})^{\frac{1}{3}}+2(244+12\sqrt{417})^{\frac{2}{3}}+46 I\sqrt{3}+6 I\sqrt{139}-2\sqrt{417}-19(} \\ t \end{array}\right.$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		d
	algebraic m	geometric k	
$\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}$	1	1	
$-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}+\frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}$	1	1	
$-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}-\frac{i\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}$	1	1	

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}$ is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_1(t)=\vec{v}_1e^{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)t}$$

$$=\left[\begin{array}{c} -\frac{4}{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)^{2}\right)\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}-\right.} \\ \frac{4}{\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)^{2}} \\ 1 \end{array}\right.$$

Therefore the final solution is

$$\vec{x}_h(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)+c_3\vec{x}_3(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right]=c_1\left[\begin{array}{c} -\frac{4\,\mathrm{e}^{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)t}}{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)^{2}\right)\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{4\right.}}\right.} \\ \frac{4\,\mathrm{e}^{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)t}}{\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)^{2}} \\ \mathrm{e}^{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)t} \end{array}\right.$$

Which becomes

Expression too large to display

9.4.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) - y, y' = x(t) + 2z(t), z'(t) = -x(t) + z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}, \left[\begin{array}{c} -\frac{4}{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{41\right.}}\right.\right.} \\ \frac{4}{\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\sqrt{4\right.}}{6}\right.} \\ 1 \end{array}\right.\right.\right.$$

- Consider eigenpair

$$\left[\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}, \left[\begin{array}{c} -\frac{4}{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)}\right.\right.} \\ \frac{4}{\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\sqrt{417}\right)}{6}\right.} \\ 1 \end{array}\right.\right.$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1=\mathrm{e}^{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)t}\cdot\left[\begin{array}{c} -\frac{4}{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{4\right.}}\right.\right.} \\ \frac{4}{\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\sqrt{\right.}}{6}\right.} \\ 1 \end{array}\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}-\frac{I\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}, \left[-\frac{}{\left(-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{}{3\left(244\right.}\right.}\right.\right.$$

- Solution from eigenpair

$$e^{\left(-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}-\frac{I\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)}{2}\right)t}\cdot\left[-\frac{}{\left(-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right.}\right.}\right.$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{\left(-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)t}\cdot\left(\cos\left(\frac{\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right)t}{2}\right)-I\sin\left(\frac{\sqrt{3}\left(\frac{\left(244+12\sqrt{\right.}{6}\right.}{}\right.\right.$$

- Simplify expression

$$e^{\left(-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)t} \cdot \left[-\frac{}{\left(-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}+\frac{4}{3\left(244+12\sqrt{41\right.}}\right.}{2}\right.} \quad 4 \quad \frac{}{-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right)}} \quad \cos \right.$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\vec{x}_2(t)=e^{\left(-\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{12}+\frac{2}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)t} \cdot \left[\frac{20736\left(61+3\sqrt{417}\right)\left(\left(244+12\sqrt{417}\right)^{\frac{5}{3}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\right.}{12\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}\right.\right.}{} \quad -\frac{72\left(244+12\sqrt{417}\right)^{\frac{2}{3}}\left(\left(244+1\right.\right.}{} \right.$$

- General solution to the system of ODEs

  $\vec{x}=c_1\vec{x}_1+c_2\vec{x}_2(t)+c_3\vec{x}_3(t)$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{2}{3}\right)t} \cdot \left[\begin{array}{c} -\frac{4}{\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\right.}\right.\right.} \\ \frac{4}{\frac{\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}{6}-\frac{4}{3\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}+\frac{5}{3}+\left(\frac{\left(244+12\right.}{6}\right.} \\ 1 \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} \frac{\left(\left(\left(\left(11187c_2\sqrt{3}+33561c_3\right)\sqrt{139}+228445\sqrt{3}\,c_3+228445c_2\right)\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}+\left(\left(-183c_2\sqrt{3}+549c_3\right)\sqrt{139}+3\right.\right.\right.}{} \\ -\frac{\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}\left(\left(\left(\frac{\left(\left(-\frac{c_2\sqrt{3}}{3}-c_3\right)\sqrt{139}-\frac{19c_2}{3}-\frac{19\sqrt{3}\,c_3}{3}\right)\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{2}{3}}}{2}\right.\right.\right.}{} \end{array}\right.$$

- Solution to the system of ODEs

$$\left\{ x(t) = \frac{\left(\left(\left(\left(11187c_2\sqrt{3}+33561c_3\right)\sqrt{139}+228445\sqrt{3}\,c_3+228445c_2\right)\left(244+12\sqrt{3}\,\sqrt{139}\right)^{\frac{1}{3}}+\left(\left(-183c_2\sqrt{3}+549c_3\right)\sqrt{139}+3737\sqrt{3}\right.\right.\right.}{} \right.$$

✓ Solution by Maple

Time used: 0.313 (sec). Leaf size: 2266

```
dsolve([diff(x(t),t)=x(t)-y(t),diff(y(t),t)=x(t)+2*z(t),diff(z(t),t)=-x(t)+z(t)],singsol=all
```

$$
\begin{aligned}
x(t) = {} & \mathrm{e}^{-\frac{\left(-8+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-8\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\right)t}{12\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}} \cos\left(\frac{\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+8\right)t\sqrt{3}\,2^{\frac{1}{3}}}{24\left(61+3\sqrt{417}\right)^{\frac{1}{3}}}\right) c_3 \\
& - \mathrm{e}^{-\frac{\left(-8+\left(244+12\sqrt{417}\right)^{\frac{2}{3}}-8\left(244+12\sqrt{417}\right)^{\frac{1}{3}}\right)t}{12\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}} \sin\left(\frac{\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+8\right)t\sqrt{3}\,2^{\frac{1}{3}}}{24\left(61+3\sqrt{417}\right)^{\frac{1}{3}}}\right) c_2 \\
& + c_1 \mathrm{e}^{\frac{\left(\left(244+12\sqrt{417}\right)^{\frac{2}{3}}+4\left(244+12\sqrt{417}\right)^{\frac{1}{3}}-8\right)t}{6\left(244+12\sqrt{417}\right)^{\frac{1}{3}}}}
\end{aligned}
$$

Expression too large to display

Expression too large to display

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 503

```
DSolve[{x'[t]==x[t]-y[t],y'[t]==x[t]+2*z[t],z'[t]==-x[t]+z[t]},{x[t],y[t],z[t]},t,IncludeSin
```

$$
\begin{aligned}
x(t) \to & -2c_3 \mathrm{RootSum}\left[\#1^3 - 2\#1^2 + 2\#1 - 3\&, \frac{e^{\#1t}}{3\#1^2 - 4\#1 + 2}\&\right] \\
& - c_2 \mathrm{RootSum}\left[\#1^3 - 2\#1^2 + 2\#1 - 3\&, \frac{\#1e^{\#1t} - e^{\#1t}}{3\#1^2 - 4\#1 + 2}\&\right] \\
& + c_1 \mathrm{RootSum}\left[\#1^3 - 2\#1^2 + 2\#1 - 3\&, \frac{\#1^2e^{\#1t} - \#1e^{\#1t}}{3\#1^2 - 4\#1 + 2}\&\right] \\
y(t) \to & \, c_1 \mathrm{RootSum}\left[\#1^3 - 2\#1^2 + 2\#1 - 3\&, \frac{\#1e^{\#1t} - 3e^{\#1t}}{3\#1^2 - 4\#1 + 2}\&\right] \\
& + 2c_3 \mathrm{RootSum}\left[\#1^3 - 2\#1^2 + 2\#1 - 3\&, \frac{\#1e^{\#1t} - e^{\#1t}}{3\#1^2 - 4\#1 + 2}\&\right] \\
& + c_2 \mathrm{RootSum}\left[\#1^3 - 2\#1^2 + 2\#1 - 3\&, \frac{\#1^2e^{\#1t} - 2\#1e^{\#1t} + e^{\#1t}}{3\#1^2 - 4\#1 + 2}\&\right] \\
z(t) \to & \, c_2 \mathrm{RootSum}\left[\#1^3 - 2\#1^2 + 2\#1 - 3\&, \frac{e^{\#1t}}{3\#1^2 - 4\#1 + 2}\&\right] \\
& - c_1 \mathrm{RootSum}\left[\#1^3 - 2\#1^2 + 2\#1 - 3\&, \frac{\#1e^{\#1t}}{3\#1^2 - 4\#1 + 2}\&\right] \\
& + c_3 \mathrm{RootSum}\left[\#1^3 - 2\#1^2 + 2\#1 - 3\&, \frac{\#1^2e^{\#1t} - \#1e^{\#1t} + e^{\#1t}}{3\#1^2 - 4\#1 + 2}\&\right]
\end{aligned}
$$

9.5 problem 5

Internal problem ID [6715]
Internal file name [`OUTPUT/5963_Sunday_June_05_2022_04_04_56_PM_20748891/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) - y + z(t) + t - 1 \\
y' &= 2x(t) + y - z(t) - 3t^2 \\
z'(t) &= x(t) + y + z(t) + t^2 - t + 2
\end{aligned}
$$

9.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} t-1 \\ -3t^2 \\ t^2-t+2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{ccc} \frac{3\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{2\,e^{2t}}{5} & -\frac{2\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{11} & -\frac{2\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{4\sqrt{11}\,e^{\frac{t}{2}}}{} \\ -\frac{e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{17\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{e^{2t}}{5} & \frac{\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{11} + e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right) & -\frac{e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{13\sqrt{11}\,e^{\frac{t}{2}}}{} \\ -\frac{3\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{3\,e^{2t}}{5} & \frac{2\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{11} & \frac{2\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{4\sqrt{11}\,e^{\frac{t}{2}}\,\mathrm{si}}{5} \end{array}\right.$$

$$= \left[\begin{array}{ccc} \frac{3\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{2\,e^{2t}}{5} & -\frac{2\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{11} & -\frac{2\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{4\sqrt{11}\,e^{\frac{t}{2}}\sin(}{55} \\ -\frac{e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{17\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{e^{2t}}{5} & \frac{e^{\frac{t}{2}}\left(\sqrt{11}\sin\left(\frac{\sqrt{11}\,t}{2}\right)+11\cos\left(\frac{\sqrt{11}\,t}{2}\right)\right)}{11} & -\frac{e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{13\sqrt{11}\,e^{\frac{t}{2}}\,\mathrm{sin}}{55} \\ -\frac{3\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{3\,e^{2t}}{5} & \frac{2\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{11} & \frac{2\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{4\sqrt{11}\,e^{\frac{t}{2}}\sin(}{55} \end{array}\right.$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \left[\begin{array}{ccc} \frac{3\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{2\,e^{2t}}{5} & -\frac{2\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{11} & -\frac{2\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{4\sqrt{11}\,e^{\frac{t}{2}}\,\mathrm{si}}{5} \\ -\frac{e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{17\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{e^{2t}}{5} & \frac{e^{\frac{t}{2}}\left(\sqrt{11}\sin\left(\frac{\sqrt{11}\,t}{2}\right)+11\cos\left(\frac{\sqrt{11}\,t}{2}\right)\right)}{11} & -\frac{e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{13\sqrt{11}\,e^{\frac{t}{2}}\,\mathrm{s}}{5} \\ -\frac{3\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{3\,e^{2t}}{5} & \frac{2\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{11} & \frac{2\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{4\sqrt{11}\,e^{\frac{t}{2}}\,\mathrm{sin}}{55} \end{array}\right.$$

$$= \left[\begin{array}{c} \left(\frac{3\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{2\,e^{2t}}{5}\right)c_1 - \frac{2\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)c_2}{11} + \left(-\frac{2\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{4\sqrt{11}\,e}{} \right. \\ \left(-\frac{e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{17\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{e^{2t}}{5}\right)c_1 + \frac{e^{\frac{t}{2}}\left(\sqrt{11}\sin\left(\frac{\sqrt{11}\,t}{2}\right)+11\cos\left(\frac{\sqrt{11}\,t}{2}\right)\right)c_2}{11} + \left(-\frac{e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \right. \\ \left(-\frac{3\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{3\,e^{2t}}{5}\right)c_1 + \frac{2\sqrt{11}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)c_2}{11} + \left(\frac{2\,e^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{4\sqrt{11}\,e}{} \right. \end{array}\right.$$

$$= \left[\begin{array}{c} -\frac{\sqrt{11}\,e^{\frac{t}{2}}\left(c_1+10c_2-4c_3\right)\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{3\,e^{\frac{t}{2}}\left(c_1-\frac{2c_3}{3}\right)\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{2\,e^{2t}\left(c_1+c_3\right)}{5} \\ -\frac{e^{\frac{t}{2}}\left(c_1-5c_2+c_3\right)\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{17\left(c_1+\frac{5c_2}{17}-\frac{13c_3}{17}\right)e^{\frac{t}{2}}\sqrt{11}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{e^{2t}\left(c_1+c_3\right)}{5} \\ \frac{\sqrt{11}\,e^{\frac{t}{2}}\left(c_1+10c_2-4c_3\right)\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} - \frac{3\,e^{\frac{t}{2}}\left(c_1-\frac{2c_3}{3}\right)\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{3\,e^{2t}\left(c_1+c_3\right)}{5} \end{array}\right]$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix}
\frac{e^{-2t}\left(\sqrt{11}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)+33\, e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)+22\right)}{55} & \frac{2\sqrt{11}\, e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{11} \\
-\frac{e^{-2t}\left(17\sqrt{11}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)+11\, e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)-11\right)}{55} & -\frac{\left(\sqrt{11}\, \sin\left(\frac{\sqrt{11}\, t}{2}\right)-11 \cos\left(\frac{\sqrt{11}\, t}{2}\right)\right) e^{-\frac{t}{2}}}{11} \\
-\frac{e^{-2t}\left(\sqrt{11}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)+33\, e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)-33\right)}{55} & -\frac{2\sqrt{11}\, e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{11}
\end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix}
\frac{3\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} - \frac{\sqrt{11}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{55} + \frac{2\, e^{2t}}{5} & -\frac{2\sqrt{11}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{11} & -\frac{2\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} + \\
-\frac{e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} + \frac{17\sqrt{11}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{55} + \frac{e^{2t}}{5} & \frac{e^{\frac{t}{2}} \left(\sqrt{11}\, \sin\left(\frac{\sqrt{11}\, t}{2}\right)+11 \cos\left(\frac{\sqrt{11}\, t}{2}\right)\right)}{11} & -\frac{e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} - \\
-\frac{3\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} + \frac{\sqrt{11}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{55} + \frac{3\, e^{2t}}{5} & \frac{2\sqrt{11}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{11} & \frac{2\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} -
\end{bmatrix} \\
&= \begin{bmatrix}
\frac{3\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} - \frac{\sqrt{11}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{55} + \frac{2\, e^{2t}}{5} & -\frac{2\sqrt{11}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{11} & -\frac{2\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} + \\
-\frac{e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} + \frac{17\sqrt{11}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{55} + \frac{e^{2t}}{5} & \frac{e^{\frac{t}{2}} \left(\sqrt{11}\, \sin\left(\frac{\sqrt{11}\, t}{2}\right)+11 \cos\left(\frac{\sqrt{11}\, t}{2}\right)\right)}{11} & -\frac{e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} - \\
-\frac{3\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} + \frac{\sqrt{11}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{55} + \frac{3\, e^{2t}}{5} & \frac{2\sqrt{11}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{11} & \frac{2\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} -
\end{bmatrix} \\
&= \begin{bmatrix}
t^2 - \frac{1}{6} \\
-\frac{1}{2}t^2 - \frac{3}{2}t - \frac{7}{4} \\
-\frac{3}{2}t^2 - \frac{1}{2}t - \frac{7}{12}
\end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix}
-\frac{\sqrt{11}\, e^{\frac{t}{2}} (c_1+10c_2-4c_3) \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{55} + \frac{3\, e^{\frac{t}{2}} \left(c_1-\frac{2c_3}{3}\right) \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} + \frac{2\, e^{2t}(c_1+c_3)}{5} + t^2 - \frac{1}{6} \\
-\frac{e^{\frac{t}{2}} (c_1-5c_2+c_3) \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} + \frac{17\left(c_1+\frac{5c_2}{17}-\frac{13c_3}{17}\right) e^{\frac{t}{2}} \sqrt{11}\, \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{55} + \frac{e^{2t}(c_1+c_3)}{5} - \frac{t^2}{2} - \frac{3t}{2} - \frac{7}{4} \\
\frac{\sqrt{11}\, e^{\frac{t}{2}} (c_1+10c_2-4c_3) \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{55} - \frac{3\, e^{\frac{t}{2}} \left(c_1-\frac{2c_3}{3}\right) \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{5} + \frac{3\, e^{2t}(c_1+c_3)}{5} - \frac{3t^2}{2} - \frac{t}{2} - \frac{7}{12}
\end{bmatrix}
\end{aligned}
$$

9.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} t-1 \\ -3t^2 \\ t^2-t+2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 & 1 \\ 2 & 1-\lambda & -1 \\ 1 & 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 5\lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}
\lambda_1 &= 2\\
\lambda_2 &= \frac{i\sqrt{11}}{2}+\frac{1}{2}\\
\lambda_3 &= \frac{1}{2}-\frac{i\sqrt{11}}{2}
\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
$\frac{1}{2}-\frac{i\sqrt{11}}{2}$	1	complex eigenvalue
$\frac{i\sqrt{11}}{2}+\frac{1}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 2 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -3 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}, v_2 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{2} - \frac{i\sqrt{11}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} - \left(\frac{1}{2} - \frac{i\sqrt{11}}{2}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{i\sqrt{11}}{2} + \frac{1}{2} & -1 & 1 \\ 2 & \frac{i\sqrt{11}}{2} + \frac{1}{2} & -1 \\ 1 & 1 & \frac{i\sqrt{11}}{2} + \frac{1}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} \frac{i\sqrt{11}}{2} + \frac{1}{2} & -1 & 1 & 0 \\ 2 & \frac{i\sqrt{11}}{2} + \frac{1}{2} & -1 & 0 \\ 1 & 1 & \frac{i\sqrt{11}}{2} + \frac{1}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{\frac{i\sqrt{11}}{2} + \frac{1}{2}} \implies \left[\begin{array}{ccc|c} \frac{i\sqrt{11}}{2} + \frac{1}{2} & -1 & 1 & 0 \\ 0 & \frac{-\sqrt{11}-i}{-\sqrt{11}+i} & \frac{\sqrt{11}-5i}{-\sqrt{11}+i} & 0 \\ 1 & 1 & \frac{i\sqrt{11}}{2} + \frac{1}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{\frac{i\sqrt{11}}{2} + \frac{1}{2}} \implies \left[\begin{array}{ccc|c} \frac{i\sqrt{11}}{2} + \frac{1}{2} & -1 & 1 & 0 \\ 0 & \frac{-\sqrt{11}-i}{-\sqrt{11}+i} & \frac{\sqrt{11}-5i}{-\sqrt{11}+i} & 0 \\ 0 & \frac{i\sqrt{11}+3}{i\sqrt{11}+1} & \frac{-\sqrt{11}-7i}{-\sqrt{11}+i} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{\left(i\sqrt{11}+3\right)\left(-\sqrt{11}+i\right)R_2}{\left(i\sqrt{11}+1\right)\left(-\sqrt{11}-i\right)} \implies \left[\begin{array}{ccc|c} \frac{i\sqrt{11}}{2} + \frac{1}{2} & -1 & 1 & 0 \\ 0 & \frac{-\sqrt{11}-i}{-\sqrt{11}+i} & \frac{\sqrt{11}-5i}{-\sqrt{11}+i} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{i\sqrt{11}}{2} + \frac{1}{2} & -1 & 1 \\ 0 & \frac{-\sqrt{11}-i}{-\sqrt{11}+i} & \frac{\sqrt{11}-5i}{-\sqrt{11}+i} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -t, v_2 = \frac{t\left(i\sqrt{11}+5\right)}{i\sqrt{11}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} -t \\ \frac{t\left(\mathrm{I}\sqrt{11}+5\right)}{\mathrm{I}\sqrt{11}-1} \\ t \end{bmatrix} = \begin{bmatrix} -t \\ \frac{t\left(i\sqrt{11}+5\right)}{i\sqrt{11}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ \frac{t\left(\mathrm{I}\sqrt{11}+5\right)}{\mathrm{I}\sqrt{11}-1} \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ \frac{i\sqrt{11}+5}{i\sqrt{11}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ \frac{t\left(\mathrm{I}\sqrt{11}+5\right)}{\mathrm{I}\sqrt{11}-1} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{i\sqrt{11}+5}{i\sqrt{11}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -t \\ \frac{t\left(\mathrm{I}\sqrt{11}+5\right)}{\mathrm{I}\sqrt{11}-1} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{i\sqrt{11}+5}{i\sqrt{11}-1} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = \frac{i\sqrt{11}}{2} + \frac{1}{2}$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} - \left(\frac{i\sqrt{11}}{2} + \frac{1}{2} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{11}}{2} & -1 & 1 \\ 2 & \frac{1}{2} - \frac{i\sqrt{11}}{2} & -1 \\ 1 & 1 & \frac{1}{2} - \frac{i\sqrt{11}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} \frac{1}{2} - \frac{i\sqrt{11}}{2} & -1 & 1 & 0 \\ 2 & \frac{1}{2} - \frac{i\sqrt{11}}{2} & -1 & 0 \\ 1 & 1 & \frac{1}{2} - \frac{i\sqrt{11}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{\frac{1}{2} - \frac{i\sqrt{11}}{2}} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{2} - \frac{i\sqrt{11}}{2} & -1 & 1 & 0 \\ 0 & \frac{\sqrt{11}-i}{i+\sqrt{11}} & \frac{-5i-\sqrt{11}}{i+\sqrt{11}} & 0 \\ 1 & 1 & \frac{1}{2} - \frac{i\sqrt{11}}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{\frac{1}{2} - \frac{i\sqrt{11}}{2}} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{2} - \frac{i\sqrt{11}}{2} & -1 & 1 & 0 \\ 0 & \frac{\sqrt{11}-i}{i+\sqrt{11}} & \frac{-5i-\sqrt{11}}{i+\sqrt{11}} & 0 \\ 0 & \frac{\sqrt{11}+3i}{i+\sqrt{11}} & \frac{\sqrt{11}-7i}{i+\sqrt{11}} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{\left(\sqrt{11} + 3i\right) R_2}{\sqrt{11} - i} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{2} - \frac{i\sqrt{11}}{2} & -1 & 1 & 0 \\ 0 & \frac{\sqrt{11}-i}{i+\sqrt{11}} & \frac{-5i-\sqrt{11}}{i+\sqrt{11}} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{11}}{2} & -1 & 1 \\ 0 & \frac{\sqrt{11}-i}{i+\sqrt{11}} & \frac{-5i-\sqrt{11}}{i+\sqrt{11}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -t, v_2 = -\frac{\left(5i+\sqrt{11}\right)t}{-\sqrt{11}+i}\right\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -\frac{\left(5\,\mathrm{I}+\sqrt{11}\right)t}{-\sqrt{11}+\mathrm{I}} \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -\frac{\left(5i+\sqrt{11}\right)t}{-\sqrt{11}+i} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -\frac{\left(5\,\mathrm{I}+\sqrt{11}\right)t}{-\sqrt{11}+\mathrm{I}} \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -\frac{5i+\sqrt{11}}{-\sqrt{11}+i} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -\frac{\left(5\,\mathrm{I}+\sqrt{11}\right)t}{-\sqrt{11}+\mathrm{I}} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{5i+\sqrt{11}}{-\sqrt{11}+i} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -t \\ -\frac{\left(5\,\mathrm{I}+\sqrt{11}\right)t}{-\sqrt{11}+\mathrm{I}} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{5i+\sqrt{11}}{-\sqrt{11}+i} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\left[\begin{array}{c} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{array}\right]$
$\frac{i\sqrt{11}}{2}+\frac{1}{2}$	1	1	No	$\left[\begin{array}{c} \frac{\frac{3i\sqrt{11}}{2}+\frac{3}{2}}{\left(\frac{i\sqrt{11}}{2}-\frac{5}{2}\right)\left(-\frac{1}{2}+\frac{i\sqrt{11}}{2}\right)} \\ -\frac{4+i\sqrt{11}}{\frac{i\sqrt{11}}{2}-\frac{5}{2}} \\ 1 \end{array}\right]$
$\frac{1}{2}-\frac{i\sqrt{11}}{2}$	1	1	No	$\left[\begin{array}{c} \frac{\frac{3}{2}-\frac{3i\sqrt{11}}{2}}{\left(-\frac{5}{2}-\frac{i\sqrt{11}}{2}\right)\left(-\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)} \\ -\frac{4-i\sqrt{11}}{-\frac{5}{2}-\frac{i\sqrt{11}}{2}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\
&= \left[\begin{array}{c} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{array}\right] e^{2t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = c_1 \left[\begin{array}{c} \frac{2\,\mathrm{e}^{2t}}{3} \\ \frac{\mathrm{e}^{2t}}{3} \\ \mathrm{e}^{2t} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{3\,\mathrm{e}^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)}{\left(\frac{i\sqrt{11}}{2}-\frac{5}{2}\right)\left(-\frac{1}{2}+\frac{i\sqrt{11}}{2}\right)} \\ -\frac{\mathrm{e}^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(4+i\sqrt{11}\right)}{\frac{i\sqrt{11}}{2}-\frac{5}{2}} \\ \mathrm{e}^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t} \end{array}\right] + c_3 \left[\begin{array}{c} \frac{3\,\mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)}{\left(-\frac{5}{2}-\frac{i\sqrt{11}}{2}\right)\left(-\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)} \\ -\frac{\mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(4-i\sqrt{11}\right)}{-\frac{5}{2}-\frac{i\sqrt{11}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t} \end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{ccc} \frac{2\,\mathrm{e}^{2t}}{3} & \frac{3\,\mathrm{e}^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)}{\left(\frac{i\sqrt{11}}{2}-\frac{5}{2}\right)\left(-\frac{1}{2}+\frac{i\sqrt{11}}{2}\right)} & \frac{3\,\mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)}{\left(-\frac{5}{2}-\frac{i\sqrt{11}}{2}\right)\left(-\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)} \\ \frac{\mathrm{e}^{2t}}{3} & -\frac{\mathrm{e}^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(4+i\sqrt{11}\right)}{\frac{i\sqrt{11}}{2}-\frac{5}{2}} & -\frac{\mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(4-i\sqrt{11}\right)}{-\frac{5}{2}-\frac{i\sqrt{11}}{2}} \\ \mathrm{e}^{2t} & \mathrm{e}^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t} & \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \left[\begin{array}{ccc} \frac{3\,\mathrm{e}^{-2t}}{5} & 0 & \frac{3\,\mathrm{e}^{-2t}}{5} \\ -\frac{\left(i\sqrt{11}-1\right)\sqrt{11}\,\mathrm{e}^{-\frac{\left(i\sqrt{11}+1\right)t}{2}}\left(-\sqrt{11}+i\right)\left(5i+\sqrt{11}\right)\left(7\sqrt{11}+19i\right)}{23760} & -\frac{i\sqrt{11}\,\mathrm{e}^{-\frac{\left(i\sqrt{11}+1\right)t}{2}}}{11} & -\frac{\left(i\sqrt{11}-1\right)\sqrt{11}\,\mathrm{e}^{-\frac{\left(i\sqrt{11}+1\right)t}{2}}(5i+}{1320} \\ -\frac{7\,\mathrm{e}^{\frac{\left(i\sqrt{11}-1\right)t}{2}}\sqrt{11}\left(i\sqrt{11}-1\right)\left(i\sqrt{11}+5\right)\left(-\sqrt{11}+i\right)\left(i\sqrt{11}+\frac{19}{7}\right)}{23760} & \frac{i\sqrt{11}\,\mathrm{e}^{\frac{\left(i\sqrt{11}-1\right)t}{2}}}{11} & -\frac{\left(i\sqrt{11}+1\right)\sqrt{11}\,\mathrm{e}^{\frac{\left(i\sqrt{11}-1\right)t}{2}}(\sqrt{11}}{1320} \end{array}\right.$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{2e^{2t}}{3} & \frac{3e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)}{\left(\frac{i\sqrt{11}}{2}-\frac{5}{2}\right)\left(-\frac{1}{2}+\frac{i\sqrt{11}}{2}\right)} & \frac{3e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)}{\left(-\frac{5}{2}-\frac{i\sqrt{11}}{2}\right)\left(-\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)} \\ \frac{e^{2t}}{3} & -\frac{e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(4+i\sqrt{11}\right)}{\frac{i\sqrt{11}}{2}-\frac{5}{2}} & -\frac{e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(4-i\sqrt{11}\right)}{-\frac{5}{2}-\frac{i\sqrt{11}}{2}} \\ e^{2t} & e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t} & e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t} \end{bmatrix} \int \left[\begin{matrix} \frac{3e^{-2t}}{5} \\ -\frac{\left(i\sqrt{11}-1\right)\sqrt{11}\,e^{-\frac{\left(i\sqrt{11}+1\right)t}{2}}\left(-\sqrt{11}+i\right)\left(5i+\sqrt{11}\right)}{23760} \\ -\frac{7e^{\frac{\left(i\sqrt{11}-1\right)t}{2}}\sqrt{11}\left(i\sqrt{11}-1\right)\left(i\sqrt{11}+5\right)\left(-\sqrt{11}+i\right.}{23760} \end{matrix}\right.$$

$$= \begin{bmatrix} \frac{2e^{2t}}{3} & \frac{3e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)}{\left(\frac{i\sqrt{11}}{2}-\frac{5}{2}\right)\left(-\frac{1}{2}+\frac{i\sqrt{11}}{2}\right)} & \frac{3e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)}{\left(-\frac{5}{2}-\frac{i\sqrt{11}}{2}\right)\left(-\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)} \\ \frac{e^{2t}}{3} & -\frac{e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(4+i\sqrt{11}\right)}{\frac{i\sqrt{11}}{2}-\frac{5}{2}} & -\frac{e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(4-i\sqrt{11}\right)}{-\frac{5}{2}-\frac{i\sqrt{11}}{2}} \\ e^{2t} & e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t} & e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t} \end{bmatrix} \int \begin{bmatrix} \frac{3e^{-2t}\left(t^2+1\right)}{5} \\ \frac{e^{-\frac{\left(i\sqrt{11}+1\right)t}{2}}\left(77+i\left(34t^2-5t+9\right)\sqrt{11}+22t^2-55t\right)}{110} \\ \frac{e^{\frac{\left(i\sqrt{11}-1\right)t}{2}}\left(77+i\left(-34t^2+5t-9\right)\sqrt{11}+22t^2-55t\right)}{110} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2e^{2t}}{3} & \frac{3e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)}{\left(\frac{i\sqrt{11}}{2}-\frac{5}{2}\right)\left(-\frac{1}{2}+\frac{i\sqrt{11}}{2}\right)} & \frac{3e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)}{\left(-\frac{5}{2}-\frac{i\sqrt{11}}{2}\right)\left(-\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)} \\ \frac{e^{2t}}{3} & -\frac{e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(4+i\sqrt{11}\right)}{\frac{i\sqrt{11}}{2}-\frac{5}{2}} & -\frac{e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(4-i\sqrt{11}\right)}{-\frac{5}{2}-\frac{i\sqrt{11}}{2}} \\ e^{2t} & e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t} & e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t} \end{bmatrix} \left[\begin{matrix} -\frac{3e^{-2t}\left(2t^2+\right.}{20} \\ \frac{\sqrt{11}\,e^{-\frac{\left(i\sqrt{11}+1\right)t}{2}}\left(30i\sqrt{11}\,t^2+9i\sqrt{11}\,t+8i\sqrt{11}-30t^2+63\right.}{7590+330i(8t-11)\sqrt{11}+} \\ \frac{e^{\frac{\left(i\sqrt{11}-1\right)t}{2}}\sqrt{11}\left(30\sqrt{11}\,t^2+9\sqrt{11}\,t+8\sqrt{11}-30it^2+63it+\right.}{-7590+330i(8t-11)\sqrt{11}-} \end{matrix}\right.$$

$$= \begin{bmatrix} \frac{360t^6-168t^5+306t^4-230t^3+125t^2+43t-31}{360t^4-168t^3+366t^2-258t+186} \\ \frac{-120t^6-304t^5-374t^4-84t^3-231t^2+115t-217}{240t^4-112t^3+244t^2-172t+124} \\ \frac{-1080t^6+144t^5-1350t^4+604t^3-727t^2+115t-217}{720t^4-336t^3+732t^2-516t+372} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{2c_1e^{2t}}{3} \\ \frac{c_1e^{2t}}{3} \\ c_1e^{2t} \end{bmatrix} + \begin{bmatrix} \frac{3c_2e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)}{\left(\frac{i\sqrt{11}}{2}-\frac{5}{2}\right)\left(-\frac{1}{2}+\frac{i\sqrt{11}}{2}\right)} \\ -\frac{c_2e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t}\left(4+i\sqrt{11}\right)}{\frac{i\sqrt{11}}{2}-\frac{5}{2}} \\ c_2e^{\left(\frac{i\sqrt{11}}{2}+\frac{1}{2}\right)t} \end{bmatrix} + \begin{bmatrix} \frac{3c_3e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)}{\left(-\frac{5}{2}-\frac{i\sqrt{11}}{2}\right)\left(-\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)} \\ -\frac{c_3e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t}\left(4-i\sqrt{11}\right)}{-\frac{5}{2}-\frac{i\sqrt{11}}{2}} \\ c_3e^{\left(\frac{1}{2}-\frac{i\sqrt{11}}{2}\right)t} \end{bmatrix} + \left[\begin{matrix} \frac{360t^6-168t^5+306t^4-}{360t^4-168t^3+3} \\ \frac{-120t^6-304t^5-374t^4-}{240t^4-112t^3+2} \\ \frac{-1080t^6+144t^5-1350t^4+}{720t^4-336t^3+7} \end{matrix}\right.$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\mathrm{e}^{-\frac{\left(i\sqrt{11}-1\right)t}{2}} c_3 + \frac{2c_1 \mathrm{e}^{2t}}{3} - \mathrm{e}^{\frac{\left(i\sqrt{11}+1\right)t}{2}} c_2 + t^2 - \frac{1}{6} \\ -\frac{7}{4} + \frac{c_3\left(1-i\sqrt{11}\right)\mathrm{e}^{-\frac{\left(i\sqrt{11}-1\right)t}{2}}}{2} + \frac{\left(i\sqrt{11}+1\right)c_2\mathrm{e}^{\frac{\left(i\sqrt{11}+1\right)t}{2}}}{2} - \frac{t^2}{2} + \frac{c_1\mathrm{e}^{2t}}{3} - \frac{3t}{2} \\ c_1\mathrm{e}^{2t} + \mathrm{e}^{\frac{\left(i\sqrt{11}+1\right)t}{2}} c_2 + \mathrm{e}^{-\frac{\left(i\sqrt{11}-1\right)t}{2}} c_3 - \frac{3t^2}{2} - \frac{t}{2} - \frac{7}{12} \end{bmatrix}$$

9.5.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) - y + z(t) + t - 1, y' = 2x(t) + y - z(t) - 3t^2, z'(t) = x(t) + y + z(t) + t^2 - t + 2]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} t-1 \\ -3t^2 \\ t^2 - t + 2 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} t-1 \\ -3t^2 \\ t^2 - t + 2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} t-1 \\ -3t^2 \\ t^2 - t + 2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}^{\prime}\left(t \right) = A \cdot \overrightarrow{x}\left(t \right)+\overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \left[\begin{array}{c} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{array}\right]\right], \left[\frac{1}{2}-\frac{\mathrm{I} \sqrt{11}}{2}, \left[\begin{array}{c} \frac{3 \left(\frac{1}{2}-\frac{\mathrm{I} \sqrt{11}}{2}\right)}{\left(-\frac{5}{2}-\frac{\mathrm{I} \sqrt{11}}{2}\right) \left(-\frac{1}{2}-\frac{\mathrm{I} \sqrt{11}}{2}\right)} \\ -\frac{4-\mathrm{I} \sqrt{11}}{-\frac{5}{2}-\frac{\mathrm{I} \sqrt{11}}{2}} \\ 1 \end{array}\right]\right], \left[\frac{\mathrm{I} \sqrt{11}}{2}+\frac{1}{2}, \left[\begin{array}{c} \frac{3 \left(\frac{\mathrm{I} \sqrt{11}}{2}+\frac{1}{2}\right)}{\left(\frac{\mathrm{I} \sqrt{11}}{2}-\frac{5}{2}\right) \left(-\frac{1}{2}+\frac{\mathrm{I} \sqrt{11}}{2}\right)} \\ -\frac{4+\mathrm{I} \sqrt{11}}{\frac{\mathrm{I} \sqrt{11}}{2}-\frac{5}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{1} = \mathrm{e}^{2 t} \cdot \left[\begin{array}{c} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2}-\frac{\mathrm{I} \sqrt{11}}{2}, \left[\begin{array}{c} \frac{3 \left(\frac{1}{2}-\frac{\mathrm{I} \sqrt{11}}{2}\right)}{\left(-\frac{5}{2}-\frac{\mathrm{I} \sqrt{11}}{2}\right) \left(-\frac{1}{2}-\frac{\mathrm{I} \sqrt{11}}{2}\right)} \\ -\frac{4-\mathrm{I} \sqrt{11}}{-\frac{5}{2}-\frac{\mathrm{I} \sqrt{11}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$
{\mathrm e}^{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{11}}{2}\right) t}\cdot \begin{bmatrix} \frac{3 \left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{11}}{2}\right)}{\left(-\frac{5}{2}-\frac{\mathrm{I}\sqrt{11}}{2}\right) \left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{11}}{2}\right)} \\ -\frac{4-\mathrm{I}\sqrt{11}}{-\frac{5}{2}-\frac{\mathrm{I}\sqrt{11}}{2}} \\ 1 \end{bmatrix}
$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$
{\mathrm e}^{\frac{t}{2}}\cdot \left(\cos \left(\frac{\sqrt{11}\, t}{2}\right)-\mathrm{I} \sin \left(\frac{\sqrt{11}\, t}{2}\right)\right)\cdot \begin{bmatrix} \frac{3 \left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{11}}{2}\right)}{\left(-\frac{5}{2}-\frac{\mathrm{I}\sqrt{11}}{2}\right) \left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{11}}{2}\right)} \\ -\frac{4-\mathrm{I}\sqrt{11}}{-\frac{5}{2}-\frac{\mathrm{I}\sqrt{11}}{2}} \\ 1 \end{bmatrix}
$$

- Simplify expression

$$
{\mathrm e}^{\frac{t}{2}}\cdot \begin{bmatrix} \frac{3 \left(\cos \left(\frac{\sqrt{11}\, t}{2}\right)-\mathrm{I} \sin \left(\frac{\sqrt{11}\, t}{2}\right)\right) \left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{11}}{2}\right)}{\left(-\frac{5}{2}-\frac{\mathrm{I}\sqrt{11}}{2}\right) \left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{11}}{2}\right)} \\ -\frac{\left(\cos \left(\frac{\sqrt{11}\, t}{2}\right)-\mathrm{I} \sin \left(\frac{\sqrt{11}\, t}{2}\right)\right) \left(4-\mathrm{I}\sqrt{11}\right)}{-\frac{5}{2}-\frac{\mathrm{I}\sqrt{11}}{2}} \\ \cos \left(\frac{\sqrt{11}\, t}{2}\right)-\mathrm{I} \sin \left(\frac{\sqrt{11}\, t}{2}\right) \end{bmatrix}
$$

- Both real and imaginary parts are solutions to the homogeneous system

$$
\left[\vec{x}_{2}\left(t \right)={\mathrm e}^{\frac{t}{2}}\cdot \begin{bmatrix} -\cos \left(\frac{\sqrt{11}\, t}{2}\right) \\ \frac{\cos \left(\frac{\sqrt{11}\, t}{2}\right)}{2}-\frac{\sqrt{11}\, \sin \left(\frac{\sqrt{11}\, t}{2}\right)}{2} \\ \cos \left(\frac{\sqrt{11}\, t}{2}\right) \end{bmatrix}, \vec{x}_{3}\left(t \right)={\mathrm e}^{\frac{t}{2}}\cdot \begin{bmatrix} \sin \left(\frac{\sqrt{11}\, t}{2}\right) \\ -\frac{\sqrt{11}\, \cos \left(\frac{\sqrt{11}\, t}{2}\right)}{2}-\frac{\sin \left(\frac{\sqrt{11}\, t}{2}\right)}{2} \\ -\sin \left(\frac{\sqrt{11}\, t}{2}\right) \end{bmatrix}\right]
$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_{p}($

  $\vec{x}\left(t \right)=c_{1} \vec{x}_{1}+c_{2} \vec{x}_{2}\left(t \right)+c_{3} \vec{x}_{3}\left(t \right)+\vec{x}_{p}\left(t \right)$

☐ Fundamental matrix

  ○ Let $\phi \left(t \right)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$
\phi \left(t \right)=\begin{bmatrix} \frac{2 \,{\mathrm e}^{2 t}}{3} & -{\mathrm e}^{\frac{t}{2}} \cos \left(\frac{\sqrt{11}\, t}{2}\right) & {\mathrm e}^{\frac{t}{2}} \sin \left(\frac{\sqrt{11}\, t}{2}\right) \\ \frac{{\mathrm e}^{2 t}}{3} & {\mathrm e}^{\frac{t}{2}} \left(\frac{\cos \left(\frac{\sqrt{11}\, t}{2}\right)}{2}-\frac{\sqrt{11}\, \sin \left(\frac{\sqrt{11}\, t}{2}\right)}{2}\right) & {\mathrm e}^{\frac{t}{2}} \left(-\frac{\sqrt{11}\, \cos \left(\frac{\sqrt{11}\, t}{2}\right)}{2}-\frac{\sin \left(\frac{\sqrt{11}\, t}{2}\right)}{2}\right) \\ {\mathrm e}^{2 t} & {\mathrm e}^{\frac{t}{2}} \cos \left(\frac{\sqrt{11}\, t}{2}\right) & -{\mathrm e}^{\frac{t}{2}} \sin \left(\frac{\sqrt{11}\, t}{2}\right) \end{bmatrix}
$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{ccc} \frac{2\,\mathrm{e}^{2t}}{3} & -\mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\,t}{2}\right) & \mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\,t}{2}\right) \\ \frac{\mathrm{e}^{2t}}{3} & \mathrm{e}^{\frac{t}{2}} \left(\frac{\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{2} - \frac{\sqrt{11}\,\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{2}\right) & \mathrm{e}^{\frac{t}{2}} \left(-\frac{\sqrt{11}\,\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{2}\right) \\ \mathrm{e}^{2t} & \mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\,t}{2}\right) & -\mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\,t}{2}\right) \end{array}\right] \cdot \frac{1}{\left[\begin{array}{ccc} \frac{2}{3} & -1 & \\ \frac{1}{3} & \frac{1}{2} & - \\ 1 & 1 & \end{array}\right.}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{ccc} \frac{3\,\mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \frac{\sqrt{11}\,\mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{2\,\mathrm{e}^{2t}}{5} & -\frac{2\sqrt{11}\,\mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\,t}{2}\right)}{11} & -\frac{2\,\mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \\ -\frac{\mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{17\sqrt{11}\,\mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{\mathrm{e}^{2t}}{5} & \frac{\left(\sqrt{11}\,\cos\left(\frac{\sqrt{11}\,t}{2}\right)+\sin\left(\frac{\sqrt{11}\,t}{2}\right)\right)\sqrt{11}\,\mathrm{e}^{\frac{t}{2}}}{11} & -\frac{\mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \\ -\frac{3\,\mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} + \frac{\sqrt{11}\,\mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55} + \frac{3\,\mathrm{e}^{2t}}{5} & \frac{2\sqrt{11}\,\mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{11}\,t}{2}\right)}{11} & \frac{2\,\mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5} - \end{array}\right.$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_{p}(t) = \Phi(t) \cdot \overrightarrow{v}(t)$
- Take the derivative of the particular solution

  $\overrightarrow{x}_{p}^{\prime}(t) = \Phi^{\prime}(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}^{\prime}(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi^{\prime}(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}^{\prime}(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}^{\prime}(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}^{\prime}(t) = \overrightarrow{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}^{\prime}(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$
- Integrate to solve for $\overrightarrow{v}(t)$

$$\overrightarrow{v}(t)=\int_{0}^{t}\frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $$\overrightarrow{x}_{p}(t)=\Phi(t)\cdot\left(\int_{0}^{t}\frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{x}_{p}(t)=\left[\begin{array}{c}\frac{3\,\mathrm{e}^{2t}}{10}-\frac{46\sqrt{11}\,\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{165}-\frac{2\,\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{15}+t^{2}-\frac{1}{6}\\ \frac{3\,\mathrm{e}^{2t}}{20}+\frac{4\sqrt{11}\,\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55}+\frac{8\,\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5}-\frac{t^{2}}{2}-\frac{3t}{2}-\frac{7}{4}\\ \frac{9\,\mathrm{e}^{2t}}{20}+\frac{46\sqrt{11}\,\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{165}+\frac{2\,\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{15}-\frac{3t^{2}}{2}-\frac{t}{2}-\frac{7}{12}\end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{x}(t)=c_{1}\overrightarrow{x}_{1}+c_{2}\overrightarrow{x}_{2}(t)+c_{3}\overrightarrow{x}_{3}(t)+\left[\begin{array}{c}\frac{3\,\mathrm{e}^{2t}}{10}-\frac{46\sqrt{11}\,\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{165}-\frac{2\,\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{15}+t^{2}-\frac{1}{6}\\ \frac{3\,\mathrm{e}^{2t}}{20}+\frac{4\sqrt{11}\,\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{55}+\frac{8\,\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{5}-\frac{t^{2}}{2}-\frac{3t}{2}-\frac{7}{4}\\ \frac{9\,\mathrm{e}^{2t}}{20}+\frac{46\sqrt{11}\,\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{165}+\frac{2\,\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{15}-\frac{3t^{2}}{2}-\frac{t}{2}-\frac{7}{12}\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t)\\ y\\ z(t)\end{array}\right]=\left[\begin{array}{c}-\frac{1}{6}+\frac{(-2-15c_{2})\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{15}+\left(c_{3}-\frac{46\sqrt{11}}{165}\right)\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)+\frac{(9+20c_{1})\mathrm{e}^{2t}}{30}+t^{2}\\ -\frac{7}{4}+\frac{\left(-\sqrt{11}\,c_{3}+c_{2}+\frac{16}{5}\right)\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{2}-\frac{\left(\left(c_{2}-\frac{8}{55}\right)\sqrt{11}+c_{3}\right)\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{2}+\frac{(9+20c_{1})\mathrm{e}^{2t}}{60}-\frac{t^{2}}{2}-\\ -\frac{7}{12}+\frac{(15c_{2}+2)\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{15}-\left(c_{3}-\frac{46\sqrt{11}}{165}\right)\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)+\frac{(9+20c_{1})\mathrm{e}^{2t}}{20}-\frac{3t^{2}}{2}-\frac{t}{2}\end{array}\right.$$

- Solution to the system of ODEs

$$\left\{x(t)=-\frac{1}{6}+\frac{(-2-15c_{2})\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)}{15}+\left(c_{3}-\frac{46\sqrt{11}}{165}\right)\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{11}\,t}{2}\right)+\frac{(9+20c_{1})\mathrm{e}^{2t}}{30}+t^{2},y=-\frac{7}{4}+\right.$$

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 172

```
dsolve([diff(x(t),t)=x(t)-y(t)+z(t)+t-1,diff(y(t),t)=2*x(t)+y(t)-z(t)-3*t^2,diff(z(t),t)=x(t
```

$$
\begin{aligned}
x(t) &= t^{2}-\frac{1}{6}+c_{1} \mathrm{e}^{2 t}+c_{2} \mathrm{e}^{\frac{t}{2}} \cos \left(\frac{\sqrt{11}\, t}{2}\right)+c_{3} \mathrm{e}^{\frac{t}{2}} \sin \left(\frac{\sqrt{11}\, t}{2}\right) \\
y(t) &= -\frac{t^{2}}{2}-\frac{7}{4}+\frac{c_{1} \mathrm{e}^{2 t}}{2}-\frac{c_{2} \mathrm{e}^{\frac{t}{2}} \cos \left(\frac{\sqrt{11}\, t}{2}\right)}{2}+\frac{c_{2} \mathrm{e}^{\frac{t}{2}} \sqrt{11}\, \sin \left(\frac{\sqrt{11}\, t}{2}\right)}{2} \\
&\quad -\frac{c_{3} \mathrm{e}^{\frac{t}{2}} \sin \left(\frac{\sqrt{11}\, t}{2}\right)}{2}-\frac{c_{3} \mathrm{e}^{\frac{t}{2}} \sqrt{11}\, \cos \left(\frac{\sqrt{11}\, t}{2}\right)}{2}-\frac{3 t}{2} \\
z(t) &= -\frac{t}{2}+\frac{3 c_{1} \mathrm{e}^{2 t}}{2}-c_{2} \mathrm{e}^{\frac{t}{2}} \cos \left(\frac{\sqrt{11}\, t}{2}\right)-c_{3} \mathrm{e}^{\frac{t}{2}} \sin \left(\frac{\sqrt{11}\, t}{2}\right)-\frac{3 t^{2}}{2}-\frac{7}{12}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 15.906 (sec). Leaf size: 304

```
DSolve[{x'[t]==x[t]-y[t]+z[t]+t-1,y'[t]==2*x[t]+y[t]-z[t]-3*t^2,z'[t]==x[t]+y[t]+z[t]+t^2-t+
```

$$
\begin{aligned}
x(t) &\to t^{2}+\frac{2}{5} c_{1} e^{2 t}+\frac{2}{5} c_{3} e^{2 t} \\
&\quad +\frac{1}{5}\left(3 c_{1}-2 c_{3}\right) e^{t / 2} \cos \left(\frac{\sqrt{11} t}{2}\right)-\frac{\left(c_{1}+10 c_{2}-4 c_{3}\right) e^{t / 2} \sin \left(\frac{\sqrt{11} t}{2}\right)}{5 \sqrt{11}}-\frac{1}{6} \\
y(t) &\to \frac{1}{220}\Bigg(-11\left(10 t^{2}+30 t-4\left(c_{1}+c_{3}\right) e^{2 t}+35\right) \\
&\quad -44\left(c_{1}-5 c_{2}+c_{3}\right) e^{t / 2} \cos \left(\frac{\sqrt{11} t}{2}\right)+4 \sqrt{11}\left(17 c_{1}+5 c_{2}-13 c_{3}\right) e^{t / 2} \sin \left(\frac{\sqrt{11} t}{2}\right)\Bigg) \\
z(t) &\to -\frac{3 t^{2}}{2}-\frac{t}{2}+\frac{3}{5}\left(c_{1}+c_{3}\right) e^{2 t} \\
&\quad -\frac{1}{5}\left(3 c_{1}-2 c_{3}\right) e^{t / 2} \cos \left(\frac{\sqrt{11} t}{2}\right)+\frac{\left(c_{1}+10 c_{2}-4 c_{3}\right) e^{t / 2} \sin \left(\frac{\sqrt{11} t}{2}\right)}{5 \sqrt{11}}-\frac{7}{12}
\end{aligned}
$$

9.6 problem 6

Internal problem ID [6716]
Internal file name [`OUTPUT/5964_Sunday_June_05_2022_04_05_01_PM_1900378/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -3x(t) + 4y + 2\sin(t)\cos(t)\,\mathrm{e}^{-t} \\ y' &= 5x(t) + 9z(t) + 8\cos(t)^2\,\mathrm{e}^{-t} - 4\,\mathrm{e}^{-t} \\ z'(t) &= y + 6z(t) - \mathrm{e}^{-t} \end{aligned}$$

9.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as Warning. Unable to find the matrix exponential.

9.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 2\sin(t)\cos(t)\,\mathrm{e}^{-t} \\ 8\cos(t)^2\,\mathrm{e}^{-t} - 4\,\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 4 & 0 \\ 5 & -\lambda & 9 \\ 0 & 1 & 6-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 - 47\lambda + 93 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{\left(-594 + 6i\sqrt{83949}\right)^{\frac{1}{3}}}{3} + \frac{50}{\left(-594 + 6i\sqrt{83949}\right)^{\frac{1}{3}}} + 1$$

$$\lambda_2 = -\frac{\left(-594 + 6i\sqrt{83949}\right)^{\frac{1}{3}}}{6} - \frac{25}{\left(-594 + 6i\sqrt{83949}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3} - \frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_3 = -\frac{\left(-594 + 6i\sqrt{83949}\right)^{\frac{1}{3}}}{6} - \frac{25}{\left(-594 + 6i\sqrt{83949}\right)^{\frac{1}{3}}} + 1 - \frac{i\sqrt{3}\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3} - \frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity
$-\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6}-\frac{25}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1+\frac{i\sqrt{3}\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2}$	1
$-\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6}-\frac{25}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1-\frac{i\sqrt{3}\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2}$	1
$\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1$	1

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{bmatrix} - \left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}\right.\right.$$

$$\begin{bmatrix} \frac{-\left(-594+6i\sqrt{83949}\right)^{\frac{2}{3}}-12\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}-150}{3\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}} & 4 & 0 \\ 5 & -\frac{\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{2}{3}}}{3}+\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}+50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}} & 9 \\ 0 & 1 & \frac{-\left(-594+6i\sqrt{83949}\right)^{\frac{2}{3}}+15(-}{3(-594+6i\sqrt{8}} \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented

matrix is

$$\begin{bmatrix} -4-\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}} & 4 & \\ 5 & -1-\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}} & \\ 0 & 1 & 5-\frac{\left(-594+6i\sqrt{83949}\right.}{3} \end{bmatrix}$$

$$R_2 = R_2 - \frac{5R_1}{-4-\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}} \Longrightarrow \begin{bmatrix} \frac{-\left(-594+6i\sqrt{83949}\right)^{\frac{2}{3}}-12\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}-150}{3\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}} & \\ 0 & \\ 0 & \end{bmatrix}$$

$$R_3 = R_3 + \frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}\left(\left(-594+6i\sqrt{83949}\right)^{\frac{2}{3}}+12\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}+150\right)R_2}{2\left(i\sqrt{83949}+276\right)\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}+30i\sqrt{83949}+52\left(-594+6i\sqrt{83949}\right)^{\frac{2}{3}}+4530} \Longrightarrow$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{-\left(-594+6i\sqrt{83949}\right)^{\frac{2}{3}}-12\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}-150}{3\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}} & 4 \\ 0 & -\frac{2\left(\left(i\sqrt{83949}+276\right)\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}+15i\sqrt{83949}+26\left(-594+6i\sqrt{83949}\right)^{\frac{2}{3}}+\right.}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}\left(\left(-594+6i\sqrt{83949}\right)^{\frac{2}{3}}+12\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}+150\right.} \\ 0 & 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{324(-594+6i\sqrt{83949})^{\frac{1}{3}} t (i\sqrt{83949}+2(-594+6i\sqrt{83949})^{\frac{2}{3}}}{(i\sqrt{83949}\,(-594+6i\sqrt{83949})^{\frac{1}{3}}+15i\sqrt{83949}+26(-594+6i\sqrt{83949})^{\frac{2}{3}}+276(-594+6i\sqrt{83949})} \right.$

Hence the solution is

$$\left[\begin{array}{c} \frac{324(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}} t (\mathrm{I}\sqrt{83949}+2(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+25(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}-99)}{(\mathrm{I}\sqrt{83949}\,(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+15\,\mathrm{I}\sqrt{83949}+26(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+276(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+2265)((-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+12(} \\ \frac{27t(\mathrm{I}\sqrt{83949}+2(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+25(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}-99)}{\mathrm{I}\sqrt{83949}\,(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+15\,\mathrm{I}\sqrt{83949}+26(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+276(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+2265} \\ t \end{array}\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} \frac{324(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}} t (\mathrm{I}\sqrt{83949}+2(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+25(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}-99)}{(\mathrm{I}\sqrt{83949}\,(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+15\,\mathrm{I}\sqrt{83949}+26(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+276(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+2265)((-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+12(} \\ \frac{27t(\mathrm{I}\sqrt{83949}+2(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+25(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}-99)}{\mathrm{I}\sqrt{83949}\,(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+15\,\mathrm{I}\sqrt{83949}+26(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+276(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+2265} \\ t \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} \frac{324(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}} t (\mathrm{I}\sqrt{83949}+2(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+25(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}-99)}{(\mathrm{I}\sqrt{83949}\,(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+15\,\mathrm{I}\sqrt{83949}+26(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+276(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+2265)((-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+12(} \\ \frac{27t(\mathrm{I}\sqrt{83949}+2(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+25(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}-99)}{\mathrm{I}\sqrt{83949}\,(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+15\,\mathrm{I}\sqrt{83949}+26(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{2}{3}}+276(-594+6\,\mathrm{I}\sqrt{83949})^{\frac{1}{3}}+2265} \\ t \end{array}\right.$$

Which is normalized to

$$\begin{bmatrix} \dfrac{324\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{1}{3}} t\Bigl(\mathrm{I}\sqrt{83949}+2\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{2}{3}}+25\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{1}{3}}-99\Bigr)}{\Bigl(\mathrm{I}\sqrt{83949}\,\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{1}{3}}+15\,\mathrm{I}\sqrt{83949}+26\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{2}{3}}+276\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{1}{3}}+2265\Bigr)\Bigl(\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{2}{3}}+12\Bigl(} \\ \dfrac{27t\Bigl(\mathrm{I}\sqrt{83949}+2\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{2}{3}}+25\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{1}{3}}-99\Bigr)}{\mathrm{I}\sqrt{83949}\,\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{1}{3}}+15\,\mathrm{I}\sqrt{83949}+26\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{2}{3}}+276\Bigl(-594+6\,\mathrm{I}\sqrt{83949}\Bigr)^{\frac{1}{3}}+2265} \\ t \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6} - \frac{25}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}} + 1 - \frac{i\sqrt{3}\Bigl(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3} - \frac{50}{\bigl(-594+6i\sqrt{83949}}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\Biggl(\begin{bmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{bmatrix} - \Biggl(-\frac{(-594+6i}{6}$$

$$\begin{bmatrix} -4 + \frac{5\sqrt{3}\,\sqrt{2}\,\cos\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6}+\frac{\pi}{6}\right)}{3} - 5\sqrt{2}\,\sin\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6}+\frac{\pi}{6}\right) & 4 \\ 5 & -1 + \frac{5\sqrt{3}\,\sqrt{2}\,\cos\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6}+\frac{\pi}{6}\right)}{3} \\ 0 & 1 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented

matrix is

$$\left[\begin{array}{cc} -4+\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6}+\frac{25}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2} & \\ 5 & -1+\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6} \\ 0 & \end{array}\right.$$

$$R_2 = R_2 - \frac{5R_1}{-4+\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6}+\frac{25}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow \left[\begin{array}{c} -4+ \\ \\ \\ \end{array}\right.$$

$$R_3 = R_3 - \frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}\left(150-24\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}+i\left(\left(-594+6i\sqrt{83949}\right)^{\frac{2}{3}}-1\right.\right.}{2\left(\left(276i\sqrt{3}-i\sqrt{83949}-3\sqrt{27983}-276\right)\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}-2265i\sqrt{3}-15i\sqrt{83949}+4\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{c} -4+\frac{5\sqrt{3}\,\sqrt{2}\,\cos\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6}+\frac{\pi}{6}\right)}{3}-5\sqrt{2}\,\sin\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6}+\frac{\pi}{6}\right) \\ 0 \quad \frac{2\left(276i\sqrt{3}-i\sqrt{83949}-3\sqrt{27983}-276\right)\left(-594+\right.}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}\left(150-24\left(-594+\right.\right.} \\ 0 \end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{162t\left(i\sqrt{3}\,\sqrt{27983}-25i\sqrt{3}\,\left(-594\right.\right.}{\left(i\sqrt{27983}\,\sqrt{3}\,\left(-594+6i\sqrt{3}\,\sqrt{27983}\right)^{\frac{1}{3}}+15i\sqrt{3}\,\sqrt{27983}-276i\sqrt{3}\,\left(-594+6i\sqrt{3}\,\sqrt{27983}\right)^{\frac{1}{3}}+3\sqrt{}\right.}\right.$

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} \frac{162t\left(\mathrm{I}\,\sqrt{3}\,\sqrt{27983}-25\,\mathrm{I}\,\sqrt{3}\,\left(-594+6\,\mathrm{I}\,\sqrt{3}\,\sqrt{27983}\right)^{\frac{1}{3}}-99\,\mathrm{I}\,\sqrt{3}-4\left(-594+6\,\mathrm{I}\right.\right.}{\left(\mathrm{I}\,\sqrt{27983}\,\sqrt{3}\,\left(-594+6\,\mathrm{I}\,\sqrt{3}\,\sqrt{27983}\right)^{\frac{1}{3}}+15\,\mathrm{I}\,\sqrt{3}\,\sqrt{27983}-276\,\mathrm{I}\left(-594+6\,\mathrm{I}\,\sqrt{3}\,\sqrt{27983}\right)^{\frac{1}{3}}\sqrt{3}+3\sqrt{27983}\,\left(-594+6\,\mathrm{I}\,\sqrt{3}\,\sqrt{27983}\right)^{\frac{1}{3}}+2265\,\mathrm{I}\right.} \\ \frac{27t\left(\mathrm{I}\,\sqrt{3}\,\sqrt{27983}-25\,\mathrm{I}\,\sqrt{3}\,\left(-594+6\,\mathrm{I}\,\sqrt{3}\,\sqrt{27983}\right)^{\frac{1}{3}}-99\,\mathrm{I}\,\sqrt{3}-4\left(-594+6\,\mathrm{I}\right.\right.}{\mathrm{I}\,\sqrt{27983}\,\sqrt{3}\,\left(-594+6\,\mathrm{I}\,\sqrt{3}\,\sqrt{27983}\right)^{\frac{1}{3}}+15\,\mathrm{I}\,\sqrt{3}\,\sqrt{27983}-276\,\mathrm{I}\left(-594+6\,\mathrm{I}\,\sqrt{3}\,\sqrt{27983}\right)^{\frac{1}{3}}\sqrt{3}+3\sqrt{27983}\,\left(-594+6\,\mathrm{I}\,\sqrt{3}\right.} \\ t \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

Considering the eigenvalue $\lambda_3 = -\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6}-\frac{25}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1+\frac{i\sqrt{3}\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6i\sqrt{83949}\right.}\right.}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{bmatrix} - \left(-\frac{\left(-594 + 6i\right.}{6}\right.\right.$$

$$\begin{bmatrix} -4 + \frac{5\sqrt{3}\,\sqrt{2}\,\cos\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6} + \frac{\pi}{6}\right)}{3} + 5\sqrt{2}\,\sin\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6} + \frac{\pi}{6}\right) & \\ 5 & -1 + \frac{5\sqrt{3}\,\sqrt{2}\,\cos\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6} + \frac{\pi}{6}\right)}{3} \\ 0 & \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} -4 + \frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6} + \frac{25}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3} - \frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2} & \\ 5 & -1 + \frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6} \\ 0 & \end{bmatrix}$$

$$R_2 = R_2 - \frac{5R_1}{-4 + \frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6} + \frac{25}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3} - \frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow \begin{bmatrix} -4 + \end{bmatrix}$$

$$R_3 = R_3 - \frac{(i(-594+6i\sqrt{3}\sqrt{27983})^{\frac{2}{3}}\sqrt{3} - 150i\sqrt{3} - (-594+6i\sqrt{3}\sqrt{27983})^{\frac{2}{3}} + 24(-594+6i}{2\,(2265 + (-594+6i\sqrt{3}\sqrt{27983})^{\frac{1}{3}}\,(276 + i\,(\sqrt{27983}+276)\,\sqrt{3} - 3\sqrt{27983}) - 52\,(-594+}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 + \frac{5\sqrt{3}\sqrt{2}\cos\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6}+\frac{\pi}{6}\right)}{3} + 5\sqrt{2}\,\sin\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6}+\frac{\pi}{6}\right) & \\ 0 & \frac{4530+2(-594+6i\sqrt{3}\sqrt{27983})^{\frac{1}{3}}(276+i(\sqrt{27}}{(i(-594+6i\sqrt{3}\sqrt{27983})^{\frac{2}{3}}\sqrt{3}-150i\sqrt{3}} \\ 0 & \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = -\frac{324(i\sqrt{3}\sqrt{2}}{\left(5\sqrt{2}\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6}\right)+15\sqrt{2}\,\cos\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6}\right)-12\right)(-3\sqrt{27983}\,(-} \right.$

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{324(\mathrm{I}\sqrt{3}\sqrt{27983}+25\,\mathrm{I}\sqrt{3}\,(-594+6\,\mathrm{I}\sqrt{3}\sqrt{27983})^{\frac{1}{3}}+9}{\left(5\sqrt{2}\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6}\right)+15\sqrt{2}\,\cos\left(\frac{\arctan\left(\frac{33\sqrt{83949}}{12358}\right)}{6}\right)-12\right)\left(-3\sqrt{27983}\,(-594+6\,\mathrm{I}\sqrt{3}\sqrt{27983})^{\frac{1}{3}}+\mathrm{I}\sqrt{27983}\sqrt{3}\,(-594+\right.} \\ \frac{27t(\mathrm{I}\sqrt{3}\sqrt{27983}+25\,\mathrm{I}\sqrt{3}\,(-594+6\,\mathrm{I}\sqrt{3}\sqrt{27983})^{\frac{1}{3}}+99}{-3\sqrt{27983}\,(-594+6\,\mathrm{I}\sqrt{3}\sqrt{27983})^{\frac{1}{3}}+\mathrm{I}\sqrt{27983}\sqrt{3}\,(-594+6\,\mathrm{I}\sqrt{3}\sqrt{27983})^{\frac{1}{3}}+15\,\mathrm{I}\sqrt{3}\sqrt{27983}+27} \\ \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity	
	algebraic m	geomet
$\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1$	1	1
$-\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6}-\frac{25}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1+\frac{i\sqrt{3}\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2}$	1	1
$-\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{6}-\frac{25}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1-\frac{i\sqrt{3}\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2}$	1	1

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \left[\begin{array}{c} \frac{324\,\mathrm{e}^{\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)t}}{\left(\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}-26+\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)^{2}\right)\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}\right.} \\ \frac{81\,\mathrm{e}^{\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)t}}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}-26+\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)} \\ \mathrm{e}^{\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)t} \end{array} \right.$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \text{Expression too large to display}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \text{Expression too large to display}$$

Hence

$$\vec{x}_p(t) = \text{Expression too large to display} \int \text{Expression too large to display} \begin{bmatrix} 2\sin(t)\cos(t)\,\mathrm{e}^{-t} \\ 8\cos(t)^2\,\mathrm{e}^{-t} - 4\,\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} \end{bmatrix} dt$$

$$= \text{Expression too large to display} \int \text{Expression too large to display}\, dt$$

$$= \text{Expression too large to display}\text{Expression too large to display}$$

$$= \text{Expression too large to display}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \Bigg[\begin{matrix} \dfrac{324c_1\mathrm{e}^{\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)t}}{\Bigg(\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}-26+\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)^{2}\Bigg)\Bigg(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+} \\ \dfrac{81c_1\mathrm{e}^{\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)t}}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}-26+\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)^{2}} \\ c_1\mathrm{e}^{\left(\frac{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6i\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)t} \end{matrix}$$

Which becomes

$$\text{Expression too large to display}$$

9.6.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -3x(t) + 4y + \frac{2\cos(t)\sin(t)}{\mathrm{e}^t}, y' = 5x(t) + 9z(t) + \frac{8\cos(t)^2}{\mathrm{e}^t} - \frac{4}{\mathrm{e}^t}, z'(t) = y + 6z(t) - \frac{1}{\mathrm{e}^t}\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -\frac{3x(t)\mathrm{e}^t - 4\,\mathrm{e}^t y - 2\cos(t)\sin(t)}{\mathrm{e}^t} + 3x(t) - 4y \\ \frac{9z(t)\mathrm{e}^t + 5x(t)\mathrm{e}^t + 8\cos(t)^2 - 4}{\mathrm{e}^t} - 5x(t) - 9z(t) \\ \frac{6z(t)\mathrm{e}^t + \mathrm{e}^t y - 1}{\mathrm{e}^t} - y - 6z(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\Bigg[\Bigg[\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}+1,\ \Bigg[\frac{\Bigg(\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}-26+\Bigg(\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)}{3}}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}-}$$

- Consider eigenpair

$$\Bigg[\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}+1,\ \Bigg[\frac{\Bigg(\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}-26+\Bigg(\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{3}}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}-2}$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1=\mathrm{e}^{\left(\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)t}\cdot\Bigg[\frac{\Bigg(\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}-26+\Bigg(\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)}{3}}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}-}$$

- Consider eigenpair

$$\left[-\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{6}-\frac{25}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}+1-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2},\right.$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2=\mathrm{e}^{\left(-\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{6}-\frac{25}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}+1-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2}\right)t}\cdot$$

- Consider eigenpair

$$\left[-\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{6}-\frac{25}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}+1+\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2},\right.$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \mathrm{e}^{\left(-\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{6}-\frac{25}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}+1+\frac{\mathrm{I}\sqrt{3}\,\left(\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{3}-\frac{50}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}\right)}{2}\right)t} \cdot \left[\left(-\frac{\left(-594+6\,\mathrm{I}\sqrt{839}\right.}{2}\right.\right.$$

- General solution to the system of ODEs

  $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$

- Substitute solutions into the general solution

$$\vec{x} = c_1\mathrm{e}^{\left(\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}{3}+\frac{50}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}+1\right)t} \cdot \left[\frac{\left(\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}-26+\left(\frac{\left(-594+6\,\mathrm{I}\sqrt{8394}\right.}{3}\right.\right.}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}+\frac{150}{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{1}{3}}}}\right.$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \left[-\frac{977724\left(-\frac{3c_2\left(\left(\left(\mathrm{I}+\frac{\sqrt{3}}{3}\right)\sqrt{27983}+\frac{823\,\mathrm{I}}{6}-\frac{823\sqrt{3}}{6}\right)\left(-594+6\,\mathrm{I}\sqrt{3}\,\sqrt{27983}\right)^{\frac{1}{3}}+\left(\left(\frac{\mathrm{I}}{12}-\frac{\sqrt{3}}{36}\right)\sqrt{27983}-\frac{133\,\mathrm{I}}{12}-\frac{133\sqrt{3}}{12}\right)\left(-594\right.\right.}{}\right.\right.}{\frac{\left(-594+6\,\mathrm{I}\sqrt{83949}\right)^{\frac{2}{3}}}{}}\right.$$

