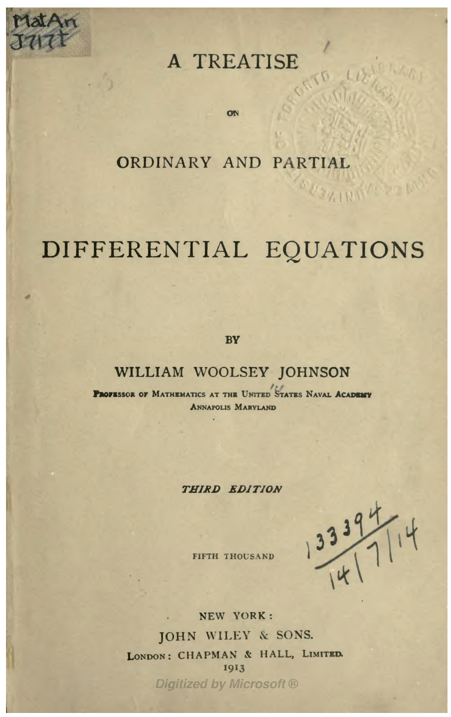
A Solution Manual For

A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913



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May 15, 2024

Contents

1 Chapter 1, Nature and meaning of a differential equation between two variables. page 12

1.1 problem 1

Internal problem ID [4681]
Internal file name [`OUTPUT/4174_Sunday_June_05_2022_12_36_38_PM_6028822/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 1, Nature and meaning of a differential equation between two variables. page 12
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + y\tan(x) = 0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -y\tan(x) \end{aligned}$$

Where $f(x) = -\tan(x)$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= -\tan(x)\,dx \\
\int \frac{1}{y}\,dy &= \int -\tan(x)\,dx \\
\ln(y) &= \ln(\cos(x)) + c_1 \\
y &= \mathrm{e}^{\ln(\cos(x))+c_1} \\
&= \cos(x)\,c_1
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\,c_1 \tag{1}$$

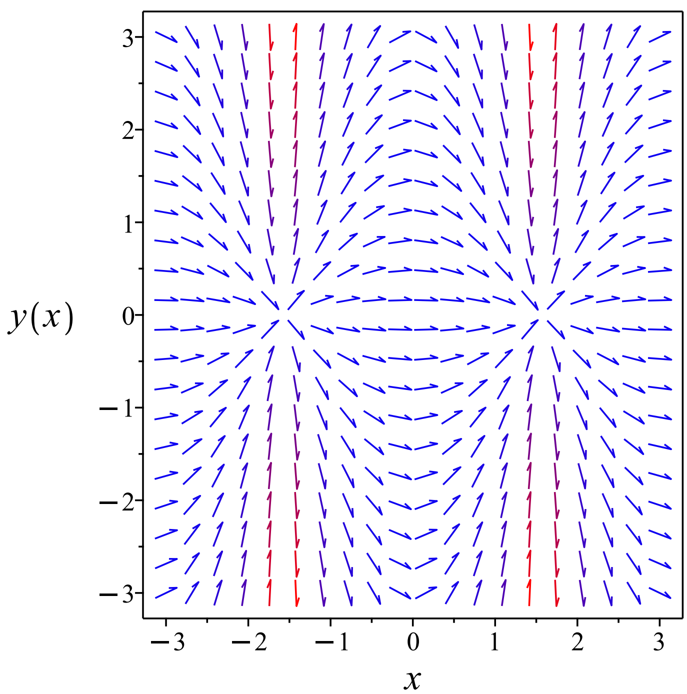


Figure 1: Slope field plot

Verification of solutions

$$y = \cos(x)\,c_1$$

Verified OK.

1.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + y\tan(x) = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \tan(x)dx} \\ &= \frac{1}{\cos(x)} \end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}(\sec(x)\, y) &= 0 \end{aligned}$$

Integrating gives

$$\sec(x)\, y = c_1$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = \cos(x)\, c_1$$

Summary

The solution(s) found are the following

$$y = \cos(x)\, c_1 \tag{1}$$

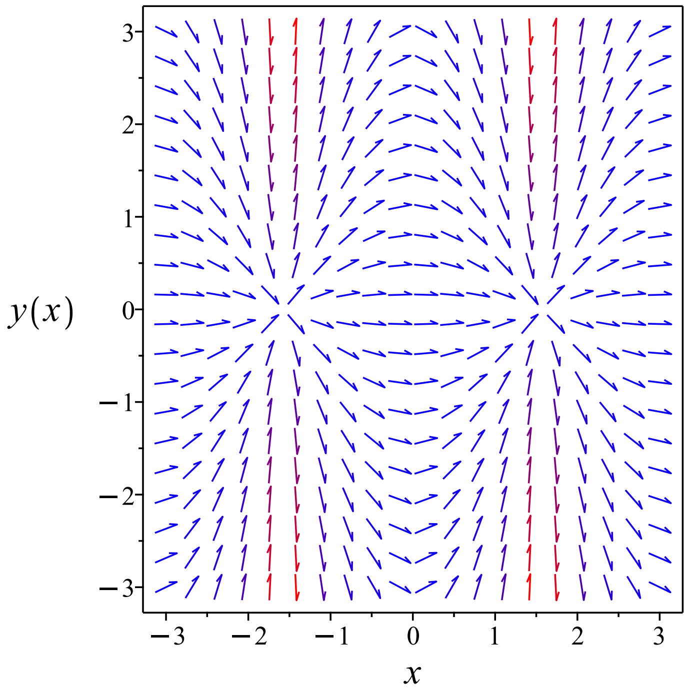


Figure 2: Slope field plot

Verification of solutions

$$y = \cos(x)\, c_1$$

Verified OK.

1.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) + u(x)\, x \tan(x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u(\tan(x)\, x + 1)}{x} \end{aligned}$$

Where $f(x) = -\frac{\tan(x)x+1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{\tan(x)\,x+1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{\tan(x)\,x+1}{x}\,dx \\
\ln(u) &= \ln(\cos(x)) - \ln(x) + c_2 \\
u &= \mathrm{e}^{\ln(\cos(x))-\ln(x)+c_2} \\
&= c_2\mathrm{e}^{\ln(\cos(x))-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2\cos(x)}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= c_2\cos(x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2\cos(x) \tag{1}
$$

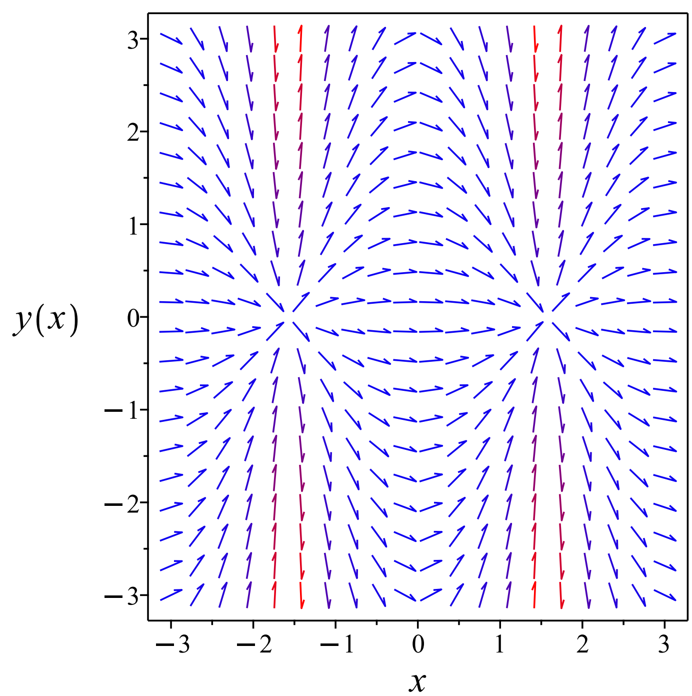


Figure 3: Slope field plot

Verification of solutions

$$y = c_2 \cos(x)$$

Verified OK.

1.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y \tan(x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \cos(x)
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y \tan(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \sec(x) \tan(x)\, y \\ S_y &= \sec(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sec(x)\, y = c_1$$

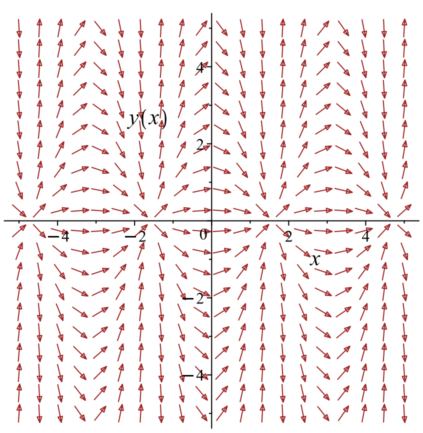
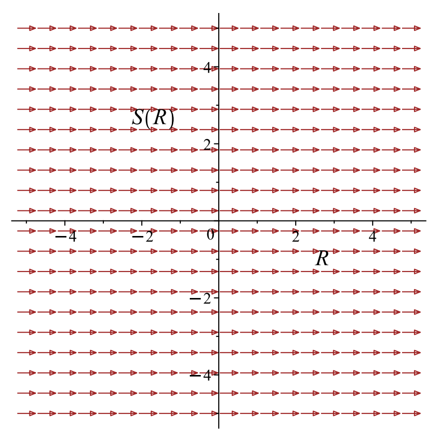
Which simplifies to

$$\sec(x)\, y = c_1$$

Which gives

$$y = \frac{c_1}{\sec(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y\tan(x)$ 	$R = x$ $S = \sec(x)\, y$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = \frac{c_1}{\sec(x)} \tag{1}$$

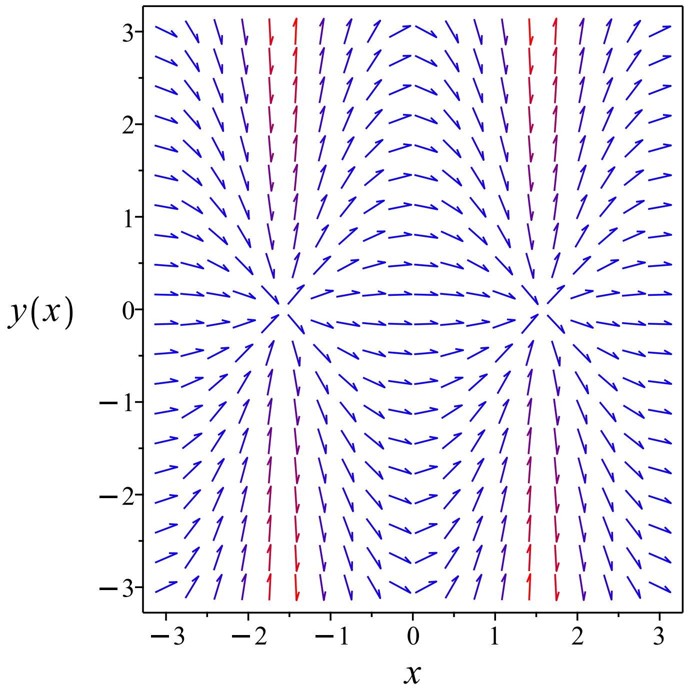


Figure 4: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sec(x)}$$

Verified OK.

1.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y}\right) \mathrm{d}y &= (\tan(x))\, \mathrm{d}x \\
(-\tan(x))\, \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\tan(x) \\
N(x, y) &= -\frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\tan(x)) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\tan(x)\,\mathrm{d}x\end{aligned}$$

$$\phi = \ln(\cos(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y}\right)\mathrm{d}y\\ f(y) &= -\ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(\cos(x)) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(\cos(x)) - \ln(y)$$

The solution becomes

$$y = \cos(x)\,e^{-c_1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \cos(x)\,e^{-c_1} \tag{1}$$

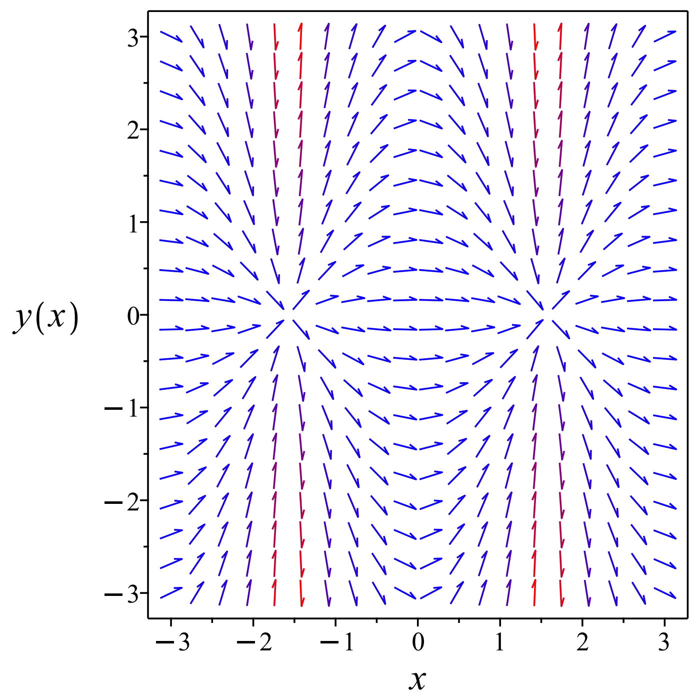


Figure 5: Slope field plot

<u>Verification of solutions</u>

$$y = \cos(x)\,e^{-c_1}$$

Verified OK.

1.1.6 Maple step by step solution

Let's solve

$y' + y\tan(x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\tan(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\tan(x)\, dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(\cos(x)) + c_1$

- Solve for y

  $y = \cos(x)\, e^{c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)+y(x)*tan(x)=0,y(x), singsol=all)
```

$$y(x) = \cos(x)\, c_1$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 15

```
DSolve[y'[x]+y[x]*Tan[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 \cos(x)$$
$$y(x) \rightarrow 0$$

1.2 problem 2

Internal problem ID [4682]
Internal file name [`OUTPUT/4175_Sunday_June_05_2022_12_36_47_PM_50821876/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 1, Nature and meaning of a differential equation between two variables. page 12
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 2xy' + 2y = 0}$$

1.2.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 2xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 2r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 2r + 2 = 0$$

Or

$$r^2 - 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^2 + c_1x$$

Summary
The solution(s) found are the following

$$y = c_2x^2 + c_1x \tag{1}$$

Verification of solutions

$$y = c_2x^2 + c_1x$$

Verified OK.

1.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -\frac{2}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{2}{x}\,dx} \\ &= \frac{1}{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\frac{y}{x}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\frac{1}{x}}$$

Or

$$y = c_1 x^2 + c_2 x$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x$$

Verified OK.

1.2.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 2xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{2}{x}dx\right)}dx \\ &= \int e^{2\ln(x)}\,dx \\ &= \int x^2 dx \\ &= \frac{x^3}{3} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{2}{x^2}}{x^4} \\ &= \frac{2}{x^6} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{x^6} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{2}{x^6} = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{3}} + c_2 \tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}$$

Verified OK.

1.2.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - 2xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{2}{x}\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{3c\sqrt{2}}{2}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1 \cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2 \sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{2}\, \sqrt{\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{2}\, \sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = x^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \tag{1}
$$

Verification of solutions

$$
y = x^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Verified OK.

1.2.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{2}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\
&= \frac{2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= \frac{2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{2}{x}}{2}} \\
&= x
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^3 v''(x) = 0
$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$
v(x) = c_1 x + c_2
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\, z(x) \\
&= (c_1 x + c_2)\,(z(x))
\end{aligned} \tag{7}
$$

But from (5)

$$
z(x) = x
$$

Hence (7) becomes

$$y = (c_1 x + c_2) x$$

Summary
The solution(s) found are the following

$$y = (c_1 x + c_2) x \tag{1}$$

Verification of solutions

$$y = (c_1 x + c_2) x$$

Verified OK.

1.2.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' - 2xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x) x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2} + \frac{2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{2}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= e^{-2\ln(x) + c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{x} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{x} + c_2\right) x^2 \\
&= (c_2 x - c_1)\, x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{x} + c_2\right) x^2 \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{x} + c_2\right) x^2
$$

Verified OK.

1.2.7 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 \\
B &= -2x \\
C &= 2 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (-2x)(-2) + (2)(-2x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x^3v'' + (0)\,v' = 0$$

Now by applying $v' = u$ the above becomes

$$-2x^3u'(x) = 0$$

Which is now solved for u. Integrating both sides gives

$$\begin{aligned}
u(x) &= \int 0 \,\mathrm{d}x \\
&= c_1
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= c_1
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int c_1 \,\mathrm{d}x \\
&= c_1x + c_2
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-2x)\left(c_1x + c_2\right) \\
&= -2(c_1x + c_2)\, x
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = -2(c_1x + c_2)\, x \tag{1}
$$

Verification of solutions

$$
y = -2(c_1x + c_2)\, x
$$

Verified OK.

1.2.8 Solving using Kovacic algorithm

Writing the ode as

$$
x^2y'' - 2xy' + 2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -2x \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 4: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2(x(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x^2 + c_1 x \tag{1}$$

Verification of solutions

$$y = c_2 x^2 + c_1 x$$

Verified OK.

1.2.9 Maple step by step solution

Let's solve

$x^2 y'' - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y'}{x} - \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{2y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - 2xy' + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

Substitute the change of variables back into the ODE

$$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)-2\frac{d}{dt}y(t)+2y(t)=0$$

- Simplify

$$\frac{d^2}{dt^2}y(t)-3\frac{d}{dt}y(t)+2y(t)=0$$

- Characteristic polynomial of ODE

$$r^2-3r+2=0$$

- Factor the characteristic polynomial

$$(r-1)(r-2)=0$$

- Roots of the characteristic polynomial

$$r=(1,2)$$

- 1st solution of the ODE

$$y_1(t)=e^t$$

- 2nd solution of the ODE

$$y_2(t)=e^{2t}$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1e^t+c_2e^{2t}$$

- Change variables back using $t=\ln(x)$

$$y=c_2x^2+c_1x$$

- Simplify

$$y=x(c_2x+c_1)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = x(c_1x + c_2)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 14

```
DSolve[x^2*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_2x + c_1)$$

1.3 problem 3

Internal problem ID [4683]
Internal file name [OUTPUT/4176_Sunday_June_05_2022_12_36_55_PM_64077409/index.tex]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 1, Nature and meaning of a differential equation between two variables. page 12
Problem number: 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _rational, _dAlembert]`

$$\boxed{y{y'}^2 + 2xy' - y = 0}$$

1.3.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y\,p^2 + 2xp - y = 0$$

Solving for y from the above results in

$$y = -\frac{2xp}{p^2 - 1} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -\frac{2p}{p^2-1} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p + \frac{2p}{p^2-1} = x\left(-\frac{2}{p^2-1} + \frac{4p^2}{(p^2-1)^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p + \frac{2p}{p^2-1} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 0 \\
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) + \frac{2p(x)}{p(x)^2-1}}{x\left(-\frac{2}{p(x)^2-1} + \frac{4p(x)^2}{\left(p(x)^2-1\right)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(-\frac{2}{p^2-1} + \frac{4p^2}{(p^2-1)^2}\right)}{p + \frac{2p}{p^2-1}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{2}{p^3 - p}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{2x(p)}{p^3 - p} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{p^3-p}dp}$$
$$= e^{-\ln(p+1)-\ln(p-1)+2\ln(p)}$$

Which simplifies to

$$\mu = \frac{p^2}{p^2 - 1}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p^2 x}{p^2 - 1}\right) = 0$$

Integrating gives

$$\frac{p^2 x}{p^2 - 1} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{p^2}{p^2-1}$ results in

$$x(p) = \frac{c_3(p^2 - 1)}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{-x + \sqrt{x^2 + y^2}}{y}$$
$$p = -\frac{x + \sqrt{x^2 + y^2}}{y}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{2c_3 x}{-x + \sqrt{x^2 + y^2}}$$
$$x = \frac{2c_3 x}{x + \sqrt{x^2 + y^2}}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$
$$y = -ix \tag{2}$$
$$y = ix \tag{3}$$
$$x = -\frac{2c_3 x}{-x + \sqrt{x^2 + y^2}} \tag{4}$$
$$x = \frac{2c_3 x}{x + \sqrt{x^2 + y^2}} \tag{5}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$x = -\frac{2c_3 x}{-x + \sqrt{x^2 + y^2}}$$

Verified OK.

$$x = \frac{2c_3 x}{x + \sqrt{x^2 + y^2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 71

```
dsolve(y(x)*diff(y(x),x)^2+2*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= -ix \\
y(x) &= ix \\
y(x) &= 0 \\
y(x) &= \sqrt{c_1 (c_1 - 2x)} \\
y(x) &= \sqrt{c_1 (c_1 + 2x)} \\
y(x) &= -\sqrt{c_1 (c_1 - 2x)} \\
y(x) &= -\sqrt{c_1 (c_1 + 2x)}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.462 (sec). Leaf size: 126

```
DSolve[y[x]*(y'[x])^2+2*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -e^{\frac{c_1}{2}} \sqrt{-2x + e^{c_1}} \\
y(x) &\to e^{\frac{c_1}{2}} \sqrt{-2x + e^{c_1}} \\
y(x) &\to -e^{\frac{c_1}{2}} \sqrt{2x + e^{c_1}} \\
y(x) &\to e^{\frac{c_1}{2}} \sqrt{2x + e^{c_1}} \\
y(x) &\to 0 \\
y(x) &\to -ix \\
y(x) &\to ix
\end{aligned}
$$

1.4 problem 4

Internal problem ID [4684]
Internal file name [`OUTPUT/4177_Sunday_June_05_2022_12_37_08_PM_24285594/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 1, Nature and meaning of a differential equation between two variables. page 12
Problem number: 4.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2(-x^2+1) = -1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1}{\sqrt{x^2-1}} \tag{1}$$

$$y' = -\frac{1}{\sqrt{x^2-1}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{\sqrt{x^2-1}}\, dx \\ &= \ln\left(x+\sqrt{x^2-1}\right)+c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln\left(x+\sqrt{x^2-1}\right)+c_1 \tag{1}$$

Verification of solutions

$$y = \ln\left(x+\sqrt{x^2-1}\right)+c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x \\ &= -\ln\left(x+\sqrt{x^2-1}\right)+c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\ln\left(x+\sqrt{x^2-1}\right)+c_2 \tag{1}$$

Verification of solutions

$$y = -\ln\left(x+\sqrt{x^2-1}\right)+c_2$$

Verified OK.

1.4.1 Maple step by step solution

Let's solve

$y'^2(-x^2+1) = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'^2(-x^2+1)\,dx = \int (-1)\,dx + c_1$

- Cannot compute integral

  $\int y'^2(-x^2+1)\,dx = -x + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 33

```
dsolve(diff(y(x),x)^2*(1-x^2)+1=0,y(x), singsol=all)
```

$$y(x) = \ln\left(x+\sqrt{x^2-1}\right)+c_1$$
$$y(x) = -\ln\left(x+\sqrt{x^2-1}\right)+c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 89

```
DSolve[y'[x]^2*(1-x^2)+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\log\left(1-\frac{x}{\sqrt{x^2-1}}\right)-\log\left(\frac{x}{\sqrt{x^2-1}}+1\right)+2c_1\right)$$
$$y(x) \to \frac{1}{2}\left(-\log\left(1-\frac{x}{\sqrt{x^2-1}}\right)+\log\left(\frac{x}{\sqrt{x^2-1}}+1\right)+2c_1\right)$$

1.5 problem 5

Internal problem ID [4685]
Internal file name [`OUTPUT/4178_Sunday_June_05_2022_12_37_17_PM_58498289/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 1, Nature and meaning of a differential equation between two variables. page 12
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - ya = e^{ax}}$$

1.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -a$$
$$q(x) = e^{ax}$$

Hence the ode is

$$y' - ya = e^{ax}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -a dx} \\ &= \mathrm{e}^{-ax}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\mathrm{e}^{ax}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-ax} y\right) &= \left(\mathrm{e}^{-ax}\right)\left(\mathrm{e}^{ax}\right) \\ \mathrm{d}\left(\mathrm{e}^{-ax} y\right) &= \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-ax} y &= \int \mathrm{d}x \\ \mathrm{e}^{-ax} y &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-ax}$ results in

$$y = x\,\mathrm{e}^{ax} + c_1 \mathrm{e}^{ax}$$

which simplifies to

$$y = \mathrm{e}^{ax}(x + c_1)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{ax}(x + c_1) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{ax}(x + c_1)$$

Verified OK.

1.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= \mathrm{e}^{ax} + ya \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= e^{ax} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{ax}} dy \end{aligned}$$

Which results in

$$S = e^{-ax} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = e^{ax} + ya$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -a\, e^{-ax} y \\ S_y &= e^{-ax} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-ax} y = x + c_1$$

Which simplifies to

$$e^{-ax} y = x + c_1$$

Which gives

$$y = e^{ax}(x + c_1)$$

Summary
The solution(s) found are the following

$$y = e^{ax}(x + c_1) \tag{1}$$

Verification of solutions

$$y = e^{ax}(x + c_1)$$

Verified OK.

1.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= (\mathrm{e}^{ax} + ya)\,\mathrm{d}x \\ (-\mathrm{e}^{ax} - ya)\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\mathrm{e}^{ax} - ya \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\mathrm{e}^{ax} - ya) \\ &= -a \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-a) - (0)) \\ &= -a \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -a \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-ax} \\ &= \mathrm{e}^{-ax} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-ax}(-\mathrm{e}^{ax} - ya) \\ &= -1 - a\,\mathrm{e}^{-ax} y \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-ax}(1) \\ &= \mathrm{e}^{-ax} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-1 - a\,\mathrm{e}^{-ax} y\right) + \left(\mathrm{e}^{-ax}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -1 - a \, \mathrm{e}^{-ax} y \, \mathrm{d}x$$
$$\phi = -x + \mathrm{e}^{-ax} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-ax} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-ax}$. Therefore equation (4) becomes

$$\mathrm{e}^{-ax} = \mathrm{e}^{-ax} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \mathrm{e}^{-ax} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \mathrm{e}^{-ax} y$$

The solution becomes

$$y = \mathrm{e}^{ax} (x + c_1)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{ax} (x + c_1) \tag{1}$$

Verification of solutions

$$y = e^{ax}(x + c_1)$$

Verified OK.

1.5.4 Maple step by step solution

Let's solve

$y' - ya = e^{ax}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = e^{ax} + ya$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - ya = e^{ax}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' - ya) = \mu(x) e^{ax}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x)(y' - ya) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x) a$

- Solve to find the integrating factor

  $\mu(x) = e^{-ax}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x) e^{ax} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int \mu(x) e^{ax} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) e^{ax} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{-ax}$

  $y = \frac{\int e^{ax} e^{-ax} dx + c_1}{e^{-ax}}$

- Evaluate the integrals on the rhs

  $y = \frac{x+c_1}{\mathrm{e}^{-ax}}$

- Simplify

  $y = \mathrm{e}^{ax}(x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=exp(a*x)+a*y(x),y(x), singsol=all)
```

$$y(x) = (x + c_1)\,\mathrm{e}^{ax}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 15

```
DSolve[y'[x]==Exp[a*x]+a*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{ax}(x + c_1)$$

1.6 problem 9

Internal problem ID [4686]
Internal file name [`OUTPUT/4179_Sunday_June_05_2022_12_37_30_PM_88618282/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 1, Nature and meaning of a differential equation between two variables. page 12
Problem number: 9.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second__order__ode__missing__x", "second__order__ode__missing__y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{\left(1+y'^2\right)^3 - a^2 y''^2 = 0}$$

1.6.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$-a^2 p'(x)^2 + \left(p(x)^5 + 3p(x)^3 + 3p(x)\right) p(x) + 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Solving the given ode for $p'(x)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$p'(x) = \frac{\left(p(x)^2+1\right)^{\frac{3}{2}}}{a} \tag{1}$$

$$p'(x) = -\frac{\left(p(x)^2+1\right)^{\frac{3}{2}}}{a} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{a}{\left(p^2+1\right)^{\frac{3}{2}}} dp = \int dx$$

$$\frac{p(x)\, a}{\sqrt{p\left(x\right)^2+1}} = x + c_1$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{a}{\left(p^2+1\right)^{\frac{3}{2}}} dp = \int dx$$

$$-\frac{p(x)\, a}{\sqrt{p\left(x\right)^2+1}} = c_2 + x$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y' a}{\sqrt{1+y'^2}} = x + c_1$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{x+c_1}{\sqrt{a^2-c_1^2-2c_1x-x^2}} \tag{1}$$

$$y' = \frac{x+c_1}{\sqrt{a^2-c_1^2-2c_1x-x^2}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{x+c_1}{\sqrt{a^2-c_1^2-2c_1x-x^2}}\,\mathrm{d}x \\
&= \frac{(c_1+a+x)\,(a-x-c_1)}{\sqrt{a^2-c_1^2-2c_1x-x^2}} + c_3
\end{aligned}
$$

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{x+c_1}{\sqrt{a^2-c_1^2-2c_1x-x^2}}\,\mathrm{d}x \\
&= -\frac{(c_1+a+x)\,(a-x-c_1)}{\sqrt{a^2-c_1^2-2c_1x-x^2}} + c_4
\end{aligned}
$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
-\frac{y'a}{\sqrt{1+y'^2}} = c_2 + x
$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$
y' = \frac{c_2+x}{\sqrt{a^2-c_2^2-2c_2x-x^2}} \tag{1}
$$

$$
y' = -\frac{c_2+x}{\sqrt{a^2-c_2^2-2c_2x-x^2}} \tag{2}
$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{c_2+x}{\sqrt{a^2-c_2^2-2c_2x-x^2}}\,\mathrm{d}x \\
&= -\frac{(c_2+a+x)\,(a-x-c_2)}{\sqrt{a^2-c_2^2-2c_2x-x^2}} + c_5
\end{aligned}
$$

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{c_2 + x}{\sqrt{a^2 - c_2^2 - 2c_2x - x^2}}\, \mathrm{d}x \\
&= \frac{(c_2 + a + x)(a - x - c_2)}{\sqrt{a^2 - c_2^2 - 2c_2x - x^2}} + c_6
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(c_1 + a + x)(a - x - c_1)}{\sqrt{a^2 - c_1^2 - 2c_1x - x^2}} + c_3 \tag{1}
$$

$$
y = -\frac{(c_1 + a + x)(a - x - c_1)}{\sqrt{a^2 - c_1^2 - 2c_1x - x^2}} + c_4 \tag{2}
$$

$$
y = -\frac{(c_2 + a + x)(a - x - c_2)}{\sqrt{a^2 - c_2^2 - 2c_2x - x^2}} + c_5 \tag{3}
$$

$$
y = \frac{(c_2 + a + x)(a - x - c_2)}{\sqrt{a^2 - c_2^2 - 2c_2x - x^2}} + c_6 \tag{4}
$$

Verification of solutions

$$
y = \frac{(c_1 + a + x)(a - x - c_1)}{\sqrt{a^2 - c_1^2 - 2c_1x - x^2}} + c_3
$$

Verified OK.

$$
y = -\frac{(c_1 + a + x)(a - x - c_1)}{\sqrt{a^2 - c_1^2 - 2c_1x - x^2}} + c_4
$$

Verified OK.

$$
y = -\frac{(c_2 + a + x)(a - x - c_2)}{\sqrt{a^2 - c_2^2 - 2c_2x - x^2}} + c_5
$$

Verified OK.

$$
y = \frac{(c_2 + a + x)(a - x - c_2)}{\sqrt{a^2 - c_2^2 - 2c_2x - x^2}} + c_6
$$

Verified OK.

1.6.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$-a^2 p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 + \left(p(y)^5 + 3p(y)^3 + 3p(y)\right) p(y) = -1$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = \frac{\left(p(y)^2 + 1\right)^{\frac{3}{2}}}{p\left(y\right) a} \tag{1}$$

$$\frac{d}{dy}p(y) = -\frac{\left(p(y)^2 + 1\right)^{\frac{3}{2}}}{p\left(y\right) a} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{pa}{\left(p^2+1\right)^{\frac{3}{2}}} dp = \int dy$$

$$-\frac{a}{\sqrt{p\left(y\right)^2 + 1}} = y + c_1$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{pa}{(p^2+1)^{\frac{3}{2}}}dp = \int dy$$

$$\frac{a}{\sqrt{p(y)^2+1}} = y + c_2$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{a}{\sqrt{1+y'^2}} = y + c_1$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{-y^2 - 2yc_1 - c_1^2 + a^2}}{y + c_1} \tag{1}$$

$$y' = -\frac{\sqrt{-y^2 - 2yc_1 - c_1^2 + a^2}}{y + c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{y + c_1}{\sqrt{a^2 - c_1^2 - 2c_1 y - y^2}}dy = \int dx$$

$$-\frac{(y + c_1 + a)(a - c_1 - y)}{\sqrt{-y^2 - 2yc_1 - c_1^2 + a^2}} = x + c_3$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y + c_1}{\sqrt{a^2 - c_1^2 - 2c_1 y - y^2}}dy = \int dx$$

$$\frac{(y + c_1 + a)(a - c_1 - y)}{\sqrt{-y^2 - 2yc_1 - c_1^2 + a^2}} = x + c_4$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{a}{\sqrt{1 + y'^2}} = y + c_2$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{-y^2 - 2c_2 y - c_2^2 + a^2}}{y + c_2} \tag{1}$$

$$y' = -\frac{\sqrt{-y^2 - 2c_2 y - c_2^2 + a^2}}{y + c_2} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{y + c_2}{\sqrt{a^2 - c_2^2 - 2c_2 y - y^2}} dy = \int dx$$

$$-\frac{(y + c_2 + a)(a - c_2 - y)}{\sqrt{-y^2 - 2c_2 y - c_2^2 + a^2}} = x + c_5$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int -\frac{y + c_2}{\sqrt{a^2 - c_2^2 - 2c_2 y - y^2}} dy = \int dx$$

$$\frac{(y + c_2 + a)(a - c_2 - y)}{\sqrt{-y^2 - 2c_2 y - c_2^2 + a^2}} = x + c_6$$

Summary

The solution(s) found are the following

$$-\frac{(y+c_1+a)(a-c_1-y)}{\sqrt{-y^2-2yc_1-c_1^2+a^2}} = x+c_3 \tag{1}$$

$$\frac{(y+c_1+a)(a-c_1-y)}{\sqrt{-y^2-2yc_1-c_1^2+a^2}} = x+c_4 \tag{2}$$

$$-\frac{(y+c_2+a)(a-c_2-y)}{\sqrt{-y^2-2c_2y-c_2^2+a^2}} = x+c_5 \tag{3}$$

$$\frac{(y+c_2+a)(a-c_2-y)}{\sqrt{-y^2-2c_2y-c_2^2+a^2}} = x+c_6 \tag{4}$$

Verification of solutions

$$-\frac{(y+c_1+a)(a-c_1-y)}{\sqrt{-y^2-2yc_1-c_1^2+a^2}} = x+c_3$$

Verified OK.

$$\frac{(y+c_1+a)(a-c_1-y)}{\sqrt{-y^2-2yc_1-c_1^2+a^2}} = x+c_4$$

Verified OK.

$$-\frac{(y+c_2+a)(a-c_2-y)}{\sqrt{-y^2-2c_2y-c_2^2+a^2}} = x+c_5$$

Verified OK.

$$\frac{(y+c_2+a)(a-c_2-y)}{\sqrt{-y^2-2c_2y-c_2^2+a^2}} = x+c_6$$

Verified OK.

1.6.3 Maple step by step solution

Let's solve

$-a^2y''^2+\left(y'^5+3y'^3+3y'\right)y'=-1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $-a^2u'(x)^2+\left(u(x)^5+3u(x)^3+3u(x)\right)u(x)=-1$

- Separate variables

  $\frac{u'(x)}{\left(u(x)^2+1\right)^{\frac{3}{2}}} = \frac{1}{a}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{\left(u(x)^2+1\right)^{\frac{3}{2}}} dx = \int \frac{1}{a} dx + c_1$

- Evaluate integral

  $\frac{u(x)}{\sqrt{u(x)^2+1}} = \frac{x}{a} + c_1$

- Solve for $u(x)$

  $\left\{u(x) = \frac{ac_1+x}{\sqrt{-c_1^2a^2-2c_1ax+a^2-x^2}}, u(x) = -\frac{ac_1+x}{\sqrt{-c_1^2a^2-2c_1ax+a^2-x^2}}\right\}$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{ac_1+x}{\sqrt{-c_1^2a^2-2c_1ax+a^2-x^2}}$

- Make substitution $u = y'$

  $y' = \frac{ac_1+x}{\sqrt{-c_1^2a^2-2c_1ax+a^2-x^2}}$

- Integrate both sides to solve for y

  $\int y'dx = \int \frac{ac_1+x}{\sqrt{-c_1^2a^2-2c_1ax+a^2-x^2}} dx + c_2$

- Compute integrals

  $y = \frac{(ac_1+a+x)(ac_1-a+x)}{\sqrt{-c_1^2a^2-2c_1ax+a^2-x^2}} + c_2$

- Solve 2nd ODE for $u(x)$

  $u(x) = -\frac{ac_1+x}{\sqrt{-c_1^2a^2-2c_1ax+a^2-x^2}}$

- Make substitution $u = y'$

  $y' = -\frac{ac_1+x}{\sqrt{-c_1^2a^2-2c_1ax+a^2-x^2}}$

- Integrate both sides to solve for y

  $\int y'dx = \int -\frac{ac_1+x}{\sqrt{-c_1^2a^2-2c_1ax+a^2-x^2}} dx + c_2$

- Compute integrals

  $y = -\frac{(ac_1+a+x)(ac_1-a+x)}{\sqrt{-c_1^2a^2-2c_1ax+a^2-x^2}} + c_2$

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      trying 2nd order, 2 integrating factors of the form mu(x,y)
      trying differential order: 2; missing variables
      `, `-> Computing symmetries using: way = 3
      -> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_b(_a)^2+1)^(3/2)/a, _b(_a), HIN
         symmetry methods on request
      `, `1st order, trying reduction of order with given symmetries:`[1, 0], [y, -_b^2-1]
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 93

```
dsolve((diff(y(x),x)^2+1)^3=a^2*(diff(y(x),x$2))^2,y(x), singsol=all)
```

$$y(x) = -ix + c_1$$
$$y(x) = ix + c_1$$
$$y(x) = \frac{(a + x + c_1)(-a + x + c_1)}{\sqrt{a^2 - c_1^2 - 2c_1x - x^2}} + c_2$$
$$y(x) = \frac{(a + x + c_1)(a - x - c_1)}{\sqrt{a^2 - c_1^2 - 2c_1x - x^2}} + c_2$$

✓ Solution by Mathematica

Time used: 0.658 (sec). Leaf size: 141

```
DSolve[(y'[x]^2+1)^3==a^2*(y''[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - i\sqrt{a^2 \left(-1 + c_1{}^2\right) - 2ac_1x + x^2}$$
$$y(x) \to i\sqrt{a^2 \left(-1 + c_1{}^2\right) - 2ac_1x + x^2} + c_2$$
$$y(x) \to c_2 - i\sqrt{a^2 \left(-1 + c_1{}^2\right) + 2ac_1x + x^2}$$
$$y(x) \to i\sqrt{a^2 \left(-1 + c_1{}^2\right) + 2ac_1x + x^2} + c_2$$

2 Chapter 2, Equations of the first order and degree. page 20

2.1 problem 1

Internal problem ID [4687]
Internal file name [`OUTPUT/4180_Sunday_June_05_2022_12_37_47_PM_84375049/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 2, Equations of the first order and degree. page 20
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x+1)\,y+(1-y)\,xy'=0}$$

2.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(x+1)\,y}{(y-1)\,x} \end{aligned}$$

Where $f(x)=\frac{x+1}{x}$ and $g(y)=\frac{y}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y-1}}\,dy = \frac{x+1}{x}\,dx$$

$$\int \frac{1}{\frac{y}{y-1}}\,dy = \int \frac{x+1}{x}\,dx$$

$$y-\ln(y) = x+\ln(x)+c_1$$

Which results in

$$y = -\operatorname{LambertW}\left(-\frac{e^{-c_1-x}}{x}\right)$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. Which simplifies to

$$y = -\operatorname{LambertW}\left(-\frac{e^{-c_1-x}}{x}\right)$$

gives

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x}}{c_1 x}\right)$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x}}{c_1 x}\right) \tag{1}$$

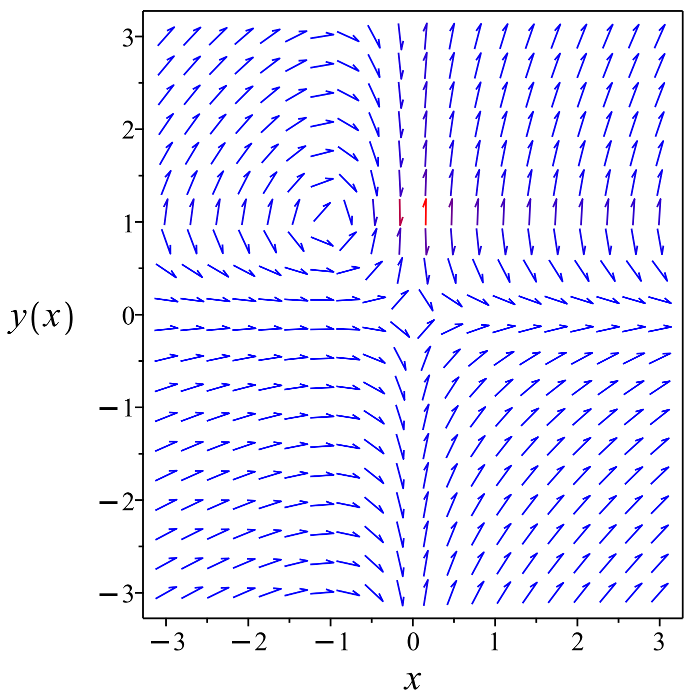


Figure 6: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-x}}{c_1 x}\right)$$

Verified OK.

2.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{(x+1)\,y}{(y-1)\,x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 11: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{x}{x+1} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x}{x+1}} dx
\end{aligned}$$

Which results in

$$S = x + \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{(x+1)\,y}{(y-1)\,x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 1 + \frac{1}{x} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y-1}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R-1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x + \ln(x) = y - \ln(y) + c_1$$

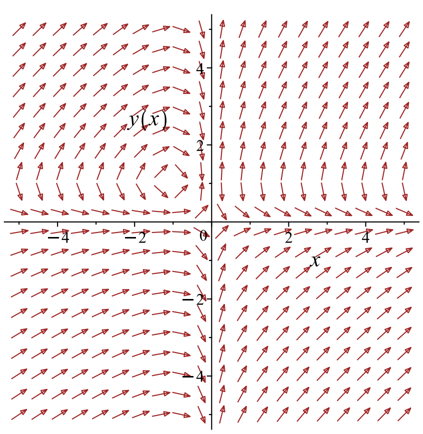
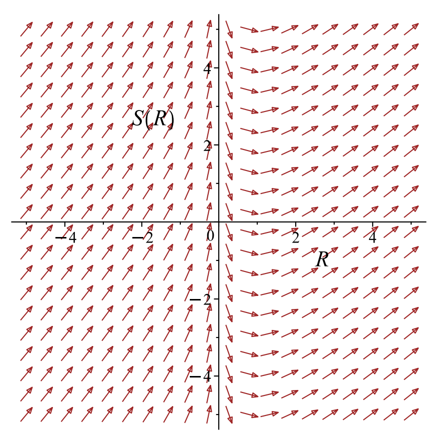
Which simplifies to

$$x + \ln(x) = y - \ln(y) + c_1$$

Which gives

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x+c_1}}{x}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(x+1)y}{(y-1)x}$ 	$R = y$ $S = x + \ln(x)$	$\frac{dS}{dR} = \frac{R-1}{R}$ 

<u>Summary</u>

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x+c_1}}{x}\right) \tag{1}$$

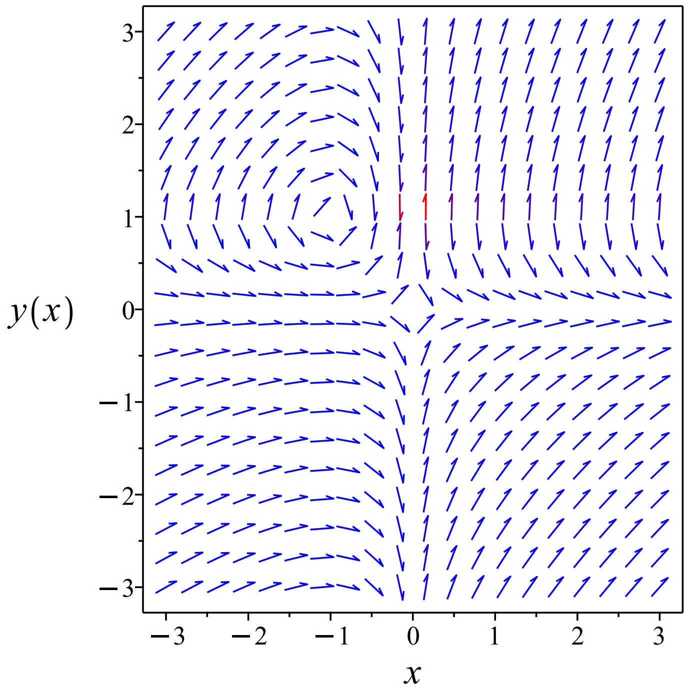


Figure 7: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x+c_1}}{x}\right)$$

Verified OK.

2.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y-1}{y}\right)\mathrm{d}y = \left(\frac{x+1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{x+1}{x}\right)\mathrm{d}x + \left(\frac{y-1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x+1}{x}$$
$$N(x,y) = \frac{y-1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+1}{x}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y-1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x+1}{x}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -x - \ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-1}{y}$. Therefore equation (4) becomes

$$
\frac{y-1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y-1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{y-1}{y}\right)\mathrm{d}y \\
f(y) &= y - \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \ln(x) + y - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \ln(x) + y - \ln(y)$$

The solution becomes

$$y = -\text{LambertW}\left(-\frac{e^{-c_1 - x}}{x}\right)$$

Summary

The solution(s) found are the following

$$y = -\text{LambertW}\left(-\frac{e^{-c_1 - x}}{x}\right) \tag{1}$$

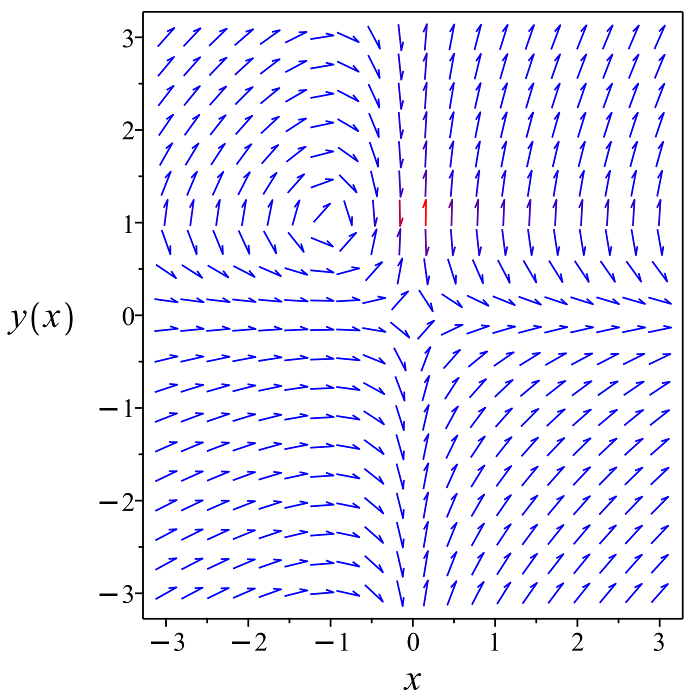


Figure 8: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{-c_1-x}}{x}\right)$$

Verified OK.

2.1.4 Maple step by step solution

Let's solve

$(x+1)\,y+(1-y)\,xy'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1-y)}{y}=-\frac{x+1}{x}$

- Integrate both sides with respect to x

  $\int\frac{y'(1-y)}{y}dx=\int-\frac{x+1}{x}dx+c_1$

- Evaluate integral

  $-y+\ln(y)=-x-\ln(x)+c_1$

- Solve for y

  $y=-\mathit{LambertW}\left(-\frac{e^{-x+c_1}}{x}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve((1+x)*y(x)+(1-y(x))*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-\frac{e^{-x}}{c_1 x}\right)$$

✓ Solution by Mathematica

Time used: 3.094 (sec). Leaf size: 28

```
DSolve[(1+x)*y[x]+(1-y[x])*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -W\left(-\frac{e^{-x-c_1}}{x}\right)$$
$$y(x) \to 0$$

2.2 problem 2

Internal problem ID [4688]
Internal file name [`OUTPUT/4181_Sunday_June_05_2022_12_37_55_PM_94198481/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 2, Equations of the first order and degree. page 20
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y^2 a x = 0}$$

2.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= y^2 a x \end{aligned}$$

Where $f(x) = ax$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = ax\,dx$$

$$\int \frac{1}{y^2}\,dy = \int ax\,dx$$

$$
-\frac{1}{y} = \frac{a\,x^2}{2} + c_1
$$

Which results in

$$
y = -\frac{2}{a\,x^2 + 2c_1}
$$

Summary
The solution(s) found are the following

$$
y = -\frac{2}{a\,x^2 + 2c_1} \tag{1}
$$

Verification of solutions

$$
y = -\frac{2}{a\,x^2 + 2c_1}
$$

Verified OK.