- Solution to the system of ODEs

$$\left\{ x(t) = -\frac{977724\left(-\frac{3c_2\left(\left(\left(\mathrm{I}+\frac{\sqrt{3}}{3}\right)\sqrt{27983}+\frac{823\,\mathrm{I}}{6}-\frac{823\sqrt{3}}{6}\right)\left(-594+6\,\mathrm{I}\sqrt{3}\,\sqrt{27983}\right)^{\frac{1}{3}}+\left(\left(\frac{\mathrm{I}}{12}-\frac{\sqrt{3}}{36}\right)\sqrt{27983}-\frac{133\,\mathrm{I}}{12}-\frac{133\sqrt{3}}{12}\right)\left(-594+6\,\mathrm{I}\sqrt{3}\,\sqrt{}\right.\right.}{823}\right.\right.}{} \right.$$

✗ Solution by Maple

```
dsolve([diff(x(t),t)=-3*x(t)+4*y(t)+exp(-t)*sin(2*t),diff(y(t),t)=5*x(t)+9*z(t)+4*exp(-t)*co
```

No solution found

✓ Solution by Mathematica

Time used: 0.649 (sec). Leaf size: 2949

```
DSolve[{x'[t]==-3*x[t]+4*y[t]+Exp[-t]*Sin[2*t],y'[t]==5*x[t]+9*z[t]+4*Exp[-t]*Cos[2*t],z'[t]
```

Too large to display

9.7 problem 7

Internal problem ID [6717]
Internal file name [`OUTPUT/5965_Sunday_June_05_2022_04_07_53_PM_80474597/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 4x(t) + 2y + \mathrm{e}^{t} \\
y' &= -x(t) + 3y - \mathrm{e}^{t}
\end{aligned}
$$

9.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{t} \\ -\mathrm{e}^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^{\frac{7t}{2}} \cos\left(\frac{\sqrt{7}\, t}{2}\right)+\frac{\sqrt{7}\, \mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{7} & \frac{4 \sqrt{7}\, \mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{7} \\ -\frac{2 \sqrt{7}\, \mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{7} & \mathrm{e}^{\frac{7t}{2}} \cos\left(\frac{\sqrt{7}\, t}{2}\right)-\frac{\sqrt{7}\, \mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{7} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{\frac{7t}{2}} \left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right)+7 \cos\left(\frac{\sqrt{7}\, t}{2}\right)\right)}{7} & \frac{4 \sqrt{7}\, \mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{7} \\ -\frac{2 \sqrt{7}\, \mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{7} & -\frac{\mathrm{e}^{\frac{7t}{2}} \left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right)-7 \cos\left(\frac{\sqrt{7}\, t}{2}\right)\right)}{7} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At} \vec{c} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{\frac{7t}{2}} \left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right)+7 \cos\left(\frac{\sqrt{7}\, t}{2}\right)\right)}{7} & \frac{4 \sqrt{7}\, \mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{7} \\ -\frac{2 \sqrt{7}\, \mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{7} & -\frac{\mathrm{e}^{\frac{7t}{2}} \left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right)-7 \cos\left(\frac{\sqrt{7}\, t}{2}\right)\right)}{7} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{\frac{7t}{2}} \left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right)+7 \cos\left(\frac{\sqrt{7}\, t}{2}\right)\right) c_1}{7}+\frac{4 \sqrt{7}\, \mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right) c_2}{7} \\ -\frac{2 \sqrt{7}\, \mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right) c_1}{7}-\frac{\mathrm{e}^{\frac{7t}{2}} \left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right)-7 \cos\left(\frac{\sqrt{7}\, t}{2}\right)\right) c_2}{7} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{\frac{7t}{2}} \left(\sqrt{7}\, (c_1+4 c_2) \sin\left(\frac{\sqrt{7}\, t}{2}\right)+7 \cos\left(\frac{\sqrt{7}\, t}{2}\right) c_1\right)}{7} \\ -\frac{2 \left(\sqrt{7}\, \left(c_1+\frac{c_2}{2}\right) \sin\left(\frac{\sqrt{7}\, t}{2}\right)-\frac{7 \cos\left(\frac{\sqrt{7}\, t}{2}\right) c_2}{2}\right) \mathrm{e}^{\frac{7t}{2}}}{7} \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} -\frac{\left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right)-7 \cos\left(\frac{\sqrt{7}\, t}{2}\right)\right) \mathrm{e}^{-\frac{7t}{2}}}{7} & -\frac{4 \sqrt{7}\, \mathrm{e}^{-\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{7} \\ \frac{2 \sqrt{7}\, \mathrm{e}^{-\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{7} & \frac{\left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right)+7 \cos\left(\frac{\sqrt{7}\, t}{2}\right)\right) \mathrm{e}^{-\frac{7t}{2}}}{7} \end{bmatrix}
\end{aligned}
$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^{\frac{7t}{2}}\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)+7\cos\left(\frac{\sqrt{7}\,t}{2}\right)\right)}{7} & \frac{4\sqrt{7}\,e^{\frac{7t}{2}}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{7} \\ -\frac{2\sqrt{7}\,e^{\frac{7t}{2}}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{7} & -\frac{e^{\frac{7t}{2}}\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)-7\cos\left(\frac{\sqrt{7}\,t}{2}\right)\right)}{7} \end{bmatrix} \int \begin{bmatrix} -\frac{\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)-7\cos\left(\frac{\sqrt{7}\,t}{2}\right)\right)e^{-\frac{7t}{2}}}{7} \\ \frac{2\sqrt{7}\,e^{-\frac{7t}{2}}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{7} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{\frac{7t}{2}}\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)+7\cos\left(\frac{\sqrt{7}\,t}{2}\right)\right)}{7} & \frac{4\sqrt{7}\,e^{\frac{7t}{2}}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{7} \\ -\frac{2\sqrt{7}\,e^{\frac{7t}{2}}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{7} & -\frac{e^{\frac{7t}{2}}\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)-7\cos\left(\frac{\sqrt{7}\,t}{2}\right)\right)}{7} \end{bmatrix} \begin{bmatrix} -\frac{e^{-\frac{5t}{2}}\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)+7\cos\left(\frac{\sqrt{7}\,t}{2}\right)\right)}{14} \\ \frac{e^{-\frac{5t}{2}}\left(-\frac{3\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{7}+\cos\left(\frac{\sqrt{7}\,t}{2}\right)\right)}{4} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^{t}}{2} \\ \frac{e^{t}}{4} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \frac{\sqrt{7}\,e^{\frac{7t}{2}}\left(c_1+4c_2\right)\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{7} + e^{\frac{7t}{2}}\cos\left(\frac{\sqrt{7}\,t}{2}\right)c_1 - \frac{e^{t}}{2} \\ -\frac{2\sqrt{7}\,e^{\frac{7t}{2}}\left(c_1+\frac{c_2}{2}\right)\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{7} + e^{\frac{7t}{2}}\cos\left(\frac{\sqrt{7}\,t}{2}\right)c_2 + \frac{e^{t}}{4} \end{bmatrix} \end{aligned}$$

9.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^2-7\lambda+14=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=\frac{7}{2}+\frac{i\sqrt{7}}{2}$$
$$\lambda_2=\frac{7}{2}-\frac{i\sqrt{7}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{7}{2}-\frac{i\sqrt{7}}{2}$	1	complex eigenvalue
$\frac{7}{2}+\frac{i\sqrt{7}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=\frac{7}{2}-\frac{i\sqrt{7}}{2}$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}-\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{i\sqrt{7}}{2}+\frac{1}{2} & 2 \\ -1 & -\frac{1}{2}+\frac{i\sqrt{7}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{i\sqrt{7}}{2}+\frac{1}{2} & 2 & 0 \\ -1 & -\frac{1}{2}+\frac{i\sqrt{7}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{\frac{i\sqrt{7}}{2}+\frac{1}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{i\sqrt{7}}{2}+\frac{1}{2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{i\sqrt{7}}{2}+\frac{1}{2} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{4t}{1+i\sqrt{7}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{4t}{1+\mathrm{I}\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4t}{1+i\sqrt{7}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{4t}{1+\mathrm{I}\sqrt{7}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{1+i\sqrt{7}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{4t}{1+\mathrm{I}\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{1+i\sqrt{7}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{4t}{1+\mathrm{I}\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{1+i\sqrt{7}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{7}{2} + \frac{i\sqrt{7}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} - \left(\frac{7}{2} + \frac{i\sqrt{7}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{7}}{2} & 2 \\ -1 & -\frac{1}{2} - \frac{i\sqrt{7}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2} - \frac{i\sqrt{7}}{2} & 2 & 0 \\ -1 & -\frac{1}{2} - \frac{i\sqrt{7}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{\frac{1}{2} - \frac{i\sqrt{7}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} - \frac{i\sqrt{7}}{2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{7}}{2} & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{-1+i\sqrt{7}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{-1+\mathrm{I}\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{-1+i\sqrt{7}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{-1+\mathrm{I}\sqrt{7}} \\ t \end{bmatrix} = t\begin{bmatrix} \frac{4}{-1+i\sqrt{7}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{4t}{-1+\mathrm{I}\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{-1+i\sqrt{7}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\left[\begin{array}{c}\frac{4t}{-1+\mathrm{I}\sqrt{7}}\\ t\end{array}\right] = \left[\begin{array}{c}\frac{4}{-1+i\sqrt{7}}\\ 1\end{array}\right]$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{7}{2}+\frac{i\sqrt{7}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{2}{-\frac{1}{2}+\frac{i\sqrt{7}}{2}}\\ 1\end{array}\right]$
$\frac{7}{2}-\frac{i\sqrt{7}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{2}{-\frac{1}{2}-\frac{i\sqrt{7}}{2}}\\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t)\\ y\end{array}\right] = c_1\left[\begin{array}{c}\frac{2\,\mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{7}}{2}}\\ \mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t}\end{array}\right] + c_2\left[\begin{array}{c}\frac{2\,\mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{7}}{2}}\\ \mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t}\end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{cc} \frac{2\,\mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{7}}{2}} & \frac{2\,\mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{7}}{2}} \\ \mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t} & \mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \left[\begin{array}{cc} \frac{i\sqrt{7}\,\mathrm{e}^{-\frac{\left(7+i\sqrt{7}\right)t}{2}}}{7} & \frac{\sqrt{7}\,\mathrm{e}^{-\frac{\left(7+i\sqrt{7}\right)t}{2}}\left(i+\sqrt{7}\right)}{14} \\ -\frac{i\sqrt{7}\,\mathrm{e}^{\frac{\left(i\sqrt{7}-7\right)t}{2}}}{7} & -\frac{\sqrt{7}\,\mathrm{e}^{\frac{\left(i\sqrt{7}-7\right)t}{2}}\left(i-\sqrt{7}\right)}{14} \end{array}\right]$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \left[\begin{array}{cc} \frac{2\,\mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{7}}{2}} & \frac{2\,\mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{7}}{2}} \\ \mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t} & \mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t} \end{array}\right] \int \left[\begin{array}{cc} \frac{i\sqrt{7}\,\mathrm{e}^{-\frac{\left(7+i\sqrt{7}\right)t}{2}}}{7} & \frac{\sqrt{7}\,\mathrm{e}^{-\frac{\left(7+i\sqrt{7}\right)t}{2}}\left(i+\sqrt{7}\right)}{14} \\ -\frac{i\sqrt{7}\,\mathrm{e}^{\frac{\left(i\sqrt{7}-7\right)t}{2}}}{7} & -\frac{\sqrt{7}\,\mathrm{e}^{\frac{\left(i\sqrt{7}-7\right)t}{2}}\left(i-\sqrt{7}\right)}{14} \end{array}\right] \left[\begin{array}{c} \mathrm{e}^{t} \\ -\mathrm{e}^{t} \end{array}\right] dt \\
&= \left[\begin{array}{cc} \frac{2\,\mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{7}}{2}} & \frac{2\,\mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{7}}{2}} \\ \mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t} & \mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t} \end{array}\right] \int \left[\begin{array}{c} \frac{\mathrm{e}^{-\frac{t\left(i\sqrt{7}+5\right)}{2}}\left(i\sqrt{7}-7\right)}{14} \\ -\frac{\mathrm{e}^{\frac{t\left(i\sqrt{7}-5\right)}{2}}\left(7+i\sqrt{7}\right)}{14} \end{array}\right] dt \\
&= \left[\begin{array}{cc} \frac{2\,\mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{7}}{2}} & \frac{2\,\mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{7}}{2}} \\ \mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t} & \mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t} \end{array}\right] \left[\begin{array}{c} -\frac{\left(1+i\sqrt{7}\right)\sqrt{7}\,\mathrm{e}^{-\frac{t\left(i\sqrt{7}+5\right)}{2}}\left(\sqrt{7}+5i\right)}{224} \\ \frac{\sqrt{7}\,\mathrm{e}^{\frac{t\left(i\sqrt{7}-5\right)}{2}}\left(i+\sqrt{7}\right)\left(i\sqrt{7}+5\right)}{224} \end{array}\right] \\
&= \left[\begin{array}{c} -\frac{\mathrm{e}^{t}}{2} \\ \frac{\mathrm{e}^{t}}{4} \end{array}\right]
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{2c_1 \mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{7}}{2}} \\ c_1 \mathrm{e}^{\left(\frac{7}{2}+\frac{i\sqrt{7}}{2}\right)t} \end{array}\right] + \left[\begin{array}{c} \frac{2c_2 \mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{7}}{2}} \\ c_2 \mathrm{e}^{\left(\frac{7}{2}-\frac{i\sqrt{7}}{2}\right)t} \end{array}\right] + \left[\begin{array}{c} -\frac{\mathrm{e}^t}{2} \\ \frac{\mathrm{e}^t}{4} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{c_1\left(-1-i\sqrt{7}\right)\mathrm{e}^{\frac{\left(7+i\sqrt{7}\right)t}{2}}}{2} + \frac{c_2\left(-1+i\sqrt{7}\right)\mathrm{e}^{-\frac{\left(i\sqrt{7}-7\right)t}{2}}}{2} - \frac{\mathrm{e}^t}{2} \\ c_1 \mathrm{e}^{\frac{\left(7+i\sqrt{7}\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(i\sqrt{7}-7\right)t}{2}} + \frac{\mathrm{e}^t}{4} \end{array}\right]$$

9.7.3 Maple step by step solution

Let's solve

$[x'(t) = 4x(t) + 2y + \mathrm{e}^t, y' = -x(t) + 3y - \mathrm{e}^t]$

- Define vector

$$\overrightarrow{x}(t) = \left[\begin{array}{c} x(t) \\ y \end{array}\right]$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \left[\begin{array}{cc} 4 & 2 \\ -1 & 3 \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} \mathrm{e}^t \\ -\mathrm{e}^t \end{array}\right]$$

- System to solve

$$\overrightarrow{x}'(t) = \left[\begin{array}{cc} 4 & 2 \\ -1 & 3 \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} \mathrm{e}^t \\ -\mathrm{e}^t \end{array}\right]$$

- Define the forcing function

$$\overrightarrow{f}(t) = \left[\begin{array}{c} \mathrm{e}^t \\ -\mathrm{e}^t \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 4 & 2 \\ -1 & 3 \end{array}\right]$$

- Rewrite the system as

  $$\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[\frac{7}{2}-\frac{\mathrm{I}\sqrt{7}}{2}, \left[\begin{array}{c} \frac{2}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}} \\ 1 \end{array}\right]\right], \left[\frac{7}{2}+\frac{\mathrm{I}\sqrt{7}}{2}, \left[\begin{array}{c} \frac{2}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[\frac{7}{2}-\frac{\mathrm{I}\sqrt{7}}{2}, \left[\begin{array}{c} \frac{2}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{\left(\frac{7}{2}-\frac{\mathrm{I}\sqrt{7}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{2}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{\frac{7t}{2}} \cdot \left(\cos\left(\frac{\sqrt{7}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{7}\,t}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{2}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

  $$\mathrm{e}^{\frac{7t}{2}} \cdot \left[\begin{array}{c} \frac{2\left(\cos\left(\frac{\sqrt{7}\,t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{7}\,t}{2}\right)\right)}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}} \\ \cos\left(\frac{\sqrt{7}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{7}\,t}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_{1}(t) = \mathrm{e}^{\frac{7t}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{7}\,t}{2}\right)}{2}+\frac{\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{2} \\ \cos\left(\frac{\sqrt{7}\,t}{2}\right) \end{array}\right], \overrightarrow{x}_{2}(t) = \mathrm{e}^{\frac{7t}{2}} \cdot \left[\begin{array}{c} \frac{\sqrt{7}\,\cos\left(\frac{\sqrt{7}\,t}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{7}\,t}{2}\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_{p}($

$\overrightarrow{x}(t)=c_1\overrightarrow{x}_1(t)+c_2\overrightarrow{x}_2(t)+\overrightarrow{x}_p(t)$

☐ Fundamental matrix

- ○ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t)=\left[\begin{array}{cc} \mathrm{e}^{\frac{7t}{2}}\left(-\frac{\cos\left(\frac{\sqrt{7}\,t}{2}\right)}{2}+\frac{\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{2}\right) & \mathrm{e}^{\frac{7t}{2}}\left(\frac{\sqrt{7}\,\cos\left(\frac{\sqrt{7}\,t}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{2}\right) \\ \mathrm{e}^{\frac{7t}{2}}\cos\left(\frac{\sqrt{7}\,t}{2}\right) & -\mathrm{e}^{\frac{7t}{2}}\sin\left(\frac{\sqrt{7}\,t}{2}\right) \end{array}\right]$$

- ○ The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where I is th

$\Phi(t)=\phi(t)\cdot\frac{1}{\phi(0)}$

- ○ Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t)=\left[\begin{array}{cc} \mathrm{e}^{\frac{7t}{2}}\left(-\frac{\cos\left(\frac{\sqrt{7}\,t}{2}\right)}{2}+\frac{\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{2}\right) & \mathrm{e}^{\frac{7t}{2}}\left(\frac{\sqrt{7}\,\cos\left(\frac{\sqrt{7}\,t}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{2}\right) \\ \mathrm{e}^{\frac{7t}{2}}\cos\left(\frac{\sqrt{7}\,t}{2}\right) & -\mathrm{e}^{\frac{7t}{2}}\sin\left(\frac{\sqrt{7}\,t}{2}\right) \end{array}\right]\cdot\frac{1}{\left[\begin{array}{cc} -\frac{1}{2} & \frac{\sqrt{7}}{2} \\ 1 & 0 \end{array}\right]}$$

- ○ Evaluate and simplify to get the fundamental matrix

$$\Phi(t)=\left[\begin{array}{cc} \frac{\mathrm{e}^{\frac{7t}{2}}\left(\sqrt{7}\,\cos\left(\frac{\sqrt{7}\,t}{2}\right)+\sin\left(\frac{\sqrt{7}\,t}{2}\right)\right)\sqrt{7}}{7} & \frac{4\sqrt{7}\,\mathrm{e}^{\frac{7t}{2}}\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{7} \\ -\frac{2\sqrt{7}\,\mathrm{e}^{\frac{7t}{2}}\sin\left(\frac{\sqrt{7}\,t}{2}\right)}{7} & -\frac{\mathrm{e}^{\frac{7t}{2}}\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)-7\cos\left(\frac{\sqrt{7}\,t}{2}\right)\right)}{7} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- ○ Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

$\overrightarrow{x}_p(t)=\Phi(t)\cdot\overrightarrow{v}(t)$

- ○ Take the derivative of the particular solution

$\overrightarrow{x}'_p(t)=\Phi'(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}'(t)$

- ○ Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}'(t)=A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\overrightarrow{f}(t)$

- ○ The fundamental matrix has columns that are solutions to the homogeneous system so its de

$A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}'(t)=A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\overrightarrow{f}(t)$

- ○ Cancel like terms

$\Phi(t)\cdot\overrightarrow{v}'(t)=\overrightarrow{f}(t)$

  - Multiply by the inverse of the fundamental matrix

    $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

  - Integrate to solve for $\vec{v}(t)$

    $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

  - Plug $\vec{v}(t)$ into the equation for the particular solution

    $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

  - Plug in the fundamental matrix and the forcing function and compute

    $$\vec{x}_p(t) = \left[\begin{array}{c} -\frac{\left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right) \mathrm{e}^{\frac{5t}{2}} - 7\cos\left(\frac{\sqrt{7}\, t}{2}\right) \mathrm{e}^{\frac{5t}{2}} + 7\right) \mathrm{e}^{t}}{14} \\ -\frac{\mathrm{e}^{t} \left(3\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right) \mathrm{e}^{\frac{5t}{2}} + 7\cos\left(\frac{\sqrt{7}\, t}{2}\right) \mathrm{e}^{\frac{5t}{2}} - 7\right)}{28} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \left[\begin{array}{c} -\frac{\left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right) \mathrm{e}^{\frac{5t}{2}} - 7\cos\left(\frac{\sqrt{7}\, t}{2}\right) \mathrm{e}^{\frac{5t}{2}} + 7\right) \mathrm{e}^{t}}{14} \\ -\frac{\mathrm{e}^{t} \left(3\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right) \mathrm{e}^{\frac{5t}{2}} + 7\cos\left(\frac{\sqrt{7}\, t}{2}\right) \mathrm{e}^{\frac{5t}{2}} - 7\right)}{28} \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} -\frac{\mathrm{e}^{\frac{7t}{2}} \left(-c_2\sqrt{7} + c_1 - 1\right) \cos\left(\frac{\sqrt{7}\, t}{2}\right)}{2} + \frac{\mathrm{e}^{\frac{7t}{2}} \left(\left(c_1 - \frac{1}{7}\right)\sqrt{7} + c_2\right) \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{2} - \frac{\mathrm{e}^{t}}{2} \\ \frac{(4c_1 - 1)\mathrm{e}^{\frac{7t}{2}} \cos\left(\frac{\sqrt{7}\, t}{2}\right)}{4} - \mathrm{e}^{\frac{7t}{2}} \left(c_2 + \frac{3\sqrt{7}}{28}\right) \sin\left(\frac{\sqrt{7}\, t}{2}\right) + \frac{\mathrm{e}^{t}}{4} \end{array}\right]$$

- Solution to the system of ODEs

  $$\Bigg\{ x(t) = -\frac{\mathrm{e}^{\frac{7t}{2}} \left(-c_2\sqrt{7} + c_1 - 1\right) \cos\left(\frac{\sqrt{7}\, t}{2}\right)}{2} + \frac{\mathrm{e}^{\frac{7t}{2}} \left(\left(c_1 - \frac{1}{7}\right)\sqrt{7} + c_2\right) \sin\left(\frac{\sqrt{7}\, t}{2}\right)}{2} - \frac{\mathrm{e}^{t}}{2}, y = \frac{(4c_1 - 1)\mathrm{e}^{\frac{7t}{2}} \cos\left(\frac{\sqrt{7}\, t}{2}\right)}{4} - \mathrm{e}^{\frac{7t}{2}} \Bigg(c$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 106

```
dsolve([diff(x(t),t)=4*x(t)+2*y(t)+exp(t),diff(y(t),t)=-x(t)+3*y(t)-exp(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right) c_2 + \mathrm{e}^{\frac{7t}{2}} \cos\left(\frac{\sqrt{7}\, t}{2}\right) c_1 - \frac{\mathrm{e}^{t}}{2}$$

$$y(t) = -\frac{\mathrm{e}^{\frac{7t}{2}} \sin\left(\frac{\sqrt{7}\, t}{2}\right) c_2}{4} + \frac{\mathrm{e}^{\frac{7t}{2}} \sqrt{7}\, \cos\left(\frac{\sqrt{7}\, t}{2}\right) c_2}{4} - \frac{\mathrm{e}^{\frac{7t}{2}} \cos\left(\frac{\sqrt{7}\, t}{2}\right) c_1}{4} - \frac{\mathrm{e}^{\frac{7t}{2}} \sqrt{7}\, \sin\left(\frac{\sqrt{7}\, t}{2}\right) c_1}{4} + \frac{\mathrm{e}^{t}}{4}$$

✓ Solution by Mathematica

Time used: 0.463 (sec). Leaf size: 129

```
DSolve[{x'[t]==4*x[t]+2*y[t]+Exp[t],y'[t]==-x[t]+3*y[t]-Exp[t]},{x[t],y[t]},t,IncludeSingula
```

$$x(t) \to -\frac{e^t}{2} + c_1 e^{7t/2} \cos\left(\frac{\sqrt{7}t}{2}\right) + \frac{(c_1 + 4c_2) e^{7t/2} \sin\left(\frac{\sqrt{7}t}{2}\right)}{\sqrt{7}}$$

$$y(t) \to \frac{e^t}{4} + c_2 e^{7t/2} \cos\left(\frac{\sqrt{7}t}{2}\right) - \frac{(2c_1 + c_2) e^{7t/2} \sin\left(\frac{\sqrt{7}t}{2}\right)}{\sqrt{7}}$$

9.8 problem 8

Internal problem ID [6718]
Internal file name [`OUTPUT/5966_Sunday_June_05_2022_04_07_58_PM_25711530/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 7x(t) + 5y - 9z(t) - 8\,\mathrm{e}^{-2t} \\
y' &= 4x(t) + y + z(t) + 2\,\mathrm{e}^{5t} \\
z'(t) &= -2y + 3z(t) + \mathrm{e}^{5t} - 3\,\mathrm{e}^{-2t}
\end{aligned}
$$

9.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} -8\,\mathrm{e}^{-2t} \\ 2\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} - 3\,\mathrm{e}^{-2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \text{Expression too large to display} \\ &= \text{Expression too large to display} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \text{Expression too large to display} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \text{Expression too large to display} \\ &= \text{Expression too large to display} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \text{Expression too large to display} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \text{Expression too large to display} \int \text{Expression too large to display} \begin{bmatrix} -8\,\mathrm{e}^{-2t} \\ 2\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} - 3\,\mathrm{e}^{-2t} \end{bmatrix} dt \\ &= \text{Expression too large to displayExpression too large to display} \\ &= \begin{bmatrix} -\frac{(3125\,\mathrm{e}^{7t}-30624)\mathrm{e}^{-2t}}{16500} \\ \frac{(575\,\mathrm{e}^{7t}-3894)\mathrm{e}^{-2t}}{1650} \\ \frac{(625\,\mathrm{e}^{7t}-1419)\mathrm{e}^{-2t}}{4125} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \text{Expression too large to display} \end{aligned}$$

9.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} -8\,\mathrm{e}^{-2t} \\ 2\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} - 3\,\mathrm{e}^{-2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 7-\lambda & 5 & -9 \\ 4 & 1-\lambda & 1 \\ 0 & -2 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 11\lambda^2 + 13\lambda - 47 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}+\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}$$

$$\lambda_2 = -\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6}-\frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}+\frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_3 = -\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6}-\frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}-\frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right)}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity
$\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}+\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}$	1
$-\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6}-\frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}-\frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right)}{2}$	1
$-\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6}-\frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}+\frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right)}{2}$	1

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}+\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{bmatrix} - \left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3} + \frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}} + \frac{11}{3}\right)\left[\right.\right.$$

$$\left[\begin{array}{ccc} \frac{-\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}+10\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}} & 5 & -9 \\ 4 & \frac{-\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-8\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}} & 1 \\ 0 & -2 & \frac{-\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-2\left(1322+6\right.}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}} \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc} \frac{10}{3}-\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}} & 5 & - \\ 4 & -\frac{8}{3}-\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}} & 1 \\ 0 & -2 & -\frac{2}{3}-\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3} \end{array}\right.$$

$$R_2 = R_2 - \frac{4R_1}{\frac{10}{3}-\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}} \Longrightarrow \left[\begin{array}{cc} \frac{-\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}+10\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}} & \\ 0 & \frac{2\left(\sqrt{33231}\right.}{\left(1322+\right.} \\ 0 & \end{array}\right.$$

$$R_3 = R_3 + \frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} \left(-\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}} + 10\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} - 82\right) R_2}{\left(\sqrt{33231}+193\right) \left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} - 2\sqrt{33231} - 16 \left(1322+6\sqrt{33231}\right)^{\frac{2}{3}} + 680} \Longrightarrow$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{-\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}+10\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}} & 5 \\ 0 & \frac{2\left(\sqrt{33231}+193\right)\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-4\sqrt{33231}-32\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}+1360}{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}\left(-\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}+10\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-82\right)} \\ 0 & 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} t \left(9\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+151\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-33\right.}{\left(16\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-193\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2\sqrt{33231}-680\right.} \right.$

Hence the solution is

$$\begin{bmatrix} \frac{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} t \left(9\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+151\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-33\sqrt{33231}+1532\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2815\right)}{\left(16\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-193\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2\sqrt{33231}-680\right)\left(\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-10\left(1322+6\sqrt{33231}\right.\right.} \\ \frac{t\left(59\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-41\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-3\sqrt{33231}-661\right)}{16\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-193\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2\sqrt{33231}-680} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\left[\begin{array}{c}\frac{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} t\left(9\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+151\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-33\sqrt{33231}+1532\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2815\right)}{\left(16\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-193\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2\sqrt{33231}-680\right)\left(\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-10\left(1322+6\sqrt{33231}\right.\right.} \\ \frac{t\left(59\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-41\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-3\sqrt{33231}-661\right)}{16\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-193\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2\sqrt{33231}-680} \\ t\end{array}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c}\frac{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} t\left(9\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+151\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-33\sqrt{33231}+1532\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2815\right)}{\left(16\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-193\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2\sqrt{33231}-680\right)\left(\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-10\left(1322+6\sqrt{33231}\right.\right.} \\ \frac{t\left(59\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-41\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-3\sqrt{33231}-661\right)}{16\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-193\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2\sqrt{33231}-680} \\ t\end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c}\frac{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} t\left(9\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+151\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-33\sqrt{33231}+1532\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2815\right)}{\left(16\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-193\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2\sqrt{33231}-680\right)\left(\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-10\left(1322+6\sqrt{33231}\right.\right.} \\ \frac{t\left(59\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-41\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-3\sqrt{33231}-661\right)}{16\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-\sqrt{33231}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-193\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+2\sqrt{33231}-680} \\ t\end{array}\right.$$

Considering the eigenvalue $\lambda_2 = -\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6}-\frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}-\frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right.}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{bmatrix} - \left(-\frac{(1322 +}{}\right.\right.$$

$$\begin{bmatrix} \frac{i\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}}\sqrt{3}+\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}}-82i\sqrt{3}+20\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{1}{3}}+82}{6\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{1}{3}}} & \\ 4 & \frac{i\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}}\sqrt{3}+\left(1322+6\sqrt{3}\,\sqrt{1\right.}{6\left(1322+6\right.} \\ 0 & - \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} \frac{10}{3}+\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6}+\frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right)}{2} & \\ 4 & -\frac{8}{3}+\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6}+ \\ 0 & \end{bmatrix}$$

$$R_2 = R_2 - \frac{4R_1}{\frac{10}{3}+\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6}+\frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow \begin{bmatrix} i\left(1322+6\sqrt{\right. \\ \\ \end{bmatrix}$$

$$R_3 = R_3 + \frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} \Big(82+20\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+i\Big(\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-82\Big)\sqrt{3}}{\left(193i\sqrt{3}+3i\sqrt{11077}-\sqrt{33231}-193\right)\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}-680i\sqrt{3}+6i\sqrt{11077}+2\sqrt{3323}}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{i\left(1322+6\sqrt{3}\sqrt{11077}\right)^{\frac{2}{3}}\sqrt{3}+\left(1322+6\sqrt{3}\sqrt{11077}\right)^{\frac{2}{3}}-82i\sqrt{3}+20\left(1322+6\sqrt{3}\sqrt{11077}\right)^{\frac{1}{3}}+82}{6\left(1322+6\sqrt{3}\sqrt{11077}\right)^{\frac{1}{3}}} & \\ 0 & \frac{2\left(193i\sqrt{3}+3i\sqrt{11077}-\sqrt{33231}-193\right)\Big(1322}{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}\Big(82+20\Big(1322} \\ 0 & \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\begin{cases} v_1 = \frac{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} t\Big(-41i\sqrt{3}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-123i\Big(1322+6\sqrt{}}{\left(i\sqrt{3}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}+\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-82i\sqrt{3}+20\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+82\right)\Big(-427i\sqrt{}} \end{cases}$

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} t\Big(-41\,\mathrm{I}\sqrt{3}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-123\,\mathrm{I}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}\sqrt{11077}+41\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}\sqrt{}}{\left(\mathrm{I}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}\sqrt{3}+\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}-82\,\mathrm{I}\sqrt{3}+20\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+82\right)\Big(-427\,\mathrm{I}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}\sqrt{3}-6\,\mathrm{I}\Big(1322+6\sqrt{33231}} \\ -\frac{t\Big(1529\,\mathrm{I}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}\sqrt{3}-3\,\mathrm{I}\sqrt{11077}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}+\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}\sqrt{33231}-270}{2\Big(-427\,\mathrm{I}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}\sqrt{3}-6\,\mathrm{I}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}\sqrt{11077}+2\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}\sqrt{33231}} \\ \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

Considering the eigenvalue $\lambda_3 = -\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6} - \frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}} + \frac{11}{3} + \frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3} - \frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right.}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{bmatrix} - \left(-\frac{(13}{}\right.\right.$$

$$\begin{bmatrix} \frac{-i\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}}\sqrt{3}+\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}}+82i\sqrt{3}+20\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{1}{3}}+82}{6\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{1}{3}}} & \\ 4 & \frac{-i\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}}\sqrt{3}+\left(1322+6\sqrt{3}\right.}{6\left(1322+\right.} \\ 0 & \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} \frac{10}{3} + \frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6} + \frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3} - \frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right)}{2} & \\ 4 & -\frac{8}{3} + \frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6} + \frac{}{3} \\ 0 & \end{bmatrix}$$

$$R_2 = R_2 - \frac{4R_1}{\frac{10}{3} + \frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6} + \frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3} - \frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow \begin{bmatrix} \frac{-i(1322+6}{} \\ \\ \\ \end{bmatrix}$$

$$R_3 = R_3 + \frac{\left(-82 - 20\left(1322 + 6\sqrt{3}\,\sqrt{11077}\right)^{\frac{1}{3}} + i\left(\left(1322 + 6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}} - 82\right)\sqrt{3} - (1322 + 6\sqrt{3}\,\sqrt{1}\right.}{193\left(1322 + 6\sqrt{3}\,\sqrt{11077}\right)^{\frac{1}{3}}\left(1 + \frac{\sqrt{11077}\left(\sqrt{3}+3i\right)}{193} + i\sqrt{3}\right) + 680 + 2\left(-\sqrt{3} + 3i\right)\sqrt{11077} - 68}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{-i\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}}\sqrt{3}+\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}}+82i\sqrt{3}+20\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{1}{3}}+82}{6\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{1}{3}}} & \\ 0 & \frac{386\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{1}{3}}\left(1+\frac{\sqrt{11077}\,(\sqrt{3}}{193}\right.}{\left(-82-20\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{1}{3}}+i((132\right.} \\ 0 & \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} t\left(41i\sqrt{3}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}+123i(1322+6\sqrt{}\right.}{\left(427i\sqrt{3}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}+6i\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}\sqrt{11077}+2\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}}\sqrt{33231}-\right.}\right.$

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\left[\begin{array}{l}
-\frac{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} t \left(41\, \mathrm{I}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}} \sqrt{3}+123\, \mathrm{I}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}} \sqrt{11077}+41\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}} \right.}{\left(427\, \mathrm{I}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}} \sqrt{3}+6\, \mathrm{I}\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}} \sqrt{11077}+2\left(1322+6\sqrt{33231}\right)^{\frac{2}{3}} \sqrt{33231}-2021\, \mathrm{I}\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}} \sqrt{3}+3\, \mathrm{I}\sqrt{11077} \right.} \\
\frac{t\left(1529\, \mathrm{I}\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}} \sqrt{3}-3\, \mathrm{I}\sqrt{11077}\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}}-\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}} \sqrt{3}\,\sqrt{11077}-27082\, \mathrm{I} \right.}{2\left(427\, \mathrm{I}\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}} \sqrt{3}+6\, \mathrm{I}\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}} \sqrt{11077}+2\left(1322+6\sqrt{3}\,\sqrt{11077}\right)^{\frac{2}{3}} \sqrt{3}\,\sqrt{11077}-2021 \right.} \\
\\
\end{array}\right.$$

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity	
eigenvalue	algebraic m	geometri
$\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}+\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}$	1	1
$-\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6}-\frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}+\frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right)}{2}$	1	1
$-\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{6}-\frac{41}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}-\frac{i\sqrt{3}\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}\right)}{2}$	1	1

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{82}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}+\frac{11}{3}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{82}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}+\frac{11}{3}\right)t} \\
&= \left[\begin{array}{c}
\frac{114(1322+6\sqrt{33231})^{\frac{1}{3}}+\frac{9348}{(1322+6\sqrt{33231})^{\frac{1}{3}}}+722}{\left(5\left(\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{82}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}+\frac{11}{3}\right)^2-\frac{58(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}-\frac{4756}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}-\frac{749}{3}\right)\Big(\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{}{3(1322}} \\
\frac{9\left(\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{82}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}+\frac{11}{3}\right)^2-12(1322+6\sqrt{33231})^{\frac{1}{3}}-\frac{984}{(1322+6\sqrt{33231})^{\frac{1}{3}}}-305}{5\left(\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{82}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}+\frac{11}{3}\right)^2-\frac{58(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}-\frac{4756}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}-\frac{749}{3}} \\
1
\end{array}\right.
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = c_1 \left[\begin{array}{c}
\frac{38\,\mathrm{e}^{\left(\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{82}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}+\frac{11}{3}\right)t}\Big(3(1322+6\sqrt{33231})^{\frac{1}{3}}+\frac{246}{(1322+6\sqrt{33231})^{\frac{1}{3}}}+}{\left(5\left(\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{82}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}+\frac{11}{3}\right)^2-\frac{58(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}-\frac{4756}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}-\frac{749}{3}\right)\Big(\frac{(1322+6\sqrt{33231})}{3}} \\
\frac{\mathrm{e}^{\left(\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{82}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}+\frac{11}{3}\right)t}\Big(9\left(\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{82}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}+\frac{11}{3}\right)^2-12(1322+6\sqrt{33231}}{5\left(\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{82}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}+\frac{11}{3}\right)^2-\frac{58(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}-\frac{4756}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}-} \\
\mathrm{e}^{\left(\frac{(1322+6\sqrt{33231})^{\frac{1}{3}}}{3}+\frac{82}{3(1322+6\sqrt{33231})^{\frac{1}{3}}}+\frac{11}{3}\right)t}
\end{array}\right.
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \text{Expression too large to display}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \text{Expression too large to display}$$

Hence

$$\vec{x}_p(t) = \text{Expression too large to display} \int \text{Expression too large to display} \begin{bmatrix} -8\,\mathrm{e}^{-2t} \\ 2\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} - 3\,\mathrm{e}^{-2t} \end{bmatrix} dt$$

$$= \text{Expression too large to display} \int \text{Expression too large to display}\, dt$$

$$= \text{Expression too large to display}\text{Expression too large to display}$$

$$= \begin{bmatrix} -\frac{\left(3125\,\mathrm{e}^{7t}-30624\right)\mathrm{e}^{-2t}}{16500} \\ \frac{\left(575\,\mathrm{e}^{7t}-3894\right)\mathrm{e}^{-2t}}{1650} \\ \frac{\left(625\,\mathrm{e}^{7t}-1419\right)\mathrm{e}^{-2t}}{4125} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \left[\begin{array}{l} \frac{38 c_1 \mathrm{e}^{\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}+\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}\right)t}\left(3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}+\frac{246}{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+1\right.}{\left(5\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}+\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}\right)^{2}-\frac{58\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{4756}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}-\frac{749}{3}\right)\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}\right.} \\ \frac{c_1 \mathrm{e}^{\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}+\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}\right)t}\left(9\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}+\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}\right)^{2}-12\left(1322+6\sqrt{33231}\right)\right.}{5\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}+\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}\right)^{2}-\frac{58\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}-\frac{4756}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}-\frac{74}{3}} \\ c_1 \mathrm{e}^{\left(\frac{\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}{3}+\frac{82}{3\left(1322+6\sqrt{33231}\right)^{\frac{1}{3}}}+\frac{11}{3}\right)t} \end{array}\right.$$

Which becomes

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✓ Solution by Maple

Time used: 5.688 (sec). Leaf size: 9313

```
dsolve([diff(x(t),t)=7*x(t)+5*y(t)-9*z(t)-8*exp(-2*t),diff(y(t),t)=4*x(t)+y(t)+z(t)+2*exp(5*
```

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Expression too large to display
Expression too large to display

✓ Solution by Mathematica

Time used: 0.206 (sec). Leaf size: 3002

```
DSolve[{x'[t]==7*x[t]+5*y[t]-9*z[t]-8*Exp[-2*t],y'[t]==4*x[t]+y[t]+z[t]+2*Exp[5*t],z'[t]==-2
```

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9.9 problem 9

Internal problem ID [6719]
Internal file name [`OUTPUT/5967_Sunday_June_05_2022_04_08_12_PM_40291562/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= x(t) - y + 2z(t) + \mathrm{e}^{-t} - 3t \\ y' &= 3x(t) - 4y + z(t) + 2\,\mathrm{e}^{-t} + t \\ z'(t) &= -2x(t) + 5y + 6z(t) + 2\,\mathrm{e}^{-t} - t \end{aligned}$$

9.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as Warning. Unable to find the matrix exponential.

9.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{-t} - 3t \\ 2\,\mathrm{e}^{-t} + t \\ 2\,\mathrm{e}^{-t} - t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\left[\begin{array}{ccc} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{array}\right] - \lambda \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) = 0$$

Therefore

$$\det\left(\left[\begin{array}{ccc} 1-\lambda & -1 & 2 \\ 3 & -4-\lambda & 1 \\ -2 & 5 & 6-\lambda \end{array}\right]\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 - 20\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{\left(2916 + 12i\sqrt{86955}\right)^{\frac{1}{3}}}{6} + \frac{46}{\left(2916 + 12i\sqrt{86955}\right)^{\frac{1}{3}}} + 1$$

$$\lambda_2 = -\frac{\left(2916 + 12i\sqrt{86955}\right)^{\frac{1}{3}}}{12} - \frac{23}{\left(2916 + 12i\sqrt{86955}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6} - \frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}\right.}{2}$$

$$\lambda_3 = -\frac{\left(2916 + 12i\sqrt{86955}\right)^{\frac{1}{3}}}{12} - \frac{23}{\left(2916 + 12i\sqrt{86955}\right)^{\frac{1}{3}}} + 1 - \frac{i\sqrt{3}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6} - \frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}\right.}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicit
$-\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{12}-\frac{23}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1-\frac{i\sqrt{3}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2}$	1
$\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1$	1
$-\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{12}-\frac{23}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1+\frac{i\sqrt{3}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2}$	1

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{bmatrix}-\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\right.$$

$$\left[\begin{matrix} -\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}+276}{6\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}} & -1 & 2 \\ 3 & \frac{-\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}-30\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}-276}{6\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}} & 1 \\ -2 & 5 & \frac{-\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}+30\left(2916+12i\sqrt{86955}\right)}{6\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}} \end{matrix}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented

matrix is

$$\begin{bmatrix} -\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}} & -1 & 2 \\ 3 & -5-\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}} & 1 \\ -2 & 5 & 5-\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6} \end{bmatrix}$$

$$R_2 = R_2 - \frac{3R_1}{-\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}} \Longrightarrow \begin{bmatrix} -\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}+276}{6\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}} & \\ 0 & -\frac{2\left(\left(i\sqrt{86955}+933\right)(2916+12}{(2916+12i} \\ -2 & \end{bmatrix}$$

$$R_3 = R_3 - \frac{12\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}} R_1}{\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}+276} \Longrightarrow \begin{bmatrix} -\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}+276}{6\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}} & \\ 0 & -\frac{2\left(\left(i\sqrt{86955}+933\right)\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}-\right.}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}\left(\right.} \\ 0 & \frac{5\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}+1}{(2916+12i} \end{bmatrix}$$

$$R_3 = R_3 + \frac{\left(5\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}+12\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}+1380\right)\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}} R_2}{2\left(i\sqrt{86955}+933\right)\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}+60i\sqrt{86955}+110\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}+27276}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{(2916+12i\sqrt{86955})^{\frac{2}{3}}+276}{6(2916+12i\sqrt{86955})^{\frac{1}{3}}} & -1 & \\ 0 & -\frac{2((i\sqrt{86955}+933)(2916+12i\sqrt{86955})^{\frac{1}{3}}+30i\sqrt{86955}+55(2916+12i\sqrt{86955})^{\frac{2}{3}}+13638)}{(2916+12i\sqrt{86955})^{\frac{1}{3}}((2916+12i\sqrt{86955})^{\frac{2}{3}}+276)} & \frac{(2916+12i}{} \\ 0 & 0 & \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{12(2916+12i\sqrt{86955})^{\frac{1}{3}} t (i\sqrt{86955}\,(2916+12i\sqrt{86955})^{\frac{1}{3}}+27i\sqrt{86955}+46(2916+12i\sqrt{8695}}{(i\sqrt{86955}\,(2916+12i\sqrt{86955})^{\frac{1}{3}}+30i\sqrt{86955}+55(2916+12i\sqrt{86955})^{\frac{2}{3}}+933(2916+12i\sqrt{86955}} \right.$

Hence the solution is

$$\begin{bmatrix} \frac{12(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}} t (\mathrm{I}\sqrt{86955}\,(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+27\,\mathrm{I}\sqrt{86955}+46(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{2}{3}}+864(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+12909}{(\mathrm{I}\sqrt{86955}\,(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+30\,\mathrm{I}\sqrt{86955}+55(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{2}{3}}+933(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+13638)((2916+12\,\mathrm{I}\sqrt{86955})^{\frac{2}{3}}+} \\ \frac{6t\left(\mathrm{I}\sqrt{86955}+3(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{2}{3}}+23(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+243\right)}{\mathrm{I}\sqrt{86955}\,(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+30\,\mathrm{I}\sqrt{86955}+55(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{2}{3}}+933(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+13638} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{12(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}} t (\mathrm{I}\sqrt{86955}\,(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+27\,\mathrm{I}\sqrt{86955}+46(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{2}{3}}+864(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+12909}{(\mathrm{I}\sqrt{86955}\,(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+30\,\mathrm{I}\sqrt{86955}+55(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{2}{3}}+933(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+13638)((2916+12\,\mathrm{I}\sqrt{86955})^{\frac{2}{3}}+} \\ \frac{6t\left(\mathrm{I}\sqrt{86955}+3(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{2}{3}}+23(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+243\right)}{\mathrm{I}\sqrt{86955}\,(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+30\,\mathrm{I}\sqrt{86955}+55(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{2}{3}}+933(2916+12\,\mathrm{I}\sqrt{86955})^{\frac{1}{3}}+13638} \\ t \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} \frac{12\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}} t\left(\mathrm{I}\sqrt{86955}\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+27\,\mathrm{I}\sqrt{86955}+46\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{2}{3}}+864\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+12909\right.}{\left(\mathrm{I}\sqrt{86955}\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+30\,\mathrm{I}\sqrt{86955}+55\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{2}{3}}+933\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+13638\right)\left(\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{2}{3}}+2\right.} \\ \frac{6t\left(\mathrm{I}\sqrt{86955}+3\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{2}{3}}+23\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+243\right)}{\mathrm{I}\sqrt{86955}\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+30\,\mathrm{I}\sqrt{86955}+55\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{2}{3}}+933\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+13638} \\ t \end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c} \frac{12\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}} t\left(\mathrm{I}\sqrt{86955}\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+27\,\mathrm{I}\sqrt{86955}+46\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{2}{3}}+864\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+12909\right.}{\left(\mathrm{I}\sqrt{86955}\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+30\,\mathrm{I}\sqrt{86955}+55\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{2}{3}}+933\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+13638\right)\left(\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{2}{3}}+2\right.} \\ \frac{6t\left(\mathrm{I}\sqrt{86955}+3\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{2}{3}}+23\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+243\right)}{\mathrm{I}\sqrt{86955}\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+30\,\mathrm{I}\sqrt{86955}+55\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{2}{3}}+933\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}+13638} \\ t \end{array}\right.$$

Considering the eigenvalue $\lambda_2 = -\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{12} - \frac{23}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}} + 1 - \frac{i\sqrt{3}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6} - \frac{46}{\left(2916+12i\sqrt{8695\right.}\right.}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{array}\right] - \left(-\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{12} - \right.\right.$$

$$\left[\begin{array}{cc} \frac{\sqrt{23}\left(\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)-3\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)\right)}{3} & -1 \\ 3 & -5+\frac{\sqrt{3}\,\sqrt{23}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)}{3}-\sqrt{23}\,\cos\left(\right. \\ -2 & 5 \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} \frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{12}+\frac{23}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2} & \\ 3 & -5+\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{12}+ \\ -2 & \end{bmatrix}$$

$$R_2 = R_2 - \frac{3R_1}{\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{12}+\frac{23}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow \begin{bmatrix} \sqrt{23}\left(\sqrt{3}\sin \right. \end{bmatrix}$$

$$R_3 = R_3 + \frac{6\sqrt{23}\,R_1}{23\left(\sqrt{3}\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)-3\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)\right)} \Longrightarrow \begin{bmatrix} \sqrt{23}\left(\sqrt{3}\sin\left(\frac{\arctan\left(\right.}{6}\right.\right. \end{bmatrix}$$

$$R_3 = R_3 - \frac{\left(-115\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)+345\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)+6}{\left(-23\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)+69\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)\right)((-6i\sqrt{28985}-1866i-}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{\sqrt{23}\left(\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)-3\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)\right)}{3} & \\ 0 & \frac{\left(-6i\sqrt{28985}-1866i-1866\sqrt{3}+2\sqrt{86955}\right)\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}((-(2916+12i\sqrt{}} \\ 0 & \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = -\frac{3\sqrt{23}\,(9\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}\sqrt{86955}+9i\sqrt{28985}\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}-276(2916+12i\sqrt{86}}{46(-468\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}\sqrt{28985}+156i\sqrt{86955}\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}+} \right.$

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

Expression too large to display $\begin{bmatrix} -\frac{3\sqrt{23}\,(9\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}\sqrt{86955}+9i\sqrt{28985}\left(2916+12i\sqrt{86955}\right)^{\frac{2}{3}}-276(2916+12i\sqrt{869}}{46(-468\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}\sqrt{28985}+156i\sqrt{86955}\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}+5i} \\ \frac{-1104i\sqrt{3}\,\sqrt{28985}+i\sqrt{28985}\left(2916+12i\sqrt{3}\,\sqrt{28985}\right)^{\frac{2}{3}}\sqrt{3}+3\sqrt{28985}\left(2916+12i\sqrt{3}\,\sqrt{28985}\right)^{\frac{2}{3}}}{-468\sqrt{28985}\left(2916+12i\sqrt{3}\,\sqrt{28985}\right)^{\frac{1}{3}}+156i\sqrt{3}\,\sqrt{28985}\left(2916+12i\sqrt{3}\,\sqrt{28985}\right)^{\frac{1}{3}}+5i\sqrt{28985}\,(2} \\ \end{bmatrix}$

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

Considering the eigenvalue $\lambda_3 = -\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{12} - \frac{23}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6} - \frac{46}{\left(2916+12i\sqrt{8695}\right.}\right.}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{bmatrix} - \left(-\frac{\left(2916 + 12i\sqrt{86955}\right)^{\frac{1}{3}}}{12} - \right.\right.$$

$$\begin{bmatrix} \frac{\sqrt{23}\left(\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)+3\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)\right)}{3} & -1 \\ 3 & -5 + \frac{\sqrt{3}\sqrt{23}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)}{3} + \sqrt{23}\,\cos\left(\right. \\ -2 & 5 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} \frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{12} + \frac{23}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6} - \frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2} & \\ 3 & -5 + \frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{12} + \\ -2 & \end{bmatrix}$$

$$R_2 = R_2 - \frac{3R_1}{\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{12} + \frac{23}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6} - \frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow \left[\frac{\sqrt{23}\left(\sqrt{3}\,\sin\right.}{} \right.$$

$$R_3 = R_3 + \frac{6\sqrt{23}\,R_1}{23\left(\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)+3\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)\right)} \Longrightarrow \left[\frac{\sqrt{23}\left(\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{8}}{46}\right.}{6}\right.\right.}{} \right.$$

$$R_3 = R_3 - \frac{\left(115\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)+345\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)-6\sqrt{2}\right.}{2\left(23\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)+69\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)\right)\left(13638+\left(2916+12i\sqrt{3}\right.\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{cc}
\frac{\sqrt{23}\left(\sqrt{3}\,\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)+3\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}+\frac{\pi}{3}\right)\right)}{3} & \\
0 & \frac{27276+2\left(2916+12i\sqrt{3}\sqrt{28985}\right)^{\frac{1}{3}}\left(933+i\left(933+\sqrt{28985}\right)\sqrt{3}-3\right.}{\left(i\left(2916+12i\sqrt{3}\sqrt{28985}\right)^{\frac{2}{3}}\sqrt{3}-276i\sqrt{3}\right.} \\
0 &
\end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\begin{cases} v_1 = \frac{9\sqrt{23}\,t\left(3i\sqrt{28985}\left(2916+12i\sqrt{3}\sqrt{28985}\right)^{\frac{2}{3}}\sqrt{3}+92i\sqrt{3}\sqrt{28985}\left(\right.\right.}{23\left(\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}\right)\sqrt{3}-3\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}\right)\right)\left(5i\sqrt{28985}\left(2916+12i\sqrt{3}\sqrt{289}\right.\right.} \end{cases}$

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

Expression too large to display $\left[\begin{array}{c}
\frac{9\sqrt{23}\left(3i\sqrt{28985}\left(2916+12i\sqrt{3}\sqrt{28985}\right)^{\frac{2}{3}}\sqrt{3}+92i\sqrt{3}\sqrt{28985}\left(29\right.\right.}{23\left(\sin\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}\right)\sqrt{3}-3\cos\left(\frac{\arctan\left(\frac{81\sqrt{86955}}{4651}\right)}{6}\right)\right)\left(5i\sqrt{28985}\left(2916+12i\sqrt{3}\sqrt{2898}\right.\right.} \\
\frac{i\sqrt{28985}\left(2916+12i\sqrt{3}\sqrt{28985}\right)^{\frac{2}{3}}\sqrt{3}+36i\sqrt{3}\sqrt{28985}\left(2916+\right.}{5i\sqrt{28985}\left(2916+12i\sqrt{3}\sqrt{28985}\right)^{\frac{2}{3}}\sqrt{3}+156i\sqrt{3}\sqrt{28985}\left(2916+\right.} \\
\
\end{array}\right.$

Let $t = 1$ the eigenvector becomes

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Which is normalized to

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The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity	
	algebraic m	geome
$\frac{\left(2916+12 i \sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12 i \sqrt{86955}\right)^{\frac{1}{3}}}+1$	1	1
$-\frac{\left(2916+12 i \sqrt{86955}\right)^{\frac{1}{3}}}{12}-\frac{23}{\left(2916+12 i \sqrt{86955}\right)^{\frac{1}{3}}}+1+\frac{i \sqrt{3}\, \left(\frac{\left(2916+12 i \sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12 i \sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2}$	1	1
$-\frac{\left(2916+12 i \sqrt{86955}\right)^{\frac{1}{3}}}{12}-\frac{23}{\left(2916+12 i \sqrt{86955}\right)^{\frac{1}{3}}}+1-\frac{i \sqrt{3}\, \left(\frac{\left(2916+12 i \sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12 i \sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2}$	1	1

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \left[\begin{array}{c} \frac{9\,\mathrm{e}^{\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)t}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{3}+\frac{92}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+\right.}{\left(\frac{7\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+21+\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)^2\right)\left(\frac{\left(2916+12i\sqrt{8695\right.}{6}\right.} \\ \frac{\mathrm{e}^{\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)t}\left(2\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)^2-\frac{2\left(2916+12i\sqrt{86955}\right.}{3}\right.}{\frac{7\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+21+\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right.} \\ \mathrm{e}^{\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)t} \end{array}\right.$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \text{Expression too large to display}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \text{Expression too large to display}$$

Hence

$$\vec{x}_p(t) = \text{Expression too large to display} \int \text{Expression too large to display} \begin{bmatrix} \mathrm{e}^{-t}-3t \\ 2\,\mathrm{e}^{-t}+t \\ 2\,\mathrm{e}^{-t}-t \end{bmatrix} dt$$

$$= \text{Expression too large to display} \int \text{Expression too large to display}\, dt$$

$$= \text{Expression too large to display}\text{Expression too large to display}$$

$$= \text{Expression too large to display}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \left[\begin{array}{c} \frac{9c_1\mathrm{e}^{\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)t}\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{3}+\frac{92}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+\right.}{\left(\frac{7\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+21+\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)^{2}\right)\left(\frac{\left(2916+12i\sqrt{86955}\right.}{6}\right.} \\ \frac{c_1\mathrm{e}^{\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)t}\left(2\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)^{2}-\frac{2\left(2916+12i\sqrt{86955}\right)}{3}\right.}{\frac{7\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+21+\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)} \\ c_1\mathrm{e}^{\left(\frac{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12i\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)t} \end{array}\right.$$

Which becomes

$$\text{Expression too large to display}$$

9.9.3 Maple step by step solution

Let's solve

$\left[x'(t) = x(t) - y + 2z(t) + \frac{1}{\mathrm{e}^t} - 3t, y' = 3x(t) - 4y + z(t) + \frac{2}{\mathrm{e}^t} + t, z'(t) = -2x(t) + 5y + 6z(t) +\right.$

- Define vector

$$\overrightarrow{x}(t) = \left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right]$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \left[\begin{array}{ccc} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} \frac{x(t)\mathrm{e}^{t}-\mathrm{e}^{t} y+2z(t)\mathrm{e}^{t}-3t\,\mathrm{e}^{t}+1}{\mathrm{e}^{t}}-x(t)+y-2z(t) \\ \frac{3x(t)\mathrm{e}^{t}-4\,\mathrm{e}^{t} y+z(t)\mathrm{e}^{t}+t\,\mathrm{e}^{t}+2}{\mathrm{e}^{t}}-3x(t)+4y-z(t) \\ -\frac{2x(t)\mathrm{e}^{t}-5\,\mathrm{e}^{t} y-6z(t)\mathrm{e}^{t}+t\,\mathrm{e}^{t}-2}{\mathrm{e}^{t}}+2x(t)-5y-6z(t) \end{array}\right]$$

- System to solve

$$\overrightarrow{x}'(t) = \left[\begin{array}{ccc} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{array}\right] \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 1 & -1 & 2 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+1, \left[\begin{array}{c} \frac{9\Bigg(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right.}{3}}{\Bigg(\frac{7\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+21+\Bigg(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right.}{6}} \\ \frac{2\Bigg(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}}{\frac{7\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+} \end{array}\right.\right.\right.$$

- Consider eigenpair

$$\left[\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+1, \left[\begin{array}{c} \frac{9\Bigg(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{}}}{3}}{\Bigg(\frac{7\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+21+\Bigg(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{}}}{6}} \\ \frac{2\Bigg(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+}{\frac{7\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+21} \end{array}\right.\right.$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)t} \cdot \left[\begin{array}{c} \frac{9\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right.}{3}\right.}{\left(\frac{7\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+21+\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right.}{6}\right.\right.} \\ \frac{2\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}\right.}{\frac{7\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+} \end{array}\right.$$

- Consider eigenpair

$$\left[-\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{12}-\frac{23}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+1-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2}, \left[\frac{}{\left(-\frac{7\left(2916+12\right.}{1}\right.}\right.\right.$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{\left(-\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{12}-\frac{23}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+1-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2}\right)t} \cdot \left[\frac{}{\left(-\frac{7\left(2916+12\,\mathrm{I}\right.}{12}\right.} \right.$$

- Consider eigenpair

$$\left[-\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{12}-\frac{23}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+1+\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2}, \left[\frac{}{\left(-\frac{7\left(2916+12\right.}{1}\right.} \right.\right.$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_3 = \mathrm{e}^{\left(-\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{12}-\frac{23}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+1+\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}-\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}\right)}{2}\right)t}\cdot\left[\begin{array}{c}\overline{\left(-\frac{7\left(2916+12\,\mathrm{I}\surd\right.}{12}\right.}\end{array}\right.$$

- General solution to the system of ODEs
  $\overrightarrow{x} = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + c_3\overrightarrow{x}_3$
- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1\mathrm{e}^{\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+1\right)t}\cdot\left[\begin{array}{c}\frac{9\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{8695}\right.}{3}\right.}{\left(\frac{7\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}+21+\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{869}\right.}{6}\right.\right.} \\ \frac{2\left(\frac{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{46}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)}\right.}{\frac{7\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}{6}+\frac{322}{\left(2916+12\,\mathrm{I}\sqrt{86955}\right)^{\frac{1}{3}}}}\end{array}\right.$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \left[-\frac{302315\left(2916+12\,\mathrm{I}\sqrt{3}\,\sqrt{28985}\right)^{\frac{2}{3}}\left(c_1\,\mathrm{e}^{\frac{\left(\left(2916+12\,\mathrm{I}\sqrt{3}\,\sqrt{28985}\right)^{\frac{2}{3}}+6\left(2916+12\,\mathrm{I}\sqrt{3}\,\sqrt{28985}\right)^{\frac{1}{3}}+276\right)t}{6\left(2916+12\,\mathrm{I}\sqrt{3}\,\sqrt{28985}\right)^{\frac{1}{3}}}}\left(-\frac{1522464}{23255}\right.\right.}{-749316c_1\left(\left(-\frac{2273\sqrt{3}\,\sqrt{28985}}{187329}+\mathrm{I}\right)\left(2916+12\,\mathrm{I}\sqrt{3}\,\sqrt{28985}\right)^{\frac{1}{3}}+\left(-\frac{277\sqrt{3}\,\sqrt{28985}}{374658}+\frac{18643\,\mathrm{I}}{374658}\right)\left(2916+12\,\mathrm{I}\sqrt{3}\,\sqrt{28985}\right)^{\frac{2}{3}}\right.} \right.$$

- Solution to the system of ODEs

$$\left\{ x(t) = -\frac{302315\left(2916+12\,\mathrm{I}\sqrt{3}\,\sqrt{28985}\right)^{\frac{2}{3}}\left(c_1\,\mathrm{e}^{\frac{\left(\left(2916+12\,\mathrm{I}\sqrt{3}\,\sqrt{28985}\right)^{\frac{2}{3}}+6\left(2916+12\,\mathrm{I}\sqrt{3}\,\sqrt{28985}\right)^{\frac{1}{3}}+276\right)t}{6\left(2916+12\,\mathrm{I}\sqrt{3}\,\sqrt{28985}\right)^{\frac{1}{3}}}}\left(-\frac{1522464\,\mathrm{I}}{23255}+\frac{\left(223547\,\mathrm{I}-\right.}{}\right.\right.}{} \right.$$

✗ Solution by Maple

```
dsolve([diff(x(t),t)=x(t)-y(t)+2*z(t)+exp(-t)-3*t,diff(y(t),t)=3*x(t)-4*y(t)+z(t)+2*exp(-t)+
```

No solution found

✓ Solution by Mathematica

Time used: 0.191 (sec). Leaf size: 3251

```
DSolve[{x'[t]==x[t]-y[t]+2*z[t]+Exp[-t]-3*t,y'[t]==3*x[t]-4*y[t]+z[t]+2*Exp[-t]+t,z'[t]==-2*
```

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9.10 problem 10

Internal problem ID [6720]
Internal file name [`OUTPUT/5968_Sunday_June_05_2022_04_10_27_PM_44911454/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= t\,\mathrm{e}^{4t} + 4\sin(t) - 4\,\mathrm{e}^{4t} - 7y + 3x(t) \\
y' &= 2t\,\mathrm{e}^{4t} + \mathrm{e}^{4t} + 8\sin(t) + y + x(t)
\end{aligned}
$$

9.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} t\,\mathrm{e}^{4t} - 4\,\mathrm{e}^{4t} + 4\sin(t) \\ 2t\,\mathrm{e}^{4t} + \mathrm{e}^{4t} + 8\sin(t) \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^{2t}\cos\left(t\sqrt{6}\right)+\frac{\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)}{6} & -\frac{7\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)}{6} \\ \frac{\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)}{6} & \mathrm{e}^{2t}\cos\left(t\sqrt{6}\right)-\frac{\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)+6\cos\left(t\sqrt{6}\right)\right)}{6} & -\frac{7\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)}{6} \\ \frac{\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)}{6} & -\frac{\mathrm{e}^{2t}\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)-6\cos\left(t\sqrt{6}\right)\right)}{6} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)+6\cos\left(t\sqrt{6}\right)\right)}{6} & -\frac{7\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)}{6} \\ \frac{\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)}{6} & -\frac{\mathrm{e}^{2t}\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)-6\cos\left(t\sqrt{6}\right)\right)}{6} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)+6\cos\left(t\sqrt{6}\right)\right)c_1}{6} - \frac{7\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)c_2}{6} \\ \frac{\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)c_1}{6} - \frac{\mathrm{e}^{2t}\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)-6\cos\left(t\sqrt{6}\right)\right)c_2}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}\left(\sqrt{6}\,(c_1-7c_2)\sin\left(t\sqrt{6}\right)+6\cos\left(t\sqrt{6}\right)c_1\right)}{6} \\ \frac{\mathrm{e}^{2t}\left(\sqrt{6}\,(c_1-c_2)\sin\left(t\sqrt{6}\right)+6\cos\left(t\sqrt{6}\right)c_2\right)}{6} \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt
$$

But

$$
\begin{aligned}
e^{-At} &= \left(e^{At}\right)^{-1} \\
&= \begin{bmatrix} -\frac{\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)-6\cos\left(t\sqrt{6}\right)\right)\mathrm{e}^{-2t}}{6} & \frac{7\sqrt{6}\,\mathrm{e}^{-2t}\sin\left(t\sqrt{6}\right)}{6} \\ -\frac{\sqrt{6}\,\mathrm{e}^{-2t}\sin\left(t\sqrt{6}\right)}{6} & \frac{\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)+6\cos\left(t\sqrt{6}\right)\right)\mathrm{e}^{-2t}}{6} \end{bmatrix}
\end{aligned}
$$

Hence

$$\vec{x}_p(t) = \left[\begin{array}{cc} \frac{e^{2t}\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)+6\cos\left(t\sqrt{6}\right)\right)}{6} & -\frac{7\sqrt{6}\,e^{2t}\sin\left(t\sqrt{6}\right)}{6} \\ \frac{\sqrt{6}\,e^{2t}\sin\left(t\sqrt{6}\right)}{6} & -\frac{e^{2t}\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)-6\cos\left(t\sqrt{6}\right)\right)}{6} \end{array}\right] \int \left[\begin{array}{cc} -\frac{\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)-6\cos\left(t\sqrt{6}\right)\right)e^{-2t}}{6} & \\ -\frac{\sqrt{6}\,e^{-2t}\sin\left(t\sqrt{6}\right)}{6} & \left(\sqrt{6}\,\mathrm{si}\right. \end{array}\right.$$

$$= \left[\begin{array}{cc} \frac{e^{2t}\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)+6\cos\left(t\sqrt{6}\right)\right)}{6} & -\frac{7\sqrt{6}\,e^{2t}\sin\left(t\sqrt{6}\right)}{6} \\ \frac{\sqrt{6}\,e^{2t}\sin\left(t\sqrt{6}\right)}{6} & -\frac{e^{2t}\left(\sqrt{6}\,\sin\left(t\sqrt{6}\right)-6\cos\left(t\sqrt{6}\right)\right)}{6} \end{array}\right] \left[\begin{array}{c} \frac{((-61200\cos(t)-166800\sin(t))e^{-2t}+(-32010t-3}{29100} \\ \frac{((-2400\cos(t)-63600\sin(t))e^{-2t}+(8730t-640}{29100} \end{array}\right.$$

$$= \left[\begin{array}{c} \frac{(-68-55t)e^{4t}}{50}-\frac{204\cos(t)}{97}-\frac{556\sin(t)}{97} \\ \frac{(-11+15t)e^{4t}}{50}-\frac{8\cos(t)}{97}-\frac{212\sin(t)}{97} \end{array}\right]$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\ &= \left[\begin{array}{c} \frac{\sqrt{6}\,(c_1-7c_2)e^{2t}\sin\left(t\sqrt{6}\right)}{6}+e^{2t}\cos\left(t\sqrt{6}\right)c_1+\frac{(-68-55t)e^{4t}}{50}-\frac{204\cos(t)}{97}-\frac{556\sin(t)}{97} \\ \frac{\sqrt{6}\,(c_1-c_2)e^{2t}\sin\left(t\sqrt{6}\right)}{6}+e^{2t}\cos\left(t\sqrt{6}\right)c_2+\frac{(-11+15t)e^{4t}}{50}-\frac{8\cos(t)}{97}-\frac{212\sin(t)}{97} \end{array}\right] \end{aligned}$$

9.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)$$

Or

$$\left[\begin{array}{c} x'(t) \\ y' \end{array}\right] = \left[\begin{array}{cc} 3 & -7 \\ 1 & 1 \end{array}\right]\left[\begin{array}{c} x(t) \\ y \end{array}\right]+\left[\begin{array}{c} t\,e^{4t}-4\,e^{4t}+4\sin(t) \\ 2t\,e^{4t}+e^{4t}+8\sin(t) \end{array}\right]$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t)+\vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}3 & -7\\ 1 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}3-\lambda & -7\\ 1 & 1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 10 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= 2 + i\sqrt{6}\\ \lambda_2 &= 2 - i\sqrt{6}\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2 + i\sqrt{6}$	1	complex eigenvalue
$2 - i\sqrt{6}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - i\sqrt{6}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{aligned}\left(\begin{bmatrix}3 & -7\\ 1 & 1\end{bmatrix} - \left(2 - i\sqrt{6}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}1 + i\sqrt{6} & -7\\ 1 & -1 + i\sqrt{6}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}1 + i\sqrt{6} & -7 & 0\\ 1 & -1 + i\sqrt{6} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{1+i\sqrt{6}} \Longrightarrow \left[\begin{array}{cc|c} 1+i\sqrt{6} & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i\sqrt{6} & -7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{7t}{1+i\sqrt{6}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{7t}{1+\mathrm{I}\sqrt{6}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{7t}{1+i\sqrt{6}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{7t}{1+\mathrm{I}\sqrt{6}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{7}{1+i\sqrt{6}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{7t}{1+\mathrm{I}\sqrt{6}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{7}{1+i\sqrt{6}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{7t}{1+\mathrm{I}\sqrt{6}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{7}{1+i\sqrt{6}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + i\sqrt{6}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -7 \\ 1 & 1 \end{bmatrix} - \left(2 + i\sqrt{6}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i\sqrt{6} & -7 \\ 1 & -1-i\sqrt{6} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-i\sqrt{6} & -7 & 0 \\ 1 & -1-i\sqrt{6} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{1-i\sqrt{6}} \Longrightarrow \left[\begin{array}{cc|c} 1-i\sqrt{6} & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i\sqrt{6} & -7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{7t}{-1+i\sqrt{6}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{7t}{-1+\mathrm{I}\sqrt{6}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7t}{-1+i\sqrt{6}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{7t}{-1+\mathrm{I}\sqrt{6}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{7}{-1+i\sqrt{6}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{7t}{-1+\mathrm{I}\sqrt{6}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7}{-1+i\sqrt{6}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{7t}{-1+\mathrm{I}\sqrt{6}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7}{-1+i\sqrt{6}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2+i\sqrt{6}$	1	1	No	$\left[\begin{array}{c} -\frac{7}{-1+i\sqrt{6}} \\ 1 \end{array}\right]$
$2-i\sqrt{6}$	1	1	No	$\left[\begin{array}{c} -\frac{7}{-1-i\sqrt{6}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{7\,\mathrm{e}^{\left(2+i\sqrt{6}\right)t}}{-1+i\sqrt{6}} \\ \mathrm{e}^{\left(2+i\sqrt{6}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{7\,\mathrm{e}^{\left(2-i\sqrt{6}\right)t}}{-1-i\sqrt{6}} \\ \mathrm{e}^{\left(2-i\sqrt{6}\right)t} \end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{cc} -\frac{7\,\mathrm{e}^{\left(2+i\sqrt{6}\right)t}}{-1+i\sqrt{6}} & -\frac{7\,\mathrm{e}^{\left(2-i\sqrt{6}\right)t}}{-1-i\sqrt{6}} \\ \mathrm{e}^{\left(2+i\sqrt{6}\right)t} & \mathrm{e}^{\left(2-i\sqrt{6}\right)t} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \left[\begin{array}{cc} -\frac{i\sqrt{6}\,\mathrm{e}^{-\left(2+i\sqrt{6}\right)t}}{12} & \frac{\sqrt{6}\,\mathrm{e}^{-\left(2+i\sqrt{6}\right)t}\left(i+\sqrt{6}\right)}{12} \\ \frac{i\sqrt{6}\,\mathrm{e}^{\left(i\sqrt{6}-2\right)t}}{12} & -\frac{\mathrm{e}^{\left(i\sqrt{6}-2\right)t}\sqrt{6}\left(i-\sqrt{6}\right)}{12} \end{array}\right]$$

Hence

$$\vec{x}_p(t) = \left[\begin{array}{cc} -\frac{7\,\mathrm{e}^{\left(2+i\sqrt{6}\right)t}}{-1+i\sqrt{6}} & -\frac{7\,\mathrm{e}^{\left(2-i\sqrt{6}\right)t}}{-1-i\sqrt{6}} \\ \mathrm{e}^{\left(2+i\sqrt{6}\right)t} & \mathrm{e}^{\left(2-i\sqrt{6}\right)t} \end{array}\right] \int \left[\begin{array}{cc} -\frac{i\sqrt{6}\,\mathrm{e}^{-\left(2+i\sqrt{6}\right)t}}{12} & \frac{\sqrt{6}\,\mathrm{e}^{-\left(2+i\sqrt{6}\right)t}\left(i+\sqrt{6}\right)}{12} \\ \frac{i\sqrt{6}\,\mathrm{e}^{\left(i\sqrt{6}-2\right)t}}{12} & -\frac{\mathrm{e}^{\left(i\sqrt{6}-2\right)t}\sqrt{6}\left(i-\sqrt{6}\right)}{12} \end{array}\right] \left[\begin{array}{c} t\,\mathrm{e}^{4t}-4\,\mathrm{e}^{4t}+4\sin\left(t\right) \\ 2t\,\mathrm{e}^{4t}+\mathrm{e}^{4t}+8\sin\left(t\right) \end{array}\right.$$

$$= \left[\begin{array}{cc} -\frac{7\,\mathrm{e}^{\left(2+i\sqrt{6}\right)t}}{-1+i\sqrt{6}} & -\frac{7\,\mathrm{e}^{\left(2-i\sqrt{6}\right)t}}{-1-i\sqrt{6}} \\ \mathrm{e}^{\left(2+i\sqrt{6}\right)t} & \mathrm{e}^{\left(2-i\sqrt{6}\right)t} \end{array}\right] \int \left[\begin{array}{c} \frac{\mathrm{e}^{-\left(i\sqrt{6}-2\right)t}\left(6+i(t+5)\sqrt{6}+12t\right)}{12}+\frac{\sin(t)\mathrm{e}^{-\left(2+i\sqrt{6}\right)t}\left(i\sqrt{6}+12\right)}{3} \\ \frac{\mathrm{e}^{\left(2+i\sqrt{6}\right)t}\left(6+i(-5-t)\sqrt{6}+12t\right)}{12}+\frac{\left(-i\sqrt{6}+12\right)\sin(t)\mathrm{e}^{\left(i\sqrt{6}-2\right)t}}{3} \end{array}\right] dt$$

$$= \left[\begin{array}{cc} -\frac{7\,\mathrm{e}^{\left(2+i\sqrt{6}\right)t}}{-1+i\sqrt{6}} & -\frac{7\,\mathrm{e}^{\left(2-i\sqrt{6}\right)t}}{-1-i\sqrt{6}} \\ \mathrm{e}^{\left(2+i\sqrt{6}\right)t} & \mathrm{e}^{\left(2-i\sqrt{6}\right)t} \end{array}\right] \left[\begin{array}{c} -\frac{\mathrm{e}^{-\left(i\sqrt{6}-2\right)t}\left(249-i(-168-25t)\sqrt{6}+600t\right)+200\,\mathrm{e}^{-\left(2+i\sqrt{6}\right)t}\left(i(5\sin(t)+\cos(t))\sqrt{6}+3\,\mathrm{co}\right.}{6\left(4\sqrt{6}+i\right)\left(2i+\sqrt{6}\right)^{3}} \\ \frac{\left(-(-168-25t)\sqrt{6}+600it+249i\right)\mathrm{e}^{\left(2+i\sqrt{6}\right)t}+600\,\mathrm{e}^{\left(i\sqrt{6}-2\right)t}\left(\frac{\sqrt{6}\,(5\sin(t)+\cos(t))}{3}+i\cos(t\right.}{6\left(-i+4\sqrt{6}\right)\left(2+i\sqrt{6}\right)^{3}} \end{array}\right.$$

$$= \left[\begin{array}{c} \frac{(-68-55t)\mathrm{e}^{4t}}{50}-\frac{204\cos(t)}{97}-\frac{556\sin(t)}{97} \\ \frac{(-11+15t)\mathrm{e}^{4t}}{50}-\frac{8\cos(t)}{97}-\frac{212\sin(t)}{97} \end{array}\right]$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t)+\vec{x}_p(t)$$

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} -\frac{7c_1\mathrm{e}^{\left(2+i\sqrt{6}\right)t}}{-1+i\sqrt{6}} \\ c_1\mathrm{e}^{\left(2+i\sqrt{6}\right)t} \end{array}\right] + \left[\begin{array}{c} -\frac{7c_2\mathrm{e}^{\left(2-i\sqrt{6}\right)t}}{-1-i\sqrt{6}} \\ c_2\mathrm{e}^{\left(2-i\sqrt{6}\right)t} \end{array}\right] + \left[\begin{array}{c} \frac{(-68-55t)\mathrm{e}^{4t}}{50}-\frac{204\cos(t)}{97}-\frac{556\sin(t)}{97} \\ \frac{(-11+15t)\mathrm{e}^{4t}}{50}-\frac{8\cos(t)}{97}-\frac{212\sin(t)}{97} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} -c_2\left(-1+i\sqrt{6}\right)\mathrm{e}^{-\left(i\sqrt{6}-2\right)t}+\left(1+i\sqrt{6}\right)c_1\mathrm{e}^{\left(2+i\sqrt{6}\right)t}+\frac{(-5335t-6596)\mathrm{e}^{4t}}{4850}-\frac{204\cos(t)}{97}-\frac{556\sin(t)}{97} \\ c_1\mathrm{e}^{\left(2+i\sqrt{6}\right)t}+c_2\mathrm{e}^{-\left(i\sqrt{6}-2\right)t}+\frac{(-11+15t)\mathrm{e}^{4t}}{50}-\frac{8\cos(t)}{97}-\frac{212\sin(t)}{97} \end{array}\right]$$

9.10.3 Maple step by step solution

Let's solve

$$\left[x'(t)=t(\mathrm{e}^{t})^{4}+4\sin(t)-4(\mathrm{e}^{t})^{4}-7y+3x(t), y'=2t(\mathrm{e}^{t})^{4}+(\mathrm{e}^{t})^{4}+8\sin(t)+y+x(t)\right]$$

- Define vector

$$\vec{x}(t)=\left[\begin{array}{c} x(t) \\ y \end{array}\right]$$

- Convert system into a vector equation

$$\vec{x}'(t)=\left[\begin{array}{cc} 3 & -7 \\ 1 & 1 \end{array}\right]\cdot\vec{x}(t)+\left[\begin{array}{c} t(\mathrm{e}^{t})^{4}-4(\mathrm{e}^{t})^{4}+4\sin(t) \\ 2t(\mathrm{e}^{t})^{4}+(\mathrm{e}^{t})^{4}+8\sin(t) \end{array}\right]$$

- System to solve

$$\vec{x}'(t)=\left[\begin{array}{cc} 3 & -7 \\ 1 & 1 \end{array}\right]\cdot\vec{x}(t)+\left[\begin{array}{c} t(\mathrm{e}^{t})^{4}-4(\mathrm{e}^{t})^{4}+4\sin(t) \\ 2t(\mathrm{e}^{t})^{4}+(\mathrm{e}^{t})^{4}+8\sin(t) \end{array}\right]$$

- Define the forcing function

$$\vec{f}(t)=\left[\begin{array}{c} t(\mathrm{e}^{t})^{4}-4(\mathrm{e}^{t})^{4}+4\sin(t) \\ 2t(\mathrm{e}^{t})^{4}+(\mathrm{e}^{t})^{4}+8\sin(t) \end{array}\right]$$

- Define the coefficient matrix

$$A=\left[\begin{array}{cc} 3 & -7 \\ 1 & 1 \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t)=A\cdot\vec{x}(t)+\vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2-\mathrm{I}\sqrt{6}, \left[\begin{array}{c} -\frac{7}{-1-\mathrm{I}\sqrt{6}} \\ 1 \end{array}\right]\right], \left[2+\mathrm{I}\sqrt{6}, \left[\begin{array}{c} -\frac{7}{-1+\mathrm{I}\sqrt{6}} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-\mathrm{I}\sqrt{6}, \left[\begin{array}{c} -\frac{7}{-1-\mathrm{I}\sqrt{6}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(2-\mathrm{I}\sqrt{6}\right)t} \cdot \left[\begin{array}{c} -\frac{7}{-1-\mathrm{I}\sqrt{6}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{2t} \cdot \left(\cos\left(t\sqrt{6}\right) - \mathrm{I}\sin\left(t\sqrt{6}\right)\right) \cdot \left[\begin{array}{c} -\frac{7}{-1-\mathrm{I}\sqrt{6}} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{2t} \cdot \left[\begin{array}{c} -\frac{7\left(\cos\left(t\sqrt{6}\right)-\mathrm{I}\sin\left(t\sqrt{6}\right)\right)}{-1-\mathrm{I}\sqrt{6}} \\ \cos\left(t\sqrt{6}\right) - \mathrm{I}\sin\left(t\sqrt{6}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \mathrm{e}^{2t} \cdot \left[\begin{array}{c} \cos\left(t\sqrt{6}\right) - \sqrt{6}\,\sin\left(t\sqrt{6}\right) \\ \cos\left(t\sqrt{6}\right) \end{array}\right], \vec{x}_2(t) = \mathrm{e}^{2t} \cdot \left[\begin{array}{c} -\sqrt{6}\,\cos\left(t\sqrt{6}\right) - \sin\left(t\sqrt{6}\right) \\ -\sin\left(t\sqrt{6}\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \left[\begin{array}{cc} \mathrm{e}^{2t}\left(\cos\left(t\sqrt{6}\right) - \sqrt{6}\,\sin\left(t\sqrt{6}\right)\right) & \mathrm{e}^{2t}\left(-\sqrt{6}\,\cos\left(t\sqrt{6}\right) - \sin\left(t\sqrt{6}\right)\right) \\ \mathrm{e}^{2t}\cos\left(t\sqrt{6}\right) & -\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right) \end{array}\right]$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc} \mathrm{e}^{2t}\left(\cos\left(t\sqrt{6}\right) - \sqrt{6}\,\sin\left(t\sqrt{6}\right)\right) & \mathrm{e}^{2t}\left(-\sqrt{6}\,\cos\left(t\sqrt{6}\right) - \sin\left(t\sqrt{6}\right)\right) \\ \mathrm{e}^{2t}\cos\left(t\sqrt{6}\right) & -\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right) \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} 1 & -\sqrt{6} \\ 1 & 0 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \left[\begin{array}{cc} \frac{\left(\sqrt{6}\, \cos\left(t \sqrt{6}\right)+\sin\left(t \sqrt{6}\right)\right) \sqrt{6}\, \mathrm{e}^{2 t}}{6} & -\frac{7 \sqrt{6}\, \mathrm{e}^{2 t} \sin\left(t \sqrt{6}\right)}{6} \\ \frac{\sqrt{6}\, \mathrm{e}^{2 t} \sin\left(t \sqrt{6}\right)}{6} & -\frac{\mathrm{e}^{2 t} \left(\sqrt{6}\, \sin\left(t \sqrt{6}\right)-6 \cos\left(t \sqrt{6}\right)\right)}{6} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_{p}(t)=\Phi(t)\cdot \overrightarrow{v}(t)$
- Take the derivative of the particular solution

  $\overrightarrow{x}_{p}^{\prime}(t)=\Phi^{\prime}(t)\cdot \overrightarrow{v}(t)+\Phi(t)\cdot \overrightarrow{v}^{\prime}(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi^{\prime}(t)\cdot \overrightarrow{v}(t)+\Phi(t)\cdot \overrightarrow{v}^{\prime}(t)=A\cdot \Phi(t)\cdot \overrightarrow{v}(t)+\overrightarrow{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot \Phi(t)\cdot \overrightarrow{v}(t)+\Phi(t)\cdot \overrightarrow{v}^{\prime}(t)=A\cdot \Phi(t)\cdot \overrightarrow{v}(t)+\overrightarrow{f}(t)$
- Cancel like terms

  $\Phi(t)\cdot \overrightarrow{v}^{\prime}(t)=\overrightarrow{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}^{\prime}(t)=\frac{1}{\Phi(t)}\cdot \overrightarrow{f}(t)$
- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t)=\int_{0}^{t}\frac{1}{\Phi(s)}\cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_{p}(t)=\Phi(t)\cdot \left(\int_{0}^{t}\frac{1}{\Phi(s)}\cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{x}_{p}(t) = \left[\begin{array}{c} \frac{8398\, \mathrm{e}^{2 t} \cos\left(t \sqrt{6}\right)}{2425}+\frac{6527 \sqrt{6}\, \mathrm{e}^{2 t} \sin\left(t \sqrt{6}\right)}{29100}+\frac{\left(-68-55 t\right) \mathrm{e}^{4 t}}{50}-\frac{204 \cos\left(t\right)}{97}-\frac{556 \sin\left(t\right)}{97} \\ \frac{1467\, \mathrm{e}^{2 t} \cos\left(t \sqrt{6}\right)}{4850}+\frac{15329 \sqrt{6}\, \mathrm{e}^{2 t} \sin\left(t \sqrt{6}\right)}{29100}+\frac{\left(-11+15 t\right) \mathrm{e}^{4 t}}{50}-\frac{8 \cos\left(t\right)}{97}-\frac{212 \sin\left(t\right)}{97} \end{array}\right]$$

● Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \left[\begin{array}{c} \frac{8398\,\mathrm{e}^{2t}\cos\left(t\sqrt{6}\right)}{2425} + \frac{6527\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)}{29100} + \frac{(-68-55t)\mathrm{e}^{4t}}{50} - \frac{204\cos(t)}{97} - \frac{556\sin(t}{97} \\ \frac{1467\,\mathrm{e}^{2t}\cos\left(t\sqrt{6}\right)}{4850} + \frac{15329\sqrt{6}\,\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right)}{29100} + \frac{(-11+15t)\mathrm{e}^{4t}}{50} - \frac{8\cos(t)}{97} - \frac{212\sin(t)}{97} \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \left(-c_2\sqrt{6} + c_1 + \frac{8398}{2425}\right)\mathrm{e}^{2t}\cos\left(t\sqrt{6}\right) - \mathrm{e}^{2t}\left(\left(c_1 - \frac{6527}{29100}\right)\sqrt{6} + c_2\right)\sin\left(t\sqrt{6}\right) + \frac{(-68-55}{50} \\ \frac{(4850c_1+1467)\mathrm{e}^{2t}\cos\left(t\sqrt{6}\right)}{4850} - \left(c_2 - \frac{15329\sqrt{6}}{29100}\right)\mathrm{e}^{2t}\sin\left(t\sqrt{6}\right) + \frac{(-11+15t)\mathrm{e}^{4t}}{50} - 8\,\mathrm{c} \end{array}\right.$$

- Solution to the system of ODEs

$$\left\{x(t) = \left(-c_2\sqrt{6} + c_1 + \frac{8398}{2425}\right)\mathrm{e}^{2t}\cos\left(t\sqrt{6}\right) - \mathrm{e}^{2t}\left(\left(c_1 - \frac{6527}{29100}\right)\sqrt{6} + c_2\right)\sin\left(t\sqrt{6}\right) + \frac{(-68-55t)\mathrm{e}^{4t}}{50} -\right.$$

✓ Solution by Maple

Time used: 1.156 (sec). Leaf size: 134

```
dsolve([diff(x(t),t)=3*x(t)-7*y(t)+4*sin(t)+(t-4)*exp(4*t),diff(y(t),t)=x(t)+y(t)+8*sin(t)+(
```

$$x(t) = \mathrm{e}^{2t}\sin\left(\sqrt{6}\,t\right)c_2 + \mathrm{e}^{2t}\cos\left(\sqrt{6}\,t\right)c_1 - \frac{11\,\mathrm{e}^{4t}t}{10} - \frac{34\,\mathrm{e}^{4t}}{25} - \frac{204\cos(t)}{97} - \frac{556\sin(t)}{97}$$

$$y(t) = \frac{3\,\mathrm{e}^{4t}t}{10} + \frac{\mathrm{e}^{2t}\sqrt{6}\,\sin\left(\sqrt{6}\,t\right)c_1}{7} - \frac{\mathrm{e}^{2t}\sqrt{6}\,\cos\left(\sqrt{6}\,t\right)c_2}{7} - \frac{11\,\mathrm{e}^{4t}}{50} + \frac{\mathrm{e}^{2t}\sin\left(\sqrt{6}\,t\right)c_2}{7} + \frac{\mathrm{e}^{2t}\cos\left(\sqrt{6}\,t\right)c_1}{7} - \frac{8\cos(t)}{97} - \frac{212\sin(t)}{97}$$

✓ Solution by Mathematica

Time used: 11.331 (sec). Leaf size: 190

```
DSolve[{x'[t]==3*x[t]-7*y[t]+4*Sin[t]+(t-4)*Exp[4*t],y'[t]==x[t]+y[t]+8*Sin[t]+(2*t+1)*Exp[4
```

$$x(t) \to -\frac{11}{10}e^{4t}t - \frac{34e^{4t}}{25} - \frac{556\sin(t)}{97} - \frac{204\cos(t)}{97} + c_1e^{2t}\cos\left(\sqrt{6}t\right) + \frac{c_1e^{2t}\sin\left(\sqrt{6}t\right)}{\sqrt{6}} - \frac{7c_2e^{2t}\sin\left(\sqrt{6}t\right)}{\sqrt{6}}$$

$$y(t) \to \frac{3}{10}e^{4t}t - \frac{11e^{4t}}{50} - \frac{212\sin(t)}{97} - \frac{8\cos(t)}{97} + c_2e^{2t}\cos\left(\sqrt{6}t\right) + \frac{c_1e^{2t}\sin\left(\sqrt{6}t\right)}{\sqrt{6}} - \frac{c_2e^{2t}\sin\left(\sqrt{6}t\right)}{\sqrt{6}}$$

9.11 problem 11

Internal problem ID [6721]
Internal file name [`OUTPUT/5969_Sunday_June_05_2022_04_10_35_PM_14599038/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 3x(t) - 4y \\ y' &= 4x(t) - 7y \end{aligned}$$

9.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(4\,\mathrm{e}^{6t}-1)\mathrm{e}^{-5t}}{3} & -\frac{2(\mathrm{e}^{6t}-1)\mathrm{e}^{-5t}}{3} \\ \frac{2(\mathrm{e}^{6t}-1)\mathrm{e}^{-5t}}{3} & -\frac{(\mathrm{e}^{6t}-4)\mathrm{e}^{-5t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(4\,\mathrm{e}^{6t}-1)\mathrm{e}^{-5t}}{3} & -\frac{2(\mathrm{e}^{6t}-1)\mathrm{e}^{-5t}}{3} \\ \frac{2(\mathrm{e}^{6t}-1)\mathrm{e}^{-5t}}{3} & -\frac{(\mathrm{e}^{6t}-4)\mathrm{e}^{-5t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(4\,\mathrm{e}^{6t}-1)\mathrm{e}^{-5t}c_1}{3} - \frac{2(\mathrm{e}^{6t}-1)\mathrm{e}^{-5t}c_2}{3} \\ \frac{2(\mathrm{e}^{6t}-1)\mathrm{e}^{-5t}c_1}{3} - \frac{(\mathrm{e}^{6t}-4)\mathrm{e}^{-5t}c_2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-5t}(2(2c_1-c_2)\mathrm{e}^{6t}-c_1+2c_2)}{3} \\ \frac{2\,\mathrm{e}^{-5t}\left(\left(c_1-\frac{c_2}{2}\right)\mathrm{e}^{6t}-c_1+2c_2\right)}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3-\lambda & -4 \\ 4 & -7-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -5$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{aligned}
\left(\begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 8 & -4 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 8 & -4 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 8 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 8 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 4 & -8 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-5	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-5t} \\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{-5t}\end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{\mathrm{e}^{-5t}}{2} \\ \mathrm{e}^{-5t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{(4c_2 \mathrm{e}^{6t} + c_1)\mathrm{e}^{-5t}}{2} \\ (c_2 \mathrm{e}^{6t} + c_1)\,\mathrm{e}^{-5t} \end{bmatrix}$$

The following is the phase plot of the system.

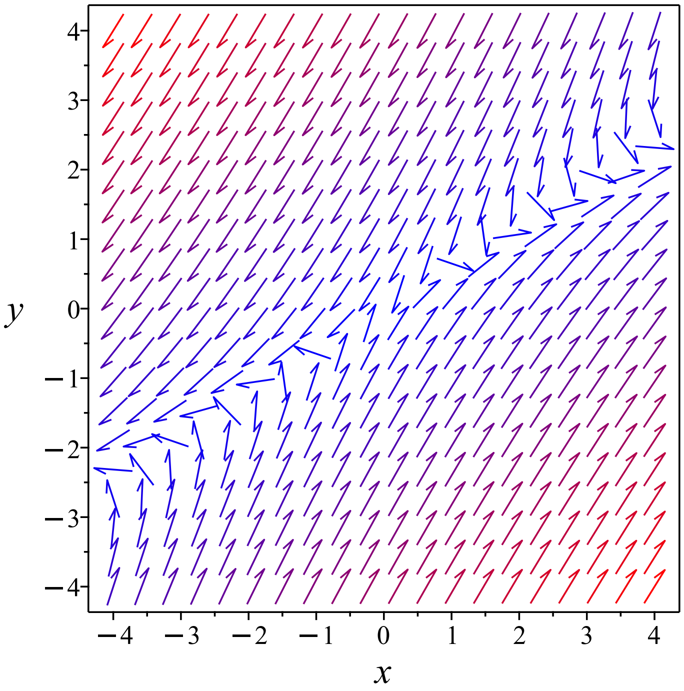


Figure 81: Phase plot

9.11.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) - 4y, y' = 4x(t) - 7y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 3 & -4 \\ 4 & -7 \end{array}\right]$$

- Rewrite the system as

  $\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

  $$\left[\left[-5, \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 2 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

  $$\left[-5, \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_{1} = \mathrm{e}^{-5t} \cdot \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

  $$\left[1, \left[\begin{array}{c} 2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_{2} = \mathrm{e}^{t} \cdot \left[\begin{array}{c} 2 \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_{1}\overrightarrow{x}_{1} + c_{2}\overrightarrow{x}_{2}$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_{1}\mathrm{e}^{-5t} \cdot \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right] + c_{2}\mathrm{e}^{t} \cdot \left[\begin{array}{c} 2 \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{(4c_2 \mathrm{e}^{6t}+c_1)\mathrm{e}^{-5t}}{2} \\ (c_2 \mathrm{e}^{6t}+c_1)\,\mathrm{e}^{-5t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x(t) = \frac{(4c_2 \mathrm{e}^{6t}+c_1)\mathrm{e}^{-5t}}{2}, y = (c_2 \mathrm{e}^{6t}+c_1)\,\mathrm{e}^{-5t}\right\}$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve([diff(x(t),t)=3*x(t)-4*y(t),diff(y(t),t)=4*x(t)-7*y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{-5t} + c_2 \mathrm{e}^{t}$$
$$y(t) = 2c_1 \mathrm{e}^{-5t} + \frac{c_2 \mathrm{e}^{t}}{2}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 73

```
DSolve[{x'[t]==3*x[t]-4*y[t],y'[t]==4*x[t]-7*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{3} e^{-5t} \left(c_1 \left(4e^{6t} - 1\right) - 2c_2 \left(e^{6t} - 1\right)\right)$$
$$y(t) \to \frac{1}{3} e^{-5t} \left(2c_1 \left(e^{6t} - 1\right) - c_2 \left(e^{6t} - 4\right)\right)$$

9.12 problem 12

Internal problem ID [6722]
Internal file name [`OUTPUT/5970_Sunday_June_05_2022_04_10_39_PM_80188498/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -2x(t) + 5y \\ y' &= -2x(t) + 4y \end{aligned}$$

9.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^{t}\cos(t) - 3\sin(t)\,\mathrm{e}^{t} & 5\sin(t)\,\mathrm{e}^{t} \\ -2\sin(t)\,\mathrm{e}^{t} & \mathrm{e}^{t}\cos(t) + 3\sin(t)\,\mathrm{e}^{t} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}(\cos(t) - 3\sin(t)) & 5\sin(t)\,\mathrm{e}^{t} \\ -2\sin(t)\,\mathrm{e}^{t} & \mathrm{e}^{t}(\cos(t) + 3\sin(t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{t}(\cos(t) - 3\sin(t)) & 5\sin(t)\,\mathrm{e}^{t} \\ -2\sin(t)\,\mathrm{e}^{t} & \mathrm{e}^{t}(\cos(t) + 3\sin(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}(\cos(t) - 3\sin(t))\,c_1 + 5\sin(t)\,\mathrm{e}^{t}c_2 \\ -2\sin(t)\,\mathrm{e}^{t}c_1 + \mathrm{e}^{t}(\cos(t) + 3\sin(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((-3c_1 + 5c_2)\sin(t) + c_1\cos(t))\,\mathrm{e}^{t} \\ \mathrm{e}^{t}(c_2\cos(t) - 2c_1\sin(t) + 3\sin(t)\,c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 5 \\ -2 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1+i$$
$$\lambda_2 = 1-i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1+i$	1	complex eigenvalue
$1-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1-i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix} - (1-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3+i & 5 \\ -2 & 3+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3+i & 5 & 0 \\ -2 & 3+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{5} - \frac{i}{5}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} -3+i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3+i & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{3}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{3}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{3}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{3}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{3}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 3+i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix} - (1+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3-i & 5 \\ -2 & 3-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3-i & 5 & 0 \\ -2 & 3-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{5} + \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -3-i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3-i & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{3}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{3}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{3}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{3}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{3}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 3-i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$1+i$	1	1	No	$\left[\begin{array}{c} \frac{3}{2}-\frac{i}{2} \\ 1 \end{array}\right]$
$1-i$	1	1	No	$\left[\begin{array}{c} \frac{3}{2}+\frac{i}{2} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} \left(\frac{3}{2}-\frac{i}{2}\right) \mathrm{e}^{(1+i)t} \\ \mathrm{e}^{(1+i)t} \end{array}\right] + c_2 \left[\begin{array}{c} \left(\frac{3}{2}+\frac{i}{2}\right) \mathrm{e}^{(1-i)t} \\ \mathrm{e}^{(1-i)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \left(\frac{3}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{(1+i)t} + \left(\frac{3}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{(1-i)t} \\ c_1 \mathrm{e}^{(1+i)t} + c_2 \mathrm{e}^{(1-i)t} \end{array}\right]$$

The following is the phase plot of the system.

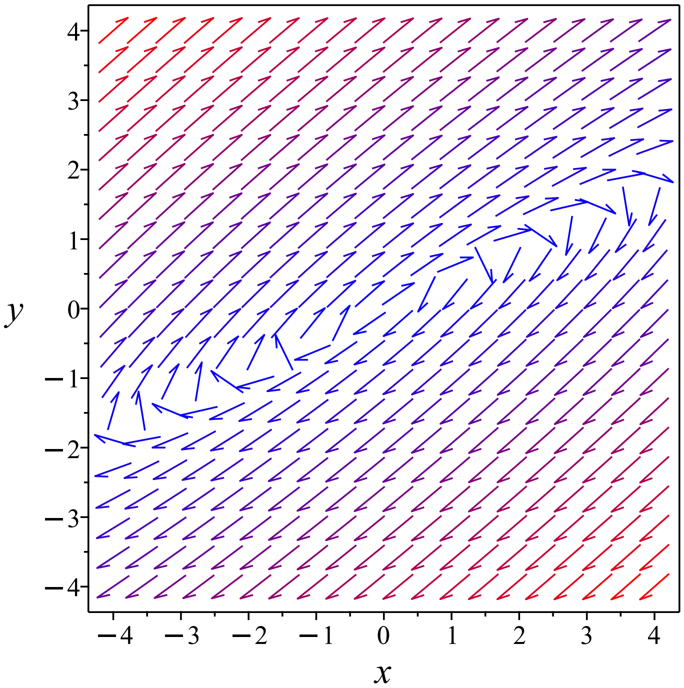


Figure 82: Phase plot

9.12.3 Maple step by step solution

Let's solve

$[x'(t) = -2x(t) + 5y, y' = -2x(t) + 4y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} -2 & 5 \\ -2 & 4 \end{array}\right]$$

- Rewrite the system as

  $\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1-\mathrm{I}, \left[\begin{array}{c} \frac{3}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[1+\mathrm{I}, \left[\begin{array}{c} \frac{3}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[1-\mathrm{I}, \left[\begin{array}{c} \frac{3}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{(1-\mathrm{I})t} \cdot \left[\begin{array}{c} \frac{3}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{t} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \left[\begin{array}{c} \frac{3}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

  $$\mathrm{e}^{t} \cdot \left[\begin{array}{c} \left(\frac{3}{2}+\frac{\mathrm{I}}{2}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_{1}(t) = \mathrm{e}^{t} \cdot \left[\begin{array}{c} \frac{3\cos(t)}{2}+\frac{\sin(t)}{2} \\ \cos(t) \end{array}\right], \overrightarrow{x}_{2}(t) = \mathrm{e}^{t} \cdot \left[\begin{array}{c} -\frac{3\sin(t)}{2}+\frac{\cos(t)}{2} \\ -\sin(t) \end{array}\right]\right]$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_{1}\overrightarrow{x}_{1}(t) + c_{2}\overrightarrow{x}_{2}(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^t \cdot \begin{bmatrix} \frac{3\cos(t)}{2} + \frac{\sin(t)}{2} \\ \cos(t) \end{bmatrix} + c_2 \mathrm{e}^t \cdot \begin{bmatrix} -\frac{3\sin(t)}{2} + \frac{\cos(t)}{2} \\ -\sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{3\left(\left(c_1 + \frac{c_2}{3}\right)\cos(t) + \frac{\sin(t)(c_1 - 3c_2)}{3}\right)\mathrm{e}^t}{2} \\ \mathrm{e}^t (c_1 \cos(t) - c_2 \sin(t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{3\left(\left(c_1 + \frac{c_2}{3}\right)\cos(t) + \frac{\sin(t)(c_1 - 3c_2)}{3}\right)\mathrm{e}^t}{2}, y = \mathrm{e}^t (c_1 \cos(t) - c_2 \sin(t))\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve([diff(x(t),t)=-2*x(t)+5*y(t),diff(y(t),t)=-2*x(t)+4*y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^t (c_1 \sin(t) + c_2 \cos(t))$$
$$y(t) = \frac{\mathrm{e}^t (3c_1 \sin(t) - c_2 \sin(t) + c_1 \cos(t) + 3c_2 \cos(t))}{5}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 51

```
DSolve[{x'[t]==-2*x[t]+5*y[t],y'[t]==-2*x[t]+4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to e^t (c_1 \cos(t) + (5c_2 - 3c_1) \sin(t))$$
$$y(t) \to e^t (c_2 (3 \sin(t) + \cos(t)) - 2c_1 \sin(t))$$

9.13 problem 13

Internal problem ID [6723]
Internal file name [`OUTPUT/5971_Sunday_June_05_2022_04_10_41_PM_42145987/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -x(t) + \frac{y}{4}$$
$$y' = x(t) - y$$

9.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{4} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-\frac{3t}{2}}}{2} + \frac{e^{-\frac{t}{2}}}{2} & \frac{e^{-\frac{t}{2}}}{4} - \frac{e^{-\frac{3t}{2}}}{4} \\ e^{-\frac{t}{2}} - e^{-\frac{3t}{2}} & \frac{e^{-\frac{3t}{2}}}{2} + \frac{e^{-\frac{t}{2}}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{\mathrm{e}^{-\frac{t}{2}}}{2} & \frac{\mathrm{e}^{-\frac{t}{2}}}{4}-\frac{\mathrm{e}^{-\frac{3t}{2}}}{4} \\ \mathrm{e}^{-\frac{t}{2}}-\mathrm{e}^{-\frac{3t}{2}} & \frac{\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{\mathrm{e}^{-\frac{t}{2}}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{\mathrm{e}^{-\frac{t}{2}}}{2}\right)c_1+\left(\frac{\mathrm{e}^{-\frac{t}{2}}}{4}-\frac{\mathrm{e}^{-\frac{3t}{2}}}{4}\right)c_2 \\ \left(\mathrm{e}^{-\frac{t}{2}}-\mathrm{e}^{-\frac{3t}{2}}\right)c_1+\left(\frac{\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{\mathrm{e}^{-\frac{t}{2}}}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1-c_2)\mathrm{e}^{-\frac{3t}{2}}}{4}+\frac{(c_1+\frac{c_2}{2})\mathrm{e}^{-\frac{t}{2}}}{2} \\ \frac{(-2c_1+c_2)\mathrm{e}^{-\frac{3t}{2}}}{2}+\left(c_1+\frac{c_2}{2}\right)\mathrm{e}^{-\frac{t}{2}} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{4} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & \frac{1}{4} \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & \frac{1}{4} \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2+2\lambda+\frac{3}{4}=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-\frac{3}{2}$$
$$\lambda_2=-\frac{1}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2}$	1	real eigenvalue
$-\frac{3}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-\frac{3}{2}$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-1 & \frac{1}{4}\\ 1 & -1\end{bmatrix}-\left(-\frac{3}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{1}{2} & \frac{1}{4}\\ 1 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{1}{2} & \frac{1}{4} & 0\\ 1 & \frac{1}{2} & 0\end{array}\right]$$

$$R_2=R_2-2R_1\Longrightarrow\left[\begin{array}{cc|c}\frac{1}{2} & \frac{1}{4} & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}\frac{1}{2} & \frac{1}{4}\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & \frac{1}{4} \\ 1 & -1 \end{bmatrix} - \left(-\frac{1}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{1}{2} & \frac{1}{4} & 0 \\ 1 & -\frac{1}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-\frac{3}{2}$	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
$-\frac{1}{2}$	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{3}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-\frac{3t}{2}} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{-\frac{3t}{2}}\end{aligned}$$

Since eigenvalue $-\frac{1}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-\frac{t}{2}} \\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{-\frac{t}{2}}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\mathrm{e}^{-\frac{3t}{2}}}{2} \\ \mathrm{e}^{-\frac{3t}{2}} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{-\frac{t}{2}}}{2} \\ \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{c_1 \mathrm{e}^{-\frac{3t}{2}}}{2} + \frac{c_2 \mathrm{e}^{-\frac{t}{2}}}{2} \\ c_1 \mathrm{e}^{-\frac{3t}{2}} + c_2 \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

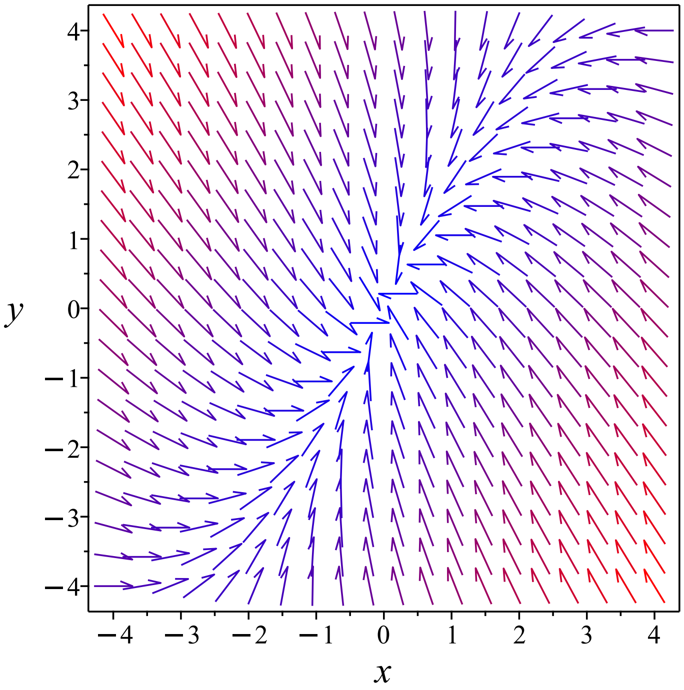


Figure 83: Phase plot

9.13.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -x(t) + \frac{y}{4}, y' = x(t) - y\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & \frac{1}{4} \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & \frac{1}{4} \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

  $$A = \left[\begin{array}{cc} -1 & \frac{1}{4} \\ 1 & -1 \end{array}\right]$$

- Rewrite the system as

  $$\overrightarrow{x}^{\prime}\left(t \right) = A \cdot \overrightarrow{x}\left(t \right)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-\frac{3}{2}, \left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}, \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

  $$\left[-\frac{3}{2}, \left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_{1} = {\mathrm e}^{-\frac{3 t}{2}} \cdot \left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

  $$\left[-\frac{1}{2}, \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_{2} = {\mathrm e}^{-\frac{t}{2}} \cdot \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

  $$\overrightarrow{x} = c_{1} \overrightarrow{x}_{1} + c_{2} \overrightarrow{x}_{2}$$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_{1} {\mathrm e}^{-\frac{3 t}{2}} \cdot \left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right] + c_{2} {\mathrm e}^{-\frac{t}{2}} \cdot \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{c_1 \mathrm{e}^{-\frac{3t}{2}}}{2} + \frac{c_2 \mathrm{e}^{-\frac{t}{2}}}{2} \\ c_1 \mathrm{e}^{-\frac{3t}{2}} + c_2 \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}$$

- Solution to the system of ODEs
  $$\left\{x(t) = -\frac{c_1 \mathrm{e}^{-\frac{3t}{2}}}{2} + \frac{c_2 \mathrm{e}^{-\frac{t}{2}}}{2}, y = c_1 \mathrm{e}^{-\frac{3t}{2}} + c_2 \mathrm{e}^{-\frac{t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=-x(t)+1/4*y(t),diff(y(t),t)=x(t)-y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{-\frac{t}{2}} + c_2 \mathrm{e}^{-\frac{3t}{2}}$$
$$y(t) = 2c_1 \mathrm{e}^{-\frac{t}{2}} - 2c_2 \mathrm{e}^{-\frac{3t}{2}}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 66

```
DSolve[{x'[t]==-x[t]+1/4*y[t],y'[t]==x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tr
```

$$x(t) \to \frac{1}{4} e^{-3t/2} \left(2c_1 \left(e^t + 1\right) + c_2 \left(e^t - 1\right)\right)$$
$$y(t) \to \frac{1}{2} e^{-3t/2} \left(2c_1 \left(e^t - 1\right) + c_2 \left(e^t + 1\right)\right)$$

9.14 problem 14

Internal problem ID [6724]
Internal file name [`OUTPUT/5972_Sunday_June_05_2022_04_10_43_PM_86939148/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 2x(t) + y \\ y' &= -x(t) \end{aligned}$$

9.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t}(t+1) & t\,\mathrm{e}^{t} \\ -t\,\mathrm{e}^{t} & \mathrm{e}^{t}(-t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{t}(t+1) & t\,\mathrm{e}^{t} \\ -t\,\mathrm{e}^{t} & \mathrm{e}^{t}(-t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}(t+1)\,c_1 + t\,\mathrm{e}^{t}c_2 \\ -t\,\mathrm{e}^{t}c_1 + \mathrm{e}^{t}(-t+1)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}(tc_1 + c_2 t + c_1) \\ -\mathrm{e}^{t}((t-1)\,c_2 + tc_1) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

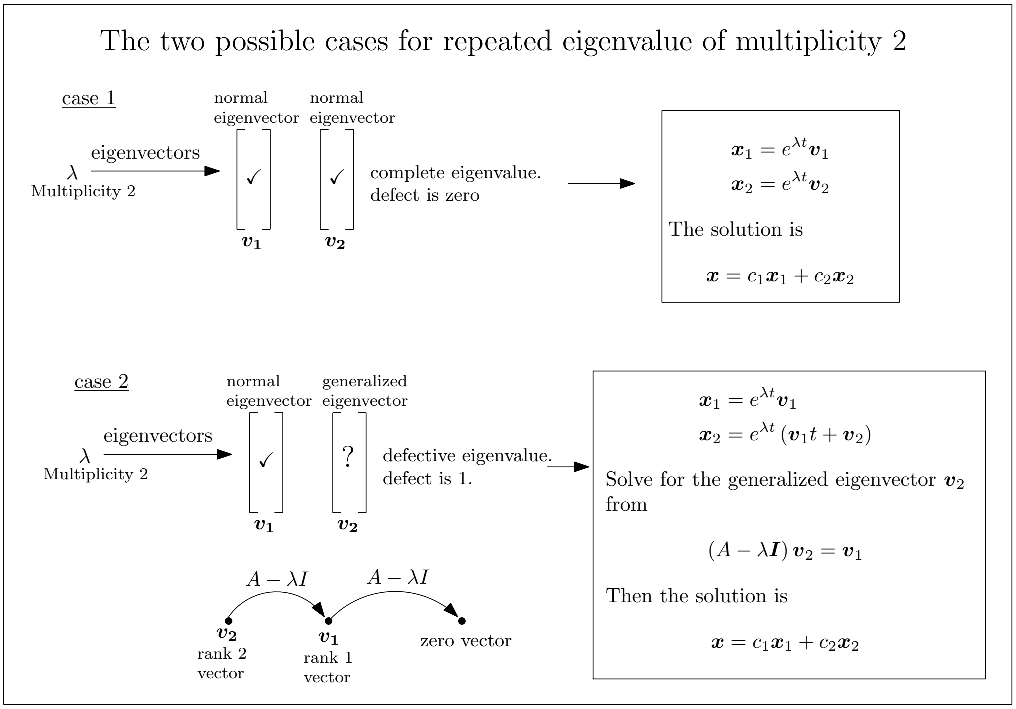


Figure 84: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 1. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{t} \\
&= \begin{bmatrix} -\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \mathrm{e}^{t} \\
&= \begin{bmatrix} -\mathrm{e}^{t}(t+2) \\ \mathrm{e}^{t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t}(-t-2) \\ \mathrm{e}^{t}(t+1) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\mathrm{e}^{t}((t+2)\, c_2 + c_1) \\ \mathrm{e}^{t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

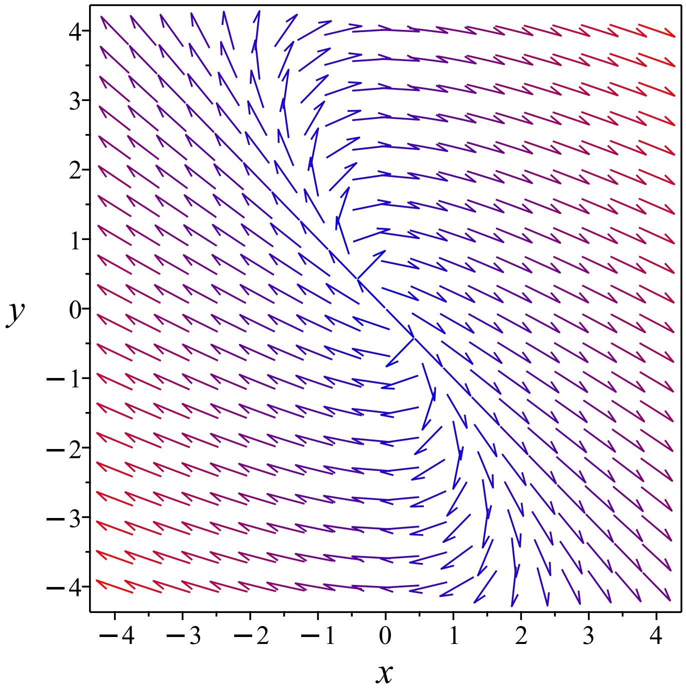


Figure 85: Phase plot

9.14.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) + y, y' = -x(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 2 & 1 \\ -1 & 0 \end{array}\right]$$

- Rewrite the system as

  $$\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[1, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 1

  $$\overrightarrow{x}_{1}(t) = \mathrm{e}^{t} \cdot \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, and

  $$\overrightarrow{x}_{2}(t) = \mathrm{e}^{\lambda t} \left(t \overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_{2}(t)$ into the homogeneous system

  $$\lambda \mathrm{e}^{\lambda t} \left(t \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t} \overrightarrow{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t \overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $$\lambda \mathrm{e}^{\lambda t} \left(t \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t} \overrightarrow{v} = \mathrm{e}^{\lambda t} \left(\lambda t \overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

  $$\lambda \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_{2}(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

  $$\left(\left[\begin{array}{cc} 2 & 1 \\ -1 & 0 \end{array}\right]-1\cdot\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p}=\left[\begin{array}{c} -1 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 1

  $$\overrightarrow{x}_2(t)=\mathrm{e}^t\cdot\left(t\cdot\left[\begin{array}{c} -1 \\ 1 \end{array}\right]+\left[\begin{array}{c} -1 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

  $\overrightarrow{x}=c_1\overrightarrow{x}_1(t)+c_2\overrightarrow{x}_2(t)$

- Substitute solutions into the general solution

  $$\overrightarrow{x}=c_1\mathrm{e}^t\cdot\left[\begin{array}{c} -1 \\ 1 \end{array}\right]+c_2\mathrm{e}^t\cdot\left(t\cdot\left[\begin{array}{c} -1 \\ 1 \end{array}\right]+\left[\begin{array}{c} -1 \\ 0 \end{array}\right]\right)$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y \end{array}\right]=\left[\begin{array}{c} \mathrm{e}^t(-tc_2-c_1-c_2) \\ \mathrm{e}^t(tc_2+c_1) \end{array}\right]$$

- Solution to the system of ODEs

  $\{x(t)=\mathrm{e}^t(-tc_2-c_1-c_2)\,,y=\mathrm{e}^t(tc_2+c_1)\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve([diff(x(t),t)=2*x(t)+y(t),diff(y(t),t)=-x(t)],singsol=all)
```

$$x(t)=-\mathrm{e}^t(c_2t+c_1+c_2)$$
$$y(t)=\mathrm{e}^t(c_2t+c_1)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 38

```
DSolve[{x'[t]==2*x[t]+y[t],y'[t]==-x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
x(t) &\to e^t(c_1(t+1)+c_2t) \\
y(t) &\to e^t(c_2-(c_1+c_2)t)
\end{aligned}
$$

9.15 problem 15

Internal problem ID [6725]
Internal file name [`OUTPUT/5973_Sunday_June_05_2022_04_10_45_PM_7356621/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= x(t) + 2y + z(t) \\ y' &= 6x(t) - y \\ z'(t) &= -x(t) - 2y - z(t) \end{aligned}$$

9.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(64\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}+27\right)\mathrm{e}^{-4t}}{84} & \frac{2\left(\mathrm{e}^{7t}-1\right)\mathrm{e}^{-4t}}{7} & \frac{\left(16\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}-9\right)\mathrm{e}^{-4t}}{84} \\ \frac{\left(16\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}-9\right)\mathrm{e}^{-4t}}{14} & \frac{\left(3\,\mathrm{e}^{7t}+4\right)\mathrm{e}^{-4t}}{7} & \frac{\left(4\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}+3\right)\mathrm{e}^{-4t}}{14} \\ -\frac{\left(64\,\mathrm{e}^{7t}-91\,\mathrm{e}^{4t}+27\right)\mathrm{e}^{-4t}}{84} & -\frac{2\left(\mathrm{e}^{7t}-1\right)\mathrm{e}^{-4t}}{7} & -\frac{\left(16\,\mathrm{e}^{7t}-91\,\mathrm{e}^{4t}-9\right)\mathrm{e}^{-4t}}{84} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{\left(64\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}+27\right)\mathrm{e}^{-4t}}{84} & \frac{2\left(\mathrm{e}^{7t}-1\right)\mathrm{e}^{-4t}}{7} & \frac{\left(16\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}-9\right)\mathrm{e}^{-4t}}{84} \\ \frac{\left(16\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}-9\right)\mathrm{e}^{-4t}}{14} & \frac{\left(3\,\mathrm{e}^{7t}+4\right)\mathrm{e}^{-4t}}{7} & \frac{\left(4\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}+3\right)\mathrm{e}^{-4t}}{14} \\ -\frac{\left(64\,\mathrm{e}^{7t}-91\,\mathrm{e}^{4t}+27\right)\mathrm{e}^{-4t}}{84} & -\frac{2\left(\mathrm{e}^{7t}-1\right)\mathrm{e}^{-4t}}{7} & -\frac{\left(16\,\mathrm{e}^{7t}-91\,\mathrm{e}^{4t}-9\right)\mathrm{e}^{-4t}}{84} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\left(64\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}+27\right)\mathrm{e}^{-4t}c_1}{84} + \frac{2\left(\mathrm{e}^{7t}-1\right)\mathrm{e}^{-4t}c_2}{7} + \frac{\left(16\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}-9\right)\mathrm{e}^{-4t}c_3}{84} \\ \frac{\left(16\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}-9\right)\mathrm{e}^{-4t}c_1}{14} + \frac{\left(3\,\mathrm{e}^{7t}+4\right)\mathrm{e}^{-4t}c_2}{7} + \frac{\left(4\,\mathrm{e}^{7t}-7\,\mathrm{e}^{4t}+3\right)\mathrm{e}^{-4t}c_3}{14} \\ -\frac{\left(64\,\mathrm{e}^{7t}-91\,\mathrm{e}^{4t}+27\right)\mathrm{e}^{-4t}c_1}{84} - \frac{2\left(\mathrm{e}^{7t}-1\right)\mathrm{e}^{-4t}c_2}{7} - \frac{\left(16\,\mathrm{e}^{7t}-91\,\mathrm{e}^{4t}-9\right)\mathrm{e}^{-4t}c_3}{84} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\left(\frac{8(-8c_1-3c_2-2c_3)\mathrm{e}^{7t}}{7}+(c_1+c_3)\mathrm{e}^{4t}-\frac{27c_1}{7}+\frac{24c_2}{7}+\frac{9c_3}{7}\right)\mathrm{e}^{-4t}}{12} \\ -\frac{\left(\frac{2(-8c_1-3c_2-2c_3)\mathrm{e}^{7t}}{7}+(c_1+c_3)\mathrm{e}^{4t}+\frac{9c_1}{7}-\frac{8c_2}{7}-\frac{3c_3}{7}\right)\mathrm{e}^{-4t}}{2} \\ \frac{13\left(\frac{8(-8c_1-3c_2-2c_3)\mathrm{e}^{7t}}{91}+(c_1+c_3)\mathrm{e}^{4t}-\frac{27c_1}{91}+\frac{24c_2}{91}+\frac{9c_3}{91}\right)\mathrm{e}^{-4t}}{12} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\left[\begin{array}{ccc}1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1\end{array}\right]-\lambda\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]\right)=0$$

Therefore

$$\det\left(\left[\begin{array}{ccc}1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda\end{array}\right]\right)=0$$

Which gives the characteristic equation

$$\lambda^{3}+\lambda^{2}-12\lambda=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=3$$
$$\lambda_2=-4$$
$$\lambda_3=0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
-4	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-4$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}1&2&1\\6&-1&0\\-1&-2&-1\end{bmatrix}-(-4)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}5&2&1\\6&3&0\\-1&-2&3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}5&2&1&0\\6&3&0&0\\-1&-2&3&0\end{array}\right]$$

$$R_2=R_2-\frac{6R_1}{5}\Longrightarrow\left[\begin{array}{ccc|c}5&2&1&0\\0&\frac{3}{5}&-\frac{6}{5}&0\\-1&-2&3&0\end{array}\right]$$

$$R_3=R_3+\frac{R_1}{5}\Longrightarrow\left[\begin{array}{ccc|c}5&2&1&0\\0&\frac{3}{5}&-\frac{6}{5}&0\\0&-\frac{8}{5}&\frac{16}{5}&0\end{array}\right]$$

$$R_3=R_3+\frac{8R_2}{3}\Longrightarrow\left[\begin{array}{ccc|c}5&2&1&0\\0&\frac{3}{5}&-\frac{6}{5}&0\\0&0&0&0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}5&2&1\\0&\frac{3}{5}&-\frac{6}{5}\\0&0&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - 6R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -13 & -6 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -13 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -13 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{13}, v_2 = -\frac{6t}{13}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{13} \\ -\frac{6t}{13} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{13} \\ -\frac{6t}{13} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{13} \\ -\frac{6t}{13} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{13} \\ -\frac{6t}{13} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{13} \\ -\frac{6t}{13} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 13 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 6 & -4 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & -2 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & -3 & -\frac{9}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -t, v_2 = -\frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -\frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -\frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
3	1	1	No	$\begin{bmatrix} -1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$
-4	1	1	No	$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} -1 \\ -\frac{3}{2} \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Since eigenvalue -4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-4t} \\ &= \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} e^{-4t}\end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{0} \\ &= \begin{bmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{3t} \\ -\frac{3\,\mathrm{e}^{3t}}{2} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-4t} \\ 2\,\mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \end{bmatrix} + c_3 \begin{bmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\left(c_1\mathrm{e}^{7t} + \frac{c_3\mathrm{e}^{4t}}{13} + c_2\right)\mathrm{e}^{-4t} \\ -\frac{\left(39c_1\mathrm{e}^{7t} + 12c_3\mathrm{e}^{4t} - 52c_2\right)\mathrm{e}^{-4t}}{26} \\ \left(c_1\mathrm{e}^{7t} + c_3\mathrm{e}^{4t} + c_2\right)\mathrm{e}^{-4t} \end{bmatrix}$$

9.15.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + 2y + z(t)\,, y' = 6x(t) - y, z'(t) = -x(t) - 2y - z(t)]$

- Define vector

$$\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \left[\begin{array}{ccc} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

- System to solve

$$\overrightarrow{x}'(t) = \left[\begin{array}{ccc} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{array}\right] \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{array}\right]$$

- Rewrite the system as

$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \left[\begin{array}{c} -1 \\ 2 \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} -1 \\ -\frac{3}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-4, \left[\begin{array}{c} -1 \\ 2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-4t} \cdot \left[\begin{array}{c} -1 \\ 2 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{2} = \left[\begin{array}{c} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c} -1 \\ -\frac{3}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{3} = \mathrm{e}^{3t} \cdot \left[\begin{array}{c} -1 \\ -\frac{3}{2} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_{1}\overrightarrow{x}_{1} + c_{2}\overrightarrow{x}_{2} + c_{3}\overrightarrow{x}_{3}$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_{1}\mathrm{e}^{-4t} \cdot \left[\begin{array}{c} -1 \\ 2 \\ 1 \end{array}\right] + c_{3}\mathrm{e}^{3t} \cdot \left[\begin{array}{c} -1 \\ -\frac{3}{2} \\ 1 \end{array}\right] + \left[\begin{array}{c} -\frac{c_{2}}{13} \\ -\frac{6c_{2}}{13} \\ c_{2} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} -\left(\frac{c_{2}\mathrm{e}^{4t}}{13} + c_{3}\mathrm{e}^{7t} + c_{1}\right)\mathrm{e}^{-4t} \\ -\frac{\left(39c_{3}\mathrm{e}^{7t} + 12c_{2}\mathrm{e}^{4t} - 52c_{1}\right)\mathrm{e}^{-4t}}{26} \\ \left(c_{3}\mathrm{e}^{7t} + c_{2}\mathrm{e}^{4t} + c_{1}\right)\mathrm{e}^{-4t} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -\left(\frac{c_{2}\mathrm{e}^{4t}}{13} + c_{3}\mathrm{e}^{7t} + c_{1}\right)\mathrm{e}^{-4t}, y = -\frac{\left(39c_{3}\mathrm{e}^{7t} + 12c_{2}\mathrm{e}^{4t} - 52c_{1}\right)\mathrm{e}^{-4t}}{26}, z(t) = \left(c_{3}\mathrm{e}^{7t} + c_{2}\mathrm{e}^{4t} + c_{1}\right)\mathrm{e}^{-4t}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 61

```
dsolve([diff(x(t),t)=x(t)+2*y(t)+z(t),diff(y(t),t)=6*x(t)-y(t),diff(z(t),t)=-x(t)-2*y(t)-z(t
```

$$
\begin{aligned}
x(t) &= -\frac{c_2 \mathrm{e}^{-4t}}{2} + \frac{2c_3 \mathrm{e}^{3t}}{3} + \frac{c_1}{6} \\
y(t) &= c_1 + c_2 \mathrm{e}^{-4t} + c_3 \mathrm{e}^{3t} \\
z(t) &= \frac{c_2 \mathrm{e}^{-4t}}{2} - \frac{2c_3 \mathrm{e}^{3t}}{3} - \frac{13c_1}{6}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 190

```
DSolve[{x'[t]==x[t]+2*y[t]+z[t],y'[t]==6*x[t]-y[t],z'[t]==-x[t]-2*y[t]-z[t]},{x[t],y[t],z[t]
```

$$
\begin{aligned}
x(t) &\to \frac{1}{84} e^{-4t} \left(c_1 \left(-7e^{4t} + 64e^{7t} + 27\right) + 24c_2 \left(e^{7t} - 1\right) + c_3 \left(-7e^{4t} + 16e^{7t} - 9\right)\right) \\
y(t) &\to \frac{1}{14} e^{-4t} \left(c_1 \left(-7e^{4t} + 16e^{7t} - 9\right) + c_2 \left(6e^{7t} + 8\right) + c_3 \left(-7e^{4t} + 4e^{7t} + 3\right)\right) \\
z(t) &\to \frac{1}{84} e^{-4t} \left(c_1 \left(91e^{4t} - 64e^{7t} - 27\right) - 24c_2 \left(e^{7t} - 1\right) - c_3 \left(-91e^{4t} + 16e^{7t} - 9\right)\right)
\end{aligned}
$$

9.16 problem 16

Internal problem ID [6726]
Internal file name [`OUTPUT/5974_Sunday_June_05_2022_04_10_47_PM_56263967/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.1. Page 332
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + z(t) \\
y' &= x(t) + y \\
z'(t) &= -2x(t) - z(t)
\end{aligned}
$$

9.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \sin(t)+\cos(t) & 0 & \sin(t) \\ \mathrm{e}^{t}-\cos(t) & \mathrm{e}^{t} & \frac{\mathrm{e}^{t}}{2}-\frac{\cos(t)}{2}-\frac{\sin(t)}{2} \\ -2\sin(t) & 0 & \cos(t)-\sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \sin(t)+\cos(t) & 0 & \sin(t) \\ \mathrm{e}^{t}-\cos(t) & \mathrm{e}^{t} & \frac{\mathrm{e}^{t}}{2}-\frac{\cos(t)}{2}-\frac{\sin(t)}{2} \\ -2\sin(t) & 0 & \cos(t)-\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} (\sin(t)+\cos(t))\, c_1 + \sin(t)\, c_3 \\ (\mathrm{e}^{t}-\cos(t))\, c_1 + \mathrm{e}^{t} c_2 + \left(\frac{\mathrm{e}^{t}}{2}-\frac{\cos(t)}{2}-\frac{\sin(t)}{2}\right) c_3 \\ -2\sin(t)\, c_1 + (\cos(t)-\sin(t))\, c_3 \end{bmatrix} \\ &= \begin{bmatrix} (c_1+c_3)\sin(t) + c_1\cos(t) \\ \frac{(2c_1+2c_2+c_3)\mathrm{e}^{t}}{2} + \frac{(-2c_1-c_3)\cos(t)}{2} - \frac{\sin(t)c_3}{2} \\ (-2c_1-c_3)\sin(t) + c_3\cos(t) \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 0 \\ -2 & 0 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \\ \lambda_3 &= 1 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -2 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} - (-i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+i & 0 & 1 \\ 1 & 1+i & 0 \\ -2 & 0 & -1+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1+i & 0 & 1 & 0 \\ 1 & 1+i & 0 & 0 \\ -2 & 0 & -1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1+i & 0 & 1 & 0 \\ 0 & 1+i & -\frac{1}{2} + \frac{i}{2} & 0 \\ -2 & 0 & -1+i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1-i) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1+i & 0 & 1 & 0 \\ 0 & 1+i & -\frac{1}{2} + \frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & 0 & 1 \\ 0 & 1+i & -\frac{1}{2}+\frac{i}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2}+\frac{i}{2}\right)t, v_2 = -\frac{it}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ -\frac{I}{2}t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right)t \\ -\frac{it}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ -\frac{I}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ -\frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ -\frac{I}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ -\frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ -\frac{I}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ -i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = i$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & 0 & 1 \\ 1 & 1-i & 0 \\ -2 & 0 & -1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1-i & 0 & 1 & 0 \\ 1 & 1-i & 0 & 0 \\ -2 & 0 & -1-i & 0 \end{array} \right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} - \frac{i}{2} \right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1-i & 0 & 1 & 0 \\ 0 & 1-i & -\frac{1}{2} - \frac{i}{2} & 0 \\ -2 & 0 & -1-i & 0 \end{array} \right]$$

$$R_3 = R_3 + (1+i) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1-i & 0 & 1 & 0 \\ 0 & 1-i & -\frac{1}{2} - \frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & 0 & 1 \\ 0 & 1-i & -\frac{1}{2} - \frac{i}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \left(-\frac{1}{2} - \frac{i}{2} \right) t, v_2 = \frac{it}{2} \right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2} \right) t \\ \frac{I}{2} t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2} \right) t \\ \frac{it}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{I}{2}\right)t \\ \frac{I}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{I}{2}\right)t \\ \frac{I}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{I}{2}\right)t \\ \frac{I}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -1-i \\ i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
i	1	1	No	$\left[\begin{array}{c} -\frac{1}{2}-\frac{i}{2} \\ \frac{i}{2} \\ 1 \end{array}\right]$
$-i$	1	1	No	$\left[\begin{array}{c} -\frac{1}{2}+\frac{i}{2} \\ -\frac{i}{2} \\ 1 \end{array}\right]$
1	1	1	No	$\left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{t} \\ &= \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right] e^{t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = c_1 \left[\begin{array}{c} \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{it} \\ \frac{i\mathrm{e}^{it}}{2} \\ \mathrm{e}^{it} \end{array}\right] + c_2 \left[\begin{array}{c} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{-it} \\ -\frac{i\mathrm{e}^{-it}}{2} \\ \mathrm{e}^{-it} \end{array}\right] + c_3 \left[\begin{array}{c} 0 \\ \mathrm{e}^{t} \\ 0 \end{array}\right]$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{it} + \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{-it} \\ \frac{i c_1 \mathrm{e}^{it}}{2} - \frac{i c_2 \mathrm{e}^{-it}}{2} + c_3 \mathrm{e}^{t} \\ c_1 \mathrm{e}^{it} + c_2 \mathrm{e}^{-it} \end{bmatrix}$$

9.16.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + z(t)\,, y' = x(t) + y, z'(t) = -2x(t) - z(t)]$

- Define vector

$$\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

- Rewrite the system as

$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c}0 \\ 1 \\ 0\end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c}-\frac{1}{2}+\frac{\mathrm{I}}{2} \\ -\frac{\mathrm{I}}{2} \\ 1\end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c}-\frac{1}{2}-\frac{\mathrm{I}}{2} \\ \frac{\mathrm{I}}{2} \\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}0 \\ 1 \\ 0\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{t} \cdot \left[\begin{array}{c}0 \\ 1 \\ 0\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c}-\frac{1}{2}+\frac{\mathrm{I}}{2} \\ -\frac{\mathrm{I}}{2} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \left[\begin{array}{c}-\frac{1}{2}+\frac{\mathrm{I}}{2} \\ -\frac{\mathrm{I}}{2} \\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(t\right)-\mathrm{I}\sin\left(t\right)\right) \cdot \left[\begin{array}{c}-\frac{1}{2}+\frac{\mathrm{I}}{2} \\ -\frac{\mathrm{I}}{2} \\ 1\end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c}\left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)\left(\cos\left(t\right)-\mathrm{I}\sin\left(t\right)\right) \\ -\frac{\mathrm{I}}{2}\left(\cos\left(t\right)-\mathrm{I}\sin\left(t\right)\right) \\ \cos\left(t\right)-\mathrm{I}\sin\left(t\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = \begin{bmatrix} -\frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ -\frac{\sin(t)}{2} \\ \cos(t) \end{bmatrix}, \vec{x}_3(t) = \begin{bmatrix} \frac{\sin(t)}{2} + \frac{\cos(t)}{2} \\ -\frac{\cos(t)}{2} \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^t \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} c_3\left(\frac{\sin(t)}{2} + \frac{\cos(t)}{2}\right) + c_2\left(-\frac{\cos(t)}{2} + \frac{\sin(t)}{2}\right) \\ -\frac{c_2\sin(t)}{2} - \frac{c_3\cos(t)}{2} \\ -c_3\sin(t) + c_2\cos(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{(-c_2+c_3)\cos(t)}{2} + \frac{(c_2+c_3)\sin(t)}{2} \\ c_1 e^t - \frac{c_3\cos(t)}{2} - \frac{c_2\sin(t)}{2} \\ -c_3\sin(t) + c_2\cos(t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(-c_2+c_3)\cos(t)}{2} + \frac{(c_2+c_3)\sin(t)}{2}, y = c_1 e^t - \frac{c_3\cos(t)}{2} - \frac{c_2\sin(t)}{2}, z(t) = -c_3\sin(t) + c_2\cos(t)\right\}$$

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 65

```
dsolve([diff(x(t),t)=x(t)+z(t),diff(y(t),t)=x(t)+y(t),diff(z(t),t)=-2*x(t)-z(t)],singsol=all
```

$$x(t) = c_2\sin(t) + c_3\cos(t)$$
$$y(t) = -\frac{c_2\cos(t)}{2} - \frac{c_3\cos(t)}{2} - \frac{c_2\sin(t)}{2} + \frac{c_3\sin(t)}{2} + c_1 e^t$$
$$z(t) = -c_3\sin(t) + c_2\cos(t) - c_2\sin(t) - c_3\cos(t)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 76

```
DSolve[{x'[t]==x[t]+z[t],y'[t]==x[t]+y[t],z'[t]==-2*x[t]-z[t]},{x[t],y[t],z[t]},t,IncludeSin
```

$$
\begin{aligned}
x(t) &\to c_1\cos(t) + (c_1 + c_3)\sin(t)\\
y(t) &\to c_2e^t + c_1\left(e^t - \cos(t)\right) - \frac{1}{2}c_3\left(-e^t + \sin(t) + \cos(t)\right)\\
z(t) &\to c_3\cos(t) - (2c_1 + c_3)\sin(t)
\end{aligned}
$$

10 CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346

10.1 problem 1

Internal problem ID [6727]
Internal file name [`OUTPUT/5975_Sunday_June_05_2022_04_10_50_PM_40790798/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + 2y \\
y' &= 4x(t) + 3y
\end{aligned}
$$

10.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,\mathrm{e}^{-t}}{3} + \frac{\mathrm{e}^{5t}}{3} & \frac{\mathrm{e}^{5t}}{3} - \frac{\mathrm{e}^{-t}}{3} \\ \frac{2\,\mathrm{e}^{5t}}{3} - \frac{2\,\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{-t}}{3} + \frac{2\,\mathrm{e}^{5t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{5t}}{3} & \frac{\mathrm{e}^{5t}}{3}-\frac{\mathrm{e}^{-t}}{3} \\ \frac{2\,\mathrm{e}^{5t}}{3}-\frac{2\,\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{-t}}{3}+\frac{2\,\mathrm{e}^{5t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{5t}}{3}\right)c_1+\left(\frac{\mathrm{e}^{5t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_2 \\ \left(\frac{2\,\mathrm{e}^{5t}}{3}-\frac{2\,\mathrm{e}^{-t}}{3}\right)c_1+\left(\frac{\mathrm{e}^{-t}}{3}+\frac{2\,\mathrm{e}^{5t}}{3}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1-c_2)\mathrm{e}^{-t}}{3}+\frac{(c_1+c_2)\mathrm{e}^{5t}}{3} \\ \frac{(-2c_1+c_2)\mathrm{e}^{-t}}{3}+\frac{2(c_1+c_2)\mathrm{e}^{5t}}{3} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{5t}\\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{5t}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t}\\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{\mathrm{e}^{5t}}{2} \\ \mathrm{e}^{5t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1 \mathrm{e}^{5t}}{2} - c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{5t} + c_2 \mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

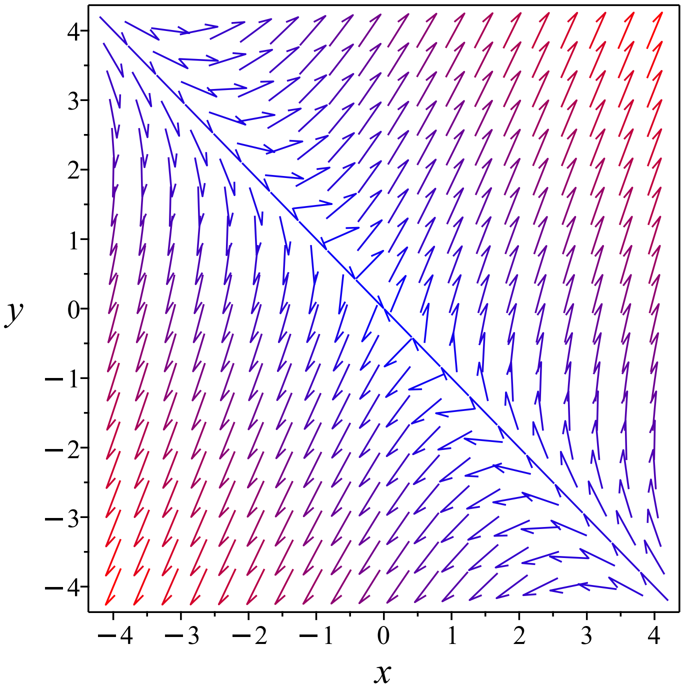


Figure 86: Phase plot

10.1.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + 2y, y' = 4x(t) + 3y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[5, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{5t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = \mathrm{e}^{-t} c_1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{5t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\mathrm{e}^{-t}c_1 + \frac{c_2\mathrm{e}^{5t}}{2} \\ \mathrm{e}^{-t}c_1 + c_2\mathrm{e}^{5t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x(t) = -\mathrm{e}^{-t}c_1 + \frac{c_2\mathrm{e}^{5t}}{2}, y = \mathrm{e}^{-t}c_1 + c_2\mathrm{e}^{5t}\right\}$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=x(t)+2*y(t),diff(y(t),t)=4*x(t)+3*y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{-t}c_1 + c_2\mathrm{e}^{5t}$$
$$y(t) = -\mathrm{e}^{-t}c_1 + 2c_2\mathrm{e}^{5t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 71

```
DSolve[{x'[t]==x[t]+2*y[t],y'[t]==4*x[t]+3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to \frac{1}{3}e^{-t}\left(c_1\left(e^{6t} + 2\right) + c_2\left(e^{6t} - 1\right)\right)$$
$$y(t) \to \frac{1}{3}e^{-t}\left(2c_1\left(e^{6t} - 1\right) + c_2\left(2e^{6t} + 1\right)\right)$$

10.2 problem 2

Internal problem ID [6728]
Internal file name [`OUTPUT/5976_Sunday_June_05_2022_04_10_52_PM_7401589/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 2x(t) + 2y \\ y' &= x(t) + 3y \end{aligned}$$

10.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,e^{t}}{3} + \frac{e^{4t}}{3} & \frac{2\,e^{4t}}{3} - \frac{2\,e^{t}}{3} \\ \frac{e^{4t}}{3} - \frac{e^{t}}{3} & \frac{e^{t}}{3} + \frac{2\,e^{4t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{t}}{3}+\frac{\mathrm{e}^{4t}}{3} & \frac{2\,\mathrm{e}^{4t}}{3}-\frac{2\,\mathrm{e}^{t}}{3} \\ \frac{\mathrm{e}^{4t}}{3}-\frac{\mathrm{e}^{t}}{3} & \frac{\mathrm{e}^{t}}{3}+\frac{2\,\mathrm{e}^{4t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,\mathrm{e}^{t}}{3}+\frac{\mathrm{e}^{4t}}{3}\right)c_1+\left(\frac{2\,\mathrm{e}^{4t}}{3}-\frac{2\,\mathrm{e}^{t}}{3}\right)c_2 \\ \left(\frac{\mathrm{e}^{4t}}{3}-\frac{\mathrm{e}^{t}}{3}\right)c_1+\left(\frac{\mathrm{e}^{t}}{3}+\frac{2\,\mathrm{e}^{4t}}{3}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1+2c_2)\mathrm{e}^{4t}}{3}+\frac{2\,\mathrm{e}^{t}(c_1-c_2)}{3} \\ \frac{(c_1+2c_2)\mathrm{e}^{4t}}{3}-\frac{\mathrm{e}^{t}(c_1-c_2)}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^2 - 5\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= 4\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{aligned}\left(\begin{bmatrix}2 & 2\\ 1 & 3\end{bmatrix} - (1)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}1 & 2 & 0\\ 1 & 2 & 0\end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c}1 & 2 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{t} \\ &= \left[\begin{array}{c} -2 \\ 1 \end{array}\right] e^{t}\end{aligned}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{4t} \\ &= \left[\begin{array}{c} 1 \\ 1 \end{array}\right] e^{4t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} -2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{array}\right] + c_2 \left[\begin{array}{c} \mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} -2c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{4t} \\ c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{4t} \end{array}\right]$$

The following is the phase plot of the system.