2.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y^2 a x \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 14: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{ax} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{ax}} dx \end{aligned}$$

Which results in

$$S = \frac{a\,x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y^2 a x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= ax \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{a\,x^2}{2} = -\frac{1}{y} + c_1$$

Which simplifies to

$$\frac{a\,x^2}{2} = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{2}{-a\,x^2 + 2c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{2}{-a\,x^2 + 2c_1} \tag{1}$$

Verification of solutions

$$y = \frac{2}{-a\,x^2 + 2c_1}$$

Verified OK.

2.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2a}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{y^2a}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= \frac{1}{y^2a}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2 a}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -x \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2 a}$. Therefore equation (4) becomes

$$
\frac{1}{y^2 a} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2 a}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2 a}\right) \mathrm{d}y$$
$$f(y) = -\frac{1}{ay} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{1}{ay} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{1}{ay}$$

The solution becomes

$$y = -\frac{2}{a\left(x^2 + 2c_1\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{2}{a\left(x^2 + 2c_1\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{2}{a\left(x^2 + 2c_1\right)}$$

Verified OK.

2.2.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 a x \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 a x$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = ax$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{axu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= a \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$axu''(x) - au'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 x^2 + c_1$$

The above shows that

$$u'(x) = 2c_2 x$$

Using the above in (1) gives the solution

$$y = -\frac{2c_2}{a\,(c_2x^2 + c_1)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2}{a\,(x^2 + c_3)}$$

Summary
The solution(s) found are the following

$$y = -\frac{2}{a\,(x^2 + c_3)} \tag{1}$$

Verification of solutions

$$y = -\frac{2}{a\,(x^2 + c_3)}$$

Verified OK.

2.2.5 Maple step by step solution

Let's solve

$y' - y^2ax = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = ax$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2}dx = \int axdx + c_1$

- Evaluate integral

  $-\frac{1}{y} = \frac{a\,x^2}{2} + c_1$

- Solve for y

$$y = -\frac{2}{a\,x^2+2c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=a*y(x)^2*x,y(x), singsol=all)
```

$$y(x) = -\frac{2}{a\,x^2 - 2c_1}$$

✓ Solution by Mathematica

Time used: 0.127 (sec). Leaf size: 24

```
DSolve[y'[x]==a*y[x]^2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{ax^2 + 2c_1}$$
$$y(x) \to 0$$

2.3 problem 3

Internal problem ID [4689]
Internal file name [`OUTPUT/4182_Sunday_June_05_2022_12_38_04_PM_68724526/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 2, Equations of the first order and degree. page 20
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 + x y^2 + \left(x^2 - y x^2\right) y' = 0}$$

2.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2(x+1)}{x^2\,(y-1)} \end{aligned}$$

Where $f(x) = \frac{x+1}{x^2}$ and $g(y) = \frac{y^2}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{y^2}{y-1}}\,dy = \frac{x+1}{x^2}\,dx$$

$$\int \frac{1}{\frac{y^2}{y-1}}\,dy = \int \frac{x+1}{x^2}\,dx$$

$$
\ln(y)+\frac{1}{y}=\ln(x)-\frac{1}{x}+c_1
$$

Which results in

$$
y = \mathrm{e}^{\frac{\ln(x)x+\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x+c_1x-1}{x}}\right)x+c_1x-1}{x}}
$$

Which simplifies to

$$
y = x\,\mathrm{e}^{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{1}{x}}\mathrm{e}^{-c_1}}{x}\right)}\mathrm{e}^{c_1}\mathrm{e}^{-\frac{1}{x}}
$$

Summary
The solution(s) found are the following

$$
y = x\,\mathrm{e}^{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{1}{x}}\mathrm{e}^{-c_1}}{x}\right)}\mathrm{e}^{c_1}\mathrm{e}^{-\frac{1}{x}} \tag{1}
$$

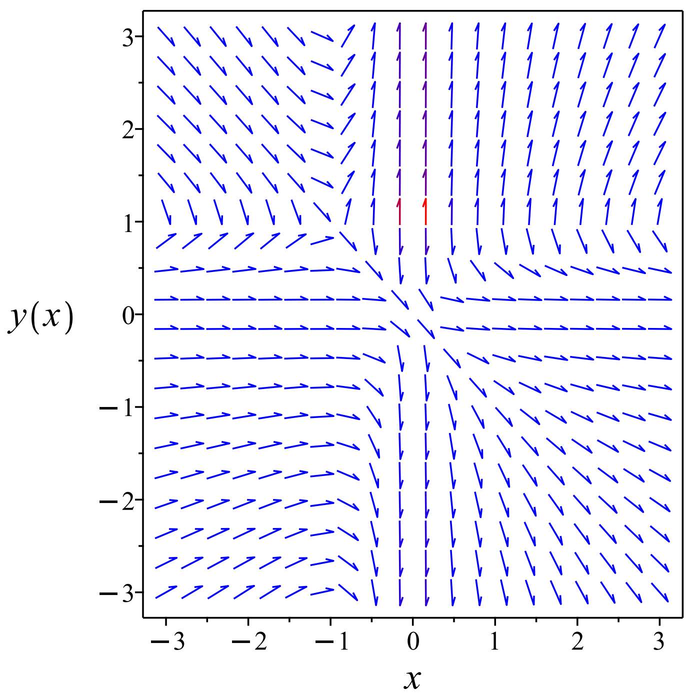


Figure 9: Slope field plot

Verification of solutions

$$
y = x\,\mathrm{e}^{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{1}{x}}\mathrm{e}^{-c_1}}{x}\right)}\mathrm{e}^{c_1}\mathrm{e}^{-\frac{1}{x}}
$$

Verified OK.

2.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2(x+1)}{x^2(y-1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 17: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{x^2}{x+1}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^2}{x+1}} dx \end{aligned}$$

Which results in

$$S = \ln(x) - \frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2(x+1)}{x^2(y-1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x+1}{x^2} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y-1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R-1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x)\,x - 1}{x} = \ln(y) + \frac{1}{y} + c_1$$

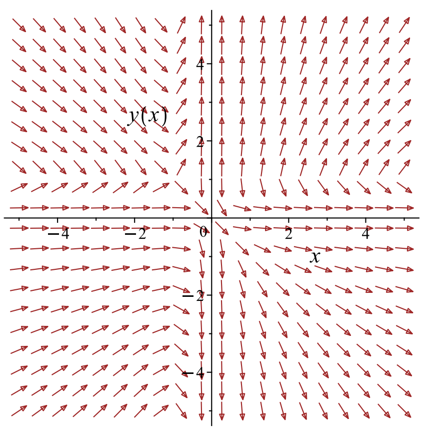
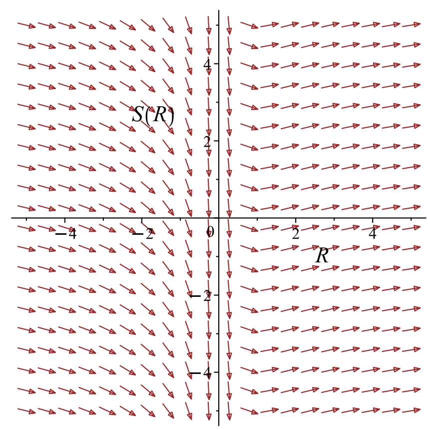
Which simplifies to

$$\frac{\ln(x)\,x - 1}{x} = \ln(y) + \frac{1}{y} + c_1$$

Which gives

$$y = \mathrm{e}^{\frac{\ln(x)x + \mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x - c_1 x - 1}{x}}\right)x - c_1 x - 1}{x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2(x+1)}{x^2(y-1)}$ 	$R = y$ $S = \frac{\ln(x)\, x - 1}{x}$	$\frac{dS}{dR} = \frac{R-1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = e^{\frac{\ln(x)x + \mathrm{LambertW}\left(-e^{-\frac{\ln(x)x - c_1 x - 1}{x}}\right)x - c_1 x - 1}{x}} \tag{1}$$

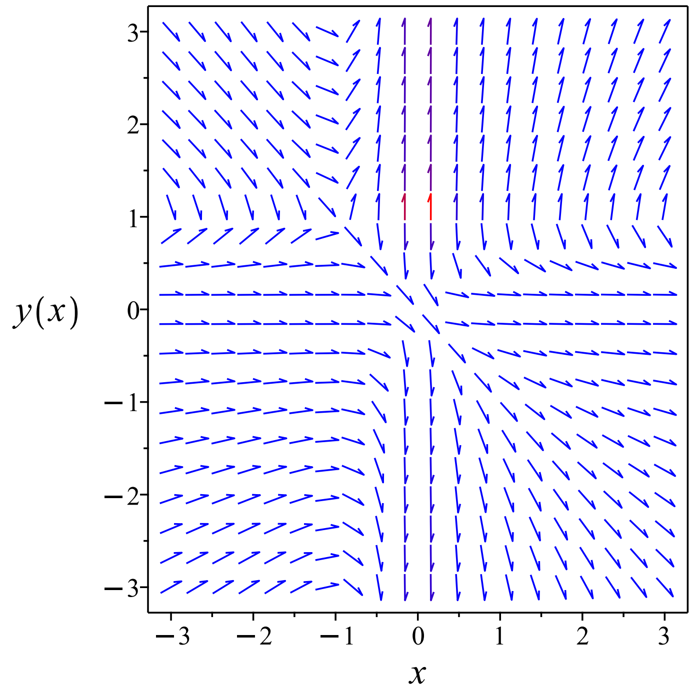


Figure 10: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{\ln(x)x + \mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x - c_1 x - 1}{x}}\right)x - c_1 x - 1}{x}}$$

Verified OK.

2.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y-1}{y^2}\right)\mathrm{d}y = \left(\frac{x+1}{x^2}\right)\mathrm{d}x$$
$$\left(-\frac{x+1}{x^2}\right)\mathrm{d}x + \left(\frac{y-1}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x+1}{x^2}$$
$$N(x, y) = \frac{y-1}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+1}{x^2}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y-1}{y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x+1}{x^2}\,\mathrm{d}x \\
\phi &= -\ln(x) + \frac{1}{x} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-1}{y^2}$. Therefore equation (4) becomes

$$
\frac{y-1}{y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y-1}{y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y-1}{y^2}\right)\mathrm{d}y$$
$$f(y) = \ln(y) + \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{1}{x} + \ln(y) + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{1}{x} + \ln(y) + \frac{1}{y}$$

The solution becomes

$$y = \mathrm{e}^{\frac{\ln(x)x+\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x+c_1x-1}{x}}\right)x+c_1x-1}{x}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\ln(x)x+\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x+c_1x-1}{x}}\right)x+c_1x-1}{x}} \tag{1}$$

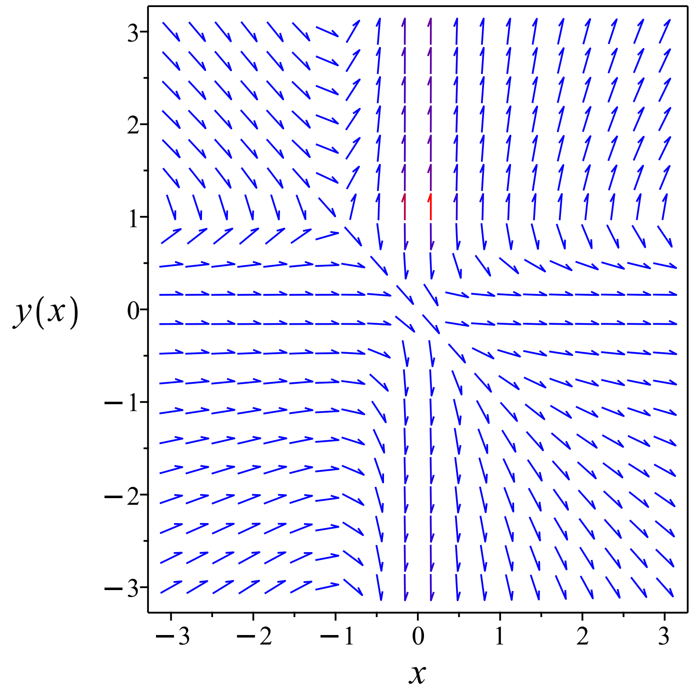


Figure 11: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{\ln(x)x+\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x+c_1x-1}{x}}\right)x+c_1x-1}{x}}$$

Verified OK.

2.3.4 Maple step by step solution

Let's solve

$y^2 + xy^2 + (x^2 - yx^2)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(y-1)}{y^2} = \frac{x+1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'(y-1)}{y^2} dx = \int \frac{x+1}{x^2} dx + c_1$

- Evaluate integral

  $\ln(y) + \frac{1}{y} = \ln(x) - \frac{1}{x} + c_1$

- Solve for y

  $$y = \mathrm{e}^{\frac{\ln(x)x + \mathit{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x + c_1 x - 1}{x}}\right)x + c_1 x - 1}{x}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve((y(x)^2+x*y(x)^2)+(x^2-y(x)*x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x\,\mathrm{e}^{\frac{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{-c_1 x + 1}{x}}}{x}\right)x + c_1 x - 1}{x}}$$

✓ Solution by Mathematica

Time used: 5.302 (sec). Leaf size: 30

```
DSolve[(y[x]^2+x*y[x]^2)+(x^2-y[x]*x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{W\left(-\frac{e^{\frac{1}{x} - c_1}}{x}\right)}$$

$$y(x) \to 0$$

2.4 problem 4

Internal problem ID [4690]
Internal file name [`OUTPUT/4183_Sunday_June_05_2022_12_38_12_PM_61438760/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 2, Equations of the first order and degree. page 20
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{xy(x^2+1)\,y'-y^2=1}$$

2.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2+1}{xy\,(x^2+1)} \end{aligned}$$

Where $f(x)=\frac{1}{x(x^2+1)}$ and $g(y)=\frac{y^2+1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+1}{y}}\,dy=\frac{1}{x\,(x^2+1)}\,dx$$

$$\int \frac{1}{\frac{y^2+1}{y}}\, dy = \int \frac{1}{x\left(x^2+1\right)}\, dx$$

$$\frac{\ln\left(y^2+1\right)}{2} = -\frac{\ln\left(x^2+1\right)}{2} + \ln\left(x\right) + c_1$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = \mathrm{e}^{-\frac{\ln\left(x^2+1\right)}{2}+\ln(x)+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = c_2 \mathrm{e}^{-\frac{\ln\left(x^2+1\right)}{2}+\ln(x)}$$

Which simplifies to

$$\sqrt{1+y^2} = \frac{c_2 x\, \mathrm{e}^{c_1}}{\sqrt{x^2+1}}$$

The solution is

$$\sqrt{1+y^2} = \frac{c_2 x\, \mathrm{e}^{c_1}}{\sqrt{x^2+1}}$$

Summary

The solution(s) found are the following

$$\sqrt{1+y^2} = \frac{c_2 x\, \mathrm{e}^{c_1}}{\sqrt{x^2+1}} \tag{1}$$

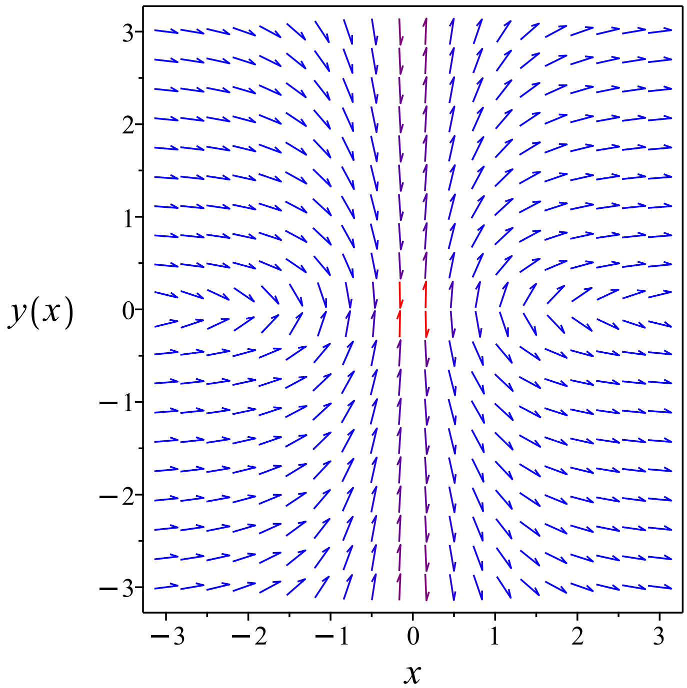


Figure 12: Slope field plot

Verification of solutions

$$\sqrt{1+y^2} = \frac{c_2 x\,\mathrm{e}^{c_1}}{\sqrt{x^2+1}}$$

Verified OK.

2.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2+1}{xy\,(x^2+1)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 20: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= x\left(x^2 + 1\right) \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x\left(x^2+1\right)} dx \end{aligned}$$

Which results in

$$S = -\frac{\ln\left(x^2+1\right)}{2} + \ln\left(x\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2+1}{xy\left(x^2+1\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x\left(x^2+1\right)} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

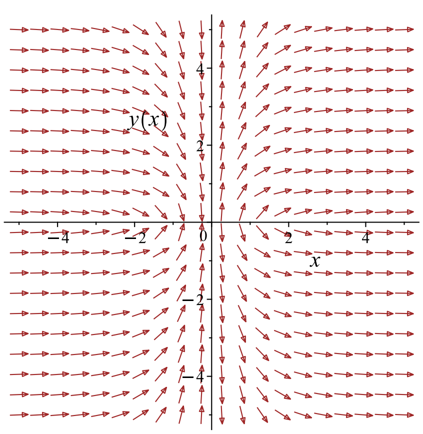
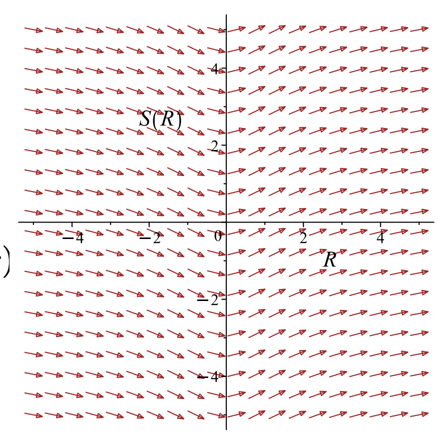
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(x^2+1)}{2} + \ln(x) = \frac{\ln(1+y^2)}{2} + c_1$$

Which simplifies to

$$-\frac{\ln(x^2+1)}{2} + \ln(x) = \frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+1}{xy(x^2+1)}$ 	$R = y$ $S = -\frac{\ln(x^2+1)}{2} + \ln(x)$	$\frac{dS}{dR} = \frac{R}{R^2+1}$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(x^2+1)}{2} + \ln(x) = \frac{\ln(1+y^2)}{2} + c_1 \tag{1}$$

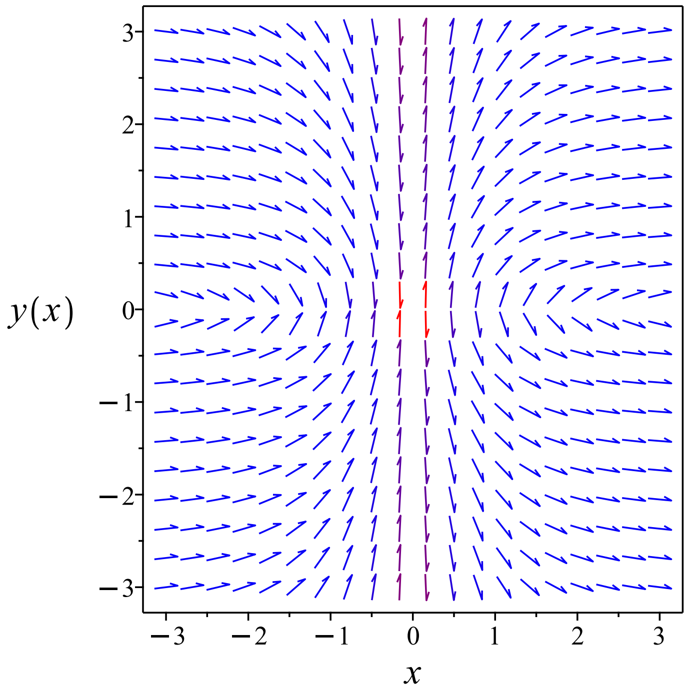


Figure 13: Slope field plot

Verification of solutions

$$-\frac{\ln(x^2+1)}{2} + \ln(x) = \frac{\ln(1+y^2)}{2} + c_1$$

Verified OK.

2.4.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2+1}{xy(x^2+1)} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x(x^2+1)}y + \frac{1}{x(x^2+1)}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x(x^2+1)} \\ f_1(x) &= \frac{1}{x(x^2+1)} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{x(x^2+1)} + \frac{1}{x(x^2+1)} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{x(x^2+1)} + \frac{1}{x(x^2+1)} \\ w' &= \frac{2w}{x(x^2+1)} + \frac{2}{x(x^2+1)} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x\left(x^2+1\right)} \\
q(x) &= \frac{2}{x\left(x^2+1\right)}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{2w(x)}{x\left(x^2+1\right)} = \frac{2}{x\left(x^2+1\right)}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{x\left(x^2+1\right)}dx} \\
&= \mathrm{e}^{\ln\left(x^2+1\right)-2\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{x^2+1}{x^2}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{2}{x\left(x^2+1\right)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\left(x^2+1\right)w}{x^2}\right) &= \left(\frac{x^2+1}{x^2}\right)\left(\frac{2}{x\left(x^2+1\right)}\right) \\
\mathrm{d}\left(\frac{\left(x^2+1\right)w}{x^2}\right) &= \left(\frac{2}{x^3}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\left(x^2+1\right)w}{x^2} &= \int \frac{2}{x^3}\,\mathrm{d}x \\
\frac{\left(x^2+1\right)w}{x^2} &= -\frac{1}{x^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{x^2+1}{x^2}$ results in

$$
w(x) = -\frac{1}{x^2+1} + \frac{c_1 x^2}{x^2+1}
$$

which simplifies to

$$
w(x) = \frac{c_1 x^2 - 1}{x^2+1}
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{c_1 x^2 - 1}{x^2 + 1}$$

Solving for y gives

$$y(x) = \frac{\sqrt{(x^2+1)(c_1 x^2 - 1)}}{x^2+1}$$
$$y(x) = -\frac{\sqrt{(x^2+1)(c_1 x^2 - 1)}}{x^2+1}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{(x^2+1)(c_1 x^2 - 1)}}{x^2+1} \tag{1}$$
$$y = -\frac{\sqrt{(x^2+1)(c_1 x^2 - 1)}}{x^2+1} \tag{2}$$

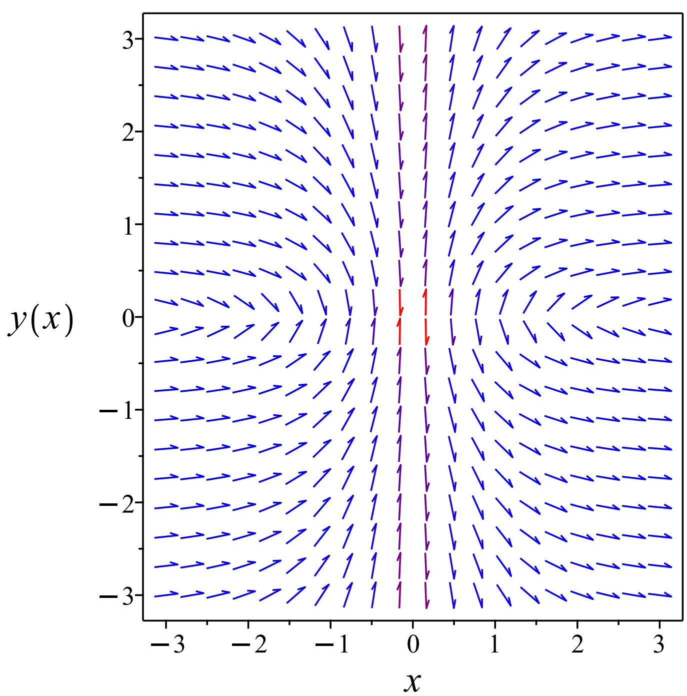


Figure 14: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{(x^2+1)(c_1x^2-1)}}{x^2+1}$$

Verified OK.

$$y = -\frac{\sqrt{(x^2+1)(c_1x^2-1)}}{x^2+1}$$

Verified OK.

2.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y}{y^2+1}\right)\mathrm{d}y = \left(\frac{1}{x\,(x^2+1)}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x\,(x^2+1)}\right)\mathrm{d}x + \left(\frac{y}{y^2+1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x\,(x^2+1)}$$
$$N(x,y) = \frac{y}{y^2+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x\,(x^2+1)}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{y^2+1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{1}{x\left(x^2+1\right)} \mathrm{d}x \\ \phi &= \frac{\ln\left(x^2+1\right)}{2} - \ln\left(x\right) + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{y^2+1}$. Therefore equation (4) becomes

$$\frac{y}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{y^2+1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) \mathrm{d}y &= \int \left(\frac{y}{y^2+1}\right) \mathrm{d}y \\ f(y) &= \frac{\ln\left(y^2+1\right)}{2} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln\left(x^2+1\right)}{2} - \ln\left(x\right) + \frac{\ln\left(y^2+1\right)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x^2+1)}{2} - \ln(x) + \frac{\ln(y^2+1)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+1)}{2} - \ln(x) + \frac{\ln(1+y^2)}{2} = c_1 \tag{1}$$

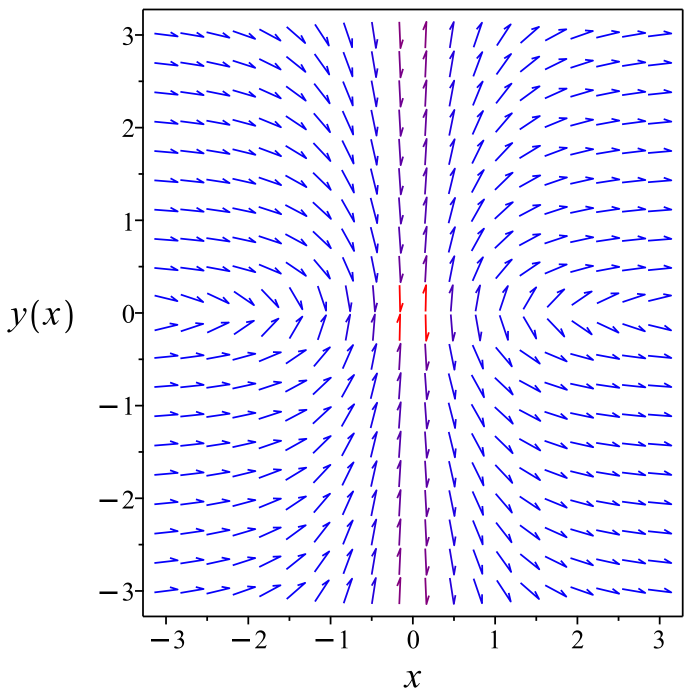


Figure 15: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+1)}{2} - \ln(x) + \frac{\ln(1+y^2)}{2} = c_1$$

Verified OK.

2.4.5 Maple step by step solution

Let's solve

$xy(x^2+1)\,y' - y^2 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{1+y^2} = \frac{1}{x(x^2+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{1+y^2}dx = \int \frac{1}{x(x^2+1)}dx + c_1$

- Evaluate integral

  $\frac{\ln(1+y^2)}{2} = -\frac{\ln(x^2+1)}{2} + \ln(x) + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{(x^2+1)\left((\mathrm{e}^{c_1})^2 x^2 - x^2 - 1\right)}}{x^2+1}, y = -\frac{\sqrt{(x^2+1)\left((\mathrm{e}^{c_1})^2 x^2 - x^2 - 1\right)}}{x^2+1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(x*y(x)*(1+x^2)*diff(y(x),x)=1+y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(x^2+1)(c_1x^2-1)}}{x^2+1}$$
$$y(x) = -\frac{\sqrt{(x^2+1)(c_1x^2-1)}}{x^2+1}$$

✓ Solution by Mathematica

Time used: 1.206 (sec). Leaf size: 131

```
DSolve[x*y[x]*(1+x^2)*y'[x]==1+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\frac{\sqrt{-1+(-1+e^{2c_1})\,x^2}}{\sqrt{x^2+1}} \\
y(x) &\to \frac{\sqrt{-1+(-1+e^{2c_1})\,x^2}}{\sqrt{x^2+1}} \\
y(x) &\to -i \\
y(x) &\to i \\
y(x) &\to -\frac{\sqrt{-x^2-1}}{\sqrt{x^2+1}} \\
y(x) &\to \frac{\sqrt{-x^2-1}}{\sqrt{x^2+1}}
\end{aligned}
$$

2.5 problem 5

Internal problem ID [4691]
Internal file name [`OUTPUT/4184_Sunday_June_05_2022_12_38_20_PM_15470054/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 2, Equations of the first order and degree. page 20
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{x}{1+y}-\frac{yy'}{x+1}=0}$$

2.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{x(x+1)}{(1+y)\,y}\end{aligned}$$

Where $f(x)=x(x+1)$ and $g(y)=\frac{1}{(1+y)y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{(1+y)y}}\,dy = x(x+1)\,\,dx$$

$$\int \frac{1}{\frac{1}{(1+y)y}}\, dy = \int x(x+1)\, dx$$

$$\frac{1}{3}y^3 + \frac{1}{2}y^2 = \frac{1}{2}x^2 + \frac{1}{3}x^3 + c_1$$

Which results in

$$y = \frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} + \frac{1}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2}$$

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} - \frac{1}{4\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} + \frac{i\sqrt{3}\left(\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5}\right.}\right.}{2}$$

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} - \frac{1}{4\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} - \frac{i\sqrt{3}\left(\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5}\right.}\right.}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} + \frac{1}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} \tag{1}$$

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} - \frac{1}{4\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} + \frac{i\sqrt{3}\left(\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} - \frac{\cdots}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5\cdots}\right.}\cdots\right.}{2} \tag{2}$$

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} - \frac{1}{4\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} - \frac{i\sqrt{3}\left(\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} - \frac{\cdots}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5\cdots}\right.}\cdots\right.}{2} \tag{3}$$

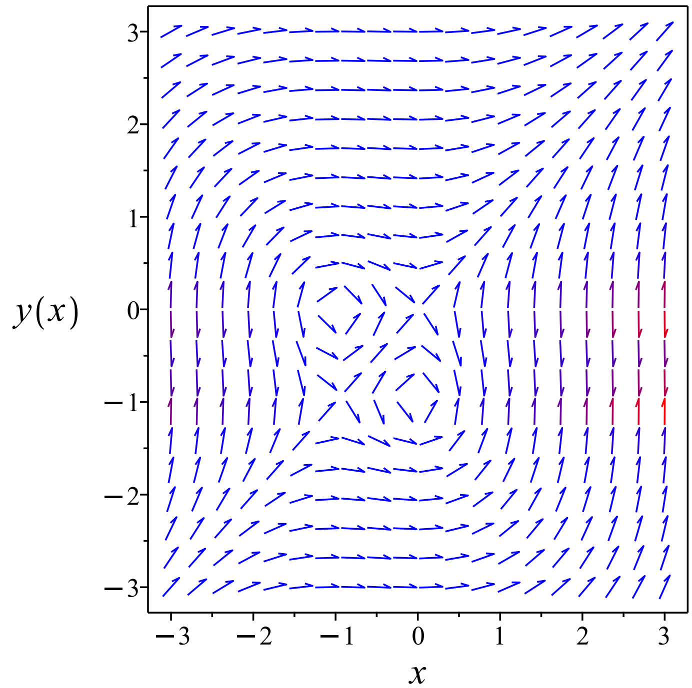


Figure 16: Slope field plot

Verification of solutions

$$y = \frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} + \frac{1}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2}$$

Verified OK.

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} - \frac{1}{4\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} + \frac{i\sqrt{3}\left(\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5}\right.}\right.}{2}$$

Verified OK.

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} - \frac{1}{4\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} - \frac{i\sqrt{3}\left(\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5}\right.}\right.}{2}$$

Verified OK.

2.5.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x(x+1)}{(1+y)\,y} \tag{1}$$

Which becomes

$$\left(y^2+y\right) dy = (x(x+1))\,dx \tag{2}$$

But the RHS is complete differential because

$$(x(x+1))\,dx = d\left(\frac{1}{2}x^2 + \frac{1}{3}x^3\right)$$

Hence (2) becomes

$$\left(y^2+y\right) dy = d\left(\frac{1}{2}x^2 + \frac{1}{3}x^3\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} + \frac{}{2}$$

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} -$$

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} -$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} + \frac{1}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} + c_1 \tag{1}$$

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} - \frac{1}{4\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} + \frac{i\sqrt{3}\left(\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} - \frac{\cdots}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5 \cdots}\right.}\right.}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} - \frac{1}{4\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} - \frac{i\sqrt{3}\left(\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} - \frac{\cdots}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5 \cdots}\right.}\right.}{2} + c_1 \tag{3}$$

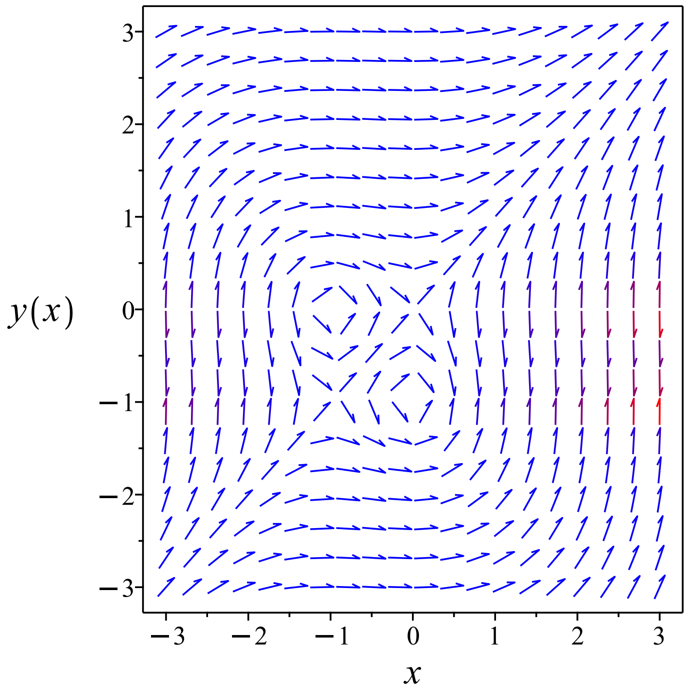


Figure 17: Slope field plot

Verification of solutions

$$y = \frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} + \frac{1}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} - \frac{1}{4\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} + \frac{i\sqrt{3}\left(\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5}\right.}\right.}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{4} - \frac{1}{4\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2} - \frac{i\sqrt{3}\left(\frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5}\right.}\right.}{2} + c_1$$

Verified OK.

2.5.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x(x+1)}{(1+y)\, y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 23: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{x\,(x+1)}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x(x+1)}} dx \end{aligned}$$

Which results in

$$S = \frac{1}{2}x^2 + \frac{1}{3}x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x(x+1)}{(1+y)\,y}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x^2 + x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (1 + y)\, y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (1 + R)\, R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{3}R^3 + \frac{1}{2}R^2 + c_1 \tag{4}$$

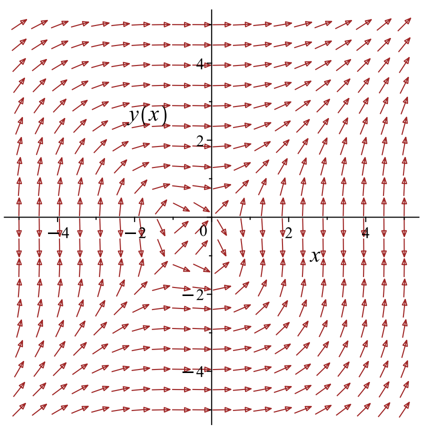
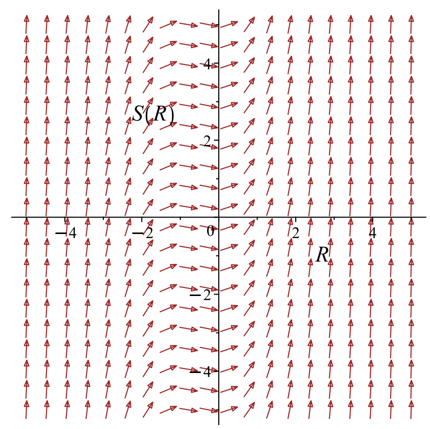
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{2}x^2 + \frac{1}{3}x^3 = \frac{y^3}{3} + \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{1}{2}x^2 + \frac{1}{3}x^3 = \frac{y^3}{3} + \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(x+1)}{(1+y)y}$ 	$R = y$ $S = \frac{1}{2}x^2 + \frac{1}{3}x^3$	$\frac{dS}{dR} = (1 + R) R$ 

Summary

The solution(s) found are the following

$$\frac{1}{2}x^2 + \frac{1}{3}x^3 = \frac{y^3}{3} + \frac{y^2}{2} + c_1 \tag{1}$$

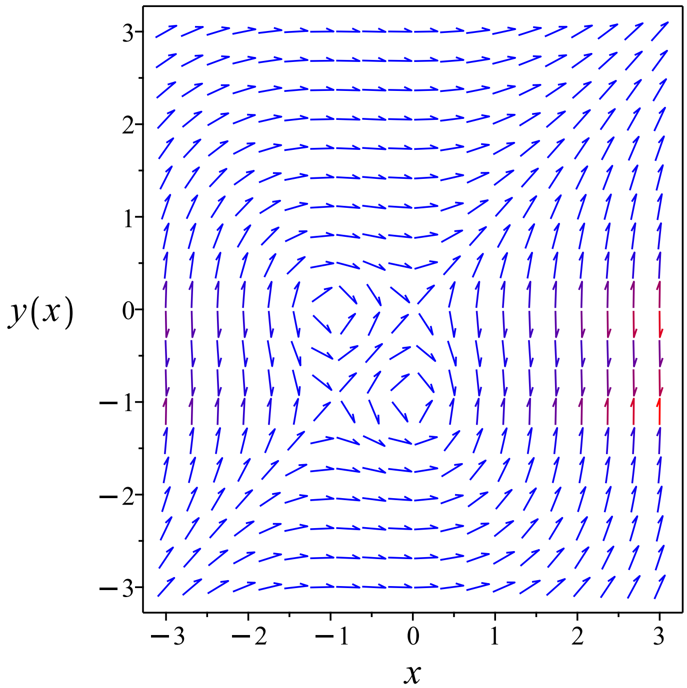


Figure 18: Slope field plot

<u>Verification of solutions</u>

$$\frac{1}{2}x^2 + \frac{1}{3}x^3 = \frac{y^3}{3} + \frac{y^2}{2} + c_1$$

Verified OK.