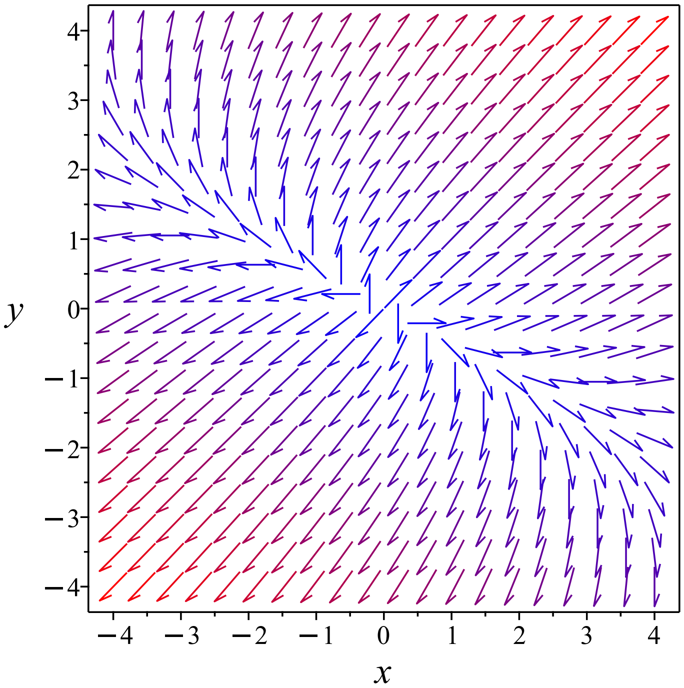


Figure 87: Phase plot

10.2.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) + 2y, y' = x(t) + 3y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{4t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_1 \mathrm{e}^{t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{4t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} -2c_1\mathrm{e}^t + c_2\mathrm{e}^{4t} \\ c_1\mathrm{e}^t + c_2\mathrm{e}^{4t} \end{array}\right]$$

- Solution to the system of ODEs

  $\{x(t) = -2c_1\mathrm{e}^t + c_2\mathrm{e}^{4t}, y = c_1\mathrm{e}^t + c_2\mathrm{e}^{4t}\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve([diff(x(t),t)=2*x(t)+2*y(t),diff(y(t),t)=x(t)+3*y(t)],singsol=all)
```

$$x(t) = c_1\mathrm{e}^t + c_2\mathrm{e}^{4t}$$
$$y(t) = -\frac{c_1\mathrm{e}^t}{2} + c_2\mathrm{e}^{4t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 67

```
DSolve[{x'[t]==2*x[t]+2*y[t],y'[t]==x[t]+3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to \frac{1}{3}e^t\left(c_1\left(e^{3t} + 2\right) + 2c_2\left(e^{3t} - 1\right)\right)$$
$$y(t) \to \frac{1}{3}e^t\left(c_1\left(e^{3t} - 1\right) + c_2\left(2e^{3t} + 1\right)\right)$$

10.3 problem 3

Internal problem ID [6729]
Internal file name [`OUTPUT/5977_Sunday_June_05_2022_04_10_53_PM_64986593/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -4x(t) + 2y \\
y' &= -\frac{5x(t)}{2} + 2y
\end{aligned}
$$

10.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{(\mathrm{e}^{4t}-5)\mathrm{e}^{-3t}}{4} & \frac{(\mathrm{e}^{4t}-1)\mathrm{e}^{-3t}}{2} \\ -\frac{5(\mathrm{e}^{4t}-1)\mathrm{e}^{-3t}}{8} & \frac{(5\,\mathrm{e}^{4t}-1)\mathrm{e}^{-3t}}{4} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{(\mathrm{e}^{4t}-5)\,\mathrm{e}^{-3t}}{4} & \frac{(\mathrm{e}^{4t}-1)\,\mathrm{e}^{-3t}}{2} \\ -\frac{5(\mathrm{e}^{4t}-1)\,\mathrm{e}^{-3t}}{8} & \frac{(5\,\mathrm{e}^{4t}-1)\,\mathrm{e}^{-3t}}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{(\mathrm{e}^{4t}-5)\,\mathrm{e}^{-3t}c_1}{4} + \frac{(\mathrm{e}^{4t}-1)\,\mathrm{e}^{-3t}c_2}{2} \\ -\frac{5(\mathrm{e}^{4t}-1)\,\mathrm{e}^{-3t}c_1}{8} + \frac{(5\,\mathrm{e}^{4t}-1)\,\mathrm{e}^{-3t}c_2}{4} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{((c_1-2c_2)\mathrm{e}^{4t}-5c_1+2c_2)\,\mathrm{e}^{-3t}}{4} \\ -\frac{5\,\mathrm{e}^{-3t}\left((c_1-2c_2)\mathrm{e}^{4t}-c_1+\frac{2c_2}{5}\right)}{8} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -4-\lambda & 2 \\ -\frac{5}{2} & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 2 \\ -\frac{5}{2} & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -\frac{5}{2} & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{5R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 \\ -\frac{5}{2} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -5 & 2 & 0 \\ -\frac{5}{2} & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix} e^t\end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-3t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{2\,\mathrm{e}^t}{5} \\ \mathrm{e}^t \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{2(c_1 \mathrm{e}^{4t} + 5c_2)\mathrm{e}^{-3t}}{5} \\ (c_1 \mathrm{e}^{4t} + c_2)\,\mathrm{e}^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

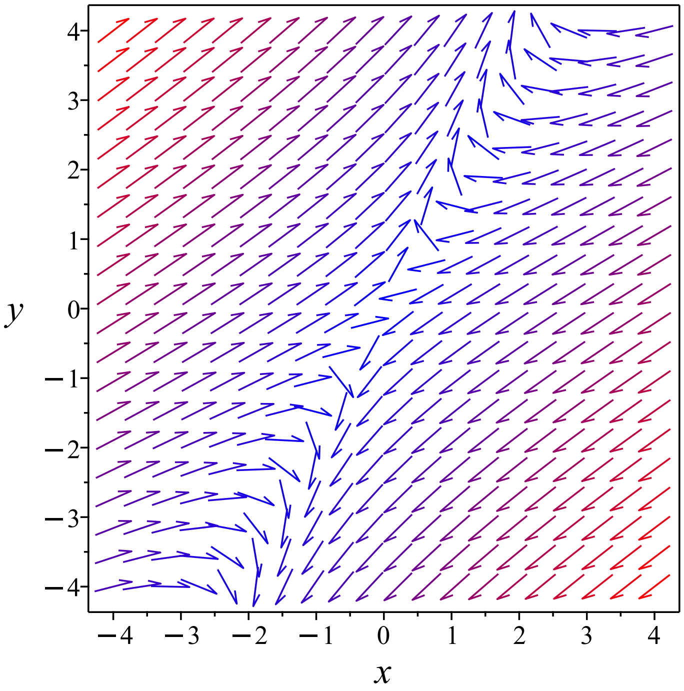


Figure 88: Phase plot

10.3.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -4x(t) + 2y, y' = -\frac{5x(t)}{2} + 2y\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-3, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-3, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_1 \mathrm{e}^{-3t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{t} \cdot \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{2(c_2 \mathrm{e}^{4t}+5c_1)\mathrm{e}^{-3t}}{5} \\ (c_2 \mathrm{e}^{4t}+c_1)\,\mathrm{e}^{-3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x(t) = \frac{2(c_2 \mathrm{e}^{4t}+5c_1)\mathrm{e}^{-3t}}{5}, y = (c_2 \mathrm{e}^{4t}+c_1)\,\mathrm{e}^{-3t}\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve([diff(x(t),t)=-4*x(t)+2*y(t),diff(y(t),t)=-5/2*x(t)+2*y(t)],singsol=all)
```

$$\begin{aligned} x(t) &= c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-3t} \\ y(t) &= \frac{5c_1 \mathrm{e}^{t}}{2} + \frac{c_2 \mathrm{e}^{-3t}}{2} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 149

```
DSolve[{x'[t]==-4*x[t]+2*y[t],y'[t]==5/2*x[t]+2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$\begin{aligned} x(t) &\to \frac{1}{28} e^{-\left(\left(1+\sqrt{14}\right)t\right)} \left(c_1\left(\left(14-3\sqrt{14}\right)e^{2\sqrt{14}t}+14+3\sqrt{14}\right)+2\sqrt{14}c_2\left(e^{2\sqrt{14}t}-1\right)\right) \\ y(t) &\to \frac{1}{56} e^{-\left(\left(1+\sqrt{14}\right)t\right)} \left(5\sqrt{14}c_1\left(e^{2\sqrt{14}t}-1\right)+2c_2\left(\left(14+3\sqrt{14}\right)e^{2\sqrt{14}t}+14-3\sqrt{14}\right)\right) \end{aligned}$$

10.4 problem 4

Internal problem ID [6730]
Internal file name [`OUTPUT/5978_Sunday_June_05_2022_04_10_55_PM_89181049/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -\frac{5x(t)}{2} + 2y \\
y' &= \frac{3x(t)}{4} - 2y
\end{aligned}
$$

10.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{3\,\mathrm{e}^{-\frac{7t}{2}}}{5} + \frac{2\,\mathrm{e}^{-t}}{5} & \frac{4\,\mathrm{e}^{-t}}{5} - \frac{4\,\mathrm{e}^{-\frac{7t}{2}}}{5} \\ \frac{3\,\mathrm{e}^{-t}}{10} - \frac{3\,\mathrm{e}^{-\frac{7t}{2}}}{10} & \frac{2\,\mathrm{e}^{-\frac{7t}{2}}}{5} + \frac{3\,\mathrm{e}^{-t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{3\,\mathrm{e}^{-\frac{7t}{2}}}{5}+\frac{2\,\mathrm{e}^{-t}}{5} & \frac{4\,\mathrm{e}^{-t}}{5}-\frac{4\,\mathrm{e}^{-\frac{7t}{2}}}{5} \\ \frac{3\,\mathrm{e}^{-t}}{10}-\frac{3\,\mathrm{e}^{-\frac{7t}{2}}}{10} & \frac{2\,\mathrm{e}^{-\frac{7t}{2}}}{5}+\frac{3\,\mathrm{e}^{-t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{3\,\mathrm{e}^{-\frac{7t}{2}}}{5}+\frac{2\,\mathrm{e}^{-t}}{5}\right)c_1+\left(\frac{4\,\mathrm{e}^{-t}}{5}-\frac{4\,\mathrm{e}^{-\frac{7t}{2}}}{5}\right)c_2 \\ \left(\frac{3\,\mathrm{e}^{-t}}{10}-\frac{3\,\mathrm{e}^{-\frac{7t}{2}}}{10}\right)c_1+\left(\frac{2\,\mathrm{e}^{-\frac{7t}{2}}}{5}+\frac{3\,\mathrm{e}^{-t}}{5}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(3c_1-4c_2)\mathrm{e}^{-\frac{7t}{2}}}{5}+\frac{2\,\mathrm{e}^{-t}(c_1+2c_2)}{5} \\ \frac{(-3c_1+4c_2)\mathrm{e}^{-\frac{7t}{2}}}{10}+\frac{3\,\mathrm{e}^{-t}(c_1+2c_2)}{10} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\frac{5}{2}-\lambda & 2 \\ \frac{3}{4} & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2+\frac{9}{2}\lambda+\frac{7}{2}=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-\frac{7}{2}$$
$$\lambda_2=-1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
$-\frac{7}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-1$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-\frac{5}{2} & 2\\ \frac{3}{4} & -2\end{bmatrix}-(-1)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}-\frac{3}{2} & 2\\ \frac{3}{4} & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}-\frac{3}{2} & 2 & 0\\ \frac{3}{4} & -1 & 0\end{array}\right]$$

$$R_2=R_2+\frac{R_1}{2}\Longrightarrow\left[\begin{array}{cc|c}-\frac{3}{2} & 2 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}-\frac{3}{2} & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix}\frac{4t}{3}\\ t\end{bmatrix} = \begin{bmatrix}\frac{4t}{3}\\ t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}\frac{4t}{3}\\ t\end{bmatrix} = t\begin{bmatrix}\frac{4}{3}\\ 1\end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix}\frac{4t}{3}\\ t\end{bmatrix} = \begin{bmatrix}\frac{4}{3}\\ 1\end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix}\frac{4t}{3}\\ t\end{bmatrix} = \begin{bmatrix}4\\ 3\end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{7}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-\frac{5}{2} & 2\\ \frac{3}{4} & -2\end{bmatrix} - \left(-\frac{7}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\ \frac{3}{4} & \frac{3}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}1 & 2 & 0\\ \frac{3}{4} & \frac{3}{2} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{cc|c}1 & 2 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{7}{2}$	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{7}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-\frac{7t}{2}} \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-\frac{7t}{2}}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -2\,\mathrm{e}^{-\frac{7t}{2}} \\ \mathrm{e}^{-\frac{7t}{2}} \end{bmatrix} + c_2 \begin{bmatrix} \frac{4\,\mathrm{e}^{-t}}{3} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -2c_1\mathrm{e}^{-\frac{7t}{2}} + \frac{4c_2\mathrm{e}^{-t}}{3} \\ c_1\mathrm{e}^{-\frac{7t}{2}} + c_2\mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

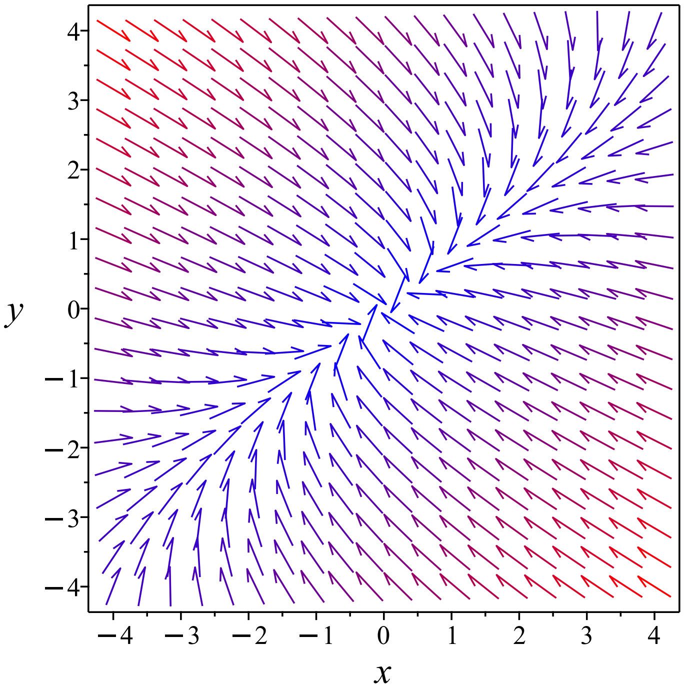


Figure 89: Phase plot

10.4.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -\frac{5x(t)}{2} + 2y, y' = \frac{3x(t)}{4} - 2y\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{array}\right]$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

  $$\left[\left[-\frac{7}{2}, \left[\begin{array}{c} -2 \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} \frac{4}{3} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

  $$\left[-\frac{7}{2}, \left[\begin{array}{c} -2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{-\frac{7t}{2}} \cdot \left[\begin{array}{c} -2 \\ 1 \end{array}\right]$$

- Consider eigenpair

  $$\left[-1, \left[\begin{array}{c} \frac{4}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} \frac{4}{3} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_1 \mathrm{e}^{-\frac{7t}{2}} \cdot \left[\begin{array}{c} -2 \\ 1 \end{array}\right] + c_2 \mathrm{e}^{-t} \cdot \left[\begin{array}{c} \frac{4}{3} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} -2c_1\mathrm{e}^{-\frac{7t}{2}} + \frac{4c_2\mathrm{e}^{-t}}{3} \\ c_1\mathrm{e}^{-\frac{7t}{2}} + c_2\mathrm{e}^{-t} \end{array}\right]$$

- Solution to the system of ODEs

  $\left\{x(t) = -2c_1\mathrm{e}^{-\frac{7t}{2}} + \frac{4c_2\mathrm{e}^{-t}}{3}, y = c_1\mathrm{e}^{-\frac{7t}{2}} + c_2\mathrm{e}^{-t}\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=-5/2*x(t)+2*y(t),diff(y(t),t)=3/4*x(t)-2*y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{-\frac{7t}{2}}c_1 + c_2\mathrm{e}^{-t}$$
$$y(t) = -\frac{\mathrm{e}^{-\frac{7t}{2}}c_1}{2} + \frac{3c_2\mathrm{e}^{-t}}{4}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 165

```
DSolve[{x'[t]==5/2*x[t]+2*y[t],y'[t]==3/4*x[t]-2*y[t]},{x[t],y[t]},t,IncludeSingularSolution
```

$$x(t) \to \frac{1}{210}e^{\frac{1}{4}\left(t-\sqrt{105}t\right)}\left(3c_1\left(\left(35+3\sqrt{105}\right)e^{\frac{\sqrt{105}t}{2}}+35-3\sqrt{105}\right)+8\sqrt{105}c_2\left(e^{\frac{\sqrt{105}t}{2}}-1\right)\right)$$
$$y(t) \to \frac{1}{70}e^{\frac{1}{4}\left(t-\sqrt{105}t\right)}\left(\sqrt{105}c_1\left(e^{\frac{\sqrt{105}t}{2}}-1\right)-c_2\left(\left(3\sqrt{105}-35\right)e^{\frac{\sqrt{105}t}{2}}-35-3\sqrt{105}\right)\right)$$

10.5 problem 5

Internal problem ID [6731]
Internal file name [`OUTPUT/5979_Sunday_June_05_2022_04_10_57_PM_66056020/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 10x(t) - 5y \\ y' &= 8x(t) - 12y \end{aligned}$$

10.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(10\,\mathrm{e}^{18t}-1)\,\mathrm{e}^{-10t}}{9} & -\frac{5(\mathrm{e}^{18t}-1)\,\mathrm{e}^{-10t}}{18} \\ \frac{4(\mathrm{e}^{18t}-1)\,\mathrm{e}^{-10t}}{9} & -\frac{(\mathrm{e}^{18t}-10)\,\mathrm{e}^{-10t}}{9} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(10\,\mathrm{e}^{18t}-1)\mathrm{e}^{-10t}}{9} & -\frac{5(\mathrm{e}^{18t}-1)\mathrm{e}^{-10t}}{18} \\ \frac{4(\mathrm{e}^{18t}-1)\mathrm{e}^{-10t}}{9} & -\frac{(\mathrm{e}^{18t}-10)\mathrm{e}^{-10t}}{9} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(10\,\mathrm{e}^{18t}-1)\mathrm{e}^{-10t}c_1}{9} - \frac{5(\mathrm{e}^{18t}-1)\mathrm{e}^{-10t}c_2}{18} \\ \frac{4(\mathrm{e}^{18t}-1)\mathrm{e}^{-10t}c_1}{9} - \frac{(\mathrm{e}^{18t}-10)\mathrm{e}^{-10t}c_2}{9} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-10t}(5(4c_1-c_2)\mathrm{e}^{18t}-2c_1+5c_2)}{18} \\ \frac{4\,\mathrm{e}^{-10t}\left(\left(c_1-\frac{c_2}{4}\right)\mathrm{e}^{18t}-c_1+\frac{5c_2}{2}\right)}{9} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 10-\lambda & -5 \\ 8 & -12-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda - 80 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= 8\\ \lambda_2 &= -10\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-10	1	real eigenvalue
8	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -10$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} - (-10)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 20 & -5 \\ 8 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 20 & -5 & 0 \\ 8 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 20 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 20 & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 8$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} - (8) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -5 \\ 8 & -20 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -5 & 0 \\ 8 & -20 & 0 \end{array}\right]$$

$$R_2 = R_2 - 4R_1 \implies \left[\begin{array}{cc|c} 2 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
8	1	1	No	$\begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$
-10	1	1	No	$\begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 8 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{8t} \\ &= \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix} e^{8t} \end{aligned}$$

Since eigenvalue -10 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{-10t} \\ &= \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} e^{-10t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{5\,\mathrm{e}^{8t}}{2} \\ \mathrm{e}^{8t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{-10t}}{4} \\ \mathrm{e}^{-10t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{(10 c_1 \mathrm{e}^{18t} + c_2)\mathrm{e}^{-10t}}{4} \\ (c_1 \mathrm{e}^{18t} + c_2)\,\mathrm{e}^{-10t} \end{bmatrix}$$

The following is the phase plot of the system.

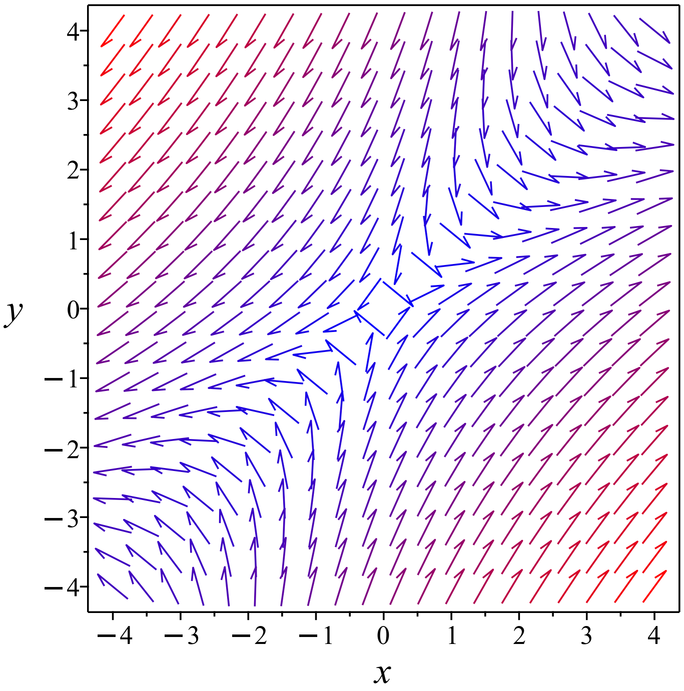


Figure 90: Phase plot

10.5.3 Maple step by step solution

Let's solve

$[x'(t) = 10x(t) - 5y, y' = 8x(t) - 12y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 10 & -5 \\ 8 & -12 \end{array}\right]$$

- Rewrite the system as

  $$\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-10, \left[\begin{array}{c} \frac{1}{4} \\ 1 \end{array}\right]\right], \left[8, \left[\begin{array}{c} \frac{5}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

  $$\left[-10, \left[\begin{array}{c} \frac{1}{4} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_{1} = \mathrm{e}^{-10t} \cdot \left[\begin{array}{c} \frac{1}{4} \\ 1 \end{array}\right]$$

- Consider eigenpair

  $$\left[8, \left[\begin{array}{c} \frac{5}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_{2} = \mathrm{e}^{8t} \cdot \left[\begin{array}{c} \frac{5}{2} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

  $$\overrightarrow{x} = c_{1}\overrightarrow{x}_{1} + c_{2}\overrightarrow{x}_{2}$$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_{1}\mathrm{e}^{-10t} \cdot \left[\begin{array}{c} \frac{1}{4} \\ 1 \end{array}\right] + c_{2}\mathrm{e}^{8t} \cdot \left[\begin{array}{c} \frac{5}{2} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{(10c_2 e^{18t}+c_1)e^{-10t}}{4} \\ (c_2 e^{18t} + c_1)e^{-10t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x(t) = \frac{(10c_2 e^{18t}+c_1)e^{-10t}}{4}, y = (c_2 e^{18t} + c_1)e^{-10t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=10*x(t)-5*y(t),diff(y(t),t)=8*x(t)-12*y(t)],singsol=all)
```

$$x(t) = c_1 e^{8t} + c_2 e^{-10t}$$
$$y(t) = \frac{2c_1 e^{8t}}{5} + 4c_2 e^{-10t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 73

```
DSolve[{x'[t]==10*x[t]-5*y[t],y'[t]==8*x[t]-12*y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{18}e^{-10t}\left(c_1\left(20e^{18t}-2\right) - 5c_2\left(e^{18t}-1\right)\right)$$
$$y(t) \to \frac{1}{9}e^{-10t}\left(4c_1\left(e^{18t}-1\right) - c_2\left(e^{18t}-10\right)\right)$$

10.6 problem 6

Internal problem ID [6732]
Internal file name [`OUTPUT/5980_Sunday_June_05_2022_04_10_59_PM_54989578/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -6x(t) + 2y$$
$$y' = -3x(t) + y$$

10.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{6\,\mathrm{e}^{-5t}}{5} - \frac{1}{5} & \frac{2}{5} - \frac{2\,\mathrm{e}^{-5t}}{5} \\ -\frac{3}{5} + \frac{3\,\mathrm{e}^{-5t}}{5} & -\frac{\mathrm{e}^{-5t}}{5} + \frac{6}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{6\,\mathrm{e}^{-5t}}{5}-\frac{1}{5} & \frac{2}{5}-\frac{2\,\mathrm{e}^{-5t}}{5} \\ -\frac{3}{5}+\frac{3\,\mathrm{e}^{-5t}}{5} & -\frac{\mathrm{e}^{-5t}}{5}+\frac{6}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{6\,\mathrm{e}^{-5t}}{5}-\frac{1}{5}\right)c_1+\left(\frac{2}{5}-\frac{2\,\mathrm{e}^{-5t}}{5}\right)c_2 \\ \left(-\frac{3}{5}+\frac{3\,\mathrm{e}^{-5t}}{5}\right)c_1+\left(-\frac{\mathrm{e}^{-5t}}{5}+\frac{6}{5}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(6c_1-2c_2)\mathrm{e}^{-5t}}{5}-\frac{c_1}{5}+\frac{2c_2}{5} \\ \frac{(3c_1-c_2)\mathrm{e}^{-5t}}{5}-\frac{3c_1}{5}+\frac{6c_2}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -6-\lambda & 2 \\ -3 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 5\lambda = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -5\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{aligned}\left(\begin{bmatrix}-6 & 2\\ -3 & 1\end{bmatrix} - (-5)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}-1 & 2\\ -3 & 6\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}-1 & 2 & 0\\ -3 & 6 & 0\end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c}-1 & 2 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}-1 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \end{array} \right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -6 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{0}\\ &= \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-5t}\\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1}{3} + 2c_2 \mathrm{e}^{-5t} \\ c_1 + c_2 \mathrm{e}^{-5t} \end{bmatrix}$$

The following is the phase plot of the system.

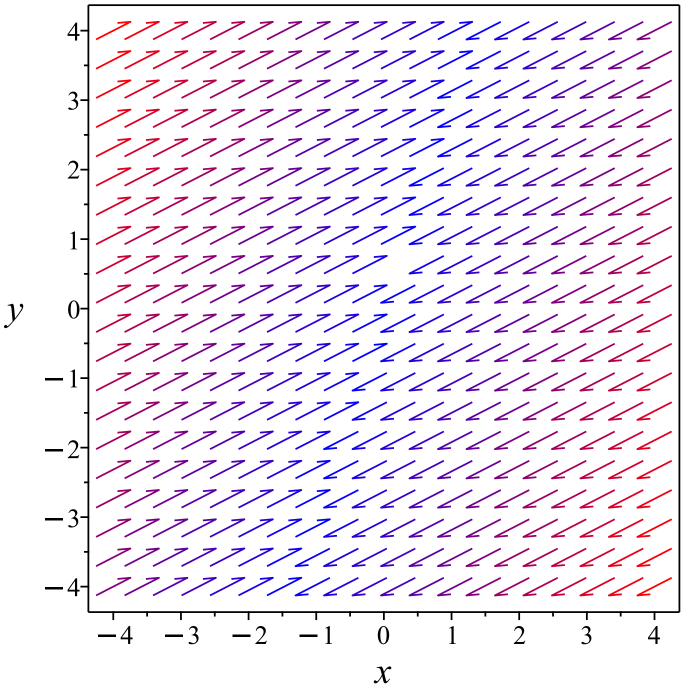


Figure 91: Phase plot

10.6.3 Maple step by step solution

Let's solve

$[x'(t) = -6x(t) + 2y, y' = -3x(t) + y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix}$$

- Rewrite the system as

  $$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

  $$\left[\left[-5, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-5, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{-5t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[0, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_1 \mathrm{e}^{-5t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{c_2}{3} \\ c_2 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 2c_1 \mathrm{e}^{-5t} + \frac{c_2}{3} \\ c_1 \mathrm{e}^{-5t} + c_2 \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x(t) = 2c_1 \mathrm{e}^{-5t} + \frac{c_2}{3}, y = c_1 \mathrm{e}^{-5t} + c_2\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve([diff(x(t),t)=-6*x(t)+2*y(t),diff(y(t),t)=-3*x(t)+y(t)],singsol=all)
```

$$x(t) = c_1 + c_2 \mathrm{e}^{-5t}$$
$$y(t) = \frac{c_2 \mathrm{e}^{-5t}}{2} + 3c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 73

```
DSolve[{x'[t]==-6*x[t]+2*y[t],y'[t]==-3*x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{5} e^{-5t} \left(2c_2 \left(e^{5t} - 1\right) - c_1 \left(e^{5t} - 6\right)\right)$$
$$y(t) \to \frac{1}{5} e^{-5t} \left(c_2 \left(6e^{5t} - 1\right) - 3c_1 \left(e^{5t} - 1\right)\right)$$

10.7 problem 7

Internal problem ID [6733]
Internal file name [`OUTPUT/5981_Sunday_June_05_2022_04_11_01_PM_7976205/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + y - z(t) \\
y' &= 2y \\
z'(t) &= y - z(t)
\end{aligned}
$$

10.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & \frac{2 e^{2t}}{3}-\frac{e^{t}}{2}-\frac{e^{-t}}{6} & -\frac{e^{t}}{2}+\frac{e^{-t}}{2} \\ 0 & e^{2t} & 0 \\ 0 & -\frac{e^{-t}}{3}+\frac{e^{2t}}{3} & e^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{t} & \frac{2 e^{2t}}{3}-\frac{e^{t}}{2}-\frac{e^{-t}}{6} & -\frac{e^{t}}{2}+\frac{e^{-t}}{2} \\ 0 & e^{2t} & 0 \\ 0 & -\frac{e^{-t}}{3}+\frac{e^{2t}}{3} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{t}c_1+\left(\frac{2 e^{2t}}{3}-\frac{e^{t}}{2}-\frac{e^{-t}}{6}\right)c_2+\left(-\frac{e^{t}}{2}+\frac{e^{-t}}{2}\right)c_3 \\ e^{2t}c_2 \\ \left(-\frac{e^{-t}}{3}+\frac{e^{2t}}{3}\right)c_2+e^{-t}c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(-c_2+3c_3)e^{-t}}{6}+\frac{2 e^{2t}c_2}{3}+\left(c_1-\frac{c_2}{2}-\frac{c_3}{2}\right)e^{t} \\ e^{2t}c_2 \\ \frac{(-c_2+3c_3)e^{-t}}{3}+\frac{e^{2t}c_2}{3} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^t \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t\end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\ &= \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{\mathrm{e}^{-t}}{2} \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2\,\mathrm{e}^{2t} \\ 3\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 \mathrm{e}^{-t}}{2} + c_2 \mathrm{e}^{t} + 2 c_3 \mathrm{e}^{2t} \\ 3 c_3 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^{-t} + c_3 \mathrm{e}^{2t} \end{bmatrix}$$

10.7.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + y - z(t), y' = 2y, z'(t) = y - z(t)]$

- Define vector

$$\vec{x}(t) = \left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right]$$

- Convert system into a vector equation

$$\vec{x}'(t) = \left[\begin{array}{ccc} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{ccc} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{array}\right] \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} \frac{1}{2} \\ 0 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right], \left[2, \left[\begin{array}{c} 2 \\ 3 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} \frac{1}{2} \\ 0 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} \frac{1}{2} \\ 0 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{t} \cdot \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} 2 \\ 3 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_3 = \mathrm{e}^{2t} \cdot \left[\begin{array}{c} 2 \\ 3 \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2 + c_3 \overrightarrow{x}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = \mathrm{e}^{-t} c_1 \cdot \left[\begin{array}{c} \frac{1}{2} \\ 0 \\ 1 \end{array}\right] + c_2 \mathrm{e}^{t} \cdot \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] + c_3 \mathrm{e}^{2t} \cdot \left[\begin{array}{c} 2 \\ 3 \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{e}^{-t}c_1}{2} + c_2\mathrm{e}^t + 2c_3\mathrm{e}^{2t} \\ 3c_3\mathrm{e}^{2t} \\ \mathrm{e}^{-t}c_1 + c_3\mathrm{e}^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x(t) = \frac{\mathrm{e}^{-t}c_1}{2} + c_2\mathrm{e}^t + 2c_3\mathrm{e}^{2t}, y = 3c_3\mathrm{e}^{2t}, z(t) = \mathrm{e}^{-t}c_1 + c_3\mathrm{e}^{2t}\right\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 50

```
dsolve([diff(x(t),t)=x(t)+y(t)-z(t),diff(y(t),t)=2*y(t),diff(z(t),t)=y(t)-z(t)],singsol=all)
```

$$\begin{aligned} x(t) &= \frac{2c_3\mathrm{e}^{2t}}{3} + c_1\mathrm{e}^t + \frac{c_2\mathrm{e}^{-t}}{2} \\ y(t) &= c_3\mathrm{e}^{2t} \\ z(t) &= \frac{c_3\mathrm{e}^{2t}}{3} + c_2\mathrm{e}^{-t} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 88

```
DSolve[{x'[t]==x[t]+y[t]-z[t],y'[t]==2*y[t],z'[t]==y[t]-z[t]},{x[t],y[t],z[t]},t,IncludeSing
```

$$\begin{aligned} x(t) &\to \frac{1}{6}e^{-t}\left(4c_2e^{3t} + (6c_1 - 3(c_2 + c_3))e^{2t} - c_2 + 3c_3\right) \\ y(t) &\to c_2e^{2t} \\ z(t) &\to \frac{1}{3}e^{-t}\left(c_2\left(e^{3t} - 1\right) + 3c_3\right) \end{aligned}$$

10.8 problem 8

Internal problem ID [6734]
Internal file name [`OUTPUT/5982_Sunday_June_05_2022_04_11_03_PM_56388582/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) - 7y \\
y' &= 5x(t) + 10y + 4z(t) \\
z'(t) &= 5y + 2z(t)
\end{aligned}
$$

10.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{4\,\mathrm{e}^{2t}}{3}-\frac{7\,\mathrm{e}^{7t}}{2}+\frac{35\,\mathrm{e}^{5t}}{6} & -\frac{7\,\mathrm{e}^{7t}}{2}+\frac{7\,\mathrm{e}^{5t}}{2} & -\frac{14\,\mathrm{e}^{7t}}{5}+\frac{14\,\mathrm{e}^{5t}}{3}-\frac{28\,\mathrm{e}^{2t}}{15} \\ -\frac{5\,\mathrm{e}^{5t}}{2}+\frac{5\,\mathrm{e}^{7t}}{2} & \frac{5\,\mathrm{e}^{7t}}{2}-\frac{3\,\mathrm{e}^{5t}}{2} & 2\,\mathrm{e}^{7t}-2\,\mathrm{e}^{5t} \\ \frac{5\,\mathrm{e}^{7t}}{2}-\frac{25\,\mathrm{e}^{5t}}{6}+\frac{5\,\mathrm{e}^{2t}}{3} & -\frac{5\,\mathrm{e}^{5t}}{2}+\frac{5\,\mathrm{e}^{7t}}{2} & \frac{7\,\mathrm{e}^{2t}}{3}+2\,\mathrm{e}^{7t}-\frac{10\,\mathrm{e}^{5t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{4\,\mathrm{e}^{2t}}{3}-\frac{7\,\mathrm{e}^{7t}}{2}+\frac{35\,\mathrm{e}^{5t}}{6} & -\frac{7\,\mathrm{e}^{7t}}{2}+\frac{7\,\mathrm{e}^{5t}}{2} & -\frac{14\,\mathrm{e}^{7t}}{5}+\frac{14\,\mathrm{e}^{5t}}{3}-\frac{28\,\mathrm{e}^{2t}}{15} \\ -\frac{5\,\mathrm{e}^{5t}}{2}+\frac{5\,\mathrm{e}^{7t}}{2} & \frac{5\,\mathrm{e}^{7t}}{2}-\frac{3\,\mathrm{e}^{5t}}{2} & 2\,\mathrm{e}^{7t}-2\,\mathrm{e}^{5t} \\ \frac{5\,\mathrm{e}^{7t}}{2}-\frac{25\,\mathrm{e}^{5t}}{6}+\frac{5\,\mathrm{e}^{2t}}{3} & -\frac{5\,\mathrm{e}^{5t}}{2}+\frac{5\,\mathrm{e}^{7t}}{2} & \frac{7\,\mathrm{e}^{2t}}{3}+2\,\mathrm{e}^{7t}-\frac{10\,\mathrm{e}^{5t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{4\,\mathrm{e}^{2t}}{3}-\frac{7\,\mathrm{e}^{7t}}{2}+\frac{35\,\mathrm{e}^{5t}}{6}\right)c_1+\left(-\frac{7\,\mathrm{e}^{7t}}{2}+\frac{7\,\mathrm{e}^{5t}}{2}\right)c_2+\left(-\frac{14\,\mathrm{e}^{7t}}{5}+\frac{14\,\mathrm{e}^{5t}}{3}-\frac{28\,\mathrm{e}^{2t}}{15}\right)c_3 \\ \left(-\frac{5\,\mathrm{e}^{5t}}{2}+\frac{5\,\mathrm{e}^{7t}}{2}\right)c_1+\left(\frac{5\,\mathrm{e}^{7t}}{2}-\frac{3\,\mathrm{e}^{5t}}{2}\right)c_2+\left(2\,\mathrm{e}^{7t}-2\,\mathrm{e}^{5t}\right)c_3 \\ \left(\frac{5\,\mathrm{e}^{7t}}{2}-\frac{25\,\mathrm{e}^{5t}}{6}+\frac{5\,\mathrm{e}^{2t}}{3}\right)c_1+\left(-\frac{5\,\mathrm{e}^{5t}}{2}+\frac{5\,\mathrm{e}^{7t}}{2}\right)c_2+\left(\frac{7\,\mathrm{e}^{2t}}{3}+2\,\mathrm{e}^{7t}-\frac{10\,\mathrm{e}^{5t}}{3}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{7(5c_1+3c_2+4c_3)\mathrm{e}^{5t}}{6}+\frac{7(-5c_1-5c_2-4c_3)\mathrm{e}^{7t}}{10}-\frac{4\,\mathrm{e}^{2t}\left(c_1+\frac{7c_3}{5}\right)}{3} \\ \frac{(-5c_1-3c_2-4c_3)\mathrm{e}^{5t}}{2}+\frac{5\,\mathrm{e}^{7t}\left(c_1+c_2+\frac{4c_3}{5}\right)}{2} \\ \frac{5(-5c_1-3c_2-4c_3)\mathrm{e}^{5t}}{6}+\frac{(5c_1+5c_2+4c_3)\mathrm{e}^{7t}}{2}+\frac{5\,\mathrm{e}^{2t}\left(c_1+\frac{7c_3}{5}\right)}{3} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 14\lambda^2 + 59\lambda - 70 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 5$$
$$\lambda_3 = 7$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
5	1	real eigenvalue
7	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -7 & 0 & 0 \\ 5 & 8 & 4 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 5 & 8 & 4 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{5R_2}{7} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 8 & 4 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 8 & 4 \\ 0 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{4t}{5}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{4t}{5} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4t}{5} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{4t}{5} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{5} \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{4t}{5} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ 0 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{4t}{5} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -7 & 0 \\ 5 & 5 & 4 \\ 0 & 5 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & -7 & 0 & 0 \\ 5 & 5 & 4 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{5R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & -7 & 0 & 0 \\ 0 & -\frac{20}{3} & 4 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_2}{4} \Longrightarrow \left[\begin{array}{ccc|c} -3 & -7 & 0 & 0 \\ 0 & -\frac{20}{3} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -7 & 0 \\ 0 & -\frac{20}{3} & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{7t}{5}, v_2 = \frac{3t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{7t}{5} \\ \frac{3t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7t}{5} \\ \frac{3t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{7t}{5} \\ \frac{3t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{7}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{7t}{5} \\ \frac{3t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{7t}{5} \\ \frac{3t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 7$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} - (7) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -7 & 0 \\ 5 & 3 & 4 \\ 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & -7 & 0 & 0 \\ 5 & 3 & 4 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -5 & -7 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{5R_2}{4} \Longrightarrow \left[\begin{array}{ccc|c} -5 & -7 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & -7 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{7t}{5}, v_2 = t\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{7t}{5} \\ t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7t}{5} \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{7t}{5} \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{7}{5} \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{7t}{5} \\ t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7}{5} \\ 1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{7t}{5} \\ t \\ t \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} -\frac{4}{5} \\ 0 \\ 1 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} -\frac{7}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}$
7	1	1	No	$\begin{bmatrix} -\frac{7}{5} \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} -\frac{4}{5} \\ 0 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{5t} \\ &= \begin{bmatrix} -\frac{7}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix} e^{5t}\end{aligned}$$

Since eigenvalue 7 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{7t}\\ &= \begin{bmatrix} -\frac{7}{5} \\ 1 \\ 1 \end{bmatrix} e^{7t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{4\,\mathrm{e}^{2t}}{5} \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{7\,\mathrm{e}^{5t}}{5} \\ \frac{3\,\mathrm{e}^{5t}}{5} \\ \mathrm{e}^{5t} \end{bmatrix} + c_3 \begin{bmatrix} -\frac{7\,\mathrm{e}^{7t}}{5} \\ \mathrm{e}^{7t} \\ \mathrm{e}^{7t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{4c_1\mathrm{e}^{2t}}{5} - \frac{7c_2\mathrm{e}^{5t}}{5} - \frac{7c_3\mathrm{e}^{7t}}{5} \\ \frac{3c_2\mathrm{e}^{5t}}{5} + c_3\mathrm{e}^{7t} \\ c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{5t} + c_3\mathrm{e}^{7t} \end{bmatrix}$$

10.8.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) - 7y, y' = 5x(t) + 10y + 4z(t)\,, z'(t) = 5y + 2z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\begin{bmatrix} \begin{bmatrix} 2, & \begin{bmatrix} -\frac{4}{5} \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}, & \begin{bmatrix} 5, & \begin{bmatrix} -\frac{7}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix} \end{bmatrix}, & \begin{bmatrix} 7, & \begin{bmatrix} -\frac{7}{5} \\ 1 \\ 1 \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, & \begin{bmatrix} -\frac{4}{5} \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{2t} \cdot \begin{bmatrix} -\frac{4}{5} \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} -\frac{7}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{5t} \cdot \begin{bmatrix} -\frac{7}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[7, \begin{bmatrix} -\frac{7}{5} \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \mathrm{e}^{7t} \cdot \begin{bmatrix} -\frac{7}{5} \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{2t} \cdot \begin{bmatrix} -\frac{4}{5} \\ 0 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{5t} \cdot \begin{bmatrix} -\frac{7}{5} \\ \frac{3}{5} \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{7t} \cdot \begin{bmatrix} -\frac{7}{5} \\ 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{4c_1 \mathrm{e}^{2t}}{5} - \frac{7c_2 \mathrm{e}^{5t}}{5} - \frac{7c_3 \mathrm{e}^{7t}}{5} \\ \frac{3c_2 \mathrm{e}^{5t}}{5} + c_3 \mathrm{e}^{7t} \\ c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{5t} + c_3 \mathrm{e}^{7t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{4c_1 \mathrm{e}^{2t}}{5} - \frac{7c_2 \mathrm{e}^{5t}}{5} - \frac{7c_3 \mathrm{e}^{7t}}{5}, y = \frac{3c_2 \mathrm{e}^{5t}}{5} + c_3 \mathrm{e}^{7t}, z(t) = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{5t} + c_3 \mathrm{e}^{7t}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 66

```
dsolve([diff(x(t),t)=2*x(t)-7*y(t),diff(y(t),t)=5*x(t)+10*y(t)+4*z(t),diff(z(t),t)=5*y(t)+2*
```

$$\begin{aligned}x(t) &= -\frac{7c_3\mathrm{e}^{7t}}{5}-\frac{7c_2\mathrm{e}^{5t}}{3}+c_1\mathrm{e}^{2t}\\y(t) &= c_2\mathrm{e}^{5t}+c_3\mathrm{e}^{7t}\\z(t) &= \frac{5c_2\mathrm{e}^{5t}}{3}+c_3\mathrm{e}^{7t}-\frac{5c_1\mathrm{e}^{2t}}{4}\end{aligned}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 190

```
DSolve[{x'[t]==2*x[t]-7*y[t],y'[t]==5*x[t]+10*y[t]+4*z[t],z'[t]==5*y[t]+2*z[t]},{x[t],y[t],z
```

$$x(t)\to -\frac{1}{30}e^{2t}\left(5c_1\left(-35e^{3t}+21e^{5t}+8\right)+7\left(-5(3c_2+4c_3)e^{3t}+3(5c_2+4c_3)e^{5t}+8c_3\right)\right)$$

$$y(t)\to \frac{1}{2}e^{5t}\left(5c_1\left(e^{2t}-1\right)+c_2\left(5e^{2t}-3\right)+4c_3\left(e^{2t}-1\right)\right)$$

$$z(t)\to \frac{1}{6}e^{2t}\left(5c_1\left(-5e^{3t}+3e^{5t}+2\right)-5(3c_2+4c_3)e^{3t}+3(5c_2+4c_3)e^{5t}+14c_3\right)$$

10.9 problem 9

Internal problem ID [6735]
Internal file name [`OUTPUT/5983_Sunday_June_05_2022_04_11_06_PM_28113920/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -x(t) + y \\
y' &= x(t) + 2y + z(t) \\
z'(t) &= 3y - z(t)
\end{aligned}
$$

10.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(\mathrm{e}^{5t}+15\,\mathrm{e}^{t}+4\right)\mathrm{e}^{-2t}}{20} & \frac{\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}}{5} & \frac{\left(\mathrm{e}^{5t}-5\,\mathrm{e}^{t}+4\right)\mathrm{e}^{-2t}}{20} \\ \frac{\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}}{5} & \frac{\left(4\,\mathrm{e}^{5t}+1\right)\mathrm{e}^{-2t}}{5} & \frac{\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}}{5} \\ \frac{3\left(\mathrm{e}^{5t}-5\,\mathrm{e}^{t}+4\right)\mathrm{e}^{-2t}}{20} & \frac{3\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}}{5} & \frac{\left(3\,\mathrm{e}^{5t}+5\,\mathrm{e}^{t}+12\right)\mathrm{e}^{-2t}}{20} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{\left(\mathrm{e}^{5t}+15\,\mathrm{e}^{t}+4\right)\mathrm{e}^{-2t}}{20} & \frac{\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}}{5} & \frac{\left(\mathrm{e}^{5t}-5\,\mathrm{e}^{t}+4\right)\mathrm{e}^{-2t}}{20} \\ \frac{\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}}{5} & \frac{\left(4\,\mathrm{e}^{5t}+1\right)\mathrm{e}^{-2t}}{5} & \frac{\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}}{5} \\ \frac{3\left(\mathrm{e}^{5t}-5\,\mathrm{e}^{t}+4\right)\mathrm{e}^{-2t}}{20} & \frac{3\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}}{5} & \frac{\left(3\,\mathrm{e}^{5t}+5\,\mathrm{e}^{t}+12\right)\mathrm{e}^{-2t}}{20} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\left(\mathrm{e}^{5t}+15\,\mathrm{e}^{t}+4\right)\mathrm{e}^{-2t}c_1}{20} + \frac{\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}c_2}{5} + \frac{\left(\mathrm{e}^{5t}-5\,\mathrm{e}^{t}+4\right)\mathrm{e}^{-2t}c_3}{20} \\ \frac{\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}c_1}{5} + \frac{\left(4\,\mathrm{e}^{5t}+1\right)\mathrm{e}^{-2t}c_2}{5} + \frac{\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}c_3}{5} \\ \frac{3\left(\mathrm{e}^{5t}-5\,\mathrm{e}^{t}+4\right)\mathrm{e}^{-2t}c_1}{20} + \frac{3\left(\mathrm{e}^{5t}-1\right)\mathrm{e}^{-2t}c_2}{5} + \frac{\left(3\,\mathrm{e}^{5t}+5\,\mathrm{e}^{t}+12\right)\mathrm{e}^{-2t}c_3}{20} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\mathrm{e}^{-2t}\left((c_1+4c_2+c_3)\mathrm{e}^{5t}+5(3c_1-c_3)\mathrm{e}^{t}+4c_1-4c_2+4c_3\right)}{20} \\ \frac{\left((c_1+4c_2+c_3)\mathrm{e}^{5t}-c_1+c_2-c_3\right)\mathrm{e}^{-2t}}{5} \\ -\frac{3\left(\frac{(-c_1-4c_2-c_3)\mathrm{e}^{5t}}{5}+\left(c_1-\frac{c_3}{3}\right)\mathrm{e}^{t}-\frac{4c_1}{5}+\frac{4c_2}{5}-\frac{4c_3}{5}\right)\mathrm{e}^{-2t}}{4} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix}-1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 3 & -1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix}-1-\lambda & 1 & 0\\ 1 & 2-\lambda & 1\\ 0 & 3 & -1-\lambda\end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^3-7\lambda-6=0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= -1\\ \lambda_3 &= 3\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \implies \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \implies \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}, v_2 = -\frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 0 & -\frac{3}{4} & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 + 4R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 0 & -\frac{3}{4} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 1 & 0 \\ 0 & -\frac{3}{4} & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}, v_2 = \frac{4t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ \frac{4t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ \frac{4t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{3t}\\ &= \begin{bmatrix}\frac{1}{3}\\ \frac{4}{3}\\ 1\end{bmatrix} e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix}x(t)\\ y\\ z(t)\end{bmatrix} = c_1\begin{bmatrix}\frac{\mathrm{e}^{-2t}}{3}\\ -\frac{\mathrm{e}^{-2t}}{3}\\ \mathrm{e}^{-2t}\end{bmatrix} + c_2\begin{bmatrix}-\mathrm{e}^{-t}\\ 0\\ \mathrm{e}^{-t}\end{bmatrix} + c_3\begin{bmatrix}\frac{\mathrm{e}^{3t}}{3}\\ \frac{4\,\mathrm{e}^{3t}}{3}\\ \mathrm{e}^{3t}\end{bmatrix}$$

Which becomes

$$\begin{bmatrix}x(t)\\ y\\ z(t)\end{bmatrix} = \begin{bmatrix}\frac{(c_3\mathrm{e}^{5t}-3c_2\mathrm{e}^{t}+c_1)\mathrm{e}^{-2t}}{3}\\ -\frac{(-4c_3\mathrm{e}^{5t}+c_1)\mathrm{e}^{-2t}}{3}\\ (c_3\mathrm{e}^{5t}+c_2\mathrm{e}^{t}+c_1)\,\mathrm{e}^{-2t}\end{bmatrix}$$

10.9.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + y, y' = x(t) + 2y + z(t)\,, z'(t) = 3y - z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix}x(t)\\ y\\ z(t)\end{bmatrix}$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \left[\begin{array}{ccc} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

- System to solve

$$\overrightarrow{x}'(t) = \left[\begin{array}{ccc} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{array}\right] \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-2t} \cdot \left[\begin{array}{c} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_3 = \mathrm{e}^{3t} \cdot \left[\begin{array}{c} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2 + c_3 \overrightarrow{x}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{-2t} \cdot \left[\begin{array}{c} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{array}\right] + c_2 \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right] + c_3 \mathrm{e}^{3t} \cdot \left[\begin{array}{c} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} \frac{(\mathrm{e}^{5t} c_3 - 3 c_2 \mathrm{e}^{t} + c_1) \mathrm{e}^{-2t}}{3} \\ -\frac{(-4\,\mathrm{e}^{5t} c_3 + c_1) \mathrm{e}^{-2t}}{3} \\ (\mathrm{e}^{5t} c_3 + c_2 \mathrm{e}^{t} + c_1) \mathrm{e}^{-2t} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(\mathrm{e}^{5t} c_3 - 3 c_2 \mathrm{e}^{t} + c_1) \mathrm{e}^{-2t}}{3}, y = -\frac{(-4\,\mathrm{e}^{5t} c_3 + c_1) \mathrm{e}^{-2t}}{3}, z(t) = (\mathrm{e}^{5t} c_3 + c_2 \mathrm{e}^{t} + c_1) \mathrm{e}^{-2t}\right\}$$

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 67

```
dsolve([diff(x(t),t)=-x(t)+y(t),diff(y(t),t)=x(t)+2*y(t)+z(t),diff(z(t),t)=3*y(t)-z(t)],sing
```

$$
\begin{aligned}
x(t) &= \frac{c_2 \mathrm{e}^{3t}}{4} + \mathrm{e}^{-t} c_1 - c_3 \mathrm{e}^{-2t} \\
y(t) &= c_2 \mathrm{e}^{3t} + c_3 \mathrm{e}^{-2t} \\
z(t) &= \frac{3 c_2 \mathrm{e}^{3t}}{4} - \mathrm{e}^{-t} c_1 - 3 c_3 \mathrm{e}^{-2t}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 158

```
DSolve[{x'[t]==-x[t]+y[t],y'[t]==x[t]+2*y[t]+z[t],z'[t]==3*y[t]-z[t]},{x[t],y[t],z[t]},t,Inc
```

$$
\begin{aligned}
x(t) &\to \frac{1}{20} e^{-2t} \left(c_1 \left(15 e^t + e^{5t} + 4\right) + 4 c_2 \left(e^{5t} - 1\right) + c_3 \left(-5 e^t + e^{5t} + 4\right)\right) \\
y(t) &\to \frac{1}{5} e^{-2t} \left(c_1 \left(e^{5t} - 1\right) + c_2 \left(4 e^{5t} + 1\right) + c_3 \left(e^{5t} - 1\right)\right) \\
z(t) &\to \frac{1}{20} e^{-2t} \left(3 c_1 \left(-5 e^t + e^{5t} + 4\right) + 12 c_2 \left(e^{5t} - 1\right) + c_3 \left(5 e^t + 3 e^{5t} + 12\right)\right)
\end{aligned}
$$

10.10 problem 10

Internal problem ID [6736]
Internal file name [`OUTPUT/5984_Sunday_June_05_2022_04_11_08_PM_44447815/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + z(t) \\
y' &= y \\
z'(t) &= x(t) + z(t)
\end{aligned}
$$

10.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{1}{2}+\frac{e^{2t}}{2} & 0 & \frac{e^{2t}}{2}-\frac{1}{2} \\ 0 & e^{t} & 0 \\ \frac{e^{2t}}{2}-\frac{1}{2} & 0 & \frac{1}{2}+\frac{e^{2t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{1}{2}+\frac{e^{2t}}{2} & 0 & \frac{e^{2t}}{2}-\frac{1}{2} \\ 0 & e^{t} & 0 \\ \frac{e^{2t}}{2}-\frac{1}{2} & 0 & \frac{1}{2}+\frac{e^{2t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{1}{2}+\frac{e^{2t}}{2}\right)c_1 + \left(\frac{e^{2t}}{2}-\frac{1}{2}\right)c_3 \\ e^{t}c_2 \\ \left(\frac{e^{2t}}{2}-\frac{1}{2}\right)c_1 + \left(\frac{1}{2}+\frac{e^{2t}}{2}\right)c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(c_1+c_3)e^{2t}}{2} + \frac{c_1}{2} - \frac{c_3}{2} \\ e^{t}c_2 \\ \frac{(c_1+c_3)e^{2t}}{2} - \frac{c_1}{2} + \frac{c_3}{2} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} - \lambda \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}1-\lambda & 0 & 1\\ 0 & 1-\lambda & 0\\ 1 & 0 & 1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 1\\ \lambda_3 &= 0\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} - (0)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
2	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{t} \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t}\end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{0} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1\mathrm{e}^{2t} - c_3 \\ c_2\mathrm{e}^{t} \\ c_1\mathrm{e}^{2t} + c_3 \end{bmatrix}$$

10.10.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + z(t), y' = y, z'(t) = x(t) + z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_3 = \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\overrightarrow{x} = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + c_3\overrightarrow{x}_3$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_2\mathrm{e}^{t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3\mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -c_1 \\ 0 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_3\mathrm{e}^{2t} - c_1 \\ c_2\mathrm{e}^{t} \\ c_3\mathrm{e}^{2t} + c_1 \end{bmatrix}$$

- Solution to the system of ODEs

$\{x(t) = c_3\mathrm{e}^{2t} - c_1, y = c_2\mathrm{e}^{t}, z(t) = c_3\mathrm{e}^{2t} + c_1\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve([diff(x(t),t)=x(t)+z(t),diff(y(t),t)=y(t),diff(z(t),t)=x(t)+z(t)],singsol=all)
```

$$
\begin{aligned}
x(t) &= c_1 + c_2 \mathrm{e}^{2t} \\
y(t) &= c_3 \mathrm{e}^{t} \\
z(t) &= c_2 \mathrm{e}^{2t} - c_1
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 128

```
DSolve[{x'[t]==x[t]+z[t],y'[t]==y[t],z'[t]==x[t]+z[t]},{x[t],y[t],z[t]},t,IncludeSingularSol
```

$$
\begin{aligned}
x(t) &\to \frac{1}{2}\left(c_1\left(e^{2t}+1\right)+c_2\left(e^{2t}-1\right)\right) \\
z(t) &\to \frac{1}{2}\left(c_1\left(e^{2t}-1\right)+c_2\left(e^{2t}+1\right)\right) \\
y(t) &\to c_3 e^{t} \\
x(t) &\to \frac{1}{2}\left(c_1\left(e^{2t}+1\right)+c_2\left(e^{2t}-1\right)\right) \\
z(t) &\to \frac{1}{2}\left(c_1\left(e^{2t}-1\right)+c_2\left(e^{2t}+1\right)\right) \\
y(t) &\to 0
\end{aligned}
$$

10.11 problem 11

Internal problem ID [6737]
Internal file name [`OUTPUT/5985_Sunday_June_05_2022_04_11_11_PM_44514396/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}x'(t) &= -x(t) - y\\ y' &= \frac{3x(t)}{4} - \frac{3y}{2} + 3z(t)\\ z'(t) &= \frac{x(t)}{8} + \frac{y}{4} - \frac{z(t)}{2}\end{aligned}$$

10.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 4\,\mathrm{e}^{-t}-\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{2}-\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2} & -2\,\mathrm{e}^{-t}+2\,\mathrm{e}^{-\frac{3t}{2}} & -6\,\mathrm{e}^{-\frac{t}{2}}+12\,\mathrm{e}^{-t}-6\,\mathrm{e}^{-\frac{3t}{2}} \\ -\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{4}+\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{4} & \mathrm{e}^{-\frac{3t}{2}} & -3\,\mathrm{e}^{-\frac{3t}{2}}+3\,\mathrm{e}^{-\frac{t}{2}} \\ -\mathrm{e}^{-t}+\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{8}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{8} & \frac{\mathrm{e}^{-t}}{2}-\frac{\mathrm{e}^{-\frac{3t}{2}}}{2} & \frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{2}-3\,\mathrm{e}^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} 4\,\mathrm{e}^{-t}-\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{2}-\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2} & -2\,\mathrm{e}^{-t}+2\,\mathrm{e}^{-\frac{3t}{2}} & -6\,\mathrm{e}^{-\frac{t}{2}}+12\,\mathrm{e}^{-t}-6\,\mathrm{e}^{-\frac{3t}{2}} \\ -\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{4}+\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{4} & \mathrm{e}^{-\frac{3t}{2}} & -3\,\mathrm{e}^{-\frac{3t}{2}}+3\,\mathrm{e}^{-\frac{t}{2}} \\ -\mathrm{e}^{-t}+\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{8}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{8} & \frac{\mathrm{e}^{-t}}{2}-\frac{\mathrm{e}^{-\frac{3t}{2}}}{2} & \frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{2}-3\,\mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(4\,\mathrm{e}^{-t}-\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{2}-\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2}\right)c_1+\left(-2\,\mathrm{e}^{-t}+2\,\mathrm{e}^{-\frac{3t}{2}}\right)c_2+\left(-6\,\mathrm{e}^{-\frac{t}{2}}+12\,\mathrm{e}^{-t}-6\,\mathrm{e}^{-\frac{3t}{2}}\right)c_3 \\ \left(-\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{4}+\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{4}\right)c_1+\mathrm{e}^{-\frac{3t}{2}}c_2+\left(-3\,\mathrm{e}^{-\frac{3t}{2}}+3\,\mathrm{e}^{-\frac{t}{2}}\right)c_3 \\ \left(-\mathrm{e}^{-t}+\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{8}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{8}\right)c_1+\left(\frac{\mathrm{e}^{-t}}{2}-\frac{\mathrm{e}^{-\frac{3t}{2}}}{2}\right)c_2+\left(\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{2}-3\,\mathrm{e}^{-t}\right)c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(-3c_1+4c_2-12c_3)\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{3(-c_1-4c_3)\mathrm{e}^{-\frac{t}{2}}}{2}+4\left(c_1-\frac{c_2}{2}+3c_3\right)\mathrm{e}^{-t} \\ \frac{(-3c_1+4c_2-12c_3)\mathrm{e}^{-\frac{3t}{2}}}{4}+\frac{3\,\mathrm{e}^{-\frac{t}{2}}(c_1+4c_3)}{4} \\ \frac{(3c_1-4c_2+12c_3)\mathrm{e}^{-\frac{3t}{2}}}{8}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}(c_1+4c_3)}{8}-\left(c_1-\frac{c_2}{2}+3c_3\right)\mathrm{e}^{-t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix}-1 & -1 & 0\\ \frac{3}{4} & -\frac{3}{2} & 3\\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2}\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix}-1-\lambda & -1 & 0\\ \frac{3}{4} & -\frac{3}{2}-\lambda & 3\\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2}-\lambda\end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^3+3\lambda^2+\frac{11}{4}\lambda+\frac{3}{4}=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-\frac{3}{2}$$
$$\lambda_2=-1$$
$$\lambda_3=-\frac{1}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
$-\frac{1}{2}$	1	real eigenvalue
$-\frac{3}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ \frac{3}{4} & -\frac{1}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ \frac{3}{4} & -\frac{1}{2} & 3 & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} \frac{3}{4} & -\frac{1}{2} & 3 & 0 \\ 0 & -1 & 0 & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{6} \Longrightarrow \left[\begin{array}{ccc|c} \frac{3}{4} & -\frac{1}{2} & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} \frac{3}{4} & -\frac{1}{2} & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -4t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -4t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -4t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -4t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -4t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{3}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} - \left(-\frac{3}{2}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & -1 & 0 \\ \frac{3}{4} & 0 & 3 \\ \frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 0 \\ \frac{3}{4} & 0 & 3 & 0 \\ \frac{1}{8} & \frac{1}{4} & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 0 \\ 0 & \frac{3}{2} & 3 & 0 \\ \frac{1}{8} & \frac{1}{4} & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 0 \\ 0 & \frac{3}{2} & 3 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 0 \\ 0 & \frac{3}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} & -1 & 0 \\ 0 & \frac{3}{2} & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -4t, v_2 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -4t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -4t \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -4t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -4t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -\frac{1}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} - \left(-\frac{1}{2}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & -1 & 0 \\ \frac{3}{4} & -1 & 3 \\ \frac{1}{8} & \frac{1}{4} & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -\frac{1}{2} & -1 & 0 & 0 \\ \frac{3}{4} & -1 & 3 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \implies \left[\begin{array}{ccc|c} -\frac{1}{2} & -1 & 0 & 0 \\ 0 & -\frac{5}{2} & 3 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{4} \implies \left[\begin{array}{ccc|c} -\frac{1}{2} & -1 & 0 & 0 \\ 0 & -\frac{5}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2} & -1 & 0 \\ 0 & -\frac{5}{2} & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{12t}{5}, v_2 = \frac{6t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{12t}{5} \\ \frac{6t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12t}{5} \\ \frac{6t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{12t}{5} \\ \frac{6t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{12t}{5} \\ \frac{6t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{12t}{5} \\ \frac{6t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{3}{2}$	1	1	No	$\begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$
$-\frac{1}{2}$	1	1	No	$\begin{bmatrix} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{3}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-\frac{3t}{2}} \\ &= \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix} e^{-\frac{3t}{2}}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue $-\frac{1}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{-\frac{t}{2}} \\
&= \left[\begin{array}{c} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{array}\right] e^{-\frac{t}{2}}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = c_1 \left[\begin{array}{c} -4\,\mathrm{e}^{-\frac{3t}{2}} \\ -2\,\mathrm{e}^{-\frac{3t}{2}} \\ \mathrm{e}^{-\frac{3t}{2}} \end{array}\right] + c_2 \left[\begin{array}{c} -4\,\mathrm{e}^{-t} \\ 0 \\ \mathrm{e}^{-t} \end{array}\right] + c_3 \left[\begin{array}{c} -\frac{12\,\mathrm{e}^{-\frac{t}{2}}}{5} \\ \frac{6\,\mathrm{e}^{-\frac{t}{2}}}{5} \\ \mathrm{e}^{-\frac{t}{2}} \end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} -4c_1\mathrm{e}^{-\frac{3t}{2}} - 4c_2\mathrm{e}^{-t} - \frac{12c_3\mathrm{e}^{-\frac{t}{2}}}{5} \\ -2c_1\mathrm{e}^{-\frac{3t}{2}} + \frac{6c_3\mathrm{e}^{-\frac{t}{2}}}{5} \\ c_1\mathrm{e}^{-\frac{3t}{2}} + c_2\mathrm{e}^{-t} + c_3\mathrm{e}^{-\frac{t}{2}} \end{array}\right]
$$

10.11.3 Maple step by step solution

Let's solve

$\left[x'(t) = -x(t) - y, y' = \frac{3x(t)}{4} - \frac{3y}{2} + 3z(t)\,, z'(t) = \frac{x(t)}{8} + \frac{y}{4} - \frac{z(t)}{2}\right]$

- Define vector

$$
\overrightarrow{x}(t) = \left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right]
$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{3}{2}, \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{3}{2}, \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-\frac{3t}{2}} \cdot \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -4 \\ 0 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{2} = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -4 \\ 0 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \left[\begin{array}{c} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{3} = \mathrm{e}^{-\frac{t}{2}} \cdot \left[\begin{array}{c} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_{1} \overrightarrow{x}_{1} + c_{2} \overrightarrow{x}_{2} + c_{3} \overrightarrow{x}_{3}$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_{1} \mathrm{e}^{-\frac{3t}{2}} \cdot \left[\begin{array}{c} -4 \\ -2 \\ 1 \end{array}\right] + c_{2} \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -4 \\ 0 \\ 1 \end{array}\right] + c_{3} \mathrm{e}^{-\frac{t}{2}} \cdot \left[\begin{array}{c} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} -4 c_{1} \mathrm{e}^{-\frac{3t}{2}} - 4 c_{2} \mathrm{e}^{-t} - \frac{12 c_{3} \mathrm{e}^{-\frac{t}{2}}}{5} \\ -2 c_{1} \mathrm{e}^{-\frac{3t}{2}} + \frac{6 c_{3} \mathrm{e}^{-\frac{t}{2}}}{5} \\ c_{1} \mathrm{e}^{-\frac{3t}{2}} + c_{2} \mathrm{e}^{-t} + c_{3} \mathrm{e}^{-\frac{t}{2}} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -4 c_{1} \mathrm{e}^{-\frac{3t}{2}} - 4 c_{2} \mathrm{e}^{-t} - \frac{12 c_{3} \mathrm{e}^{-\frac{t}{2}}}{5}, y = -2 c_{1} \mathrm{e}^{-\frac{3t}{2}} + \frac{6 c_{3} \mathrm{e}^{-\frac{t}{2}}}{5}, z(t) = c_{1} \mathrm{e}^{-\frac{3t}{2}} + c_{2} \mathrm{e}^{-t} + c_{3} \mathrm{e}^{-\frac{t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 67

```
dsolve([diff(x(t),t)=-x(t)-y(t),diff(y(t),t)=3/4*x(t)-3/2*y(t)+3*z(t),diff(z(t),t)=1/8*x(t)+
```

$$
\begin{aligned}
x(t) &= c_1 \mathrm{e}^{-\frac{3t}{2}} + c_2 \mathrm{e}^{-t} + c_3 \mathrm{e}^{-\frac{t}{2}} \\
y(t) &= \frac{c_1 \mathrm{e}^{-\frac{3t}{2}}}{2} - \frac{c_3 \mathrm{e}^{-\frac{t}{2}}}{2} \\
z(t) &= -\frac{c_1 \mathrm{e}^{-\frac{3t}{2}}}{4} - \frac{c_2 \mathrm{e}^{-t}}{4} - \frac{5 c_3 \mathrm{e}^{-\frac{t}{2}}}{12}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 168

```
DSolve[{x'[t]==-x[t]-y[t],y'[t]==3/4*x[t]-3/2*y[t]+3*z[t],z'[t]==1/8x[t]+1/4*y[t]-1/2*z[t]},
```

$$
\begin{aligned}
x(t) &\to \frac{1}{2} e^{-3t/2} \left(c_1 \left(8e^{t/2} - 3e^t - 3\right) - 4\left(e^{t/2} - 1\right) \left(3c_3 \left(e^{t/2} - 1\right) + c_2\right)\right) \\
y(t) &\to \frac{1}{4} e^{-3t/2} \left(3c_1 \left(e^t - 1\right) + 4\left(3c_3 \left(e^t - 1\right) + c_2\right)\right) \\
z(t) &\to \frac{1}{8} e^{-3t/2} \left(c_1 \left(-8e^{t/2} + 5e^t + 3\right) + 4c_2 \left(e^{t/2} - 1\right) + 4c_3 \left(-6e^{t/2} + 5e^t + 3\right)\right)
\end{aligned}
$$

10.12 problem 11

Internal problem ID [6738]
Internal file name [`OUTPUT/5986_Sunday_June_05_2022_04_11_13_PM_26450596/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -x(t) - y \\ y' &= \frac{3x(t)}{4} - \frac{3y}{2} + 3z(t) \\ z'(t) &= \frac{x(t)}{8} + \frac{y}{4} - \frac{z(t)}{2} \end{aligned}$$

10.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 4\,\mathrm{e}^{-t}-\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{2}-\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2} & -2\,\mathrm{e}^{-t}+2\,\mathrm{e}^{-\frac{3t}{2}} & -6\,\mathrm{e}^{-\frac{t}{2}}+12\,\mathrm{e}^{-t}-6\,\mathrm{e}^{-\frac{3t}{2}} \\ -\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{4}+\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{4} & \mathrm{e}^{-\frac{3t}{2}} & -3\,\mathrm{e}^{-\frac{3t}{2}}+3\,\mathrm{e}^{-\frac{t}{2}} \\ -\mathrm{e}^{-t}+\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{8}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{8} & \frac{\mathrm{e}^{-t}}{2}-\frac{\mathrm{e}^{-\frac{3t}{2}}}{2} & \frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{2}-3\,\mathrm{e}^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} 4\,\mathrm{e}^{-t}-\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{2}-\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2} & -2\,\mathrm{e}^{-t}+2\,\mathrm{e}^{-\frac{3t}{2}} & -6\,\mathrm{e}^{-\frac{t}{2}}+12\,\mathrm{e}^{-t}-6\,\mathrm{e}^{-\frac{3t}{2}} \\ -\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{4}+\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{4} & \mathrm{e}^{-\frac{3t}{2}} & -3\,\mathrm{e}^{-\frac{3t}{2}}+3\,\mathrm{e}^{-\frac{t}{2}} \\ -\mathrm{e}^{-t}+\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{8}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{8} & \frac{\mathrm{e}^{-t}}{2}-\frac{\mathrm{e}^{-\frac{3t}{2}}}{2} & \frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{2}-3\,\mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(4\,\mathrm{e}^{-t}-\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{2}-\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2}\right)c_1+\left(-2\,\mathrm{e}^{-t}+2\,\mathrm{e}^{-\frac{3t}{2}}\right)c_2+\left(-6\,\mathrm{e}^{-\frac{t}{2}}+12\,\mathrm{e}^{-t}-6\,\mathrm{e}^{-\frac{3t}{2}}\right)c_3 \\ \left(-\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{4}+\frac{3\,\mathrm{e}^{-\frac{t}{2}}}{4}\right)c_1+\mathrm{e}^{-\frac{3t}{2}}c_2+\left(-3\,\mathrm{e}^{-\frac{3t}{2}}+3\,\mathrm{e}^{-\frac{t}{2}}\right)c_3 \\ \left(-\mathrm{e}^{-t}+\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{8}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{8}\right)c_1+\left(\frac{\mathrm{e}^{-t}}{2}-\frac{\mathrm{e}^{-\frac{3t}{2}}}{2}\right)c_2+\left(\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}}{2}-3\,\mathrm{e}^{-t}\right)c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(-3c_1+4c_2-12c_3)\mathrm{e}^{-\frac{3t}{2}}}{2}+\frac{3(-c_1-4c_3)\mathrm{e}^{-\frac{t}{2}}}{2}+4\left(c_1-\frac{c_2}{2}+3c_3\right)\mathrm{e}^{-t} \\ \frac{(-3c_1+4c_2-12c_3)\mathrm{e}^{-\frac{3t}{2}}}{4}+\frac{3\,\mathrm{e}^{-\frac{t}{2}}(c_1+4c_3)}{4} \\ \frac{(3c_1-4c_2+12c_3)\mathrm{e}^{-\frac{3t}{2}}}{8}+\frac{5\,\mathrm{e}^{-\frac{t}{2}}(c_1+4c_3)}{8}-\left(c_1-\frac{c_2}{2}+3c_3\right)\mathrm{e}^{-t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2}-\lambda & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2}-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 3\lambda^2 + \frac{11}{4}\lambda + \frac{3}{4} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = -\frac{3}{2}$$
$$\lambda_3 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
$-\frac{1}{2}$	1	real eigenvalue
$-\frac{3}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ \frac{3}{4} & -\frac{1}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ \frac{3}{4} & -\frac{1}{2} & 3 & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} \frac{3}{4} & -\frac{1}{2} & 3 & 0 \\ 0 & -1 & 0 & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{6} \Longrightarrow \left[\begin{array}{ccc|c} \frac{3}{4} & -\frac{1}{2} & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} \frac{3}{4} & -\frac{1}{2} & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -4t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -4t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -4t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -4t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -4t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{3}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} - \left(-\frac{3}{2}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & -1 & 0 \\ \frac{3}{4} & 0 & 3 \\ \frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 0 \\ \frac{3}{4} & 0 & 3 & 0 \\ \frac{1}{8} & \frac{1}{4} & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 0 \\ 0 & \frac{3}{2} & 3 & 0 \\ \frac{1}{8} & \frac{1}{4} & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 0 \\ 0 & \frac{3}{2} & 3 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 0 \\ 0 & \frac{3}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} & -1 & 0 \\ 0 & \frac{3}{2} & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -4t, v_2 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -4t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -4t \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -4t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -4t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -\frac{1}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} - \left(-\frac{1}{2}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & -1 & 0 \\ \frac{3}{4} & -1 & 3 \\ \frac{1}{8} & \frac{1}{4} & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -\frac{1}{2} & -1 & 0 & 0 \\ \frac{3}{4} & -1 & 3 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -\frac{1}{2} & -1 & 0 & 0 \\ 0 & -\frac{5}{2} & 3 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -\frac{1}{2} & -1 & 0 & 0 \\ 0 & -\frac{5}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2} & -1 & 0 \\ 0 & -\frac{5}{2} & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{12t}{5}, v_2 = \frac{6t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{12t}{5} \\ \frac{6t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12t}{5} \\ \frac{6t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{12t}{5} \\ \frac{6t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{12t}{5} \\ \frac{6t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{12t}{5} \\ \frac{6t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{2}$	1	1	No	$\left[\begin{array}{c} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{array}\right]$
$-\frac{3}{2}$	1	1	No	$\left[\begin{array}{c} -4 \\ -2 \\ 1 \end{array}\right]$
-1	1	1	No	$\left[\begin{array}{c} -4 \\ 0 \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-\frac{t}{2}} \\ &= \left[\begin{array}{c} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{array}\right] e^{-\frac{t}{2}}\end{aligned}$$

Since eigenvalue $-\frac{3}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-\frac{3t}{2}} \\ &= \left[\begin{array}{c} -4 \\ -2 \\ 1 \end{array}\right] e^{-\frac{3t}{2}}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-t} \\ &= \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{12\,\mathrm{e}^{-\frac{t}{2}}}{5} \\ \frac{6\,\mathrm{e}^{-\frac{t}{2}}}{5} \\ \mathrm{e}^{-\frac{t}{2}} \end{bmatrix} + c_2 \begin{bmatrix} -4\,\mathrm{e}^{-\frac{3t}{2}} \\ -2\,\mathrm{e}^{-\frac{3t}{2}} \\ \mathrm{e}^{-\frac{3t}{2}} \end{bmatrix} + c_3 \begin{bmatrix} -4\,\mathrm{e}^{-t} \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{12c_1\mathrm{e}^{-\frac{t}{2}}}{5} - 4c_2\mathrm{e}^{-\frac{3t}{2}} - 4c_3\mathrm{e}^{-t} \\ \frac{6c_1\mathrm{e}^{-\frac{t}{2}}}{5} - 2c_2\mathrm{e}^{-\frac{3t}{2}} \\ c_1\mathrm{e}^{-\frac{t}{2}} + c_2\mathrm{e}^{-\frac{3t}{2}} + c_3\mathrm{e}^{-t} \end{bmatrix}$$

10.12.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -x(t) - y, y' = \frac{3x(t)}{4} - \frac{3y}{2} + 3z(t)\,, z'(t) = \frac{x(t)}{8} + \frac{y}{4} - \frac{z(t)}{2}\right]$$

- Define vector

$$\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \left[\begin{array}{ccc} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

- System to solve

$$\overrightarrow{x}'(t) = \left[\begin{array}{ccc} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{array}\right] \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} -1 & -1 & 0 \\ \frac{3}{4} & -\frac{3}{2} & 3 \\ \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{3}{2}, \left[\begin{array}{c} -4 \\ -2 \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} -4 \\ 0 \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}, \left[\begin{array}{c} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-\frac{3}{2}, \left[\begin{array}{c} -4 \\ -2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-\frac{3t}{2}} \cdot \left[\begin{array}{c} -4 \\ -2 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -4 \\ 0 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{2} = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -4 \\ 0 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \left[\begin{array}{c} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{3} = \mathrm{e}^{-\frac{t}{2}} \cdot \left[\begin{array}{c} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_{1} \overrightarrow{x}_{1} + c_{2} \overrightarrow{x}_{2} + c_{3} \overrightarrow{x}_{3}$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_{1} \mathrm{e}^{-\frac{3t}{2}} \cdot \left[\begin{array}{c} -4 \\ -2 \\ 1 \end{array}\right] + c_{2} \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -4 \\ 0 \\ 1 \end{array}\right] + c_{3} \mathrm{e}^{-\frac{t}{2}} \cdot \left[\begin{array}{c} -\frac{12}{5} \\ \frac{6}{5} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} -4 c_{1} \mathrm{e}^{-\frac{3t}{2}} - 4 c_{2} \mathrm{e}^{-t} - \frac{12 c_{3} \mathrm{e}^{-\frac{t}{2}}}{5} \\ -2 c_{1} \mathrm{e}^{-\frac{3t}{2}} + \frac{6 c_{3} \mathrm{e}^{-\frac{t}{2}}}{5} \\ c_{1} \mathrm{e}^{-\frac{3t}{2}} + c_{2} \mathrm{e}^{-t} + c_{3} \mathrm{e}^{-\frac{t}{2}} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -4 c_{1} \mathrm{e}^{-\frac{3t}{2}} - 4 c_{2} \mathrm{e}^{-t} - \frac{12 c_{3} \mathrm{e}^{-\frac{t}{2}}}{5}, y = -2 c_{1} \mathrm{e}^{-\frac{3t}{2}} + \frac{6 c_{3} \mathrm{e}^{-\frac{t}{2}}}{5}, z(t) = c_{1} \mathrm{e}^{-\frac{3t}{2}} + c_{2} \mathrm{e}^{-t} + c_{3} \mathrm{e}^{-\frac{t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 67

```
dsolve([diff(x(t),t)=-x(t)-y(t),diff(y(t),t)=3/4*x(t)-3/2*y(t)+3*z(t),diff(z(t),t)=1/8*x(t)+
```

$$
\begin{aligned}
x(t) &= c_1 \mathrm{e}^{-\frac{3t}{2}} + c_2 \mathrm{e}^{-t} + c_3 \mathrm{e}^{-\frac{t}{2}} \\
y(t) &= \frac{c_1 \mathrm{e}^{-\frac{3t}{2}}}{2} - \frac{c_3 \mathrm{e}^{-\frac{t}{2}}}{2} \\
z(t) &= -\frac{c_1 \mathrm{e}^{-\frac{3t}{2}}}{4} - \frac{c_2 \mathrm{e}^{-t}}{4} - \frac{5 c_3 \mathrm{e}^{-\frac{t}{2}}}{12}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 168

```
DSolve[{x'[t]==-x[t]-y[t],y'[t]==3/4*x[t]-3/2*y[t]+3*z[t],z'[t]==1/8x[t]+1/4*y[t]-1/2*z[t]},
```

$$
\begin{aligned}
x(t) &\to \frac{1}{2} e^{-3t/2} \left(c_1 \left(8e^{t/2} - 3e^t - 3\right) - 4\left(e^{t/2} - 1\right) \left(3c_3 \left(e^{t/2} - 1\right) + c_2\right)\right) \\
y(t) &\to \frac{1}{4} e^{-3t/2} \left(3c_1 \left(e^t - 1\right) + 4\left(3c_3 \left(e^t - 1\right) + c_2\right)\right) \\
z(t) &\to \frac{1}{8} e^{-3t/2} \left(c_1 \left(-8e^{t/2} + 5e^t + 3\right) + 4c_2 \left(e^{t/2} - 1\right) + 4c_3 \left(-6e^{t/2} + 5e^t + 3\right)\right)
\end{aligned}
$$

10.13 problem 12

Internal problem ID [6739]
Internal file name [`OUTPUT/5987_Sunday_June_05_2022_04_11_17_PM_96085296/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -x(t) + 4y + 2z(t) \\
y' &= 4x(t) - y - 2z(t) \\
z'(t) &= 6z(t)
\end{aligned}
$$

10.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(e^{8t}+1)e^{-5t}}{2} & \frac{(e^{8t}-1)e^{-5t}}{2} & \frac{2(e^{11t}-1)e^{-5t}}{11} \\ \frac{(e^{8t}-1)e^{-5t}}{2} & \frac{(e^{8t}+1)e^{-5t}}{2} & -\frac{2(e^{11t}-1)e^{-5t}}{11} \\ 0 & 0 & e^{6t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{(e^{8t}+1)e^{-5t}}{2} & \frac{(e^{8t}-1)e^{-5t}}{2} & \frac{2(e^{11t}-1)e^{-5t}}{11} \\ \frac{(e^{8t}-1)e^{-5t}}{2} & \frac{(e^{8t}+1)e^{-5t}}{2} & -\frac{2(e^{11t}-1)e^{-5t}}{11} \\ 0 & 0 & e^{6t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(e^{8t}+1)e^{-5t}c_1}{2} + \frac{(e^{8t}-1)e^{-5t}c_2}{2} + \frac{2(e^{11t}-1)e^{-5t}c_3}{11} \\ \frac{(e^{8t}-1)e^{-5t}c_1}{2} + \frac{(e^{8t}+1)e^{-5t}c_2}{2} - \frac{2(e^{11t}-1)e^{-5t}c_3}{11} \\ e^{6t}c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{-5t}\left((c_1+c_2)e^{8t}+\frac{4\,e^{11t}c_3}{11}+c_1-c_2-\frac{4c_3}{11}\right)}{2} \\ \frac{\left((c_1+c_2)e^{8t}-\frac{4\,e^{11t}c_3}{11}-c_1+c_2+\frac{4c_3}{11}\right)e^{-5t}}{2} \\ e^{6t}c_3 \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}-1 & 4 & 2\\ 4 & -1 & -2\\ 0 & 0 & 6\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-1-\lambda & 4 & 2\\ 4 & -1-\lambda & -2\\ 0 & 0 & 6-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 4\lambda^2 - 27\lambda + 90 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 6$$
$$\lambda_3 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue
-5	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-1 & 4 & 2\\ 4 & -1 & -2\\ 0 & 0 & 6\end{bmatrix} - (-5)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}4 & 4 & 2\\ 4 & 4 & -2\\ 0 & 0 & 11\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & -2 & 0 \\ 0 & 0 & 11 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 11 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{11R_2}{4} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 4 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 4 & 2 \\ 4 & -4 & -2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 4 & 2 & 0 \\ 4 & -4 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -4 & 4 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 4 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 4 & 2 \\ 4 & -7 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -7 & 4 & 2 & 0 \\ 4 & -7 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{4R_1}{7} \Longrightarrow \left[\begin{array}{ccc|c} -7 & 4 & 2 & 0 \\ 0 & -\frac{33}{7} & -\frac{6}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -7 & 4 & 2 \\ 0 & -\frac{33}{7} & -\frac{6}{7} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{11}, v_2 = -\frac{2t}{11}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{11} \\ -\frac{2t}{11} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{11} \\ -\frac{2t}{11} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{11} \\ -\frac{2t}{11} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{11} \\ -\frac{2}{11} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{11} \\ -\frac{2t}{11} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{11} \\ -\frac{2}{11} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{11} \\ -\frac{2t}{11} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 11 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} \frac{2}{11} \\ -\frac{2}{11} \\ 1 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{6t} \\ &= \begin{bmatrix} \frac{2}{11} \\ -\frac{2}{11} \\ 1 \end{bmatrix} e^{6t} \end{aligned}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-5t}\\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,\mathrm{e}^{6t}}{11} \\ -\frac{2\,\mathrm{e}^{6t}}{11} \\ \mathrm{e}^{6t} \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{(2c_2\mathrm{e}^{11t}+11c_1\mathrm{e}^{8t}-11c_3)\mathrm{e}^{-5t}}{11} \\ -\frac{(2c_2\mathrm{e}^{11t}-11c_1\mathrm{e}^{8t}-11c_3)\mathrm{e}^{-5t}}{11} \\ c_2\mathrm{e}^{6t} \end{bmatrix}$$

10.13.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + 4y + 2z(t)\,, y' = 4x(t) - y - 2z(t)\,, z'(t) = 6z(t)]$

- Define vector

$$\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{bmatrix} \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[3, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right], \left[6, \begin{bmatrix} \frac{2}{11} \\ -\frac{2}{11} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-5, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-5t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{2} = \mathrm{e}^{3t} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right]$$

- Consider eigenpair

$$\left[6, \left[\begin{array}{c} \frac{2}{11} \\ -\frac{2}{11} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{3} = \mathrm{e}^{6t} \cdot \left[\begin{array}{c} \frac{2}{11} \\ -\frac{2}{11} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_{1} \overrightarrow{x}_{1} + c_{2} \overrightarrow{x}_{2} + c_{3} \overrightarrow{x}_{3}$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_{1} \mathrm{e}^{-5t} \cdot \left[\begin{array}{c} -1 \\ 1 \\ 0 \end{array}\right] + \mathrm{e}^{3t} c_{2} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right] + c_{3} \mathrm{e}^{6t} \cdot \left[\begin{array}{c} \frac{2}{11} \\ -\frac{2}{11} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} -\left(-c_{2} \mathrm{e}^{8t} - \frac{2 c_{3} \mathrm{e}^{11t}}{11} + c_{1}\right) \mathrm{e}^{-5t} \\ -\frac{\left(2 c_{3} \mathrm{e}^{11t} - 11 c_{2} \mathrm{e}^{8t} - 11 c_{1}\right) \mathrm{e}^{-5t}}{11} \\ c_{3} \mathrm{e}^{6t} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -\left(-c_{2} \mathrm{e}^{8t} - \frac{2 c_{3} \mathrm{e}^{11t}}{11} + c_{1}\right) \mathrm{e}^{-5t}, y = -\frac{\left(2 c_{3} \mathrm{e}^{11t} - 11 c_{2} \mathrm{e}^{8t} - 11 c_{1}\right) \mathrm{e}^{-5t}}{11}, z(t) = c_{3} \mathrm{e}^{6t}\right\}$$

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 58

```
dsolve([diff(x(t),t)=-x(t)+4*y(t)+2*z(t),diff(y(t),t)=4*x(t)-y(t)-2*z(t),diff(z(t),t)=6*z(t)
```

$$
\begin{aligned}
x(t) &= c_2 \mathrm{e}^{3t} + c_1 \mathrm{e}^{-5t} + \frac{2c_3 \mathrm{e}^{6t}}{11} \\
y(t) &= -\frac{2c_3 \mathrm{e}^{6t}}{11} - c_1 \mathrm{e}^{-5t} + c_2 \mathrm{e}^{3t} \\
z(t) &= c_3 \mathrm{e}^{6t}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 105

```
DSolve[{x'[t]==-x[t]+4*y[t]+2*z[t],y'[t]==4*x[t]-y[t]-2*z[t],z'[t]==6*z[t]},{x[t],y[t],z[t]}
```

$$
\begin{aligned}
x(t) &\to \frac{1}{22} e^{-5t} \left(11c_1 \left(e^{8t}+1\right) + 11c_2 \left(e^{8t}-1\right) + 4c_3 \left(e^{11t}-1\right)\right) \\
y(t) &\to \frac{1}{22} e^{-5t} \left(11c_1 \left(e^{8t}-1\right) + 11c_2 \left(e^{8t}+1\right) - 4c_3 \left(e^{11t}-1\right)\right) \\
z(t) &\to c_3 e^{6t}
\end{aligned}
$$

10.14 problem 13

Internal problem ID [6740]
Internal file name [`OUTPUT/5988_Sunday_June_05_2022_04_11_20_PM_13215236/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{x(t)}{2} \\
y' &= x(t) - \frac{y}{2}
\end{aligned}
$$

With initial conditions

$$[x(0) = 4, y(0) = 5]$$

10.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{\frac{t}{2}} & 0 \\ \mathrm{e}^{\frac{t}{2}} - \mathrm{e}^{-\frac{t}{2}} & \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{x}_0\\ &= \begin{bmatrix} e^{\frac{t}{2}} & 0 \\ e^{\frac{t}{2}} - e^{-\frac{t}{2}} & e^{-\frac{t}{2}} \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}\\ &= \begin{bmatrix} 4\,e^{\frac{t}{2}} \\ 4\,e^{\frac{t}{2}} + e^{-\frac{t}{2}} \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} \frac{1}{2} - \lambda & 0 \\ 1 & -\frac{1}{2} - \lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(\frac{1}{2} - \lambda)(-\frac{1}{2} - \lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{1}{2}$	1	real eigenvalue
$-\frac{1}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} - \left(-\frac{1}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} - \left(\frac{1}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{1}{2}$	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$-\frac{1}{2}$	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{1}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\frac{t}{2}} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{\frac{t}{2}}\end{aligned}$$

Since eigenvalue $-\frac{1}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-\frac{t}{2}} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-\frac{t}{2}}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{\frac{t}{2}} \\ \mathrm{e}^{\frac{t}{2}} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{\frac{t}{2}} \\ c_1 \mathrm{e}^{\frac{t}{2}} + c_2 \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 4 \\ y(0) = 5 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 4 \\ c_2 = 1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} 4\,\mathrm{e}^{\frac{t}{2}} \\ 4\,\mathrm{e}^{\frac{t}{2}} + \mathrm{e}^{-\frac{t}{2}} \end{array}\right]$$

The following is the phase plot of the system.

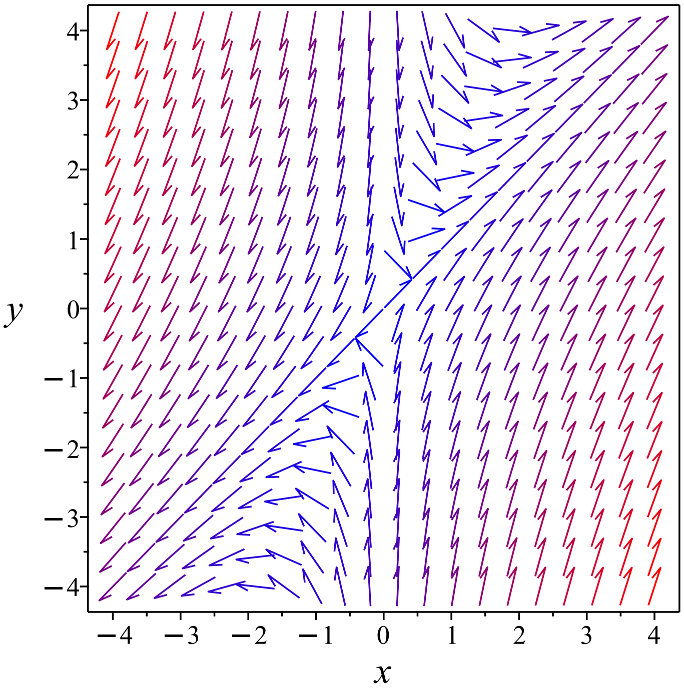
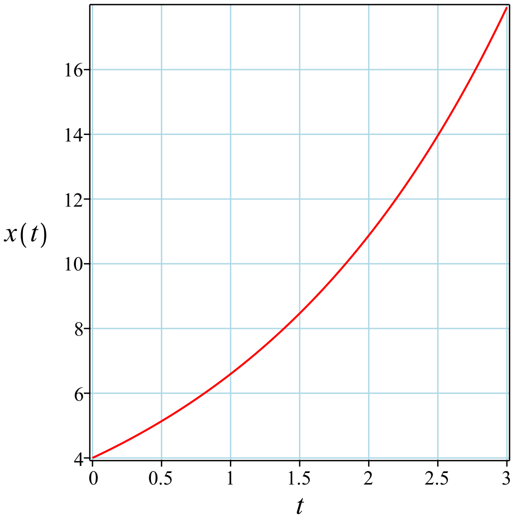
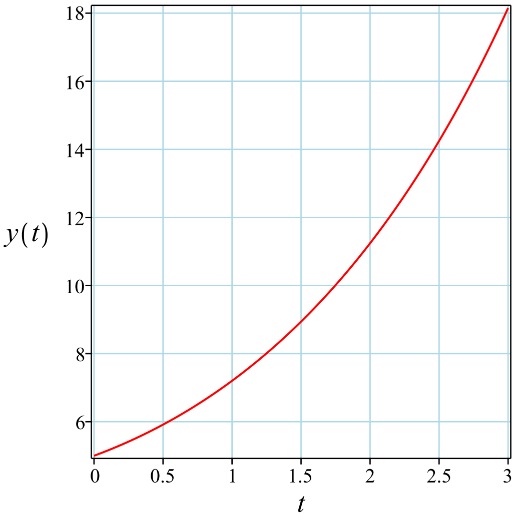


Figure 92: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve([diff(x(t),t) = 1/2*x(t), diff(y(t),t) = x(t)-1/2*y(t), x(0) = 4, y(0) = 5], singsol=
```

$$x(t) = 4\,\mathrm{e}^{\frac{t}{2}}$$
$$y(t) = 4\,\mathrm{e}^{\frac{t}{2}} + \mathrm{e}^{-\frac{t}{2}}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 32

```
DSolve[{x'[t]==1/2*x[t],y'[t]==x[t]-1/2*y[t]},{x[0]==4,y[0]==5},{x[t],y[t]},t,IncludeSingula
```

$$x(t) \to 4e^{t/2}$$
$$y(t) \to e^{-t/2}\left(4e^t + 1\right)$$

10.15 problem 14

Internal problem ID [6741]
Internal file name [`OUTPUT/5989_Sunday_June_05_2022_04_11_22_PM_13291157/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= x(t) + y + 4z(t) \\ y' &= 2y \\ z'(t) &= x(t) + y + z(t) \end{aligned}$$

With initial conditions

$$[x(0) = 1, y(0) = 3, z(0) = 0]$$

10.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}}{2}+\frac{e^{3t}}{2} & \frac{3\,e^{3t}}{2}-\frac{5\,e^{2t}}{3}+\frac{e^{-t}}{6} & e^{3t}-e^{-t} \\ 0 & e^{2t} & 0 \\ \frac{e^{3t}}{4}-\frac{e^{-t}}{4} & \frac{3\,e^{3t}}{4}-\frac{e^{-t}}{12}-\frac{2\,e^{2t}}{3} & \frac{e^{-t}}{2}+\frac{e^{3t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{e^{-t}}{2}+\frac{e^{3t}}{2} & \frac{3\,e^{3t}}{2}-\frac{5\,e^{2t}}{3}+\frac{e^{-t}}{6} & e^{3t}-e^{-t} \\ 0 & e^{2t} & 0 \\ \frac{e^{3t}}{4}-\frac{e^{-t}}{4} & \frac{3\,e^{3t}}{4}-\frac{e^{-t}}{12}-\frac{2\,e^{2t}}{3} & \frac{e^{-t}}{2}+\frac{e^{3t}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t}+5\,e^{3t}-5\,e^{2t} \\ 3\,e^{2t} \\ \frac{5\,e^{3t}}{2}-\frac{e^{-t}}{2}-2\,e^{2t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & 4 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= -1 \\ \lambda_3 &= 2 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{6} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -1 & 1 & 4 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{2}, v_2 = -\frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{2} \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{2} \\ -\frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{2} \\ -\frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{5t}{2} \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{5t}{2} \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_2}{2} \implies \left[\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} \frac{5}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{2t}\\ &= \begin{bmatrix} \frac{5}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ 0 \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -2\,\mathrm{e}^{-t} \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{5\,\mathrm{e}^{2t}}{2} \\ -\frac{3\,\mathrm{e}^{2t}}{2} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} 2c_1\mathrm{e}^{3t} - 2c_2\mathrm{e}^{-t} + \frac{5c_3\mathrm{e}^{2t}}{2} \\ -\frac{3c_3\mathrm{e}^{2t}}{2} \\ c_1\mathrm{e}^{3t} + c_2\mathrm{e}^{-t} + c_3\mathrm{e}^{2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 3 \\ z(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c_1 - 2c_2 + \frac{5c_3}{2} \\ -\frac{3c_3}{2} \\ c_1 + c_2 + c_3 \end{bmatrix}$$

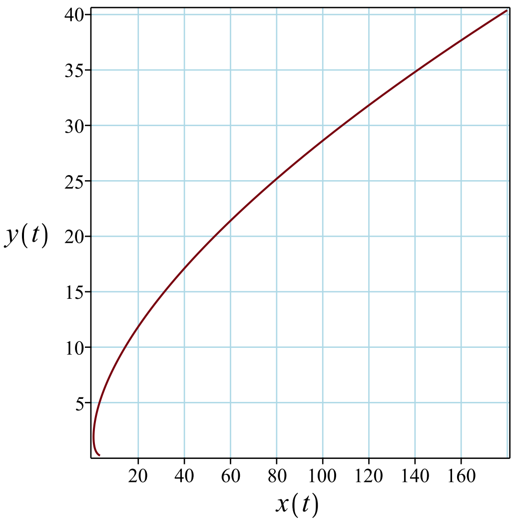
Solving for the constants of integrations gives

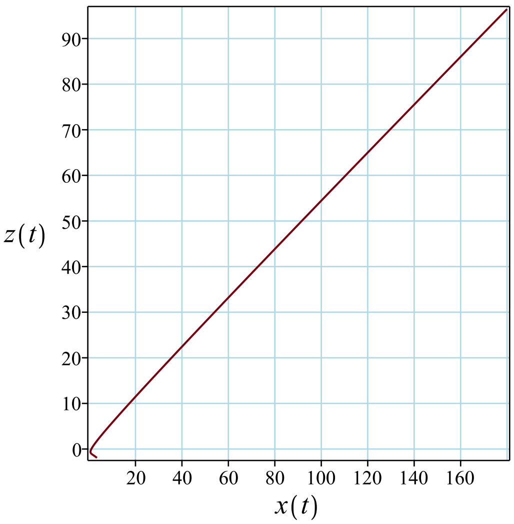
$$\begin{bmatrix} c_1 = \frac{5}{2} \\ c_2 = -\frac{1}{2} \\ c_3 = -2 \end{bmatrix}$$

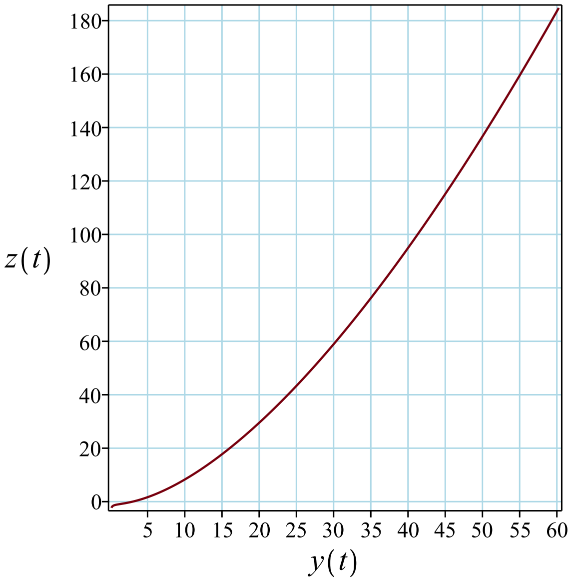
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} e^{-t} + 5\,e^{3t} - 5\,e^{2t} \\ 3\,e^{2t} \\ \frac{5\,e^{3t}}{2} - \frac{e^{-t}}{2} - 2\,e^{2t} \end{bmatrix}$$

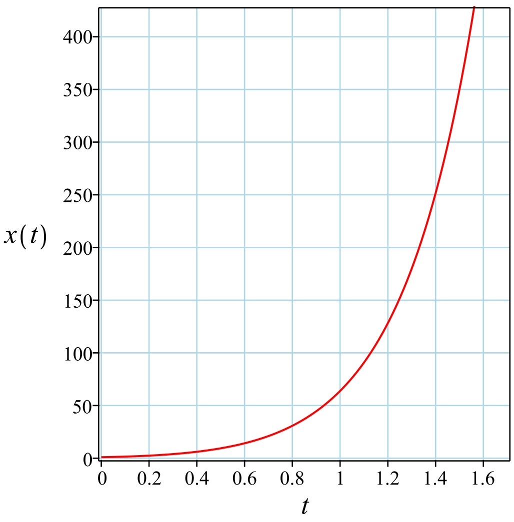
The following are plots of each solution against another.

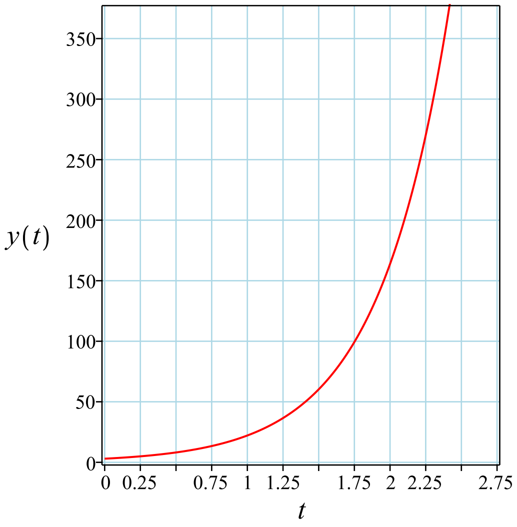


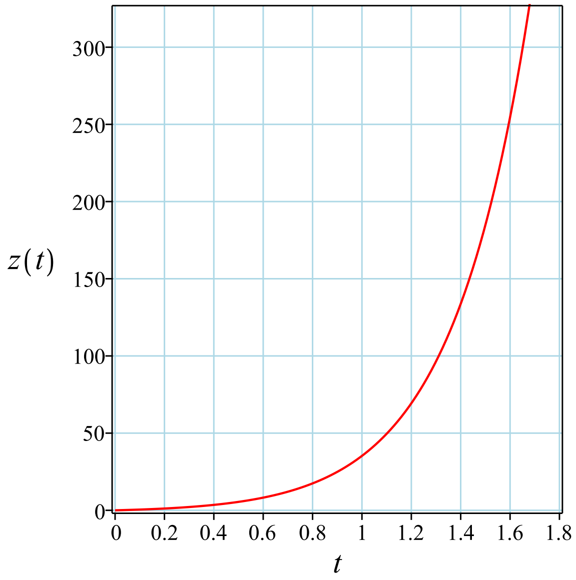




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 53

```
dsolve([diff(x(t),t) = x(t)+y(t)+4*z(t), diff(y(t),t) = 2*y(t), diff(z(t),t) = x(t)+y(t)+z(t
```

$$x(t) = e^{-t} + 5\,e^{3t} - 5\,e^{2t}$$
$$y(t) = 3\,e^{2t}$$
$$z(t) = -\frac{e^{-t}}{2} + \frac{5\,e^{3t}}{2} - 2\,e^{2t}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 63

```
DSolve[{x'[t]==x[t]+y[t]+4*z[t],y'[t]==2*y[t],z'[t]==x[t]+y[t]+z[t]},{x[0]==1,y[0]==3,z[0]==
```

$$x(t) \to e^{-t} - 5e^{2t} + 5e^{3t}$$
$$y(t) \to 3e^{2t}$$
$$z(t) \to \frac{1}{2}e^{-t}\left(-4e^{3t} + 5e^{4t} - 1\right)$$

10.16 problem 15

Internal problem ID [6742]
Internal file name [`OUTPUT/5990_Sunday_June_05_2022_04_11_24_PM_42613464/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{9x(t)}{10} + \frac{21y}{10} + \frac{16z(t)}{5} \\
y' &= \frac{7x(t)}{10} + \frac{13y}{2} + \frac{21z(t)}{5} \\
z'(t) &= \frac{11x(t)}{10} + \frac{17y}{10} + \frac{17z(t)}{5}
\end{aligned}
$$

10.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as Warning. Unable to find the matrix exponential.

10.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} \frac{9}{10} & \frac{21}{10} & \frac{16}{5} \\ \frac{7}{10} & \frac{13}{2} & \frac{21}{5} \\ \frac{11}{10} & \frac{17}{10} & \frac{17}{5} \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} \frac{9}{10} & \frac{21}{10} & \frac{16}{5} \\ \frac{7}{10} & \frac{13}{2} & \frac{21}{5} \\ \frac{11}{10} & \frac{17}{10} & \frac{17}{5} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} \frac{9}{10} - \lambda & \frac{21}{10} & \frac{16}{5} \\ \frac{7}{10} & \frac{13}{2} - \lambda & \frac{21}{5} \\ \frac{11}{10} & \frac{17}{10} & \frac{17}{5} - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \frac{54}{5}\lambda^2 + \frac{472}{25}\lambda + \frac{113}{125} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{\left(329940 + 60i\sqrt{29760999}\right)^{\frac{1}{3}}}{30} + \frac{200}{\left(329940 + 60i\sqrt{29760999}\right)^{\frac{1}{3}}} + \frac{18}{5}$$

$$\lambda_2 = -\frac{\left(329940 + 60i\sqrt{29760999}\right)^{\frac{1}{3}}}{60} - \frac{100}{\left(329940 + 60i\sqrt{29760999}\right)^{\frac{1}{3}}} + \frac{18}{5} + \frac{i\sqrt{3}\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{6} - \right.}{10}$$

$$\lambda_3 = -\frac{\left(329940 + 60i\sqrt{29760999}\right)^{\frac{1}{3}}}{60} - \frac{100}{\left(329940 + 60i\sqrt{29760999}\right)^{\frac{1}{3}}} + \frac{18}{5} - \frac{i\sqrt{3}\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{6} - \right.}{10}$$

This table summarises the above result

eigenvalue	alge
$\frac{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}{30}+\frac{200}{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}+\frac{18}{5}$	1
$-\frac{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}{60}-\frac{100}{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}+\frac{18}{5}+\frac{i\sqrt{3}\left(\frac{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}{6}-\frac{1000}{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}\right)}{10}$	1
$-\frac{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}{60}-\frac{100}{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}+\frac{18}{5}-\frac{i\sqrt{3}\left(\frac{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}{6}-\frac{1000}{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}\right)}{10}$	1

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=\frac{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}{30}+\frac{200}{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}+\frac{18}{5}$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}\frac{9}{10} & \frac{21}{10} & \frac{16}{5}\\ \frac{7}{10} & \frac{13}{2} & \frac{21}{5}\\ \frac{11}{10} & \frac{17}{10} & \frac{17}{5}\end{bmatrix}-\left(\frac{(329940+60i\sqrt{29760999})^{\frac{1}{3}}}{30}\right.\right.$$

$$\begin{bmatrix}\frac{-(329940+60i\sqrt{29760999})^{\frac{2}{3}}-81(329940+60i\sqrt{29760999})^{\frac{1}{3}}-6000}{30(329940+60i\sqrt{29760999})^{\frac{1}{3}}} & \frac{21}{10}\\ \frac{7}{10} & \frac{-(329940+60i\sqrt{29760999})^{\frac{2}{3}}+87(329940+60i\sqrt{29760999})^{\frac{1}{3}}-6000}{30(329940+60i\sqrt{29760999})^{\frac{1}{3}}}\\ \frac{11}{10} & \frac{17}{10}\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented

matrix is

$$\begin{bmatrix} -\frac{27}{10}-\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{30}-\frac{200}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}} & \frac{21}{10} \\ \frac{7}{10} & \frac{29}{10}-\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{30}-\frac{200}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}} \\ \frac{11}{10} & \frac{17}{10} \end{bmatrix}$$

$$R_2 = R_2 - \frac{7R_1}{10\left(-\frac{27}{10}-\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{30}-\frac{200}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}\right)} \Longrightarrow \begin{bmatrix} \frac{-\left(329940+60i\sqrt{29760999}\right)^{\frac{2}{3}}-81\left(32\right.}{30\left(329940+60i\sqrt{2}\right.} \\ 0 \\ \frac{11}{10} \end{bmatrix}$$

$$R_3 = R_3 - \frac{33\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}R_1}{-\left(329940+60i\sqrt{29760999}\right)^{\frac{2}{3}}-81\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}-6000} \Longrightarrow \begin{bmatrix} -\left(329940+60i\sqrt{29}\right. \\ \\ \\ \end{bmatrix}$$

$$R_3 = R_3 + \frac{\left(17\left(329940+60i\sqrt{29760999}\right)^{\frac{2}{3}}+2070\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}+102000\right)\left(329940+60\right.}{2\left(10\left(329940+60i\sqrt{29760999}\right)^{\frac{2}{3}}+810\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}+60000\right)\left(\left(i\sqrt{29760999}\right.\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{cc} \frac{-(329940+60i\sqrt{29760999})^{\frac{2}{3}}-81(329940+60i\sqrt{29760999})^{\frac{1}{3}}-6000}{30(329940+60i\sqrt{29760999})^{\frac{1}{3}}} & \frac{21}{10} \\ 0 & -\frac{2((i\sqrt{29760999}+4899)(329940+60i\sqrt{29760999})^{\frac{1}{3}}-6i\sqrt{2976099}}{(329940+60i\sqrt{29760999})^{\frac{1}{3}}((329940+60i\sqrt{29760999})^{\frac{2}{3}}+81} \\ 0 & 0 \end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{3(329940+60i\sqrt{29760999})^{\frac{1}{3}} t (320i\sqrt{29760999}\,(329940+60i\sqrt{29760999})^{\frac{1}{3}}+2454}{5(2i\sqrt{29760999}\,(329940+60i\sqrt{29760999})^{\frac{1}{3}}-12i\sqrt{29760999}+121(329940+60i\sqrt{29760999})^{\frac{2}{3}}+} \right.$

Hence the solution is

$$\left[\begin{array}{c} \frac{3(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{1}{3}} t (320\,\mathrm{I}\sqrt{29760999}\,(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{1}{3}}+24540\,\mathrm{I}\sqrt{29760999}+62137(329940+60\,\mathrm{I}\sqrt{29}}{5(2\,\mathrm{I}\sqrt{29760999}\,(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{1}{3}}-12\,\mathrm{I}\sqrt{29760999}+121(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{2}{3}}+9798(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{1}{3}}+1134} \\ \frac{21t(60\,\mathrm{I}\sqrt{29760999}+97(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{2}{3}}+6000(329940+60\,\mathrm{I}\sqrt{2976}}{5(2\,\mathrm{I}\sqrt{29760999}\,(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{1}{3}}-12\,\mathrm{I}\sqrt{29760999}+121(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{2}{3}}+979} \\ t \end{array}\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} \frac{3(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{1}{3}} t (320\,\mathrm{I}\sqrt{29760999}\,(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{1}{3}}+24540\,\mathrm{I}\sqrt{29760999}+62137(329940+60\,\mathrm{I}\sqrt{29}}{5(2\,\mathrm{I}\sqrt{29760999}\,(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{1}{3}}-12\,\mathrm{I}\sqrt{29760999}+121(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{2}{3}}+9798(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{1}{3}}+1134} \\ \frac{21t(60\,\mathrm{I}\sqrt{29760999}+97(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{2}{3}}+6000(329940+60\,\mathrm{I}\sqrt{2976}}{5(2\,\mathrm{I}\sqrt{29760999}\,(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{1}{3}}-12\,\mathrm{I}\sqrt{29760999}+121(329940+60\,\mathrm{I}\sqrt{29760999})^{\frac{2}{3}}+979} \\ t \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} \frac{3\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{1}{3}} t \Bigl(320\ \mathrm{I}\sqrt{29760999}\,\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{1}{3}}+24540\ \mathrm{I}\sqrt{29760999}+62137\bigl(329940+60\ \mathrm{I}\sqrt{29}}{5\Bigl(2\ \mathrm{I}\sqrt{29760999}\,\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{1}{3}}-12\ \mathrm{I}\sqrt{29760999}+121\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{2}{3}}+9798\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{1}{3}}+1134} \\ \frac{21t\Bigl(60\ \mathrm{I}\sqrt{29760999}+97\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{2}{3}}+6000\bigl(329940+60\ \mathrm{I}\sqrt{2976}}{5\Bigl(2\ \mathrm{I}\sqrt{29760999}\,\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{1}{3}}-12\ \mathrm{I}\sqrt{29760999}+121\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{2}{3}}+9798} \\ t \end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c} \frac{3\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{1}{3}} t \Bigl(320\ \mathrm{I}\sqrt{29760999}\,\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{1}{3}}+24540\ \mathrm{I}\sqrt{29760999}+62137\bigl(329940+60\ \mathrm{I}\sqrt{29}}{5\Bigl(2\ \mathrm{I}\sqrt{29760999}\,\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{1}{3}}-12\ \mathrm{I}\sqrt{29760999}+121\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{2}{3}}+9798\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{1}{3}}+1134} \\ \frac{21t\Bigl(60\ \mathrm{I}\sqrt{29760999}+97\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{2}{3}}+6000\bigl(329940+60\ \mathrm{I}\sqrt{2976}}{5\Bigl(2\ \mathrm{I}\sqrt{29760999}\,\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{1}{3}}-12\ \mathrm{I}\sqrt{29760999}+121\bigl(329940+60\ \mathrm{I}\sqrt{29760999}\bigr)^{\frac{2}{3}}+9798} \\ t \end{array}\right.$$

Considering the eigenvalue $\lambda_2 = -\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{60} - \frac{100}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}} + \frac{18}{5} - \frac{i\sqrt{3}\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{6} - \frac{1000}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}\right)}{10}$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{9}{10} & \frac{21}{10} & \frac{16}{5} \\ \frac{7}{10} & \frac{13}{2} & \frac{21}{5} \\ \frac{11}{10} & \frac{17}{10} & \frac{17}{5} \end{bmatrix} - \Bigg(-\frac{\left(329940 + 60i\sqrt{29760999}\right)^{\frac{1}{3}}}{60} - \right.$$

$$\left[\begin{array}{cc} -\frac{27}{10} + \frac{2\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right)}{3} - 2\sqrt{5}\,\sin\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) & \\ \frac{7}{10} & \frac{29}{10} + \frac{2\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right)}{3} \\ \frac{11}{10} & \end{array} \right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc} -\frac{27}{10} + \frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{60} + \frac{100}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{6} - \frac{1000}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}\right)}{10} & \\ \frac{7}{10} & \frac{29}{10} + \\ \frac{11}{10} & \end{array} \right.$$

$$R_2 = R_2 - \frac{7R_1}{10\left(-\frac{27}{10} + \frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{60} + \frac{100}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{6} - \frac{1000}{\left(329940+60i\sqrt{2\right.}}\right)}{10}\right.}$$

$$R_3 = R_3 - \frac{11R_1}{10\left(-\frac{27}{10} + \frac{2\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right)}{3} - 2\sqrt{5}\,\sin\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right)\right)} \Longrightarrow \left[\begin{array}{l} -\frac{27}{10} + \frac{2\sqrt{3}}{} \end{array}\right.$$

$$R_3 = R_3 - \frac{\left(34\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) - 102\sqrt{5}\,\sin\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) -\right.}{2\left(20\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) - 60\sqrt{5}\,\sin\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) - 81\right)\left(\left(4899i\sqrt{3} -\right.\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{cc} -\frac{27}{10} + \frac{2\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right)}{3} - 2\sqrt{5}\,\sin\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) & \\ 0 & \frac{2\left(4899i\sqrt{3}-i\sqrt{29760999}-3\sqrt{9920333}-4899\right.}{\left(-162\left(329940+60i\sqrt{2}\right.\right.} \\ 0 & \end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation Expression too large to display

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

Expression too large to displayExpression too large to display

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

Considering the eigenvalue $\lambda_3 = -\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{60} - \frac{100}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}} + \frac{18}{5} + \frac{i\sqrt{3}\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{6} - \frac{1000}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}\right)}{10}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc} \frac{9}{10} & \frac{21}{10} & \frac{16}{5} \\ \frac{7}{10} & \frac{13}{2} & \frac{21}{5} \\ \frac{11}{10} & \frac{17}{10} & \frac{17}{5} \end{array}\right] - \left(-\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{60} - \right.\right.$$

$$\left[\begin{array}{cc} -\frac{27}{10} + \frac{2\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right)}{3} + 2\sqrt{5}\,\sin\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) & \\ \frac{7}{10} & \frac{29}{10} + \frac{2\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right)}{3} \\ \frac{11}{10} & \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc} -\frac{27}{10} + \frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{60} + \frac{100}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{6} - \frac{1000}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}\right)}{10} & \\ \frac{7}{10} & \frac{29}{10} + \\ \frac{11}{10} & \end{array}\right.$$

$$R_2 = R_2 - \frac{7R_1}{10\left(-\frac{27}{10} + \frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{60} + \frac{100}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{6} - \frac{1000}{\left(329940+60i\sqrt{2}\right.}\right.}{10}\right.}$$

$$R_3 = R_3 - \frac{11R_1}{10\left(-\frac{27}{10} + \frac{2\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right)}{3} + 2\sqrt{5}\,\sin\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right)\right)} \implies \left[-\frac{27}{10} + \frac{2\sqrt{3}}{}\right.$$

$$R_3 = R_3 - \frac{\left(34\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) + 102\sqrt{5}\,\sin\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) -\right.}{2\left(20\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) + 60\sqrt{5}\,\sin\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) - 81\right)\left(\left(4899i\sqrt{3} + \right.\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{cc}
-\frac{27}{10}+\frac{2\sqrt{3}\,\sqrt{5}\,\cos\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right)}{3}+2\sqrt{5}\,\sin\left(\frac{\arctan\left(\frac{1833\sqrt{29760999}}{79667}\right)}{6}\right) & \\
0 & \frac{2\left(4899i\sqrt{3}+i\sqrt{29760999}-3\sqrt{9920333}+4899\right.}{\left(162\left(329940+60i\sqrt{29}\right.\right.} \\
0 &
\end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation Expression too large to display

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

Expression too large to displayExpression too large to display

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	
	alge
$\frac{\left(329940+60 i \sqrt{29760999}\right)^{\frac{1}{3}}}{30}+\frac{200}{\left(329940+60 i \sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}$	
$-\frac{\left(329940+60 i \sqrt{29760999}\right)^{\frac{1}{3}}}{60}-\frac{100}{\left(329940+60 i \sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}+\frac{i \sqrt{3}\left(\frac{\left(329940+60 i \sqrt{29760999}\right)^{\frac{1}{3}}}{6}-\frac{1000}{\left(329940+60 i \sqrt{29760999}\right)^{\frac{1}{3}}}\right)}{10}$	
$-\frac{\left(329940+60 i \sqrt{29760999}\right)^{\frac{1}{3}}}{60}-\frac{100}{\left(329940+60 i \sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}-\frac{i \sqrt{3}\left(\frac{\left(329940+60 i \sqrt{29760999}\right)^{\frac{1}{3}}}{6}-\frac{1000}{\left(329940+60 i \sqrt{29760999}\right)^{\frac{1}{3}}}\right)}{10}$	

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \left[\begin{array}{c} \frac{1411\,\mathrm{e}^{\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{30}+\frac{200}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}\right)t}\left(\frac{16\left(329940+60i\sqrt{29}\right.}{3}\right.}{\left(-\frac{334\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{3}-\frac{668000}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}-47413+1050\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{30}+\frac{200}{\left(329940+60i\sqrt{2976}\right.}\right.\right.} \\ -\frac{\mathrm{e}^{\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{30}+\frac{200}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}\right)t}\left(1600\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{30}+\frac{200}{\left(329940+60i\sqrt{29760999}\right)}\right.\right.}{-\frac{334\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{3}-\frac{668000}{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}-47413+1050\left(\frac{\left(329940+60i\sqrt{}\right.}{30}\right.} \\ \mathrm{e}^{\left(\frac{\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}{30}+\frac{200}{\left(329940+60i\sqrt{29760999}\right)}\right.} \end{array}\right.$$

Which becomes

Expression too large to display

10.16.3 Maple step by step solution

Let's solve

$$\left[x'(t) = \frac{9x(t)}{10} + \frac{21y}{10} + \frac{16z(t)}{5}, y' = \frac{7x(t)}{10} + \frac{13y}{2} + \frac{21z(t)}{5}, z'(t) = \frac{11x(t)}{10} + \frac{17y}{10} + \frac{17z(t)}{5}\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} \frac{9}{10} & \frac{21}{10} & \frac{16}{5} \\ \frac{7}{10} & \frac{13}{2} & \frac{21}{5} \\ \frac{11}{10} & \frac{17}{10} & \frac{17}{5} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} \frac{9}{10} & \frac{21}{10} & \frac{16}{5} \\ \frac{7}{10} & \frac{13}{2} & \frac{21}{5} \\ \frac{11}{10} & \frac{17}{10} & \frac{17}{5} \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} \frac{9}{10} & \frac{21}{10} & \frac{16}{5} \\ \frac{7}{10} & \frac{13}{2} & \frac{21}{5} \\ \frac{11}{10} & \frac{17}{10} & \frac{17}{5} \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{30}+\frac{200}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}, \left[\begin{matrix} \overline{\left(-\frac{334\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{3}-\frac{668000}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)}\right.} \\ -\frac{1600\left(\frac{\left(329940+60\,\mathrm{I}\sqrt{2976}\right.}{30}\right.}{-\frac{334\left(329940+60\,\mathrm{I}\sqrt{2976}\right.}{3}} \end{matrix}\right.\right.\right.$$

- Consider eigenpair

$$\left[\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{30}+\frac{200}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5},\ \left[\left(-\frac{334\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{3}-\frac{668000}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}\right.\right.\right.$$

$$-\frac{1600\left(\frac{\left(329940+60\,\mathrm{I}\sqrt{297609}\right.}{30}\right.}{-\frac{334\left(329940+60\,\mathrm{I}\sqrt{297609}\right.}{3}}$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1=\mathrm{e}^{\left(\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{30}+\frac{200}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}\right)t}\cdot\left[\left(-\frac{334\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{3}-\frac{668000}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}\right.\right.$$

$$-\frac{1600\left(\frac{\left(329940+60\,\mathrm{I}\sqrt{29760}\right.}{30}\right.}{-\frac{334\left(329940+60\,\mathrm{I}\sqrt{297609}\right.}{3}}$$

- Consider eigenpair

$$\left[-\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{60}-\frac{100}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{6}-\frac{1000}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)}\right.}{10}\right.$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2=\mathrm{e}^{\left(-\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{60}-\frac{100}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{6}-\frac{1000}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}\right)}{10}\right)t}$$

- Consider eigenpair

$$\left[-\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{60}-\frac{100}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}+\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{6}-\frac{1000}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)}\right.}{10}\right.$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_3=\mathrm{e}^{\left(-\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{60}-\frac{100}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}+\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{6}-\frac{1000}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}\right)}{10}\right)t}$$

- General solution to the system of ODEs

  $\overrightarrow{x}=c_1\overrightarrow{x}_1+c_2\overrightarrow{x}_2+c_3\overrightarrow{x}_3$
- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{\left(\frac{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{30}+\frac{200}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}+\frac{18}{5}\right)t} \cdot \left[\begin{array}{l} \frac{}{\left(-\frac{334\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{1}{3}}}{3}-\frac{668000}{\left(329940+60\,\mathrm{I}\sqrt{29760999}\right.}\right.} \\ -\frac{1600\left(\frac{\left(329940+60\,\mathrm{I}\sqrt{2976\right.}}{30}\right.}{-\frac{334\left(329940+60\,\mathrm{I}\sqrt{2976\right.}}{3}} \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{l} -\frac{10217142\left(c_1\left(\left(\frac{102037659}{567619}-\frac{209431\,\mathrm{I}\sqrt{29760999}}{5108571}\right)\left(329940+60\,\mathrm{I}\sqrt{29760999}\right)^{\frac{2}{3}}+\left(\frac{79667}{1833}+\mathrm{I}\sqrt{29760999}\right)\left(329940+60\right.\right.\right.}{\frac{\left(-2363719890\,\mathrm{I}+200\left(-30483501\,\mathrm{I}+10999\sqrt{3}\,\sqrt{9920333}\right)\left(329940+60\,\mathrm{I}\sqrt{3}\,\sqrt{9920333}\right)\right.}{}} \end{array}\right.$$

- Solution to the system of ODEs

$$\begin{cases} x(t) = -\frac{10217142 \Bigg(c_1 \bigg(\Big(\frac{102037659}{567619} - \frac{209431\,\mathrm{I}\sqrt{29760999}}{5108571} \Big) \Big(329940+60\,\mathrm{I}\sqrt{29760999} \Big)^{\frac{2}{3}} + \Big(\frac{79667}{1833} + \mathrm{I}\sqrt{29760999} \Big) \Big(329940+60\,\mathrm{I}\sqrt{2976}}{} \end{cases}$$

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 1014

```
dsolve([diff(x(t),t)=9/10*x(t)+21/10*y(t)+32/10*z(t),diff(y(t),t)=7/10*x(t)+65/10*y(t)+42/10
```

$$x(t) = c_1 \mathrm{e}^{-\frac{\left(i\sqrt{3}\,\left(329940+60i\sqrt{29760999}\right)^{\frac{2}{3}}-6000i\sqrt{3}+\left(329940+60i\sqrt{29760999}\right)^{\frac{2}{3}}-216\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}+6000\right)t}{60\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}}$$

$$+ c_2 \mathrm{e}^{\frac{\left(i\sqrt{3}\,\left(329940+60i\sqrt{29760999}\right)^{\frac{2}{3}}-6000i\sqrt{3}-\left(329940+60i\sqrt{29760999}\right)^{\frac{2}{3}}+216\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}-6000\right)t}{60\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}}$$

$$+ c_3 \mathrm{e}^{\frac{\left(\left(329940+60i\sqrt{29760999}\right)^{\frac{2}{3}}+108\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}+6000\right)t}{30\left(329940+60i\sqrt{29760999}\right)^{\frac{1}{3}}}}$$

$y(t)$

$$= \frac{(8i\sqrt{3}\,\left(329940+60i\sqrt{29760999}\right)^{\frac{4}{3}} - 8\left(329940+60i\sqrt{29760999}\right)^{\frac{4}{3}} + 207000i\sqrt{3}\,(329940+60i\sqrt{29760}}{}$$

$$- \frac{(8i\sqrt{3}\,\left(329940+60i\sqrt{29760999}\right)^{\frac{4}{3}} + 8\left(329940+60i\sqrt{29760999}\right)^{\frac{4}{3}} + 207000i\sqrt{3}\,(329940+60i\sqrt{297}}{}$$

$$+ \frac{(16\left(329940+60i\sqrt{29760999}\right)^{\frac{4}{3}} + 20757\left(329940+60i\sqrt{29760999}\right)^{\frac{2}{3}} + 4140i\sqrt{29760999} + 414000(3}{88731\,(329940+60i\sqrt{}}$$

$z(t) =$

$$- \frac{(7i\sqrt{3}\,\left(329940+60i\sqrt{29760999}\right)^{\frac{4}{3}} - 7\left(329940+60i\sqrt{29760999}\right)^{\frac{4}{3}} - 3516000i\sqrt{3}\,(329940+60i\sqrt{29}}{}$$

$$+ \frac{(7i\sqrt{3}\,\left(329940+60i\sqrt{29760999}\right)^{\frac{4}{3}} + 7\left(329940+60i\sqrt{29760999}\right)^{\frac{4}{3}} - 3516000i\sqrt{3}\,(329940+60i\sqrt{29}}{}$$

$$- \frac{(7\left(329940+60i\sqrt{29760999}\right)^{\frac{4}{3}} - 40830\left(329940+60i\sqrt{29760999}\right)^{\frac{2}{3}} - 35160i\sqrt{29760999} - 3516000(}{59154\,(329940+60i\sqrt{}}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 616

```
DSolve[{x'[t]==9/10*x[t]+21/10*y[t]+32/10*z[t],y'[t]==7/10*x[t]+65/10*y[t]+42/10*z[t],z'[t]=
```

$$
\begin{aligned}
x(t) \to{}& 2c_3\text{RootSum}\left[\#1^3-108\#1^2+1888\#1+904\&,\frac{16\#1e^{\frac{\#1t}{10}}-599e^{\frac{\#1t}{10}}}{3\#1^2-216\#1+1888}\&\right]\\
&+c_2\text{RootSum}\left[\#1^3-108\#1^2+1888\#1+904\&,\frac{21\#1e^{\frac{\#1t}{10}}-170e^{\frac{\#1t}{10}}}{3\#1^2-216\#1+1888}\&\right]\\
&+c_1\text{RootSum}\left[\#1^3-108\#1^2+1888\#1\right.\\
&\left.+904\&,\frac{\#1^2e^{\frac{\#1t}{10}}-99\#1e^{\frac{\#1t}{10}}+1496e^{\frac{\#1t}{10}}}{3\#1^2-216\#1+1888}\&\right]\\
y(t) \to{}& 7c_1\text{RootSum}\left[\#1^3-108\#1^2+1888\#1+904\&,\frac{\#1e^{\frac{\#1t}{10}}+32e^{\frac{\#1t}{10}}}{3\#1^2-216\#1+1888}\&\right]\\
&+14c_3\text{RootSum}\left[\#1^3-108\#1^2+1888\#1+904\&,\frac{3\#1e^{\frac{\#1t}{10}}-11e^{\frac{\#1t}{10}}}{3\#1^2-216\#1+1888}\&\right]\\
&+c_2\text{RootSum}\left[\#1^3-108\#1^2+1888\#1\right.\\
&\left.+904\&,\frac{\#1^2e^{\frac{\#1t}{10}}-43\#1e^{\frac{\#1t}{10}}-46e^{\frac{\#1t}{10}}}{3\#1^2-216\#1+1888}\&\right]\\
z(t) \to{}& c_1\text{RootSum}\left[\#1^3-108\#1^2+1888\#1+904\&,\frac{11\#1e^{\frac{\#1t}{10}}-596e^{\frac{\#1t}{10}}}{3\#1^2-216\#1+1888}\&\right]\\
&+c_2\text{RootSum}\left[\#1^3-108\#1^2+1888\#1+904\&,\frac{17\#1e^{\frac{\#1t}{10}}+78e^{\frac{\#1t}{10}}}{3\#1^2-216\#1+1888}\&\right]\\
&+c_3\text{RootSum}\left[\#1^3-108\#1^2+1888\#1\right.\\
&\left.+904\&,\frac{\#1^2e^{\frac{\#1t}{10}}-74\#1e^{\frac{\#1t}{10}}+438e^{\frac{\#1t}{10}}}{3\#1^2-216\#1+1888}\&\right]
\end{aligned}
$$

10.17 problem 16

Internal problem ID [6743]
Internal file name [`OUTPUT/5991_Sunday_June_05_2022_04_12_11_PM_19657200/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + 2x_3(t) - \frac{9x_4(t)}{5} \\
x_2'(t) &= \frac{51x_2(t)}{10} - x_4(t) + 3x_5(t) \\
x_3'(t) &= x_1(t) + 2x_2(t) - 3x_3(t) \\
x_4'(t) &= x_2(t) - \frac{31x_3(t)}{10} + 4x_4(t) \\
x_5'(t) &= -\frac{14x_1(t)}{5} + \frac{3x_4(t)}{2} - x_5(t)
\end{aligned}
$$

10.17.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as Warning. Unable to find the matrix exponential.