2.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
((1 + y)\, y)\, \mathrm{d}y &= (x(x + 1))\, \mathrm{d}x \\
(-x(x + 1))\, \mathrm{d}x + ((1 + y)\, y)\, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x(x + 1) \\
N(x, y) &= (1 + y)\, y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x(x + 1)) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}((1 + y)\, y) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int -x(x+1) \, dx$$

$$\phi = -\frac{1}{3}x^3 - \frac{1}{2}x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (1+y)\,y$. Therefore equation (4) becomes

$$(1+y)\,y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = (1+y)\,y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int ((1+y)\,y) \, dy$$

$$f(y) = \frac{1}{3}y^3 + \frac{1}{2}y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{3}y^3 + \frac{1}{2}y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{3}y^3 + \frac{1}{2}y^2$$

Summary
The solution(s) found are the following

$$-\frac{x^3}{3} + \frac{y^3}{3} - \frac{x^2}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

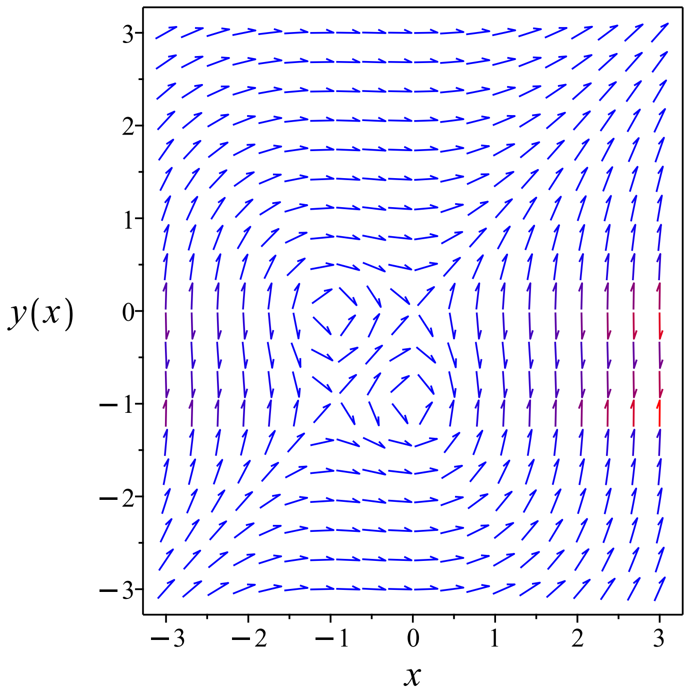


Figure 19: Slope field plot

Verification of solutions

$$-\frac{x^3}{3} + \frac{y^3}{3} - \frac{x^2}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

2.5.5 Maple step by step solution

Let's solve

$\frac{x}{1+y} - \frac{yy'}{x+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(1+y)\,y = x(x+1)$

- Integrate both sides with respect to x

  $\int y'(1+y)\,y dx = \int x(x+1)\,dx + c_1$

- Evaluate integral

  $\frac{y^3}{3} + \frac{y^2}{2} = \frac{1}{2}x^2 + \frac{1}{3}x^3 + c_1$

- Solve for y

  $y = \frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} + \frac{}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5}\right.}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 498

```
dsolve(x/(1+y(x))=y(x)/(1+x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}}{2} + \frac{1}{2\left(-1+4x^3+6x^2+12c_1+2\sqrt{4x^6+12x^5+24c_1x^3+9x^4+36c_1x^2-2x^3+36c_1^2-3x^2-6c_1}\right)^{\frac{1}{3}}} - \frac{1}{2}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(4x^3+6x^2+2\sqrt{\left(2x^3+3x^2+6c_1\right)\left(2x^3+3x^2+6c_1-1\right)}+12c_1-1\right)^{\frac{2}{3}} - i\sqrt{3}+2\left(4x^3+\right.}{4\left(4x^3+6x^2+2\sqrt{\left(2x^3+3x^2+6c_1\right)\left(2x^3+3x^2+6c\right.}\right.}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(4x^3+6x^2+2\sqrt{\left(2x^3+3x^2+6c_1\right)\left(2x^3+3x^2+6c_1-1\right)}+12c_1-1\right)^{\frac{2}{3}} - i\sqrt{3}-2\left(4x^3+6x\right.}{4\left(4x^3+6x^2+2\sqrt{\left(2x^3+3x^2+6c_1\right)\left(2x^3+3x^2+6c_1\right.}\right.}$$

✓ Solution by Mathematica

Time used: 4.125 (sec). Leaf size: 346

```
DSolve[x/(1+y[x])==y[x]/(1+x)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\sqrt[3]{4x^3+6x^2+\sqrt{-1+(4x^3+6x^2-1+12c_1)^2}-1+12c_1} + \frac{1}{\sqrt[3]{4x^3+6x^2+\sqrt{-1+(4x^3+6x^2-1+12c_1)^2}-1+12c_1}} - 1\right)$$

$$y(x) \to \frac{1}{8}\left(2i\left(\sqrt{3}+i\right)\sqrt[3]{4x^3+6x^2+\sqrt{-1+(4x^3+6x^2-1+12c_1)^2}-1+12c_1} + \frac{-2-2i\sqrt{3}}{\sqrt[3]{4x^3+6x^2+\sqrt{-1+(4x^3+6x^2-1+12c_1)^2}-1+12c_1}} - 4\right)$$

$$y(x) \to \frac{1}{8}\left(-2\left(1+i\sqrt{3}\right)\sqrt[3]{4x^3+6x^2+\sqrt{-1+(4x^3+6x^2-1+12c_1)^2}-1+12c_1} + \frac{2i\left(\sqrt{3}+i\right)}{\sqrt[3]{4x^3+6x^2+\sqrt{-1+(4x^3+6x^2-1+12c_1)^2}-1+12c_1}} - 4\right)$$

2.6 problem 6

Internal problem ID [4692]
Internal file name [`OUTPUT/4185_Sunday_June_05_2022_12_38_28_PM_87658595/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 2, Equations of the first order and degree. page 20
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y^2 b^2 = a^2}$$

2.6.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^2 b^2 + a^2} dy = \int dx$$
$$-\frac{\ln (by - a)}{2ab} + \frac{\ln (by + a)}{2ab} = x + c_1$$

The above can be written as

$$\left(-\frac{1}{2ab}\right) (\ln (by - a) - \ln (by + a)) = x + c_1$$
$$\ln (by - a) - \ln (by + a) = (-2ab) (x + c_1)$$
$$= -2ab(x + c_1)$$

Raising both side to exponential gives

$$e^{\ln(by-a)-\ln(by+a)} = -2abc_1 e^{-2abx}$$

Which simplifies to

$$\frac{by-a}{by+a} = c_2 \mathrm{e}^{-2abx}$$

Summary

The solution(s) found are the following

$$y = -\frac{a\left(c_2 \mathrm{e}^{-2abx}+1\right)}{b\left(c_2 \mathrm{e}^{-2abx}-1\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{a\left(c_2 \mathrm{e}^{-2abx}+1\right)}{b\left(c_2 \mathrm{e}^{-2abx}-1\right)}$$

Verified OK.

2.6.2 Maple step by step solution

Let's solve

$y' + y^2 b^2 = a^2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2 b^2 + a^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2 b^2 + a^2} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{\ln(by-a)}{2ab} + \frac{\ln(by+a)}{2ab} = x + c_1$

- Solve for y

  $y = -\frac{a\left(\mathrm{e}^{-2c_1 ab - 2abx}+1\right)}{b\left(\mathrm{e}^{-2c_1 ab - 2abx}-1\right)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x)+b^2*y(x)^2=a^2,y(x), singsol=all)
```

$$y(x) = -\frac{a\left(e^{-2ba(x+c_1)} + 1\right)}{b\left(e^{-2ba(x+c_1)} - 1\right)}$$

✓ Solution by Mathematica

Time used: 3.208 (sec). Leaf size: 37

```
DSolve[y'[x]+b^2*y[x]^2==a^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{a \tanh(ab(x + c_1))}{b}$$
$$y(x) \to -\frac{a}{b}$$
$$y(x) \to \frac{a}{b}$$

2.7 problem 7

Internal problem ID [4693]
Internal file name [`OUTPUT/4186_Sunday_June_05_2022_12_38_38_PM_93334403/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 2, Equations of the first order and degree. page 20
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{1+y^2}{x^2+1} = 0}$$

2.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2+1}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\frac{1}{y^2+1}\,dy = \frac{1}{x^2+1}\,dx$$

$$\int \frac{1}{y^2+1}\,dy = \int \frac{1}{x^2+1}\,dx$$

$$\arctan(y) = \arctan(x) + c_1$$

Which results in

$$y = \tan(\arctan(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \tan(\arctan(x) + c_1) \tag{1}$$

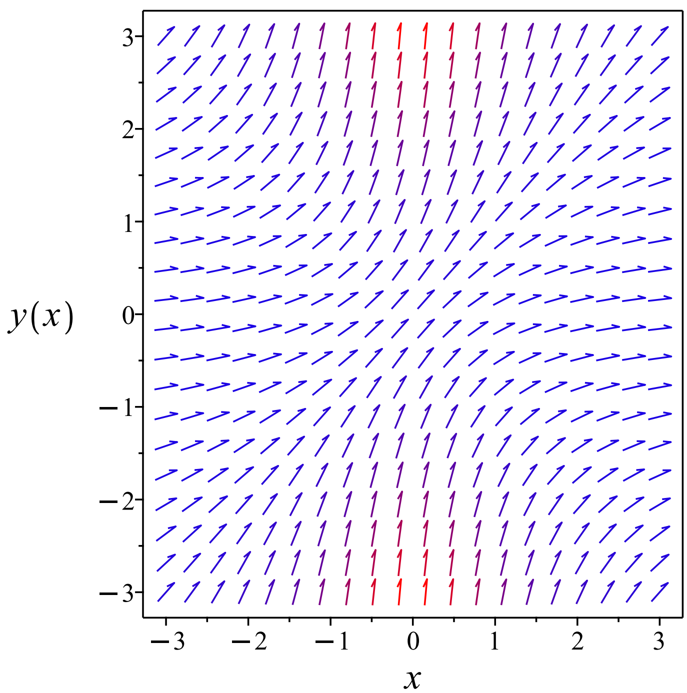


Figure 20: Slope field plot

Verification of solutions

$$y = \tan(\arctan(x) + c_1)$$

Verified OK.

2.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2+1}{x^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 27: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= x^2+1\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta=0$ then in this special case

$$R=y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi}dx\\ &= \int \frac{1}{x^2+1}dx\end{aligned}$$

Which results in

$$S=\arctan(x)$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{y^2+1}{x^2+1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2+1} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2+1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \arctan(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\arctan(x) = \arctan(y) + c_1
$$

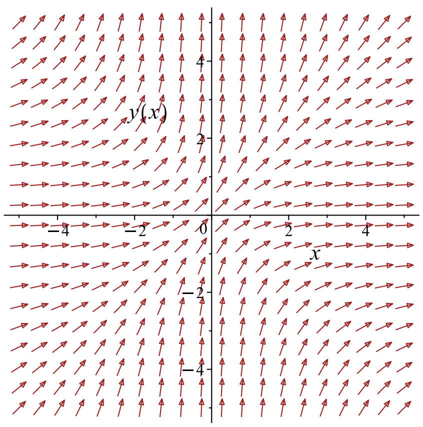
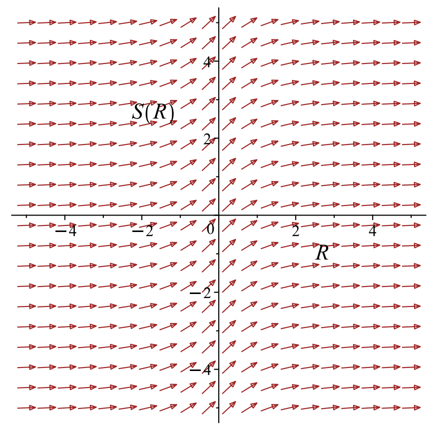
Which simplifies to

$$
\arctan(x) = \arctan(y) + c_1
$$

Which gives

$$
y = -\tan(-\arctan(x) + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$
</td><td>$R = y$
$S = \arctan(x)$</td><td>$\frac{dS}{dR} = \frac{1}{R^2+1}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\tan(-\arctan(x) + c_1) \tag{1}$$

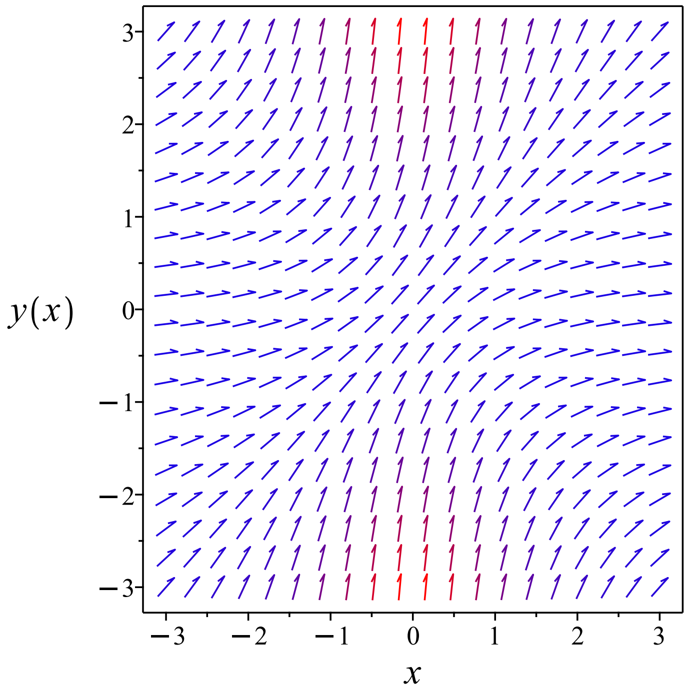


Figure 21: Slope field plot

Verification of solutions

$$y = -\tan(-\arctan(x) + c_1)$$

Verified OK.

2.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2+1}\right) \mathrm{d}y = \left(\frac{1}{x^2+1}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x^2+1}\right) \mathrm{d}x + \left(\frac{1}{y^2+1}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x^2+1}$$
$$N(x, y) = \frac{1}{y^2+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2+1}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x^2+1} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\arctan(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$
\frac{1}{y^2+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y^2+1}\right) \mathrm{d}y \\
f(y) &= \arctan(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan(x) + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan(x) + \arctan(y)$$

Summary

The solution(s) found are the following

$$-\arctan(x) + \arctan(y) = c_1 \tag{1}$$

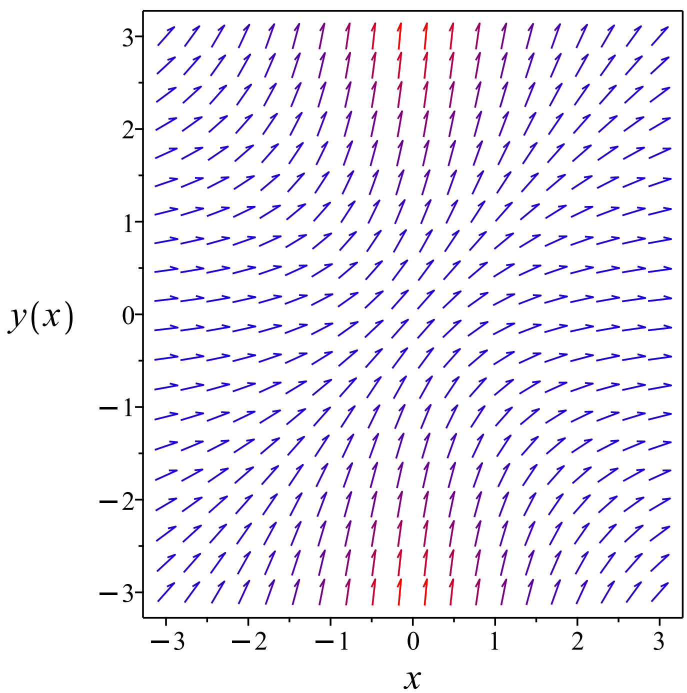


Figure 22: Slope field plot

Verification of solutions

$$-\arctan(x) + \arctan(y) = c_1$$

Verified OK.

2.7.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2+1}{x^2+1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x^2+1} + \frac{1}{x^2+1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{x^2+1}$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x^2+1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2+1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2x}{(x^2+1)^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{1}{(x^2+1)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2+1} + \frac{2xu'(x)}{(x^2+1)^2} + \frac{u(x)}{(x^2+1)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 x + c_2}{\sqrt{x^2+1}}$$

The above shows that

$$u'(x) = \frac{-c_2 x + c_1}{\left(x^2+1\right)^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = -\frac{-c_2 x + c_1}{c_1 x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x - c_3}{c_3 x + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x - c_3}{c_3 x + 1} \tag{1}$$

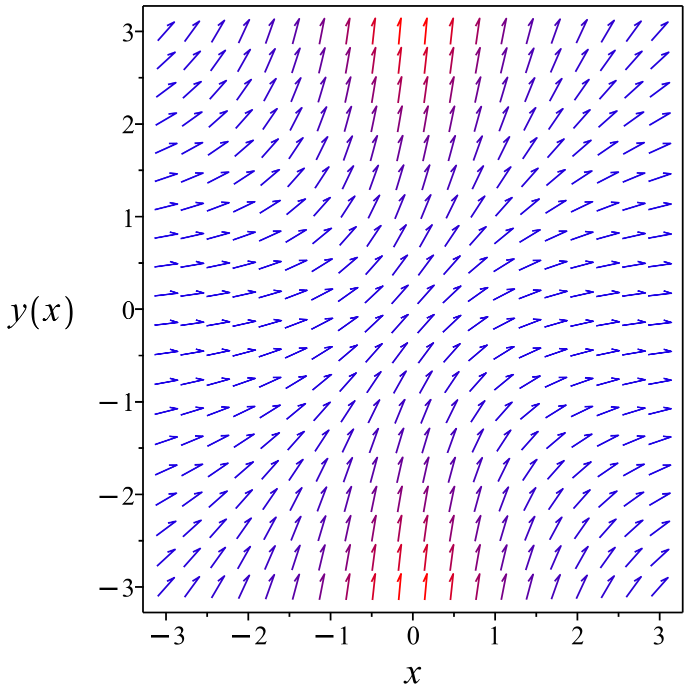


Figure 23: Slope field plot

Verification of solutions

$$y = \frac{x - c_3}{c_3 x + 1}$$

Verified OK.

2.7.5 Maple step by step solution

Let's solve

$y' - \frac{1+y^2}{x^2+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = \frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int \frac{1}{x^2+1} dx + c_1$

- Evaluate integral

  $\arctan(y) = \arctan(x) + c_1$

- Solve for y

  $y = \tan(\arctan(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 9

```
dsolve(diff(y(x),x)=(y(x)^2+1)/(x^2+1),y(x), singsol=all)
```

$$y(x) = \tan (\arctan (x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.237 (sec). Leaf size: 25

```
DSolve[y'[x]==(y[x]^2+1)/(x^2+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan(\arctan(x) + c_1)$$
$$y(x) \to -i$$
$$y(x) \to i$$

2.8 problem 8

Internal problem ID [4694]
Internal file name [`OUTPUT/4187_Sunday_June_05_2022_12_38_49_PM_78703214/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 2, Equations of the first order and degree. page 20
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sin(x)\cos(y) - \cos(x)\sin(y)\,y' = 0}$$

2.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\sin(x)\cot(y)}{\cos(x)} \end{aligned}$$

Where $f(x) = \frac{\sin(x)}{\cos(x)}$ and $g(y) = \cot(y)$. Integrating both sides gives

$$\frac{1}{\cot(y)}\,dy = \frac{\sin(x)}{\cos(x)}\,dx$$

$$\int \frac{1}{\cot(y)}\,dy = \int \frac{\sin(x)}{\cos(x)}\,dx$$

$$-\ln(\cos(y)) = -\ln(\cos(x)) + c_1$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = e^{-\ln(\cos(x))+c_1}$$

Which simplifies to

$$\sec(y) = \frac{c_2}{\cos(x)}$$

Summary

The solution(s) found are the following

$$y = \operatorname{arcsec}\left(\frac{c_2 e^{c_1}}{\cos(x)}\right) \tag{1}$$

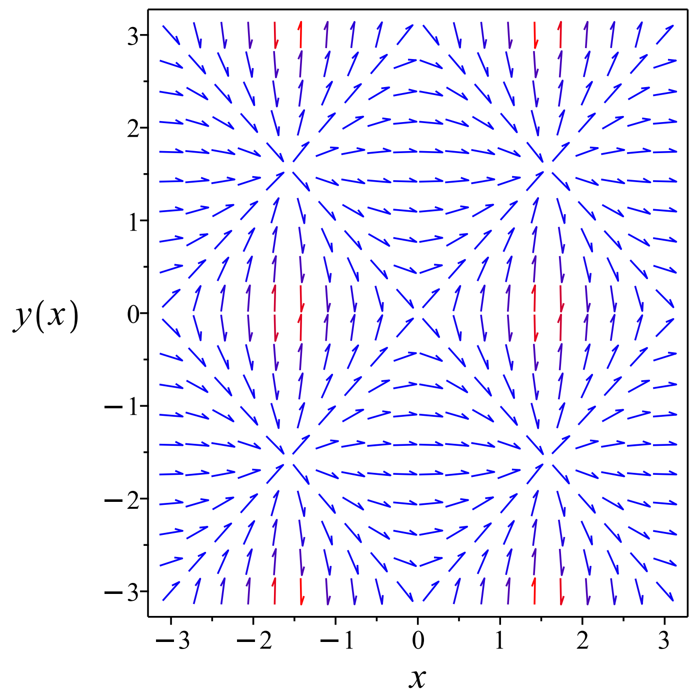


Figure 24: Slope field plot

Verification of solutions

$$y = \operatorname{arcsec}\left(\frac{c_2 e^{c_1}}{\cos(x)}\right)$$

Verified OK.

2.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sin(x)\cos(y)}{\cos(x)\sin(y)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 30: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{\cos(x)}{\sin(x)} \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{\cos(x)}{\sin(x)}} dx
\end{aligned}
$$

Which results in

$$
S = -\ln(\cos(x))
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \qquad (2)
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{\sin(x)\cos(y)}{\cos(x)\sin(y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \tan(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \tan(y) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \tan(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(\cos(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(\cos(x)) = -\ln(\cos(y)) + c_1
$$

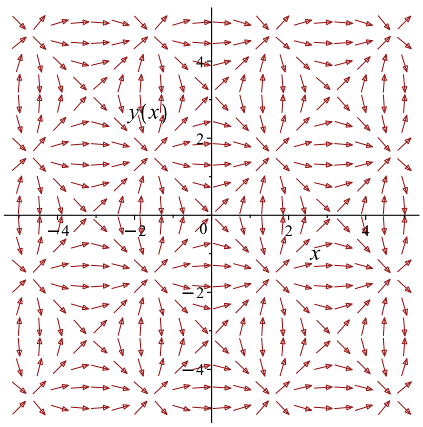
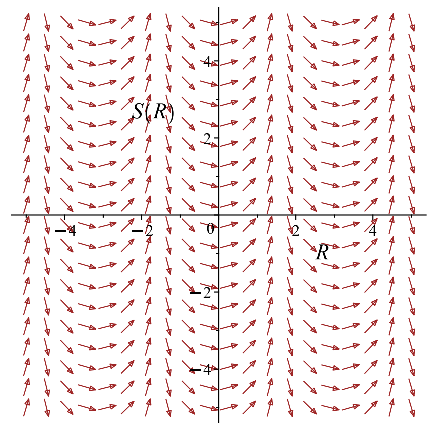
Which simplifies to

$$
-\ln(\cos(x)) = -\ln(\cos(y)) + c_1
$$

Which gives

$$
y = \arccos\left(\cos(x)\, e^{c_1}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sin(x)\cos(y)}{\cos(x)\sin(y)}$ 	$R = y$ $S = -\ln(\cos(x))$	$\frac{dS}{dR} = \tan(R)$ 

Summary

The solution(s) found are the following

$$y = \arccos\left(\cos(x)\, e^{c_1}\right) \tag{1}$$

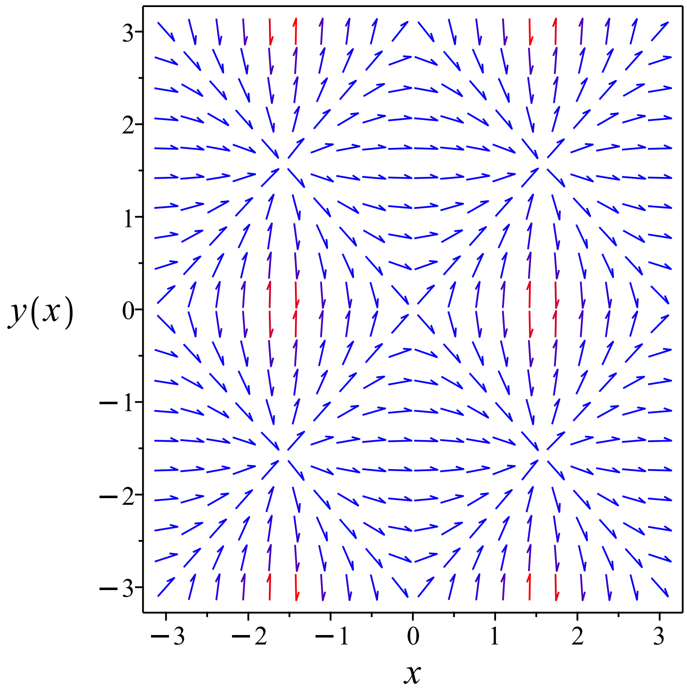


Figure 25: Slope field plot

Verification of solutions

$$y = \arccos\left(\cos\left(x\right) e^{c_1}\right)$$

Verified OK.

2.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y = \left(\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x$$
$$\left(-\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x + \left(\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{\sin(x)}{\cos(x)}$$
$$N(x, y) = \frac{\sin(y)}{\cos(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{\cos(x)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\sin(y)}{\cos(y)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\sin(x)}{\cos(x)}\,\mathrm{d}x\\ \phi &= \ln(\cos(x)) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\sin(y)}{\cos(y)}$. Therefore equation (4) becomes

$$\frac{\sin(y)}{\cos(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= \frac{\sin(y)}{\cos(y)}\\ &= \tan(y)\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\tan(y))\,\mathrm{d}y$$
$$f(y) = -\ln(\cos(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(\cos(x)) - \ln(\cos(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(\cos(x)) - \ln(\cos(y))$$

Summary

The solution(s) found are the following

$$\ln(\cos(x)) - \ln(\cos(y)) = c_1 \tag{1}$$

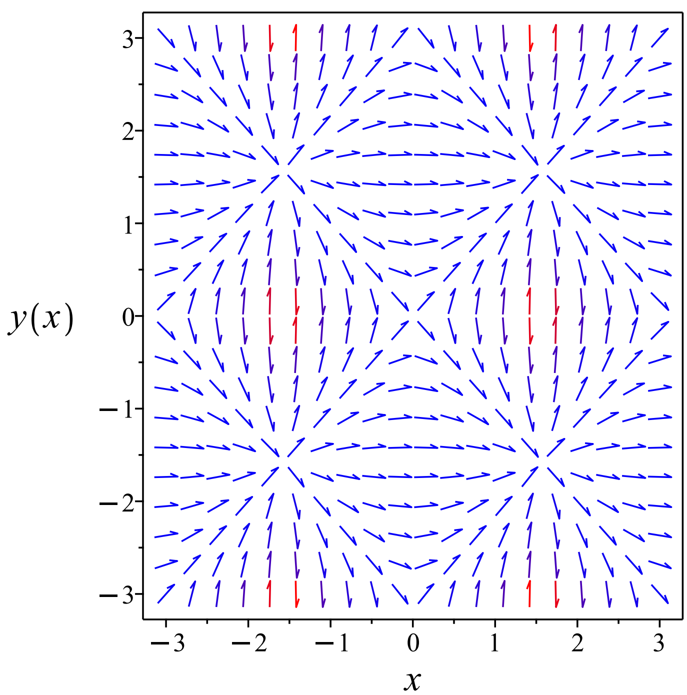


Figure 26: Slope field plot

Verification of solutions

$$\ln(\cos(x)) - \ln(\cos(y)) = c_1$$

Verified OK.

2.8.4 Maple step by step solution

Let's solve

$\sin(x)\cos(y) - \cos(x)\sin(y)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'\sin(y)}{\cos(y)} = \frac{\sin(x)}{\cos(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'\sin(y)}{\cos(y)} dx = \int \frac{\sin(x)}{\cos(x)} dx + c_1$

- Evaluate integral

  $-\ln(\cos(y)) = -\ln(\cos(x)) + c_1$

- Solve for y

  $y = \arccos\left(\frac{\cos(x)}{e^{c_1}}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 11

```
dsolve(sin(x)*cos(y(x))=cos(x)*sin(y(x))*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{\cos(x)}{c_1}\right)$$

✓ Solution by Mathematica

Time used: 5.183 (sec). Leaf size: 47

```
DSolve[Sin[x]*Cos[y[x]]==Cos[x]*Sin[y[x]]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arccos\left(\frac{1}{2}c_1\cos(x)\right)$$
$$y(x) \to \arccos\left(\frac{1}{2}c_1\cos(x)\right)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

2.9 problem 9

Internal problem ID [4695]
Internal file name [`OUTPUT/4188_Sunday_June_05_2022_12_38_59_PM_67758324/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter 2, Equations of the first order and degree. page 20
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{axy' + 2y - xyy' = 0}$$

2.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{2y}{x(y-a)} \end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(y) = \frac{y}{y-a}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y-a}}\, dy = \frac{2}{x}\, dx$$

$$\int \frac{1}{\frac{y}{y-a}}\, dy = \int \frac{2}{x}\, dx$$

$$y - \ln(y)\, a = 2\ln(x) + c_1$$

Which results in

$$y = e^{-\frac{a \operatorname{LambertW}\left(-\frac{e^{-\frac{2\ln(x)+c_1}{a}}}{a}\right)+2\ln(x)+c_1}{a}}$$

Which simplifies to

$$y = e^{-\operatorname{LambertW}\left(-\frac{x^{-\frac{2}{a}} e^{-\frac{c_1}{a}}}{a}\right)} x^{-\frac{2}{a}} e^{-\frac{c_1}{a}}$$

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(-\frac{x^{-\frac{2}{a}} e^{-\frac{c_1}{a}}}{a}\right)} x^{-\frac{2}{a}} e^{-\frac{c_1}{a}} \tag{1}$$

Verification of solutions

$$y = e^{-\operatorname{LambertW}\left(-\frac{x^{-\frac{2}{a}} e^{-\frac{c_1}{a}}}{a}\right)} x^{-\frac{2}{a}} e^{-\frac{c_1}{a}}$$

Verified OK.

2.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2y}{x(y-a)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 33: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= \frac{x}{2}\\ \eta(x, y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x}{2}} dx \end{aligned}$$

Which results in

$$S = 2\ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y}{x(y-a)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{2}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y-a}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R-a}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R - \ln(R)\, a + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\ln(x) = y - \ln(y)\, a + c_1$$

Which simplifies to

$$2\ln(x) = y - \ln(y)\, a + c_1$$

Which gives

$$y = \mathrm{e}^{-\frac{a\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{2\ln(x)-c_1}{a}}}{a}\right)+2\ln(x)-c_1}{a}}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{a\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{2\ln(x)-c_1}{a}}}{a}\right)+2\ln(x)-c_1}{a}} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{a\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{2\ln(x)-c_1}{a}}}{a}\right)+2\ln(x)-c_1}{a}}$$

Verified OK.

2.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{y-a}{2 y}\right) \mathrm{d} y &= \left(\frac{1}{x}\right) \mathrm{d} x \\
\left(-\frac{1}{x}\right) \mathrm{d} x+\left(\frac{y-a}{2 y}\right) \mathrm{d} y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= \frac{y-a}{2 y}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y-a}{2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-a}{2y}$. Therefore equation (4) becomes

$$
\frac{y-a}{2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{a-y}{2y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{y-a}{2y}\right) \mathrm{d}y \\
f(y) &= \frac{y}{2} - \frac{\ln(y)\, a}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{y}{2} - \frac{\ln(y)\, a}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{y}{2} - \frac{\ln(y)\, a}{2}$$

The solution becomes

$$y = \mathrm{e}^{-\frac{a \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{2(\ln(x)+c_1)}{a}}}{a}\right)+2\ln(x)+2c_1}{a}}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{a \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{2(\ln(x)+c_1)}{a}}}{a}\right)+2\ln(x)+2c_1}{a}} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{a \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{2(\ln(x)+c_1)}{a}}}{a}\right)+2\ln(x)+2c_1}{a}}$$

Verified OK.

2.9.4 Maple step by step solution

Let's solve

$axy' + 2y - xyy' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(a-y)}{y} = -\frac{2}{x}$

- Integrate both sides with respect to x

$$\int \frac{y'(a-y)}{y}dx = \int -\frac{2}{x}dx + c_1$$

- Evaluate integral

$$-y + \ln(y)\, a = -2\ln(x) + c_1$$

- Solve for y

$$y = e^{-\frac{a\,\mathit{LambertW}\left(-\frac{e^{-\frac{2\ln(x)-c_1}{a}}}{a}\right)+2\ln(x)-c_1}{a}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(a*x*diff(y(x),x)+2*y(x)=x*y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = x^{-\frac{2}{a}} e^{\frac{-a\,\mathrm{LambertW}\left(-\frac{x^{-\frac{2}{a}} e^{-\frac{2c_1}{a}}}{a}\right)-2c_1}{a}}$$

✓ Solution by Mathematica

Time used: 60.019 (sec). Leaf size: 29

```
DSolve[a*x*y'[x]+2*y[x]==x*y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -aW\left(-\frac{e^{\frac{c_1}{a}} x^{-2/a}}{a}\right)$$

3 Chapter VII, Solutions in series. Examples XIV. page 177

3.1 problem 1

Internal problem ID [4696]
Internal file name [`OUTPUT/4189_Sunday_June_05_2022_12_39_07_PM_53153814/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (x+n)\,y' + (1+n)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (x+n)\,y' + (1+n)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x+n}{x}$$
$$q(x) = \frac{1+n}{x}$$

Table 36: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+n}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1+n}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (x + n) y' + (1 + n) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) x + (x + n) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (1 + n) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} n\, x^{n+r-1} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (1+n)\, a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} (1+n)\, a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} (1+n)\, x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n+r-1} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (1+n)\, x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + n\, x^{n+r-1} a_n (n+r) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + n\, x^{-1+r} a_0 r = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + n\, x^{-1+r} r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r} (-1+r+n) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r+n)=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -n+1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1+r+n)=0$$

Solving for r gives the roots of the indicial equation as Assuming the roots differ by non-integer Since $r_1 - r_2 = n-1$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-n+1} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)\,(n+r-1)+a_{n-1}(n+r-1)+na_n(n+r)+a_{n-1}(1+n)=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+n+r)}{nn+nr+n^2+2nr+r^2-n-r} \tag{4}$$

Which for the root $r=0$ becomes

$$a_n = -\frac{a_{n-1}(n+n)}{n\,(n+n-1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1 - n - r}{(r + 1)(r + n)}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{-n - 1}{n}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-n-r}{(r+1)(r+n)}$	$\frac{-n-1}{n}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{n + 2 + r}{(r + 1)(r + n)(r + 2)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{n + 2}{2n}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-n-r}{(r+1)(r+n)}$	$\frac{-n-1}{n}$
a_2	$\frac{n+2+r}{(r+1)(r+n)(r+2)}$	$\frac{n+2}{2n}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-n - 3 - r}{(r + 3)(r + 2)(r + 1)(r + n)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{-n-3}{6n}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-n-r}{(r+1)(r+n)}$	$\frac{-n-1}{n}$
a_2	$\frac{n+2+r}{(r+1)(r+n)(r+2)}$	$\frac{n+2}{2n}$
a_3	$\frac{-n-3-r}{(r+3)(r+2)(r+1)(r+n)}$	$\frac{-n-3}{6n}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{n+4+r}{(r+3)(r+2)(r+1)(r+n)(r+4)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{n+4}{24n}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-n-r}{(r+1)(r+n)}$	$\frac{-n-1}{n}$
a_2	$\frac{n+2+r}{(r+1)(r+n)(r+2)}$	$\frac{n+2}{2n}$
a_3	$\frac{-n-3-r}{(r+3)(r+2)(r+1)(r+n)}$	$\frac{-n-3}{6n}$
a_4	$\frac{n+4+r}{(r+3)(r+2)(r+1)(r+n)(r+4)}$	$\frac{n+4}{24n}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-n-5-r}{(r+5)(r+4)(r+3)(r+2)(r+1)(r+n)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{-n-5}{120n}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-n-r}{(r+1)(r+n)}$	$\frac{-n-1}{n}$
a_2	$\frac{n+2+r}{(r+1)(r+n)(r+2)}$	$\frac{n+2}{2n}$
a_3	$\frac{-n-3-r}{(r+3)(r+2)(r+1)(r+n)}$	$\frac{-n-3}{6n}$
a_4	$\frac{n+4+r}{(r+3)(r+2)(r+1)(r+n)(r+4)}$	$\frac{n+4}{24n}$
a_5	$\frac{-n-5-r}{(r+5)(r+4)(r+3)(r+2)(r+1)(r+n)}$	$\frac{-n-5}{120n}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 + \frac{(-n-1)\,x}{n} + \frac{(n+2)\,x^2}{2n} + \frac{(-n-3)\,x^3}{6n} + \frac{(n+4)\,x^4}{24n} + \frac{(-n-5)\,x^5}{120n} + O(x^6)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)\,(n+r-1) + b_{n-1}(n+r-1) + n b_n(n+r) + b_{n-1}(1+n) = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}(n+n+r)}{nn + nr + n^2 + 2nr + r^2 - n - r} \tag{4}
$$

Which for the root $r = -n + 1$ becomes

$$
b_n = \frac{b_{n-1}(n+1)}{n\,(n-n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -n + 1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-1 - n - r}{(r + 1)(r + n)}$$

Which for the root $r = -n + 1$ becomes

$$b_1 = \frac{2}{-2 + n}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-n-r}{(r+1)(r+n)}$	$\frac{2}{-2+n}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{n + 2 + r}{(r + 1)(r + n)(r + 2)}$$

Which for the root $r = -n + 1$ becomes

$$b_2 = \frac{3}{(-2 + n)(n - 3)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-n-r}{(r+1)(r+n)}$	$\frac{2}{-2+n}$
b_2	$\frac{n+2+r}{(r+1)(r+n)(r+2)}$	$\frac{3}{(-2+n)(n-3)}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-n - 3 - r}{(r + 3)(r + 2)(r + 1)(r + n)}$$

Which for the root $r = -n + 1$ becomes

$$b_3 = \frac{4}{(-2 + n)(n - 3)(n - 4)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-n-r}{(r+1)(r+n)}$	$\frac{2}{-2+n}$
b_2	$\frac{n+2+r}{(r+1)(r+n)(r+2)}$	$\frac{3}{(-2+n)(n-3)}$
b_3	$\frac{-n-3-r}{(r+3)(r+2)(r+1)(r+n)}$	$\frac{4}{(-2+n)(n-3)(n-4)}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{n+4+r}{(r+3)(r+2)(r+1)(r+n)(r+4)}$$

Which for the root $r = -n+1$ becomes

$$b_4 = \frac{5}{(n-4)(n-3)(-2+n)(n-5)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-n-r}{(r+1)(r+n)}$	$\frac{2}{-2+n}$
b_2	$\frac{n+2+r}{(r+1)(r+n)(r+2)}$	$\frac{3}{(-2+n)(n-3)}$
b_3	$\frac{-n-3-r}{(r+3)(r+2)(r+1)(r+n)}$	$\frac{4}{(-2+n)(n-3)(n-4)}$
b_4	$\frac{n+4+r}{(r+3)(r+2)(r+1)(r+n)(r+4)}$	$\frac{5}{(n-4)(n-3)(-2+n)(n-5)}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-n-5-r}{(r+5)(r+4)(r+3)(r+2)(r+1)(r+n)}$$

Which for the root $r = -n+1$ becomes

$$b_5 = \frac{6}{(n-4)(n-3)(-2+n)(n-5)(n-6)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-n-r}{(r+1)(r+n)}$	$\frac{2}{-2+n}$
b_2	$\frac{n+2+r}{(r+1)(r+n)(r+2)}$	$\frac{3}{(-2+n)(n-3)}$
b_3	$\frac{-n-3-r}{(r+3)(r+2)(r+1)(r+n)}$	$\frac{4}{(-2+n)(n-3)(n-4)}$
b_4	$\frac{n+4+r}{(r+3)(r+2)(r+1)(r+n)(r+4)}$	$\frac{5}{(n-4)(n-3)(-2+n)(n-5)}$
b_5	$\frac{-n-5-r}{(r+5)(r+4)(r+3)(r+2)(r+1)(r+n)}$	$\frac{6}{(n-4)(n-3)(-2+n)(n-5)(n-6)}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{-n+1}\left(1 + \frac{2x}{-2+n} + \frac{3x^2}{(-2+n)(n-3)} + \frac{4x^3}{(-2+n)(n-3)(n-4)} + \frac{5x^4}{(n-4)(n-3)(-2+n)(\right.
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + \frac{(-n-1)x}{n} + \frac{(n+2)x^2}{2n} + \frac{(-n-3)x^3}{6n} + \frac{(n+4)x^4}{24n} + \frac{(-n-5)x^5}{120n}\right. \\
&\quad \left. + O(x^6)\right) + c_2 x^{-n+1}\left(1 + \frac{2x}{-2+n} + \frac{3x^2}{(-2+n)(n-3)}\right. \\
&\quad + \frac{4x^3}{(-2+n)(n-3)(n-4)} + \frac{5x^4}{(n-4)(n-3)(-2+n)(n-5)} \\
&\quad \left. + \frac{6x^5}{(n-4)(n-3)(-2+n)(n-5)(n-6)} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 + \frac{(-n-1)x}{n} + \frac{(n+2)x^2}{2n} + \frac{(-n-3)x^3}{6n} + \frac{(n+4)x^4}{24n} + \frac{(-n-5)x^5}{120n} + O(x^6)\right) \\
&\quad + c_2 x^{-n+1}\left(1 + \frac{2x}{-2+n} + \frac{3x^2}{(-2+n)(n-3)} + \frac{4x^3}{(-2+n)(n-3)(n-4)}\right. \\
&\quad + \frac{5x^4}{(n-4)(n-3)(-2+n)(n-5)} + \frac{6x^5}{(n-4)(n-3)(-2+n)(n-5)(n-6)} \\
&\quad \left. + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1+\frac{(-n-1)\,x}{n}+\frac{(n+2)\,x^2}{2n}+\frac{(-n-3)\,x^3}{6n}+\frac{(n+4)\,x^4}{24n}+\frac{(-n-5)\,x^5}{120n}+O\left(x^6\right)\right)+c_2 x^{-n+1}\left(1+\frac{2x}{-2+n}+\frac{3x^2}{(-2+n)\,(n-3)}+\frac{4x^3}{(-2+n)\,(n-3)\,(n-4)}+\frac{5x^4}{(n-4)\,(n-3)\,(-2+n)\,(n-5)}+\frac{6x^5}{(n-4)\,(n-3)\,(-2+n)\,(n-5)\,(n-6)}+O\left(x^6\right)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1+\frac{(-n-1)\,x}{n}+\frac{(n+2)\,x^2}{2n}+\frac{(-n-3)\,x^3}{6n}+\frac{(n+4)\,x^4}{24n}+\frac{(-n-5)\,x^5}{120n}+O\left(x^6\right)\right)+c_2 x^{-n+1}\left(1+\frac{2x}{-2+n}+\frac{3x^2}{(-2+n)\,(n-3)}+\frac{4x^3}{(-2+n)\,(n-3)\,(n-4)}+\frac{5x^4}{(n-4)\,(n-3)\,(-2+n)\,(n-5)}+\frac{6x^5}{(n-4)\,(n-3)\,(-2+n)\,(n-5)\,(n-6)}+O\left(x^6\right)\right)
$$

Verified OK.