10.17.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \\ x_5'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & -\frac{9}{5} & 0 \\ 0 & \frac{51}{10} & 0 & -1 & 3 \\ 1 & 2 & -3 & 0 & 0 \\ 0 & 1 & -\frac{31}{10} & 4 & 0 \\ -\frac{14}{5} & 0 & 0 & \frac{3}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 2 & -\frac{9}{5} & 0 \\ 0 & \frac{51}{10} & 0 & -1 & 3 \\ 1 & 2 & -3 & 0 & 0 \\ 0 & 1 & -\frac{31}{10} & 4 & 0 \\ -\frac{14}{5} & 0 & 0 & \frac{3}{2} & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 2 & -\frac{9}{5} & 0 \\ 0 & \frac{51}{10}-\lambda & 0 & -1 & 3 \\ 1 & 2 & -3-\lambda & 0 & 0 \\ 0 & 1 & -\frac{31}{10} & 4-\lambda & 0 \\ -\frac{14}{5} & 0 & 0 & \frac{3}{2} & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^5 - \frac{61}{10}\lambda^4 - \frac{89}{10}\lambda^3 + \frac{3511}{50}\lambda^2 + \frac{20779}{500}\lambda - \frac{81879}{500} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 1\right)$$
$$\lambda_2 = \text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 2\right)$$
$$\lambda_3 = \text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 3\right)$$
$$\lambda_4 = \text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 4\right)$$
$$\lambda_5 = \text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 5\right)$$

This table summarises the above result

eigenvalue	algebraic multi
$\text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 3\right)$	1
$\text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 2\right)$	1
$\text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 1\right)$	1
$\text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 5\right)$	1
$\text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 4\right)$	1

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 8187\right.$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left[\begin{array}{cc} 1 - \text{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 1\right) & \\ 0 & \frac{51}{10} - \text{Root} \\ 1 & \\ 0 & \\ -\frac{14}{5} & \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} 1-\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=1\right) & \\ 0 & \frac{51}{10}-\mathrm{RootC} \\ 1 & \\ 0 & \\ -\frac{14}{5} & \end{bmatrix}$$

$$R_3 = R_3 - \frac{R_1}{1-\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=1\right)} =$$

$$R_5 = R_5 + \frac{14R_1}{5\left(1-\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=1\right)\right)}$$

$$R_3 = R_3 - \frac{2R_2}{\frac{51}{10}-\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=1\right)} =$$

$$R_4 = R_4 - \frac{R_2}{\frac{51}{10} - \mathrm{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \mathrm{index} = 1\right)} =$$

$$R_4 = R_4 + \frac{}{10\left(\frac{100\,\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=1\right)^4}{3299} - \frac{510\,\mathrm{RootOf}\left(500\_Z^5-305\right.}{}\right.}$$

$$R_5 = R_5 - \frac{\left(-\frac{280\,\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=1\right)^4}{3299} + \frac{1428\,\mathrm{RootOf}\left(500\_Z^5-305\right.}{}\right.}{\frac{100\,\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=1\right)^4}{3299} - \frac{510\,\mathrm{RootOf}\left(500\_Z^5-30\right.}{}}$$

$$R_5 = R_5 - \frac{\left(\frac{162168785240\,\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=1\right)^4}{151160503941} - \frac{766665001004\,\mathrm{Root}}{}\right.}{\frac{14024369740\,\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=1\right)^4}{50386834647} - \frac{124603878473\,\mathrm{Ro}}{}}$$

Therefore the system in Echelon form is

$$\text{Expression too large to display}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation Expression too large to display

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

Expression too large to displayExpression too large to display

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

Considering the eigenvalue $\lambda_2 = \operatorname{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 8187$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{bmatrix} 1 - \operatorname{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \text{index} = 2\right) & \\ 0 & \frac{51}{10} - \text{Root} \\ 1 & \\ 0 & \\ -\frac{14}{5} & \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc}1-\mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=2\right) & \\ 0 & \frac{51}{10}-\mathrm{RootO} \\ 1 & \\ 0 & \\ -\frac{14}{5} & \end{array}\right.$$

$$R_3=R_3-\frac{R_1}{1-\mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=2\right)}=$$

$$R_5=R_5+\frac{14R_1}{5\left(1-\mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=2\right)\right)}$$

$$R_3=R_3-\frac{2R_2}{\frac{51}{10}-\mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=2\right)}=$$

$$R_4 = R_4 - \frac{R_2}{\frac{51}{10} - \mathrm{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \mathrm{index} = 2\right)} =$$

$$R_4 = R_4 + \frac{}{10\left(\frac{100\,\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=2\right)^4}{3299} - \frac{510\,\mathrm{RootOf}\left(500\_Z^5-305\right.}{}\right.}$$

$$R_5 = R_5 - \frac{\left(-\frac{280\,\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=2\right)^4}{3299} + \frac{1428\,\mathrm{RootOf}\left(500\_Z^5-305\right.}{}\right.}{\frac{100\,\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=2\right)^4}{3299} - \frac{510\,\mathrm{RootOf}\left(500\_Z^5-30\right.}{}}$$

$$R_5 = R_5 - \frac{\left(\frac{162168785240\,\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=2\right)^4}{151160503941} - \frac{766665001004\,\mathrm{Root}}{}\right.}{\frac{14024369740\,\mathrm{RootOf}\left(500\_Z^5-3050\_Z^4-4450\_Z^3+35110\_Z^2+20779\_Z-81879,\mathrm{index}=2\right)^4}{50386834647} - \frac{124603878473\,\mathrm{Ro}}{}}$$

Therefore the system in Echelon form is

$$\text{Expression too large to display}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation Expression too large to display

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$$\left[\begin{array}{cc} 1 - \mathrm{RootOf}\left(500\_Z^5 - 3050\_Z^4 - 4450\_Z^3 + 35110\_Z^2 + 20779\_Z - 81879, \mathrm{index} = 3\right) & \\ 0 & \frac{51}{10} - \mathrm{Root} \\ 1 & \\ 0 & \\ -\frac{14}{5} & \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

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$$R_3 = R_3 - \frac{R_1}{1-\mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=3\right)} =$$

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$$R_5 = R_5 - \frac{\left(-\frac{280\,\mathrm{RootOf}\left(500\_\_Z^5-3050\_\_Z^4-4450\_\_Z^3+35110\_\_Z^2+20779\_\_Z-81879,\mathrm{index}=3\right)^4}{3299} + \frac{1428\,\mathrm{RootOf}\left(500\_\_Z^5-305\right.}{}\right.}{\frac{100\,\mathrm{RootOf}\left(500\_\_Z^5-3050\_\_Z^4-4450\_\_Z^3+35110\_\_Z^2+20779\_\_Z-81879,\mathrm{index}=3\right)^4}{3299} - \frac{510\,\mathrm{RootOf}\left(500\_\_Z^5-30\right.}{}}$$

$$R_5 = R_5 - \frac{\left(\frac{162168785240\,\mathrm{RootOf}\left(500\_\_Z^5-3050\_\_Z^4-4450\_\_Z^3+35110\_\_Z^2+20779\_\_Z-81879,\mathrm{index}=3\right)^4}{151160503941} - \frac{766665001004\,\mathrm{RootC}}{}\right.}{\frac{14024369740\,\mathrm{RootOf}\left(500\_\_Z^5-3050\_\_Z^4-4450\_\_Z^3+35110\_\_Z^2+20779\_\_Z-81879,\mathrm{index}=3\right)^4}{50386834647} - \frac{124603878473\,\mathrm{Ro}}{}}$$

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Expression too large to displayExpression too large to display

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multipli	
	algebraic m	g
$\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=1\right)$	1	
$\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=2\right)$	1	
$\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=3\right)$	1	
$\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=4\right)$	1	
$\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=5\right)$	1	

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\right.$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=1\right)t}\\ &= \text{Expression too large to display} e^{\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=1\right)t}\end{aligned}$$

Since eigenvalue $\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=2\right)$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=2\right)t}\\ &= \text{Expression too large to display} e^{\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=2\right)t}\end{aligned}$$

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$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=3\right)t}\\ &= \text{Expression too large to display} e^{\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=3\right)t}\end{aligned}$$

Since eigenvalue $\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=4\right)$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_4(t) &= \vec{v}_4 e^{\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=4\right)t}\\ &= \text{Expression too large to display} e^{\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=4\right)t}\end{aligned}$$

Since eigenvalue $\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=5\right)$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_5(t) &= \vec{v}_5 e^{\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=5\right)t}\\ &= \text{Expression too large to display} e^{\operatorname{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \text{index}=5\right)t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t) + c_5\vec{x}_5(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} = c_1\text{Expression too large to display} + c_2\text{Expression too large to display} + c_3\text{Expression too la}$$

Which becomes

$$\text{Expression too large to display}$$

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 1389

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dsolve([diff(x__1(t),t)=x__1(t)+2*x__3(t)-18/10*x__4(t),diff(x__2(t),t)=51/10*x__2(t)-x__4(t
```

$$x_1(t) = \frac{1334393\Bigg(\sum_{\_a=1}^{5} \mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a\right)}{1052220} + \frac{3565\Bigg(\sum_{\_a=1}^{5} \mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a\right)^{4}}{105222} - \frac{29663\Bigg(\sum_{\_a=1}^{5} \mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a\right)^{3}}{210444} - \frac{49145\Bigg(\sum_{\_a=1}^{5} \mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a\right)^{2}}{105222} + \frac{20174057\left(\sum_{\_a=1}^{5} e^{\mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a\right)t}\_C\_a\right)}{10522200}$$

$$x_2(t) = -\frac{16027\Bigg(\sum_{\_a=1}^{5} \mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a\right)^{3}}{420888} + \frac{10385\Bigg(\sum_{\_a=1}^{5} \mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a\right)^{4}}{210444} - \frac{210025\Bigg(\sum_{\_a=1}^{5} \mathrm{RootOf}\Big(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a}{210444} - \frac{371003\Bigg(\sum_{\_a=1}^{5} \mathrm{RootOf}\Big(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a}{2104440} + \frac{92973853\left(\sum_{\_a=1}^{5} e^{\mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a\right)t}\_C\_a\right)}{21044400}$$

$$x_3(t) = -\frac{33875\Bigg(\sum_{\_a=1}^{5} \mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a\right)^{2}}{105222} - \frac{2585\Bigg(\sum_{\_a=1}^{5} \mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a\right)^{3}}{210444}$$

$$79853\Bigg(\sum_{\_a=1}^{5} \mathrm{RootOf}\left(500\_Z^{5}-3050\_Z^{4}-4450\_Z^{3}+35110\_Z^{2}+20779\_Z-81879, \mathrm{index}=\_a\right)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 2839

```
DSolve[{x1'[t]==x1[t]+2*x3[t]-18/10*x4[t],x2'[t]==51/10*x2[t]-x4[t]+3*x5[t],x3'[t]==x1[t]+2
```

Too large to display

10.18 problem 19

Internal problem ID [6744]
Internal file name [`OUTPUT/5992_Sunday_June_05_2022_04_12_55_PM_33839197/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 3x(t) - y \\ y' &= 9x(t) - 3y \end{aligned}$$

10.18.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c}\\ &= \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\\ &= \begin{bmatrix} (1+3t)\,c_1 - tc_2 \\ 9tc_1 + (1-3t)\,c_2 \end{bmatrix}\\ &= \begin{bmatrix} (3c_1 - c_2)\,t + c_1 \\ (9c_1 - 3c_2)\,t + c_2 \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.18.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 9 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	2	1	Yes	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

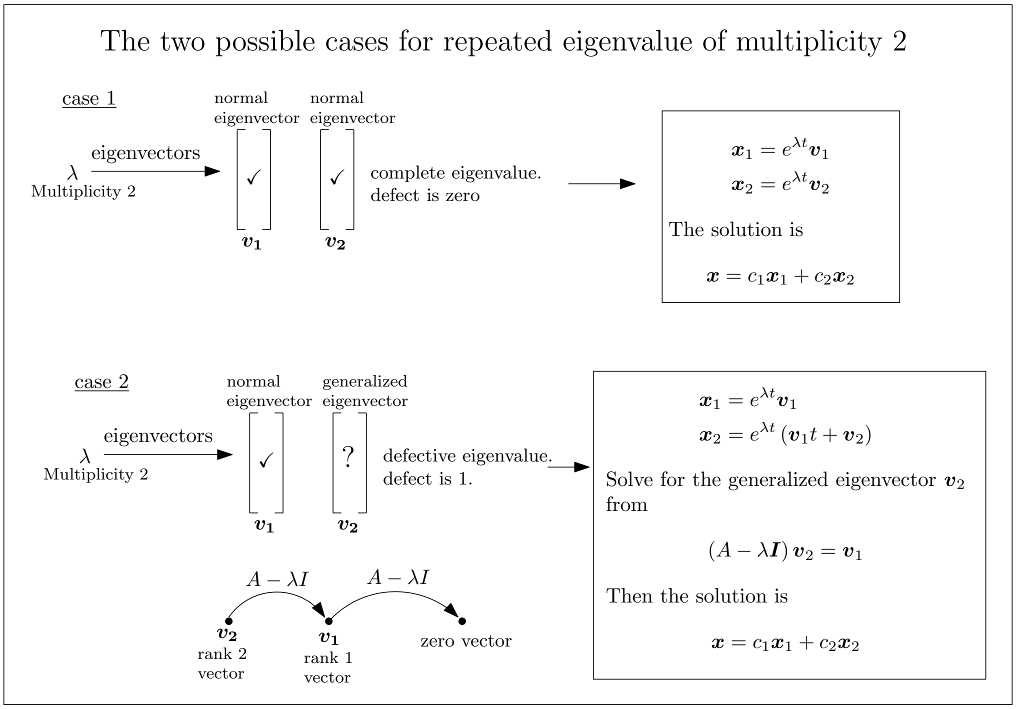


Figure 93: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{8}{3} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix} 1\\ &= \begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix} t + \begin{bmatrix}1\\ \frac{8}{3}\end{bmatrix}\right) 1\\ &= \begin{bmatrix}\frac{t}{3}+1\\ t+\frac{8}{3}\end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix}x(t)\\ y\end{bmatrix} = c_1 \begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix} + c_2 \begin{bmatrix}\frac{t}{3}+1\\ t+\frac{8}{3}\end{bmatrix}$$

Which becomes

$$\begin{bmatrix}x(t)\\ y\end{bmatrix} = \begin{bmatrix}\frac{1}{3}c_1 + \frac{1}{3}c_2 t + c_2\\ c_1 + c_2 t + \frac{8}{3}c_2\end{bmatrix}$$

The following is the phase plot of the system.

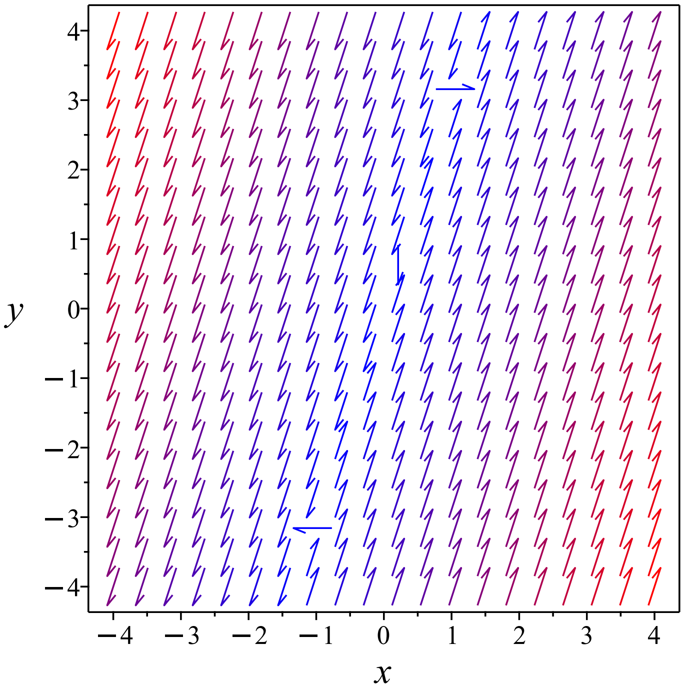


Figure 94: Phase plot

10.18.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) - y, y' = 9x(t) - 3y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 3 & -1 \\ 9 & -3 \end{array}\right]$$

- Rewrite the system as

  $$\vec{x}^{\prime}\left(t \right) = A \cdot \vec{x}\left(t \right)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[0, \left[\begin{array}{c} \frac{1}{3} \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

  $$\left[0, \left[\begin{array}{c} \frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_{1} = \left[\begin{array}{c} \frac{1}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

  $$\left[0, \left[\begin{array}{c} 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_{2} = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

- General solution to the system of ODEs

  $$\vec{x} = c_{1} \vec{x}_{1} + c_{2} \vec{x}_{2}$$

- Substitute solutions into the general solution

  $$\vec{x} = \left[\begin{array}{c} \frac{c_{1}}{3} \\ c_{1} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1}{3} \\ c_1 \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x(t) = \frac{c_1}{3}, y = c_1\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve([diff(x(t),t)=3*x(t)-y(t),diff(y(t),t)=9*x(t)-3*y(t)],singsol=all)
```

$$x(t) = c_1 t + c_2$$
$$y(t) = 3c_1 t - c_1 + 3c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 34

```
DSolve[{x'[t]==3*x[t]-y[t],y'[t]==9*x[t]-3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to 3c_1 t - c_2 t + c_1$$
$$y(t) \to 9c_1 t - 3c_2 t + c_2$$

10.19 problem 20

Internal problem ID [6745]
Internal file name [`OUTPUT/5993_Sunday_June_05_2022_04_12_57_PM_25650586/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -6x(t) + 5y \\ y' &= -5x(t) + 4y \end{aligned}$$

10.19.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t}(1-5t) & 5t\,\mathrm{e}^{-t} \\ -5t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1+5t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-t}(1-5t) & 5t\,\mathrm{e}^{-t} \\ -5t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1+5t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t}(1-5t)\,c_1 + 5t\,\mathrm{e}^{-t}c_2 \\ -5t\,\mathrm{e}^{-t}c_1 + \mathrm{e}^{-t}(1+5t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-5t) + 5c_2t)\,\mathrm{e}^{-t} \\ (c_2(1+5t) - 5tc_1)\,\mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.19.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -5 & 5 & 0 \\ -5 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -5 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

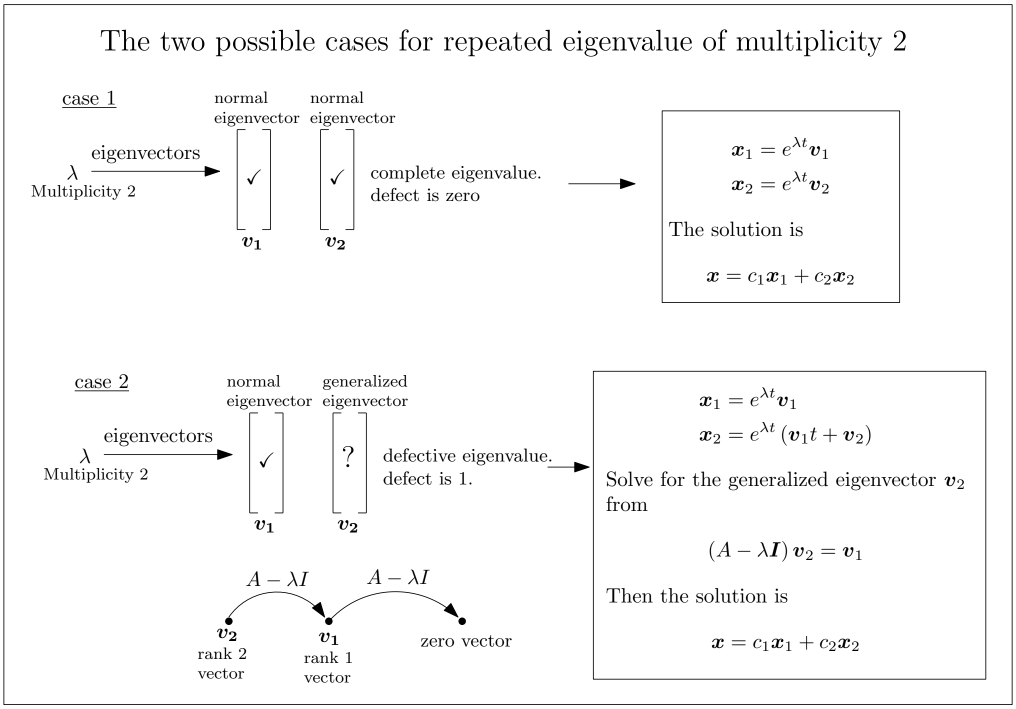


Figure 95: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\begin{aligned}\left(\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}\end{aligned}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{4}{5} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{-t}\\ &= \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{4}{5} \\ 1 \end{bmatrix}\right) \mathrm{e}^{-t}\\ &= \begin{bmatrix} \frac{\mathrm{e}^{-t}(5t+4)}{5} \\ (t+1)\,\mathrm{e}^{-t} \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t}\left(t + \frac{4}{5}\right) \\ (t+1)\,\mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-t}\left(c_1 + c_2 t + \frac{4}{5} c_2\right) \\ \mathrm{e}^{-t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

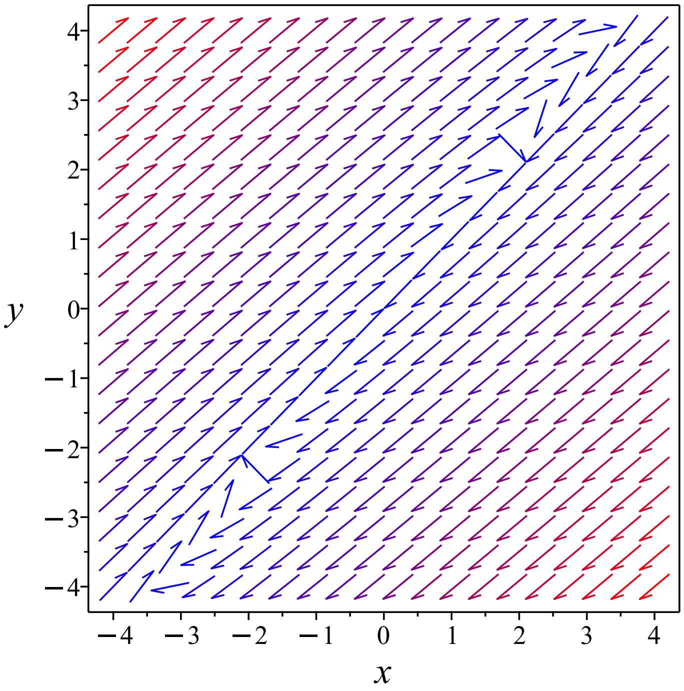


Figure 96: Phase plot

10.19.3 Maple step by step solution

Let's solve

$[x'(t) = -6x(t) + 5y, y' = -5x(t) + 4y]$

- Define vector

$$\vec{x}(t) = \left[\begin{array}{c} x(t) \\ y \end{array}\right]$$

- Convert system into a vector equation

$$\vec{x}'(t) = \left[\begin{array}{cc} -6 & 5 \\ -5 & 4 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{cc} -6 & 5 \\ -5 & 4 \end{array}\right] \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} -6 & 5 \\ -5 & 4 \end{array}\right]$$

- Rewrite the system as

  $\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} 0 \\ 0 \end{array}\right]\right]\right]$$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-1, \left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right]$$
- First solution from eigenvalue -1

  $$\overrightarrow{x}_{1}(t) = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$
- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

  $\overrightarrow{x}_{2}(t) = \mathrm{e}^{\lambda t} \left(t \overrightarrow{v} + \overrightarrow{p}\right)$
- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_{2}(t)$ into the homogeneous system

  $\lambda \,\mathrm{e}^{\lambda t} \left(t \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t} \overrightarrow{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t \overrightarrow{v} + \overrightarrow{p}\right)$
- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda \,\mathrm{e}^{\lambda t} \left(t \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t} \overrightarrow{v} = \mathrm{e}^{\lambda t} \left(\lambda t \overrightarrow{v} + A \cdot \overrightarrow{p}\right)$
- Simplify equation

  $\lambda \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$
- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$
- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_{2}(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{cc} -6 & 5 \\ -5 & 4 \end{array}\right]-(-1)\cdot\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c} -\frac{1}{5} \\ 0 \end{array}\right]$$

- Second solution from eigenvalue -1

$$\overrightarrow{x}_{2}(t)=\mathrm{e}^{-t}\cdot\left(t\cdot\left[\begin{array}{c} 1 \\ 1 \end{array}\right]+\left[\begin{array}{c} -\frac{1}{5} \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x}=c_{1}\overrightarrow{x}_{1}(t)+c_{2}\overrightarrow{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x}=\mathrm{e}^{-t}c_{1}\cdot\left[\begin{array}{c} 1 \\ 1 \end{array}\right]+c_{2}\mathrm{e}^{-t}\cdot\left(t\cdot\left[\begin{array}{c} 1 \\ 1 \end{array}\right]+\left[\begin{array}{c} -\frac{1}{5} \\ 0 \end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right]=\left[\begin{array}{c} \mathrm{e}^{-t}\left(c_{1}+tc_{2}-\frac{1}{5}c_{2}\right) \\ \mathrm{e}^{-t}(tc_{2}+c_{1}) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t)=\mathrm{e}^{-t}\left(c_{1}+tc_{2}-\frac{1}{5}c_{2}\right), y=\mathrm{e}^{-t}(tc_{2}+c_{1})\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve([diff(x(t),t)=-6*x(t)+5*y(t),diff(y(t),t)=-5*x(t)+4*y(t)],singsol=all)
```

$$x(t)=\mathrm{e}^{-t}(c_{2}t+c_{1})$$

$$y(t)=\frac{\mathrm{e}^{-t}(5c_{2}t+5c_{1}+c_{2})}{5}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{x'[t]==-6*x[t]+5*y[t],y'[t]==-5*x[t]+4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$
\begin{aligned}
x(t) &\to e^{-t}(-5c_1t + 5c_2t + c_1) \\
y(t) &\to e^{-t}(-5c_1t + 5c_2t + c_2)
\end{aligned}
$$

10.20 problem 21

Internal problem ID [6746]
Internal file name [`OUTPUT/5994_Sunday_June_05_2022_04_12_58_PM_51456075/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -x(t) + 3y \\
y' &= -3x(t) + 5y
\end{aligned}
$$

10.20.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{2t}(1-3t) & 3t\,\mathrm{e}^{2t} \\ -3t\,\mathrm{e}^{2t} & \mathrm{e}^{2t}(1+3t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{2t}(1-3t) & 3t\,\mathrm{e}^{2t} \\ -3t\,\mathrm{e}^{2t} & \mathrm{e}^{2t}(1+3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{2t}(1-3t)\,c_1 + 3t\,\mathrm{e}^{2t}c_2 \\ -3t\,\mathrm{e}^{2t}c_1 + \mathrm{e}^{2t}(1+3t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-3t) + 3c_2t)\,\mathrm{e}^{2t} \\ (c_2(1+3t) - 3tc_1)\,\mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.20.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

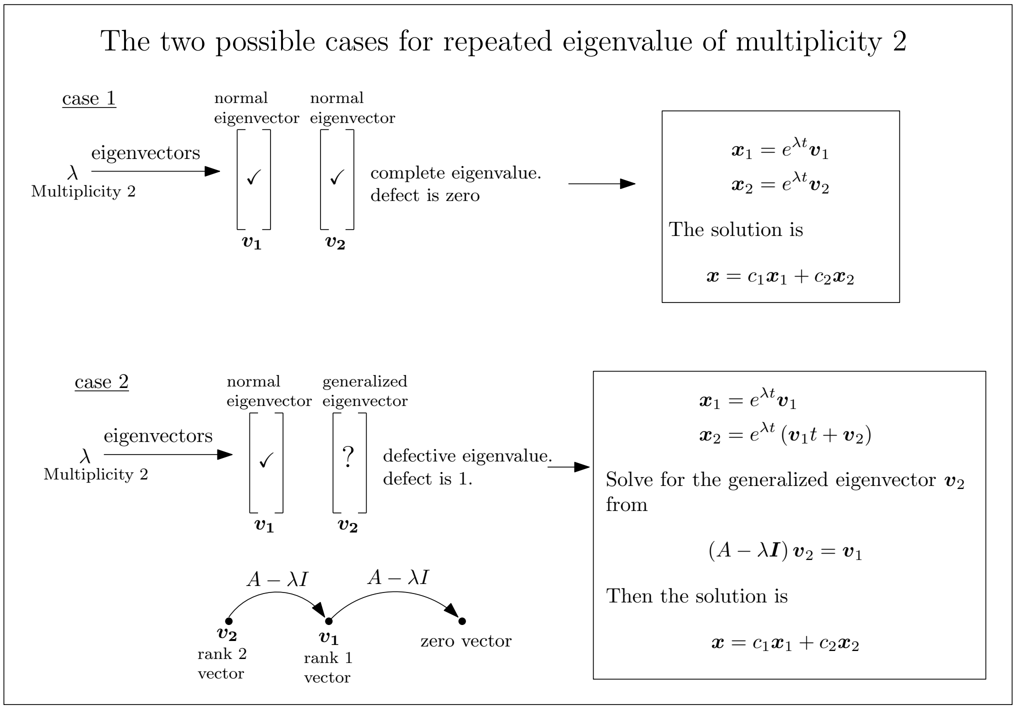


Figure 97: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix}1\\1\end{bmatrix} \mathrm{e}^{2t}\\ &= \begin{bmatrix}\mathrm{e}^{2t}\\ \mathrm{e}^{2t}\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix}1\\1\end{bmatrix} t + \begin{bmatrix}\frac{2}{3}\\1\end{bmatrix}\right) \mathrm{e}^{2t}\\ &= \begin{bmatrix}\frac{\mathrm{e}^{2t}(2+3t)}{3}\\ \mathrm{e}^{2t}(t+1)\end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix}x(t)\\ y\end{bmatrix} = c_1 \begin{bmatrix}\mathrm{e}^{2t}\\ \mathrm{e}^{2t}\end{bmatrix} + c_2 \begin{bmatrix}\mathrm{e}^{2t}\left(t+\frac{2}{3}\right)\\ \mathrm{e}^{2t}(t+1)\end{bmatrix}$$

Which becomes

$$\begin{bmatrix}x(t)\\ y\end{bmatrix} = \begin{bmatrix}\mathrm{e}^{2t}\left(c_1 + c_2 t + \frac{2}{3}c_2\right)\\ \mathrm{e}^{2t}(c_2 t + c_1 + c_2)\end{bmatrix}$$

The following is the phase plot of the system.

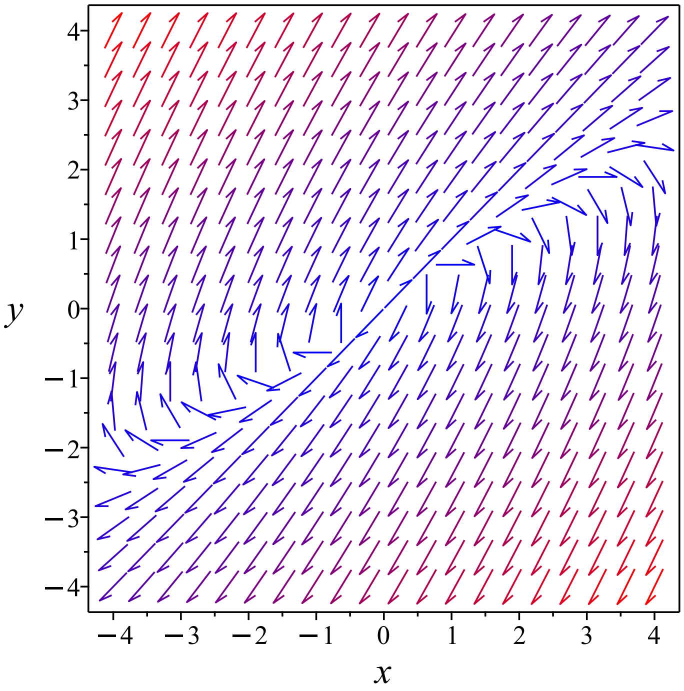


Figure 98: Phase plot

10.20.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + 3y, y' = -3x(t) + 5y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix}$$

- Rewrite the system as

  $$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

  $$\vec{x}_1(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and

  $$\vec{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system

  $$\lambda \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $$\lambda \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

  $$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\left[\begin{array}{cc}-1 & 3 \\ -3 & 5\end{array}\right]-2 \cdot\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]\right) \cdot \overrightarrow{p}=\left[\begin{array}{c}1 \\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c}-\frac{1}{3} \\ 0\end{array}\right]$$

- Second solution from eigenvalue 2

$$\overrightarrow{x}_{2}(t)=\mathrm{e}^{2t} \cdot\left(t \cdot\left[\begin{array}{c}1 \\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{1}{3} \\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x}=c_{1} \overrightarrow{x}_{1}(t)+c_{2} \overrightarrow{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x}=c_{1} \mathrm{e}^{2t} \cdot\left[\begin{array}{c}1 \\ 1\end{array}\right]+c_{2} \mathrm{e}^{2t} \cdot\left(t \cdot\left[\begin{array}{c}1 \\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{1}{3} \\ 0\end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right]=\left[\begin{array}{c}\mathrm{e}^{2t}\left(c_{1}+t c_{2}-\frac{1}{3} c_{2}\right) \\ \mathrm{e}^{2t}\left(t c_{2}+c_{1}\right)\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t)=\mathrm{e}^{2t}\left(c_{1}+t c_{2}-\frac{1}{3} c_{2}\right), y=\mathrm{e}^{2t}\left(t c_{2}+c_{1}\right)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve([diff(x(t),t)=-x(t)+3*y(t),diff(y(t),t)=-3*x(t)+5*y(t)],singsol=all)
```

$$x(t)=\mathrm{e}^{2t}\left(c_{2} t+c_{1}\right)$$
$$y(t)=\frac{\mathrm{e}^{2t}\left(3 c_{2} t+3 c_{1}+c_{2}\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{x'[t]==-x[t]+3*y[t],y'[t]==-3*x[t]+5*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$
\begin{aligned}
x(t) &\rightarrow e^{2t}(-3c_1t + 3c_2t + c_1) \\
y(t) &\rightarrow e^{2t}(-3c_1t + 3c_2t + c_2)
\end{aligned}
$$

10.21 problem 22

Internal problem ID [6747]
Internal file name [`OUTPUT/5995_Sunday_June_05_2022_04_13_00_PM_45701359/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 12x(t) - 9y \\
y' &= 4x(t)
\end{aligned}
$$

10.21.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{6t}(1+6t) & -9t\,\mathrm{e}^{6t} \\ 4t\,\mathrm{e}^{6t} & \mathrm{e}^{6t}(1-6t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{6t}(1+6t) & -9t\,e^{6t} \\ 4t\,e^{6t} & e^{6t}(1-6t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{6t}(1+6t)\,c_1 - 9t\,e^{6t}c_2 \\ 4t\,e^{6t}c_1 + e^{6t}(1-6t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{6t}(6tc_1 - 9c_2t + c_1) \\ e^{6t}(4tc_1 - 6c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.21.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 12-\lambda & -9 \\ 4 & -\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 12\lambda + 36 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 6 & -9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix}\frac{3t}{2}\\ t\end{bmatrix}=\begin{bmatrix}\frac{3t}{2}\\ t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}\frac{3t}{2}\\ t\end{bmatrix}=t\begin{bmatrix}\frac{3}{2}\\ 1\end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix}\frac{3t}{2}\\ t\end{bmatrix}=\begin{bmatrix}\frac{3}{2}\\ 1\end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix}\frac{3t}{2}\\ t\end{bmatrix}=\begin{bmatrix}3\\ 2\end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m>k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
6	2	1	Yes	$\begin{bmatrix}\frac{3}{2}\\ 1\end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 6 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

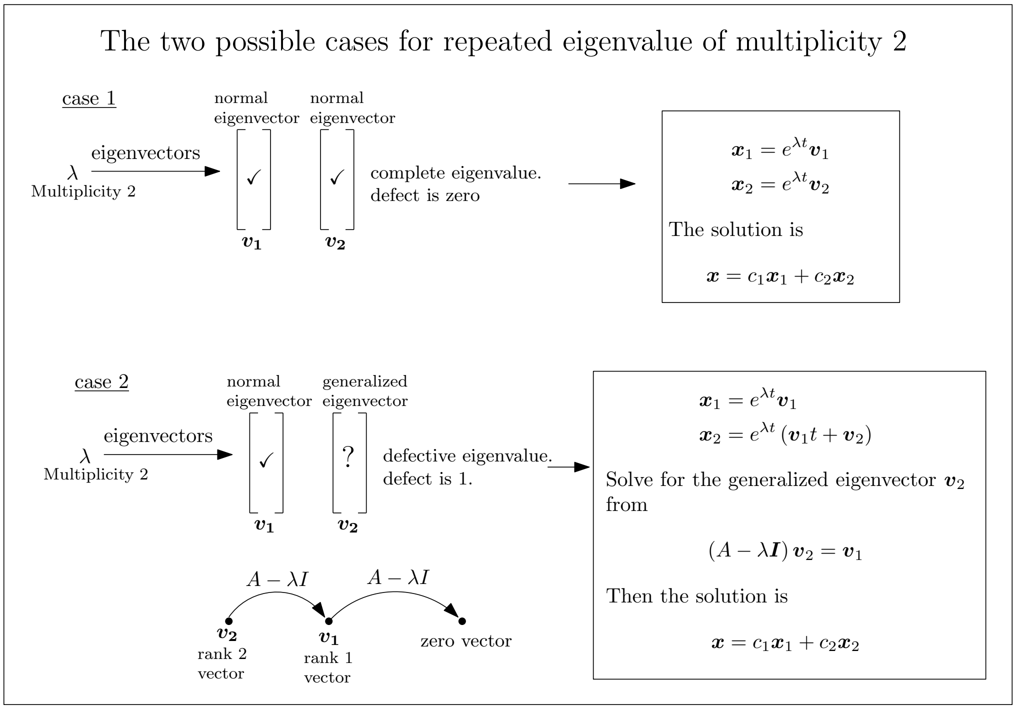


Figure 99: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{7}{4} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 6. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \left[\begin{array}{c}\frac{3}{2} \\ 1\end{array}\right] \mathrm{e}^{6t} \\ &= \left[\begin{array}{c}\frac{3\,\mathrm{e}^{6t}}{2} \\ \mathrm{e}^{6t}\end{array}\right]\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\left[\begin{array}{c}\frac{3}{2} \\ 1\end{array}\right] t + \left[\begin{array}{c}\frac{7}{4} \\ 1\end{array}\right]\right) \mathrm{e}^{6t} \\ &= \left[\begin{array}{c}\frac{\mathrm{e}^{6t}(6t+7)}{4} \\ \mathrm{e}^{6t}(t+1)\end{array}\right]\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = c_1 \left[\begin{array}{c}\frac{3\,\mathrm{e}^{6t}}{2} \\ \mathrm{e}^{6t}\end{array}\right] + c_2 \left[\begin{array}{c}\mathrm{e}^{6t}\left(\frac{3t}{2} + \frac{7}{4}\right) \\ \mathrm{e}^{6t}(t+1)\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = \left[\begin{array}{c}\frac{((6t+7)c_2+6c_1)\mathrm{e}^{6t}}{4} \\ \mathrm{e}^{6t}(c_2 t + c_1 + c_2)\end{array}\right]$$

The following is the phase plot of the system.

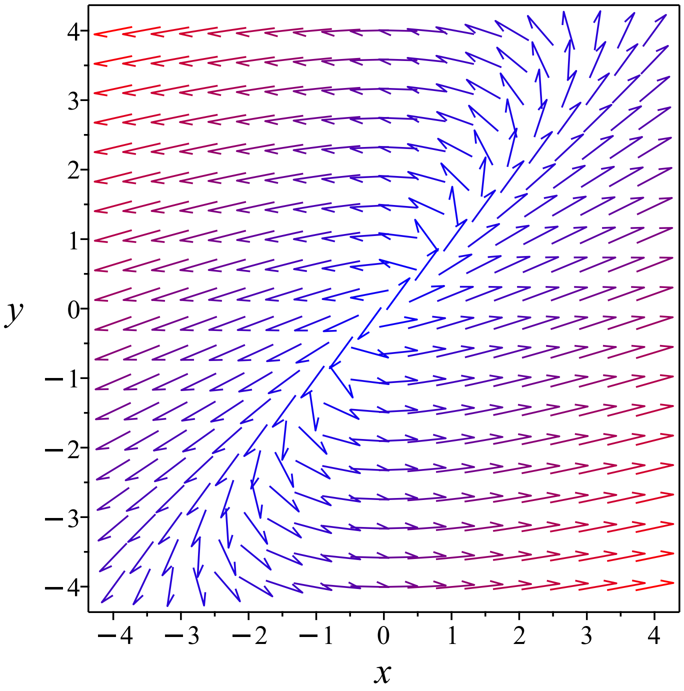


Figure 100: Phase plot

10.21.3 Maple step by step solution

Let's solve

$[x'(t) = 12x(t) - 9y, y' = 4x(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix}$$

- Rewrite the system as

  $$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[6, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[6, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 6

  $$\overrightarrow{x}_1(t) = \mathrm{e}^{6t} \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 6$ is the eigenvalue, and

  $$\overrightarrow{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

  $$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 6

  $$\left(\left[\begin{array}{cc} 12 & -9 \\ 4 & 0 \end{array}\right]-6\cdot\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c} \frac{3}{2} \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p}=\left[\begin{array}{c} \frac{1}{4} \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 6

  $$\overrightarrow{x}_2(t)=\mathrm{e}^{6t}\cdot\left(t\cdot\left[\begin{array}{c} \frac{3}{2} \\ 1 \end{array}\right]+\left[\begin{array}{c} \frac{1}{4} \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

  $$\overrightarrow{x}=c_1\overrightarrow{x}_1(t)+c_2\overrightarrow{x}_2(t)$$

- Substitute solutions into the general solution

  $$\overrightarrow{x}=c_1\mathrm{e}^{6t}\cdot\left[\begin{array}{c} \frac{3}{2} \\ 1 \end{array}\right]+c_2\mathrm{e}^{6t}\cdot\left(t\cdot\left[\begin{array}{c} \frac{3}{2} \\ 1 \end{array}\right]+\left[\begin{array}{c} \frac{1}{4} \\ 0 \end{array}\right]\right)$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y \end{array}\right]=\left[\begin{array}{c} \frac{\mathrm{e}^{6t}(6tc_2+6c_1+c_2)}{4} \\ \mathrm{e}^{6t}(tc_2+c_1) \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t)=\frac{\mathrm{e}^{6t}(6tc_2+6c_1+c_2)}{4}, y=\mathrm{e}^{6t}(tc_2+c_1)\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve([diff(x(t),t)=12*x(t)-9*y(t),diff(y(t),t)=4*x(t)],singsol=all)
```

$$x(t)=\frac{\mathrm{e}^{6t}(6c_2t+6c_1+c_2)}{4}$$
$$y(t)=\mathrm{e}^{6t}(c_2t+c_1)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{x'[t]==12*x[t]-9*y[t],y'[t]==4*x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \rightarrow e^{6t}(6c_1t - 9c_2t + c_1)$$
$$y(t) \rightarrow e^{6t}(4c_1t - 6c_2t + c_2)$$

10.22 problem 23

Internal problem ID [6748]
Internal file name [`OUTPUT/5996_Sunday_June_05_2022_04_13_03_PM_19960611/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 3x(t) - y - z(t) \\
y' &= x(t) + y - z(t) \\
z'(t) &= x(t) - y + z(t)
\end{aligned}
$$

10.22.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 2\,\mathrm{e}^{2t}-\mathrm{e}^{t} & -\mathrm{e}^{2t}+\mathrm{e}^{t} & -\mathrm{e}^{2t}+\mathrm{e}^{t} \\ \mathrm{e}^{2t}-\mathrm{e}^{t} & \mathrm{e}^{t} & -\mathrm{e}^{2t}+\mathrm{e}^{t} \\ \mathrm{e}^{2t}-\mathrm{e}^{t} & -\mathrm{e}^{2t}+\mathrm{e}^{t} & \mathrm{e}^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} 2\,\mathrm{e}^{2t}-\mathrm{e}^{t} & -\mathrm{e}^{2t}+\mathrm{e}^{t} & -\mathrm{e}^{2t}+\mathrm{e}^{t} \\ \mathrm{e}^{2t}-\mathrm{e}^{t} & \mathrm{e}^{t} & -\mathrm{e}^{2t}+\mathrm{e}^{t} \\ \mathrm{e}^{2t}-\mathrm{e}^{t} & -\mathrm{e}^{2t}+\mathrm{e}^{t} & \mathrm{e}^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} (2\,\mathrm{e}^{2t}-\mathrm{e}^{t})\,c_1 + (-\mathrm{e}^{2t}+\mathrm{e}^{t})\,c_2 + (-\mathrm{e}^{2t}+\mathrm{e}^{t})\,c_3 \\ (\mathrm{e}^{2t}-\mathrm{e}^{t})\,c_1 + \mathrm{e}^{t}c_2 + (-\mathrm{e}^{2t}+\mathrm{e}^{t})\,c_3 \\ (\mathrm{e}^{2t}-\mathrm{e}^{t})\,c_1 + (-\mathrm{e}^{2t}+\mathrm{e}^{t})\,c_2 + \mathrm{e}^{t}c_3 \end{bmatrix} \\ &= \begin{bmatrix} (2c_1-c_2-c_3)\,\mathrm{e}^{2t} - \mathrm{e}^{t}(c_1-c_2-c_3) \\ (c_1-c_3)\,\mathrm{e}^{2t} - \mathrm{e}^{t}(c_1-c_2-c_3) \\ (-c_2+c_1)\,\mathrm{e}^{2t} - \mathrm{e}^{t}(c_1-c_2-c_3) \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.22.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t + s\}$

Hence the solution is

$$\begin{bmatrix} t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} t+s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t+s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	2	No	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. $\underline{\text{eigenvalue } 2}$ is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

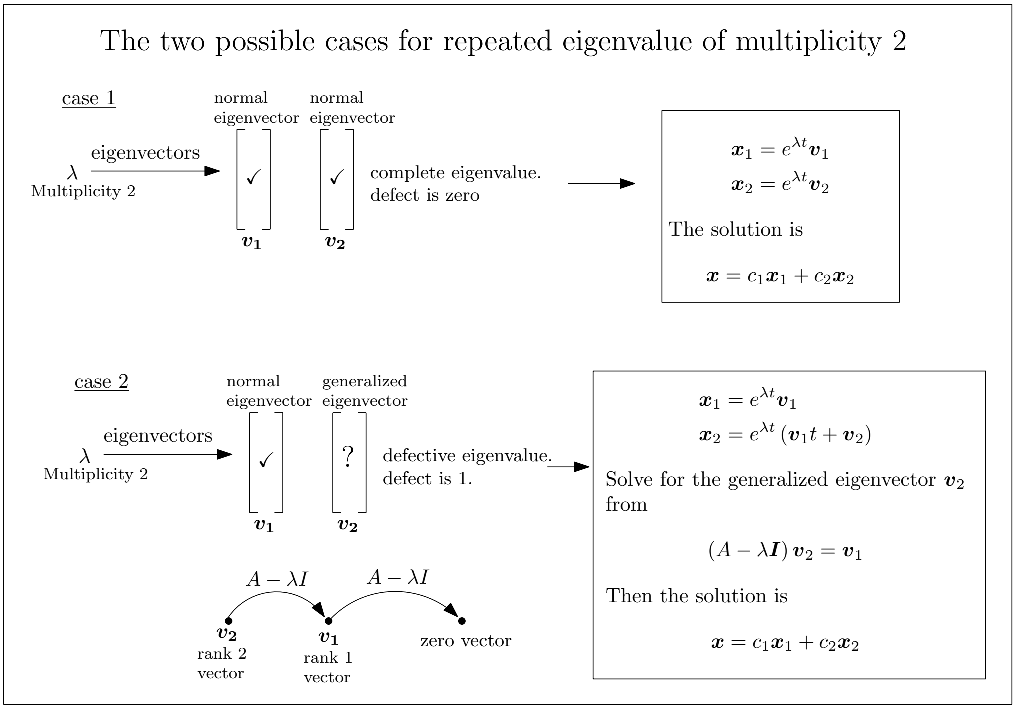


Figure 101: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^t \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} (c_1 + c_2)\,\mathrm{e}^{2t} + c_3\mathrm{e}^{t} \\ c_2\mathrm{e}^{2t} + c_3\mathrm{e}^{t} \\ c_1\mathrm{e}^{2t} + c_3\mathrm{e}^{t} \end{bmatrix}$$

10.22.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) - y - z(t)\,, y' = x(t) + y - z(t)\,, z'(t) = x(t) - y + z(t)]$

- Define vector

$$\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

  $$\overrightarrow{x}_2(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and

  $$\overrightarrow{x}_3(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_3(t)$ into the homogeneous system

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

  $$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_3(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

  $$\left(\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{x}_3(t) = \mathrm{e}^{2t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{2t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} ((t+1) c_3 + c_2) \mathrm{e}^{2t} + c_1 \mathrm{e}^{t} \\ c_1 \mathrm{e}^{t} \\ (t c_3 + c_2) \mathrm{e}^{2t} + c_1 \mathrm{e}^{t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = ((t+1) c_3 + c_2) \mathrm{e}^{2t} + c_1 \mathrm{e}^{t}, y = c_1 \mathrm{e}^{t}, z(t) = (t c_3 + c_2) \mathrm{e}^{2t} + c_1 \mathrm{e}^{t}\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
dsolve([diff(x(t),t)=3*x(t)-y(t)-z(t),diff(y(t),t)=x(t)+y(t)-z(t),diff(z(t),t)=x(t)-y(t)+z(t
```

$$\begin{aligned} x(t) &= c_2 \mathrm{e}^{t} + c_3 \mathrm{e}^{2t} \\ y(t) &= c_2 \mathrm{e}^{t} + c_3 \mathrm{e}^{2t} + c_1 \mathrm{e}^{2t} \\ z(t) &= c_2 \mathrm{e}^{t} - c_1 \mathrm{e}^{2t} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 86

```
DSolve[{x'[t]==3*x[t]-y[t]-z[t],y'[t]==x[t]+y[t]-z[t],z'[t]==x[t]-y[t]+z[t]},{x[t],y[t],z[t]
```

$$
\begin{aligned}
x(t) &\to e^t\left(c_1\left(2e^t-1\right)-(c_2+c_3)\left(e^t-1\right)\right)\\
y(t) &\to e^t\left(c_1\left(e^t-1\right)-c_3e^t+c_2+c_3\right)\\
z(t) &\to e^t\left(c_1\left(e^t-1\right)-c_2e^t+c_2+c_3\right)
\end{aligned}
$$

10.23 problem 24

Internal problem ID [6749]
Internal file name [`OUTPUT/5997_Sunday_June_05_2022_04_13_05_PM_12102809/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 3x(t) + 2y + 4z(t) \\ y' &= 2x(t) + 2z(t) \\ z'(t) &= 4x(t) + 2y + 3z(t) \end{aligned}$$

10.23.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{5\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & -\frac{4\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} \\ -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & \frac{8\,e^{-t}}{9}+\frac{e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} \\ -\frac{4\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & \frac{5\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{5\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & -\frac{4\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} \\ -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & \frac{8\,e^{-t}}{9}+\frac{e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} \\ -\frac{4\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} & -\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9} & \frac{5\,e^{-t}}{9}+\frac{4\,e^{8t}}{9} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{5\,e^{-t}}{9}+\frac{4\,e^{8t}}{9}\right)c_1+\left(-\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9}\right)c_2+\left(-\frac{4\,e^{-t}}{9}+\frac{4\,e^{8t}}{9}\right)c_3 \\ \left(-\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9}\right)c_1+\left(\frac{8\,e^{-t}}{9}+\frac{e^{8t}}{9}\right)c_2+\left(-\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9}\right)c_3 \\ \left(-\frac{4\,e^{-t}}{9}+\frac{4\,e^{8t}}{9}\right)c_1+\left(-\frac{2\,e^{-t}}{9}+\frac{2\,e^{8t}}{9}\right)c_2+\left(\frac{5\,e^{-t}}{9}+\frac{4\,e^{8t}}{9}\right)c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(5c_1-2c_2-4c_3)e^{-t}}{9}+\frac{4\left(c_1+\frac{c_2}{2}+c_3\right)e^{8t}}{9} \\ \frac{(-2c_1+8c_2-2c_3)e^{-t}}{9}+\frac{2\left(c_1+\frac{c_2}{2}+c_3\right)e^{8t}}{9} \\ \frac{(-4c_1-2c_2+5c_3)e^{-t}}{9}+\frac{4\left(c_1+\frac{c_2}{2}+c_3\right)e^{8t}}{9} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.23.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix}3-\lambda&2&4\\2&-\lambda&2\\4&2&3-\lambda\end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^3-6\lambda^2-15\lambda-8=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=8$$
$$\lambda_2=-1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
8	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-1$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}-(-1)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}4&2&4\\2&1&2\\4&2&4\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2} - s\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this

eigenvalue. The above can be written as

$$\begin{aligned}\begin{bmatrix}-\frac{t}{2}-s\\ t\\ s\end{bmatrix} &= \begin{bmatrix}-\frac{t}{2}\\ t\\ 0\end{bmatrix}+\begin{bmatrix}-s\\ 0\\ s\end{bmatrix}\\ &= t\begin{bmatrix}-\frac{1}{2}\\ 1\\ 0\end{bmatrix}+s\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}\end{aligned}$$

By letting $t=1$ and $s=1$ then the above becomes

$$\begin{bmatrix}-\frac{t}{2}-s\\ t\\ s\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\ 1\\ 0\end{bmatrix}+\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix}-\frac{1}{2}\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}\right)$$

Which are normalized to

$$\left(\begin{bmatrix}-1\\ 2\\ 0\end{bmatrix},\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}\right)$$

Considering the eigenvalue $\lambda_2=8$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}3&2&4\\ 2&0&2\\ 4&2&3\end{bmatrix}-(8)\begin{bmatrix}1&0&0\\ 0&1&0\\ 0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-5&2&4\\ 2&-8&2\\ 4&2&-5\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{4R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 0 & \frac{18}{5} & -\frac{9}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 2 & 4 \\ 0 & -\frac{36}{5} & \frac{18}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
8	1	1	No	$\begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$
-1	2	2	No	$\begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 8 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{8t}\\ &= \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} e^{8t}\end{aligned}$$

<u>eigenvalue -1</u> is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

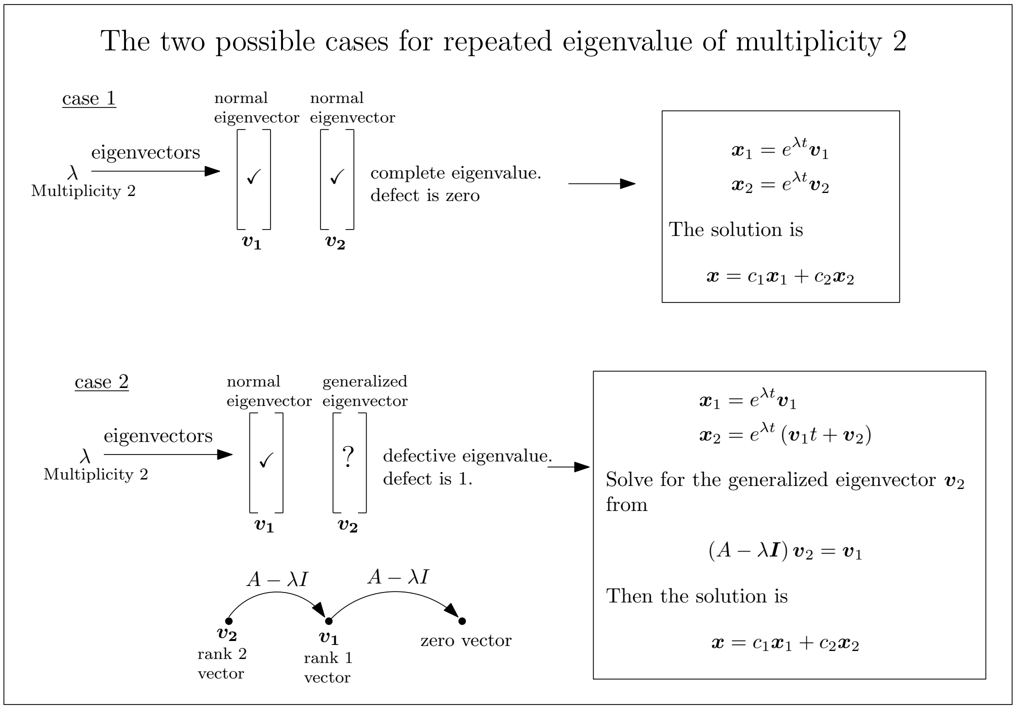


Figure 102: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above.

Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} e^{-t}\end{aligned}$$

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{8t} \\ \frac{\mathrm{e}^{8t}}{2} \\ \mathrm{e}^{8t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{-t} \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{(-c_2 - 2c_3)\mathrm{e}^{-t}}{2} + c_1 \mathrm{e}^{8t} \\ \frac{c_1 \mathrm{e}^{8t}}{2} + c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{8t} + c_3 \mathrm{e}^{-t} \end{bmatrix}$$

10.23.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) + 2y + 4z(t), y' = 2x(t) + 2z(t), z'(t) = 4x(t) + 2y + 3z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[8, \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\overrightarrow{x}_1(t) = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -\frac{1}{2} \\ 1 \\ 0 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

  $$\overrightarrow{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

  $$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

  $$\left(\left[\begin{array}{ccc} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{array}\right] - (-1) \cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} -\frac{1}{2} \\ 1 \\ 0 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p} = \left[\begin{array}{c} -\frac{1}{8} \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue -1

$$\overrightarrow{x}_2(t) = \mathrm{e}^{-t} \cdot \left(t \cdot \left[\begin{array}{c} -\frac{1}{2} \\ 1 \\ 0 \end{array}\right] + \left[\begin{array}{c} -\frac{1}{8} \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

$$\left[8, \left[\begin{array}{c} 1 \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_3 = \mathrm{e}^{8t} \cdot \left[\begin{array}{c} 1 \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t) + c_3 \overrightarrow{x}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = \mathrm{e}^{-t} c_1 \cdot \left[\begin{array}{c} -\frac{1}{2} \\ 1 \\ 0 \end{array}\right] + c_2 \mathrm{e}^{-t} \cdot \left(t \cdot \left[\begin{array}{c} -\frac{1}{2} \\ 1 \\ 0 \end{array}\right] + \left[\begin{array}{c} -\frac{1}{8} \\ 0 \\ 0 \end{array}\right]\right) + c_3 \mathrm{e}^{8t} \cdot \left[\begin{array}{c} 1 \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} \frac{((-4t-1)c_2-4c_1)\mathrm{e}^{-t}}{8} + c_3 \mathrm{e}^{8t} \\ \mathrm{e}^{-t}(tc_2 + c_1) + \frac{c_3 \mathrm{e}^{8t}}{2} \\ c_3 \mathrm{e}^{8t} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{((-4t-1)c_2-4c_1)\mathrm{e}^{-t}}{8} + c_3 \mathrm{e}^{8t}, y = \mathrm{e}^{-t}(tc_2 + c_1) + \frac{c_3 \mathrm{e}^{8t}}{2}, z(t) = c_3 \mathrm{e}^{8t}\right\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 67

```
dsolve([diff(x(t),t)=3*x(t)+2*y(t)+4*z(t),diff(y(t),t)=2*x(t)+2*z(t),diff(z(t),t)=4*x(t)+2*y
```

$$
\begin{aligned}
x(t) &= 2c_2\mathrm{e}^{8t} + 2c_3\mathrm{e}^{-t} + \mathrm{e}^{-t}c_1 \\
y(t) &= c_2\mathrm{e}^{8t} + c_3\mathrm{e}^{-t} \\
z(t) &= 2c_2\mathrm{e}^{8t} - \frac{5c_3\mathrm{e}^{-t}}{2} - \mathrm{e}^{-t}c_1
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 135

```
DSolve[{x'[t]==3*x[t]+2*y[t]+4*z[t],y'[t]==2*x[t]+2*z[t],z'[t]==4*x[t]+2*y[t]+3*z[t]},{x[t],
```

$$
\begin{aligned}
x(t) &\to \frac{1}{9}e^{-t}\left(c_1\left(4e^{9t}+5\right)+2(c_2+2c_3)\left(e^{9t}-1\right)\right) \\
y(t) &\to \frac{1}{9}e^{-t}\left(2c_1\left(e^{9t}-1\right)+c_2\left(e^{9t}+8\right)+2c_3\left(e^{9t}-1\right)\right) \\
z(t) &\to \frac{1}{9}e^{-t}\left(4c_1\left(e^{9t}-1\right)+2c_2\left(e^{9t}-1\right)+c_3\left(4e^{9t}+5\right)\right)
\end{aligned}
$$

10.24 problem 25

Internal problem ID [6750]
Internal file name [`OUTPUT/5998_Sunday_June_05_2022_04_13_07_PM_60057513/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 5x(t) - 4y \\ y' &= x(t) + 2z(t) \\ z'(t) &= 2y + 5z(t) \end{aligned}$$

10.24.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{4}{25}+\frac{(-20t+29)\mathrm{e}^{5t}}{25} & -\frac{4\,\mathrm{e}^{5t}}{5}+\frac{4}{5} & -\frac{8}{25}+\frac{8(1-5t)\mathrm{e}^{5t}}{25} \\ \frac{\mathrm{e}^{5t}}{5}-\frac{1}{5} & 1 & \frac{2\,\mathrm{e}^{5t}}{5}-\frac{2}{5} \\ \frac{2}{25}+\frac{2\,\mathrm{e}^{5t}(-1+5t)}{25} & \frac{2\,\mathrm{e}^{5t}}{5}-\frac{2}{5} & \frac{4}{25}+\frac{(20t+21)\mathrm{e}^{5t}}{25} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} -\frac{4}{25}+\frac{(-20t+29)\mathrm{e}^{5t}}{25} & -\frac{4\,\mathrm{e}^{5t}}{5}+\frac{4}{5} & -\frac{8}{25}+\frac{8(1-5t)\mathrm{e}^{5t}}{25} \\ \frac{\mathrm{e}^{5t}}{5}-\frac{1}{5} & 1 & \frac{2\,\mathrm{e}^{5t}}{5}-\frac{2}{5} \\ \frac{2}{25}+\frac{2\,\mathrm{e}^{5t}(-1+5t)}{25} & \frac{2\,\mathrm{e}^{5t}}{5}-\frac{2}{5} & \frac{4}{25}+\frac{(20t+21)\mathrm{e}^{5t}}{25} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(-\frac{4}{25}+\frac{(-20t+29)\mathrm{e}^{5t}}{25}\right)c_1+\left(-\frac{4\,\mathrm{e}^{5t}}{5}+\frac{4}{5}\right)c_2+\left(-\frac{8}{25}+\frac{8(1-5t)\mathrm{e}^{5t}}{25}\right)c_3 \\ \left(\frac{\mathrm{e}^{5t}}{5}-\frac{1}{5}\right)c_1+c_2+\left(\frac{2\,\mathrm{e}^{5t}}{5}-\frac{2}{5}\right)c_3 \\ \left(\frac{2}{25}+\frac{2\,\mathrm{e}^{5t}(-1+5t)}{25}\right)c_1+\left(\frac{2\,\mathrm{e}^{5t}}{5}-\frac{2}{5}\right)c_2+\left(\frac{4}{25}+\frac{(20t+21)\mathrm{e}^{5t}}{25}\right)c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{((-20t+29)c_1+(-40t+8)c_3-20c_2)\mathrm{e}^{5t}}{25}-\frac{4c_1}{25}+\frac{4c_2}{5}-\frac{8c_3}{25} \\ \frac{(c_1+2c_3)\mathrm{e}^{5t}}{5}-\frac{c_1}{5}+c_2-\frac{2c_3}{5} \\ \frac{((10t-2)c_1+(20t+21)c_3+10c_2)\mathrm{e}^{5t}}{25}+\frac{2c_1}{25}-\frac{2c_2}{5}+\frac{4c_3}{25} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.24.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 10\lambda^2 + 25\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & -4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & -4 & 0 & 0 \\ 0 & \frac{4}{5} & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{5R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} 5 & -4 & 0 & 0 \\ 0 & \frac{4}{5} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -4 & 0 \\ 0 & \frac{4}{5} & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t, v_2 = -\frac{5t}{2}\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ -\frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ -\frac{5t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ -\frac{5t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -\frac{5}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ -\frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -\frac{5}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -2t \\ -\frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -4 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 1 & -5 & 2 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \implies \left[\begin{array}{ccc|c} 1 & -5 & 2 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -5 & 2 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
5	2	1	Yes	$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} -2 \\ -\frac{5}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

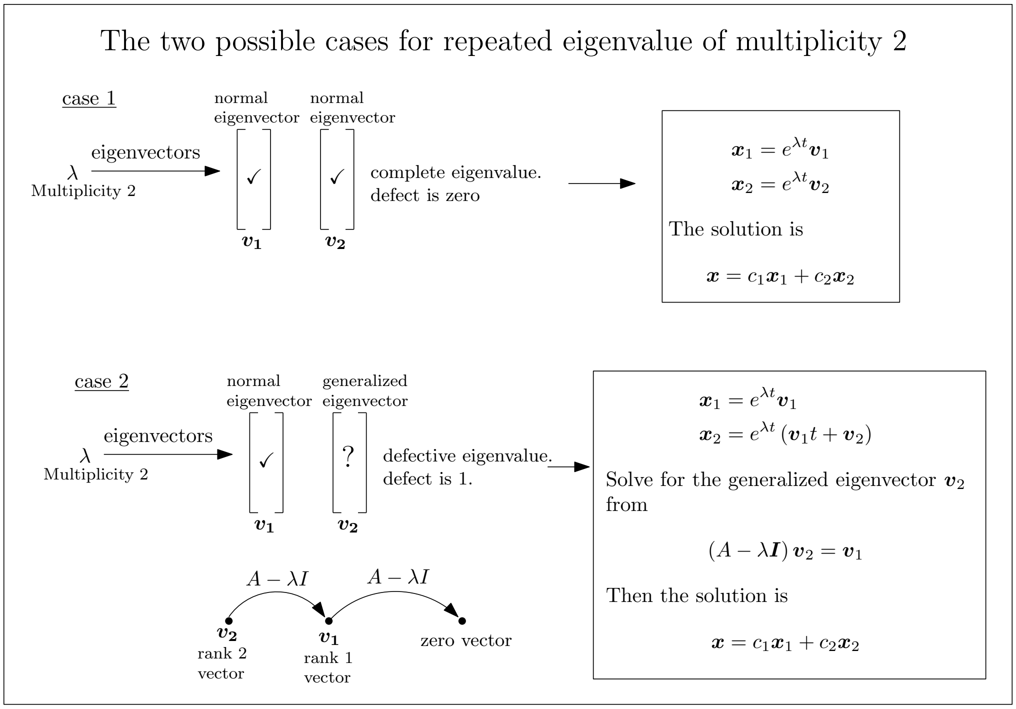


Figure 103: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A-\lambda I)\,\vec{v}_2=\vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\begin{aligned}\left(\begin{bmatrix}5 & -4 & 0\\ 1 & 0 & 2\\ 0 & 2 & 5\end{bmatrix}-(5)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}&=\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix}\\ \begin{bmatrix}0 & -4 & 0\\ 1 & -5 & 2\\ 0 & 2 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}&=\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix}\end{aligned}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ 1\end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 5. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t)&=\vec{v}_1e^{\lambda t}\\ &=\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix}\mathrm{e}^{5t}\\ &=\begin{bmatrix}-2\,\mathrm{e}^{5t}\\ 0\\ \mathrm{e}^{5t}\end{bmatrix}\end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\left[\begin{array}{c} -2 \\ 0 \\ 1 \end{array}\right] t + \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{array}\right] \right) \mathrm{e}^{5t} \\
&= \left[\begin{array}{c} -\frac{\mathrm{e}^{5t}(4t-1)}{2} \\ \frac{\mathrm{e}^{5t}}{2} \\ \mathrm{e}^{5t}(t+1) \end{array}\right]
\end{aligned}
$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{0} \\
&= \left[\begin{array}{c} -2 \\ -\frac{5}{2} \\ 1 \end{array}\right] e^{0}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = c_1 \left[\begin{array}{c} -2\,\mathrm{e}^{5t} \\ 0 \\ \mathrm{e}^{5t} \end{array}\right] + c_2 \left[\begin{array}{c} \mathrm{e}^{5t}\left(-2t + \frac{1}{2}\right) \\ \frac{\mathrm{e}^{5t}}{2} \\ \mathrm{e}^{5t}(t+1) \end{array}\right] + c_3 \left[\begin{array}{c} -2 \\ -\frac{5}{2} \\ 1 \end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} \frac{((1-4t)c_2 - 4c_1)\mathrm{e}^{5t}}{2} - 2c_3 \\ \frac{c_2 \mathrm{e}^{5t}}{2} - \frac{5c_3}{2} \\ \left((t+1)\, c_2 + c_1\right) \mathrm{e}^{5t} + c_3 \end{array}\right]
$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 63

```
dsolve([diff(x(t),t)=5*x(t)-4*y(t),diff(y(t),t)=x(t)+2*z(t),diff(z(t),t)=2*y(t)+5*z(t)],sing
```

$$
\begin{aligned}
x(t) &= -4\,\mathrm{e}^{5t}c_3t+\mathrm{e}^{5t}c_1+\frac{4c_2}{5}\\
y(t) &= c_2+c_3\mathrm{e}^{5t}\\
z(t) &= 2\,\mathrm{e}^{5t}c_3t-\frac{\mathrm{e}^{5t}c_1}{2}+\frac{5c_3\mathrm{e}^{5t}}{2}-\frac{2c_2}{5}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 141