3.1.1 Maple step by step solution

Let's solve

$y''x+(x+n)\,y'+(1+n)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(1+n)y}{x}-\frac{(x+n)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x+n)y'}{x}+\frac{(1+n)y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x+n}{x}, P_3(x)=\frac{1+n}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\Big|_{x=0} = n$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x + (x + n)\, y' + (1 + n)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\, (k + r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1 + r + n)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k + 1 + r)\, (k + r + n) + a_k(k + r + n + 1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-1 + r + n) = 0$$

- Values of r that satisfy the indicial equation

  $r \in \{0, -n+1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+n) + a_k(k+r+n+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+n+1)}{(k+1+r)(k+r+n)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+1+n)}{(k+1)(k+n)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(k+1+n)}{(k+1)(k+n)}\right]$

- Recursion relation for $r = -n+1$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+2-n)(k+1)}$

- Solution for $r = -n+1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-n+1}, a_{k+1} = -\frac{a_k(k+2)}{(k+2-n)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-n+1}\right), a_{k+1} = -\frac{a_k(k+1+n)}{(k+1)(k+n)}, b_{k+1} = -\frac{b_k(k+2)}{(k+2-n)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 248

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(x+n)*diff(y(x),x)+(n+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{1-n}\left(1 + 2\frac{1}{n-2}x + 3\frac{1}{(-3+n)(n-2)}x^2 + 4\frac{1}{(-4+n)(-3+n)(n-2)}x^3 + 5\frac{1}{(-5+n)(-4+n)(-3+n)(n-2)}x^4 + 6\frac{1}{(-6+n)(-5+n)(-4+n)(-3+n)(n-2)}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 + \frac{-1-n}{n}x + \frac{1}{2}\frac{n+2}{n}x^2 - \frac{1}{6}\frac{n+3}{n}x^3 + \frac{1}{24}\frac{n+4}{n}x^4 - \frac{1}{120}\frac{n+5}{n}x^5 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 519

```
AsymptoticDSolveValue[x*y''[x]+(x+n)*y'[x]+(n+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{(-n-1)(n+2)(n+3)(n+4)(n+5)x^5}{n(2n+2)(3n+6)(4n+12)(5n+20)} - \frac{(-n-1)(n+2)(n+3)(n+4)x^4}{n(2n+2)(3n+6)(4n+12)} + \frac{(-n-1)(n+2)(n+3)x^3}{n(2n+2)(3n+6)} - \frac{(-n-1)(n+2)x^2}{n(2n+2)} + \frac{(-n-1)x}{n} + 1\right)$$
$$+c_1\left(-\frac{720x^5}{((1-n)(2-n)+n(2-n))((2-n)(3-n)+n(3-n))((3-n)(4-n)+n(4-n))((4-n)(5}\right.$$
$$+\frac{120x^4}{((1-n)(2-n)+n(2-n))((2-n)(3-n)+n(3-n))((3-n)(4-n)+n(4-n))((4-n)(5-n)}$$
$$-\frac{24x^3}{((1-n)(2-n)+n(2-n))((2-n)(3-n)+n(3-n))((3-n)(4-n)+n(4-n))}$$
$$+\frac{6x^2}{((1-n)(2-n)+n(2-n))((2-n)(3-n)+n(3-n))}$$
$$\left.-\frac{2x}{(1-n)(2-n)+n(2-n)} + 1\right)x^{1-n}$$

3.2 problem 2

Internal problem ID [4697]
Internal file name [`OUTPUT/4190_Sunday_June_05_2022_12_39_12_PM_34227600/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{52}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{53}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -xy \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -y - xy' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2y' + yx^2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x(xy' + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -yx^3 + 6xy' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -2y'(0) \\
F_3 &= 0 \\
F_4 &= 4y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x - \frac{1}{12}x^4\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n \\
\sum_{n=0}^{\infty} x^{1+n} a_n &= \sum_{n=1}^{\infty} a_{n-1} x^n
\end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{6}\right) a_0 + \left(x - \frac{1}{12} x^4\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O\left(x^6\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O\left(x^6\right) \tag{2}$$

<u>Verification of solutions</u>

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O\left(x^6\right)$$

Verified OK.

3.2.1 Maple step by step solution

Let's solve

$y'' = -xy$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- > k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_{k-1})\, x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + a_{k-1} = 0$

- Shift index using $k- > k+1$

$$\left((k+1)^2+3k+5\right)a_{k+3}+a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=-\frac{a_k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{x^3}{6}\right)y(0)+\left(x-\frac{1}{12}x^4\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(x-\frac{x^4}{12}\right)+c_1\left(1-\frac{x^3}{6}\right)$$

3.3 problem 3

Internal problem ID [4698]
Internal file name [`OUTPUT/4191_Sunday_June_05_2022_12_39_19_PM_55110410/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{2x^2y'' - xy' + (-x^2 + 1)\, y = x^2}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + (-x^2 + 1)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2 - 1}{2x^2}$$

Table 39: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = x^2$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2 + 1)\, y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&2x^2\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) \\
&-x\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\left(-x^2+1\right)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)\,(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right) \\
&+\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)\,(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right) \\
&+\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r}\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r}a_n(n+r)\,(n+r-1)-x^{n+r}a_n(n+r)+a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
2x^r a_0 r(-1+r)-x^r a_0 r+a_0 x^r=0
$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= \frac{1}{2} \end{aligned}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = x^2$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right)\\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\boxed{
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{a_0}{2r^2+5r+3} \\
a_3 &= 0 \\
a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\
a_5 &= 0
\end{aligned}
}
$$

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$
(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^2
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{3} \\
m &= 2
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+2}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{3}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{3}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$
\begin{array}{|l|}
\hline
c_0 = \frac{1}{3} \\
c_1 = 0 \\
c_2 = \frac{1}{63} \\
c_3 = 0 \\
c_4 = \frac{1}{3465} \\
c_5 = 0 \\
\hline
\end{array}
$$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^2 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^2 \left(\frac{1}{3} + \frac{1}{63} x^2 + \frac{1}{3465} x^4\right) \\
&= \frac{1}{3} x^2 + \frac{1}{63} x^4 + \frac{1}{3465} x^6
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = \frac{x^2}{3} + \frac{x^4}{63} + \frac{x^6}{3465} + O(x^6)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x^2}{3} + \frac{x^4}{63} + \frac{x^6}{3465} + O(x^6) + c_1 x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2 \sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{x^2}{3} + \frac{x^4}{63} + \frac{x^6}{3465} + O(x^6) + c_1 x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \\
& + c_2 \sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = \frac{x^2}{3} + \frac{x^4}{63} + \frac{x^6}{3465} + O(x^6) + c_1 x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2 \sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+(1-x^2)*y(x)=x^2,y(x),type='series',x=0);
```

$$y(x) = c_1 \sqrt{x} \left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 x \left(1 + \frac{1}{10}x^2 + \frac{1}{360}x^4 + \mathrm{O}\left(x^6\right)\right) + x^2 \left(\frac{1}{3} + \frac{1}{63}x^2 + \mathrm{O}\left(x^4\right)\right)$$

✓ <u>Solution by Mathematica</u>

Time used: 0.024 (sec). Leaf size: 160

```
AsymptoticDSolveValue[2*x^2*y''[x]-x*y'[x]+(1-x^2)*y[x]==x^2,y[x],{x,0,5}]
```

$$y(x) \to c_2 x \left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)+c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+\sqrt{x}\left(-\frac{x^{11/2}}{1980}-\frac{x^{7/2}}{35}-\frac{2x^{3/2}}{3}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{x^5}{840}+\frac{x^3}{18}+x\right)\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)$$

3.4 problem 4

Internal problem ID [4699]
Internal file name [OUTPUT/4192_Sunday_June_05_2022_12_39_26_PM_66133488/index.tex]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{xy'' + 2y' + a^3x^2y = 2}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y' + a^3x^2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = a^3x$$

Table 40: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = a^3 x$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y' + a^3x^2y = 2$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $xy'' + 2y' + a^3x^2y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right)x + 2\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right) + a^3x^2\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty}2(n+r)\,a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty}a^3x^{2+n+r}a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}a^3x^{2+n+r}a_n = \sum_{n=3}^{\infty}a_{n-3}a^3x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty}2(n+r)\,a_n x^{n+r-1}\right) + \left(\sum_{n=3}^{\infty}a_{n-3}a^3x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)\,(n+r-1) + 2(n+r)\,a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0r(-1+r) + 2ra_0x^{-1+r} = 0$$

Or

$$\left(x^{-1+r}r(-1+r) + 2r\,x^{-1+r}\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}(1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -1$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(x^{-1+m}m(-1+m) + 2m\,x^{-1+m}\right) c_0 = 2$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = Cy_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

For $3 \le n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_n(n+r) + a_{n-3}a^3 = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-3}a^3}{n^2 + 2nr + r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-3}a^3}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{a^3}{r^2 + 7r + 12}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{a^3}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{a^3}{r^2+7r+12}$	$-\frac{a^3}{12}$

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{a^3}{r^2+7r+12}$	$-\frac{a^3}{12}$
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{a^3}{r^2+7r+12}$	$-\frac{a^3}{12}$
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - \frac{a^3 x^3}{12} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -1} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = 0$$

For $3 \leq n$ the recursive equation is

$$b_n(n + r)(n + r - 1) + 2(n + r) b_n + b_{n-3}a^3 = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n - 1)(n - 2) + 2(n - 1) b_n + b_{n-3}a^3 = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-3}a^3}{n^2 + 2nr + r^2 + n + r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-3}a^3}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{a^3}{r^2 + 7r + 12}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{a^3}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{a^3}{r^2+7r+12}$	$-\frac{a^3}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{a^3}{r^2+7r+12}$	$-\frac{a^3}{6}$
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{a^3}{r^2+7r+12}$	$-\frac{a^3}{6}$
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots\right) \\ &= \frac{1 - \frac{a^3 x^3}{6} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 - \frac{a^3 x^3}{12} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{a^3 x^3}{6} + O(x^6)\right)}{x} \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(x^{-1+m} m(-1+m) + 2m\, x^{-1+m}\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\boxed{\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= -\frac{a_0 a^3}{r^2+7r+12} \\ a_4 &= 0 \\ a_5 &= 0 \end{aligned}}$$

Now we determine the particular solution y_p associated with $F = 2$ by solving the balance equation

$$\left(x^{-1+m} m(-1+m) + 2m\, x^{-1+m}\right) c_0 = 2$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= 1 \\ m &= 1 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+1} \end{aligned}$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 1$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 1$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$
\boxed{\begin{aligned}
c_0 &= 1\\
c_1 &= 0\\
c_2 &= 0\\
c_3 &= -\tfrac{a^3}{20}\\
c_4 &= 0\\
c_5 &= 0
\end{aligned}}
$$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n\\
&= x \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x\left(1 - \frac{a^3 x^3}{20}\right)\\
&= x - \frac{1}{20} a^3 x^4
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = x - \frac{a^3 x^4}{20} + O(x^6)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p\\
&= x - \frac{a^3 x^4}{20} + O(x^6) + c_1\left(1 - \frac{a^3 x^3}{12} + O(x^6)\right) + \frac{c_2\left(1 - \frac{a^3 x^3}{6} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x - \frac{a^3 x^4}{20} + O(x^6) + c_1\left(1 - \frac{a^3 x^3}{12} + O(x^6)\right) + \frac{c_2\left(1 - \frac{a^3 x^3}{6} + O(x^6)\right)}{x} \tag{1}
$$

Verification of solutions

$$y = x - \frac{a^3x^4}{20} + O(x^6) + c_1\left(1 - \frac{a^3x^3}{12} + O(x^6)\right) + \frac{c_2\left(1 - \frac{a^3x^3}{6} + O(x^6)\right)}{x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
Order:=6;
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+a^3*x^2*y(x)=2,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{1}{12}a^3x^3 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{1}{6}a^3x^3 + \mathrm{O}\left(x^6\right)\right)}{x} + x\left(1 - \frac{1}{20}a^3x^3 + \mathrm{O}\left(x^5\right)\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 136

```
AsymptoticDSolveValue[x*y''[x]+2*y'[x]+a^3*x^2*y[x]==2,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{a^6x^6}{504} - \frac{a^3x^3}{12} + 1\right) + \frac{c_2\left(\frac{a^6x^6}{180} - \frac{a^3x^3}{6} + 1\right)}{x} + \left(2x - \frac{a^3x^4}{12}\right)\left(\frac{a^6x^6}{504} - \frac{a^3x^3}{12} + 1\right) + \frac{\left(\frac{a^3x^5}{30} - x^2\right)\left(\frac{a^6x^6}{180} - \frac{a^3x^3}{6} + 1\right)}{x}$$

3.5 problem 5

Internal problem ID [4700]
Internal file name [`OUTPUT/4193_Sunday_June_05_2022_12_39_32_PM_6838053/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + a\,x^2 y = x + 1}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (57)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (58)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -a\,x^2 y + x + 1 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -2yax - a\,x^2 y' + 1 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= a\left(ya\,x^4 - x^3 - 4xy' - x^2 - 2y\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= a\left(\left(a\,x^4 - 6\right) y' + 8ya\,x^3 - 7x^2 - 6x\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= a\left(-ya^2 x^6 + a\,x^5 + 12a\,x^3 y' + a\,x^4 + 30a\,x^2 y - 20x - 12\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 1 \\
F_2 &= -2y(0)\,a \\
F_3 &= -6y'(0)\,a \\
F_4 &= -12a
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{a\,x^4}{12}\right) y(0) + \left(x - \frac{1}{20} a\,x^5\right) y'(0) + \frac{x^2}{2} + \frac{x^3}{6} - \frac{a\,x^6}{60} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -a\, x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) + x + 1 \tag{1}$$

Expanding $x + 1$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} x + 1 &= x + 1 + \dots \\ &= x + 1 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + a\, x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) = x + 1$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a\, x^{n+2} a_n\right) = x + 1 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n \\ \sum_{n=0}^{\infty} a\, x^{n+2} a_n &= \sum_{n=2}^{\infty} a a_{n-2} x^n \end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=2}^{\infty} a a_{n-2} x^n\right) = x+1 \tag{3}$$

$n = 0$ gives

$$\begin{aligned}(2a_2)\, 1 &= 1\\ 2a_2 &= 1\end{aligned}$$

Or

$$a_2 = \frac{1}{2}$$

$n = 1$ gives

$$\begin{aligned}(6a_3)\, x &= x\\ 6a_3 &= 1\end{aligned}$$

Which after substituting earlier equations, simplifies to

$$6a_3 = 1$$

Or

$$a_3 = \frac{1}{6}$$

For $2 \leq n$, the recurrence equation is

$$((n+2)\, a_{n+2}(n+1) + a a_{n-2})\, x^n = x+1 \tag{4}$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(a a_0 + 12a_4)\, x^2 &= 0\\ a a_0 + 12a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$(aa_1 + 20a_5)\, x^3 = 0$$
$$aa_1 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{aa_1}{20}$$

For $n = 4$ the recurrence equation gives

$$(aa_2 + 30a_6)\, x^4 = 0$$
$$aa_2 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a}{60}$$

For $n = 5$ the recurrence equation gives

$$(aa_3 + 42a_7)\, x^5 = 0$$
$$aa_3 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a}{252}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}aa_0x^4 - \frac{1}{20}aa_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1-\frac{a\,x^4}{12}\right) a_0 + \left(x-\frac{1}{20}a\,x^5\right) a_1 + \frac{x^2}{2} + \frac{x^3}{6} + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1-\frac{a\,x^4}{12}\right) c_1 + \left(x-\frac{1}{20}a\,x^5\right) c_2 + \frac{x^2}{2} + \frac{x^3}{6} + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1-\frac{a\,x^4}{12}\right) y(0) + \left(x-\frac{1}{20}a\,x^5\right) y'(0) + \frac{x^2}{2} + \frac{x^3}{6} - \frac{a\,x^6}{60} + O\left(x^6\right) \tag{1}$$

$$y = \left(1-\frac{a\,x^4}{12}\right) c_1 + \left(x-\frac{1}{20}a\,x^5\right) c_2 + \frac{x^2}{2} + \frac{x^3}{6} + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{a\,x^4}{12}\right) y(0) + \left(x-\frac{1}{20}a\,x^5\right) y'(0) + \frac{x^2}{2} + \frac{x^3}{6} - \frac{a\,x^6}{60} + O\left(x^6\right)$$

Verified OK.

$$y = \left(1-\frac{a\,x^4}{12}\right) c_1 + \left(x-\frac{1}{20}a\,x^5\right) c_2 + \frac{x^2}{2} + \frac{x^3}{6} + O\left(x^6\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
Order:=6;
dsolve(diff(y(x),x$2)+a*x^2*y(x)=1+x,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{a\,x^4}{12}\right) y(0) + \left(x - \frac{1}{20} a\,x^5\right) D(y)\,(0) + \frac{x^2}{2} + \frac{x^3}{6} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 44

```
AsymptoticDSolveValue[y''[x]+a*x^2*y[x]==1+x,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{ax^5}{20}\right) + c_1\left(1 - \frac{ax^4}{12}\right) + \frac{x^3}{6} + \frac{x^2}{2}$$

3.6 problem 7

Internal problem ID [4701]
Internal file name [`OUTPUT/4194_Sunday_June_05_2022_12_39_38_PM_27103984/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^4y'' + xy' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^4y'' + xy' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x^3}$$
$$q(x) = \frac{1}{x^4}$$

Table 41: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x^3}$	
singularity	type
$x = 0$	"irregular"

$q(x) = \frac{1}{x^4}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^4*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 49

```
AsymptoticDSolveValue[x^4*y''[x]+x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1(1-x^2)}{x} + c_2 e^{\frac{1}{2x^2}} \left(420x^6 + 45x^4 + 6x^2 + 1\right) x^4$$

3.7 problem 8

Internal problem ID [4702]
Internal file name [`OUTPUT/4195_Sunday_June_05_2022_12_39_44_PM_57910761/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + (2x^2 + x)\, y' - 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (2x^2 + x)\, y' - 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2x + 1}{x}$$
$$q(x) = -\frac{4}{x^2}$$

Table 42: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x+1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (2x^2 + x) y' - 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + (2x^2 + x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-4 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2 a_{n-1} (n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2 a_{n-1} (n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-4 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 4 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 4 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 2 \\
r_2 &= -2
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(r^2 - 4)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2} \right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) + a_n(n+r) - 4a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{2a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - 4} \tag{4}
$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{2a_{n-1}(1+n)}{n\,(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2r}{r^2+2r-3}$$

Which for the root $r = 2$ becomes

$$a_1 = -\frac{4}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+2r-3}$	$-\frac{4}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r+4}{r^3+6r^2+5r-12}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+2r-3}$	$-\frac{4}{5}$
a_2	$\frac{4r+4}{r^3+6r^2+5r-12}$	$\frac{2}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-16 - 8r}{r^4 + 11r^3 + 35r^2 + 13r - 60}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{16}{105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+2r-3}$	$-\frac{4}{5}$
a_2	$\frac{4r+4}{r^3+6r^2+5r-12}$	$\frac{2}{5}$
a_3	$\frac{-16-8r}{r^4+11r^3+35r^2+13r-60}$	$-\frac{16}{105}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{r^4 + 14r^3 + 59r^2 + 46r - 120}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+2r-3}$	$-\frac{4}{5}$
a_2	$\frac{4r+4}{r^3+6r^2+5r-12}$	$\frac{2}{5}$
a_3	$\frac{-16-8r}{r^4+11r^3+35r^2+13r-60}$	$-\frac{16}{105}$
a_4	$\frac{16}{r^4+14r^3+59r^2+46r-120}$	$\frac{1}{21}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{r^5 + 20r^4 + 140r^3 + 370r^2 + 99r - 630}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{4}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+2r-3}$	$-\frac{4}{5}$
a_2	$\frac{4r+4}{r^3+6r^2+5r-12}$	$\frac{2}{5}$
a_3	$\frac{-16-8r}{r^4+11r^3+35r^2+13r-60}$	$-\frac{16}{105}$
a_4	$\frac{16}{r^4+14r^3+59r^2+46r-120}$	$\frac{1}{21}$
a_5	$-\frac{32}{r^5+20r^4+140r^3+370r^2+99r-630}$	$-\frac{4}{315}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2 \left(1 - \frac{4x}{5} + \frac{2x^2}{5} - \frac{16x^3}{105} + \frac{x^4}{21} - \frac{4x^5}{315} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{16}{r^4 + 14r^3 + 59r^2 + 46r - 120} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{16}{r^4 + 14r^3 + 59r^2 + 46r - 120} &= \lim_{r \to -2} \frac{16}{r^4 + 14r^3 + 59r^2 + 46r - 120} \\ &= -\frac{2}{9} \end{aligned}$$

The limit is $-\frac{2}{9}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-2}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + 2b_{n-1}(n+r-1) + b_n(n+r) - 4b_n = 0 \tag{4}
$$

Which for for the root $r = -2$ becomes

$$
b_n(n-2)(n-3) + 2b_{n-1}(n-3) + b_n(n-2) - 4b_n = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{2b_{n-1}(n+r-1)}{n^2+2nr+r^2-4} \tag{5}
$$

Which for the root $r = -2$ becomes

$$
b_n = -\frac{2b_{n-1}(n-3)}{n^2-4n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{2r}{r^2+2r-3}
$$

Which for the root $r = -2$ becomes

$$
b_1 = -\frac{4}{3}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+2r-3}$	$-\frac{4}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r + 4}{(r^2 + 2r - 3)(r + 4)}$$

Which for the root $r = -2$ becomes

$$b_2 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+2r-3}$	$-\frac{4}{3}$
b_2	$\frac{4r+4}{r^3+6r^2+5r-12}$	$\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8(2 + r)}{(r + 5)(r^2 + 2r - 3)(r + 4)}$$

Which for the root $r = -2$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+2r-3}$	$-\frac{4}{3}$
b_2	$\frac{4r+4}{r^3+6r^2+5r-12}$	$\frac{2}{3}$
b_3	$\frac{-16-8r}{(r+5)(3+r)(-1+r)(r+4)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(r+6)(r+4)(-1+r)(r+5)}$$

Which for the root $r = -2$ becomes

$$b_4 = -\frac{2}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+2r-3}$	$-\frac{4}{3}$
b_2	$\frac{4r+4}{r^3+6r^2+5r-12}$	$\frac{2}{3}$
b_3	$\frac{-16-8r}{(r+5)(3+r)(-1+r)(r+4)}$	0
b_4	$\frac{16}{(r+6)(r+4)(-1+r)(r+5)}$	$-\frac{2}{9}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{(r+6)(-1+r)(r+5)(r^2+10r+21)}$$

Which for the root $r = -2$ becomes

$$b_5 = \frac{8}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+2r-3}$	$-\frac{4}{3}$
b_2	$\frac{4r+4}{r^3+6r^2+5r-12}$	$\frac{2}{3}$
b_3	$\frac{-16-8r}{(r+5)(3+r)(-1+r)(r+4)}$	0
b_4	$\frac{16}{(r+6)(r+4)(-1+r)(r+5)}$	$-\frac{2}{9}$
b_5	$-\frac{32}{(r+7)(3+r)(r+5)(-1+r)(r+6)}$	$\frac{8}{45}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^2 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \right) \\
&= \frac{1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{2x^4}{9} + \frac{8x^5}{45} + O(x^6)}{x^2}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 - \frac{4x}{5} + \frac{2x^2}{5} - \frac{16x^3}{105} + \frac{x^4}{21} - \frac{4x^5}{315} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{2x^4}{9} + \frac{8x^5}{45} + O(x^6)\right)}{x^2}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - \frac{4x}{5} + \frac{2x^2}{5} - \frac{16x^3}{105} + \frac{x^4}{21} - \frac{4x^5}{315} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{2x^4}{9} + \frac{8x^5}{45} + O(x^6)\right)}{x^2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = c_1 x^2 \left(1 - \frac{4x}{5} + \frac{2x^2}{5} - \frac{16x^3}{105} + \frac{x^4}{21} - \frac{4x^5}{315} + O(x^6)\right) \\
+ \frac{c_2 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{2x^4}{9} + \frac{8x^5}{45} + O(x^6)\right)}{x^2}
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y = c_1 x^2 \left(1 - \frac{4x}{5} + \frac{2x^2}{5} - \frac{16x^3}{105} + \frac{x^4}{21} - \frac{4x^5}{315} + O(x^6)\right) \\
+ \frac{c_2 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{2x^4}{9} + \frac{8x^5}{45} + O(x^6)\right)}{x^2}
\end{aligned}$$

Verified OK.

3.7.1 Maple step by step solution

Let's solve

$x^2y'' + (2x^2 + x)\,y' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y}{x^2} - \frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{x} - \frac{4y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2x+1}{x}, P_3(x) = -\frac{4}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -4$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(2x + 1)\,y' - 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\,x^{k+r-1+m}$

- ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-2) + 2a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+2)\,(k+r-2) + 2a_{k-1}(k+r-1) = 0$$

- Shift index using $k- >k+1$

  $$a_{k+1}(k+3+r)\,(k+r-1) + 2a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{2a_k(k+r)}{(k+3+r)(k+r-1)}$$

- Recursion relation for $r = -2$; series terminates at $k = 2$

  $$a_{k+1} = -\frac{2a_k(k-2)}{(k+1)(k-3)}$$

- Apply recursion relation for $k = 0$

  $$a_1 = -\frac{4a_0}{3}$$

- Apply recursion relation for $k = 1$

  $$a_2 = -\frac{a_1}{2}$$

- Express in terms of a_0

  $$a_2 = \frac{2a_0}{3}$$

- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second

  $$y = a_0 \cdot \left(1 - \frac{4}{3}x + \frac{2}{3}x^2\right)$$

- Recursion relation for $r = 2$

  $$a_{k+1} = -\frac{2a_k(k+2)}{(k+5)(k+1)}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+5)(k+1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = a_0 \cdot \left(1 - \frac{4}{3}x + \frac{2}{3}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{2b_k(k+2)}{(k+5)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+(x+2*x^2)*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 - \frac{4}{5}x + \frac{2}{5}x^2 - \frac{16}{105}x^3 + \frac{1}{21}x^4 - \frac{4}{315}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(-144 + 192x - 96x^2 + 32x^4 - \frac{128}{5}x^5 + \mathrm{O}\left(x^6\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 208

```
AsymptoticDSolveValue[x^2*y''[x]+(x+2*x^2)*y'[x]-4*y[x]==2,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(\frac{2x^2}{3}-\frac{4x}{3}+1\right)}{x^2}+c_2\left(-\frac{4x^5}{315}+\frac{x^4}{21}-\frac{16x^3}{105}+\frac{2x^2}{5}-\frac{4x}{5}+1\right)x^2 + \left(-\frac{4x^5}{315}+\frac{x^4}{21}-\frac{16x^3}{105}+\frac{2x^2}{5}-\frac{4x}{5}+1\right)\left(\frac{7x^6}{2430}+\frac{19x^5}{2025}+\frac{5x^4}{216}+\frac{2x^3}{45}+\frac{x^2}{18} - \frac{1}{4x^2}-\frac{1}{3x}\right)x^2+\frac{\left(\frac{2x^2}{3}-\frac{4x}{3}+1\right)\left(-\frac{x^6}{84}-\frac{4x^5}{105}-\frac{x^4}{10}-\frac{x^3}{5}-\frac{x^2}{4}\right)}{x^2}$$

3.8 problem 9

Internal problem ID [4703]
Internal file name [OUTPUT/4196_Sunday_June_05_2022_12_39_52_PM_74022088/index.tex]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(-x^2+x)\, y''+3y'+2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^2+x)\, y''+3y'+2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x\,(x-1)} \\ q(x) &= -\frac{2}{x\,(x-1)} \end{aligned}$$

Table 44: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{2}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-1) + 3y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) + 3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) + 3a_n(n+r) + 2a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n + r - 3)\, a_{n-1}}{n + 2 + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(n - 3)\, a_{n-1}}{n + 2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2 + r}{3 + r}$$

Which for the root $r = 0$ becomes

$$a_1 = -\frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{3+r}$	$-\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2 - 3r + 2}{(3 + r)\,(4 + r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{3+r}$	$-\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(r^2 - 3r + 2)\, r}{(4 + r)\,(5 + r)\,(3 + r)}$$

Which for the root $r = 0$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{3+r}$	$-\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$
a_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-1 + r)\,(-2 + r)\, r(1 + r)}{(3 + r)\,(4 + r)\,(5 + r)\,(6 + r)}$$

Which for the root $r = 0$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{3+r}$	$-\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$
a_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	0
a_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r^5 - 5r^3 + 4r}{(3 + r)\,(4 + r)\,(5 + r)\,(6 + r)\,(7 + r)}$$

Which for the root $r = 0$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{3+r}$	$-\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$
a_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	0
a_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	0
a_5	$\frac{r^5-5r^3+4r}{(3+r)(4+r)(5+r)(6+r)(7+r)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - \frac{2x}{3} + \frac{x^2}{6} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{r^2 - 3r + 2}{(3+r)(4+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{r^2 - 3r + 2}{(3+r)(4+r)} &= \lim_{r \to -2} \frac{r^2 - 3r + 2}{(3+r)(4+r)} \\ &= 6 \end{aligned}$$

The limit is 6. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-2}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
-b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) + 3(n+r)b_n + 2b_{n-1} = 0 \quad (4)
$$

Which for for the root $r = -2$ becomes

$$
-b_{n-1}(n-3)(n-4) + b_n(n-2)(n-3) + 3(n-2)b_n + 2b_{n-1} = 0 \qquad (4\text{A})
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = \frac{(n+r-3)\, b_{n-1}}{n+2+r} \tag{5}
$$

Which for the root $r = -2$ becomes

$$
b_n = \frac{(n-5)\, b_{n-1}}{n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{-2+r}{3+r}
$$

Which for the root $r = -2$ becomes

$$
b_1 = -4
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{3+r}$	-4

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2 - 3r + 2}{(3 + r)(4 + r)}$$

Which for the root $r = -2$ becomes

$$b_2 = 6$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{3+r}$	-4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(r^2 - 3r + 2)\, r}{(4 + r)(5 + r)(3 + r)}$$

Which for the root $r = -2$ becomes

$$b_3 = -4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{3+r}$	-4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6
b_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	-4

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r^2 - 3r + 2)\, r(1 + r)}{(5 + r)\,(3 + r)\,(6 + r)\,(4 + r)}$$

Which for the root $r = -2$ becomes

$$b_4 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{3+r}$	-4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6
b_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	-4
b_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	1

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{(r^2 - 3r + 2)\, r(1 + r)\,(2 + r)}{(6 + r)\,(3 + r)\,(4 + r)\,(5 + r)\,(7 + r)}$$

Which for the root $r = -2$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{3+r}$	-4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6
b_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	-4
b_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	1
b_5	$\frac{(-2+r)(-1+r)r(1+r)(2+r)}{(6+r)(3+r)(4+r)(5+r)(7+r)}$	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - 4x + 6x^2 - 4x^3 + x^4 + O(x^6)}{x^2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 - \frac{2x}{3} + \frac{x^2}{6} + O(x^6)\right) + \frac{c_2(1 - 4x + 6x^2 - 4x^3 + x^4 + O(x^6))}{x^2}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - \frac{2x}{3} + \frac{x^2}{6} + O(x^6)\right) + \frac{c_2(1 - 4x + 6x^2 - 4x^3 + x^4 + O(x^6))}{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \left(1 - \frac{2x}{3} + \frac{x^2}{6} + O(x^6)\right) + \frac{c_2(1 - 4x + 6x^2 - 4x^3 + x^4 + O(x^6))}{x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \left(1 - \frac{2x}{3} + \frac{x^2}{6} + O(x^6)\right) + \frac{c_2(1 - 4x + 6x^2 - 4x^3 + x^4 + O(x^6))}{x^2}
$$

Verified OK.

3.8.1 Maple step by step solution

Let's solve

$-y''x(x-1) + 3y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x(x-1)} + \frac{3y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x(x-1)} - \frac{2y}{x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x) = -\frac{3}{x(x-1)}, P_3(x) = -\frac{2}{x(x-1)}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x(x-1) - 3y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  ○ Shift index using $k- >k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

  ○ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  ○ Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$$-a_0 r(2+r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(k+3+r)+a_k(k+1+r)\,(k+r-2)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-2,0\}$

- Each term in the series must be 0, giving the recursion relation

  $((-k-3-r)\,a_{k+1}+a_k(k+r-2))\,(k+1+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-2)}{k+3+r}$

- Recursion relation for $r=-2$; series terminates at $k=4$

  $a_{k+1}=\frac{a_k(k-4)}{k+1}$

- Apply recursion relation for $k=0$

  $a_1=-4a_0$

- Apply recursion relation for $k=1$

  $a_2=-\frac{3a_1}{2}$

- Express in terms of a_0

  $a_2=6a_0$

- Apply recursion relation for $k=2$

  $a_3=-\frac{2a_2}{3}$

- Express in terms of a_0

  $a_3=-4a_0$

- Apply recursion relation for $k=3$

  $a_4=-\frac{a_3}{4}$

- Express in terms of a_0

  $a_4=a_0$

- Terminating series solution of the ODE for $r=-2$. Use reduction of order to find the second

  $y=a_0\cdot(1-4x+6x^2-4x^3+x^4)$

- Recursion relation for $r=0$; series terminates at $k=2$

$a_{k+1} = \frac{a_k(k-2)}{k+3}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{2a_0}{3}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{6}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{2}{3}x + \frac{1}{6}x^2\right)$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot (1 - 4x + 6x^2 - 4x^3 + x^4) + b_0 \cdot \left(1 - \frac{2}{3}x + \frac{1}{6}x^2\right)\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
Order:=6;
dsolve((x-x^2)*diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{2}{3}x + \frac{1}{6}x^2 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2(-2 + 8x - 12x^2 + 8x^3 - 2x^4 + \mathrm{O}\left(x^6\right))}{x^2}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 40

```
AsymptoticDSolveValue[(x-x^2)*y''[x]+3*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(x^2 + \frac{1}{x^2} - 4x - \frac{4}{x} + 6\right) + c_2\left(\frac{x^2}{6} - \frac{2x}{3} + 1\right)$$

3.9 problem 10

Internal problem ID [4704]
Internal file name [`OUTPUT/4197_Sunday_June_05_2022_12_39_59_PM_6718125/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(4x^3-14x^2-2x\right)y''-\left(6x^2-7x+1\right)y'+(-1+6x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3-14x^2-2x\right)y''+\left(-6x^2+7x-1\right)y'+(-1+6x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=-\frac{6x^2-7x+1}{2x\left(2x^2-7x-1\right)}\\ q(x)&=\frac{-1+6x}{2\left(2x^2-7x-1\right)x}\end{aligned}$$

Table 46: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{6x^2-7x+1}{2x(2x^2-7x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{7}{4} - \frac{\sqrt{57}}{4}$	"regular"
$x = \frac{7}{4} + \frac{\sqrt{57}}{4}$	"regular"

$q(x) = \frac{-1+6x}{2(2x^2-7x-1)x}$	
singularity	type
$x = 0$	"regular"
$x = \frac{7}{4} - \frac{\sqrt{57}}{4}$	"regular"
$x = \frac{7}{4} + \frac{\sqrt{57}}{4}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{7}{4} - \frac{\sqrt{57}}{4}, \frac{7}{4} + \frac{\sqrt{57}}{4}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2y''x(2x^2 - 7x - 1) + (-6x^2 + 7x - 1)\, y' + (-1 + 6x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) x(2x^2 - 7x - 1) + (-6x^2 + 7x - 1) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-1 + 6x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-14x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 7x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) \\
&+ \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 6x^{1+n+r} a_n\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)\, x^{n+r-1}
$$

$$
\sum_{n=0}^{\infty} \left(-14x^{n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-14a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right)
$$

$$
\sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-6a_{n-2}(n+r-2)\, x^{n+r-1}\right)
$$

$$
\sum_{n=0}^{\infty} 7x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} 7a_{n-1}(n+r-1)\, x^{n+r-1}
$$

$$
\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)
$$

$$
\sum_{n=0}^{\infty} 6x^{1+n+r} a_n = \sum_{n=2}^{\infty} 6a_{n-2} x^{n+r-1}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n + r - 1$.

$$\left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-14a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r-1}a_n(n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-6a_{n-2}(n+r-2)x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 7a_{n-1}(n+r-1)x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-(n+r)a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} 6a_{n-2}x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$-2x^{n+r-1}a_n(n+r)(n+r-1) - (n+r)a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$-2x^{-1+r}a_0 r(-1+r) - r a_0 x^{-1+r} = 0$$

Or

$$\left(-2x^{-1+r}r(-1+r) - r\, x^{-1+r}\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(-2r^2 + r\right)x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation becomes

$$-2r^2 + r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(-2r^2 + r\right)x^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-14r^2 + 21r - 1}{2r^2 + 3r + 1}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &4a_{n-2}(n+r-2)(n-3+r) - 14a_{n-1}(n+r-1)(n+r-2) - 2a_n(n+r)(n+r-1) \\ &- 6a_{n-2}(n+r-2) + 7a_{n-1}(n+r-1) - a_n(n+r) - a_{n-1} + 6a_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{4n^2a_{n-2} - 14n^2a_{n-1} + 8nra_{n-2} - 28nra_{n-1} + 4r^2a_{n-2} - 14r^2a_{n-1} - 26na_{n-2} + 49na_{n-1} - 26ra_{n-2}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{(4a_{n-2} - 14a_{n-1})\, n^2 + (-22a_{n-2} + 35a_{n-1})\, n + 30a_{n-2} - 15a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-14r^2+21r-1}{2r^2+3r+1}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{204r^4 - 204r^3 - 231r^2 + 141r}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-14r^2+21r-1}{2r^2+3r+1}$	2
a_2	$\frac{204r^4-204r^3-231r^2+141r}{4r^4+20r^3+35r^2+25r+6}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-1484r^4 + 4577r^2 - 2103r}{4r^4 + 36r^3 + 119r^2 + 171r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-14r^2+21r-1}{2r^2+3r+1}$	2
a_2	$\frac{204r^4-204r^3-231r^2+141r}{4r^4+20r^3+35r^2+25r+6}$	0
a_3	$\frac{-1484r^4+4577r^2-2103r}{4r^4+36r^3+119r^2+171r+90}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{10796r^4 + 10796r^3 - 54371r^2 + 23137r}{4r^4 + 52r^3 + 251r^2 + 533r + 420}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-14r^2+21r-1}{2r^2+3r+1}$	2
a_2	$\frac{204r^4-204r^3-231r^2+141r}{4r^4+20r^3+35r^2+25r+6}$	0
a_3	$\frac{-1484r^4+4577r^2-2103r}{4r^4+36r^3+119r^2+171r+90}$	0
a_4	$\frac{10796r^4+10796r^3-54371r^2+23137r}{4r^4+52r^3+251r^2+533r+420}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-78540r^4 - 157080r^3 + 548853r^2 - 225339r}{4r^4 + 68r^3 + 431r^2 + 1207r + 1260}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-14r^2+21r-1}{2r^2+3r+1}$	2
a_2	$\frac{204r^4-204r^3-231r^2+141r}{4r^4+20r^3+35r^2+25r+6}$	0
a_3	$\frac{-1484r^4+4577r^2-2103r}{4r^4+36r^3+119r^2+171r+90}$	0
a_4	$\frac{10796r^4+10796r^3-54371r^2+23137r}{4r^4+52r^3+251r^2+533r+420}$	0
a_5	$\frac{-78540r^4-157080r^3+548853r^2-225339r}{4r^4+68r^3+431r^2+1207r+1260}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= \sqrt{x}\left(1 + 2x + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{-14r^2 + 21r - 1}{2r^2 + 3r + 1}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 4b_{n-2}(n+r-2)(n-3+r) - 14b_{n-1}(n+r-1)(n+r-2) - 2b_n(n+r)(n+r-1) \\ - 6b_{n-2}(n+r-2) + 7b_{n-1}(n+r-1) - (n+r)b_n - b_{n-1} + 6b_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{4n^2b_{n-2} - 14n^2b_{n-1} + 8nrb_{n-2} - 28nrb_{n-1} + 4r^2b_{n-2} - 14r^2b_{n-1} - 26nb_{n-2} + 49nb_{n-1} - 26rb_{n-2} +}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{(4b_{n-2} - 14b_{n-1})n^2 + (-26b_{n-2} + 49b_{n-1})n + 42b_{n-2} - 36b_{n-1}}{2n^2 - n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-14r^2+21r-1}{2r^2+3r+1}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{204r^4 - 204r^3 - 231r^2 + 141r}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-14r^2+21r-1}{2r^2+3r+1}$	-1
b_2	$\frac{204r^4-204r^3-231r^2+141r}{4r^4+20r^3+35r^2+25r+6}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-1484r^4 + 4577r^2 - 2103r}{4r^4 + 36r^3 + 119r^2 + 171r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-14r^2+21r-1}{2r^2+3r+1}$	-1
b_2	$\frac{204r^4-204r^3-231r^2+141r}{4r^4+20r^3+35r^2+25r+6}$	0
b_3	$\frac{-1484r^4+4577r^2-2103r}{4r^4+36r^3+119r^2+171r+90}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{10796r^4 + 10796r^3 - 54371r^2 + 23137r}{4r^4 + 52r^3 + 251r^2 + 533r + 420}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-14r^2+21r-1}{2r^2+3r+1}$	-1
b_2	$\frac{204r^4-204r^3-231r^2+141r}{4r^4+20r^3+35r^2+25r+6}$	0
b_3	$\frac{-1484r^4+4577r^2-2103r}{4r^4+36r^3+119r^2+171r+90}$	0
b_4	$\frac{10796r^4+10796r^3-54371r^2+23137r}{4r^4+52r^3+251r^2+533r+420}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-78540r^4 - 157080r^3 + 548853r^2 - 225339r}{4r^4 + 68r^3 + 431r^2 + 1207r + 1260}$$

Which for the root $r = 0$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-14r^2+21r-1}{2r^2+3r+1}$	-1
b_2	$\frac{204r^4-204r^3-231r^2+141r}{4r^4+20r^3+35r^2+25r+6}$	0
b_3	$\frac{-1484r^4+4577r^2-2103r}{4r^4+36r^3+119r^2+171r+90}$	0
b_4	$\frac{10796r^4+10796r^3-54371r^2+23137r}{4r^4+52r^3+251r^2+533r+420}$	0
b_5	$\frac{-78540r^4-157080r^3+548853r^2-225339r}{4r^4+68r^3+431r^2+1207r+1260}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - x + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x}\left(1 + 2x + O(x^6)\right) + c_2\left(1 - x + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x}\left(1 + 2x + O(x^6)\right) + c_2\left(1 - x + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\left(1 + 2x + O(x^6)\right) + c_2\left(1 - x + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x}\left(1 + 2x + O(x^6)\right) + c_2\left(1 - x + O(x^6)\right)$$

Verified OK.

3.9.1 Maple step by step solution

Let's solve

$2y''x(2x^2-7x-1)+(-6x^2+7x-1)\,y'+(-1+6x)\,y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{(-1+6x)y}{2x(2x^2-7x-1)}+\frac{(6x^2-7x+1)y'}{2x(2x^2-7x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''-\frac{(6x^2-7x+1)y'}{2x(2x^2-7x-1)}+\frac{(-1+6x)y}{2x(2x^2-7x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{6x^2-7x+1}{2x(2x^2-7x-1)}, P_3(x)=\frac{-1+6x}{2(2x^2-7x-1)x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$2y''x(2x^2-7x-1)+(-6x^2+7x-1)\,y'+(-1+6x)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-1+2r)\, x^{-1+r} + (-a_1(1+r)\,(1+2r) - a_0(14r^2-21r+1))\, x^r + \left(\sum_{k=1}^{\infty} (-a_{k+1}(k+1+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term must be 0

  $$-a_1(1+r)\,(1+2r) - a_0(14r^2-21r+1) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-14a_k + 4a_{k-1} - 2a_{k+1})\, k^2 + ((-28a_k + 8a_{k-1} - 4a_{k+1})\, r + 21a_k - 18a_{k-1} - 3a_{k+1})\, k + (-14a$$

- Shift index using $k->k+1$

  $$(-14a_{k+1} + 4a_k - 2a_{k+2})\,(k+1)^2 + ((-28a_{k+1} + 8a_k - 4a_{k+2})\, r + 21a_{k+1} - 18a_k - 3a_{k+2})\,(k+$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} + 8kra_k - 28kra_{k+1} + 4r^2a_k - 14r^2a_{k+1} - 10ka_k - 7ka_{k+1} - 10ra_k - 7ra_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 4kr + 2r^2 + 7k + 7r + 6}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{4k^2 a_k - 14k^2 a_{k+1} - 10k a_k - 7k a_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}, -a_1 - a_0 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = \frac{4k^2 a_k - 14k^2 a_{k+1} - 6k a_k - 21k a_{k+1} + 2a_k - a_{k+1}}{2k^2 + 9k + 10}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4k^2 a_k - 14k^2 a_{k+1} - 6k a_k - 21k a_{k+1} + 2a_k - a_{k+1}}{2k^2 + 9k + 10}, -3a_1 + 6a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4k^2 a_k - 14k^2 a_{k+1} - 10k a_k - 7k a_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}, -a_1 - a_0 = 0, b_k$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 28

```
Order:=6;
dsolve((4*x^3-14*x^2-2*x)*diff(y(x),x$2)-(6*x^2-7*x+1)*diff(y(x),x)+(6*x-1)*y(x)=0,y(x),type
```

$$y(x) = c_1 \sqrt{x}\left(1 + 2x + O\left(x^6\right)\right) + c_2\left(1 - x + O\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 25

```
AsymptoticDSolveValue[(4*x^3-14*x^2-2*x)*y''[x]-(6*x^2-7*x+1)*y'[x]+(6*x-1)*y[x]==0,y[x],{x,
```

$$y(x) \to c_1\sqrt{x}(2x+1) + c_2(1-x)$$

3.10 problem 11

Internal problem ID [4705]
Internal file name [`OUTPUT/4198_Sunday_June_05_2022_12_40_07_PM_21174301/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x^2y' + (-2 + x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + x^2y' + (-2 + x)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 1$$
$$q(x) = \frac{-2 + x}{x^2}$$

Table 48: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = \frac{-2+x}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + x^2 y' + (-2 + x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\, (n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\, (n+r-1)\, a_n x^{n+r-2}\right) + x^2 \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-2+x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)\right) \\ + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) \\ + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - r - 2)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_{n-1}(n + r - 1) - 2a_n + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r)}{n^2 + 2nr + r^2 - n - r - 2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-1}(n+2)}{n\,(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1 - r}{r^2 + r - 2}$$

Which for the root $r = 2$ becomes

$$a_1 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+r-2}$	$-\frac{3}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1 + r}{(r+3)\,r\,(-1+r)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{3}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+r-2}$	$-\frac{3}{4}$
a_2	$\frac{1+r}{(r+3)r(-1+r)}$	$\frac{3}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(r+4)\,r\,(-1+r)}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+r-2}$	$-\frac{3}{4}$
a_2	$\frac{1+r}{(r+3)r(-1+r)}$	$\frac{3}{10}$
a_3	$-\frac{1}{(r+4)r(-1+r)}$	$-\frac{1}{12}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r\,(-1+r)\,(r+5)\,(2+r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{56}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+r-2}$	$-\frac{3}{4}$
a_2	$\frac{1+r}{(r+3)r(-1+r)}$	$\frac{3}{10}$
a_3	$-\frac{1}{(r+4)r(-1+r)}$	$-\frac{1}{12}$
a_4	$\frac{1}{r(-1+r)(r+5)(2+r)}$	$\frac{1}{56}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{r\,(-1+r)\,(2+r)\,(r+6)\,(r+3)}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{1}{320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+r-2}$	$-\frac{3}{4}$
a_2	$\frac{1+r}{(r+3)r(-1+r)}$	$\frac{3}{10}$
a_3	$-\frac{1}{(r+4)r(-1+r)}$	$-\frac{1}{12}$
a_4	$\frac{1}{r(-1+r)(r+5)(2+r)}$	$\frac{1}{56}$
a_5	$-\frac{1}{r(-1+r)(2+r)(r+6)(r+3)}$	$-\frac{1}{320}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2 \left(1 - \frac{3x}{4} + \frac{3x^2}{10} - \frac{x^3}{12} + \frac{x^4}{56} - \frac{x^5}{320} + O\left(x^6\right)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_3 \\
&= -\frac{1}{(r+4)\, r\, (-1+r)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r \to r_2} -\frac{1}{(r+4)\, r\, (-1+r)} &= \lim_{r \to -1} -\frac{1}{(r+4)\, r\, (-1+r)} \\
&= -\frac{1}{6}
\end{aligned}$$

The limit is $-\frac{1}{6}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) - 2b_n + b_{n-1} = 0 \tag{4}
$$

Which for for the root $r = -1$ becomes

$$
b_n(n-1)(n-2) + b_{n-1}(n-2) - 2b_n + b_{n-1} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}(n+r)}{n^2 + 2nr + r^2 - n - r - 2} \tag{5}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{b_{n-1}(n-1)}{n^2 - 3n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{1+r}{r^2 + r - 2}
$$

Which for the root $r = -1$ becomes

$$
b_1 = 0
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r^2+r-2}$	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1+r}{(r+3)\,r\,(-1+r)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r^2+r-2}$	0
b_2	$\frac{1+r}{(r+3)r(-1+r)}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{(r+4)\,r\,(-1+r)}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r^2+r-2}$	0
b_2	$\frac{1+r}{(r+3)r(-1+r)}$	0
b_3	$-\frac{1}{(r+4)r(-1+r)}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{r(-1+r)(r^2+7r+10)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r^2+r-2}$	0
b_2	$\frac{1+r}{(r+3)r(-1+r)}$	0
b_3	$-\frac{1}{(r+4)r(-1+r)}$	$-\frac{1}{6}$
b_4	$\frac{1}{r(-1+r)(r+5)(2+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{r(-1+r)(2+r)(r^2+9r+18)}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{1}{20}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r^2+r-2}$	0
b_2	$\frac{1+r}{(r+3)r(-1+r)}$	0
b_3	$-\frac{1}{(r+4)r(-1+r)}$	$-\frac{1}{6}$
b_4	$\frac{1}{r(-1+r)(r+5)(2+r)}$	$\frac{1}{8}$
b_5	$-\frac{1}{r(-1+r)(2+r)(r+6)(r+3)}$	$-\frac{1}{20}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^2\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\
&= \frac{1 - \frac{x^3}{6} + \frac{x^4}{8} - \frac{x^5}{20} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x^2\left(1 - \frac{3x}{4} + \frac{3x^2}{10} - \frac{x^3}{12} + \frac{x^4}{56} - \frac{x^5}{320} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^3}{6} + \frac{x^4}{8} - \frac{x^5}{20} + O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1x^2\left(1 - \frac{3x}{4} + \frac{3x^2}{10} - \frac{x^3}{12} + \frac{x^4}{56} - \frac{x^5}{320} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^3}{6} + \frac{x^4}{8} - \frac{x^5}{20} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1x^2\left(1 - \frac{3x}{4} + \frac{3x^2}{10} - \frac{x^3}{12} + \frac{x^4}{56} - \frac{x^5}{320} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^3}{6} + \frac{x^4}{8} - \frac{x^5}{20} + O(x^6)\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1x^2\left(1 - \frac{3x}{4} + \frac{3x^2}{10} - \frac{x^3}{12} + \frac{x^4}{56} - \frac{x^5}{320} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^3}{6} + \frac{x^4}{8} - \frac{x^5}{20} + O(x^6)\right)}{x}
$$

Verified OK.

3.10.1 Maple step by step solution

Let's solve

$x^2y'' + x^2y' + (-2 + x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -y' - \frac{(-2+x)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{(-2+x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = 1, P_3(x) = \frac{-2+x}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -2$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x^2y' + (-2+x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$

- ◦ Shift index using $k- >k - 1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- ◦ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r) = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+2+r)\,(k-1+r) + a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+1)}{(k+2+r)(k-1+r)}$$

- Recursion relation for $r = -1$

$$a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$$

- Recursion relation for $r = 2$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 43

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)+(x-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 - \frac{3}{4}x + \frac{3}{10}x^2 - \frac{1}{12}x^3 + \frac{1}{56}x^4 - \frac{1}{320}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2 \left(12 - 2x^3 + \frac{3}{2}x^4 - \frac{3}{5}x^5 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x^2*y''[x]+x^2*y'[x]+(x-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^3}{8} - \frac{x^2}{6} + \frac{1}{x}\right) + c_2 \left(\frac{x^6}{56} - \frac{x^5}{12} + \frac{3x^4}{10} - \frac{3x^3}{4} + x^2\right)$$

3.11 problem 13

Internal problem ID [4706]
Internal file name [`OUTPUT/4199_Sunday_June_05_2022_12_40_15_PM_52626658/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x^2y' + (-2 + x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - x^2y' + (-2 + x)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -1 \\ q(x) &= \frac{-2 + x}{x^2} \end{aligned}$$

Table 50: Table $p(x), q(x)$ singularites.

$p(x) = -1$	
singularity	type

$q(x) = \frac{-2+x}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - x^2y' + (-2 + x)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) - x^2 \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-2 + x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2 a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2 a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 2 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 2 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 2 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - r - 2)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1} \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 2a_n + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{1+n+r} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{a_{n-1}}{3+n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2+r}$$

Which for the root $r = 2$ becomes

$$a_1 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{1}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(2+r)(3+r)(4+r)}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{20}$
a_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{120}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(3+r)(4+r)(5+r)(2+r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{20}$
a_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{120}$
a_4	$\frac{1}{(3+r)(4+r)(5+r)(2+r)}$	$\frac{1}{840}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(4+r)(5+r)(2+r)(6+r)(3+r)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{1}{6720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{20}$
a_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{120}$
a_4	$\frac{1}{(3+r)(4+r)(5+r)(2+r)}$	$\frac{1}{840}$
a_5	$\frac{1}{(4+r)(5+r)(2+r)(6+r)(3+r)}$	$\frac{1}{6720}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \right) \\
&= x^2 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_3 \\
&= \frac{1}{(2 + r)(3 + r)(4 + r)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r \to r_2} \frac{1}{(2 + r)(3 + r)(4 + r)} &= \lim_{r \to -1} \frac{1}{(2 + r)(3 + r)(4 + r)} \\
&= \frac{1}{6}
\end{aligned}$$

The limit is $\frac{1}{6}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) - 2b_n + b_{n-1} = 0 \tag{4}
$$

Which for for the root $r = -1$ becomes

$$
b_n(n-1)(n-2) - b_{n-1}(n-2) - 2b_n + b_{n-1} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = \frac{b_{n-1}}{1+n+r} \tag{5}
$$

Which for the root $r = -1$ becomes

$$
b_n = \frac{b_{n-1}}{n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{1}{2+r}
$$

Which for the root $r = -1$ becomes

$$
b_1 = 1
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{(2+r)(3+r)(4+r)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(3+r)(4+r)(5+r)(2+r)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(3+r)(4+r)(5+r)(2+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(4+r)(5+r)(2+r)(6+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(3+r)(4+r)(5+r)(2+r)}$	$\frac{1}{24}$
b_5	$\frac{1}{(4+r)(5+r)(2+r)(6+r)(3+r)}$	$\frac{1}{120}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= x^2\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)}{x}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O(x^6)\right) \\ &\quad + \frac{c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1 x^2 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O(x^6)\right) \\ &\quad + \frac{c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x}\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y &= c_1 x^2 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O(x^6)\right) \\ &\quad + \frac{c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x}\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}y &= c_1 x^2 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O(x^6)\right) \\ &\quad + \frac{c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x}\end{aligned}$$

Verified OK.

3.11.1 Maple step by step solution

Let's solve

$x^2y'' - x^2y' + (-2+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = y' - \frac{(-2+x)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - y' + \frac{(-2+x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -1, P_3(x) = \frac{-2+x}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2y'' - x^2y' + (-2+x)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$$

- Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)\,(a_k(k+r+1) - a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$(k-1+r)\,(a_{k+1}(k+2+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+2+r}$$

- Recursion relation for $r = -1$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+4}$$

- Solution for $r = 2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=\frac{a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x^2*diff(y(x),x)+(x-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1x^2\left(1+\frac{1}{4}x+\frac{1}{20}x^2+\frac{1}{120}x^3+\frac{1}{840}x^4+\frac{1}{6720}x^5+\mathrm{O}\left(x^6\right)\right)+\frac{c_2\left(12+12x+6x^2+2x^3+\frac{1}{2}x^4+\frac{1}{10}x^5+\mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 66

```
AsymptoticDSolveValue[x^2*y''[x]-x^2*y'[x]+(x-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^3}{24}+\frac{x^2}{6}+\frac{x}{2}+\frac{1}{x}+1\right)+c_2\left(\frac{x^6}{840}+\frac{x^5}{120}+\frac{x^4}{20}+\frac{x^3}{4}+x^2\right)$$

3.12 problem 14

Internal problem ID [4707]
Internal file name [`OUTPUT/4200_Sunday_June_05_2022_12_40_23_PM_76393033/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-4x)\,y''+\left((-n+1)\,x-(6-4n)\,x^2\right)y'+n(-n+1)\,xy=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-4x^3+x^2\right)y''+\left((4n-6)\,x^2+(-n+1)\,x\right)y'+\left(-n^2+n\right)xy=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{4nx-n-6x+1}{x\,(4x-1)}$$
$$q(x)=\frac{n(n-1)}{x\,(4x-1)}$$

Table 52: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{4nx-n-6x+1}{x(4x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{4}$	"regular"

$q(x) = \frac{n(n-1)}{x(4x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{4}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{4}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-x^2(4x-1)\,y'' + \left((4n-6)\,x^2 + (-n+1)\,x\right) y' + \left(-n^2+n\right) xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &-x^2(4x-1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ \left((4n-6)\,x^2 + (-n+1)\,x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) \\ &+ \left(-n^2+n\right) x \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{gathered}\sum_{n=0}^{\infty}\left(-4x^{1+n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)\\+\left(\sum_{n=0}^{\infty}4x^{1+n+r}(n+r)\left(n-\frac{3}{2}\right)a_n\right)\\+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n-1)(n+r)\right)+\sum_{n=0}^{\infty}\left(-x^n x^r a_n n(n-1)x\right)=0\end{gathered} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-4x^{1+n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-4a_{n-1}(n+r-1)(n+r-2)x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}4x^{1+n+r}(n+r)\left(n-\frac{3}{2}\right)a_n=\sum_{n=1}^{\infty}4a_{n-1}\left(n-\frac{3}{2}\right)(n+r-1)x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-x^n x^r a_n n(n-1)x\right)=\sum_{n=1}^{\infty}\left(-na_{n-1}(n-1)x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{gathered}\sum_{n=1}^{\infty}\left(-4a_{n-1}(n+r-1)(n+r-2)x^{n+r}\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)\\+\left(\sum_{n=1}^{\infty}4a_{n-1}\left(n-\frac{3}{2}\right)(n+r-1)x^{n+r}\right)\\+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n-1)(n+r)\right)+\sum_{n=1}^{\infty}\left(-na_{n-1}(n-1)x^{n+r}\right)=0\end{gathered} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r}a_n(n+r)(n+r-1)-x^{n+r}a_n(n-1)(n+r)=0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 (n-1)\, r = 0$$

Or

$$\left(x^r r(-1+r) - x^r (n-1)\, r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^r (r-n) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(r-n) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= n \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^r (r-n) = 0$$

Solving for r gives the roots of the indicial equation as Assuming the roots differ by non-integer Since $r_1 - r_2 = -n$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+n} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &-4a_{n-1}(n+r-1)\,(n+r-2) + a_n(n+r)\,(n+r-1) \\ &+ 4a_{n-1}\left(n-\frac{3}{2}\right)(n+r-1) - a_n(n-1)\,(n+r) - n a_{n-1}(n-1) = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 - 4nn - 4nr + 4n^2 + 8nr + 4r^2 + 3n - 6n - 6r + 2)}{nn + nr - n^2 - 2nr - r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}(n - 2n + 2)(n - 2n + 1)}{n(n - n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{(n - 2r)(n - 2r - 1)}{(1 + r)(n - r - 1)}$$

Which for the root $r = 0$ becomes

$$a_1 = -n$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(n-2r)(n-2r-1)}{(1+r)(n-r-1)}$	$-n$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(n - 2r - 2)(n - 2r - 3)(n - 2r)(n - 2r - 1)}{(1 + r)(n - r - 1)(r + 2)(-r + n - 2)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{(n - 3)n}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(n-2r)(n-2r-1)}{(1+r)(n-r-1)}$	$-n$
a_2	$\frac{(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)}$	$\frac{(n-3)n}{2}$

For $n=3$, using the above recursive equation gives

$$a_3 = -\frac{(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)}$$

Which for the root $r=0$ becomes

$$a_3 = -\frac{(n-4)(n-5)n}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(n-2r)(n-2r-1)}{(1+r)(n-r-1)}$	$-n$
a_2	$\frac{(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)}$	$\frac{(n-3)n}{2}$
a_3	$-\frac{(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)}$	$-\frac{(n-4)(n-5)n}{6}$

For $n=4$, using the above recursive equation gives

$$a_4 = \frac{(n-2r-6)(n-2r-7)(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)(r+4)(-r+n-4)}$$

Which for the root $r=0$ becomes

$$a_4 = \frac{(n-6)(n-7)(n-5)n}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(n-2r)(n-2r-1)}{(1+r)(n-r-1)}$	$-n$
a_2	$\frac{(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)}$	$\frac{(n-3)n}{2}$
a_3	$-\frac{(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)}$	$-\frac{(n-4)(n-5)n}{6}$
a_4	$\frac{(n-2r-6)(n-2r-7)(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)(r+4)(-r+n-4)}$	$\frac{(n-6)(n-7)(n-5)n}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(n-2r-8)(n-2r-9)(n-2r-6)(n-2r-7)(n-2r-4)(n-2r-5)(n-2r-2)(n-2r}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)(r+4)(-r+n-4)(r+}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{(n-8)(n-9)(n-6)(n-7)n}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(n-2r)(n-2r-1)}{(1+r)(n-r-1)}$	$-n$
a_2	$\frac{(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)}$	$\frac{(n-3)n}{2}$
a_3	$-\frac{(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)}$	$-\frac{(n-4)(n-5)n}{6}$
a_4	$\frac{(n-2r-6)(n-2r-7)(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)(r+4)(-r+n-4)}$	$\frac{(n-6)(n-7)(n-5)n}{24}$
a_5	$-\frac{(n-2r-8)(n-2r-9)(n-2r-6)(n-2r-7)(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)(r+4)(-r+n-4)(r+5)(-r+n-5)}$	$-\frac{(n-8)(n-9)(n-6)(n-7)n}{120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - nx + \frac{(n-3)\,n\,x^2}{2} - \frac{(n-4)(n-5)\,n\,x^3}{6} + \frac{(n-6)(n-7)(n-5)\,n\,x^4}{24} - \frac{(n-8)(n-9)(n}{12} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &-4b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) \\ &+ 4b_{n-1}\left(n-\frac{3}{2}\right)(n+r-1) - b_n(n-1)(n+r) - nb_{n-1}(n-1) = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n^2 - 4nn - 4nr + 4n^2 + 8nr + 4r^2 + 3n - 6n - 6r + 2)}{nn + nr - n^2 - 2nr - r^2} \tag{4}$$

Which for the root $r = n$ becomes

$$b_n = \frac{b_{n-1}(n + 2n - 1)\,(n + 2n - 2)}{n\,(n + n)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = n$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{(n - 2r)\,(n - 2r - 1)}{(1 + r)\,(n - r - 1)}$$

Which for the root $r = n$ becomes

$$b_1 = n$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(n-2r)(n-2r-1)}{(1+r)(n-r-1)}$	n

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(n - 2r - 2)\,(n - 2r - 3)\,(n - 2r)\,(n - 2r - 1)}{(1 + r)\,(n - r - 1)\,(r + 2)\,(-r + n - 2)}$$

Which for the root $r = n$ becomes

$$b_2 = \frac{(n + 3)\,n}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(n-2r)(n-2r-1)}{(1+r)(n-r-1)}$	n
b_2	$\frac{(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)}$	$\frac{(n+3)n}{2}$

For $n=3$, using the above recursive equation gives

$$b_3 = -\frac{(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)}$$

Which for the root $r=n$ becomes

$$b_3 = \frac{n(n+5)(n+4)}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(n-2r)(n-2r-1)}{(1+r)(n-r-1)}$	n
b_2	$\frac{(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)}$	$\frac{(n+3)n}{2}$
b_3	$-\frac{(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)}$	$\frac{n(n+5)(n+4)}{6}$

For $n=4$, using the above recursive equation gives

$$b_4 = \frac{(n-2r-6)(n-2r-7)(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)(r+4)(-r+n-4)}$$

Which for the root $r=n$ becomes

$$b_4 = \frac{(n+6)(n+7)(n+5)n}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(n-2r)(n-2r-1)}{(1+r)(n-r-1)}$	n
b_2	$\frac{(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)}$	$\frac{(n+3)n}{2}$
b_3	$-\frac{(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)}$	$\frac{n(n+5)(n+4)}{6}$
b_4	$\frac{(n-2r-6)(n-2r-7)(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)(r+4)(-r+n-4)}$	$\frac{(n+6)(n+7)(n+5)n}{24}$

For $n=5$, using the above recursive equation gives

$$b_5 = -\frac{(n-2r-8)(n-2r-9)(n-2r-6)(n-2r-7)(n-2r-4)(n-2r-5)(n-2r-2)(n-2r}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)(r+4)(-r+n-4)(r+}$$

Which for the root $r = n$ becomes

$$b_5 = \frac{n(n+7)(n+6)(n+9)(n+8)}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(n-2r)(n-2r-1)}{(1+r)(n-r-1)}$	n
b_2	$\frac{(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)}$	$\frac{(n+3)n}{2}$
b_3	$-\frac{(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)}$	$\frac{n(n+5)(n+4)}{6}$
b_4	$\frac{(n-2r-6)(n-2r-7)(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)(r+4)(-r+n-4)}$	$\frac{(n+6)(n+7)(n+5)n}{24}$
b_5	$-\frac{(n-2r-8)(n-2r-9)(n-2r-6)(n-2r-7)(n-2r-4)(n-2r-5)(n-2r-2)(n-2r-3)(n-2r)(n-2r-1)}{(1+r)(n-r-1)(r+2)(-r+n-2)(r+3)(-r+n-3)(r+4)(-r+n-4)(r+5)(-r+n-5)}$	$\frac{n(n+7)(n+6)(n+9)(n+8)}{120}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^n \left(1 + nx + \frac{(n+3)\, n\, x^2}{2} + \frac{n(n+5)(n+4)\, x^3}{6} + \frac{(n+6)(n+7)(n+5)\, n\, x^4}{24} + n(n+7)(n+6 \right. \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 - nx + \frac{(n-3)\, n\, x^2}{2} - \frac{(n-4)(n-5)\, n\, x^3}{6} + \frac{(n-6)(n-7)(n-5)\, n\, x^4}{24} \right. \\ &\quad \left. - \frac{(n-8)(n-9)(n-6)(n-7)\, n\, x^5}{120} + O(x^6)\right) \\ &\quad + c_2 x^n \left(1 + nx + \frac{(n+3)\, n\, x^2}{2} + \frac{n(n+5)(n+4)\, x^3}{6} \right. \\ &\quad \left. + \frac{(n+6)(n+7)(n+5)\, n\, x^4}{24} + \frac{n(n+7)(n+6)(n+9)(n+8)\, x^5}{120} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$y = y_h$$

$$
= c_1 \left(1 - nx + \frac{(n-3)\, n\, x^2}{2} - \frac{(n-4)\,(n-5)\, n\, x^3}{6} + \frac{(n-6)\,(n-7)\,(n-5)\, n\, x^4}{24} - \frac{(n-8)\,(n-9)\,(n-6)\,(n-7)\, n\, x^5}{120} + O(x^6)\right) + c_2 x^n \left(1 + nx + \frac{(n+3)\, n\, x^2}{2} + \frac{n(n+5)\,(n+4)\, x^3}{6} + \frac{(n+6)\,(n+7)\,(n+5)\, n\, x^4}{24} + \frac{n(n+7)\,(n+6)\,(n+9)\,(n+8)\, x^5}{120} + O(x^6)\right)
$$

Summary

The solution(s) found are the following

$$
y = c_1 \left(1 - nx + \frac{(n-3)\, n\, x^2}{2} - \frac{(n-4)\,(n-5)\, n\, x^3}{6} + \frac{(n-6)\,(n-7)\,(n-5)\, n\, x^4}{24} - \frac{(n-8)\,(n-9)\,(n-6)\,(n-7)\, n\, x^5}{120} + O(x^6)\right) + c_2 x^n \left(1 + nx + \frac{(n+3)\, n\, x^2}{2} + \frac{n(n+5)\,(n+4)\, x^3}{6} + \frac{(n+6)\,(n+7)\,(n+5)\, n\, x^4}{24} + \frac{n(n+7)\,(n+6)\,(n+9)\,(n+8)\, x^5}{120} + O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 \left(1 - nx + \frac{(n-3)\, n\, x^2}{2} - \frac{(n-4)\,(n-5)\, n\, x^3}{6} + \frac{(n-6)\,(n-7)\,(n-5)\, n\, x^4}{24} - \frac{(n-8)\,(n-9)\,(n-6)\,(n-7)\, n\, x^5}{120} + O(x^6)\right) + c_2 x^n \left(1 + nx + \frac{(n+3)\, n\, x^2}{2} + \frac{n(n+5)\,(n+4)\, x^3}{6} + \frac{(n+6)\,(n+7)\,(n+5)\, n\, x^4}{24} + \frac{n(n+7)\,(n+6)\,(n+9)\,(n+8)\, x^5}{120} + O(x^6)\right)
$$

Verified OK.

3.12.1 Maple step by step solution

Let's solve

$-x^2(4x-1)\,y''+\left((4n-6)\,x^2+(-n+1)\,x\right)y'+\left(-n^2+n\right)xy=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{n(n-1)y}{x(4x-1)}+\frac{(4nx-n-6x+1)y'}{x(4x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(4nx-n-6x+1)y'}{x(4x-1)}+\frac{n(n-1)y}{x(4x-1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{4nx-n-6x+1}{x(4x-1)},P_3(x)=\frac{n(n-1)}{x(4x-1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-n+1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x(4x-1)+(-4nx+n+6x-1)\,y'+n(n-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{m}\cdot y''$ to series expansion for $m=1..2$

  $$x^{m}\cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^{m}\cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(r-n)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k+1+r-n) + a_k(2r+1-n+2k)\,(2r-n+2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(r-n)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\{0,n\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-a_{k+1}(k+1+r)\,(k+1+r-n)+4\left(r-\frac{n}{2}+k\right)\left(r+\frac{1}{2}-\frac{n}{2}+k\right)a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=\frac{(2r-n+2k)(2r+1-n+2k)a_k}{(k+1+r)(k+1+r-n)}$$

- Recursion relation for $r=0$

  $$a_{k+1}=\frac{(-n+2k)(1-n+2k)a_k}{(k+1)(k+1-n)}$$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+1}=\frac{(-n+2k)(1-n+2k)a_k}{(k+1)(k+1-n)}\right]$$

- Recursion relation for $r=n$

  $$a_{k+1}=\frac{(n+2k)(n+1+2k)a_k}{(k+1+n)(k+1)}$$

- Solution for $r=n$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k+n}, a_{k+1}=\frac{(n+2k)(n+1+2k)a_k}{(k+1+n)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+n}\right), a_{k+1}=\frac{(-n+2k)(1-n+2k)a_k}{(k+1)(k+1-n)}, b_{k+1}=\frac{(n+2k)(n+1+2k)b_k}{(k+1+n)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 471

```
Order:=6;
dsolve(x^2*(1-4*x)*diff(y(x),x$2)+((1-n)*x-(6-4*n)*x^2)*diff(y(x),x)+n*(1-n)*x*y(x)=0,y(x),t
```

$$y(x)=c_1x^n\left(1+nx+\frac{1}{2}n(n+3)\,x^2+\frac{1}{6}(n+5)\,(n+4)\,nx^3+\frac{1}{24}n(n+5)\,(n+7)\,(n+6)\,x^4+\frac{1}{120}(n+9)\,(n+8)\,(n+7)\,(n+6)\,nx^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(1-nx+\frac{1}{2}n(-3+n)\,x^2-\frac{1}{6}(-4+n)\,(-5+n)\,nx^3+\frac{1}{24}n(-5+n)\,(-6+n)\,(n-7)\,x^4-\frac{1}{120}(-6+n)\,(n-7)\,(n-8)\,(n-9)\,nx^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 2114

```
AsymptoticDSolveValue[x^2*(1-4*x)*y''[x]+((1-n)*x-(6-4*n)*x^2)*y'[x]+n*(1-n)*x*y[x]==0,y[x],
```

$$y(x) \to \left(\frac{\left(512n - 256(n-n^2) - \frac{(n^2+n)(64(n-n^2)-128(n+1))}{(1-n)(n+1)+n(n+1)} - \frac{(16(n-n^2)-32(n+2))\left(8n-4(n-n^2)-\frac{(n^2+n)(-n^2+n-2(n+1))}{(1-n)(n+1)+n(n+1)}\right)}{(1-n)(n+2)+(n+1)(n+2)} \right.}{} \right.$$

$$+ \frac{\left(128n - 64(n-n^2) - \frac{(n^2+n)(16(n-n^2)-32(n+1))}{(1-n)(n+1)+n(n+1)} - \frac{(4(n-n^2)-8(n+2))\left(8n-4(n-n^2)-\frac{(n^2+n)(-n^2+n-2(n+1))}{(1-n)(n+1)+n(n+1)}\right)}{(1-n)(n+2)+(n+1)(n+2)} - \right.}{(1-n)(n+4)+}$$

$$+ \frac{\left(32n - 16(n-n^2) - \frac{(n^2+n)(4(n-n^2)-8(n+1))}{(1-n)(n+1)+n(n+1)} - \frac{(-n^2+n-2(n+2))\left(8n-4(n-n^2)-\frac{(n^2+n)(-n^2+n-2(n+1))}{(1-n)(n+1)+n(n+1)}\right)}{(1-n)(n+2)+(n+1)(n+2)}\right)x^3}{(1-n)(n+3)+(n+2)(n+3)}$$

3.13 problem 15

Internal problem ID [4708]
Internal file name [`OUTPUT/4201_Sunday_June_05_2022_12_40_31_PM_19405499/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + (x^2 + x) y' + (x - 9) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (x^2 + x) y' + (x - 9) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x+1}{x}$$
$$q(x) = \frac{x-9}{x^2}$$

Table 54: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x-9}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^2 + x)\, y' + (x - 9)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (x^2 + x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (x-9) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-9 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-9 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 9 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 9 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 9 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 9\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 9 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = -3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 9)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^3}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-3}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_n(n+r) + a_{n-1} - 9a_n = 0 \qquad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r)}{n^2+2nr+r^2-9} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-1}(n+3)}{n\,(n+6)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1-r}{r^2+2r-8}$$

Which for the root $r = 3$ becomes

$$a_1 = -\frac{4}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{4}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(2+r)(1+r)}{r^4+6r^3-5r^2-42r+40}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{5}{28}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{4}{7}$
a_2	$\frac{(2+r)(1+r)}{r^4+6r^3-5r^2-42r+40}$	$\frac{5}{28}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(2+r)(1+r)(3+r)}{(r^4+6r^3-5r^2-42r+40)\, r\,(r+6)}$$

Which for the root $r = 3$ becomes

$$a_3 = -\frac{5}{126}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{4}{7}$
a_2	$\frac{(2+r)(1+r)}{r^4+6r^3-5r^2-42r+40}$	$\frac{5}{28}$
a_3	$-\frac{(2+r)(1+r)(3+r)}{(r^4+6r^3-5r^2-42r+40)r(r+6)}$	$-\frac{5}{126}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(2+r)(3+r)}{(r+7)(r+6)\, r\,(r+5)(-1+r)(r-2)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{4}{7}$
a_2	$\frac{(2+r)(1+r)}{r^4+6r^3-5r^2-42r+40}$	$\frac{5}{28}$
a_3	$-\frac{(2+r)(1+r)(3+r)}{(r^4+6r^3-5r^2-42r+40)r(r+6)}$	$-\frac{5}{126}$
a_4	$\frac{(2+r)(3+r)}{(r+7)(r+6)r(r+5)(-1+r)(r-2)}$	$\frac{1}{144}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r-3}{(r+8)(r-2)(-1+r)\, r\,(r+6)(r+7)}$$

Which for the root $r = 3$ becomes

$$a_5 = -\frac{1}{990}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{4}{7}$
a_2	$\frac{(2+r)(1+r)}{r^4+6r^3-5r^2-42r+40}$	$\frac{5}{28}$
a_3	$-\frac{(2+r)(1+r)(3+r)}{(r^4+6r^3-5r^2-42r+40)r(r+6)}$	$-\frac{5}{126}$
a_4	$\frac{(2+r)(3+r)}{(r+7)(r+6)r(r+5)(-1+r)(r-2)}$	$\frac{1}{144}$
a_5	$\frac{-r-3}{(r+8)(r-2)(-1+r)r(r+6)(r+7)}$	$-\frac{1}{990}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{1}{(r+9)(r+7)r(-1+r)(r-2)(r+8)}$$

Which for the root $r = 3$ becomes

$$a_6 = \frac{1}{7920}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{4}{7}$
a_2	$\frac{(2+r)(1+r)}{r^4+6r^3-5r^2-42r+40}$	$\frac{5}{28}$
a_3	$-\frac{(2+r)(1+r)(3+r)}{(r^4+6r^3-5r^2-42r+40)r(r+6)}$	$-\frac{5}{126}$
a_4	$\frac{(2+r)(3+r)}{(r+7)(r+6)r(r+5)(-1+r)(r-2)}$	$\frac{1}{144}$
a_5	$\frac{-r-3}{(r+8)(r-2)(-1+r)r(r+6)(r+7)}$	$-\frac{1}{990}$
a_6	$\frac{1}{(r+9)(r+7)r(-1+r)(r-2)(r+8)}$	$\frac{1}{7920}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \dots\right) \\ &= x^3\left(1 - \frac{4x}{7} + \frac{5x^2}{28} - \frac{5x^3}{126} + \frac{x^4}{144} - \frac{x^5}{990} + \frac{x^6}{7920} + O(x^7)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_6 \\ &= \frac{1}{(r+9)(r+7)r(-1+r)(r-2)(r+8)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1}{(r+9)(r+7)r(-1+r)(r-2)(r+8)} &= \lim_{r \to -3} \frac{1}{(r+9)(r+7)r(-1+r)(r-2)(r+8)} \\ &= -\frac{1}{7200} \end{aligned}$$

The limit is $-\frac{1}{7200}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-3} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) + b_n(n+r) + b_{n-1} - 9b_n = 0 \tag{4}$$

Which for for the root $r = -3$ becomes

$$b_n(n-3)(n-4) + b_{n-1}(n-4) + b_n(n-3) + b_{n-1} - 9b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n+r)}{n^2+2nr+r^2-9} \tag{5}$$

Which for the root $r = -3$ becomes

$$b_n = -\frac{b_{n-1}(n-3)}{n^2-6n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1+r}{r^2+2r-8}$$

Which for the root $r = -3$ becomes

$$b_1 = -\frac{2}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(2+r)(1+r)}{(r^2+2r-8)(r^2+4r-5)}$$

Which for the root $r = -3$ becomes

$$b_2 = \frac{1}{20}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{2}{5}$
b_2	$\frac{(2+r)(1+r)}{r^4+6r^3-5r^2-42r+40}$	$\frac{1}{20}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(2+r)(1+r)(3+r)}{(r^2+2r-8)(r^2+4r-5)r(r+6)}$$

Which for the root $r = -3$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{2}{5}$
b_2	$\frac{(2+r)(1+r)}{r^4+6r^3-5r^2-42r+40}$	$\frac{1}{20}$
b_3	$-\frac{(2+r)(1+r)(3+r)}{r^6+12r^5+31r^4-72r^3-212r^2+240r}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(2+r)(3+r)}{(r+7)(r+6)r(r^2+4r-5)(r-2)}$$

Which for the root $r = -3$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{2}{5}$
b_2	$\frac{(2+r)(1+r)}{r^4+6r^3-5r^2-42r+40}$	$\frac{1}{20}$
b_3	$-\frac{(2+r)(1+r)(3+r)}{r^6+12r^5+31r^4-72r^3-212r^2+240r}$	0
b_4	$\frac{(2+r)(3+r)}{(r+7)(r+6)r(r+5)(-1+r)(r-2)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{3+r}{(r+8)(r-2)(-1+r)\,r\,(r+6)(r+7)}$$

Which for the root $r = -3$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{2}{5}$
b_2	$\frac{(2+r)(1+r)}{r^4+6r^3-5r^2-42r+40}$	$\frac{1}{20}$
b_3	$-\frac{(2+r)(1+r)(3+r)}{r^6+12r^5+31r^4-72r^3-212r^2+240r}$	0
b_4	$\frac{(2+r)(3+r)}{(r+7)(r+6)r(r+5)(-1+r)(r-2)}$	0
b_5	$\frac{-r-3}{(r+8)(r-2)(-1+r)r(r+6)(r+7)}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{1}{(r+9)(r+7)\,r\,(-1+r)(r-2)(r+8)}$$

Which for the root $r = -3$ becomes

$$b_6 = -\frac{1}{7200}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r^2+2r-8}$	$-\frac{2}{5}$
b_2	$\frac{(2+r)(1+r)}{r^4+6r^3-5r^2-42r+40}$	$\frac{1}{20}$
b_3	$-\frac{(2+r)(1+r)(3+r)}{r^6+12r^5+31r^4-72r^3-212r^2+240r}$	0
b_4	$\frac{(2+r)(3+r)}{(r+7)(r+6)r(r+5)(-1+r)(r-2)}$	0
b_5	$\frac{-r-3}{(r+8)(r-2)(-1+r)r(r+6)(r+7)}$	0
b_6	$\frac{1}{(r+9)(r+7)r(-1+r)(r-2)(r+8)}$	$-\frac{1}{7200}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^3\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 \dots\right) \\ &= \frac{1 - \frac{2x}{5} + \frac{x^2}{20} - \frac{x^6}{7200} + O(x^7)}{x^3} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^3 \left(1 - \frac{4x}{7} + \frac{5x^2}{28} - \frac{5x^3}{126} + \frac{x^4}{144} - \frac{x^5}{990} + \frac{x^6}{7920} + O(x^7)\right) \\ &+ \frac{c_2\left(1 - \frac{2x}{5} + \frac{x^2}{20} - \frac{x^6}{7200} + O(x^7)\right)}{x^3} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^3 \left(1 - \frac{4x}{7} + \frac{5x^2}{28} - \frac{5x^3}{126} + \frac{x^4}{144} - \frac{x^5}{990} + \frac{x^6}{7920} + O(x^7)\right) \\ &+ \frac{c_2\left(1 - \frac{2x}{5} + \frac{x^2}{20} - \frac{x^6}{7200} + O(x^7)\right)}{x^3} \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x^3 \left(1 - \frac{4x}{7} + \frac{5x^2}{28} - \frac{5x^3}{126} + \frac{x^4}{144} - \frac{x^5}{990} + \frac{x^6}{7920} + O(x^7)\right) \\ &+ \frac{c_2\left(1 - \frac{2x}{5} + \frac{x^2}{20} - \frac{x^6}{7200} + O(x^7)\right)}{x^3} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1 x^3 \left(1 - \frac{4x}{7} + \frac{5x^2}{28} - \frac{5x^3}{126} + \frac{x^4}{144} - \frac{x^5}{990} + \frac{x^6}{7920} + O(x^7)\right) \\ &+ \frac{c_2\left(1 - \frac{2x}{5} + \frac{x^2}{20} - \frac{x^6}{7200} + O(x^7)\right)}{x^3} \end{aligned}$$

Verified OK.

3.13.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 + x)\, y' + (x - 9)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-9)y}{x^2} - \frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+1)y'}{x} + \frac{(x-9)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x+1}{x}, P_3(x) = \frac{x-9}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -9$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x + 1)\, y' + (x - 9)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 (3+r)\,(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+3)\,(k+r-3) + a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k (k+r+3)\,(k+r-3) + a_{k-1}(k+r) = 0$

- Shift index using $k->k+1$

  $a_{k+1} (k+4+r)\,(k-2+r) + a_k (k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k (k+r+1)}{(k+4+r)(k-2+r)}$

- Recursion relation for $r = -3$; series terminates at $k = 2$

  $a_{k+1} = -\frac{a_k (k-2)}{(k+1)(k-5)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{2a_0}{5}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{8}$

- Express in terms of a_0

$a_2 = \frac{a_0}{20}$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - \frac{2}{5}x + \frac{1}{20}x^2\right)$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+7)(k+1)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a_k(k+4)}{(k+7)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - \frac{2}{5}x + \frac{1}{20}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), b_{k+1} = -\frac{b_k(k+4)}{(k+7)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+(x+x^2)*diff(y(x),x)+(x-9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 \left(1 - \frac{4}{7}x + \frac{5}{28}x^2 - \frac{5}{126}x^3 + \frac{1}{144}x^4 - \frac{1}{990}x^5 + O\left(x^6\right)\right) + \frac{c_2(-86400 + 34560x - 4320x^2 + O\left(x^6\right))}{x^3}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x^2*y''[x]+(x+x^2)*y'[x]+(x-9)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{x^3} - \frac{2}{5x^2} + \frac{1}{20x}\right) + c_2\left(\frac{x^7}{144} - \frac{5x^6}{126} + \frac{5x^5}{28} - \frac{4x^4}{7} + x^3\right)$$

3.14 problem 16

Internal problem ID [4709]
Internal file name [`OUTPUT/4202_Sunday_June_05_2022_12_40_39_PM_13480000/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{\left(a^{2}+x^{2}\right) y''+x y'-y n^{2}=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (67) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (68) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{-xy' + yn^2}{a^2 + x^2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left((n^2+2)\, x^2 + a^2(n^2-1)\right) y' - 3n^2 xy}{\left(a^2+x^2\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{3(2(-n^2-1)\, x^3 + a^2(-2n^2+3)\, x)\, y' + yn^2((n^2+11)\, x^2 + n^2a^2 - 4a^2)}{\left(a^2+x^2\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left((n^4+35n^2+24)\, x^4 + 2\left(n^4 + \frac{25}{2}n^2 - 36\right) a^2x^2 + a^4n^4 - 10a^4n^2 + 9a^4\right) y' - 10xyn^2\left((n^2+5)\, x^2 + a\right.}{\left(a^2+x^2\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{-15x((n^4+15n^2+8)\, x^4 + 2a^2(n^4+n^2-20)\, x^2 + a^4(n^4-13n^2+15))\, y' + yn^2\left((n^4+85n^2+274)\right.}{\left(a^2+x^2\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)\, n^2}{a^2} \\
F_1 &= \frac{y'(0)\, n^2 - y'(0)}{a^2} \\
F_2 &= \frac{y(0)\, n^4 - 4y(0)\, n^2}{a^4} \\
F_3 &= \frac{y'(0)\, n^4 - 10y'(0)\, n^2 + 9y'(0)}{a^4} \\
F_4 &= \frac{y(0)\, n^6 - 20y(0)\, n^4 + 64y(0)\, n^2}{a^6}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1+\frac{n^{2} x^{2}}{2 a^{2}}+\frac{x^{4} n^{4}}{24 a^{4}}-\frac{x^{4} n^{2}}{6 a^{4}}+\frac{x^{6} n^{6}}{720 a^{6}}-\frac{x^{6} n^{4}}{36 a^{6}}+\frac{4 x^{6} n^{2}}{45 a^{6}}\right) y(0) \\
&+\left(x+\frac{x^{3} n^{2}}{6 a^{2}}-\frac{x^{3}}{6 a^{2}}+\frac{x^{5} n^{4}}{120 a^{4}}-\frac{x^{5} n^{2}}{12 a^{4}}+\frac{3 x^{5}}{40 a^{4}}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(a^{2}+x^{2}\right) y''+x y'-y n^{2}=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &=\sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &=\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(a^{2}+x^{2}\right)\left(\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}\right)+x\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)-\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) n^{2}=0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} n\, x^{n-2} a^{2} a_{n}(n-1)\right)+\left(\sum_{n=1}^{\infty} n a_{n} x^{n}\right)+\sum_{n=0}^{\infty}\left(-n^{2} a_{n} x^{n}\right)=0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} n\, x^{n-2} a^{2} a_{n}(n-1)=\sum_{n=0}^{\infty}(n+2) a_{n+2} a^{2}(n+1) x^{n}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=0}^{\infty}(n+2) a_{n+2} a^2(n+1) x^n\right) \\ +\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-n^2 a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$2 a_2 a^2-a_0 n^2=0$$

$$a_2=\frac{a_0 n^2}{2 a^2}$$

$n=1$ gives

$$6 a_3 a^2-a_1 n^2+a_1=0$$

Which after substituting earlier equations, simplifies to

$$a_3=\frac{a_1\left(n^2-1\right)}{6 a^2}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1)+(n+2) a_{n+2} a^2(n+1)+n a_n-a_n n^2=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2}=\frac{a_n\left(n^2-n^2\right)}{(n+2) a^2(n+1)} \tag{5}$$

For $n=2$ the recurrence equation gives

$$12 a_4 a^2-a_2 n^2+4 a_2=0$$

Which after substituting the earlier terms found becomes

$$a_4=\frac{a_0 n^2\left(n^2-4\right)}{24 a^4}$$

For $n = 3$ the recurrence equation gives

$$20a_5a^2 - a_3n^2 + 9a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1(n^2-1)(n^2-9)}{120a^4}$$

For $n = 4$ the recurrence equation gives

$$30a_6a^2 - a_4n^2 + 16a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0n^2(n^2-4)(n^2-16)}{720a^6}$$

For $n = 5$ the recurrence equation gives

$$42a_7a^2 - a_5n^2 + 25a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1(n^2-1)(n^2-9)(n^2-25)}{5040a^6}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \frac{a_0n^2x^2}{2a^2} + \frac{a_1(n^2-1)x^3}{6a^2} + \frac{a_0n^2(n^2-4)x^4}{24a^4} + \frac{a_1(n^2-1)(n^2-9)x^5}{120a^4} + \dots$$