```
DSolve[{x'[t]==5*x[t]-4*y[t],y'[t]==x[t]+2*z[t],z'[t]==2*y[t]+5*z[t]},{x[t],y[t],z[t]},t,Inc
```

$$
\begin{aligned}
x(t) &\to \frac{1}{25}\left(c_1\left(e^{5t}(29-20t)-4\right)-4\left(5c_2\left(e^{5t}-1\right)+2c_3\left(e^{5t}(5t-1)+1\right)\right)\right)\\
y(t) &\to \frac{1}{5}c_1\left(e^{5t}-1\right)+\frac{2}{5}c_3\left(e^{5t}-1\right)+c_2\\
z(t) &\to \frac{1}{25}\left(2c_1\left(e^{5t}(5t-1)+1\right)+10c_2\left(e^{5t}-1\right)+c_3\left(e^{5t}(20t+21)+4\right)\right)
\end{aligned}
$$

10.25 problem 26

Internal problem ID [6751]
Internal file name [`OUTPUT/5999_Sunday_June_05_2022_04_13_10_PM_33850252/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= x(t) \\ y' &= 3y + z(t) \\ z'(t) &= -y + z(t) \end{aligned}$$

10.25.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ 0 & \mathrm{e}^{2t}(t+1) & t\,\mathrm{e}^{2t} \\ 0 & -t\,\mathrm{e}^{2t} & \mathrm{e}^{2t}(-t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ 0 & \mathrm{e}^{2t}(t+1) & t\,\mathrm{e}^{2t} \\ 0 & -t\,\mathrm{e}^{2t} & \mathrm{e}^{2t}(-t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{t}c_1 \\ \mathrm{e}^{2t}(t+1)\,c_2 + t\,\mathrm{e}^{2t}c_3 \\ -t\,\mathrm{e}^{2t}c_2 + \mathrm{e}^{2t}(-t+1)\,c_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{t}c_1 \\ \mathrm{e}^{2t}(tc_2 + c_3t + c_2) \\ -((t-1)\,c_3 + tc_2)\,\mathrm{e}^{2t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.25.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 1\\ 0 & -1 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}1-\lambda & 0 & 0\\ 0 & 3-\lambda & 1\\ 0 & -1 & 1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 1\\ 0 & -1 & 1\end{bmatrix} - (1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1, 2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{array}\right]$$

Since the current pivot $A(2, 3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array} \right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

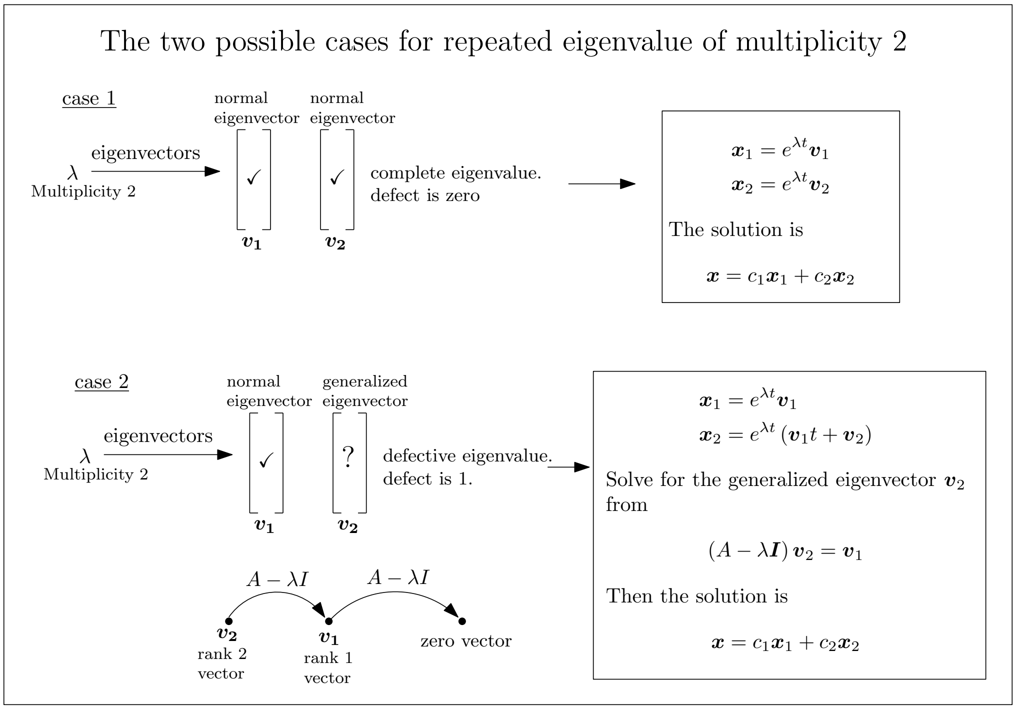


Figure 104: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\ &= \begin{bmatrix} 0 \\ -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\right) \mathrm{e}^{2t} \\ &= \begin{bmatrix} 0 \\ -\mathrm{e}^{2t}(t+2) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{2t}(-t-2) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_3 \mathrm{e}^{t} \\ -\left(\left(t+2\right) c_2 + c_1\right) \mathrm{e}^{2t} \\ \mathrm{e}^{2t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

10.25.3 Maple step by step solution

Let's solve

$[x'(t) = x(t)\,, y' = 3y + z(t)\,, z'(t) = -y + z(t)]$

- Define vector

  $$\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

  $$\vec{x}_2(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and

  $$\vec{x}_3(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_3(t)$ into the homogeneous system

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

  $$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_3(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

  $$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{x}_3(t) = e^{2t} \cdot \left(t \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution to the system of ODEs

$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{t} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{2t} \cdot \left(t \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{t} \\ e^{2t}(-tc_3 - c_2) \\ (tc_3 + c_2)\, e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

$\{x(t) = c_1 e^{t}, y = e^{2t}(-tc_3 - c_2)\,, z(t) = (tc_3 + c_2)\, e^{2t}\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve([diff(x(t),t)=x(t),diff(y(t),t)=3*y(t)+z(t),diff(z(t),t)=-y(t)+z(t)],singsol=all)
```

$$x(t) = c_3 e^{t}$$
$$y(t) = e^{2t}(c_2 t + c_1)$$
$$z(t) = -e^{2t}(c_2 t + c_1 - c_2)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 96

```
DSolve[{x'[t]==x[t],y'[t]==3*y[t]+z[t],z'[t]==-y[t]+z[t]},{x[t],y[t],z[t]},t,IncludeSingular
```

$$
\begin{aligned}
x(t) &\to c_1 e^t \\
y(t) &\to e^{2t}(c_2(t+1)+c_3 t) \\
z(t) &\to e^{2t}(c_3-(c_2+c_3)t) \\
x(t) &\to 0 \\
y(t) &\to e^{2t}(c_2(t+1)+c_3 t) \\
z(t) &\to e^{2t}(c_3-(c_2+c_3)t)
\end{aligned}
$$

10.26 problem 27

Internal problem ID [6752]
Internal file name [`OUTPUT/6000_Sunday_June_05_2022_04_13_12_PM_68367667/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= x(t) \\ y' &= 2x(t) + 2y - z(t) \\ z'(t) &= y \end{aligned}$$

10.26.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ \mathrm{e}^{t} t(t+2) & \mathrm{e}^{t}(t+1) & -t\,\mathrm{e}^{t} \\ \mathrm{e}^{t} t^{2} & t\,\mathrm{e}^{t} & \mathrm{e}^{t}(-t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{t} & 0 & 0 \\ e^{t}t(t+2) & e^{t}(t+1) & -t\,e^{t} \\ e^{t}t^{2} & t\,e^{t} & e^{t}(-t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{t}c_1 \\ e^{t}t(t+2)\,c_1 + e^{t}(t+1)\,c_2 - t\,e^{t}c_3 \\ e^{t}t^{2}c_1 + t\,e^{t}c_2 + e^{t}(-t+1)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{t}c_1 \\ (t^{2}c_1 + (2c_1 + c_2 - c_3)\,t + c_2)\,e^{t} \\ (t^{2}c_1 + (c_2 - c_3)\,t + c_3)\,e^{t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.26.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	3	1	Yes	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

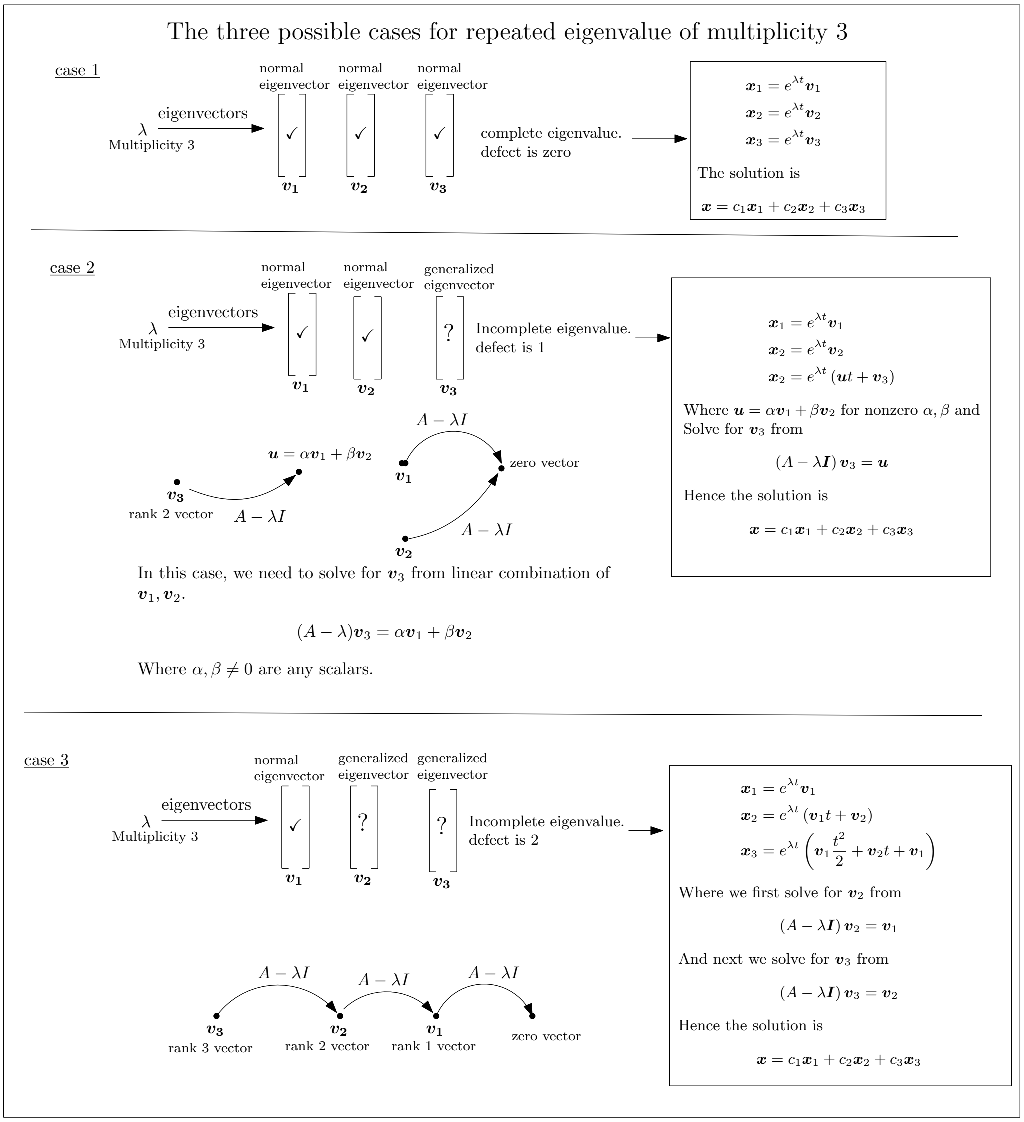


Figure 105: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}1 & 0 & 0\\ 2 & 2 & -1\\ 0 & 1 & 0\end{bmatrix} - (1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 0\\ 2 & 1 & -1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}0\\ 2\\ 1\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}1 & 0 & 0\\ 2 & 2 & -1\\ 0 & 1 & 0\end{bmatrix} - (1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 2\\ 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 0\\ 2 & 1 & -1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 2\\ 1\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}\frac{1}{2}\\ 2\\ 1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 1. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix}0\\1\\1\end{bmatrix} \mathrm{e}^{t}\\ &= \begin{bmatrix}0\\ \mathrm{e}^{t}\\ \mathrm{e}^{t}\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2)\\ &= \mathrm{e}^{t}\left(\begin{bmatrix}0\\1\\1\end{bmatrix} t + \begin{bmatrix}0\\2\\1\end{bmatrix}\right)\\ &= \begin{bmatrix}0\\ \mathrm{e}^{t}(t+2)\\ \mathrm{e}^{t}(t+1)\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t}\\ &= \left(\begin{bmatrix}0\\1\\1\end{bmatrix} \frac{t^2}{2} + \begin{bmatrix}0\\2\\1\end{bmatrix} t + \begin{bmatrix}\frac{1}{2}\\2\\1\end{bmatrix}\right) \mathrm{e}^{t}\\ &= \begin{bmatrix}\frac{\mathrm{e}^{t}}{2}\\ \frac{\mathrm{e}^{t}(t+2)^2}{2}\\ \frac{\mathrm{e}^{t}(t^2+2t+2)}{2}\end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{t}(t+2) \\ \mathrm{e}^{t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} \frac{\mathrm{e}^{t}}{2} \\ \mathrm{e}^{t}\left(\frac{1}{2}t^2 + 2t + 2\right) \\ \mathrm{e}^{t}\left(\frac{1}{2}t^2 + t + 1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{c_3 \mathrm{e}^{t}}{2} \\ \frac{\left((t+2)^2 c_3 + 2c_2 t + 2c_1 + 4c_2\right)\mathrm{e}^{t}}{2} \\ \frac{\left((t^2+2t+2)c_3 + 2c_2 t + 2c_1 + 2c_2\right)\mathrm{e}^{t}}{2} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 46

```
dsolve([diff(x(t),t)=x(t),diff(y(t),t)=2*x(t)+2*y(t)-z(t),diff(z(t),t)=y(t)],singsol=all)
```

$$\begin{aligned} x(t) &= c_3 \mathrm{e}^{t} \\ y(t) &= \mathrm{e}^{t}\left(c_3 t^2 + c_1 t + 2c_3 t + c_1 + c_2\right) \\ z(t) &= \mathrm{e}^{t}\left(c_3 t^2 + c_1 t + c_2\right) \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 65

```
DSolve[{x'[t]==x[t],y'[t]==2*x[t]+2*y[t]-z[t],z'[t]==y[t]},{x[t],y[t],z[t]},t,IncludeSingula
```

$$\begin{aligned} x(t) &\to c_1 e^{t} \\ y(t) &\to e^{t}\left(c_1 t^2 + (2c_1 + c_2 - c_3)t + c_2\right) \\ z(t) &\to e^{t}\left(c_1 t^2 + (c_2 - c_3)t + c_3\right) \end{aligned}$$

10.27 problem 28

Internal problem ID [6753]
Internal file name [`OUTPUT/6001_Sunday_June_05_2022_04_13_14_PM_6833603/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 4x(t) + y \\ y' &= 4y + z(t) \\ z'(t) &= 4z(t) \end{aligned}$$

10.27.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{4t} & t\,e^{4t} & \frac{t^2 e^{4t}}{2} \\ 0 & e^{4t} & t\,e^{4t} \\ 0 & 0 & e^{4t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{4t} & t\,e^{4t} & \frac{t^2 e^{4t}}{2} \\ 0 & e^{4t} & t\,e^{4t} \\ 0 & 0 & e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{4t}c_1 + t\,e^{4t}c_2 + \frac{t^2 e^{4t} c_3}{2} \\ e^{4t}c_2 + t\,e^{4t}c_3 \\ e^{4t}c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{4t}\left(c_1 + c_2 t + \frac{1}{2}c_3 t^2\right) \\ e^{4t}(c_3 t + c_2) \\ e^{4t}c_3 \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.27.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(4-\lambda)(4-\lambda)(4-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
4	3	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 4 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

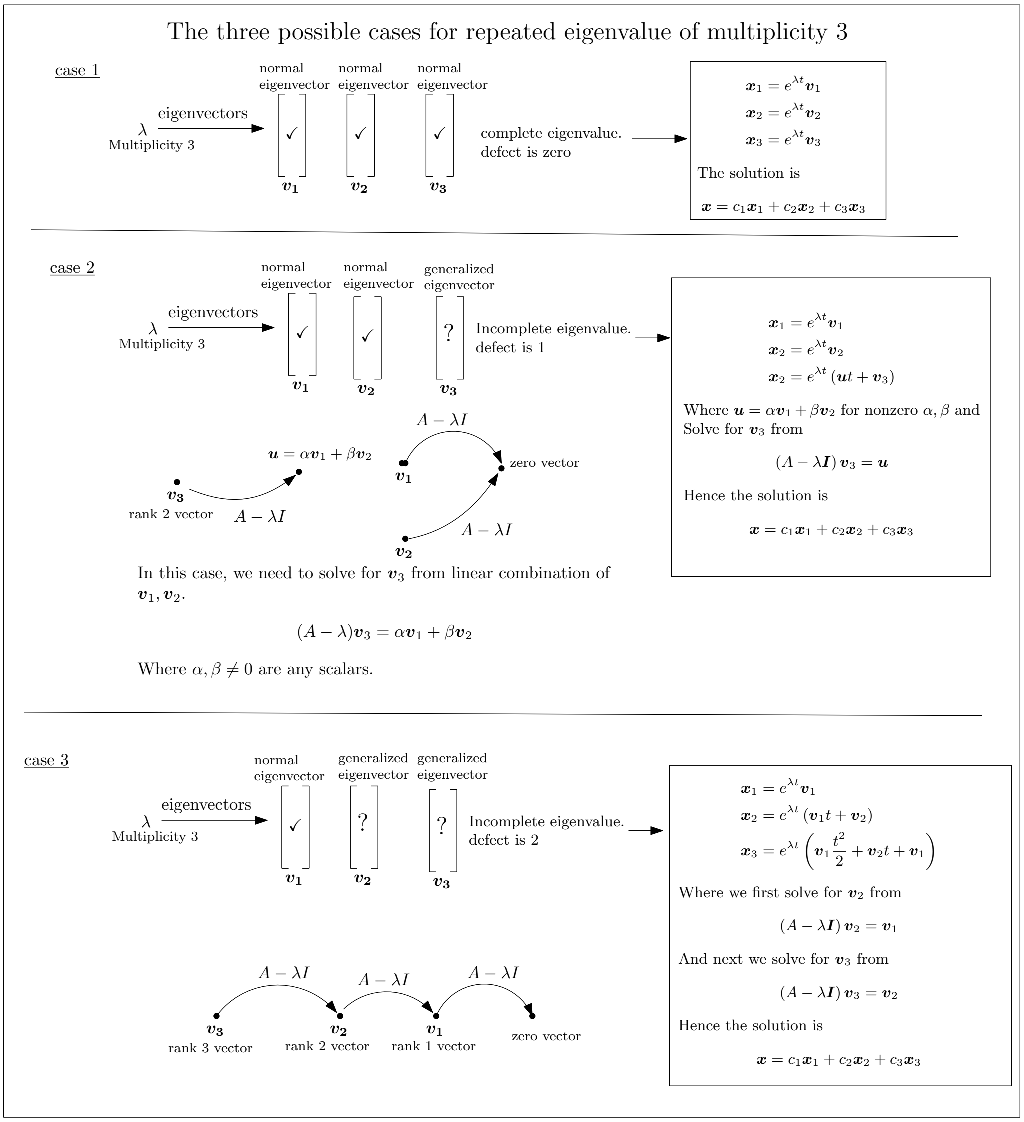


Figure 106: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}4 & 1 & 0\\0 & 4 & 1\\0 & 0 & 4\end{bmatrix}-(4)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A-\lambda I)\,\vec{v}_3=\vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}4 & 1 & 0\\0 & 4 & 1\\0 & 0 & 4\end{bmatrix}-(4)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 4. Therefore the three basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{4t} \\
&= \begin{bmatrix} \mathrm{e}^{4t} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{4t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} \mathrm{e}^{4t}(t+1) \\ \mathrm{e}^{4t} \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3 \right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{4t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{4t}(t^2+2t+2)}{2} \\ \mathrm{e}^{4t}(t+1) \\ \mathrm{e}^{4t} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{4t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{4t}(t+1) \\ \mathrm{e}^{4t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{4t}\left(\frac{1}{2}t^2+t+1\right) \\ \mathrm{e}^{4t}(t+1) \\ \mathrm{e}^{4t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{((t^2+2t+2)c_3+2c_2t+2c_1+2c_2)\mathrm{e}^{4t}}{2} \\ \mathrm{e}^{4t}(c_3t+c_2+c_3) \\ c_3\mathrm{e}^{4t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 46

```
dsolve([diff(x(t),t)=4*x(t)+y(t),diff(y(t),t)=4*y(t)+z(t),diff(z(t),t)=4*z(t)],singsol=all)
```

$$x(t) = \frac{(c_3t^2+2c_2t+2c_1)\,\mathrm{e}^{4t}}{2}$$
$$y(t) = (c_3t+c_2)\,\mathrm{e}^{4t}$$
$$z(t) = c_3\mathrm{e}^{4t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 57

```
DSolve[{x'[t]==4*x[t]+y[t],y'[t]==4*y[t]+z[t],z'[t]==4*z[t]},{x[t],y[t],z[t]},t,IncludeSingu
```

$$x(t) \to \frac{1}{2}e^{4t}(t(c_3t+2c_2)+2c_1)$$
$$y(t) \to e^{4t}(c_3t+c_2)$$
$$z(t) \to c_3e^{4t}$$

10.28 problem 29

Internal problem ID [6754]
Internal file name [`OUTPUT/6002_Sunday_June_05_2022_04_13_16_PM_45995531/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 2x(t) + 4y \\ y' &= -x(t) + 6y \end{aligned}$$

With initial conditions

$$[x(0) = -1, y(0) = 6]$$

10.28.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{4t}(1-2t) & 4t\,\mathrm{e}^{4t} \\ -t\,\mathrm{e}^{4t} & \mathrm{e}^{4t}(2t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \mathrm{e}^{4t}(1-2t) & 4t\,\mathrm{e}^{4t} \\ -t\,\mathrm{e}^{4t} & \mathrm{e}^{4t}(2t+1) \end{bmatrix} \begin{bmatrix} -1 \\ 6 \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{e}^{4t}(1-2t) + 24t\,\mathrm{e}^{4t} \\ t\,\mathrm{e}^{4t} + 6\,\mathrm{e}^{4t}(2t+1) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{4t}(-1+26t) \\ \mathrm{e}^{4t}(6+13t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.28.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 8\lambda + 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
4	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

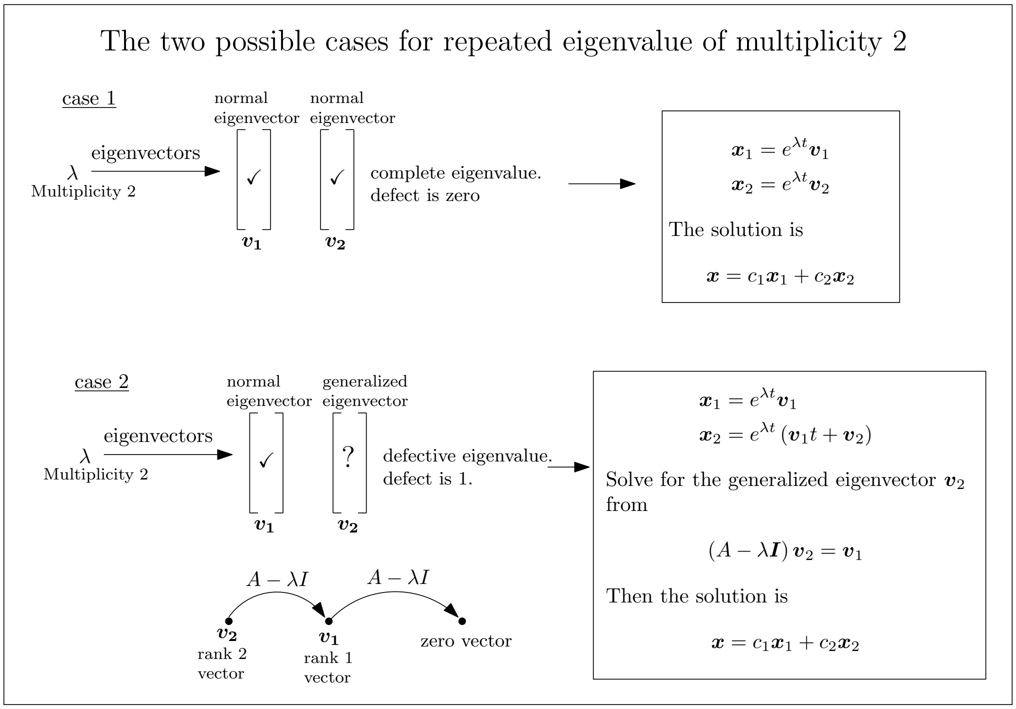


Figure 107: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\, \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 4. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^{4t}\\ &= \begin{bmatrix} 2\,\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \mathrm{e}^{4t}\\ &= \begin{bmatrix} \mathrm{e}^{4t}(2t+1) \\ \mathrm{e}^{4t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{4t}(2t+1) \\ \mathrm{e}^{4t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{4t}(2c_2 t + 2c_1 + c_2) \\ \mathrm{e}^{4t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = -1 \\ y(0) = 6 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -7 \\ c_2 = 13 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} e^{4t}(-1 + 26t) \\ e^{4t}(6 + 13t) \end{bmatrix}$$

The following is the phase plot of the system.

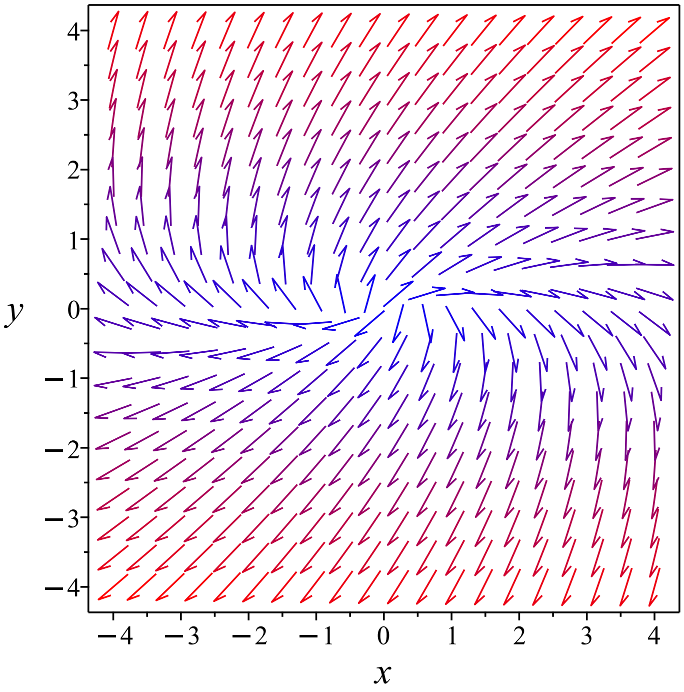
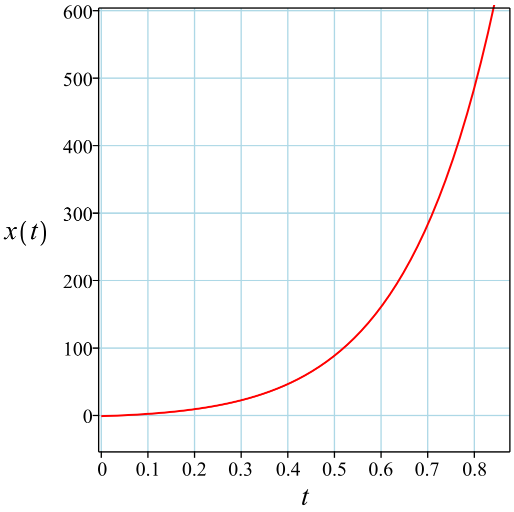
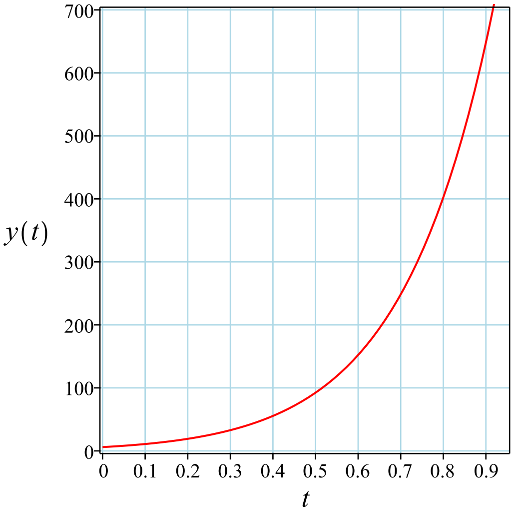


Figure 108: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 29

```
dsolve([diff(x(t),t) = 2*x(t)+4*y(t), diff(y(t),t) = -x(t)+6*y(t), x(0) = -1, y(0) = 6], sin
```

$$x(t) = \mathrm{e}^{4t}(26t - 1)$$
$$y(t) = \frac{\mathrm{e}^{4t}(52t + 24)}{4}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[{x'[t]==2*x[t]+4*y[t],y'[t]==-x[t]+6*y[t]},{x[0]==-1,y[0]==6},{x[t],y[t]},t,IncludeSi
```

$$x(t) \to e^{4t}(26t - 1)$$
$$y(t) \to e^{4t}(13t + 6)$$

10.29 problem 30

Internal problem ID [6755]
Internal file name [`OUTPUT/6003_Sunday_June_05_2022_04_13_19_PM_45609686/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= z(t) \\ y' &= y \\ z'(t) &= x(t) \end{aligned}$$

With initial conditions

$$[x(0) = 1, y(0) = 2, z(0) = 5]$$

10.29.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}}{2}+\frac{e^{t}}{2} & 0 & \frac{e^{t}}{2}-\frac{e^{-t}}{2} \\ 0 & e^{t} & 0 \\ \frac{e^{t}}{2}-\frac{e^{-t}}{2} & 0 & \frac{e^{-t}}{2}+\frac{e^{t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{e^{-t}}{2}+\frac{e^{t}}{2} & 0 & \frac{e^{t}}{2}-\frac{e^{-t}}{2} \\ 0 & e^{t} & 0 \\ \frac{e^{t}}{2}-\frac{e^{-t}}{2} & 0 & \frac{e^{-t}}{2}+\frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -2\,e^{-t}+3\,e^{t} \\ 2\,e^{t} \\ 3\,e^{t}+2\,e^{-t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.29.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-\lambda & 0 & 1\\ 0 & 1-\lambda & 0\\ 1 & 0 & -\lambda\end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^3-\lambda^2-\lambda+1=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-1$$
$$\lambda_2=1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-1$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}-(-1)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}1&0&1\\0&2&0\\1&0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}1&0&1&0\\0&2&0&0\\1&0&1&0\end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = s\}$

Hence the solution is

$$\begin{bmatrix} s \\ t \\ s \end{bmatrix} = \begin{bmatrix} s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this

eigenvalue. The above can be written as

$$\begin{aligned}\begin{bmatrix} s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} \\ &= t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
1	2	2	No	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

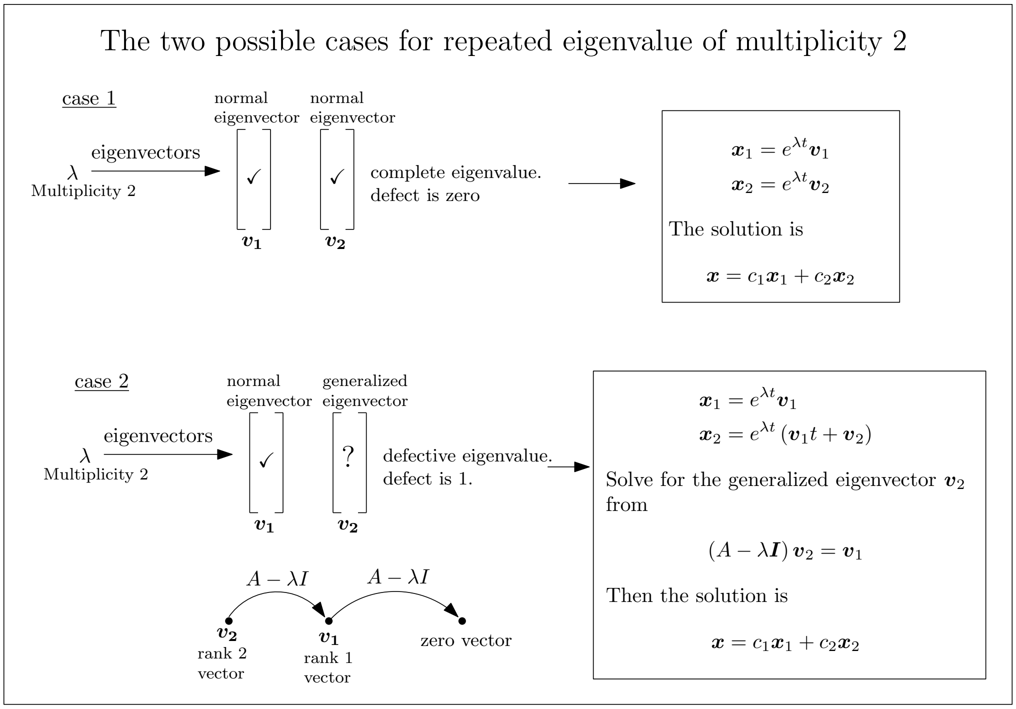


Figure 109: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above.

Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^t \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t\end{aligned}$$

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^t \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-t} \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ \mathrm{e}^{t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ \mathrm{e}^{t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{t} \\ c_3\mathrm{e}^{t} \\ c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 2 \\ z(0) = 5 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 \\ c_3 \\ c_1 + c_2 \end{bmatrix}$$

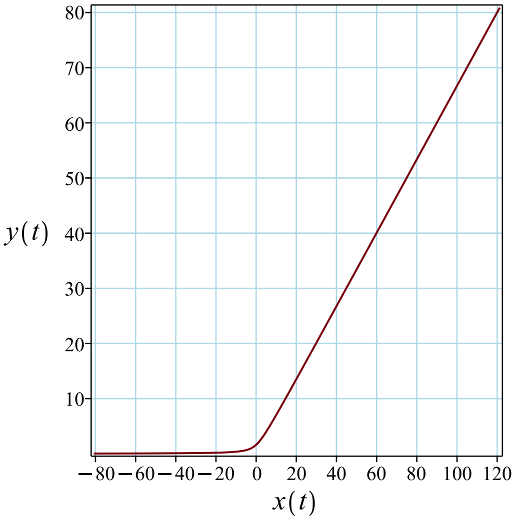
Solving for the constants of integrations gives

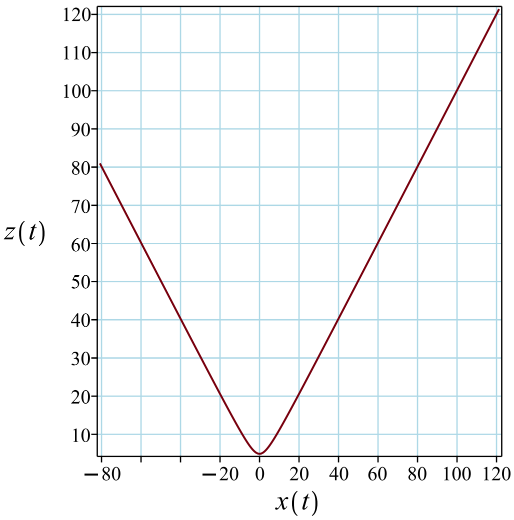
$$\begin{bmatrix} c_1 = 2 \\ c_2 = 3 \\ c_3 = 2 \end{bmatrix}$$

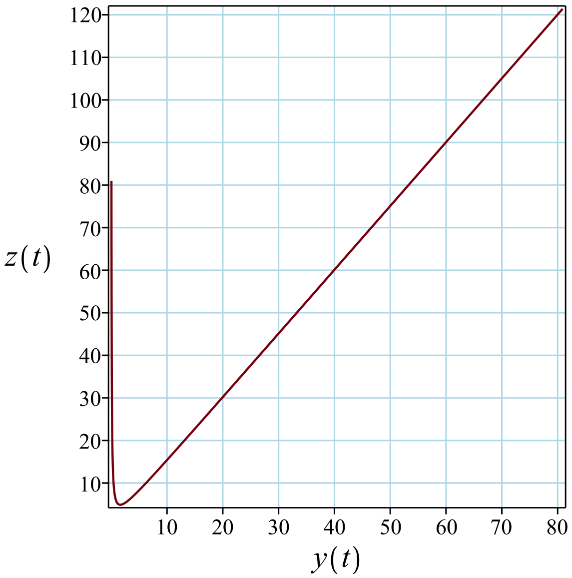
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -2\,\mathrm{e}^{-t} + 3\,\mathrm{e}^{t} \\ 2\,\mathrm{e}^{t} \\ 3\,\mathrm{e}^{t} + 2\,\mathrm{e}^{-t} \end{bmatrix}$$

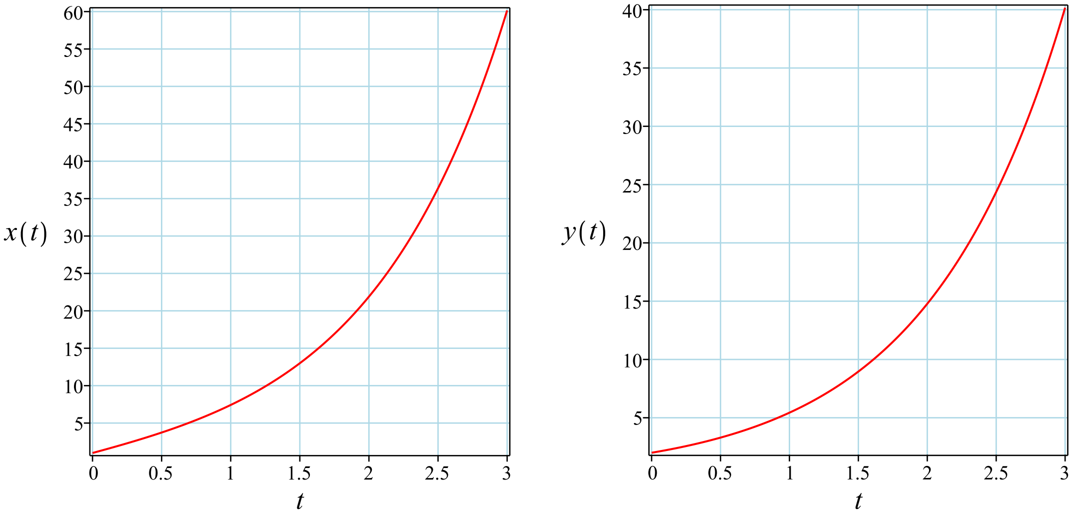
The following are plots of each solution against another.

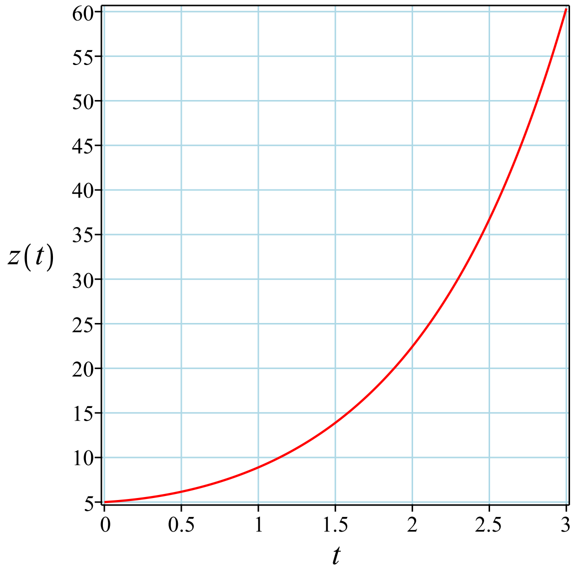






The following are plots of each solution.





✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 37

```
dsolve([diff(x(t),t) = z(t), diff(y(t),t) = y(t), diff(z(t),t) = x(t), x(0) = 1, y(0) = 2, z
```

$$
\begin{aligned}
x(t) &= 3\,\mathrm{e}^{t}-2\,\mathrm{e}^{-t}\\
y(t) &= 2\,\mathrm{e}^{t}\\
z(t) &= 3\,\mathrm{e}^{t}+2\,\mathrm{e}^{-t}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 42

```
DSolve[{x'[t]==z[t],y'[t]==y[t],z'[t]==x[t]},{x[0]==1,y[0]==2,z[0]==5},{x[t],y[t],z[t]},t,In
```

$$
\begin{aligned}
x(t) &\to 3e^{t}-2e^{-t}\\
z(t) &\to 2e^{-t}+3e^{t}\\
y(t) &\to 2e^{t}
\end{aligned}
$$

10.30 problem 33

Internal problem ID [6756]
Internal file name [`OUTPUT/6004_Sunday_June_05_2022_04_13_22_PM_98488337/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 6x(t) - y \\ y' &= 5x(t) + 2y \end{aligned}$$

10.30.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{4t}\cos(t)+2\,e^{4t}\sin(t) & -e^{4t}\sin(t) \\ 5\,e^{4t}\sin(t) & e^{4t}\cos(t)-2\,e^{4t}\sin(t) \end{bmatrix} \\
&= \begin{bmatrix} e^{4t}(\cos(t)+2\sin(t)) & -e^{4t}\sin(t) \\ 5\,e^{4t}\sin(t) & e^{4t}(\cos(t)-2\sin(t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{4t}(\cos(t)+2\sin(t)) & -e^{4t}\sin(t) \\ 5\,e^{4t}\sin(t) & e^{4t}(\cos(t)-2\sin(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{4t}(\cos(t)+2\sin(t))\,c_1 - e^{4t}\sin(t)\,c_2 \\ 5\,e^{4t}\sin(t)\,c_1 + e^{4t}(\cos(t)-2\sin(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{4t}((2c_1-c_2)\sin(t)+c_1\cos(t)) \\ ((5c_1-2c_2)\sin(t)+c_2\cos(t))\,e^{4t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.30.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 8\lambda + 17 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4 + i$$
$$\lambda_2 = 4 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$4+i$	1	complex eigenvalue
$4-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 4 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} - (4 - i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+i & -1 \\ 5 & -2+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i & -1 & 0 \\ 5 & -2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-2 + i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{2}{5} - \frac{i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{5} - \frac{i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} - (4+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-2-i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{2}{5} + \frac{i}{5}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{5} + \frac{i}{5}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$4+i$	1	1	No	$\begin{bmatrix} \frac{2}{5}+\frac{i}{5} \\ 1 \end{bmatrix}$
$4-i$	1	1	No	$\begin{bmatrix} \frac{2}{5}-\frac{i}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{2}{5}+\frac{i}{5}\right) \mathrm{e}^{(4+i)t} \\ \mathrm{e}^{(4+i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{2}{5}-\frac{i}{5}\right) \mathrm{e}^{(4-i)t} \\ \mathrm{e}^{(4-i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{5}+\frac{i}{5}\right) c_1\mathrm{e}^{(4+i)t} + \left(\frac{2}{5}-\frac{i}{5}\right) c_2\mathrm{e}^{(4-i)t} \\ c_1\mathrm{e}^{(4+i)t} + c_2\mathrm{e}^{(4-i)t} \end{bmatrix}$$

The following is the phase plot of the system.

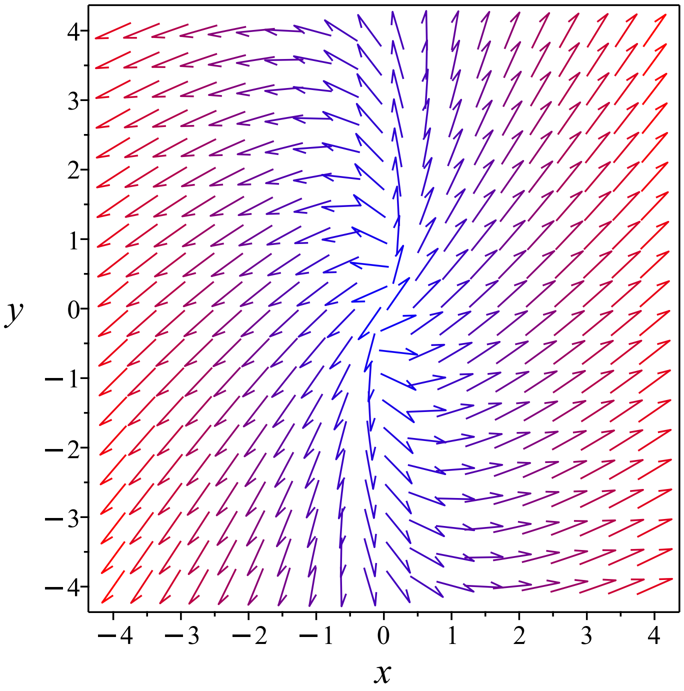


Figure 110: Phase plot

10.30.3 Maple step by step solution

Let's solve

$[x'(t) = 6x(t) - y, y' = 5x(t) + 2y]$

- Define vector

$$\vec{x}(t) = \left[\begin{array}{c} x(t) \\ y \end{array}\right]$$

- Convert system into a vector equation

$$\vec{x}'(t) = \left[\begin{array}{cc} 6 & -1 \\ 5 & 2 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{cc} 6 & -1 \\ 5 & 2 \end{array}\right] \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[4-\mathrm{I}, \begin{bmatrix} \frac{2}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[4+\mathrm{I}, \begin{bmatrix} \frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$
- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[4-\mathrm{I}, \begin{bmatrix} \frac{2}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$
- Solution from eigenpair

  $$\mathrm{e}^{(4-\mathrm{I})t} \cdot \begin{bmatrix} \frac{2}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}$$
- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{4t} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} \frac{2}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}$$
- Simplify expression

  $$\mathrm{e}^{4t} \cdot \begin{bmatrix} \left(\frac{2}{5}-\frac{\mathrm{I}}{5}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$
- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_1(t) = \mathrm{e}^{4t} \cdot \begin{bmatrix} \frac{2\cos(t)}{5} - \frac{\sin(t)}{5} \\ \cos(t) \end{bmatrix}, \overrightarrow{x}_2(t) = \mathrm{e}^{4t} \cdot \begin{bmatrix} -\frac{2\sin(t)}{5} - \frac{\cos(t)}{5} \\ -\sin(t) \end{bmatrix}\right]$$
- General solution to the system of ODEs

  $\overrightarrow{x} = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t)$
- Substitute solutions into the general solution

$$\vec{x} = \mathrm{e}^{4t} c_1 \cdot \left[\begin{array}{c} \frac{2\cos(t)}{5} - \frac{\sin(t)}{5} \\ \cos(t) \end{array}\right] + c_2 \mathrm{e}^{4t} \cdot \left[\begin{array}{c} -\frac{2\sin(t)}{5} - \frac{\cos(t)}{5} \\ -\sin(t) \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{2\left(\left(c_1 - \frac{c_2}{2}\right)\cos(t) - \frac{\sin(t)(c_1+2c_2)}{2}\right)\mathrm{e}^{4t}}{5} \\ \mathrm{e}^{4t}(c_1 \cos(t) - c_2 \sin(t)) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{2\left(\left(c_1 - \frac{c_2}{2}\right)\cos(t) - \frac{\sin(t)(c_1+2c_2)}{2}\right)\mathrm{e}^{4t}}{5}, y = \mathrm{e}^{4t}(c_1 \cos(t) - c_2 \sin(t))\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
dsolve([diff(x(t),t)=6*x(t)-y(t),diff(y(t),t)=5*x(t)+2*y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{4t}(c_1 \sin(t) + c_2 \cos(t))$$
$$y(t) = \mathrm{e}^{4t}(-c_1 \cos(t) + c_2 \sin(t) + 2c_1 \sin(t) + 2c_2 \cos(t))$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 55

```
DSolve[{x'[t]==6*x[t]-y[t],y'[t]==5*x[t]+2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to e^{4t}(c_1(2\sin(t) + \cos(t)) - c_2 \sin(t))$$
$$y(t) \to e^{4t}(c_2 \cos(t) + (5c_1 - 2c_2)\sin(t))$$

10.31 problem 34

Internal problem ID [6757]
Internal file name [`OUTPUT/6005_Sunday_June_05_2022_04_13_25_PM_13874635/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= x(t) + y \\ y' &= -2x(t) - y \end{aligned}$$

10.31.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \sin(t) + \cos(t) & \sin(t) \\ -2\sin(t) & \cos(t) - \sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \sin(t)+\cos(t) & \sin(t) \\ -2\sin(t) & \cos(t)-\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (\sin(t)+\cos(t))\, c_1 + \sin(t)\, c_2 \\ -2\sin(t)\, c_1 + (\cos(t)-\sin(t))\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1+c_2)\sin(t) + c_1\cos(t) \\ (-2c_1-c_2)\sin(t) + c_2\cos(t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.31.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+i & 1 & 0 \\ -2 & -1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1-i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2}+\frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} - (i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 1-i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
i	1	1	No	$\left[\begin{array}{c} -\frac{1}{2}-\frac{i}{2} \\ 1 \end{array}\right]$
$-i$	1	1	No	$\left[\begin{array}{c} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{it} \\ \mathrm{e}^{it} \end{array}\right] + c_2 \left[\begin{array}{c} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \left(-\frac{1}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{it} + \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{-it} \\ c_1 \mathrm{e}^{it} + c_2 \mathrm{e}^{-it} \end{array}\right]$$

The following is the phase plot of the system.

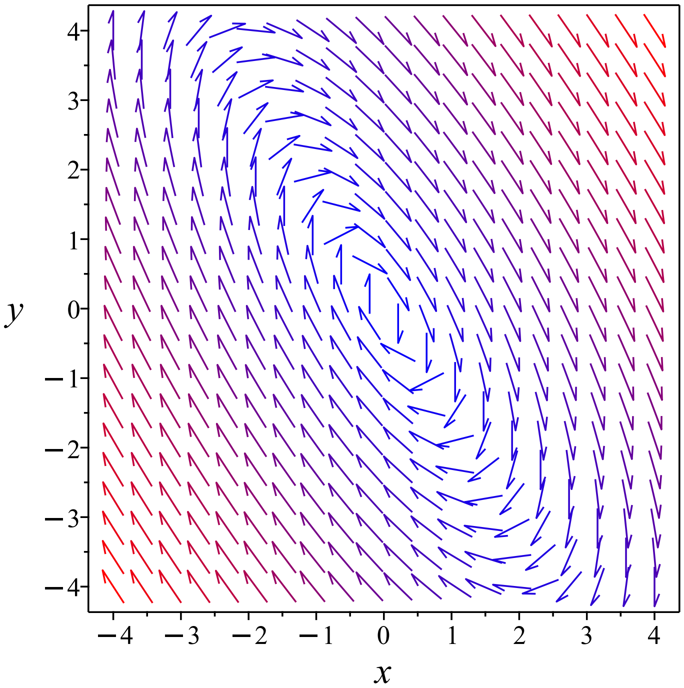


Figure 111: Phase plot

10.31.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + y, y' = -2x(t) - y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 1 & 1 \\ -2 & -1 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} \left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{x}_{1}(t) = \left[\begin{array}{c} -\frac{\cos(t)}{2}+\frac{\sin(t)}{2} \\ \cos(t) \end{array}\right], \overrightarrow{x}_{2}(t) = \left[\begin{array}{c} \frac{\sin(t)}{2}+\frac{\cos(t)}{2} \\ -\sin(t) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_{1}\overrightarrow{x}_{1}(t) + c_{2}\overrightarrow{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\vec{x} = \begin{bmatrix} c_2\left(\frac{\sin(t)}{2}+\frac{\cos(t)}{2}\right)+c_1\left(-\frac{\cos(t)}{2}+\frac{\sin(t)}{2}\right) \\ c_1\cos(t)-c_2\sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{(-c_1+c_2)\cos(t)}{2}+\frac{(c_1+c_2)\sin(t)}{2} \\ c_1\cos(t)-c_2\sin(t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t)=\frac{(-c_1+c_2)\cos(t)}{2}+\frac{(c_1+c_2)\sin(t)}{2}, y=c_1\cos(t)-c_2\sin(t)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve([diff(x(t),t)=x(t)+y(t),diff(y(t),t)=-2*x(t)-y(t)],singsol=all)
```

$$x(t) = c_1\sin(t)+c_2\cos(t)$$
$$y(t) = c_1\cos(t)-c_2\sin(t)-c_1\sin(t)-c_2\cos(t)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 39

```
DSolve[{x'[t]==x[t]+y[t],y'[t]==-2*x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True
```

$$x(t) \to c_1\cos(t)+(c_1+c_2)\sin(t)$$
$$y(t) \to c_2\cos(t)-(2c_1+c_2)\sin(t)$$

10.32 problem 35

Internal problem ID [6758]
Internal file name [`OUTPUT/6006_Sunday_June_05_2022_04_13_27_PM_77480091/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 5x(t) + y \\ y' &= -2x(t) + 3y \end{aligned}$$

10.32.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^{4t}\cos(t)+\mathrm{e}^{4t}\sin(t) & \mathrm{e}^{4t}\sin(t) \\ -2\,\mathrm{e}^{4t}\sin(t) & \mathrm{e}^{4t}\cos(t)-\mathrm{e}^{4t}\sin(t) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{4t}(\sin(t)+\cos(t)) & \mathrm{e}^{4t}\sin(t) \\ -2\,\mathrm{e}^{4t}\sin(t) & \mathrm{e}^{4t}(\cos(t)-\sin(t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{4t}(\sin(t)+\cos(t)) & \mathrm{e}^{4t}\sin(t) \\ -2\,\mathrm{e}^{4t}\sin(t) & \mathrm{e}^{4t}(\cos(t)-\sin(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{4t}(\sin(t)+\cos(t))\,c_1+\mathrm{e}^{4t}\sin(t)\,c_2 \\ -2\,\mathrm{e}^{4t}\sin(t)\,c_1+\mathrm{e}^{4t}(\cos(t)-\sin(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{4t}((c_1+c_2)\sin(t)+c_1\cos(t)) \\ \mathrm{e}^{4t}(c_2\cos(t)-2c_1\sin(t)-\sin(t)\,c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.32.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 8\lambda + 17 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4 + i$$
$$\lambda_2 = 4 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$4+i$	1	complex eigenvalue
$4-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 4 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} - (4-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+i & 1 & 0 \\ -2 & -1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1-i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} - (4+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 1-i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$4+i$	1	1	No	$\begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$
$4-i$	1	1	No	$\begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(4+i)t} \\ \mathrm{e}^{(4+i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(4-i)t} \\ \mathrm{e}^{(4-i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{(4+i)t} + \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{(4-i)t} \\ c_1 \mathrm{e}^{(4+i)t} + c_2 \mathrm{e}^{(4-i)t} \end{bmatrix}$$

The following is the phase plot of the system.

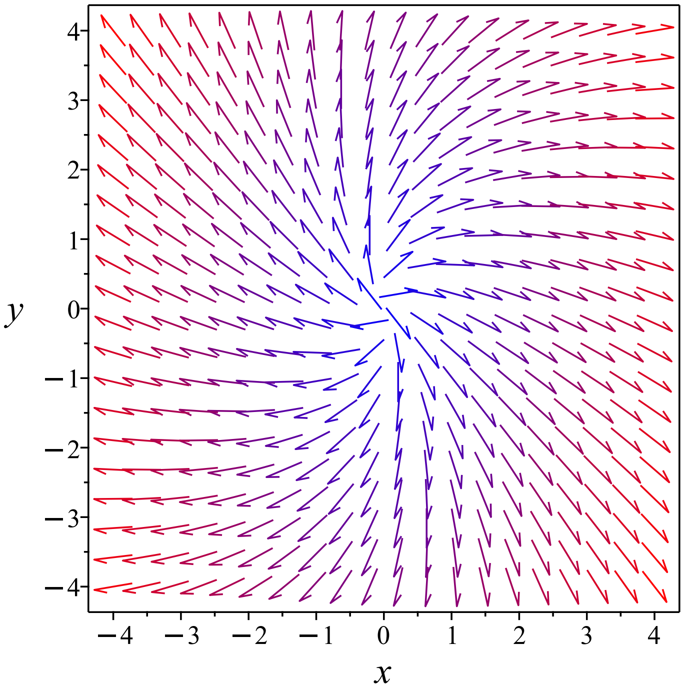


Figure 112: Phase plot

10.32.3 Maple step by step solution

Let's solve

$[x'(t) = 5x(t) + y, y' = -2x(t) + 3y]$

- Define vector

$$\vec{x}(t) = \left[\begin{array}{c} x(t) \\ y \end{array}\right]$$

- Convert system into a vector equation

$$\vec{x}'(t) = \left[\begin{array}{cc} 5 & 1 \\ -2 & 3 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{cc} 5 & 1 \\ -2 & 3 \end{array}\right] \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 5 & 1 \\ -2 & 3 \end{array}\right]$$

- Rewrite the system as

  $\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[4-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[4+\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[4-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{(4-\mathrm{I})t} \cdot \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{4t} \cdot \left(\cos\left(t\right) - \mathrm{I}\sin\left(t\right)\right) \cdot \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

  $$\mathrm{e}^{4t} \cdot \left[\begin{array}{c} \left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)\left(\cos\left(t\right) - \mathrm{I}\sin\left(t\right)\right) \\ \cos\left(t\right) - \mathrm{I}\sin\left(t\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_{1}(t) = \mathrm{e}^{4t} \cdot \left[\begin{array}{c} -\frac{\cos(t)}{2}+\frac{\sin(t)}{2} \\ \cos\left(t\right) \end{array}\right], \overrightarrow{x}_{2}(t) = \mathrm{e}^{4t} \cdot \left[\begin{array}{c} \frac{\sin(t)}{2}+\frac{\cos(t)}{2} \\ -\sin\left(t\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_{1}\overrightarrow{x}_{1}(t) + c_{2}\overrightarrow{x}_{2}(t)$

- Substitute solutions into the general solution

$$\vec{x} = \mathrm{e}^{4t} c_1 \cdot \begin{bmatrix} -\frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ \cos(t) \end{bmatrix} + c_2 \mathrm{e}^{4t} \cdot \begin{bmatrix} \frac{\sin(t)}{2} + \frac{\cos(t)}{2} \\ -\sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{((c_1 - c_2)\cos(t) - (c_1 + c_2)\sin(t))\mathrm{e}^{4t}}{2} \\ \mathrm{e}^{4t}(c_1 \cos(t) - c_2 \sin(t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{((c_1 - c_2)\cos(t) - (c_1 + c_2)\sin(t))\mathrm{e}^{4t}}{2}, y = \mathrm{e}^{4t}(c_1 \cos(t) - c_2 \sin(t))\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 46

```
dsolve([diff(x(t),t)=5*x(t)+y(t),diff(y(t),t)=-2*x(t)+3*y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{4t}(c_1 \sin(t) + c_2 \cos(t))$$
$$y(t) = -\mathrm{e}^{4t}(c_1 \sin(t) + c_2 \sin(t) - c_1 \cos(t) + c_2 \cos(t))$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 51

```
DSolve[{x'[t]==5*x[t]+y[t],y'[t]==-2*x[t]+3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to e^{4t}(c_1 \cos(t) + (c_1 + c_2)\sin(t))$$
$$y(t) \to e^{4t}(c_2 \cos(t) - (2c_1 + c_2)\sin(t))$$

10.33 problem 36

Internal problem ID [6759]
Internal file name [`OUTPUT/6007_Sunday_June_05_2022_04_13_29_PM_73816838/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 4x(t) + 5y \\ y' &= -2x(t) + 6y \end{aligned}$$

10.33.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned}
e^{At} &= \begin{bmatrix} e^{5t}\cos(3t)-\frac{e^{5t}\sin(3t)}{3} & \frac{5\,e^{5t}\sin(3t)}{3} \\ -\frac{2\,e^{5t}\sin(3t)}{3} & e^{5t}\cos(3t)+\frac{e^{5t}\sin(3t)}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{5t}(3\cos(3t)-\sin(3t))}{3} & \frac{5\,e^{5t}\sin(3t)}{3} \\ -\frac{2\,e^{5t}\sin(3t)}{3} & \frac{e^{5t}(3\cos(3t)+\sin(3t))}{3} \end{bmatrix}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{5t}(3\cos(3t)-\sin(3t))}{3} & \frac{5\,e^{5t}\sin(3t)}{3} \\ -\frac{2\,e^{5t}\sin(3t)}{3} & \frac{e^{5t}(3\cos(3t)+\sin(3t))}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{5t}(3\cos(3t)-\sin(3t))c_1}{3}+\frac{5\,e^{5t}\sin(3t)c_2}{3} \\ -\frac{2\,e^{5t}\sin(3t)c_1}{3}+\frac{e^{5t}(3\cos(3t)+\sin(3t))c_2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{5t}(-c_1+5c_2)\sin(3t)}{3}+e^{5t}\cos(3t)\,c_1 \\ -\frac{2\,e^{5t}\left(\left(c_1-\frac{c_2}{2}\right)\sin(3t)-\frac{3c_2\cos(3t)}{2}\right)}{3} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.33.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 10\lambda + 34 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 5+3i \\ \lambda_2 &= 5-3i \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$5-3i$	1	complex eigenvalue
$5+3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 5-3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{aligned} \left(\begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} - (5-3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1+3i & 5 \\ -2 & 1+3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+3i & 5 & 0 \\ -2 & 1+3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{5} - \frac{3i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1+3i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+3i & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{3i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{3i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{3i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{3i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1+3i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5 + 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} - (5+3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1-3i & 5 \\ -2 & 1-3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1-3i & 5 & 0 \\ -2 & 1-3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{5} + \frac{3i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1-3i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-3i & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{3i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{3i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{3i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{3i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$5+3i$	1	1	No	$\begin{bmatrix} \frac{1}{2}-\frac{3i}{2} \\ 1 \end{bmatrix}$
$5-3i$	1	1	No	$\begin{bmatrix} \frac{1}{2}+\frac{3i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}-\frac{3i}{2}\right) \mathrm{e}^{(5+3i)t} \\ \mathrm{e}^{(5+3i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}+\frac{3i}{2}\right) \mathrm{e}^{(5-3i)t} \\ \mathrm{e}^{(5-3i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}-\frac{3i}{2}\right) c_1 \mathrm{e}^{(5+3i)t} + \left(\frac{1}{2}+\frac{3i}{2}\right) c_2 \mathrm{e}^{(5-3i)t} \\ c_1 \mathrm{e}^{(5+3i)t} + c_2 \mathrm{e}^{(5-3i)t} \end{bmatrix}$$

The following is the phase plot of the system.

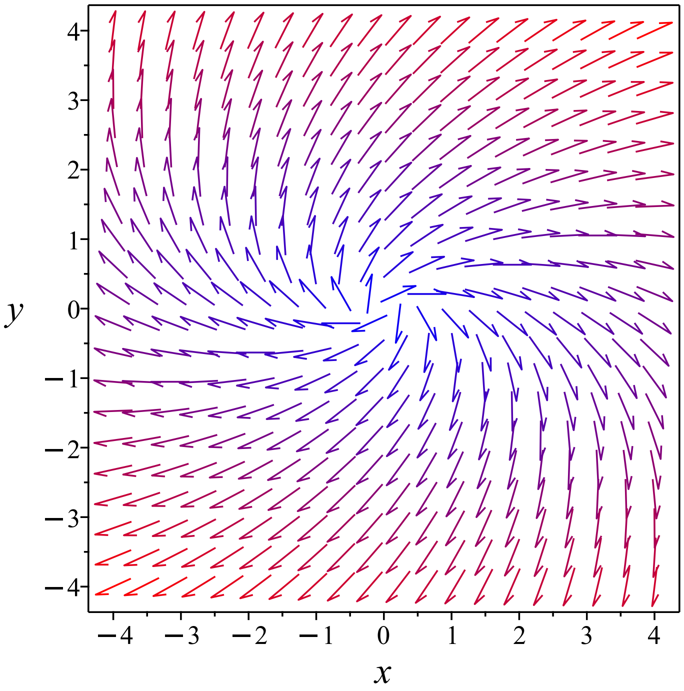


Figure 113: Phase plot

10.33.3 Maple step by step solution

Let's solve

$[x'(t) = 4x(t) + 5y, y' = -2x(t) + 6y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 4 & 5 \\ -2 & 6 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[5-3\,\mathrm{I}, \left[\begin{array}{c} \frac{1}{2}+\frac{3\,\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[5+3\,\mathrm{I}, \left[\begin{array}{c} \frac{1}{2}-\frac{3\,\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[5-3\,\mathrm{I}, \left[\begin{array}{c} \frac{1}{2}+\frac{3\,\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(5-3\,\mathrm{I})t} \cdot \left[\begin{array}{c} \frac{1}{2}+\frac{3\,\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{5t} \cdot \left(\cos(3t) - \mathrm{I}\sin(3t)\right) \cdot \left[\begin{array}{c} \frac{1}{2}+\frac{3\,\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{5t} \cdot \left[\begin{array}{c} \left(\frac{1}{2}+\frac{3\,\mathrm{I}}{2}\right)\left(\cos(3t) - \mathrm{I}\sin(3t)\right) \\ \cos(3t) - \mathrm{I}\sin(3t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{x}_{1}(t) = \mathrm{e}^{5t} \cdot \left[\begin{array}{c} \frac{\cos(3t)}{2}+\frac{3\sin(3t)}{2} \\ \cos(3t) \end{array}\right], \overrightarrow{x}_{2}(t) = \mathrm{e}^{5t} \cdot \left[\begin{array}{c} -\frac{\sin(3t)}{2}+\frac{3\cos(3t)}{2} \\ -\sin(3t) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_{1}\overrightarrow{x}_{1}(t) + c_{2}\overrightarrow{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{5t} \cdot \begin{bmatrix} \frac{\cos(3t)}{2} + \frac{3\sin(3t)}{2} \\ \cos(3t) \end{bmatrix} + c_2 e^{5t} \cdot \begin{bmatrix} -\frac{\sin(3t)}{2} + \frac{3\cos(3t)}{2} \\ -\sin(3t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{\left((c_1+3c_2)\cos(3t)+3\sin(3t)\left(c_1-\frac{c_2}{3}\right)\right)e^{5t}}{2} \\ e^{5t}(c_1\cos(3t) - c_2\sin(3t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{\left((c_1+3c_2)\cos(3t)+3\sin(3t)\left(c_1-\frac{c_2}{3}\right)\right)e^{5t}}{2}, y = e^{5t}(c_1\cos(3t) - c_2\sin(3t))\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
dsolve([diff(x(t),t)=4*x(t)+5*y(t),diff(y(t),t)=-2*x(t)+6*y(t)],singsol=all)
```

$$x(t) = e^{5t}(c_1\sin(3t) + c_2\cos(3t))$$
$$y(t) = \frac{e^{5t}(c_1\sin(3t) - 3c_2\sin(3t) + 3c_1\cos(3t) + c_2\cos(3t))}{5}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 69

```
DSolve[{x'[t]==4*x[t]+5*y[t],y'[t]==-2*x[t]+6*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -
```

$$x(t) \to \frac{1}{3}e^{5t}(3c_1\cos(3t) - (c_1 - 5c_2)\sin(3t))$$
$$y(t) \to \frac{1}{3}e^{5t}(3c_2\cos(3t) + (c_2 - 2c_1)\sin(3t))$$

10.34 problem 37

Internal problem ID [6760]
Internal file name [`OUTPUT/6008_Sunday_June_05_2022_04_13_32_PM_73659519/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 4x(t) - 5y \\ y' &= 5x(t) - 4y \end{aligned}$$

10.34.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(3t) + \frac{4\sin(3t)}{3} & -\frac{5\sin(3t)}{3} \\ \frac{5\sin(3t)}{3} & \cos(3t) - \frac{4\sin(3t)}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(3t) + \frac{4\sin(3t)}{3} & -\frac{5\sin(3t)}{3} \\ \frac{5\sin(3t)}{3} & \cos(3t) - \frac{4\sin(3t)}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\cos(3t) + \frac{4\sin(3t)}{3}\right) c_1 - \frac{5\sin(3t)c_2}{3} \\ \frac{5\sin(3t)c_1}{3} + \left(\cos(3t) - \frac{4\sin(3t)}{3}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(4c_1-5c_2)\sin(3t)}{3} + c_1\cos(3t) \\ \frac{(5c_1-4c_2)\sin(3t)}{3} + c_2\cos(3t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.34.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & -5 \\ 5 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-3i$	1	complex eigenvalue
$3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} - (-3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4+3i & -5 \\ 5 & -4+3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4+3i & -5 & 0 \\ 5 & -4+3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{4}{5} + \frac{3i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 4+3i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4+3i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{4}{5} - \frac{3i}{5}\right) t\right\}$

Hence the solution is

$$\left[\begin{array}{c} \left(\frac{4}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{array}\right] = \left[\begin{array}{c} \left(\frac{4}{5} - \frac{3i}{5}\right) t \\ t \end{array}\right]$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} \left(\frac{4}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{array}\right] = t \left[\begin{array}{c} \frac{4}{5} - \frac{3i}{5} \\ 1 \end{array}\right]$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} \left(\frac{4}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{array}\right] = \left[\begin{array}{c} \frac{4}{5} - \frac{3i}{5} \\ 1 \end{array}\right]$$

Which is normalized to

$$\left[\begin{array}{c} \left(\frac{4}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{array}\right] = \left[\begin{array}{c} 4 - 3i \\ 5 \end{array}\right]$$

Considering the eigenvalue $\lambda_2 = 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\left[\begin{array}{cc} 4 & -5 \\ 5 & -4 \end{array}\right] - (3i)\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right)\left[\begin{array}{c} v_1 \\ v_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

$$\left[\begin{array}{cc} 4 - 3i & -5 \\ 5 & -4 - 3i \end{array}\right]\left[\begin{array}{c} v_1 \\ v_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 - 3i & -5 & 0 \\ 5 & -4 - 3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{4}{5} - \frac{3i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 - 3i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4-3i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{4}{5} + \frac{3i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{4}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{4}{5} + \frac{3i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{4}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{5} + \frac{3i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{4}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{5} + \frac{3i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{4}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 4+3i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$3i$	1	1	No	$\left[\begin{array}{c}\frac{4}{5}+\frac{3i}{5} \\ 1\end{array}\right]$
$-3i$	1	1	No	$\left[\begin{array}{c}\frac{4}{5}-\frac{3i}{5} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} \left(\frac{4}{5}+\frac{3i}{5}\right) \mathrm{e}^{3it} \\ \mathrm{e}^{3it} \end{array}\right] + c_2 \left[\begin{array}{c} \left(\frac{4}{5}-\frac{3i}{5}\right) \mathrm{e}^{-3it} \\ \mathrm{e}^{-3it} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \left(\frac{4}{5}+\frac{3i}{5}\right) c_1 \mathrm{e}^{3it} + \left(\frac{4}{5}-\frac{3i}{5}\right) c_2 \mathrm{e}^{-3it} \\ c_1 \mathrm{e}^{3it} + c_2 \mathrm{e}^{-3it} \end{array}\right]$$

The following is the phase plot of the system.