Collecting terms, the solution becomes

$$\begin{aligned} y = {}& \left(1 + \frac{n^2x^2}{2a^2} + \frac{n^2(n^2-4)x^4}{24a^4}\right)a_0 \\ &+ \left(x + \frac{(n^2-1)x^3}{6a^2} + \frac{(n^2-1)(n^2-9)x^5}{120a^4}\right)a_1 + O(x^6) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1+\frac{n^{2} x^{2}}{2 a^{2}}+\frac{n^{2} \left(n^{2}-4\right) x^{4}}{24 a^{4}}\right) c_{1}+\left(x+\frac{\left(n^{2}-1\right) x^{3}}{6 a^{2}}+\frac{\left(n^{2}-1\right) \left(n^{2}-9\right) x^{5}}{120 a^{4}}\right) c_{2}+O\left(x^{6}\right)$$

Summary
The solution(s) found are the following

$$\begin{aligned} y = &\left(1+\frac{n^{2} x^{2}}{2 a^{2}}+\frac{x^{4} n^{4}}{24 a^{4}}-\frac{x^{4} n^{2}}{6 a^{4}}+\frac{x^{6} n^{6}}{720 a^{6}}-\frac{x^{6} n^{4}}{36 a^{6}}+\frac{4 x^{6} n^{2}}{45 a^{6}}\right) y(0) \\ &+\left(x+\frac{x^{3} n^{2}}{6 a^{2}}-\frac{x^{3}}{6 a^{2}}+\frac{x^{5} n^{4}}{120 a^{4}}-\frac{x^{5} n^{2}}{12 a^{4}}+\frac{3 x^{5}}{40 a^{4}}\right) y'(0)+O\left(x^{6}\right) \end{aligned} \tag{1}$$

$$\begin{aligned} y = &\left(1+\frac{n^{2} x^{2}}{2 a^{2}}+\frac{n^{2} \left(n^{2}-4\right) x^{4}}{24 a^{4}}\right) c_{1} \\ &+\left(x+\frac{\left(n^{2}-1\right) x^{3}}{6 a^{2}}+\frac{\left(n^{2}-1\right) \left(n^{2}-9\right) x^{5}}{120 a^{4}}\right) c_{2}+O\left(x^{6}\right) \end{aligned} \tag{2}$$

Verification of solutions

$$\begin{aligned} y = &\left(1+\frac{n^{2} x^{2}}{2 a^{2}}+\frac{x^{4} n^{4}}{24 a^{4}}-\frac{x^{4} n^{2}}{6 a^{4}}+\frac{x^{6} n^{6}}{720 a^{6}}-\frac{x^{6} n^{4}}{36 a^{6}}+\frac{4 x^{6} n^{2}}{45 a^{6}}\right) y(0) \\ &+\left(x+\frac{x^{3} n^{2}}{6 a^{2}}-\frac{x^{3}}{6 a^{2}}+\frac{x^{5} n^{4}}{120 a^{4}}-\frac{x^{5} n^{2}}{12 a^{4}}+\frac{3 x^{5}}{40 a^{4}}\right) y'(0)+O\left(x^{6}\right) \end{aligned}$$

Verified OK.

$$y = \left(1+\frac{n^{2} x^{2}}{2 a^{2}}+\frac{n^{2} \left(n^{2}-4\right) x^{4}}{24 a^{4}}\right) c_{1}+\left(x+\frac{\left(n^{2}-1\right) x^{3}}{6 a^{2}}+\frac{\left(n^{2}-1\right) \left(n^{2}-9\right) x^{5}}{120 a^{4}}\right) c_{2}+O\left(x^{6}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 72

```
Order:=6;
dsolve((a^2+x^2)*diff(y(x),x$2)+x*diff(y(x),x)-n^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{n^2x^2}{2a^2}+\frac{n^2(n^2-4)\,x^4}{24a^4}\right)y(0) + \left(x+\frac{(n^2-1)\,x^3}{6a^2}+\frac{(n^4-10n^2+9)\,x^5}{120a^4}\right)D(y)\,(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 112

```
AsymptoticDSolveValue[(a^2+x^2)*y''[x]+x*y'[x]-n^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{n^4x^5}{120a^4}-\frac{n^2x^5}{12a^4}+\frac{3x^5}{40a^4}+\frac{n^2x^3}{6a^2}-\frac{x^3}{6a^2}+x\right)+c_1\left(\frac{n^4x^4}{24a^4}-\frac{n^2x^4}{6a^4}+\frac{n^2x^2}{2a^2}+1\right)$$

3.15 problem 18

Internal problem ID [4710]
Internal file name [OUTPUT/4203_Sunday_June_05_2022_12_40_46_PM_57402417/index.tex]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XIV. page 177
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Gegenbauer, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{\left(-x^2+1\right)y''-xy'+ya^2=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f\big|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'\big|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (70) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (71) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{ya^2 - xy'}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(\left(a^2+2\right) x^2 - a^2 + 1\right) y' - 3ya^2 x}{\left(x^2-1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\left(-6a^2x^3 + 6a^2x - 6x^3 - 9x\right) y' + y\left(\left(a^2+11\right) x^2 - a^2 + 4\right) a^2}{\left(x^2-1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(\left(a^4 + 35a^2 + 24\right) x^4 + \left(-2a^4 - 25a^2 + 72\right) x^2 + a^4 - 10a^2 + 9\right) y' - 10y\left(\left(a^2+5\right) x^2 - a^2 + \frac{11}{2}\right) x\, a^2}{\left(x^2-1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(x+1)\left(-15x\left(\left(a^4 + 15a^2 + 8\right) x^4 + \left(-2a^4 - 2a^2 + 40\right) x^2 + a^4 - 13a^2 + 15\right) y' + y\left(\left(a^4 + 85a^2 + 27\right.\right.\right.}{\left(x^2-1\right)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)\, a^2 \\
F_1 &= -y'(0)\, a^2 + y'(0) \\
F_2 &= y(0)\, a^4 - 4y(0)\, a^2 \\
F_3 &= y'(0)\, a^4 - 10y'(0)\, a^2 + 9y'(0) \\
F_4 &= -y(0)\, a^6 + 20y(0)\, a^4 - 64y(0)\, a^2
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{2}a^2x^2 + \frac{1}{24}a^4x^4 - \frac{1}{6}a^2x^4 - \frac{1}{720}x^6a^6 + \frac{1}{36}x^6a^4 - \frac{4}{45}a^2x^6\right) y(0) + \left(x - \frac{1}{6}a^2x^3 + \frac{1}{6}x^3 + \frac{1}{120}a^4x^5 - \frac{1}{12}a^2x^5 + \frac{3}{40}x^5\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-x^2 + 1)\, y'' - xy' + ya^2 = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-x^2 + 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) a^2 = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} a^2 a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^{n}\right) + \sum_{n=1}^{\infty} \left(-n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} a^{2} a_{n} x^{n}\right)=0 \tag{3}$$

$n=0$ gives

$$a_{0} a^{2}+2 a_{2}=0$$

$$a_{2}=-\frac{a_{0} a^{2}}{2}$$

$n=1$ gives

$$a_{1} a^{2}-a_{1}+6 a_{3}=0$$

Which after substituting earlier equations, simplifies to

$$a_{3}=-\frac{1}{6} a_{1} a^{2}+\frac{1}{6} a_{1}$$

For $2 \leq n$, the recurrence equation is

$$-n a_{n}(n-1)+(n+2) a_{n+2}(n+1)-n a_{n}+a_{n} a^{2}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2}=-\frac{a_{n}\left(a^{2}-n^{2}\right)}{(n+2)(n+1)} \tag{5}$$

For $n=2$ the recurrence equation gives

$$a_{2} a^{2}-4 a_{2}+12 a_{4}=0$$

Which after substituting the earlier terms found becomes

$$a_{4}=\frac{1}{24} a^{4} a_{0}-\frac{1}{6} a_{0} a^{2}$$

For $n = 3$ the recurrence equation gives

$$a_3 a^2 - 9a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{120} a^4 a_1 - \frac{1}{12} a_1 a^2 + \frac{3}{40} a_1$$

For $n = 4$ the recurrence equation gives

$$a_4 a^2 - 16a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1}{720} a^6 a_0 + \frac{1}{36} a^4 a_0 - \frac{4}{45} a_0 a^2$$

For $n = 5$ the recurrence equation gives

$$a_5 a^2 - 25a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1}{5040} a^6 a_1 + \frac{1}{144} a^4 a_1 - \frac{37}{720} a_1 a^2 + \frac{5}{112} a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x - \frac{a_0 a^2 x^2}{2} + \left(-\frac{1}{6} a_1 a^2 + \frac{1}{6} a_1\right) x^3 \\ + \left(\frac{1}{24} a^4 a_0 - \frac{1}{6} a_0 a^2\right) x^4 + \left(\frac{1}{120} a^4 a_1 - \frac{1}{12} a_1 a^2 + \frac{3}{40} a_1\right) x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{a^2x^2}{2} + \left(\frac{1}{24}a^4 - \frac{1}{6}a^2\right)x^4\right)a_0 + \left(x + \left(-\frac{a^2}{6} + \frac{1}{6}\right)x^3 + \left(\frac{1}{120}a^4 - \frac{1}{12}a^2 + \frac{3}{40}\right)x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{a^2x^2}{2} + \left(\frac{1}{24}a^4 - \frac{1}{6}a^2\right)x^4\right)c_1 + \left(x + \left(-\frac{a^2}{6} + \frac{1}{6}\right)x^3 + \left(\frac{1}{120}a^4 - \frac{1}{12}a^2 + \frac{3}{40}\right)x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}a^2x^2 + \frac{1}{24}a^4x^4 - \frac{1}{6}a^2x^4 - \frac{1}{720}x^6a^6 + \frac{1}{36}x^6a^4 - \frac{4}{45}a^2x^6\right)y(0) + \left(x - \frac{1}{6}a^2x^3 + \frac{1}{6}x^3 + \frac{1}{120}a^4x^5 - \frac{1}{12}a^2x^5 + \frac{3}{40}x^5\right)y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{a^2x^2}{2} + \left(\frac{1}{24}a^4 - \frac{1}{6}a^2\right)x^4\right)c_1 + \left(x + \left(-\frac{a^2}{6} + \frac{1}{6}\right)x^3 + \left(\frac{1}{120}a^4 - \frac{1}{12}a^2 + \frac{3}{40}\right)x^5\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}a^2x^2 + \frac{1}{24}a^4x^4 - \frac{1}{6}a^2x^4 - \frac{1}{720}x^6a^6 + \frac{1}{36}x^6a^4 - \frac{4}{45}a^2x^6\right)y(0) + \left(x - \frac{1}{6}a^2x^3 + \frac{1}{6}x^3 + \frac{1}{120}a^4x^5 - \frac{1}{12}a^2x^5 + \frac{3}{40}x^5\right)y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{a^2x^2}{2} + \left(\frac{1}{24}a^4 - \frac{1}{6}a^2\right)x^4\right)c_1 + \left(x + \left(-\frac{a^2}{6} + \frac{1}{6}\right)x^3 + \left(\frac{1}{120}a^4 - \frac{1}{12}a^2 + \frac{3}{40}\right)x^5\right)c_2 + O(x^6)$$

Verified OK.

3.15.1 Maple step by step solution

Let's solve

$\left(-x^2+1\right) y''-x y'+y a^2=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{x y'}{x^2-1}+\frac{a^2 y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{x y'}{x^2-1}-\frac{a^2 y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{x}{x^2-1}, P_3(x)=-\frac{a^2}{x^2-1}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{1}{2}$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$y''(x^2-1)+x y'-y a^2=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)-a^2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (-a_{1+k}(1+k+r)\,(1+2k+2r) - a_k(a+k+r)\,(a-k-r))\, u^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2\left(\frac{1}{2}+k+r\right)(1+k+r)\, a_{1+k} - a_k(a+k+r)\,(a-k-r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{1+k} = -\frac{a_k(a+k+r)(a-k-r)}{(1+2k+2r)(1+k+r)}$

- Recursion relation for $r = 0$

  $a_{1+k} = -\frac{a_k(a+k)(a-k)}{(1+2k)(1+k)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{1+k} = -\frac{a_k(a+k)(a-k)}{(1+2k)(1+k)}\right]$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{1+k} = -\frac{a_k(a+k)(a-k)}{(1+2k)(1+k)}\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{1+k}=-\frac{a_k\left(a+k+\frac{1}{2}\right)\left(a-k-\frac{1}{2}\right)}{(2+2k)\left(\frac{3}{2}+k\right)}$

- Solution for $r=\frac{1}{2}$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{1+k}=-\frac{a_k\left(a+k+\frac{1}{2}\right)\left(a-k-\frac{1}{2}\right)}{(2+2k)\left(\frac{3}{2}+k\right)}\right]$

- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{1}{2}}, a_{1+k}=-\frac{a_k\left(a+k+\frac{1}{2}\right)\left(a-k-\frac{1}{2}\right)}{(2+2k)\left(\frac{3}{2}+k\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} b_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty} c_k(x+1)^{k+\frac{1}{2}}\right), b_{k+1}=-\frac{b_k(a+k)(a-k)}{(1+2k)(k+1)}, c_{k+1}=-\frac{c_k\left(a+k+\frac{1}{2}\right)\left(a-k-\frac{1}{2}\right)}{(2+2k)\left(\frac{3}{2}+k\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 71

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-x*diff(y(x),x)+a^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{x^2a^2}{2}+\frac{a^2(a^2-4)\,x^4}{24}\right)y(0)+\left(x-\frac{(a^2-1)\,x^3}{6}+\frac{(a^4-10a^2+9)\,x^5}{120}\right)D(y)\,(0)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 88

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-x*y'[x]+a^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{a^4x^5}{120} - \frac{a^2x^5}{12} - \frac{a^2x^3}{6} + \frac{3x^5}{40} + \frac{x^3}{6} + x\right) + c_1\left(\frac{a^4x^4}{24} - \frac{a^2x^4}{6} - \frac{a^2x^2}{2} + 1\right)$$

4 Chapter VII, Solutions in series. Examples XV. page 194

4.1 problem 1

Internal problem ID [4711]
Internal file name [`OUTPUT/4204_Sunday_June_05_2022_12_40_52_PM_93659287/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{xy'' + y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{1}{x}$$

Table 57: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r) (n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad (2A)$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{(1+r)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)^2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)^2}$	-1
a_2	$\frac{1}{(1+r)^2(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(1+r)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)^2}$	-1
a_2	$\frac{1}{(1+r)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$-\frac{1}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)^2 (r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)^2}$	-1
a_2	$\frac{1}{(1+r)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$-\frac{1}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$
a_4	$\frac{1}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(1+r)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{14400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)^2}$	-1
a_2	$\frac{1}{(1+r)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$-\frac{1}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$
a_4	$\frac{1}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{576}$
a_5	$-\frac{1}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$-\frac{1}{14400}$

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{1}{(1+r)^2}$	-1	$\frac{2}{(1+r)^3}$	2
b_2	$\frac{1}{(1+r)^2 (r+2)^2}$	$\frac{1}{4}$	$\frac{-4r-6}{(1+r)^3 (r+2)^3}$	$-\frac{3}{4}$
b_3	$-\frac{1}{(1+r)^2 (r+2)^2 (r+3)^2}$	$-\frac{1}{36}$	$\frac{6r^2+24r+22}{(1+r)^3 (r+2)^3 (r+3)^3}$	$\frac{11}{108}$
b_4	$\frac{1}{(1+r)^2 (r+2)^2 (r+3)^2 (r+4)^2}$	$\frac{1}{576}$	$\frac{-8r^3-60r^2-140r-100}{(1+r)^3 (r+2)^3 (r+3)^3 (r+4)^3}$	$-\frac{25}{3456}$
b_5	$-\frac{1}{(1+r)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$	$-\frac{1}{14400}$	$\frac{10r^4+120r^3+510r^2+900r+548}{(1+r)^3 (r+2)^3 (r+3)^3 (r+4)^3 (r+5)^3}$	$\frac{137}{432000}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= \left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \ln(x) \\ &\quad + 2x - \frac{3x^2}{4} + \frac{11x^3}{108} - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \\
&\quad + c_2 \left(\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \ln(x) + 2x - \frac{3x^2}{4} + \frac{11x^3}{108} \right. \\
&\quad \left. - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \\
&\quad + c_2 \left(\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \ln(x) + 2x - \frac{3x^2}{4} + \frac{11x^3}{108} \right. \\
&\quad \left. - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \\
&\quad + c_2 \left(\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \ln(x) + 2x - \frac{3x^2}{4} + \frac{11x^3}{108} \right. \\
&\quad \left. - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \\
&\quad + c_2 \left(\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \ln(x) + 2x - \frac{3x^2}{4} + \frac{11x^3}{108} \right. \\
&\quad \left. - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

4.1.1 Maple step by step solution

Let's solve

$y''x + y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- ○ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- ○ Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{(k+1)^2}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{(k+1)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & (c_2 \ln(x) + c_1)\left(1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \frac{1}{576}x^4 - \frac{1}{14400}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + \left(2x - \frac{3}{4}x^2 + \frac{11}{108}x^3 - \frac{25}{3456}x^4 + \frac{137}{432000}x^5 + \mathrm{O}\left(x^6\right)\right) c_2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 111

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to {} & c_1\left(-\frac{x^5}{14400} + \frac{x^4}{576} - \frac{x^3}{36} + \frac{x^2}{4} - x + 1\right) + c_2\left(\frac{137x^5}{432000} - \frac{25x^4}{3456} + \frac{11x^3}{108} - \frac{3x^2}{4}\right. \\
& \left. + \left(-\frac{x^5}{14400} + \frac{x^4}{576} - \frac{x^3}{36} + \frac{x^2}{4} - x + 1\right)\log(x) + 2x\right)
\end{aligned}
$$

4.2 problem 2

Internal problem ID [4712]
Internal file name [`OUTPUT/4205_Sunday_June_05_2022_12_41_00_PM_56004095/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + y' + pxy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + pxy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = p$$

Table 59: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = p$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + pxy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + px\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} p\, x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} p\, x^{1+n+r} a_n = \sum_{n=2}^{\infty} p a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} p a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_n(n + r) + p a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{p a_{n-2}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{p a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{p}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{p}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{p}{(r+2)^2}$	$-\frac{p}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{p}{(r+2)^2}$	$-\frac{p}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{p^2}{(r+2)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{p^2}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{p}{(r+2)^2}$	$-\frac{p}{4}$
a_3	0	0
a_4	$\frac{p^2}{(r+2)^2(r+4)^2}$	$\frac{p^2}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{p}{(r+2)^2}$	$-\frac{p}{4}$
a_3	0	0
a_4	$\frac{p^2}{(r+2)^2(r+4)^2}$	$\frac{p^2}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \ldots \\ &= 1 - \frac{p\,x^2}{4} + \frac{p^2x^4}{64} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{p}{(r+2)^2}$	$-\frac{p}{4}$	$\frac{2p}{(r+2)^3}$	$\frac{p}{4}$
b_3	0	0	0	0
b_4	$\frac{p^2}{(r+2)^2(r+4)^2}$	$\frac{p^2}{64}$	$-\frac{4p^2(3+r)}{(r+2)^3(r+4)^3}$	$-\frac{3p^2}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 - \frac{p\,x^2}{4} + \frac{p^2 x^4}{64} + O(x^6)\right)\ln(x) + \frac{p\,x^2}{4} - \frac{3p^2 x^4}{128} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - \frac{p\,x^2}{4} + \frac{p^2 x^4}{64} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 - \frac{p\,x^2}{4} + \frac{p^2 x^4}{64} + O(x^6)\right)\ln(x) + \frac{p\,x^2}{4} - \frac{3p^2 x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 - \frac{p\,x^2}{4} + \frac{p^2 x^4}{64} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 - \frac{p\,x^2}{4} + \frac{p^2 x^4}{64} + O(x^6)\right)\ln(x) + \frac{p\,x^2}{4} - \frac{3p^2 x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(1-\frac{p\,x^2}{4}+\frac{p^2 x^4}{64}+O\left(x^6\right)\right) \\
&+ c_2 \left(\left(1-\frac{p\,x^2}{4}+\frac{p^2 x^4}{64}+O\left(x^6\right)\right)\ln\left(x\right)+\frac{p\,x^2}{4}-\frac{3p^2 x^4}{128}+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \left(1-\frac{p\,x^2}{4}+\frac{p^2 x^4}{64}+O\left(x^6\right)\right) \\
&+ c_2 \left(\left(1-\frac{p\,x^2}{4}+\frac{p^2 x^4}{64}+O\left(x^6\right)\right)\ln\left(x\right)+\frac{p\,x^2}{4}-\frac{3p^2 x^4}{128}+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

4.2.1 Maple step by step solution

Let's solve

$y''x+y'+pxy=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y'}{x}-py$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}+py=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{1}{x}, P_3(x)=p\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

- ○ $x=0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$y''x + y' + pxy = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k- >k - 1$

$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert y' to series expansion

$y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

$y' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\,(k + r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r^2 x^{-1+r} + a_1 (1 + r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1} (k + r + 1)^2 + p a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

  $a_1(1+r)^2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + pa_{k-1} = 0$

- Shift index using $k- > k+1$

  $a_{k+2}(k+2)^2 + pa_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{pa_k}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{pa_k}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{pa_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+p*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - \frac{1}{4}p\,x^2 + \frac{1}{64}p^2x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{p}{4}x^2 - \frac{3}{128}p^2x^4 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 72

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+p*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1\left(\frac{p^2x^4}{64} - \frac{px^2}{4} + 1\right) + c_2\left(-\frac{3}{128}p^2x^4 + \left(\frac{p^2x^4}{64} - \frac{px^2}{4} + 1\right)\log(x) + \frac{px^2}{4}\right)$$

4.3 problem 3

Internal problem ID [4713]
Internal file name [`OUTPUT/4206_Sunday_June_05_2022_12_41_08_PM_96597171/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{1}{x}$$

Table 61: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}}{(n+1)n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{(1 + r) r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1 + r)^2 r (2 + r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(1 + r)^2 r (2 + r)^2 (3 + r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{1}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{1}{2880}$
a_5	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$	$-\frac{1}{86400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= -\frac{1}{(1+r)\, r}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} -\frac{1}{(1+r)\, r} &= \lim_{r \to 0} -\frac{1}{(1+r)\, r} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$
$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$
$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$
$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $xy'' + y = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left((y_1''(x)\,x + y_1(x))\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x + y_1(x) = 0$$

Eq (7) simplifes to

$$\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)xC + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^n a_n (n+1)\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} + \left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1)\right) + \sum_{n=0}^{\infty} \left(-C\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-C\, x^n a_n\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=1}^{\infty} 2 C a_{n-1} n\, x^{n-1}\right)+\sum_{n=1}^{\infty}\left(-C a_{n-1} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right)+\left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C+1=0$$

Which is solved for C. Solving for C gives

$$C=-1$$

For $n=2$, Eq (2B) gives

$$3 C a_1+b_1+2 b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 b_2+\frac{3}{2}=0$$

Solving the above for b_2 gives

$$b_2=-\frac{3}{4}$$

For $n=3$, Eq (2B) gives

$$5 C a_2+b_2+6 b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6 b_3-\frac{7}{6}=0$$

Solving the above for b_3 gives

$$b_3=\frac{7}{36}$$

For $n=4$, Eq (2B) gives

$$7 C a_3+b_3+12 b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12 b_4+\frac{35}{144}=0$$

Solving the above for b_4 gives

$$b_4 = -\frac{35}{1728}$$

For $n = 5$, Eq (2B) gives

$$9Ca_4 + b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$20b_5 - \frac{101}{4320} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{101}{86400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = (-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\right)\ln(x) \\ + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right) \\ &\quad + c_2\left((-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\right)\ln(x) + 1 \right. \\ &\quad \left. - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right) \\ &\quad + c_2\left(-x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\ln(x) + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} \right. \\ &\quad \left. - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right) \\
& + c_2 \left(-x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right) \ln(x)+1-\frac{3x^2}{4} \right. \\
& \left. +\frac{7x^3}{36}-\frac{35x^4}{1728}+\frac{101x^5}{86400}+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right) \\
& + c_2 \left(-x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right) \ln(x)+1-\frac{3x^2}{4}+\frac{7x^3}{36} \right. \\
& \left. -\frac{35x^4}{1728}+\frac{101x^5}{86400}+O(x^6)\right)
\end{aligned}
$$

Verified OK.

4.3.1 Maple step by step solution

Let's solve

$y''x+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=0, P_3(x)=\frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0}=0$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x=0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

  - Shift index using $k->k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r) + a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r=0$

  $a_{k+1} = -\frac{a_k}{(k+1)k}$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1)k}\right]$$

- Recursion relation for $r = 1$

  $$a_{k+1} = -\frac{a_k}{(k+2)(k+1)}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{(k+2)(k+1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{a_k}{(k+1)k}, b_{k+1} = -\frac{b_k}{(k+2)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 58

```
Order:=6;
dsolve(x*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{144}x^3 + \frac{1}{2880}x^4 - \frac{1}{86400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(-x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{144}x^4 - \frac{1}{2880}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \frac{101}{86400}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1\left(\frac{1}{144}x\left(x^3 - 12x^2 + 72x - 144\right)\log(x) + \frac{-47x^4 + 480x^3 - 2160x^2 + 1728x + 1728}{1728}\right) + c_2\left(\frac{x^5}{2880} - \frac{x^4}{144} + \frac{x^3}{12} - \frac{x^2}{2} + x\right)$$

4.4 problem 4

Internal problem ID [4714]
Internal file name [`OUTPUT/4207_Sunday_June_05_2022_12_41_17_PM_88905230/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__bessel__ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' - (2x-1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^3y'' + (1-2x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = -\frac{2x-1}{x^3}$$

Table 63: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{2x-1}{x^3}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

4.4.1 Maple step by step solution

Let's solve

$y''x^3 + (1 - 2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2x-1)y}{x^3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y}{x^3} = 0$

- Multiply by denominators of the ODE

  $y''x^3 - (2x - 1)\, y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

- ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^3 - (2x-1)\, y(t) = 0$

- Simplify

  $x\left(\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t)\right) + (1-2x)\, y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{(2x-1)y(t)}{x} + \frac{d}{dt}y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - \frac{(2x-1)y(t)}{x} = 0$

- Characteristic polynomial of ODE

  $r^2 - r - \frac{2x-1}{x} = 0$

- Factor the characteristic polynomial

  $\frac{r^2x - rx - 2x + 1}{x} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{x+\sqrt{9x^2-4x}}{2x}, -\frac{-x+\sqrt{9x^2-4x}}{2x}\right)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{\frac{\left(x+\sqrt{9x^2-4x}\right)t}{2x}}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-\frac{\left(-x+\sqrt{9x^2-4x}\right)t}{2x}}$

- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$

- Substitute in solutions

$$y(t) = c_1 \mathrm{e}^{\frac{\left(x+\sqrt{9x^2-4x}\right)t}{2x}} + c_2 \mathrm{e}^{-\frac{\left(-x+\sqrt{9x^2-4x}\right)t}{2x}}$$

- Change variables back using $t = \ln(x)$

$$y = c_1 \mathrm{e}^{\frac{\left(x+\sqrt{9x^2-4x}\right)\ln(x)}{2x}} + c_2 \mathrm{e}^{-\frac{\left(-x+\sqrt{9x^2-4x}\right)\ln(x)}{2x}}$$

- Simplify

$$y = c_1 x^{\frac{x+\sqrt{9x^2-4x}}{2x}} + c_2 x^{-\frac{-x+\sqrt{9x^2-4x}}{2x}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^3*diff(y(x),x$2)-(2*x-1)*y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 222

```
AsymptoticDSolveValue[x^3*y''[x]-(2*x-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 e^{-\frac{2i}{\sqrt{x}}} x^{3/4} \left(-\frac{1159525191825 i x^{9/2}}{8796093022208} + \frac{218243025 i x^{7/2}}{4294967296} - \frac{405405 i x^{5/2}}{8388608} + \frac{3465 i x^{3/2}}{8192} + \frac{75369137468625 x^5}{281474976710656} - \frac{41247931725 x^4}{549755813888} + \frac{11486475 x^3}{268435456} - \frac{45045 x^2}{524288} - \frac{945 x}{512} - \frac{35 i \sqrt{x}}{16} + 1 \right) + c_2 e^{\frac{2i}{\sqrt{x}}} x^{3/4} \left(\frac{1159525191825 i x^{9/2}}{8796093022208} - \frac{218243025 i x^{7/2}}{4294967296} + \frac{405405 i x^{5/2}}{8388608} - \frac{3465 i x^{3/2}}{8192} + \frac{75369137468625}{28147497671065} \right.$$

4.5 problem 5

Internal problem ID [4715]
Internal file name [OUTPUT/4208_Sunday_June_05_2022_12_41_22_PM_94141571/index.tex]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2 y'' + x(x+1)\, y' + (3x-1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2 y'' + \left(x^2 + x\right) y' + (3x-1)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x+1}{x}$$
$$q(x) = \frac{3x-1}{x^2}$$

Table 65: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3x-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + \left(x^2 + x\right) y' + (3x - 1) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(x^2 + x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (3x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_n(n+r) + 3a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r+2)}{n^2+2nr+r^2-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}(n+3)}{n\,(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-3-r}{r\,(r+2)}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{4}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4+r}{(r+1)\,r\,(r+2)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{5}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$
a_2	$\frac{4+r}{(r+1)r(r+2)}$	$\frac{5}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-5-r}{(r+2)^2 (r+1) r}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$
a_2	$\frac{4+r}{(r+1)r(r+2)}$	$\frac{5}{6}$
a_3	$\frac{-5-r}{(r+2)^2(r+1)r}$	$-\frac{1}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{6+r}{(3+r)(r+2)^2 (r+1) r}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{7}{72}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$
a_2	$\frac{4+r}{(r+1)r(r+2)}$	$\frac{5}{6}$
a_3	$\frac{-5-r}{(r+2)^2(r+1)r}$	$-\frac{1}{3}$
a_4	$\frac{6+r}{(3+r)(r+2)^2(r+1)r}$	$\frac{7}{72}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-7-r}{(4+r)(3+r)(r+2)^2 (r+1) r}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-r}{r(r+2)}$	$-\frac{4}{3}$
a_2	$\frac{4+r}{(r+1)r(r+2)}$	$\frac{5}{6}$
a_3	$\frac{-5-r}{(r+2)^2(r+1)r}$	$-\frac{1}{3}$
a_4	$\frac{6+r}{(3+r)(r+2)^2(r+1)r}$	$\frac{7}{72}$
a_5	$\frac{-7-r}{(4+r)(3+r)(r+2)^2(r+1)r}$	$-\frac{1}{45}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{4x}{3} + \frac{5x^2}{6} - \frac{x^3}{3} + \frac{7x^4}{72} - \frac{x^5}{45} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{4+r}{(r+1)\, r\, (r+2)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{4+r}{(r+1)\, r\, (r+2)} &= \lim_{r \to -1} \frac{4+r}{(r+1)\, r\, (r+2)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2) (-1 + n + r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2 y'' + (x^2 + x) y' + (3x - 1) y = 0$ gives

$$\begin{aligned}
& x^2 \left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \right. \\
& \left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2}\right)\right) \\
& + (x^2 + x) \left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x}\right)\right) \\
& + (3x - 1) \left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(x^2 y_1''(x) + (x^2+x)\, y_1'(x) + (3x-1)\, y_1(x)\right) \ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(x^2+x)\, y_1(x)}{x}\right) C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (x^2+x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (3x-1)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) + (x^2+x)\, y_1'(x) + (3x-1)\, y_1(x) = 0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(x^2+x)\, y_1(x)}{x}\right) C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (x^2+x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (3x-1)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right) x\right) C + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + (x^2+x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) + (3x-1)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = -1$ then the above becomes

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^n a_n(1+n)\right) x + \left(\sum_{n=0}^{\infty} a_n x^{1+n}\right) x\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-3+n} b_n(n-1)(-2+n)\right) x^2 \\
&+ (x^2+x)\left(\sum_{n=0}^{\infty} x^{-2+n} b_n(n-1)\right) + (3x-1)\left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n(1+n)\right) + \left(\sum_{n=0}^{\infty} C\, x^{n+2} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n(n^2-3n+2)\right) + \left(\sum_{n=0}^{\infty} x^n b_n(n-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n(n-1)\right) + \left(\sum_{n=0}^{\infty} 3b_n x^n\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n(1+n) = \sum_{n=2}^{\infty} 2C a_{-2+n}(n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} C\, x^{n+2} a_n = \sum_{n=3}^{\infty} C a_{-3+n} x^{n-1}$$

$$\sum_{n=0}^{\infty} x^n b_n(n-1) = \sum_{n=1}^{\infty} b_{n-1}(-2+n)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} 3b_n x^n = \sum_{n=1}^{\infty} 3b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=2}^{\infty} 2Ca_{-2+n}(n-1)\, x^{n-1}\right) + \left(\sum_{n=3}^{\infty} Ca_{-3+n}x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{n-1}b_n\left(n^2-3n+2\right)\right) + \left(\sum_{n=1}^{\infty} b_{n-1}(-2+n)\, x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{n-1}b_n(n-1)\right) + \left(\sum_{n=1}^{\infty} 3b_{n-1}x^{n-1}\right) + \sum_{n=0}^{\infty}\left(-b_n x^{n-1}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$2b_0 - b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 - b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 2$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C + 6 = 0$$

Which is solved for C. Solving for C gives

$$C = -3$$

For $n=3$, Eq (2B) gives

$$(a_0 + 4a_1)\, C + 4b_2 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$13 + 3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{13}{3}$$

For $n = 4$, Eq (2B) gives

$$(a_1 + 6a_2)\,C + 5b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{98}{3} + 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{49}{12}$$

For $n = 5$, Eq (2B) gives

$$(a_2 + 8a_3)\,C + 6b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$30 + 15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -2$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -3$ and all b_n, then the second solution becomes

$$y_2(x) = (-3)\left(x\left(1 - \frac{4x}{3} + \frac{5x^2}{6} - \frac{x^3}{3} + \frac{7x^4}{72} - \frac{x^5}{45} + O(x^6)\right)\right)\ln(x) + \frac{1 + 2x - \frac{13x^3}{3} + \frac{49x^4}{12} - 2x^5 + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{4x}{3} + \frac{5x^2}{6} - \frac{x^3}{3} + \frac{7x^4}{72} - \frac{x^5}{45} + O(x^6)\right) \\ &\quad + c_2\left((-3)\left(x\left(1 - \frac{4x}{3} + \frac{5x^2}{6} - \frac{x^3}{3} + \frac{7x^4}{72} - \frac{x^5}{45} + O(x^6)\right)\right)\ln(x) \right. \\ &\quad \left. + \frac{1 + 2x - \frac{13x^3}{3} + \frac{49x^4}{12} - 2x^5 + O(x^6)}{x}\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 - \frac{4x}{3} + \frac{5x^2}{6} - \frac{x^3}{3} + \frac{7x^4}{72} - \frac{x^5}{45} + O(x^6)\right) \\
&\quad + c_2 \left(-3x \left(1 - \frac{4x}{3} + \frac{5x^2}{6} - \frac{x^3}{3} + \frac{7x^4}{72} - \frac{x^5}{45} + O(x^6)\right) \ln(x) \right. \\
&\quad \left. + \frac{1 + 2x - \frac{13x^3}{3} + \frac{49x^4}{12} - 2x^5 + O(x^6)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1 - \frac{4x}{3} + \frac{5x^2}{6} - \frac{x^3}{3} + \frac{7x^4}{72} - \frac{x^5}{45} + O(x^6)\right) \\
&\quad + c_2 \left(-3x \left(1 - \frac{4x}{3} + \frac{5x^2}{6} - \frac{x^3}{3} + \frac{7x^4}{72} - \frac{x^5}{45} + O(x^6)\right) \ln(x) \right. \\
&\quad \left. + \frac{1 + 2x - \frac{13x^3}{3} + \frac{49x^4}{12} - 2x^5 + O(x^6)}{x}\right) \qquad (1)
\end{aligned}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1 - \frac{4x}{3} + \frac{5x^2}{6} - \frac{x^3}{3} + \frac{7x^4}{72} - \frac{x^5}{45} + O(x^6)\right) \\
&\quad + c_2 \left(-3x \left(1 - \frac{4x}{3} + \frac{5x^2}{6} - \frac{x^3}{3} + \frac{7x^4}{72} - \frac{x^5}{45} + O(x^6)\right) \ln(x) \right. \\
&\quad \left. + \frac{1 + 2x - \frac{13x^3}{3} + \frac{49x^4}{12} - 2x^5 + O(x^6)}{x}\right)
\end{aligned}
$$

Verified OK.

4.5.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 + x)\, y' + (3x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x-1)y}{x^2} - \frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+1)y'}{x} + \frac{(3x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x+1}{x}, P_3(x) = \frac{3x-1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x + 1)\, y' + (3x - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (1+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+1)\,(k+r-1) + a_{k-1}(k+2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k (k+r+1)\,(k+r-1) + a_{k-1}(k+2+r) = 0$$

- Shift index using $k- >k+1$

$$a_{k+1} (k+2+r)\,(k+r) + a_k (k+r+3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k (k+r+3)}{(k+2+r)(k+r)}$$

- Recursion relation for $r = -1$

$$a_{k+1} = -\frac{a_k (k+2)}{(k+1)(k-1)}$$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 1$

$$a_{k+1} = -\frac{a_k (k+2)}{(k+1)(k-1)}$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k (k+4)}{(k+3)(k+1)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k(k+4)}{(k+3)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 65

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(x+1)*diff(y(x),x)+(3*x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{4}{3}x + \frac{5}{6}x^2 - \frac{1}{3}x^3 + \frac{7}{72}x^4 - \frac{1}{45}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(6x^2 - 8x^3 + 5x^4 - 2x^5 + \mathrm{O}\left(x^6\right)\right) + (-2}{x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x^2*y''[x]+x*(x+1)*y'[x]+(3*x-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{13x^4 - 12x^3 - 4x^2 + 8x + 4}{4x} - \frac{1}{2}x\left(5x^2 - 8x + 6\right)\log(x)\right) + c_2 \left(\frac{7x^5}{72} - \frac{x^4}{3} + \frac{5x^3}{6} - \frac{4x^2}{3} + x\right)$$

4.6 problem 6

Internal problem ID [4716]
Internal file name [`OUTPUT/4209_Sunday_June_05_2022_12_41_32_PM_77856753/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(-x^2+x\right)y''-y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^2+x\right)y''-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=0\\ q(x)&=\frac{1}{x\left(x-1\right)}\end{aligned}$$

Table 67: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-1) - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x(x-1) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2)\,x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)(n+r-1)=0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0r(-1+r)=0$$

Or

$$x^{-1+r}a_0r(-1+r)=0$$

Since $a_0\neq 0$ then the above simplifies to

$$x^{-1+r}r(-1+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r)=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=1$$
$$r_2=0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n^2 + 2nr + r^2 - 3n - 3r + 3)}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}(n^2 - n + 1)}{(n+1)\,n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r^2 - r + 1}{(1 + r) r}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-r+1}{(1+r)r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^4 + r^2 + 1}{(1 + r)^2 r (2 + r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-r+1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{r^4+r^2+1}{(1+r)^2 r(2+r)}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r^6 + 3r^5 + 4r^4 + 3r^3 + 4r^2 + 3r + 3}{(1 + r)^2 r (2 + r)^2 (3 + r)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{7}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-r+1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{r^4+r^2+1}{(1+r)^2 r(2+r)}$	$\frac{1}{4}$
a_3	$\frac{r^6+3r^5+4r^4+3r^3+4r^2+3r+3}{(1+r)^2 r(2+r)^2(3+r)}$	$\frac{7}{48}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^8 + 8r^7 + 26r^6 + 44r^5 + 47r^4 + 44r^3 + 46r^2 + 36r + 21}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{91}{960}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-r+1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{r^4+r^2+1}{(1+r)^2 r(2+r)}$	$\frac{1}{4}$
a_3	$\frac{r^6+3r^5+4r^4+3r^3+4r^2+3r+3}{(1+r)^2 r(2+r)^2(3+r)}$	$\frac{7}{48}$
a_4	$\frac{r^8+8r^7+26r^6+44r^5+47r^4+44r^3+46r^2+36r+21}{(1+r)^2 r(2+r)^2(3+r)^2(4+r)}$	$\frac{91}{960}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(r^2 + r + 1)(r^2 - r + 1)(r^2 + 3r + 3)(r^2 + 5r + 7)(r^2 + 7r + 13)}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{637}{9600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2-r+1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{r^4+r^2+1}{(1+r)^2r(2+r)}$	$\frac{1}{4}$
a_3	$\frac{r^6+3r^5+4r^4+3r^3+4r^2+3r+3}{(1+r)^2r(2+r)^2(3+r)}$	$\frac{7}{48}$
a_4	$\frac{r^8+8r^7+26r^6+44r^5+47r^4+44r^3+46r^2+36r+21}{(1+r)^2r(2+r)^2(3+r)^2(4+r)}$	$\frac{91}{960}$
a_5	$\frac{(r^2+r+1)(r^2-r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)}{(1+r)^2r(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$\frac{637}{9600}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{7x^3}{48} + \frac{91x^4}{960} + \frac{637x^5}{9600} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= \frac{r^2 - r + 1}{(1+r)\,r}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{r^2 - r + 1}{(1+r)\,r} &= \lim_{r\to 0} \frac{r^2 - r + 1}{(1+r)\,r} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $-y''x(x-1) - y = 0$ gives

$$
\begin{aligned}
&-\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x(x-1) \\
&- Cy_1(x)\ln(x) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(-y_1''(x)\, x(x-1) - y_1(x)\right)\ln(x) - \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x(x-1)\right) C \\
&- \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x(x-1) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
-y_1''(x)\, x(x-1) - y_1(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&-\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x(x-1)\,C \\
&-\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\frac{\left(-2x(x-1)\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)+(x-1)\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} \\
&+\frac{(-x^3+x^2)\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x}=0
\end{aligned}
\tag{9}
$$

Since $r_1=1$ and $r_2=0$ then the above becomes

$$
\begin{aligned}
&\frac{\left(-2x(x-1)\left(\sum_{n=0}^{\infty}x^{n}a_n(n+1)\right)+(x-1)\left(\sum_{n=0}^{\infty}a_n x^{n+1}\right)\right)C}{x} \\
&+\frac{(-x^3+x^2)\left(\sum_{n=0}^{\infty}x^{-2+n}b_n n(n-1)\right)-\left(\sum_{n=0}^{\infty}b_n x^{n}\right)x}{x}=0
\end{aligned}
\tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(-2C\,x^{n+1}a_n(n+1)\right)+\left(\sum_{n=0}^{\infty}2C\,x^{n}a_n(n+1)\right) \\
&+\left(\sum_{n=0}^{\infty}C\,x^{n+1}a_n\right)+\sum_{n=0}^{\infty}\left(-Ca_n x^{n}\right)+\sum_{n=0}^{\infty}\left(-x^{n}b_n n(n-1)\right) \\
&+\left(\sum_{n=0}^{\infty}n\,x^{n-1}b_n(n-1)\right)+\sum_{n=0}^{\infty}\left(-b_n x^{n}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2C\,x^{n+1}a_n(n+1)\right)=\sum_{n=2}^{\infty}\left(-2Ca_{-2+n}(n-1)\,x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}2C\,x^{n}a_n(n+1)=\sum_{n=1}^{\infty}2Ca_{n-1}n\,x^{n-1}$$

$$\sum_{n=0}^{\infty}C\,x^{n+1}a_n=\sum_{n=2}^{\infty}Ca_{-2+n}x^{n-1}$$

$$\sum_{n=0}^{\infty}\left(-Ca_nx^{n}\right)=\sum_{n=1}^{\infty}\left(-Ca_{n-1}x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-x^{n}b_nn(n-1)\right)=\sum_{n=1}^{\infty}\left(-(n-1)\,b_{n-1}(-2+n)\,x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-b_nx^{n}\right)=\sum_{n=1}^{\infty}\left(-b_{n-1}x^{n-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}\sum_{n=2}^{\infty}\left(-2Ca_{-2+n}(n-1)\,x^{n-1}\right)+\left(\sum_{n=1}^{\infty}2Ca_{n-1}n\,x^{n-1}\right)+\left(\sum_{n=2}^{\infty}Ca_{-2+n}x^{n-1}\right)\\+\sum_{n=1}^{\infty}\left(-Ca_{n-1}x^{n-1}\right)+\sum_{n=1}^{\infty}\left(-(n-1)\,b_{n-1}(-2+n)\,x^{n-1}\right)\\+\left(\sum_{n=0}^{\infty}n\,x^{n-1}b_n(n-1)\right)+\sum_{n=1}^{\infty}\left(-b_{n-1}x^{n-1}\right)=0\end{aligned}\tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C-1=0$$

Which is solved for C. Solving for C gives

$$C=1$$

For $n = 2$, Eq (2B) gives

$$(-a_0 + 3a_1) C - b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{2} + 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{1}{4}$$

For $n = 3$, Eq (2B) gives

$$(-3a_1 + 5a_2) C - 3b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{2} + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{12}$$

For $n = 4$, Eq (2B) gives

$$(-5a_2 + 7a_3) C - 7b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{17}{48} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{17}{576}$$

For $n = 5$, Eq (2B) gives

$$(-7a_3 + 9a_4) C - 13b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{311}{1440} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{311}{28800}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = 1&\left(x\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{7x^3}{48}+\frac{91x^4}{960}+\frac{637x^5}{9600}+O(x^6)\right)\right)\ln(x) \\ &+1-\frac{x^2}{4}-\frac{x^3}{12}-\frac{17x^4}{576}-\frac{311x^5}{28800}+O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{7x^3}{48}+\frac{91x^4}{960}+\frac{637x^5}{9600}+O(x^6)\right) \\ &\quad + c_2\left(1\left(x\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{7x^3}{48}+\frac{91x^4}{960}+\frac{637x^5}{9600}+O(x^6)\right)\right)\ln(x)+1-\frac{x^2}{4}\right. \\ &\quad \left.-\frac{x^3}{12}-\frac{17x^4}{576}-\frac{311x^5}{28800}+O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{7x^3}{48}+\frac{91x^4}{960}+\frac{637x^5}{9600}+O(x^6)\right) \\ &\quad + c_2\left(x\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{7x^3}{48}+\frac{91x^4}{960}+\frac{637x^5}{9600}+O(x^6)\right)\ln(x)+1-\frac{x^2}{4}-\frac{x^3}{12}-\frac{17x^4}{576}\right. \\ &\quad \left.-\frac{311x^5}{28800}+O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1x&\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{7x^3}{48}+\frac{91x^4}{960}+\frac{637x^5}{9600}+O(x^6)\right) \\ &+ c_2\left(x\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{7x^3}{48}+\frac{91x^4}{960}+\frac{637x^5}{9600}+O(x^6)\right)\ln(x)+1-\frac{x^2}{4}-\frac{x^3}{12}\right. \\ &\left.-\frac{17x^4}{576}-\frac{311x^5}{28800}+O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1x\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{7x^3}{48}+\frac{91x^4}{960}+\frac{637x^5}{9600}+O(x^6)\right)$$
$$+ c_2\left(x\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{7x^3}{48}+\frac{91x^4}{960}+\frac{637x^5}{9600}+O(x^6)\right)\ln(x)+1-\frac{x^2}{4}-\frac{x^3}{12}\right.$$
$$\left.-\frac{17x^4}{576}-\frac{311x^5}{28800}+O(x^6)\right)$$

Verified OK.

4.6.1 Maple step by step solution

Let's solve

$-y''x(x-1)-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y}{x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=0, P_3(x)=\frac{1}{x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$$y''x(x-1)+y=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y''$ to series expansion for $m=1..2$

$$x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- ○ Shift index using $k->k+2-m$

$$x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(-1+r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(k+r)+a_k(k^2+2kr+r^2-k-r+1)\right)x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{0,1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2+(2r-1)\,k+r^2-r+1)\,a_k-a_{k+1}(k+1+r)\,(k+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{(k^2+2kr+r^2-k-r+1)\,a_k}{(k+1+r)(k+r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{(k^2-k+1)\,a_k}{(k+1)k}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{(k^2-k+1)\,a_k}{(k+1)k}\right]$$

- Recursion relation for $r=1$

$$a_{k+1}=\frac{(k^2+k+1)\,a_k}{(k+2)(k+1)}$$

- Solution for $r=1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{(k^2+k+1) a_k}{(k+2)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = \frac{(k^2-k+1) a_k}{(k+1)k}, b_{k+1} = \frac{(k^2+k+1) b_k}{(k+2)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
Order:=6;
dsolve((x-x^2)*diff(y(x),x$2)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{7}{48}x^3 + \frac{91}{960}x^4 + \frac{637}{9600}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(x + \frac{1}{2}x^2 + \frac{1}{4}x^3 + \frac{7}{48}x^4 + \frac{91}{960}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - \frac{1}{4}x^2 - \frac{1}{12}x^3 - \frac{17}{576}x^4 - \frac{311}{28800}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 87

```
AsymptoticDSolveValue[(x-x^2)*y''[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{48}x\left(7x^3 + 12x^2 + 24x + 48\right)\log(x) + \frac{1}{576}\left(-185x^4 - 336x^3 - 720x^2 - 1152x + 576\right)\right) + c_2\left(\frac{91x^5}{960} + \frac{7x^4}{48} + \frac{x^3}{4} + \frac{x^2}{2} + x\right)$$

4.7 problem 7

Internal problem ID [4717]
Internal file name [`OUTPUT/4210_Sunday_June_05_2022_12_41_42_PM_96059987/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_elliptic, _class_I]]`

$$\boxed{x(-x^2+1)\,y''+(-3x^2+1)\,y'-xy=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^3+x)\,y''+(-3x^2+1)\,y'-xy=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3x^2-1}{x\,(x^2-1)}$$
$$q(x)=\frac{1}{x^2-1}$$

Table 69: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x^2-1}{x(x^2-1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{1}{x^2-1}$	
singularity	type
$x = -1$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x^2-1) + (-3x^2+1)\,y' - xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned}&-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x(x^2-1) \\ &+ (-3x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0\end{aligned} \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1}a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-3x^{1+n+r}a_n(n+r)\right) + \left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n(n+r)(n+r-1)\right) = \sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-3x^{1+n+r}a_n(n+r)\right) = \sum_{n=2}^{\infty}\left(-3a_{n-2}(n+r-2)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right) = \sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r)\,x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1}a_n(n+r)(n+r-1)\right) + \sum_{n=2}^{\infty}\left(-3a_{n-2}(n+r-2)\,x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)(n+r-1) + (n+r)\,a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\,x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case

$n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &-a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ &- 3a_{n-2}(n+r-2) + a_n(n+r) - a_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}(n^2 + 2nr + r^2 - 2n - 2r + 1)}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-2}(n-1)^2}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(1+r)^2}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{(1+r)^2}{(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{(1+r)^2}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(1+r)^2 (r+3)^2}{(r+2)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{9}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{(1+r)^2}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{(1+r)^2(r+3)^2}{(r+2)^2(r+4)^2}$	$\frac{9}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{(1+r)^2}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{(1+r)^2(r+3)^2}{(r+2)^2(r+4)^2}$	$\frac{9}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned}y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 + \frac{x^2}{4} + \frac{9x^4}{64} + O(x^6)\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{(1+r)^2}{(r+2)^2}$	$\frac{1}{4}$	$\frac{2+2r}{(r+2)^3}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{(1+r)^2(r+3)^2}{(r+2)^2(r+4)^2}$	$\frac{9}{64}$	$\frac{4(1+r)(r+3)(r^2+5r+7)}{(r+2)^3(r+4)^3}$	$\frac{21}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\
&= \left(1 + \frac{x^2}{4} + \frac{9x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} + \frac{21x^4}{128} + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + \frac{x^2}{4} + \frac{9x^4}{64} + O(x^6)\right) + c_2\left(\left(1 + \frac{x^2}{4} + \frac{9x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} + \frac{21x^4}{128} \right. \\
&\qquad \left. + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\left(1+\frac{x^2}{4}+\frac{9x^4}{64}+O(x^6)\right)+c_2\left(\left(1+\frac{x^2}{4}+\frac{9x^4}{64}+O(x^6)\right)\ln(x)+\frac{x^2}{4}+\frac{21x^4}{128}+O(x^6)\right)
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned}
y = {} & c_1\left(1 + \frac{x^2}{4} + \frac{9x^4}{64} + O(x^6)\right) \\
& + c_2\left(\left(1 + \frac{x^2}{4} + \frac{9x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} + \frac{21x^4}{128} + O(x^6)\right)
\end{aligned} \tag{1}$$

<u>Verification of solutions</u>

$$y=c_1\left(1+\frac{x^2}{4}+\frac{9x^4}{64}+O(x^6)\right)+c_2\left(\left(1+\frac{x^2}{4}+\frac{9x^4}{64}+O(x^6)\right)\ln(x)+\frac{x^2}{4}+\frac{21x^4}{128}+O(x^6)\right)$$

Verified OK.

4.7.1 Maple step by step solution

Let's solve

$-y''x(x^2-1)+(-3x^2+1)\,y'-xy=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x^2-1}-\frac{(3x^2-1)y'}{x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3x^2-1)y'}{x(x^2-1)}+\frac{y}{x^2-1}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{3x^2-1}{x(x^2-1)},P_3(x)=\frac{1}{x^2-1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=1$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $y''x(x^2-1)+(3x^2-1)\,y'+xy=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-3u^2+2u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^2-6u+2)\left(\frac{d}{du}y(u)\right)+(u-1)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0r^2u^{-1+r} + \left(2a_1(1+r)^2 - a_0(3r^2+3r+1)\right) u^r + \left(\sum_{k=1}^{\infty} \left(2a_{k+1}(k+r+1)^2 - a_k(3k^2+6kr+3\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$2r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term must be 0

  $$2a_1(1+r)^2 - a_0(3r^2+3r+1) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-3a_k + a_{k-1} + 2a_{k+1})\, k^2 + (-3a_k + 4a_{k+1})\, k - a_k + 2a_{k+1} = 0$$

- Shift index using $k- >k + 1$

  $$(-3a_{k+1} + a_k + 2a_{k+2})\,(k+1)^2 + (-3a_{k+1} + 4a_{k+2})\,(k+1) - a_{k+1} + 2a_{k+2} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 2k a_k - 9k a_{k+1} + a_k - 7a_{k+1}}{2(k^2+4k+4)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 2k a_k - 9k a_{k+1} + a_k - 7a_{k+1}}{2(k^2+4k+4)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 2k a_k - 9k a_{k+1} + a_k - 7a_{k+1}}{2(k^2+4k+4)}, 2a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 2k a_k - 9k a_{k+1} + a_k - 7a_{k+1}}{2(k^2+4k+4)}, 2a_1 - a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   <- elliptic successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
Order:=6;
dsolve(x*(1-x^2)*diff(y(x),x$2)+(1-3*x^2)*diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + \frac{1}{4}x^2 + \frac{9}{64}x^4 + O(x^6)\right) + \left(\frac{1}{4}x^2 + \frac{21}{128}x^4 + O(x^6)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x*(1-x^2)*y''[x]+(1-3*x^2)*y'[x]-x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{9x^4}{64} + \frac{x^2}{4} + 1\right) + c_2\left(\frac{21x^4}{128} + \frac{x^2}{4} + \left(\frac{9x^4}{64} + \frac{x^2}{4} + 1\right)\log(x)\right)$$

4.8 problem 8

Internal problem ID [4718]
Internal file name [`OUTPUT/4211_Sunday_June_05_2022_12_41_52_PM_98307444/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + \frac{ay}{x^{\frac{3}{2}}} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + \frac{ay}{x^{\frac{3}{2}}} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{a}{x^{\frac{3}{2}}}$$

Table 71: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{a}{x^{\frac{3}{2}}}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : []

Irregular singular points : $[0, \infty]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(diff(y(x),x$2)+a/x^(3/2)*y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.244 (sec). Leaf size: 576

```
AsymptoticDSolveValue[y''[x]+a/x^(3/2)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \rightarrow & -\frac{16x^5(126a^{10}c_2\log(x) - 252\pi a^{10}c_1 + 504\gamma a^{10}c_2 - 1423a^{10}c_2 + 252a^{10}c_2\log(a) + 504a^{10}c_2\log(2))}{281302875\pi} \\
& + \frac{32x^{9/2}(1260a^9c_2\log(x) - 2520\pi a^9c_1 + 5040\gamma a^9c_2 - 13663a^9c_2 + 2520a^9c_2\log(a) + 5040a^9c_2\log(2))}{281302875\pi} \\
& - \frac{8x^4(140a^8c_2\log(x) - 280\pi a^8c_1 + 560\gamma a^8c_2 - 1447a^8c_2 + 280a^8c_2\log(a) + 560a^8c_2\log(2))}{496125\pi} \\
& + \frac{128x^{7/2}(105a^7c_2\log(x) - 210\pi a^7c_1 + 420\gamma a^7c_2 - 1024a^7c_2 + 210a^7c_2\log(a) + 420a^7c_2\log(2))}{496125\pi} \\
& - \frac{32x^3(15a^6c_2\log(x) - 30\pi a^6c_1 + 60\gamma a^6c_2 - 136a^6c_2 + 30a^6c_2\log(a) + 60a^6c_2\log(2))}{2025\pi} \\
& + \frac{32x^{5/2}(30a^5c_2\log(x) - 60\pi a^5c_1 + 120\gamma a^5c_2 - 247a^5c_2 + 60a^5c_2\log(a) + 120a^5c_2\log(2))}{675\pi} \\
& - \frac{8x^2(6a^4c_2\log(x) - 12\pi a^4c_1 + 24\gamma a^4c_2 - 43a^4c_2 + 12a^4c_2\log(a) + 24a^4c_2\log(2))}{9\pi} \\
& + \frac{32x^{3/2}(3a^3c_2\log(x) - 6\pi a^3c_1 + 12\gamma a^3c_2 - 17a^3c_2 + 6a^3c_2\log(a) + 12a^3c_2\log(2))}{9\pi} \\
& - \frac{8x(a^2c_2\log(x) - 2\pi a^2c_1 + 4\gamma a^2c_2 - 3a^2c_2 + 2a^2c_2\log(a) + 4a^2c_2\log(2))}{\pi} \\
& + \frac{8ac_2\sqrt{x}}{\pi} + \frac{2c_2}{\pi}
\end{aligned}
$$

4.9 problem 9

Internal problem ID [4719]
Internal file name [OUTPUT/4212_Sunday_June_05_2022_12_41_57_PM_92909546/index.tex]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(x^2 + 4x\right)y' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 - 4x\right)y' + 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x + 4}{x}$$
$$q(x) = \frac{4}{x^2}$$

Table 72: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+4}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-x^2 - 4x\right) y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + \left(-x^2 - 4x\right) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 4x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 4x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 4x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 5r + 4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 5r + 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 4 \\
r_2 &= 1
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(r^2 - 5r + 4\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^4 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + x \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+4} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{1+n} \right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 4a_n(n+r) + 4a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}}{n+r-4} \tag{4}
$$

Which for the root $r = 4$ becomes

$$a_n = \frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{-3 + r}$$

Which for the root $r = 4$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-3+r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(-3 + r)(-2 + r)}$$

Which for the root $r = 4$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-3+r}$	1
a_2	$\frac{1}{(-3+r)(-2+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(-3+r)(-2+r)(-1+r)}$$

Which for the root $r = 4$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-3+r}$	1
a_2	$\frac{1}{(-3+r)(-2+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(-3+r)(-2+r)(-1+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(-3+r)(-2+r)(-1+r)\,r}$$

Which for the root $r = 4$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-3+r}$	1
a_2	$\frac{1}{(-3+r)(-2+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(-3+r)(-2+r)(-1+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(-3+r)(-2+r)(-1+r)r}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{r^5 - 5r^4 + 5r^3 + 5r^2 - 6r}$$

Which for the root $r = 4$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-3+r}$	1
a_2	$\frac{1}{(-3+r)(-2+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(-3+r)(-2+r)(-1+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(-3+r)(-2+r)(-1+r)r}$	$\frac{1}{24}$
a_5	$\frac{1}{r^5-5r^4+5r^3+5r^2-6r}$	$\frac{1}{120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^4\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^4\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= \frac{1}{(-3+r)(-2+r)(-1+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} \frac{1}{(-3+r)(-2+r)(-1+r)} &= \lim_{r\rightarrow 1} \frac{1}{(-3+r)(-2+r)(-1+r)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y'' + (-x^2-4x)\,y' + 4y = 0$ gives

$$
\begin{aligned}
&x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (-x^2-4x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ 4Cy_1(x)\ln(x) + 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x^2y_1''(x) + (-x^2-4x)\,y_1'(x) + 4y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right. \\
&\left.+ \frac{(-x^2-4x)\,y_1(x)}{x}\right)C + x^2\left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (-x^2-4x)\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) + \left(-x^2 - 4x\right) y_1'(x) + 4y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned} &\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{\left(-x^2-4x\right) y_1(x)}{x}\right) C \\ &+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\ &+ \left(-x^2-4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned} &\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) x - (5+x)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\ &+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) x^2 \\ &+ \left(-x^2-4x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) + 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \end{aligned} \tag{9}$$

Since $r_1 = 4$ and $r_2 = 1$ then the above becomes

$$\begin{aligned} &\left(2\left(\sum_{n=0}^{\infty} x^{3+n} a_n(n+4)\right) x - (5+x)\left(\sum_{n=0}^{\infty} a_n x^{n+4}\right)\right) C \\ &+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n(1+n) n\right) x^2 \\ &+ \left(-x^2-4x\right)\left(\sum_{n=0}^{\infty} x^n b_n(1+n)\right) + 4\left(\sum_{n=0}^{\infty} b_n x^{1+n}\right) = 0 \end{aligned} \tag{10}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4)\right) + \sum_{n=0}^{\infty} \left(-5C\, x^{n+4} a_n\right) + \sum_{n=0}^{\infty} \left(-C\, x^{n+5} a_n\right) + \left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (1+n)\right) + \sum_{n=0}^{\infty} \left(-x^{n+2} b_n (1+n)\right) + \sum_{n=0}^{\infty} \left(-4x^{1+n} b_n (1+n)\right) + \left(\sum_{n=0}^{\infty} 4b_n x^{1+n}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $1+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{1+n} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4) &= \sum_{n=3}^{\infty} 2C a_{-3+n} (1+n)\, x^{1+n} \\
\sum_{n=0}^{\infty} \left(-5C\, x^{n+4} a_n\right) &= \sum_{n=3}^{\infty} \left(-5C a_{-3+n} x^{1+n}\right) \\
\sum_{n=0}^{\infty} \left(-C\, x^{n+5} a_n\right) &= \sum_{n=4}^{\infty} \left(-C a_{-4+n} x^{1+n}\right) \\
\sum_{n=0}^{\infty} \left(-x^{n+2} b_n (1+n)\right) &= \sum_{n=1}^{\infty} \left(-b_{n-1} n\, x^{1+n}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $1+n$.

$$
\left(\sum_{n=3}^{\infty} 2C a_{-3+n} (1+n)\, x^{1+n}\right) + \sum_{n=3}^{\infty} \left(-5C a_{-3+n} x^{1+n}\right) + \sum_{n=4}^{\infty} \left(-C a_{-4+n} x^{1+n}\right) + \left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (1+n)\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} n\, x^{1+n}\right) + \sum_{n=0}^{\infty} \left(-4x^{1+n} b_n (1+n)\right) + \left(\sum_{n=0}^{\infty} 4b_n x^{1+n}\right) = 0 \tag{2B}
$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-2b_1 - b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_1 - 1 = 0$$

Solving the above for b_1 gives

$$b_1 = -\frac{1}{2}$$

For $n = 2$, Eq (2B) gives

$$-2b_2 - 2b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_2 + 1 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{2}$$

For $n = N$, where $N = 3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n = 3$, Eq (2B) gives

$$3C - \frac{3}{2} = 0$$

Which is solved for C. Solving for C gives

$$C = \frac{1}{2}$$

For $n = 4$, Eq (2B) gives

$$(-a_0 + 5a_1) C - 4b_3 + 4b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 + 4b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1}{2}$$

For $n = 5$, Eq (2B) gives

$$(-a_1 + 7a_2) C - 5b_4 + 10b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{15}{4}+10b_5=0$$

Solving the above for b_5 gives

$$b_5=-\frac{3}{8}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)$$

Using the above value found for $C=\frac{1}{2}$ and all b_n, then the second solution becomes

$$\begin{aligned}y_2(x)=&\frac{1}{2}\left(x^4\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)\right)\ln(x)\\&+x\left(1-\frac{x}{2}+\frac{x^2}{2}-\frac{x^4}{2}-\frac{3x^5}{8}+O(x^6)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x)&=c_1y_1(x)+c_2y_2(x)\\&=c_1x^4\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)\\&\quad+c_2\left(\frac{1}{2}\left(x^4\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)\right)\ln(x)\right.\\&\qquad\left.+x\left(1-\frac{x}{2}+\frac{x^2}{2}-\frac{x^4}{2}-\frac{3x^5}{8}+O(x^6)\right)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y&=y_h\\&=c_1x^4\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)\\&\quad+c_2\left(\frac{x^4\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)\ln(x)}{2}\right.\\&\qquad\left.+x\left(1-\frac{x}{2}+\frac{x^2}{2}-\frac{x^4}{2}-\frac{3x^5}{8}+O(x^6)\right)\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^4 \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O\left(x^6\right)\right) \\
+ c_2 \left(\frac{x^4 \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O\left(x^6\right)\right) \ln\left(x\right)}{2} \right. \\
\left. + x\left(1-\frac{x}{2}+\frac{x^2}{2}-\frac{x^4}{2}-\frac{3x^5}{8}+O\left(x^6\right)\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^4 \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O\left(x^6\right)\right) \\
+ c_2 \left(\frac{x^4 \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O\left(x^6\right)\right) \ln\left(x\right)}{2} \right. \\
\left. + x\left(1-\frac{x}{2}+\frac{x^2}{2}-\frac{x^4}{2}-\frac{3x^5}{8}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Verified OK.

4.9.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(-x^2-4x\right) y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2} + \frac{(x+4)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+4)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{x+4}{x}, P_3(x) = \frac{4}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -4$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(x+4)\,y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-4+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-4) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 4\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-1)(a_k(k+r-4)-a_{k-1})=0$

- Shift index using $k->k+1$

  $(k+r)(a_{k+1}(k-3+r)-a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k-3+r}$

- Recursion relation for $r=1$

  $a_{k+1}=\frac{a_k}{k-2}$

- Series not valid for $r=1$, division by 0 in the recursion relation at $k=2$

  $a_{k+1}=\frac{a_k}{k-2}$

- Recursion relation for $r=4$

  $a_{k+1}=\frac{a_k}{k+1}$

- Solution for $r=4$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1}=\frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 61

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-(x^2+4*x)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(c_1 x^3 \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(6x^3 + 6x^4 + 3x^5 + \mathrm{O}\left(x^6\right)\right) + \left(12 - 6x + 6x^2 + 11x^3 + 5x^4 + x^5 + \mathrm{O}\left(x^6\right)\right)\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 74

```
AsymptoticDSolveValue[x^2*y''[x]-(x^2+4*x)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{2}(x+1)x^4 \log(x) + \frac{1}{4}\left(x^4+3x^3+2x^2-2x+4\right) x\right) + c_2 \left(\frac{x^8}{24} + \frac{x^7}{6} + \frac{x^6}{2} + x^5 + x^4\right)$$

4.10 problem 10

Internal problem ID [4720]
Internal file name [`OUTPUT/4213_Sunday_June_05_2022_12_42_07_PM_46948170/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_elliptic, _class_II]]`

$$\boxed{x(-x^2+1)\,y''+(-x^2+1)\,y'+xy=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^3+x)\,y''+(-x^2+1)\,y'+xy=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{1}{x}$$
$$q(x)=-\frac{1}{x^2-1}$$

Table 74: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x^2-1}$	
singularity	type
$x = -1$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -1, 1, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x^2-1) + (-x^2+1)\, y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x^2-1) + (-x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty}(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n (n+r)(n+r-1)\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n (n+r)\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n=\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right)+\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2) x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1)+(n+r) a_n x^{n+r-1}=0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case

$n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} -a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ -a_{n-2}(n+r-2) + a_n(n+r) + a_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}(n^2 + 2nr + r^2 - 4n - 4r + 3)}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-2}(n^2 - 4n + 3)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2 - 1}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r^2-1}{(r+2)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r^2-1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r+3)(-1+r)(1+r)^2}{(r+2)^2(r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = -\frac{3}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r^2-1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{(r+3)(-1+r)(1+r)^2}{(r+2)^2(r+4)^2}$	$-\frac{3}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r^2-1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{(r+3)(-1+r)(1+r)^2}{(r+2)^2(r+4)^2}$	$-\frac{3}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 - \frac{x^2}{4} - \frac{3x^4}{64} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{r^2-1}{(r+2)^2}$	$-\frac{1}{4}$	$\frac{4r+2}{(r+2)^3}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{(r+3)(-1+r)(1+r)^2}{(r+2)^2(r+4)^2}$	$-\frac{3}{64}$	$\frac{8r^4+52r^3+108r^2+68r+4}{(r+2)^3(r+4)^3}$	$\frac{1}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 - \frac{x^2}{4} - \frac{3x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} + \frac{x^4}{128} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - \frac{x^2}{4} - \frac{3x^4}{64} + O(x^6)\right) + c_2\left(\left(1 - \frac{x^2}{4} - \frac{3x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} + \frac{x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 - \frac{x^2}{4} - \frac{3x^4}{64} + O(x^6)\right) + c_2\left(\left(1 - \frac{x^2}{4} - \frac{3x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} + \frac{x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\left(1 - \frac{x^2}{4} - \frac{3x^4}{64} + O(x^6)\right) \\
& + c_2\left(\left(1 - \frac{x^2}{4} - \frac{3x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} + \frac{x^4}{128} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1 - \frac{x^2}{4} - \frac{3x^4}{64} + O(x^6)\right) + c_2\left(\left(1 - \frac{x^2}{4} - \frac{3x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} + \frac{x^4}{128} + O(x^6)\right)
$$

Verified OK.

4.10.1 Maple step by step solution

Let's solve

$-y''x(x^2-1)+(-x^2+1)\,y'+xy=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{x^2-1}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}-\frac{y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1}{x},P_3(x)=-\frac{1}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=0$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $y''x(x^2-1)+(x^2-1)\,y'-xy=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-3u^2+2u)\left(\frac{d^2}{du^2}y(u)\right)+(u^2-2u)\left(\frac{d}{du}y(u)\right)+(-u+1)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-1+r)\, u^{-1+r} + (2a_1(1+r)\, r - a_0(3r^2 - r - 1))\, u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+r+1)\,(k+r) - a_k($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 1\}$$

- Each term must be 0

$$2a_1(1+r)\, r - a_0(3r^2 - r - 1) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-3a_k + a_{k-1} + 2a_{k+1})\, k^2 + ((-6a_k + 2a_{k-1} + 4a_{k+1})\, r + a_k - 2a_{k-1} + 2a_{k+1})\, k + (-3a_k + a_{k-1}$$

- Shift index using $k- >k+1$

$$(-3a_{k+1} + a_k + 2a_{k+2})\,(k+1)^2 + ((-6a_{k+1} + 2a_k + 4a_{k+2})\, r + a_{k+1} - 2a_k + 2a_{k+2})\,(k+1) + (-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 2kr a_k - 6kr a_{k+1} + r^2 a_k - 3r^2 a_{k+1} - 5k a_{k+1} - 5r a_{k+1} - a_k - a_{k+1}}{2(k^2 + 2kr + r^2 + 3k + 3r + 2)}$$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} - 5k a_{k+1} - a_k - a_{k+1}}{2(k^2+3k+2)}$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} - 5k a_{k+1} - a_k - a_{k+1}}{2(k^2+3k+2)}, a_0 = 0\right]$$

- Revert the change of variables $u = x + 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} - 5k a_{k+1} - a_k - a_{k+1}}{2(k^2+3k+2)}, a_0 = 0\right]$$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 2k a_k - 11k a_{k+1} - 9a_{k+1}}{2(k^2+5k+6)}$

- Solution for $r = 1$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 2k a_k - 11k a_{k+1} - 9a_{k+1}}{2(k^2+5k+6)}, 4a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+1}, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 2k a_k - 11k a_{k+1} - 9a_{k+1}}{2(k^2+5k+6)}, 4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+1}\right), a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} - 5k a_{k+1} - a_k - a_{k+1}}{2(k^2+3k+2)}, a_0 = 0, b_{k+2}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   <- elliptic successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
Order:=6;
dsolve(x*(1-x^2)*diff(y(x),x$2)+(1-x^2)*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - \frac{1}{4}x^2 - \frac{3}{64}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{4}x^2 + \frac{1}{128}x^4 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x*(1-x^2)*y''[x]+(1-x^2)*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{3x^4}{64} - \frac{x^2}{4} + 1\right) + c_2\left(\frac{x^4}{128} + \frac{x^2}{4} + \left(-\frac{3x^4}{64} - \frac{x^2}{4} + 1\right)\log(x)\right)$$

4.11 problem 11

Internal problem ID [4721]
Internal file name [`OUTPUT/4214_Sunday_June_05_2022_12_42_14_PM_60822516/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{4x(1-x)\,y''-4y'-y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-4x^2+4x\right)y''-4y'-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{1}{x\,(x-1)}$$
$$q(x)=\frac{1}{4x\,(x-1)}$$

Table 76: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{1}{4x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-4y''x(x-1) - 4y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-4\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x(x-1) - 4\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-4x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}4x^{n+r-1}a_n(n+r)(n+r-1)\right) \\ +\sum_{n=0}^{\infty}\left(-4(n+r)a_n x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-4x^{n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-4a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-4a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty}4x^{n+r-1}a_n(n+r)(n+r-1)\right) \\ +\sum_{n=0}^{\infty}\left(-4(n+r)a_n x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1}a_n(n+r)(n+r-1)-4(n+r)a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$4x^{-1+r}a_0 r(-1+r)-4ra_0x^{-1+r}=0$$

Or

$$\left(4x^{-1+r}r(-1+r)-4r\,x^{-1+r}\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$4r\,x^{-1+r}(-2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$4r\,x^{-1+r}(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-4a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) - 4a_n(n+r) - a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(4n^2 + 8nr + 4r^2 - 12n - 12r + 9)}{4n^2 + 8nr + 4r^2 - 8n - 8r} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{\left(n + \frac{1}{2}\right)^2 a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{(2r-1)^2}{4r^2 - 4}$$

Which for the root $r = 2$ becomes

$$a_1 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(2r-1)^2}{4r^2-4}$	$\frac{3}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16r^4 - 8r^2 + 1}{16r^4 + 32r^3 - 16r^2 - 32r}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{75}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(2r-1)^2}{4r^2-4}$	$\frac{3}{4}$
a_2	$\frac{16r^4-8r^2+1}{16r^4+32r^3-16r^2-32r}$	$\frac{75}{128}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(2r+1)^2 (2r+3)^2 (2r-1)^2}{64r(-1+r)(r+2)(1+r)^2 (r+3)}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{245}{512}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(2r-1)^2}{4r^2-4}$	$\frac{3}{4}$
a_2	$\frac{16r^4-8r^2+1}{16r^4+32r^3-16r^2-32r}$	$\frac{75}{128}$
a_3	$\frac{(2r+1)^2(2r+3)^2(2r-1)^2}{64r(-1+r)(r+2)(1+r)^2(r+3)}$	$\frac{245}{512}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(2r+1)^2 (2r+3)^2 (2r-1)^2 (4r^2+20r+25)}{256r(-1+r)(r+2)^2 (1+r)^2 (r+3)(r+4)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{6615}{16384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(2r-1)^2}{4r^2-4}$	$\frac{3}{4}$
a_2	$\frac{16r^4-8r^2+1}{16r^4+32r^3-16r^2-32r}$	$\frac{75}{128}$
a_3	$\frac{(2r+1)^2(2r+3)^2(2r-1)^2}{64r(-1+r)(r+2)(1+r)^2(r+3)}$	$\frac{245}{512}$
a_4	$\frac{(2r+1)^2(2r+3)^2(2r-1)^2(4r^2+20r+25)}{256r(-1+r)(r+2)^2(1+r)^2(r+3)(r+4)}$	$\frac{6615}{16384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(2r+7)^2 (2r+1)^2 (2r+3)^2 (2r-1)^2 (2r+5)^2}{1024r(-1+r)(r+2)^2 (1+r)^2 (r+3)^2 (r+4)(r+5)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{22869}{65536}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(2r-1)^2}{4r^2-4}$	$\frac{3}{4}$
a_2	$\frac{16r^4-8r^2+1}{16r^4+32r^3-16r^2-32r}$	$\frac{75}{128}$
a_3	$\frac{(2r+1)^2(2r+3)^2(2r-1)^2}{64r(-1+r)(r+2)(1+r)^2(r+3)}$	$\frac{245}{512}$
a_4	$\frac{(2r+1)^2(2r+3)^2(2r-1)^2(4r^2+20r+25)}{256r(-1+r)(r+2)^2(1+r)^2(r+3)(r+4)}$	$\frac{6615}{16384}$
a_5	$\frac{(2r+7)^2(2r+1)^2(2r+3)^2(2r-1)^2(2r+5)^2}{1024r(-1+r)(r+2)^2(1+r)^2(r+3)^2(r+4)(r+5)}$	$\frac{22869}{65536}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2 \left(1 + \frac{3x}{4} + \frac{75x^2}{128} + \frac{245x^3}{512} + \frac{6615x^4}{16384} + \frac{22869x^5}{65536} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_2\\ &= \frac{16r^4 - 8r^2 + 1}{16r^4 + 32r^3 - 16r^2 - 32r}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} \frac{16r^4 - 8r^2 + 1}{16r^4 + 32r^3 - 16r^2 - 32r} &= \lim_{r\to 0} \frac{16r^4 - 8r^2 + 1}{16r^4 + 32r^3 - 16r^2 - 32r}\\ &= \textit{undefined}\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}\frac{d}{dx}y_2(x) &= Cy_1'(x) \ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n + r_2)}{x}\right)\\ &= Cy_1'(x) \ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n + r_2)\right)\\ \frac{d^2}{dx^2}y_2(x) &= Cy_1''(x) \ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n + r_2)}{x^2}\right)\\ &= Cy_1''(x) \ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)\end{aligned}$$

Substituting these back into the given ode $-4y''x(x-1)-4y'-y=0$ gives

$$
\begin{aligned}
&-4\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)-4Cy_1'(x)\ln(x)\\
&-\frac{4Cy_1(x)}{x}-4\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-Cy_1(x)\ln(x)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(-4y_1''(x)\,x(x-1)-4y_1'(x)-y_1(x)\right)\ln(x)-4\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x(x-1)\right.\\
&\left.-\frac{4y_1(x)}{x}\right)C-4\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)\\
&-4\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
-4y_1''(x)\,x(x-1)-4y_1'(x)-y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(-4\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x(x-1)-\frac{4y_1(x)}{x}\right)C\\
&-4\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)\\
&-4\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\frac{\left(-8x(x-1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) + 4(-2+x)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \frac{(-4x^3+4x^2)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) - 4\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{x} = 0 \tag{9}
$$

Since $r_1 = 2$ and $r_2 = 0$ then the above becomes

$$
\frac{\left(-8x(x-1)\left(\sum_{n=0}^{\infty} x^{1+n} a_n (n+2)\right) + 4(-2+x)\left(\sum_{n=0}^{\infty} a_n x^{n+2}\right)\right) C}{x} + \frac{(-4x^3+4x^2)\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) - 4\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x - \left(\sum_{n=0}^{\infty} b_n x^{n}\right) x}{x} = 0 \tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left(-8C\, x^{n+2} a_n (n+2)\right) + \left(\sum_{n=0}^{\infty} 8C\, x^{1+n} a_n (n+2)\right) \\
&+ \sum_{n=0}^{\infty} \left(-8C\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 4C\, x^{n+2} a_n\right) + \sum_{n=0}^{\infty} \left(-4x^n b_n n(n-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 4n\, x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-4x^{n-1} b_n n\right) + \sum_{n=0}^{\infty} \left(-b_n x^n\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-8C\, x^{n+2} a_n (n+2)\right) = \sum_{n=3}^{\infty} \left(-8C a_{n-3} (n-1)\, x^{n-1}\right)
$$

$$
\sum_{n=0}^{\infty} 8C\, x^{1+n} a_n (n+2) = \sum_{n=2}^{\infty} 8C a_{-2+n} n\, x^{n-1}
$$

$$\sum_{n=0}^{\infty} \left(-8C\, x^{1+n} a_n\right) = \sum_{n=2}^{\infty} \left(-8C a_{-2+n} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 4C\, x^{n+2} a_n = \sum_{n=3}^{\infty} 4C a_{n-3} x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-4x^n b_n n(n-1)\right) = \sum_{n=1}^{\infty} \left(-4(n-1)\, b_{n-1}(-2+n)\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-b_n x^n\right) = \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\sum_{n=3}^{\infty} \left(-8C a_{n-3}(n-1)\, x^{n-1}\right) + \left(\sum_{n=2}^{\infty} 8C a_{-2+n} n\, x^{n-1}\right) \\
&+ \sum_{n=2}^{\infty} \left(-8C a_{-2+n} x^{n-1}\right) + \left(\sum_{n=3}^{\infty} 4C a_{n-3} x^{n-1}\right) \\
&+ \sum_{n=1}^{\infty} \left(-4(n-1)\, b_{n-1}(-2+n)\, x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 4n\, x^{n-1} b_n (n-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-4x^{n-1} b_n n\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-4b_1 - b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_1 - 1 = 0$$

Solving the above for b_1 gives

$$b_1 = -\frac{1}{4}$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$8C + \frac{1}{4} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{32}$$

For $n = 3$, Eq (2B) gives

$$(-12a_0 + 16a_1) C - 9b_2 + 12b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$(-20a_1 + 24a_2) C - 25b_3 + 32b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{15}{512} + 32b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{15}{16384}$$

For $n = 5$, Eq (2B) gives

$$(-28a_2 + 32a_3) C - 49b_4 + 60b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1295}{16384} + 60b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{259}{196608}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{32}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{32}\left(x^2\left(1+\frac{3x}{4}+\frac{75x^2}{128}+\frac{245x^3}{512}+\frac{6615x^4}{16384}+\frac{22869x^5}{65536}+O(x^6)\right)\right)\ln(x) + 1 - \frac{x}{4} - \frac{15x^4}{16384} - \frac{259x^5}{196608} + O(x^6)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2\left(1+\frac{3x}{4}+\frac{75x^2}{128}+\frac{245x^3}{512}+\frac{6615x^4}{16384}+\frac{22869x^5}{65536}+O(x^6)\right) \\ &\quad + c_2\left(-\frac{1}{32}\left(x^2\left(1+\frac{3x}{4}+\frac{75x^2}{128}+\frac{245x^3}{512}+\frac{6615x^4}{16384}+\frac{22869x^5}{65536}+O(x^6)\right)\right)\ln(x) + 1 - \frac{x}{4} - \frac{15x^4}{16384} - \frac{259x^5}{196608} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^2\left(1+\frac{3x}{4}+\frac{75x^2}{128}+\frac{245x^3}{512}+\frac{6615x^4}{16384}+\frac{22869x^5}{65536}+O(x^6)\right) \\ &\quad + c_2\left(-\frac{x^2\left(1+\frac{3x}{4}+\frac{75x^2}{128}+\frac{245x^3}{512}+\frac{6615x^4}{16384}+\frac{22869x^5}{65536}+O(x^6)\right)\ln(x)}{32} + 1 - \frac{x}{4} - \frac{15x^4}{16384} - \frac{259x^5}{196608} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x^2\left(1+\frac{3x}{4}+\frac{75x^2}{128}+\frac{245x^3}{512}+\frac{6615x^4}{16384}+\frac{22869x^5}{65536}+O(x^6)\right) \\ &\quad + c_2\left(-\frac{x^2\left(1+\frac{3x}{4}+\frac{75x^2}{128}+\frac{245x^3}{512}+\frac{6615x^4}{16384}+\frac{22869x^5}{65536}+O(x^6)\right)\ln(x)}{32} + 1 - \frac{x}{4} - \frac{15x^4}{16384} - \frac{259x^5}{196608} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1x^2\left(1+\frac{3x}{4}+\frac{75x^2}{128}+\frac{245x^3}{512}+\frac{6615x^4}{16384}+\frac{22869x^5}{65536}+O(x^6)\right) + c_2\left(-\frac{x^2\left(1+\frac{3x}{4}+\frac{75x^2}{128}+\frac{245x^3}{512}+\frac{6615x^4}{16384}+\frac{22869x^5}{65536}+O(x^6)\right)\ln(x)}{32}+1-\frac{x}{4} - \frac{15x^4}{16384}-\frac{259x^5}{196608}+O(x^6)\right)$$

Verified OK.

4.11.1 Maple step by step solution

Let's solve

$-4y''x(x-1)-4y'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4x(x-1)} - \frac{y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x(x-1)} + \frac{y}{4x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1}{x(x-1)}, P_3(x)=\frac{1}{4x(x-1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0} = -1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0} = 0$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $4y''x(x-1)+4y'+y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$

  - Shift index using $k->k+1$

    $y'=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,x^{k+r}$

  - Convert $x^m\cdot y''$ to series expansion for $m=1..2$

    $x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-4a_0r(-2+r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-4a_{k+1}(k+1+r)\,(k+r-1)+a_k(2k+2r-1)^2\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-4r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- Each term in the series must be 0, giving the recursion relation

  $-4a_{k+1}(k+1+r)\,(k+r-1)+a_k(2k+2r-1)^2=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(2k+2r-1)^2}{4(k+1+r)(k+r-1)}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k(2k-1)^2}{4(k+1)(k-1)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = \frac{a_k(2k-1)^2}{4(k+1)(k-1)}$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(2k+3)^2}{4(k+3)(k+1)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(2k+3)^2}{4(k+3)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 60

```
Order:=6;
dsolve(4*x*(1-x)*diff(y(x),x$2)-4*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 + \frac{3}{4}x + \frac{75}{128}x^2 + \frac{245}{512}x^3 + \frac{6615}{16384}x^4 + \frac{22869}{65536}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(\frac{1}{16}x^2 + \frac{3}{64}x^3 + \frac{75}{2048}x^4 + \frac{245}{8192}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(-2 + \frac{1}{2}x + \frac{1}{2}x^2 + \frac{3}{8}x^3 + \frac{2415}{8192}x^4 + \frac{23779}{98304}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 86

```
AsymptoticDSolveValue[4*x*(1-x)*y''[x]-4*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{135x^4 + 192x^3 + 256x^2 - 4096x + 16384}{16384} - \frac{x^2(75x^2 + 96x + 128)\log(x)}{4096}\right) + c_2 \left(\frac{6615x^6}{16384} + \frac{245x^5}{512} + \frac{75x^4}{128} + \frac{3x^3}{4} + x^2\right)$$

4.12 problem 12

Internal problem ID [4722]
Internal file name [`OUTPUT/4215_Sunday_June_05_2022_12_42_24_PM_92371578/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + y = x^{\frac{3}{2}}}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^3y'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{1}{x^3}$$

Table 78: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x^3}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^3*diff(y(x),x$2)+y(x)=x^(3/2),y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.251 (sec). Leaf size: 688

```
AsymptoticDSolveValue[x^3*y''[x]+y[x]==x^(3/2),y[x],{x,0,5}]
```

$$y(x) \to \frac{e^{\frac{2i}{\sqrt{x}}} x^{3/4} \left(\frac{468131288625ix^{9/2}}{8796093022208} - \frac{66891825ix^{7/2}}{4294967296} + \frac{72765ix^{5/2}}{8388608} - \frac{105ix^{3/2}}{8192} + \frac{33424574007825x^5}{281474976710656} - \frac{14783093325x^4}{549755813888} + \frac{2837835x^3}{268435456} \right.}{}$$

$$+ \frac{e^{-\frac{2i}{\sqrt{x}}} x^{3/4} \left(-\frac{468131288625ix^{9/2}}{8796093022208} + \frac{66891825ix^{7/2}}{4294967296} - \frac{72765ix^{5/2}}{8388608} + \frac{105ix^{3/2}}{8192} + \frac{33424574007825x^5}{281474976710656} - \frac{14783093325x^4}{549755813888} + \frac{28378}{26843} \right.}{}$$

$$+ c_1 e^{-\frac{2i}{\sqrt{x}}} x^{3/4} \left(-\frac{468131288625ix^{9/2}}{8796093022208} + \frac{66891825ix^{7/2}}{4294967296} - \frac{72765ix^{5/2}}{8388608} + \frac{105ix^{3/2}}{8192} + \frac{33424574007825x^5}{281474976710656} - \frac{14}{54} \right.$$

4.13 problem 13

Internal problem ID [4723]
Internal file name [`OUTPUT/4216_Sunday_June_05_2022_12_42_30_PM_62461942/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{2x^2y'' - (2+3x)\,y' + \frac{(2x-1)\,y}{x} = \sqrt{x}}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + (-3x-2)\,y' + \left(2 - \frac{1}{x}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2+3x}{2x^2}$$
$$q(x) = \frac{2x-1}{2x^3}$$

Table 79: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2+3x}{2x^2}$	
singularity	type
$x = 0$	"irregular"

$q(x) = \frac{2x-1}{2x^3}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
      Solution has integrals. Trying a special function solution free of integrals...
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         <- Kummer successful
      <- special function solution successful
         Solution using Kummer functions still has integrals. Trying a hypergeometric soluti
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         -> Trying to convert hypergeometric functions to elementary form...
         <- elementary form for at least one hypergeometric solution is achieved - returning
      <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)-(3*x+2)*diff(y(x),x)+(2*x-1)/x*y(x)=x^(1/2),y(x),type='series',x
```