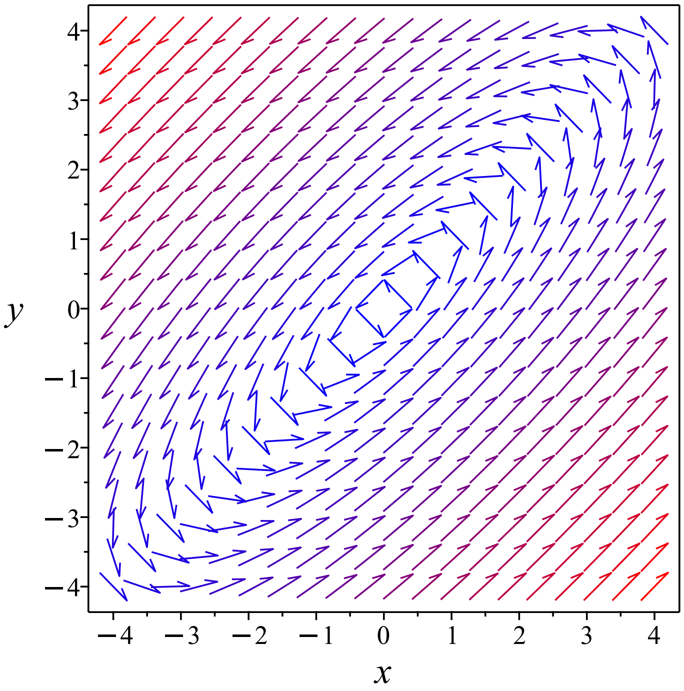


Figure 114: Phase plot

10.34.3 Maple step by step solution

Let's solve

$[x'(t) = 4x(t) - 5y, y' = 5x(t) - 4y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 4 & -5 \\ 5 & -4 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3\,\mathrm{I}, \left[\begin{array}{c} \frac{4}{5}-\frac{3\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right], \left[3\,\mathrm{I}, \left[\begin{array}{c} \frac{4}{5}+\frac{3\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3\,\mathrm{I}, \left[\begin{array}{c} \frac{4}{5}-\frac{3\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-3\,\mathrm{I} t} \cdot \left[\begin{array}{c} \frac{4}{5}-\frac{3\,\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(3 t\right)-\mathrm{I} \sin\left(3 t\right)\right) \cdot \left[\begin{array}{c} \frac{4}{5}-\frac{3\,\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} \left(\frac{4}{5}-\frac{3\,\mathrm{I}}{5}\right) \left(\cos\left(3 t\right)-\mathrm{I} \sin\left(3 t\right)\right) \\ \cos\left(3 t\right)-\mathrm{I} \sin\left(3 t\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{x}_{1}(t) = \left[\begin{array}{c} \frac{4 \cos\left(3 t\right)}{5}-\frac{3 \sin\left(3 t\right)}{5} \\ \cos\left(3 t\right) \end{array}\right], \overrightarrow{x}_{2}(t) = \left[\begin{array}{c} -\frac{4 \sin\left(3 t\right)}{5}-\frac{3 \cos\left(3 t\right)}{5} \\ -\sin\left(3 t\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_{1} \overrightarrow{x}_{1}(t) + c_{2} \overrightarrow{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\vec{x} = \left[\begin{array}{c} c_2\left(-\frac{4\sin(3t)}{5}-\frac{3\cos(3t)}{5}\right)+c_1\left(\frac{4\cos(3t)}{5}-\frac{3\sin(3t)}{5}\right) \\ c_1\cos(3t)-c_2\sin(3t) \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{(4c_1-3c_2)\cos(3t)}{5}-\frac{3\left(c_1+\frac{4c_2}{3}\right)\sin(3t)}{5} \\ c_1\cos(3t)-c_2\sin(3t) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(4c_1-3c_2)\cos(3t)}{5}-\frac{3\left(c_1+\frac{4c_2}{3}\right)\sin(3t)}{5}, y = c_1\cos(3t)-c_2\sin(3t)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
dsolve([diff(x(t),t)=4*x(t)-5*y(t),diff(y(t),t)=5*x(t)-4*y(t)],singsol=all)
```

$$x(t) = c_1\sin(3t)+c_2\cos(3t)$$
$$y(t) = -\frac{3c_1\cos(3t)}{5}+\frac{3c_2\sin(3t)}{5}+\frac{4c_1\sin(3t)}{5}+\frac{4c_2\cos(3t)}{5}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 58

```
DSolve[{x'[t]==4*x[t]-5*y[t],y'[t]==5*x[t]-4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to c_1\cos(3t)+\frac{1}{3}(4c_1-5c_2)\sin(3t)$$
$$y(t) \to c_2\cos(3t)+\frac{1}{3}(5c_1-4c_2)\sin(3t)$$

10.35 problem 38

Internal problem ID [6761]
Internal file name [`OUTPUT/6009_Sunday_June_05_2022_04_13_34_PM_30740301/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= x(t) - 8y \\ y' &= x(t) - 3y \end{aligned}$$

10.35.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \cos(2t)\,\mathrm{e}^{-t}+\sin(2t)\,\mathrm{e}^{-t} & -4\sin(2t)\,\mathrm{e}^{-t} \\ \frac{\sin(2t)\mathrm{e}^{-t}}{2} & \cos(2t)\,\mathrm{e}^{-t}-\sin(2t)\,\mathrm{e}^{-t} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t}(\cos(2t)+\sin(2t)) & -4\sin(2t)\,\mathrm{e}^{-t} \\ \frac{\sin(2t)\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t}(\cos(2t)-\sin(2t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-t}(\cos(2t)+\sin(2t)) & -4\sin(2t)\,\mathrm{e}^{-t} \\ \frac{\sin(2t)\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t}(\cos(2t)-\sin(2t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t}(\cos(2t)+\sin(2t))\,c_1-4\sin(2t)\,\mathrm{e}^{-t}c_2 \\ \frac{\sin(2t)\mathrm{e}^{-t}c_1}{2}+\mathrm{e}^{-t}(\cos(2t)-\sin(2t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((c_1-4c_2)\sin(2t)+c_1\cos(2t))\,\mathrm{e}^{-t} \\ \frac{((c_1-2c_2)\sin(2t)+2c_2\cos(2t))\mathrm{e}^{-t}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.35.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -8 \\ 1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= -1+2i \\ \lambda_2 &= -1-2i \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1-2i$	1	complex eigenvalue
$-1+2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1-2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} - (-1-2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2+2i & -8 \\ 1 & -2+2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+2i & -8 & 0 \\ 1 & -2+2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{4} + \frac{i}{4}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+2i & -8 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+2i & -8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2-2i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2-2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2-2i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2-2\,\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2-2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2-2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2-2i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1+2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} - (-1+2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-2i & -8 \\ 1 & -2-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-2i & -8 & 0 \\ 1 & -2-2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{4} - \frac{i}{4}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2-2i & -8 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-2i & -8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2+2i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2+2\ \mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2+2i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2+2\ \mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2+2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2+2\ \mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2+2i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-1+2i$	1	1	No	$\begin{bmatrix} 2+2i \\ 1 \end{bmatrix}$
$-1-2i$	1	1	No	$\begin{bmatrix} 2-2i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} (2+2i)\,\mathrm{e}^{(-1+2i)t} \\ \mathrm{e}^{(-1+2i)t} \end{array}\right] + c_2 \left[\begin{array}{c} (2-2i)\,\mathrm{e}^{(-1-2i)t} \\ \mathrm{e}^{(-1-2i)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} (2+2i)\,c_1\mathrm{e}^{(-1+2i)t} + (2-2i)\,c_2\mathrm{e}^{(-1-2i)t} \\ c_1\mathrm{e}^{(-1+2i)t} + c_2\mathrm{e}^{(-1-2i)t} \end{array}\right]$$

The following is the phase plot of the system.

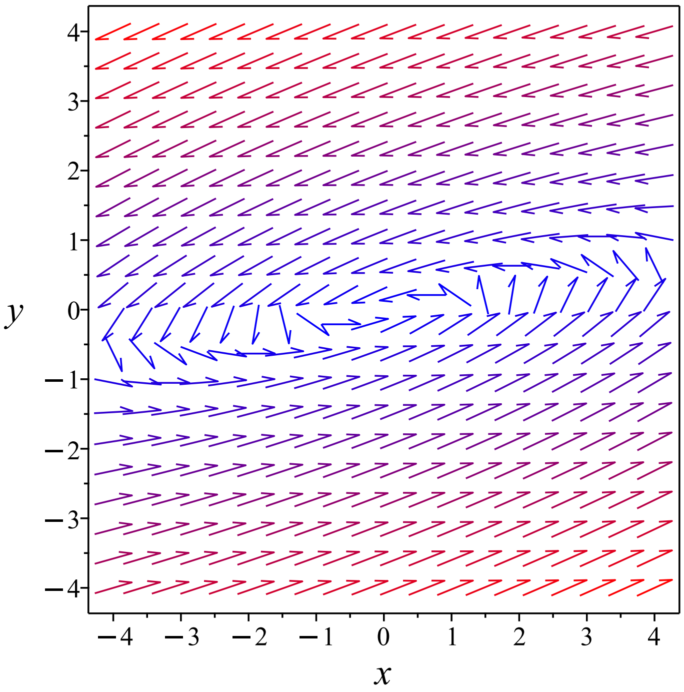


Figure 115: Phase plot

10.35.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) - 8y, y' = x(t) - 3y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1 - 2\,\mathrm{I}, \begin{bmatrix} 2 - 2\,\mathrm{I} \\ 1 \end{bmatrix}\right], \left[-1 + 2\,\mathrm{I}, \begin{bmatrix} 2 + 2\,\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - 2\,\mathrm{I}, \begin{bmatrix} 2 - 2\,\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-2\,\mathrm{I})t} \cdot \begin{bmatrix} 2 - 2\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} 2-2\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-t} \cdot \begin{bmatrix} (2-2\,\mathrm{I})(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = e^{-t} \cdot \begin{bmatrix} -2\sin(2t) + 2\cos(2t) \\ \cos(2t) \end{bmatrix}, \vec{x}_2(t) = e^{-t} \cdot \begin{bmatrix} -2\cos(2t) - 2\sin(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = e^{-t}c_1 \cdot \begin{bmatrix} -2\sin(2t) + 2\cos(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} -2\cos(2t) - 2\sin(2t) \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 2((c_1 - c_2)\cos(2t) - \sin(2t)(c_1 + c_2))\,e^{-t} \\ e^{-t}(c_1\cos(2t) - c_2\sin(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = 2((c_1 - c_2)\cos(2t) - \sin(2t)(c_1 + c_2))\,e^{-t}, y = e^{-t}(c_1\cos(2t) - c_2\sin(2t))\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
dsolve([diff(x(t),t)=x(t)-8*y(t),diff(y(t),t)=x(t)-3*y(t)],singsol=all)
```

$$x(t) = e^{-t}(c_1\sin(2t) + c_2\cos(2t))$$
$$y(t) = \frac{e^{-t}(c_1\sin(2t) + c_2\sin(2t) - c_1\cos(2t) + c_2\cos(2t))}{4}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 64

```
DSolve[{x'[t]==x[t]-8*y[t],y'[t]==x[t]-3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tru
```

$$
\begin{aligned}
x(t) &\to e^{-t}(c_1\cos(2t)+(c_1-4c_2)\sin(2t))\\
y(t) &\to \frac{1}{2}e^{-t}(2c_2\cos(2t)+(c_1-2c_2)\sin(2t))
\end{aligned}
$$

10.36 problem 39

Internal problem ID [6762]
Internal file name [`OUTPUT/6010_Sunday_June_05_2022_04_13_36_PM_50562756/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= z(t) \\ y' &= -z(t) \\ z'(t) &= y \end{aligned}$$

10.36.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & -\cos(t)+1 & \sin(t) \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} 1 & -\cos(t)+1 & \sin(t) \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} c_1 + (-\cos(t)+1)\,c_2 + \sin(t)\,c_3 \\ \cos(t)\,c_2 - \sin(t)\,c_3 \\ \sin(t)\,c_2 + \cos(t)\,c_3 \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.36.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \lambda = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \\ \lambda_3 &= 0 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & 0 & 1 \\ 0 & i & -1 \\ 0 & 1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} i & 0 & 1 & 0 \\ 0 & i & -1 & 0 \\ 0 & 1 & i & 0 \end{array}\right]$$

$$R_3 = iR_2 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} i & 0 & 1 & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & 0 & 1 \\ 0 & i & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_2 = -it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ \text{-}\mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ \text{-}\mathrm{I}t \\ t \end{bmatrix} = t\begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ \text{-}\mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} - (i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 0 & 1 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -i & 0 & 1 & 0 \\ 0 & -i & -1 & 0 \\ 0 & 1 & -i & 0 \end{array}\right]$$

$$R_3 = -iR_2 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} -i & 0 & 1 & 0 \\ 0 & -i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & 0 & 1 \\ 0 & -i & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_2 = it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ \text{I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ \text{I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ \text{I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{0} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{0}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -i \mathrm{e}^{it} \\ i \mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} i \mathrm{e}^{-it} \\ -i \mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -ic_1\mathrm{e}^{it} + ic_2\mathrm{e}^{-it} + c_3 \\ i(c_1\mathrm{e}^{it} - c_2\mathrm{e}^{-it}) \\ c_1\mathrm{e}^{it} + c_2\mathrm{e}^{-it} \end{bmatrix}$$

10.36.3 Maple step by step solution

Let's solve

$[x'(t) = z(t), y' = -z(t), z'(t) = y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix}1\\0\\0\end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix}\mathrm{I}\\-\mathrm{I}\\1\end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix}-\mathrm{I}\\\mathrm{I}\\1\end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}1\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix}\mathrm{I}\\-\mathrm{I}\\1\end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix}\mathrm{I}\\-\mathrm{I}\\1\end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix}\mathrm{I}\\-\mathrm{I}\\1\end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix}\mathrm{I}(\cos(t) - \mathrm{I}\sin(t))\\-\mathrm{I}(\cos(t) - \mathrm{I}\sin(t))\\\cos(t) - \mathrm{I}\sin(t)\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = \begin{bmatrix} \sin(t) \\ -\sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_3(t) = \begin{bmatrix} \cos(t) \\ -\cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$
- Substitute solutions into the general solution

  $$\vec{x} = \begin{bmatrix} c_3\cos(t) + c_2\sin(t) + c_1 \\ -c_3\cos(t) - c_2\sin(t) \\ -c_3\sin(t) + c_2\cos(t) \end{bmatrix}$$
- Substitute in vector of dependent variables

  $$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_3\cos(t) + c_2\sin(t) + c_1 \\ -c_3\cos(t) - c_2\sin(t) \\ -c_3\sin(t) + c_2\cos(t) \end{bmatrix}$$
- Solution to the system of ODEs

  $\{x(t) = c_3\cos(t) + c_2\sin(t) + c_1, y = -c_3\cos(t) - c_2\sin(t), z(t) = -c_3\sin(t) + c_2\cos(t)\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
dsolve([diff(x(t),t)=z(t),diff(y(t),t)=-z(t),diff(z(t),t)=y(t)],singsol=all)
```

$$\begin{aligned} x(t) &= c_3\sin(t) - c_2\cos(t) + c_1 \\ y(t) &= -c_3\sin(t) + c_2\cos(t) \\ z(t) &= c_2\sin(t) + c_3\cos(t) \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 50

```
DSolve[{x'[t]==z[t],y'[t]==-z[t],z'[t]==y[t]},{x[t],y[t],z[t]},t,IncludeSingularSolutions ->
```

$$\begin{aligned} x(t) &\to -c_2\cos(t) + c_3\sin(t) + c_1 + c_2 \\ y(t) &\to c_2\cos(t) - c_3\sin(t) \\ z(t) &\to c_3\cos(t) + c_2\sin(t) \end{aligned}$$

10.37 problem 40

Internal problem ID [6763]
Internal file name [`OUTPUT/6011_Sunday_June_05_2022_04_13_39_PM_50218011/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}x'(t) &= 2x(t) + y + 2z(t)\\ y' &= 3x(t) + 6z(t)\\ z'(t) &= -4x(t) - 3z(t)\end{aligned}$$

10.37.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t} \cos\left(2t\right)+\frac{\mathrm{e}^{t} \sin\left(2t\right)}{2} & \frac{\mathrm{e}^{t} \sin\left(2t\right)}{2} & \mathrm{e}^{t} \sin\left(2t\right) \\ \frac{6 \,\mathrm{e}^{t} \cos\left(2t\right)}{5}-\frac{9 \,\mathrm{e}^{t} \sin\left(2t\right)}{10}-\frac{6 \,\mathrm{e}^{-3t}}{5} & \frac{3 \,\mathrm{e}^{t} \cos\left(2t\right)}{5}+\frac{3 \,\mathrm{e}^{t} \sin\left(2t\right)}{10}+\frac{2 \,\mathrm{e}^{-3t}}{5} & \frac{6 \,\mathrm{e}^{t} \cos\left(2t\right)}{5}+\frac{3 \,\mathrm{e}^{t} \sin\left(2t\right)}{5}-\frac{6 \,\mathrm{e}^{-3t}}{5} \\ -\frac{3 \,\mathrm{e}^{t} \cos\left(2t\right)}{5}-\frac{4 \,\mathrm{e}^{t} \sin\left(2t\right)}{5}+\frac{3 \,\mathrm{e}^{-3t}}{5} & \frac{\mathrm{e}^{t} \cos\left(2t\right)}{5}-\frac{2 \,\mathrm{e}^{t} \sin\left(2t\right)}{5}-\frac{\mathrm{e}^{-3t}}{5} & \frac{3 \,\mathrm{e}^{-3t}}{5}+\frac{2 \,\mathrm{e}^{t} \cos\left(2t\right)}{5}-\frac{4 \,\mathrm{e}^{t} \sin\left(2t\right)}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left(2 \cos\left(2t\right)+\sin\left(2t\right)\right) \mathrm{e}^{t}}{2} & \frac{\mathrm{e}^{t} \sin\left(2t\right)}{2} & \mathrm{e}^{t} \sin\left(2t\right) \\ \frac{6 \,\mathrm{e}^{-3t} \left(-1+\left(\cos\left(2t\right)-\frac{3 \sin\left(2t\right)}{4}\right) \mathrm{e}^{4t}\right)}{5} & \frac{\left(6 \cos\left(2t\right)+3 \sin\left(2t\right)\right) \mathrm{e}^{-3t} \mathrm{e}^{4t}}{10}+\frac{2 \,\mathrm{e}^{-3t}}{5} & \frac{6 \,\mathrm{e}^{-3t} \left(-1+\left(\cos\left(2t\right)+\frac{\sin\left(2t\right)}{2}\right) \mathrm{e}^{4t}\right)}{5} \\ -\frac{3 \,\mathrm{e}^{-3t} \left(-1+\left(\cos\left(2t\right)+\frac{4 \sin\left(2t\right)}{3}\right) \mathrm{e}^{4t}\right)}{5} & \frac{\left(-1+\left(\cos\left(2t\right)-2 \sin\left(2t\right)\right) \mathrm{e}^{4t}\right) \mathrm{e}^{-3t}}{5} & \frac{2 \left(\frac{3}{2}+\left(\cos\left(2t\right)-2 \sin\left(2t\right)\right) \mathrm{e}^{4t}\right) \mathrm{e}^{-3t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{\left(2 \cos\left(2t\right)+\sin\left(2t\right)\right) \mathrm{e}^{t}}{2} & \frac{\mathrm{e}^{t} \sin\left(2t\right)}{2} & \mathrm{e}^{t} \sin\left(2t\right) \\ \frac{6 \,\mathrm{e}^{-3t} \left(-1+\left(\cos\left(2t\right)-\frac{3 \sin\left(2t\right)}{4}\right) \mathrm{e}^{4t}\right)}{5} & \frac{\left(6 \cos\left(2t\right)+3 \sin\left(2t\right)\right) \mathrm{e}^{-3t} \mathrm{e}^{4t}}{10}+\frac{2 \,\mathrm{e}^{-3t}}{5} & \frac{6 \,\mathrm{e}^{-3t} \left(-1+\left(\cos\left(2t\right)+\frac{\sin\left(2t\right)}{2}\right) \mathrm{e}^{4t}\right)}{5} \\ -\frac{3 \,\mathrm{e}^{-3t} \left(-1+\left(\cos\left(2t\right)+\frac{4 \sin\left(2t\right)}{3}\right) \mathrm{e}^{4t}\right)}{5} & \frac{\left(-1+\left(\cos\left(2t\right)-2 \sin\left(2t\right)\right) \mathrm{e}^{4t}\right) \mathrm{e}^{-3t}}{5} & \frac{2 \left(\frac{3}{2}+\left(\cos\left(2t\right)-2 \sin\left(2t\right)\right) \mathrm{e}^{4t}\right) \mathrm{e}^{-3t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left(2 \cos\left(2t\right)+\sin\left(2t\right)\right) \mathrm{e}^{t} c_1}{2}+\frac{\mathrm{e}^{t} \sin\left(2t\right) c_2}{2}+\mathrm{e}^{t} \sin\left(2t\right) c_3 \\ \frac{6 \,\mathrm{e}^{-3t} \left(-1+\left(\cos\left(2t\right)-\frac{3 \sin\left(2t\right)}{4}\right) \mathrm{e}^{4t}\right) c_1}{5}+\left(\frac{\left(6 \cos\left(2t\right)+3 \sin\left(2t\right)\right) \mathrm{e}^{-3t} \mathrm{e}^{4t}}{10}+\frac{2 \,\mathrm{e}^{-3t}}{5}\right) c_2+\frac{6 \,\mathrm{e}^{-3t} \left(-1+\left(\cos\left(2t\right)+\frac{\sin\left(2t\right)}{2}\right) \mathrm{e}^{4t}\right) c_3}{5} \\ -\frac{3 \,\mathrm{e}^{-3t} \left(-1+\left(\cos\left(2t\right)+\frac{4 \sin\left(2t\right)}{3}\right) \mathrm{e}^{4t}\right) c_1}{5}+\frac{\left(-1+\left(\cos\left(2t\right)-2 \sin\left(2t\right)\right) \mathrm{e}^{4t}\right) \mathrm{e}^{-3t} c_2}{5}+\frac{2 \left(\frac{3}{2}+\left(\cos\left(2t\right)-2 \sin\left(2t\right)\right) \mathrm{e}^{4t}\right) \mathrm{e}^{-3t} c_3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\left(\frac{c_1}{2}+\frac{c_2}{2}+c_3\right) \sin\left(2t\right)+c_1 \cos\left(2t\right)\right) \mathrm{e}^{t} \\ \frac{6 \left(\left(\left(c_1+\frac{c_2}{2}+c_3\right) \cos\left(2t\right)-\frac{3 \left(c_1-\frac{c_2}{3}-\frac{2 c_3}{3}\right) \sin\left(2t\right)}{4}\right) \mathrm{e}^{4t}-c_1+\frac{c_2}{3}-c_3\right) \mathrm{e}^{-3t}}{5} \\ -\frac{3 \,\mathrm{e}^{-3t} \left(\left(\left(c_1-\frac{c_2}{3}-\frac{2 c_3}{3}\right) \cos\left(2t\right)+\frac{4 \left(c_1+\frac{c_2}{2}+c_3\right) \sin\left(2t\right)}{3}\right) \mathrm{e}^{4t}-c_1+\frac{c_2}{3}-c_3\right)}{5} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.37.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & 2 \\ 3 & -\lambda & 6 \\ -4 & 0 & -3-\lambda \end{bmatrix} \right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \lambda^2 - \lambda + 15 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$
$$\lambda_3 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
$1-2i$	1	complex eigenvalue
$1+2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & 2 \\ 3 & 3 & 6 \\ -4 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & 1 & 2 & 0 \\ 3 & 3 & 6 & 0 \\ -4 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 1 & 2 & 0 \\ 0 & \frac{12}{5} & \frac{24}{5} & 0 \\ -4 & 0 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{4R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 1 & 2 & 0 \\ 0 & \frac{12}{5} & \frac{24}{5} & 0 \\ 0 & \frac{4}{5} & \frac{8}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 1 & 2 & 0 \\ 0 & \frac{12}{5} & \frac{24}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 1 & 2 \\ 0 & \frac{12}{5} & \frac{24}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -2t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{bmatrix} - (1 - 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 + 2i & 1 & 2 \\ 3 & -1 + 2i & 6 \\ -4 & 0 & -4 + 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1+2i & 1 & 2 & 0 \\ 3 & -1+2i & 6 & 0 \\ -4 & 0 & -4+2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{5} + \frac{6i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1+2i & 1 & 2 & 0 \\ 0 & -\frac{8}{5} + \frac{16i}{5} & \frac{24}{5} + \frac{12i}{5} & 0 \\ -4 & 0 & -4+2i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{4}{5} - \frac{8i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1+2i & 1 & 2 & 0 \\ 0 & -\frac{8}{5} + \frac{16i}{5} & \frac{24}{5} + \frac{12i}{5} & 0 \\ 0 & \frac{4}{5} - \frac{8i}{5} & -\frac{12}{5} - \frac{6i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} 1+2i & 1 & 2 & 0 \\ 0 & -\frac{8}{5} + \frac{16i}{5} & \frac{24}{5} + \frac{12i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+2i & 1 & 2 \\ 0 & -\frac{8}{5} + \frac{16i}{5} & \frac{24}{5} + \frac{12i}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{1}{2}it - t, v_2 = \frac{3it}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{\mathrm{I}t}{2} - t \\ \frac{3\,\mathrm{I}}{2}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}it - t \\ \frac{3it}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{\mathrm{I}\,t}{2}-t \\ \frac{3\,\mathrm{I}}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{\frac{1}{2}it-t}{t} \\ \frac{3i}{2} \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} \frac{\mathrm{I}\,t}{2}-t \\ \frac{3\,\mathrm{I}}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -1+\frac{i}{2} \\ \frac{3i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{\mathrm{I}\,t}{2}-t \\ \frac{3\,\mathrm{I}}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -2+i \\ 3i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1+2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{bmatrix} - (1+2i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-2i & 1 & 2 \\ 3 & -1-2i & 6 \\ -4 & 0 & -4-2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1-2i & 1 & 2 & 0 \\ 3 & -1-2i & 6 & 0 \\ -4 & 0 & -4-2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{5} - \frac{6i}{5}\right)R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1-2i & 1 & 2 & 0 \\ 0 & -\frac{8}{5}-\frac{16i}{5} & \frac{24}{5}-\frac{12i}{5} & 0 \\ -4 & 0 & -4-2i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{4}{5} + \frac{8i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1-2i & 1 & 2 & 0 \\ 0 & -\frac{8}{5} - \frac{16i}{5} & \frac{24}{5} - \frac{12i}{5} & 0 \\ 0 & \frac{4}{5} + \frac{8i}{5} & -\frac{12}{5} + \frac{6i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} 1-2i & 1 & 2 & 0 \\ 0 & -\frac{8}{5} - \frac{16i}{5} & \frac{24}{5} - \frac{12i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-2i & 1 & 2 \\ 0 & -\frac{8}{5} - \frac{16i}{5} & \frac{24}{5} - \frac{12i}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{1}{2}it - t, v_2 = -\frac{3it}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{\mathrm{I}t}{2} - t \\ -\frac{3\,\mathrm{I}}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}it - t \\ -\frac{3it}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{\mathrm{I}t}{2} - t \\ -\frac{3\,\mathrm{I}}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{-\frac{1}{2}it - t}{t} \\ -\frac{3i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{\mathrm{I}t}{2} - t \\ -\frac{3\,\mathrm{I}}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -1 - \frac{i}{2} \\ -\frac{3i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{\mathrm{I}t}{2}-t \\ -\frac{3\,\mathrm{I}}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -2-i \\ -3i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$1+2i$	1	1	No	$\begin{bmatrix} -1-\frac{i}{2} \\ -\frac{3i}{2} \\ 1 \end{bmatrix}$
$1-2i$	1	1	No	$\begin{bmatrix} -1+\frac{i}{2} \\ \frac{3i}{2} \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-3t} \\ &= \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^{-3t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-1-\frac{i}{2}\right) \mathrm{e}^{(1+2i)t} \\ -\frac{3i\mathrm{e}^{(1+2i)t}}{2} \\ \mathrm{e}^{(1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(-1+\frac{i}{2}\right) \mathrm{e}^{(1-2i)t} \\ \frac{3i\mathrm{e}^{(1-2i)t}}{2} \\ \mathrm{e}^{(1-2i)t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -2\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \left(-1-\frac{i}{2}\right) c_1\mathrm{e}^{(1+2i)t} + \left(-1+\frac{i}{2}\right) c_2\mathrm{e}^{(1-2i)t} \\ -\frac{3ic_1\mathrm{e}^{(1+2i)t}}{2} + \frac{3ic_2\mathrm{e}^{(1-2i)t}}{2} - 2c_3\mathrm{e}^{-3t} \\ c_1\mathrm{e}^{(1+2i)t} + c_2\mathrm{e}^{(1-2i)t} + c_3\mathrm{e}^{-3t} \end{bmatrix}$$

10.37.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) + y + 2z(t)\,, y' = 3x(t) + 6z(t)\,, z'(t) = -4x(t) - 3z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{array}\right]$$

- Rewrite the system as

  $\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \left[\begin{array}{c} 0 \\ -2 \\ 1 \end{array}\right]\right], \left[1-2\mathrm{I}, \left[\begin{array}{c} -1+\frac{\mathrm{I}}{2} \\ \frac{3\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[1+2\mathrm{I}, \left[\begin{array}{c} -1-\frac{\mathrm{I}}{2} \\ -\frac{3\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-3, \left[\begin{array}{c} 0 \\ -2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-3t} \cdot \left[\begin{array}{c} 0 \\ -2 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-2\mathrm{I}, \left[\begin{array}{c} -1+\frac{\mathrm{I}}{2} \\ \frac{3\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-2\mathrm{I})t} \cdot \left[\begin{array}{c} -1+\frac{\mathrm{I}}{2} \\ \frac{3\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{t}\cdot\left(\cos\left(2t\right)-\mathrm{I}\sin\left(2t\right)\right)\cdot\left[\begin{array}{c}-1+\frac{\mathrm{I}}{2}\\ \frac{3\,\mathrm{I}}{2}\\ 1\end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{t}\cdot\left[\begin{array}{c}\left(-1+\frac{\mathrm{I}}{2}\right)\left(\cos\left(2t\right)-\mathrm{I}\sin\left(2t\right)\right)\\ \frac{3\,\mathrm{I}}{2}\left(\cos\left(2t\right)-\mathrm{I}\sin\left(2t\right)\right)\\ \cos\left(2t\right)-\mathrm{I}\sin\left(2t\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_{2}\left(t\right)=\mathrm{e}^{t}\cdot\left[\begin{array}{c}-\cos\left(2t\right)+\frac{\sin\left(2t\right)}{2}\\ \frac{3\sin\left(2t\right)}{2}\\ \cos\left(2t\right)\end{array}\right],\vec{x}_{3}\left(t\right)=\mathrm{e}^{t}\cdot\left[\begin{array}{c}\sin\left(2t\right)+\frac{\cos\left(2t\right)}{2}\\ \frac{3\cos\left(2t\right)}{2}\\ -\sin\left(2t\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{x}=c_{1}\vec{x}_{1}+c_{2}\vec{x}_{2}\left(t\right)+c_{3}\vec{x}_{3}\left(t\right)$$

- Substitute solutions into the general solution

$$\vec{x}=c_{1}\mathrm{e}^{-3t}\cdot\left[\begin{array}{c}0\\ -2\\ 1\end{array}\right]+c_{2}\mathrm{e}^{t}\cdot\left[\begin{array}{c}-\cos\left(2t\right)+\frac{\sin\left(2t\right)}{2}\\ \frac{3\sin\left(2t\right)}{2}\\ \cos\left(2t\right)\end{array}\right]+c_{3}\mathrm{e}^{t}\cdot\left[\begin{array}{c}\sin\left(2t\right)+\frac{\cos\left(2t\right)}{2}\\ \frac{3\cos\left(2t\right)}{2}\\ -\sin\left(2t\right)\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x\left(t\right)\\ y\\ z\left(t\right)\end{array}\right]=\left[\begin{array}{c}-\left(\left(c_{2}-\frac{c_{3}}{2}\right)\cos\left(2t\right)-\frac{\sin\left(2t\right)\left(c_{2}+2c_{3}\right)}{2}\right)\mathrm{e}^{t}\\ \frac{3\left(c_{2}\sin\left(2t\right)+c_{3}\cos\left(2t\right)\right)\mathrm{e}^{-3t}\mathrm{e}^{4t}}{2}-2c_{1}\mathrm{e}^{-3t}\\ \left(\mathrm{e}^{4t}\cos\left(2t\right)c_{2}-\mathrm{e}^{4t}\sin\left(2t\right)c_{3}+c_{1}\right)\mathrm{e}^{-3t}\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x\left(t\right)=-\left(\left(c_{2}-\frac{c_{3}}{2}\right)\cos\left(2t\right)-\frac{\sin\left(2t\right)\left(c_{2}+2c_{3}\right)}{2}\right)\mathrm{e}^{t},y=\frac{3\left(c_{2}\sin\left(2t\right)+c_{3}\cos\left(2t\right)\right)\mathrm{e}^{-3t}\mathrm{e}^{4t}}{2}-2c_{1}\mathrm{e}^{-3t},z\left(t\right)=\left(\mathrm{e}^{4t}\right.\right.$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 92

```
dsolve([diff(x(t),t)=2*x(t)+y(t)+2*z(t),diff(y(t),t)=3*x(t)+6*z(t),diff(z(t),t)=-4*x(t)-3*z(
```

$$
\begin{aligned}
x(t) &= -\frac{\mathrm{e}^{t}(2c_2\sin(2t) - c_3\sin(2t) + c_2\cos(2t) + 2c_3\cos(2t))}{2} \\
y(t) &= -2c_1\mathrm{e}^{-3t} - \frac{3c_2\mathrm{e}^{t}\cos(2t)}{2} + \frac{3\sin(2t)\,\mathrm{e}^{t}c_3}{2} \\
z(t) &= c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{t}\sin(2t) + \mathrm{e}^{t}c_3\cos(2t)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 176

```
DSolve[{x'[t]==2*x[t]+y[t]+2*z[t],y'[t]==3*x[t]+6*z[t],z'[t]==-4*x[t]-3*z[t]},{x[t],y[t],z[t
```

$$
\begin{aligned}
x(t) &\to \frac{1}{2}e^{t}(2c_1\cos(2t) + (c_1 + c_2 + 2c_3)\sin(2t)) \\
y(t) &\to \frac{2}{5}(-3c_1 + c_2 - 3c_3)e^{-3t} + \frac{3}{5}(2c_1 + c_2 + 2c_3)e^{t}\cos(2t) - \frac{3}{5}(3c_1 - c_2 \\
&\qquad - 2c_3)e^{t}\sin(t)\cos(t) \\
z(t) &\to \frac{1}{5}e^{-3t}\left(-(3c_1 - c_2 - 2c_3)e^{4t}\cos(2t) - 2(2c_1 + c_2 + 2c_3)e^{4t}\sin(2t) + 3c_1 - c_2 + 3c_3\right)
\end{aligned}
$$

10.38 problem 45

Internal problem ID [6764]
Internal file name [`OUTPUT/6012_Sunday_June_05_2022_04_13_42_PM_62728912/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= x(t) - 12y - 14z(t) \\ y' &= x(t) + 2y - 3z(t) \\ z'(t) &= x(t) + y - 2z(t) \end{aligned}$$

With initial conditions

$$[x(0) = 4, y(0) = 6, z(0) = -7]$$

10.38.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(5t) + \frac{\sin(5t)}{5} & -\frac{25\,e^{t}}{13} + \frac{25\cos(5t)}{13} - \frac{131\sin(5t)}{65} & -\frac{25\cos(5t)}{13} - \frac{207\sin(5t)}{65} + \frac{25\,e^{t}}{13} \\ \frac{\sin(5t)}{5} & \frac{7\,e^{t}}{13} + \frac{6\cos(5t)}{13} + \frac{19\sin(5t)}{65} & \frac{7\cos(5t)}{13} - \frac{32\sin(5t)}{65} - \frac{7\,e^{t}}{13} \\ \frac{\sin(5t)}{5} & \frac{6\cos(5t)}{13} + \frac{19\sin(5t)}{65} - \frac{6\,e^{t}}{13} & \frac{6\,e^{t}}{13} + \frac{7\cos(5t)}{13} - \frac{32\sin(5t)}{65} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \cos(5t) + \frac{\sin(5t)}{5} & -\frac{25\,e^{t}}{13} + \frac{25\cos(5t)}{13} - \frac{131\sin(5t)}{65} & -\frac{25\cos(5t)}{13} - \frac{207\sin(5t)}{65} + \frac{25\,e^{t}}{13} \\ \frac{\sin(5t)}{5} & \frac{7\,e^{t}}{13} + \frac{6\cos(5t)}{13} + \frac{19\sin(5t)}{65} & \frac{7\cos(5t)}{13} - \frac{32\sin(5t)}{65} - \frac{7\,e^{t}}{13} \\ \frac{\sin(5t)}{5} & \frac{6\cos(5t)}{13} + \frac{19\sin(5t)}{65} - \frac{6\,e^{t}}{13} & \frac{6\,e^{t}}{13} + \frac{7\cos(5t)}{13} - \frac{32\sin(5t)}{65} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix} \\ &= \begin{bmatrix} 29\cos(5t) + 11\sin(5t) - 25\,e^{t} \\ 6\sin(5t) + 7\,e^{t} - \cos(5t) \\ 6\sin(5t) - \cos(5t) - 6\,e^{t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.38.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -12 & -14 \\ 1 & 2-\lambda & -3 \\ 1 & 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \lambda^2 + 25\lambda - 25 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= 5i \\ \lambda_3 &= -5i\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
$5i$	1	complex eigenvalue
$-5i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -12 & -14 \\ 1 & 1 & -3 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -12 & -14 & 0 \\ 1 & 1 & -3 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & -12 & -14 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & -12 & -14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & -12 & -14 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{25t}{6}, v_2 = -\frac{7t}{6}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{25t}{6} \\ -\frac{7t}{6} \\ t \end{bmatrix} = \begin{bmatrix} \frac{25t}{6} \\ -\frac{7t}{6} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{25t}{6} \\ -\frac{7t}{6} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{25}{6} \\ -\frac{7}{6} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{25t}{6} \\ -\frac{7t}{6} \\ t \end{bmatrix} = \begin{bmatrix} \frac{25}{6} \\ -\frac{7}{6} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{25t}{6} \\ -\frac{7t}{6} \\ t \end{bmatrix} = \begin{bmatrix} 25 \\ -7 \\ 6 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -5i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{bmatrix} - (-5i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+5i & -12 & -14 \\ 1 & 2+5i & -3 \\ 1 & 1 & -2+5i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1+5i & -12 & -14 & 0 \\ 1 & 2+5i & -3 & 0 \\ 1 & 1 & -2+5i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{26} + \frac{5i}{26}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1+5i & -12 & -14 & 0 \\ 0 & \frac{32}{13} + \frac{35i}{13} & -\frac{32}{13} - \frac{35i}{13} & 0 \\ 1 & 1 & -2+5i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{26} + \frac{5i}{26}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1+5i & -12 & -14 & 0 \\ 0 & \frac{32}{13} + \frac{35i}{13} & -\frac{32}{13} - \frac{35i}{13} & 0 \\ 0 & \frac{19}{13} - \frac{30i}{13} & -\frac{19}{13} + \frac{30i}{13} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{34}{173} + \frac{125i}{173}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1+5i & -12 & -14 & 0 \\ 0 & \frac{32}{13} + \frac{35i}{13} & -\frac{32}{13} - \frac{35i}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+5i & -12 & -14 \\ 0 & \frac{32}{13}+\frac{35i}{13} & -\frac{32}{13}-\frac{35i}{13} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1-5i)\,t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} (1-5\ \mathrm{I})\,t \\ t \\ t \end{bmatrix} = \begin{bmatrix} (1-5i)\,t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1-5\ \mathrm{I})\,t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1-5i \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1-5\ \mathrm{I})\,t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1-5i \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 5i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{bmatrix} - (5i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-5i & -12 & -14 \\ 1 & 2-5i & -3 \\ 1 & 1 & -2-5i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 1 & 2-5i & -3 & 0 \\ 1 & 1 & -2-5i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{26} - \frac{5i}{26}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 0 & \frac{32}{13} - \frac{35i}{13} & -\frac{32}{13} + \frac{35i}{13} & 0 \\ 1 & 1 & -2-5i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{26} - \frac{5i}{26}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 0 & \frac{32}{13} - \frac{35i}{13} & -\frac{32}{13} + \frac{35i}{13} & 0 \\ 0 & \frac{19}{13} + \frac{30i}{13} & -\frac{19}{13} - \frac{30i}{13} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{34}{173} - \frac{125i}{173}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 0 & \frac{32}{13} - \frac{35i}{13} & -\frac{32}{13} + \frac{35i}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-5i & -12 & -14 \\ 0 & \frac{32}{13} - \frac{35i}{13} & -\frac{32}{13} + \frac{35i}{13} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1+5i)\,t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} (1+5\ \mathrm{I})\,t \\ t \\ t \end{bmatrix} = \begin{bmatrix} (1+5i)\,t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+5\ \mathrm{I})\,t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1+5i \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1+5\ \mathrm{I})\,t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1+5i \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} \frac{25}{6} \\ -\frac{7}{6} \\ 1 \end{bmatrix}$
$5i$	1	1	No	$\begin{bmatrix} 1+5i \\ 1 \\ 1 \end{bmatrix}$
$-5i$	1	1	No	$\begin{bmatrix} 1-5i \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} \frac{25}{6} \\ -\frac{7}{6} \\ 1 \end{bmatrix} e^t\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{25\,\mathrm{e}^t}{6} \\ -\frac{7\,\mathrm{e}^t}{6} \\ \mathrm{e}^t \end{bmatrix} + c_2 \begin{bmatrix} (1+5i)\,\mathrm{e}^{5it} \\ \mathrm{e}^{5it} \\ \mathrm{e}^{5it} \end{bmatrix} + c_3 \begin{bmatrix} (1-5i)\,\mathrm{e}^{-5it} \\ \mathrm{e}^{-5it} \\ \mathrm{e}^{-5it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{25c_1\mathrm{e}^t}{6} + (1+5i)\,c_2\mathrm{e}^{5it} + (1-5i)\,c_3\mathrm{e}^{-5it} \\ -\frac{7c_1\mathrm{e}^t}{6} + c_2\mathrm{e}^{5it} + c_3\mathrm{e}^{-5it} \\ c_1\mathrm{e}^t + c_2\mathrm{e}^{5it} + c_3\mathrm{e}^{-5it} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 4 \\ y(0) = 6 \\ z(0) = -7 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix} = \begin{bmatrix} (1+5i)\,c_2 + (1-5i)\,c_3 + \frac{25c_1}{6} \\ -\frac{7c_1}{6} + c_2 + c_3 \\ c_1 + c_2 + c_3 \end{bmatrix}$$

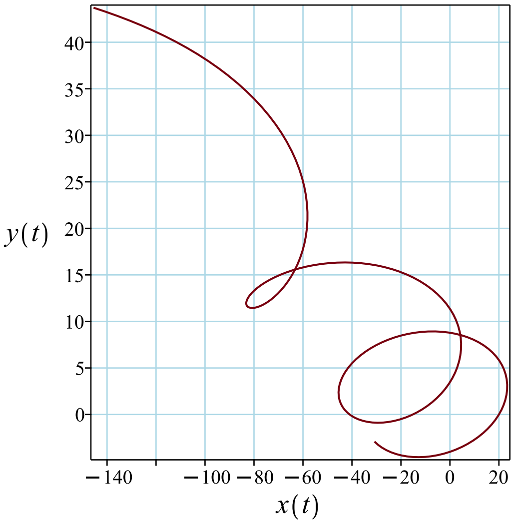
Solving for the constants of integrations gives

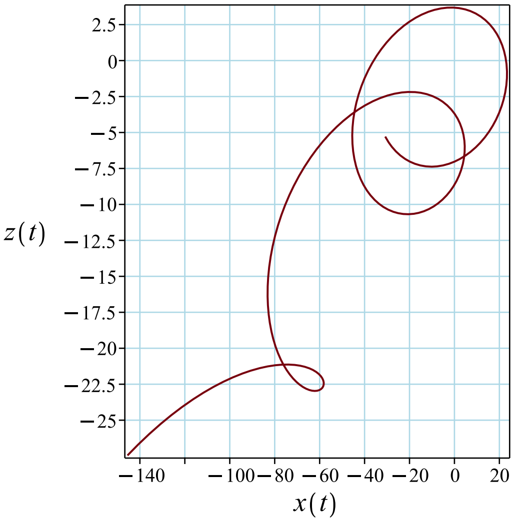
$$\begin{bmatrix} c_1 = -6 \\ c_2 = -\frac{1}{2} - 3i \\ c_3 = -\frac{1}{2} + 3i \end{bmatrix}$$

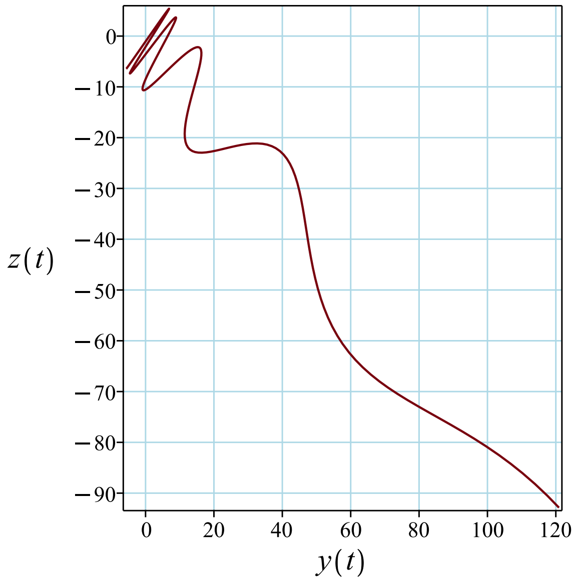
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -25\,\mathrm{e}^{t} + \left(\frac{29}{2} - \frac{11i}{2}\right) \mathrm{e}^{5it} + \left(\frac{29}{2} + \frac{11i}{2}\right) \mathrm{e}^{-5it} \\ 7\,\mathrm{e}^{t} + \left(-\frac{1}{2} - 3i\right) \mathrm{e}^{5it} + \left(-\frac{1}{2} + 3i\right) \mathrm{e}^{-5it} \\ -6\,\mathrm{e}^{t} + \left(-\frac{1}{2} - 3i\right) \mathrm{e}^{5it} + \left(-\frac{1}{2} + 3i\right) \mathrm{e}^{-5it} \end{bmatrix}$$

The following are plots of each solution against another.







The following are plots of each solution.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 62

```
dsolve([diff(x(t),t) = x(t)-12*y(t)-14*z(t), diff(y(t),t) = x(t)+2*y(t)-3*z(t), diff(z(t),t)
```

$$
\begin{aligned}
x(t) &= -25\,\mathrm{e}^{t}+11\sin\left(5t\right)+29\cos\left(5t\right)\\
y(t) &= 7\,\mathrm{e}^{t}+6\sin\left(5t\right)-\cos\left(5t\right)\\
z(t) &= -6\,\mathrm{e}^{t}-\cos\left(5t\right)+6\sin\left(5t\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 65

```
DSolve[{x'[t]==x[t]-12*y[t]-14*z[t],y'[t]==x[t]+2*y[t]-3*z[t],z'[t]==x[t]+y[t]-2*z[t]},{x[0]
```

$$
\begin{aligned}
x(t) &\to -25e^{t}+11\sin(5t)+29\cos(5t)\\
y(t) &\to 7e^{t}+6\sin(5t)-\cos(5t)\\
z(t) &\to -6e^{t}+6\sin(5t)-\cos(5t)
\end{aligned}
$$

11 CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.3. Page 354

11.1 problem 1

Internal problem ID [6765]
Internal file name [`OUTPUT/6013_Sunday_June_05_2022_04_13_45_PM_33149453/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.3. Page 354
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2x(t) + 3y - 7$$
$$y' = -x(t) - 2y + 5$$

11.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} \\ -\frac{e^{t}}{2} + \frac{e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} \\ -\frac{e^{t}}{2} + \frac{e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(-\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2}\right) c_1 + \left(\frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2}\right) c_2 \\ \left(-\frac{e^{t}}{2} + \frac{e^{-t}}{2}\right) c_1 + \left(\frac{3\,e^{-t}}{2} - \frac{e^{t}}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(-c_1-3c_2)e^{-t}}{2} + \frac{3\,e^{t}(c_1+c_2)}{2} \\ \frac{(c_1+3c_2)e^{-t}}{2} - \frac{e^{t}(c_1+c_2)}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} & -\frac{3\,e^{t}}{2} + \frac{3\,e^{-t}}{2} \\ \frac{e^{t}}{2} - \frac{e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} \\ -\frac{e^{t}}{2} + \frac{e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} & -\frac{3\,e^{t}}{2} + \frac{3\,e^{-t}}{2} \\ \frac{e^{t}}{2} - \frac{e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} \end{bmatrix} \begin{bmatrix} -7 \\ 5 \end{bmatrix} dt \\ &= \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} \\ -\frac{e^{t}}{2} + \frac{e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} 3\,e^{-t} - 4\,e^{t} \\ 4\,e^{t} - e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 3 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \left[\begin{array}{c} \frac{(-c_1-3c_2)\mathrm{e}^{-t}}{2} - 1 + \frac{(3c_1+3c_2)\mathrm{e}^{t}}{2} \\ \frac{(c_1+3c_2)\mathrm{e}^{-t}}{2} + 3 + \frac{(-c_1-c_2)\mathrm{e}^{t}}{2} \end{array}\right]
\end{aligned}
$$

11.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\left[\begin{array}{c} x'(t) \\ y' \end{array}\right] = \left[\begin{array}{cc} 2 & 3 \\ -1 & -2 \end{array}\right] \left[\begin{array}{c} x(t) \\ y \end{array}\right] + \left[\begin{array}{c} -7 \\ 5 \end{array}\right]$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\left[\begin{array}{cc} 2 & 3 \\ -1 & -2 \end{array}\right] - \lambda \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right) = 0$$

Therefore

$$\det\left(\left[\begin{array}{cc} 2-\lambda & 3 \\ -1 & -2-\lambda \end{array}\right]\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ -1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -3t\}$

Hence the solution is

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^t \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -3\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -3\,\mathrm{e}^{t} & -\mathrm{e}^{-t} \\ \mathrm{e}^{t} & \mathrm{e}^{-t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2} & -\frac{\mathrm{e}^{-t}}{2} \\ \frac{\mathrm{e}^{t}}{2} & \frac{3\,\mathrm{e}^{t}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -3\,\mathrm{e}^{t} & -\mathrm{e}^{-t} \\ \mathrm{e}^{t} & \mathrm{e}^{-t} \end{bmatrix} \int \begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2} & -\frac{\mathrm{e}^{-t}}{2} \\ \frac{\mathrm{e}^{t}}{2} & \frac{3\,\mathrm{e}^{t}}{2} \end{bmatrix} \begin{bmatrix} -7 \\ 5 \end{bmatrix} dt \\
&= \begin{bmatrix} -3\,\mathrm{e}^{t} & -\mathrm{e}^{-t} \\ \mathrm{e}^{t} & \mathrm{e}^{-t} \end{bmatrix} \int \begin{bmatrix} \mathrm{e}^{-t} \\ 4\,\mathrm{e}^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} -3\,\mathrm{e}^{t} & -\mathrm{e}^{-t} \\ \mathrm{e}^{t} & \mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} -\mathrm{e}^{-t} \\ 4\,\mathrm{e}^{t} \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 3 \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y \end{bmatrix} &= \begin{bmatrix} -3c_1\mathrm{e}^{t} \\ c_1\mathrm{e}^{t} \end{bmatrix} + \begin{bmatrix} -c_2\mathrm{e}^{-t} \\ c_2\mathrm{e}^{-t} \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -3c_1\mathrm{e}^{t} - c_2\mathrm{e}^{-t} - 1 \\ c_1\mathrm{e}^{t} + c_2\mathrm{e}^{-t} + 3 \end{bmatrix}$$

The following is the phase plot of the system.

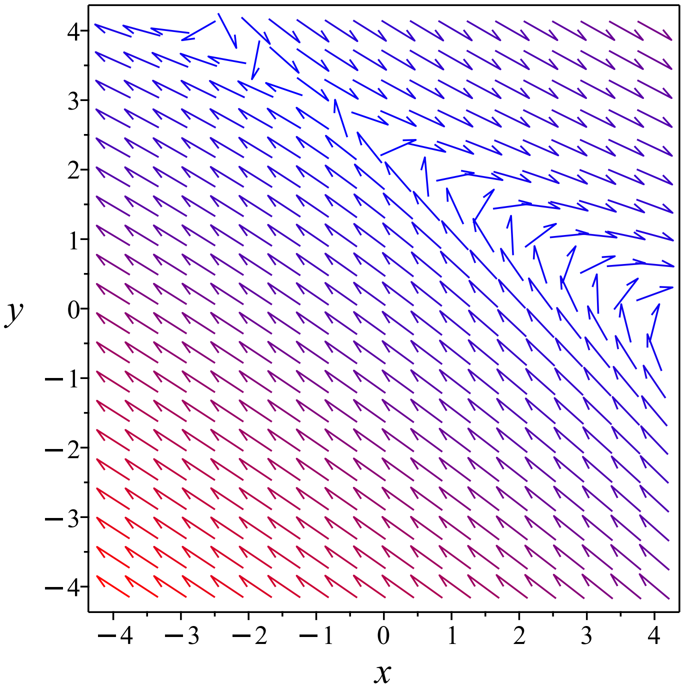


Figure 116: Phase plot

11.1.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) + 3y - 7, y' = -x(t) - 2y + 5]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \left[\begin{array}{c} -7 \\ 5 \end{array}\right]$$

- Define the coefficient matrix

  $$A = \left[\begin{array}{cc} 2 & 3 \\ -1 & -2 \end{array}\right]$$

- Rewrite the system as

  $$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} -3 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

  $$\left[-1, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_1 = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

  $$\left[1, \left[\begin{array}{c} -3 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_2 = \mathrm{e}^{t} \cdot \left[\begin{array}{c} -3 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \vec{x}_p(t)$$

☐ Fundamental matrix

  ○ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \left[\begin{array}{cc} -\mathrm{e}^{-t} & -3\,\mathrm{e}^{t} \\ \mathrm{e}^{-t} & \mathrm{e}^{t} \end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \left[\begin{array}{cc} -\mathrm{e}^{-t} & -3\,\mathrm{e}^{t} \\ \mathrm{e}^{-t} & \mathrm{e}^{t} \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} -1 & -3 \\ 1 & 1 \end{array}\right]}$$
- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \left[\begin{array}{cc} -\frac{\mathrm{e}^{-t}}{2} + \frac{3\,\mathrm{e}^{t}}{2} & \frac{3\,\mathrm{e}^{t}}{2} - \frac{3\,\mathrm{e}^{-t}}{2} \\ -\frac{\mathrm{e}^{t}}{2} + \frac{\mathrm{e}^{-t}}{2} & \frac{3\,\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{t}}{2} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_{p}(t) = \Phi(t) \cdot \overrightarrow{v}(t)$
- Take the derivative of the particular solution

  $\overrightarrow{x}_{p}^{\prime}(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}^{\prime}(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}^{\prime}(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}^{\prime}(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}^{\prime}(t) = \overrightarrow{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}^{\prime}(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$
- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_{0}^{t} \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

$$\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$$

  - Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{x}_p(t) = \left[\begin{array}{c} 4\,\mathrm{e}^{-t} - 1 - 3\,\mathrm{e}^{t} \\ -4\,\mathrm{e}^{-t} + 3 + \mathrm{e}^{t} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2 + \left[\begin{array}{c} 4\,\mathrm{e}^{-t} - 1 - 3\,\mathrm{e}^{t} \\ -4\,\mathrm{e}^{-t} + 3 + \mathrm{e}^{t} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} (-c_1 + 4)\,\mathrm{e}^{-t} - 1 + (-3c_2 - 3)\,\mathrm{e}^{t} \\ (c_1 - 4)\,\mathrm{e}^{-t} + 3 + (c_2 + 1)\,\mathrm{e}^{t} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = (-c_1 + 4)\,\mathrm{e}^{-t} - 1 + (-3c_2 - 3)\,\mathrm{e}^{t}, y = (c_1 - 4)\,\mathrm{e}^{-t} + 3 + (c_2 + 1)\,\mathrm{e}^{t}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve([diff(x(t),t)=2*x(t)+3*y(t)-7,diff(y(t),t)=-x(t)-2*y(t)+5],singsol=all)
```

$$x(t) = c_2 \mathrm{e}^{t} + \mathrm{e}^{-t} c_1 - 1$$
$$y(t) = -\frac{c_2 \mathrm{e}^{t}}{3} - \mathrm{e}^{-t} c_1 + 3$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 76

```
DSolve[{x'[t]==2*x[t]+3*y[t]-7,y'[t]==-x[t]-2*y[t]+5},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{2} e^{-t} \left(-2e^{t} + 3(c_1 + c_2)e^{2t} - c_1 - 3c_2\right)$$
$$y(t) \to \frac{1}{2} e^{-t} \left(6e^{t} - (c_1 + c_2)e^{2t} + c_1 + 3c_2\right)$$

11.2 problem 2

Internal problem ID [6766]
Internal file name [`OUTPUT/6014_Sunday_June_05_2022_04_13_48_PM_98186038/index.tex`]

Book: DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition.
Section: CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.3. Page 354
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 5x(t) + 9y + 2 \\ y' &= -x(t) + 11y + 6 \end{aligned}$$

11.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ -1 & 11 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At}=\begin{bmatrix} \mathrm{e}^{8t}(1-3t) & 9t\,\mathrm{e}^{8t} \\ -t\,\mathrm{e}^{8t} & \mathrm{e}^{8t}(1+3t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{8t}(1-3t) & 9t\,\mathrm{e}^{8t} \\ -t\,\mathrm{e}^{8t} & \mathrm{e}^{8t}(1+3t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{8t}(1-3t)\,c_1+9t\,\mathrm{e}^{8t}c_2 \\ -t\,\mathrm{e}^{8t}c_1+\mathrm{e}^{8t}(1+3t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-3t)+9c_2t)\,\mathrm{e}^{8t} \\ \mathrm{e}^{8t}(-tc_1+3c_2t+c_2) \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t)=e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \mathrm{e}^{-8t}(1+3t) & -9t\,\mathrm{e}^{-8t} \\ t\,\mathrm{e}^{-8t} & \mathrm{e}^{-8t}(1-3t) \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \mathrm{e}^{8t}(1-3t) & 9t\,\mathrm{e}^{8t} \\ -t\,\mathrm{e}^{8t} & \mathrm{e}^{8t}(1+3t) \end{bmatrix}\int\begin{bmatrix} \mathrm{e}^{-8t}(1+3t) & -9t\,\mathrm{e}^{-8t} \\ t\,\mathrm{e}^{-8t} & \mathrm{e}^{-8t}(1-3t) \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix}dt \\
&= \begin{bmatrix} \mathrm{e}^{8t}(1-3t) & 9t\,\mathrm{e}^{8t} \\ -t\,\mathrm{e}^{8t} & \mathrm{e}^{8t}(1+3t) \end{bmatrix}\begin{bmatrix} \frac{\mathrm{e}^{-8t}(1+12t)}{2} \\ \frac{\mathrm{e}^{-8t}(4t-1)}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} (c_1(1-3t) + 9c_2 t)\,\mathrm{e}^{8t} + \frac{1}{2} \\ -\frac{1}{2} + (c_2(1+3t) - tc_1)\,\mathrm{e}^{8t} \end{bmatrix}
\end{aligned}
$$

11.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ -1 & 11 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & 9 \\ -1 & 11 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & 9 \\ -1 & 11-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 16\lambda + 64 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 8$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
8	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 8$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 9 \\ -1 & 11 \end{bmatrix} - (8)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 9 & 0 \\ -1 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} -3 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
8	2	1	Yes	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 8 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

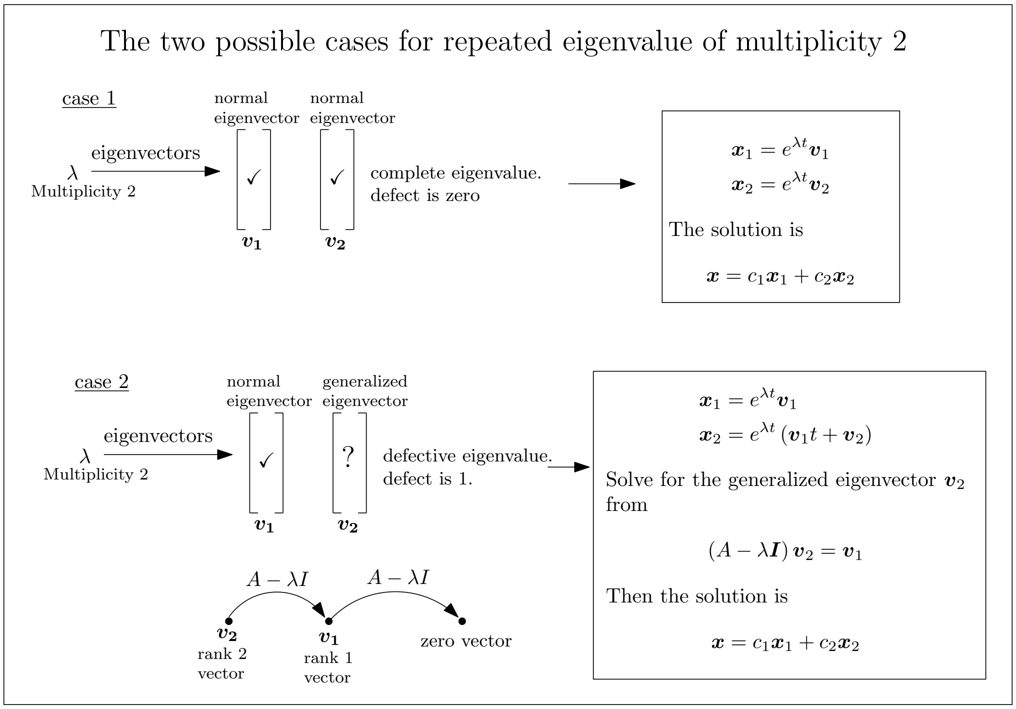


Figure 117: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 5 & 9 \\ -1 & 11 \end{bmatrix} - (8)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 8. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \mathrm{e}^{8t}\\ &= \begin{bmatrix} 3\,\mathrm{e}^{8t} \\ \mathrm{e}^{8t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) \mathrm{e}^{8t}\\ &= \begin{bmatrix} \mathrm{e}^{8t}(2+3t) \\ \mathrm{e}^{8t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 3\,\mathrm{e}^{8t} \\ \mathrm{e}^{8t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{8t}(2+3t) \\ \mathrm{e}^{8t}(t+1) \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 3\,\mathrm{e}^{8t} & \mathrm{e}^{8t}(2+3t) \\ \mathrm{e}^{8t} & \mathrm{e}^{8t}(t+1) \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \mathrm{e}^{-8t}(t+1) & (-2-3t)\,\mathrm{e}^{-8t} \\ -\mathrm{e}^{-8t} & 3\,\mathrm{e}^{-8t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 3\,\mathrm{e}^{8t} & \mathrm{e}^{8t}(2+3t) \\ \mathrm{e}^{8t} & \mathrm{e}^{8t}(t+1) \end{bmatrix} \int \begin{bmatrix} \mathrm{e}^{-8t}(t+1) & (-2-3t)\,\mathrm{e}^{-8t} \\ -\mathrm{e}^{-8t} & 3\,\mathrm{e}^{-8t} \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} dt \\
&= \begin{bmatrix} 3\,\mathrm{e}^{8t} & \mathrm{e}^{8t}(2+3t) \\ \mathrm{e}^{8t} & \mathrm{e}^{8t}(t+1) \end{bmatrix} \int \begin{bmatrix} (-16t-10)\,\mathrm{e}^{-8t} \\ 16\,\mathrm{e}^{-8t} \end{bmatrix} dt \\
&= \begin{bmatrix} 3\,\mathrm{e}^{8t} & \mathrm{e}^{8t}(2+3t) \\ \mathrm{e}^{8t} & \mathrm{e}^{8t}(t+1) \end{bmatrix} \begin{bmatrix} \frac{\mathrm{e}^{-8t}(4t+3)}{2} \\ -2\,\mathrm{e}^{-8t} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y \end{bmatrix} &= \begin{bmatrix} 3c_1\mathrm{e}^{8t} \\ c_1\mathrm{e}^{8t} \end{bmatrix} + \begin{bmatrix} c_2\mathrm{e}^{8t}(2+3t) \\ c_2\mathrm{e}^{8t}(t+1) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + ((2+3t)\,c_2 + 3c_1)\,\mathrm{e}^{8t} \\ -\frac{1}{2} + ((t+1)\,c_2 + c_1)\,\mathrm{e}^{8t} \end{bmatrix}$$

The following is the phase plot of the system.

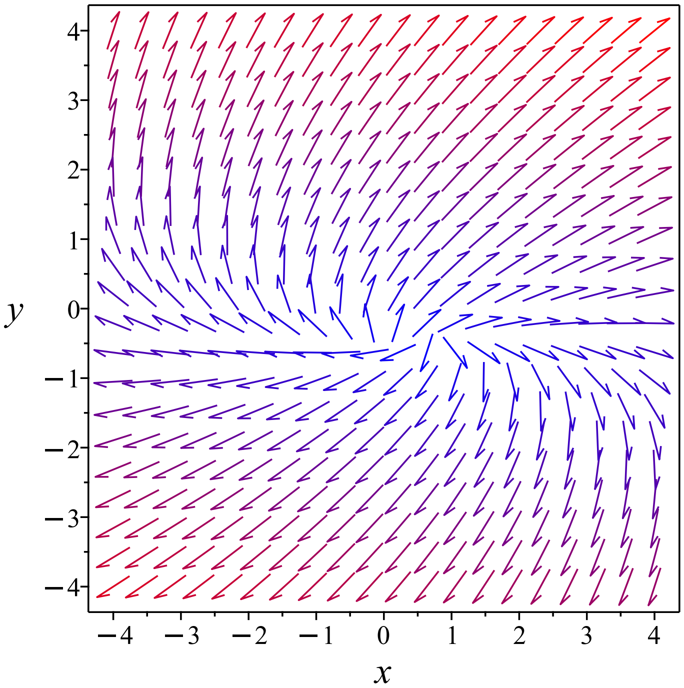


Figure 118: Phase plot

11.2.3 Maple step by step solution

Let's solve

$[x'(t) = 5x(t) + 9y + 2, y' = -x(t) + 11y + 6]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 5 & 9 \\ -1 & 11 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 5 & 9 \\ -1 & 11 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \left[\begin{array}{c} 2 \\ 6 \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 5 & 9 \\ -1 & 11 \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[8, \left[\begin{array}{c} 3 \\ 1 \end{array}\right]\right], \left[8, \left[\begin{array}{c} 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[8, \left[\begin{array}{c} 3 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 8

$$\vec{x}_1(t) = \mathrm{e}^{8t} \cdot \left[\begin{array}{c} 3 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 8$ is the eigenvalue, an

$$\vec{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{x}_2(t)$ into the homogeneous system

$$\lambda \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$
- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 8

  $$\left(\begin{bmatrix} 5 & 9 \\ -1 & 11 \end{bmatrix} - 8 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
- Second solution from eigenvalue 8

  $$\overrightarrow{x}_2(t) = \mathrm{e}^{8t} \cdot \left(t \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$
- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t) + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \begin{bmatrix} 3\,\mathrm{e}^{8t} & \mathrm{e}^{8t}(-1+3t) \\ \mathrm{e}^{8t} & t\,\mathrm{e}^{8t} \end{bmatrix}$$
- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \begin{bmatrix} 3\,\mathrm{e}^{8t} & \mathrm{e}^{8t}(-1+3t) \\ \mathrm{e}^{8t} & t\,\mathrm{e}^{8t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}}$$
- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \begin{bmatrix} \mathrm{e}^{8t}(1-3t) & 9t\,\mathrm{e}^{8t} \\ -t\,\mathrm{e}^{8t} & \mathrm{e}^{8t}(1+3t) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$
- Take the derivative of the particular solution

  $\overrightarrow{x}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$
- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{x}_p(t) = \left[\begin{array}{c} 6t\,\mathrm{e}^{8t} - \frac{\mathrm{e}^{8t}}{2} + \frac{1}{2} \\ 2t\,\mathrm{e}^{8t} + \frac{\mathrm{e}^{8t}}{2} - \frac{1}{2} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t) + \left[\begin{array}{c} 6t\,\mathrm{e}^{8t} - \frac{\mathrm{e}^{8t}}{2} + \frac{1}{2} \\ 2t\,\mathrm{e}^{8t} + \frac{\mathrm{e}^{8t}}{2} - \frac{1}{2} \end{array}\right]$$
- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{1}{2} + \frac{((6t-2)c_2+12t+6c_1-1)\mathrm{e}^{8t}}{2} \\ -\frac{1}{2} + \frac{((2c_2+4)t+2c_1+1)\mathrm{e}^{8t}}{2} \end{array}\right]$$
- Solution to the system of ODEs

$$\left\{x(t) = \frac{1}{2} + \frac{((6t-2)c_2+12t+6c_1-1)\mathrm{e}^{8t}}{2}, y = -\frac{1}{2} + \frac{((2c_2+4)t+2c_1+1)\mathrm{e}^{8t}}{2}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve([diff(x(t),t)=5*x(t)+9*y(t)+2,diff(y(t),t)=-x(t)+11*y(t)+6],singsol=all)
```

$$\begin{aligned} x(t) &= \frac{1}{2} + \mathrm{e}^{8t}(c_1t + c_2) \\ y(t) &= -\frac{1}{2} + \frac{\mathrm{e}^{8t}(3c_1t + c_1 + 3c_2)}{9} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 54

```
DSolve[{x'[t]==5*x[t]+9*y[t]+2,y'[t]==-x[t]+11*y[t]+6},{x[t],y[t]},t,IncludeSingularSolution
```

$$\begin{aligned} x(t) &\to \frac{1}{2} + e^{8t}(-3c_1t + 9c_2t + c_1) \\ y(t) &\to -\frac{1}{2} + e^{8t}(c_1(-t) + 3c_2t + c_2) \end{aligned}$$