No solution found

✓ Solution by Mathematica

Time used: 0.158 (sec). Leaf size: 222

```
AsymptoticDSolveValue[2*x^2*y''[x]-(3*x+2)*y'[x]+(2*x-1)/x*y[x]==x^(1/2),y[x],{x,0,5}]
```

$$y(x) \to \frac{1}{256} e^{-1/x} \left(-\frac{405405x^5}{16} + \frac{45045x^4}{16} - \frac{693x^3}{2} + \frac{189x^2}{4} - 7x + 1 \right) x^4 \left(\frac{2e^{\frac{1}{x}}(15663375x^7 + 20072325x^6 + 10329540x^5 + 4131816x^4 + 2754544x^3 + 5509088x^2 - 64x}{x^{3/2}} - 11018112\sqrt{\pi}\operatorname{erfi}\left(\frac{1}{\sqrt{x}}\right) \right) + c_2 e^{-1/x} \left(-\frac{405405x^5}{16} + \frac{45045x^4}{16} - \frac{693x^3}{2} + \frac{189x^2}{4} - 7x + 1 \right) x^4 + \frac{\left(\frac{5x}{2}+1\right)\left(-\frac{15015x^6}{64} + \frac{693x^5}{20} - \frac{189x^4}{32} + 7}{\sqrt{x}}$$

4.14 problem 14

Internal problem ID [4724]
Internal file name [`OUTPUT/4217_Sunday_June_05_2022_12_42_35_PM_77828568/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XV. page 194
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{(-x^2 + x)\, y'' + 3y' + 2y = 3x^2}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^2 + x)\, y'' + 3y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{3}{x\,(x-1)}$$
$$q(x) = -\frac{2}{x\,(x-1)}$$

Table 80: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{2}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-1) + 3y' + 2y = 3x^2$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $-y''x(x-1) + 3y' + 2y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x(x-1) \\
&+3\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right)+2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 3(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\,(n+r-1)\right)=\sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r-1}\right)
$$

$$
\sum_{n=0}^{\infty} 2a_n x^{n+r}=\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 3(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n (n+r)\,(n+r-1)+3(n+r)\,a_n x^{n+r-1}=0
$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -2 \end{aligned}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(x^{-1+m} m(-1+m) + 3m\, x^{-1+m}\right) c_0 = 3x^2$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2} \end{aligned}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) + 3a_n(n+r) + 2a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r-3)\, a_{n-1}}{n+2+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(n-3)\, a_{n-1}}{n+2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2+r}{3+r}$$

Which for the root $r = 0$ becomes

$$a_1 = -\frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{3+r}$	$-\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2 - 3r + 2}{(3 + r)(4 + r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{3+r}$	$-\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(r^2 - 3r + 2) r}{(4 + r)(5 + r)(3 + r)}$$

Which for the root $r = 0$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{3+r}$	$-\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$
a_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-1 + r)(-2 + r) r(1 + r)}{(3 + r)(4 + r)(5 + r)(6 + r)}$$

Which for the root $r = 0$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{3+r}$	$-\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$
a_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	0
a_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r^5 - 5r^3 + 4r}{(3+r)(4+r)(5+r)(6+r)(7+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{3+r}$	$-\frac{2}{3}$
a_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	$\frac{1}{6}$
a_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	0
a_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	0
a_5	$\frac{r^5-5r^3+4r}{(3+r)(4+r)(5+r)(6+r)(7+r)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 - \frac{2x}{3} + \frac{x^2}{6} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{r^2 - 3r + 2}{(3+r)(4+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{r^2 - 3r + 2}{(3+r)(4+r)} &= \lim_{r\to -2} \frac{r^2 - 3r + 2}{(3+r)(4+r)} \\ &= 6 \end{aligned}$$

The limit is 6. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-2} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$-b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) + 3(n+r)b_n + 2b_{n-1} = 0 \quad (4)$$

Which for for the root $r = -2$ becomes

$$-b_{n-1}(n-3)(n-4) + b_n(n-2)(n-3) + 3(n-2)b_n + 2b_{n-1} = 0 \qquad (4A)$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{(n+r-3)\, b_{n-1}}{n+2+r} \qquad (5)$$

Which for the root $r = -2$ becomes

$$b_n = \frac{(n-5)\, b_{n-1}}{n} \qquad (6)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-2 + r}{3 + r}$$

Which for the root $r = -2$ becomes

$$b_1 = -4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{3+r}$	-4

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2 - 3r + 2}{(3 + r)(4 + r)}$$

Which for the root $r = -2$ becomes

$$b_2 = 6$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{3+r}$	-4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(r^2 - 3r + 2) r}{(4 + r)(5 + r)(3 + r)}$$

Which for the root $r = -2$ becomes

$$b_3 = -4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{3+r}$	-4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6
b_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	-4

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r^2 - 3r + 2)\, r(1 + r)}{(5 + r)\,(3 + r)\,(4 + r)\,(6 + r)}$$

Which for the root $r = -2$ becomes

$$b_4 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{3+r}$	-4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6
b_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	-4
b_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	1

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{(r^2 - 3r + 2)\, r(1 + r)\,(2 + r)}{(6 + r)\,(3 + r)\,(4 + r)\,(7 + r)\,(5 + r)}$$

Which for the root $r = -2$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{3+r}$	-4
b_2	$\frac{r^2-3r+2}{(3+r)(4+r)}$	6
b_3	$\frac{(r^2-3r+2)r}{(4+r)(5+r)(3+r)}$	-4
b_4	$\frac{(-1+r)(-2+r)r(1+r)}{(3+r)(4+r)(5+r)(6+r)}$	1
b_5	$\frac{(-2+r)(-1+r)r(1+r)(2+r)}{(6+r)(3+r)(4+r)(7+r)(5+r)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - 4x + 6x^2 - 4x^3 + x^4 + O(x^6)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 - \frac{2x}{3} + \frac{x^2}{6} + O(x^6)\right) + \frac{c_2 (1 - 4x + 6x^2 - 4x^3 + x^4 + O(x^6))}{x^2} \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(x^{-1+m} m(-1+m) + 3m\, x^{-1+m}\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{array}{|l|}
\hline
a_1 = \frac{(-2+r)a_0}{3+r} \\
a_2 = \frac{(-1+r)(-2+r)a_0}{(3+r)(4+r)} \\
a_3 = \frac{r(-1+r)(-2+r)a_0}{(3+r)(4+r)(5+r)} \\
a_4 = \frac{r(-2+r)a_0(r^2-1)}{(3+r)(4+r)(5+r)(6+r)} \\
a_5 = \frac{a_0(r^5-5r^3+4r)}{(3+r)(4+r)(5+r)(6+r)(7+r)} \\
\hline
\end{array}
$$

Now we determine the particular solution y_p associated with $F = 3x^2$ by solving the balance equation

$$
\left(x^{-1+m}m(-1+m) + 3m\,x^{-1+m}\right) c_0 = 3x^2
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{5} \\
m &= 3
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+3}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{5}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{5}$ and $r = m$ or $r = 3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$
\begin{array}{|l|}
\hline
c_0 = \frac{1}{5} \\
c_1 = \frac{1}{30} \\
c_2 = \frac{1}{105} \\
c_3 = \frac{1}{280} \\
c_4 = \frac{1}{630} \\
c_5 = \frac{1}{1260} \\
\hline
\end{array}
$$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3 \left(\frac{1}{5}+\frac{1}{30}x+\frac{1}{105}x^2+\frac{1}{280}x^3+\frac{1}{630}x^4+\frac{1}{1260}x^5\right) \\
&= \frac{1}{5}x^3+\frac{1}{30}x^4+\frac{1}{105}x^5+\frac{1}{280}x^6+\frac{1}{630}x^7+\frac{1}{1260}x^8
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = \frac{x^3}{5}+\frac{x^4}{30}+\frac{x^5}{105}+\frac{x^6}{280}+\frac{x^7}{630}+\frac{x^8}{1260}+O(x^6)
$$

Truncating the particular solution to the order of series requested gives

$$
y_p = \frac{x^3}{5}+\frac{x^4}{30}+\frac{x^5}{105}+O(x^6)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x^3}{5}+\frac{x^4}{30}+\frac{x^5}{105}+O(x^6)+c_1\left(1-\frac{2x}{3}+\frac{x^2}{6}+O(x^6)\right) \\
&\quad + \frac{c_2(1-4x+6x^2-4x^3+x^4+O(x^6))}{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\frac{x^3}{5}+\frac{x^4}{30}+\frac{x^5}{105}+O(x^6)+c_1\left(1-\frac{2x}{3}+\frac{x^2}{6}+O(x^6)\right) \\
&+ \frac{c_2(1-4x+6x^2-4x^3+x^4+O(x^6))}{x^2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = \frac{x^3}{5} + \frac{x^4}{30} + \frac{x^5}{105} + O(x^6) + c_1\left(1 - \frac{2x}{3} + \frac{x^2}{6} + O(x^6)\right) + \frac{c_2(1 - 4x + 6x^2 - 4x^3 + x^4 + O(x^6))}{x^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
Order:=6;
dsolve((x-x^2)*diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=3*x^2,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{2}{3}x + \frac{1}{6}x^2 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2(-2 + 8x - 12x^2 + 8x^3 - 2x^4 + \mathrm{O}\left(x^6\right))}{x^2} + x^3\left(\frac{1}{5} + \frac{1}{30}x + \frac{1}{105}x^2 + \mathrm{O}\left(x^3\right)\right)$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 91

```
AsymptoticDSolveValue[(x-x^2)*y''[x]+3*y'[x]+2*y[x]==3*x^2,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^2}{6} - \frac{2x}{3} + 1\right) + \frac{c_2(1 - 4x)}{x^2} + \frac{(1 - 4x)\left(-\frac{5x^6}{6} - \frac{3x^5}{10}\right)}{x^2} + \left(\frac{x^2}{6} - \frac{2x}{3} + 1\right)\left(-5x^6 - \frac{9x^5}{5} + \frac{x^3}{2}\right)$$

5 Chapter VII, Solutions in series. Examples XVI. page 220

5.1 problem 5

Internal problem ID [4725]
Internal file name [OUTPUT/4218_Sunday_June_05_2022_12_42_42_PM_72682560/index.tex]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XVI. page 220
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[_Jacobi]
```

$$\boxed{x(1-x)\,y''+\left(\frac{3}{2}-2x\right)y'-\frac{y}{4}=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^{2}+x\right)y''+\left(\frac{3}{2}-2x\right)y'-\frac{y}{4}=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{-3+4x}{2x\,(x-1)}\\ q(x)&=\frac{1}{4x\,(x-1)}\end{aligned}$$

Table 81: Table $p(x), q(x)$ singularites.

$p(x) = \frac{-3+4x}{2x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{1}{4x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-1) + \left(\frac{3}{2} - 2x\right) y' - \frac{y}{4} = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) + \left(\frac{3}{2} - 2x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \frac{\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)}{4} = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-2 x^{n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} \frac{3(n+r) a_n x^{n+r-1}}{2}\right)+\sum_{n=0}^{\infty}\left(-\frac{a_n x^{n+r}}{4}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)(n+r-1)\right) &= \sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}\right) \\
\sum_{n=0}^{\infty}\left(-2 x^{n+r} a_n(n+r)\right) &= \sum_{n=1}^{\infty}\left(-2 a_{n-1}(n+r-1) x^{n+r-1}\right) \\
\sum_{n=0}^{\infty}\left(-\frac{a_n x^{n+r}}{4}\right) &= \sum_{n=1}^{\infty}\left(-\frac{a_{n-1} x^{n+r-1}}{4}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-2 a_{n-1}(n+r-1) x^{n+r-1}\right) \\
&+\left(\sum_{n=0}^{\infty} \frac{3(n+r) a_n x^{n+r-1}}{2}\right)+\sum_{n=1}^{\infty}\left(-\frac{a_{n-1} x^{n+r-1}}{4}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n(n+r)(n+r-1)+\frac{3(n+r) a_n x^{n+r-1}}{2}=0
$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + \frac{3 r a_0 x^{-1+r}}{2} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + \frac{3 r\, x^{-1+r}}{2}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r} \left(\frac{1}{2} + r\right) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + \frac{1}{2} r = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -\frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r} \left(\frac{1}{2} + r\right) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &-a_{n-1}(n+r-1)(n+r-2)+a_n(n+r)(n+r-1) \\ &-2a_{n-1}(n+r-1)+\frac{3a_n(n+r)}{2}-\frac{a_{n-1}}{4}=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(4n^2+8nr+4r^2-4n-4r+1)}{4n^2+8nr+4r^2+2n+2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(2n-1)^2 a_{n-1}}{4n^2+2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{(1+2r)^2}{4r^2+10r+6}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(1+2r)^2}{4r^2+10r+6}$	$\frac{1}{6}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(1+2r)^2(2r+3)}{8r^3+44r^2+76r+40}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{3}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(1+2r)^2}{4r^2+10r+6}$	$\frac{1}{6}$
a_2	$\frac{(1+2r)^2(2r+3)}{8r^3+44r^2+76r+40}$	$\frac{3}{40}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(1+2r)^2 (2r+3)(2r+5)}{8(2r^2+13r+21)(r^2+3r+2)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{5}{112}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(1+2r)^2}{4r^2+10r+6}$	$\frac{1}{6}$
a_2	$\frac{(1+2r)^2(2r+3)}{8r^3+44r^2+76r+40}$	$\frac{3}{40}$
a_3	$\frac{(1+2r)^2(2r+3)(2r+5)}{8(2r^2+13r+21)(r^2+3r+2)}$	$\frac{5}{112}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(1+2r)^2 (2r+3)(2r+5)(2r+7)}{32r^5+464r^4+2560r^3+6640r^2+7968r+3456}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{35}{1152}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(1+2r)^2}{4r^2+10r+6}$	$\frac{1}{6}$
a_2	$\frac{(1+2r)^2(2r+3)}{8r^3+44r^2+76r+40}$	$\frac{3}{40}$
a_3	$\frac{(1+2r)^2(2r+3)(2r+5)}{8(2r^2+13r+21)(r^2+3r+2)}$	$\frac{5}{112}$
a_4	$\frac{(1+2r)^2(2r+3)(2r+5)(2r+7)}{32r^5+464r^4+2560r^3+6640r^2+7968r+3456}$	$\frac{35}{1152}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(1+2r)^2 (2r+3)(2r+5)(2r+7)(2r+9)}{32(2r^2+21r+55)(r^4+10r^3+35r^2+50r+24)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{63}{2816}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(1+2r)^2}{4r^2+10r+6}$	$\frac{1}{6}$
a_2	$\frac{(1+2r)^2(2r+3)}{8r^3+44r^2+76r+40}$	$\frac{3}{40}$
a_3	$\frac{(1+2r)^2(2r+3)(2r+5)}{8(2r^2+13r+21)(r^2+3r+2)}$	$\frac{5}{112}$
a_4	$\frac{(1+2r)^2(2r+3)(2r+5)(2r+7)}{32r^5+464r^4+2560r^3+6640r^2+7968r+3456}$	$\frac{35}{1152}$
a_5	$\frac{(1+2r)^2(2r+3)(2r+5)(2r+7)(2r+9)}{32(2r^2+21r+55)(r^4+10r^3+35r^2+50r+24)}$	$\frac{63}{2816}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 + \frac{x}{6} + \frac{3x^2}{40} + \frac{5x^3}{112} + \frac{35x^4}{1152} + \frac{63x^5}{2816} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&-b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) \\
&- 2b_{n-1}(n+r-1) + \frac{3(n+r)\,b_n}{2} - \frac{b_{n-1}}{4} = 0
\end{aligned} \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}(4n^2 + 8nr + 4r^2 - 4n - 4r + 1)}{4n^2 + 8nr + 4r^2 + 2n + 2r} \tag{4}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_n = \frac{2b_{n-1}(n-1)^2}{2n^2 - n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{(1+2r)^2}{4r^2 + 10r + 6}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_1 = 0
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{(1+2r)^2}{4r^2+10r+6}$	0

For $n = 2$, using the above recursive equation gives

$$
b_2 = \frac{(1+2r)^2 (2r+3)}{8r^3 + 44r^2 + 76r + 40}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{(1+2r)^2}{4r^2+10r+6}$	0
b_2	$\frac{(1+2r)^2(2r+3)}{8r^3+44r^2+76r+40}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(1+2r)^2(2r+3)(2r+5)}{8(2r^2+13r+21)(r^2+3r+2)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{(1+2r)^2}{4r^2+10r+6}$	0
b_2	$\frac{(1+2r)^2(2r+3)}{8r^3+44r^2+76r+40}$	0
b_3	$\frac{(1+2r)^2(2r+3)(2r+5)}{8(2r^2+13r+21)(r^2+3r+2)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(1+2r)^2(2r+3)(2r+5)(2r+7)}{32r^5+464r^4+2560r^3+6640r^2+7968r+3456}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{(1+2r)^2}{4r^2+10r+6}$	0
b_2	$\frac{(1+2r)^2(2r+3)}{8r^3+44r^2+76r+40}$	0
b_3	$\frac{(1+2r)^2(2r+3)(2r+5)}{8(2r^2+13r+21)(r^2+3r+2)}$	0
b_4	$\frac{(1+2r)^2(2r+3)(2r+5)(2r+7)}{32r^5+464r^4+2560r^3+6640r^2+7968r+3456}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{(1+2r)^2 (2r+3)(2r+5)(2r+7)(2r+9)}{32(2r^2+21r+55)(r^4+10r^3+35r^2+50r+24)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{(1+2r)^2}{4r^2+10r+6}$	0
b_2	$\frac{(1+2r)^2(2r+3)}{8r^3+44r^2+76r+40}$	0
b_3	$\frac{(1+2r)^2(2r+3)(2r+5)}{8(2r^2+13r+21)(r^2+3r+2)}$	0
b_4	$\frac{(1+2r)^2(2r+3)(2r+5)(2r+7)}{32r^5+464r^4+2560r^3+6640r^2+7968r+3456}$	0
b_5	$\frac{(1+2r)^2(2r+3)(2r+5)(2r+7)(2r+9)}{32(2r^2+21r+55)(r^4+10r^3+35r^2+50r+24)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 + \frac{x}{6} + \frac{3x^2}{40} + \frac{5x^3}{112} + \frac{35x^4}{1152} + \frac{63x^5}{2816} + O(x^6)\right) + \frac{c_2(1 + O(x^6))}{\sqrt{x}} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1+\frac{x}{6}+\frac{3x^2}{40}+\frac{5x^3}{112}+\frac{35x^4}{1152}+\frac{63x^5}{2816}+O\left(x^6\right)\right)+\frac{c_2(1+O(x^6))}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1+\frac{x}{6}+\frac{3x^2}{40}+\frac{5x^3}{112}+\frac{35x^4}{1152}+\frac{63x^5}{2816}+O\left(x^6\right)\right)+\frac{c_2(1+O(x^6))}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1+\frac{x}{6}+\frac{3x^2}{40}+\frac{5x^3}{112}+\frac{35x^4}{1152}+\frac{63x^5}{2816}+O\left(x^6\right)\right)+\frac{c_2(1+O(x^6))}{\sqrt{x}}
$$

Verified OK.

5.1.1 Maple step by step solution

Let's solve

$-y''x(x-1)+\left(\frac{3}{2}-2x\right)y'-\frac{y}{4}=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(-3+4x)y'}{2x(x-1)}-\frac{y}{4x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(-3+4x)y'}{2x(x-1)}+\frac{y}{4x(x-1)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{-3+4x}{2x(x-1)}, P_3(x)=\frac{1}{4x(x-1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=\frac{3}{2}$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4y''x(x-1) + (8x-6)\,y' + y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0r(1+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(2k+3+2r) + a_k(2k+2r+1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+1)^2 - 4a_{k+1}(k+1+r)\left(k+\frac{3}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(2k+2r+1)^2}{2(k+1+r)(2k+3+2r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(2k+1)^2}{2(k+1)(2k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(2k+1)^2}{2(k+1)(2k+3)}\right]$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{2a_k k^2}{(k+\frac{1}{2})(2k+2)}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{(k+\frac{1}{2})(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = \frac{a_k(2k+1)^2}{2(k+1)(2k+3)}, b_{k+1} = \frac{2b_k k^2}{(k+\frac{1}{2})(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 34

```
Order:=6;
dsolve(x*(1-x)*diff(y(x),x$2)+(3/2-2*x)*diff(y(x),x)-1/4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1(1 + \mathrm{O}(x^6))}{\sqrt{x}} + c_2\left(1 + \frac{1}{6}x + \frac{3}{40}x^2 + \frac{5}{112}x^3 + \frac{35}{1152}x^4 + \frac{63}{2816}x^5 + \mathrm{O}(x^6)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 50

```
AsymptoticDSolveValue[x*(1-x)*y''[x]+(3/2-2*x)*y'[x]-1/4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{63x^5}{2816} + \frac{35x^4}{1152} + \frac{5x^3}{112} + \frac{3x^2}{40} + \frac{x}{6} + 1\right) + \frac{c_2}{\sqrt{x}}$$

5.2 problem 6

Internal problem ID [4726]
Internal file name [`OUTPUT/4219_Sunday_June_05_2022_12_42_52_PM_2605663/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XVI. page 220
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x(1-x)\,y''+xy'-y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^2+2x\right)y''+xy'-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{1}{2\,(x-1)}$$
$$q(x)=\frac{1}{2x\,(x-1)}$$

Table 83: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2(x-1)}$	
singularity	type
$x = 1$	"regular"

$q(x) = \frac{1}{2x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[1, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-2y''x(x-1) + xy' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x(x-1) + x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r-1}\right)$$
$$\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)=\sum_{n=1}^{\infty}a_{n-1}(n+r-1)\,x^{n+r-1}$$
$$\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty}a_{n-1}(n+r-1)\,x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1}a_n(n+r)(n+r-1)=0$$

When $n=0$ the above becomes

$$2x^{-1+r}a_0r(-1+r)=0$$

Or

$$2x^{-1+r}a_0r(-1+r)=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2x^{-1+r}r(-1+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$2r(-1+r)=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^{-1+r}r(-1+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-2a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(2n^2 + 4nr + 2r^2 - 7n - 7r + 6)}{2(n+r)(n+r-1)} \quad (4)$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}(2n^2 - 3n + 1)}{2(n+1)n} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2r^2 - 3r + 1}{2(1+r)r}$$

Which for the root $r = 1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-3r+1}{2(1+r)r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^3 - 4r^2 - r + 1}{4(1+r)^2(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-3r+1}{2(1+r)r}$	0
a_2	$\frac{4r^3-4r^2-r+1}{4(1+r)^2(2+r)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^4 + 4r^3 - 14r^2 - r + 3}{8(3+r)(2+r)^2(1+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-3r+1}{2(1+r)r}$	0
a_2	$\frac{4r^3-4r^2-r+1}{4(1+r)^2(2+r)}$	0
a_3	$\frac{8r^4+4r^3-14r^2-r+3}{8(3+r)(2+r)^2(1+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^5 + 48r^4 - 8r^3 - 72r^2 + r + 15}{16(4+r)(1+r)(2+r)(3+r)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-3r+1}{2(1+r)r}$	0
a_2	$\frac{4r^3-4r^2-r+1}{4(1+r)^2(2+r)}$	0
a_3	$\frac{8r^4+4r^3-14r^2-r+3}{8(3+r)(2+r)^2(1+r)}$	0
a_4	$\frac{16r^5+48r^4-8r^3-72r^2+r+15}{16(4+r)(1+r)(2+r)(3+r)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32r^6 + 208r^5 + 320r^4 - 200r^3 - 502r^2 + 37r + 105}{32(5+r)(3+r)(2+r)(1+r)(4+r)^2}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-3r+1}{2(1+r)r}$	0
a_2	$\frac{4r^3-4r^2-r+1}{4(1+r)^2(2+r)}$	0
a_3	$\frac{8r^4+4r^3-14r^2-r+3}{8(3+r)(2+r)^2(1+r)}$	0
a_4	$\frac{16r^5+48r^4-8r^3-72r^2+r+15}{16(4+r)(1+r)(2+r)(3+r)^2}$	0
a_5	$\frac{32r^6+208r^5+320r^4-200r^3-502r^2+37r+105}{32(5+r)(3+r)(2+r)(1+r)(4+r)^2}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x\left(1 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{2r^2 - 3r + 1}{2(1+r)r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} \frac{2r^2 - 3r + 1}{2(1+r)r} &= \lim_{r\rightarrow 0} \frac{2r^2 - 3r + 1}{2(1+r)r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $-2y''x(x-1)+xy'-y=0$ gives

$$\begin{aligned}&-2\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.\\&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)\\&+x\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\&-Cy_1(x)\ln(x)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}$$

Which can be written as

$$\begin{aligned}&\left(\left(-2y_1''(x)\,x(x-1)+y_1'(x)\,x-y_1(x)\right)\ln(x)-2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x(x-1)\right.\\&\left.+y_1(x)\right)C-2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)\\&+x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$-2y_1''(x)\,x(x-1)+y_1'(x)\,x-y_1(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(-2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x(x-1)+y_1(x)\right)C\\&-2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)\\&+x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(-4x(x-1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) + (3x-2)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \frac{2(-x^3+x^2)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x^2 - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(-4x(x-1)\left(\sum_{n=0}^{\infty} x^{n} a_n (n+1)\right) + (3x-2)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} + \frac{2(-x^3+x^2)\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x^2 - \left(\sum_{n=0}^{\infty} b_n x^{n}\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{aligned}\sum_{n=0}^{\infty}\left(-4C\,x^{n+1} a_n (n+1)\right) &+ \left(\sum_{n=0}^{\infty} 4C\,x^{n} a_n (n+1)\right) \\ &+ \left(\sum_{n=0}^{\infty} 3C\,x^{n+1} a_n\right) + \sum_{n=0}^{\infty}\left(-2C a_n x^{n}\right) + \sum_{n=0}^{\infty}\left(-2x^{n} b_n n(n-1)\right) \\ &+ \left(\sum_{n=0}^{\infty} 2n\,x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} x^{n} b_n n\right) + \sum_{n=0}^{\infty}\left(-b_n x^{n}\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-4C\,x^{n+1} a_n (n+1)\right) = \sum_{n=2}^{\infty}\left(-4C a_{-2+n} (n-1)\,x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 4C\,x^{n} a_n (n+1) = \sum_{n=1}^{\infty} 4C a_{n-1} n\,x^{n-1}$$

$$\sum_{n=0}^{\infty} 3C\, x^{n+1} a_n = \sum_{n=2}^{\infty} 3C a_{-2+n} x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-2C a_n x^n\right) = \sum_{n=1}^{\infty} \left(-2C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2x^n b_n n(n-1)\right) = \sum_{n=1}^{\infty} \left(-2(n-1)\, b_{n-1}(-2+n)\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} x^n b_n n = \sum_{n=1}^{\infty} (n-1)\, b_{n-1} x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-b_n x^n\right) = \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left(-4C a_{-2+n}(n-1)\, x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 4C a_{n-1} n\, x^{n-1}\right) \\
&+ \left(\sum_{n=2}^{\infty} 3C a_{-2+n} x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-2C a_{n-1} x^{n-1}\right) \\
&+ \sum_{n=1}^{\infty} \left(-2(n-1)\, b_{n-1}(-2+n)\, x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 2n\, x^{n-1} b_n (n-1)\right) \\
&+ \left(\sum_{n=1}^{\infty} (n-1)\, b_{n-1} x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$2C - 1 = 0$$

Which is solved for C. Solving for C gives

$$C = \frac{1}{2}$$

For $n = 2$, Eq (2B) gives

$$(-a_0 + 6a_1)\, C + 4b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1}{2}+4b_2=0$$

Solving the above for b_2 gives

$$b_2=\frac{1}{8}$$

For $n=3$, Eq (2B) gives

$$(-5a_1+10a_2)\,C-3b_2+12b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{3}{8}+12b_3=0$$

Solving the above for b_3 gives

$$b_3=\frac{1}{32}$$

For $n=4$, Eq (2B) gives

$$(-9a_2+14a_3)\,C-10b_3+24b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{5}{16}+24b_4=0$$

Solving the above for b_4 gives

$$b_4=\frac{5}{384}$$

For $n=5$, Eq (2B) gives

$$(-13a_3+18a_4)\,C-21b_4+40b_5=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{35}{128}+40b_5=0$$

Solving the above for b_5 gives

$$b_5=\frac{7}{1024}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_nx^{n+r_2}\right)$$

Using the above value found for $C = \frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = \frac{1}{2}\left(x\left(1+O\left(x^6\right)\right)\right)\ln\left(x\right) + 1 + \frac{x^2}{8} + \frac{x^3}{32} + \frac{5x^4}{384} + \frac{7x^5}{1024} + O\left(x^6\right)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1+O\left(x^6\right)\right) + c_2\left(\frac{1}{2}\left(x\left(1+O\left(x^6\right)\right)\right)\ln\left(x\right) + 1 + \frac{x^2}{8} + \frac{x^3}{32} + \frac{5x^4}{384} + \frac{7x^5}{1024} + O\left(x^6\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1+O\left(x^6\right)\right) + c_2\left(\frac{x(1+O(x^6))\ln\left(x\right)}{2} + 1 + \frac{x^2}{8} + \frac{x^3}{32} + \frac{5x^4}{384} + \frac{7x^5}{1024} + O\left(x^6\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\left(1+O\left(x^6\right)\right) + c_2\left(\frac{x(1+O(x^6))\ln\left(x\right)}{2} + 1 + \frac{x^2}{8} + \frac{x^3}{32} + \frac{5x^4}{384} + \frac{7x^5}{1024} + O\left(x^6\right)\right) \quad (1)$$

Verification of solutions

$$y = c_1 x\left(1+O\left(x^6\right)\right) + c_2\left(\frac{x(1+O(x^6))\ln\left(x\right)}{2} + 1 + \frac{x^2}{8} + \frac{x^3}{32} + \frac{5x^4}{384} + \frac{7x^5}{1024} + O\left(x^6\right)\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
Order:=6;
dsolve(2*x*(1-x)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(\frac{1}{2}x + \mathrm{O}\left(x^6\right)\right)\ln\left(x\right)c_2 + c_1 x\left(1 + \mathrm{O}\left(x^6\right)\right) + \left(1 - \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{32}x^3 + \frac{5}{384}x^4 + \frac{7}{1024}x^5 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 43

```
AsymptoticDSolveValue[2*x*(1-x)*y''[x]+x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{384}\left(5x^4 + 12x^3 + 48x^2 - 768x + 384\right) + \frac{1}{2}x\log(x)\right) + c_2 x$$

5.3 problem 8

Internal problem ID [4727]
Internal file name [`OUTPUT/4220_Sunday_June_05_2022_12_43_02_PM_64376133/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XVI. page 220
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{2x(1-x)\,y''+(1-11x)\,y'-10y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-2x^2+2x)\,y''+(1-11x)\,y'-10y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{-1+11x}{2x\,(x-1)}$$
$$q(x)=\frac{5}{x\,(x-1)}$$

Table 84: Table $p(x), q(x)$ singularites.

$p(x) = \frac{-1+11x}{2x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{5}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-2y''x(x-1) + (1-11x)\,y' - 10y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned}&-2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) \\ &+ (1-11x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 10\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0\end{aligned} \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-11x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-10 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-11x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-11a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-10 a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-10 a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-11a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-10 a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$2x^{-1+r}a_0r(-1+r) + ra_0x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r}r(-1+r) + r\,x^{-1+r}\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}(2r-1) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{2} \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2r-1) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &-2a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) \\ &- 11a_{n-1}(n+r-1) + a_n(n+r) - 10a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(2n^2 + 4nr + 2r^2 + 5n + 5r + 3)}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}(2n^2 + 7n + 6)}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2r^2 + 9r + 10}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+9r+10}{2r^2+3r+1}$	5

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^3 + 36r^2 + 107r + 105}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = 14$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+9r+10}{2r^2+3r+1}$	5
a_2	$\frac{4r^3+36r^2+107r+105}{4r^3+12r^2+11r+3}$	14

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4r^3 + 48r^2 + 191r + 252}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = 30$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+9r+10}{2r^2+3r+1}$	5
a_2	$\frac{4r^3+36r^2+107r+105}{4r^3+12r^2+11r+3}$	14
a_3	$\frac{4r^3+48r^2+191r+252}{4r^3+12r^2+11r+3}$	30

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^3 + 60r^2 + 299r + 495}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = 55$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+9r+10}{2r^2+3r+1}$	5
a_2	$\frac{4r^3+36r^2+107r+105}{4r^3+12r^2+11r+3}$	14
a_3	$\frac{4r^3+48r^2+191r+252}{4r^3+12r^2+11r+3}$	30
a_4	$\frac{4r^3+60r^2+299r+495}{4r^3+12r^2+11r+3}$	55

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{4r^3 + 72r^2 + 431r + 858}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = 91$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+9r+10}{2r^2+3r+1}$	5
a_2	$\frac{4r^3+36r^2+107r+105}{4r^3+12r^2+11r+3}$	14
a_3	$\frac{4r^3+48r^2+191r+252}{4r^3+12r^2+11r+3}$	30
a_4	$\frac{4r^3+60r^2+299r+495}{4r^3+12r^2+11r+3}$	55
a_5	$\frac{4r^3+72r^2+431r+858}{4r^3+12r^2+11r+3}$	91

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + 5x + 14x^2 + 30x^3 + 55x^4 + 91x^5 + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&-2b_{n-1}(n+r-1)(n+r-2) + 2b_n(n+r)(n+r-1) \\
&- 11b_{n-1}(n+r-1) + (n+r)b_n - 10b_{n-1} = 0
\end{aligned} \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}(2n^2 + 4nr + 2r^2 + 5n + 5r + 3)}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{b_{n-1}(2n^2 + 5n + 3)}{2n^2 - n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2r^2 + 9r + 10}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = 10$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2+9r+10}{2r^2+3r+1}$	10

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^3 + 36r^2 + 107r + 105}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = 0$ becomes

$$b_2 = 35$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2+9r+10}{2r^2+3r+1}$	10
b_2	$\frac{4r^3+36r^2+107r+105}{4r^3+12r^2+11r+3}$	35

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{4r^3 + 48r^2 + 191r + 252}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = 0$ becomes

$$b_3 = 84$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2+9r+10}{2r^2+3r+1}$	10
b_2	$\frac{4r^3+36r^2+107r+105}{4r^3+12r^2+11r+3}$	35
b_3	$\frac{4r^3+48r^2+191r+252}{4r^3+12r^2+11r+3}$	84

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^3 + 60r^2 + 299r + 495}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = 0$ becomes

$$b_4 = 165$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2+9r+10}{2r^2+3r+1}$	10
b_2	$\frac{4r^3+36r^2+107r+105}{4r^3+12r^2+11r+3}$	35
b_3	$\frac{4r^3+48r^2+191r+252}{4r^3+12r^2+11r+3}$	84
b_4	$\frac{4r^3+60r^2+299r+495}{4r^3+12r^2+11r+3}$	165

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{4r^3 + 72r^2 + 431r + 858}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = 0$ becomes

$$b_5 = 286$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2+9r+10}{2r^2+3r+1}$	10
b_2	$\frac{4r^3+36r^2+107r+105}{4r^3+12r^2+11r+3}$	35
b_3	$\frac{4r^3+48r^2+191r+252}{4r^3+12r^2+11r+3}$	84
b_4	$\frac{4r^3+60r^2+299r+495}{4r^3+12r^2+11r+3}$	165
b_5	$\frac{4r^3+72r^2+431r+858}{4r^3+12r^2+11r+3}$	286

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 + 10x + 35x^2 + 84x^3 + 165x^4 + 286x^5 + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x} \left(1 + 5x + 14x^2 + 30x^3 + 55x^4 + 91x^5 + O(x^6)\right) \\
&\quad + c_2 \left(1 + 10x + 35x^2 + 84x^3 + 165x^4 + 286x^5 + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x} \left(1 + 5x + 14x^2 + 30x^3 + 55x^4 + 91x^5 + O(x^6)\right) \\
&\quad + c_2 \left(1 + 10x + 35x^2 + 84x^3 + 165x^4 + 286x^5 + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \sqrt{x} \left(1 + 5x + 14x^2 + 30x^3 + 55x^4 + 91x^5 + O(x^6)\right) \\
&\quad + c_2 \left(1 + 10x + 35x^2 + 84x^3 + 165x^4 + 286x^5 + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \sqrt{x} \left(1 + 5x + 14x^2 + 30x^3 + 55x^4 + 91x^5 + O(x^6)\right) \\
&\quad + c_2 \left(1 + 10x + 35x^2 + 84x^3 + 165x^4 + 286x^5 + O(x^6)\right)
\end{aligned}
$$

Verified OK.

5.3.1 Maple step by step solution

Let's solve

$-2y''x(x-1)+(1-11x)\,y'-10y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(-1+11x)y'}{2x(x-1)}-\frac{5y}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(-1+11x)y'}{2x(x-1)}+\frac{5y}{x(x-1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{-1+11x}{2x(x-1)}, P_3(x)=\frac{5}{x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2y''x(x-1)+(-1+11x)\,y'+10y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^{m}\cdot y''$ to series expansion for $m=1..2$

  $$x^{m}\cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^{m}\cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_{0}r(-1+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_{k}(2k+2r+5)\,(k+r+2)\right) x^{k-}\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2(k+1+r)\left(k+r+\frac{1}{2}\right)a_{k+1} + 2a_{k}\left(k+r+\frac{5}{2}\right)(k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_{k}(2k+2r+5)(k+r+2)}{(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r=0$

  $$a_{k+1} = \frac{a_{k}(2k+5)(k+2)}{(k+1)(2k+1)}$$

- Solution for $r=0$

  $$\left[y = \sum_{k=0}^{\infty} a_{k}x^{k}, a_{k+1} = \frac{a_{k}(2k+5)(k+2)}{(k+1)(2k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

  $$a_{k+1} = \frac{a_{k}(2k+6)\left(k+\frac{5}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$$

- Solution for $r=\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_{k}x^{k+\frac{1}{2}}, a_{k+1} = \frac{a_{k}(2k+6)\left(k+\frac{5}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k(2k+5)(k+2)}{(k+1)(2k+1)}, b_{k+1}=\frac{b_k(2k+6)\left(k+\frac{5}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
Order:=6;
dsolve(2*x*(1-x)*diff(y(x),x$2)+(1-11*x)*diff(y(x),x)-10*y(x)=0,y(x),type='series',x=0);
```

$$\begin{aligned} y(x) = & c_1\sqrt{x}\left(1+5x+14x^2+30x^3+55x^4+91x^5+\mathrm{O}\left(x^6\right)\right) \\ & + c_2\left(1+10x+35x^2+84x^3+165x^4+286x^5+\mathrm{O}\left(x^6\right)\right) \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 65

```
AsymptoticDSolveValue[2*x*(1-x)*y''[x]+(1-11*x)*y'[x]-10*y[x]==0,y[x],{x,0,5}]
```

$$\begin{aligned} y(x) \to & c_1\sqrt{x}\left(91x^5+55x^4+30x^3+14x^2+5x+1\right) \\ & + c_2\left(286x^5+165x^4+84x^3+35x^2+10x+1\right) \end{aligned}$$

5.4 problem 9

Internal problem ID [4728]
Internal file name [OUTPUT/4221_Sunday_June_05_2022_12_43_11_PM_13817898/index.tex]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XVI. page 220
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[_Jacobi]
```

$$\boxed{x(1-x)\,y''+\frac{(1-2x)\,y'}{3}+\frac{20y}{9}=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^2+x)\,y''+\left(-\frac{2x}{3}+\frac{1}{3}\right)y'+\frac{20y}{9}=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x-1}{3x\,(x-1)}$$
$$q(x)=-\frac{20}{9x\,(x-1)}$$

Table 86: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x-1}{3x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{20}{9x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-1) + \left(-\frac{2x}{3} + \frac{1}{3}\right) y' + \frac{20y}{9} = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) + \left(-\frac{2x}{3} + \frac{1}{3}\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \frac{20\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)}{9} = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r) (n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \sum_{n=0}^{\infty} \left(-\frac{2x^{n+r} a_n (n+r)}{3}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{(n+r) a_n x^{n+r-1}}{3}\right) + \left(\sum_{n=0}^{\infty} \frac{20 a_n x^{n+r}}{9}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r) (n+r-1)\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) (n+r-2) x^{n+r-1}\right) \\
\sum_{n=0}^{\infty} \left(-\frac{2x^{n+r} a_n (n+r)}{3}\right) &= \sum_{n=1}^{\infty} \left(-\frac{2a_{n-1} (n+r-1) x^{n+r-1}}{3}\right) \\
\sum_{n=0}^{\infty} \frac{20 a_n x^{n+r}}{9} &= \sum_{n=1}^{\infty} \frac{20 a_{n-1} x^{n+r-1}}{9}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) (n+r-2) x^{n+r-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \sum_{n=1}^{\infty} \left(-\frac{2a_{n-1} (n+r-1) x^{n+r-1}}{3}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{(n+r) a_n x^{n+r-1}}{3}\right) + \left(\sum_{n=1}^{\infty} \frac{20 a_{n-1} x^{n+r-1}}{9}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n (n+r) (n+r-1) + \frac{(n+r) a_n x^{n+r-1}}{3} = 0
$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + \frac{r a_0 x^{-1+r}}{3} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + \frac{r\, x^{-1+r}}{3}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r} \left(-\frac{2}{3} + r\right) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{2}{3} r = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{2}{3} \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r} \left(-\frac{2}{3} + r\right) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&-a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\
&- \frac{2a_{n-1}(n+r-1)}{3} + \frac{a_n(n+r)}{3} + \frac{20a_{n-1}}{9} = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}(9n^2 + 18nr + 9r^2 - 21n - 21r - 8)}{9n^2 + 18nr + 9r^2 - 6n - 6r} \tag{4}
$$

Which for the root $r = \frac{2}{3}$ becomes

$$
a_n = \frac{3a_{n-1}(n^2 - n - 2)}{n(3n+2)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = \frac{9r^2 - 3r - 20}{9r^2 + 12r + 3}
$$

Which for the root $r = \frac{2}{3}$ becomes

$$
a_1 = -\frac{6}{5}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9r^2-3r-20}{9r^2+12r+3}$	$-\frac{6}{5}$

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{27r^3 - 117r + 70}{27r^3 + 90r^2 + 81r + 18}
$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9r^2-3r-20}{9r^2+12r+3}$	$-\frac{6}{5}$
a_2	$\frac{27r^3-117r+70}{27r^3+90r^2+81r+18}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{27r^3 + 27r^2 - 180r + 100}{27(r+3)(r+2)(1+r)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9r^2-3r-20}{9r^2+12r+3}$	$-\frac{6}{5}$
a_2	$\frac{27r^3-117r+70}{27r^3+90r^2+81r+18}$	0
a_3	$\frac{27r^3+27r^2-180r+100}{27(r+3)(r+2)(1+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r^4 + 270r^3 - 513r^2 - 582r + 520}{81(1+r)(r+2)(r+3)(r+4)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9r^2-3r-20}{9r^2+12r+3}$	$-\frac{6}{5}$
a_2	$\frac{27r^3-117r+70}{27r^3+90r^2+81r+18}$	0
a_3	$\frac{27r^3+27r^2-180r+100}{27(r+3)(r+2)(1+r)}$	0
a_4	$\frac{81r^4+270r^3-513r^2-582r+520}{81(1+r)(r+2)(r+3)(r+4)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{243r^5 + 1620r^4 + 675r^3 - 6390r^2 - 3288r + 4480}{243\,(r+5)\,(1+r)\,(r+2)\,(r+3)\,(r+4)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9r^2-3r-20}{9r^2+12r+3}$	$-\frac{6}{5}$
a_2	$\frac{27r^3-117r+70}{27r^3+90r^2+81r+18}$	0
a_3	$\frac{27r^3+27r^2-180r+100}{27(r+3)(r+2)(1+r)}$	0
a_4	$\frac{81r^4+270r^3-513r^2-582r+520}{81(1+r)(r+2)(r+3)(r+4)}$	0
a_5	$\frac{243r^5+1620r^4+675r^3-6390r^2-3288r+4480}{243(r+5)(1+r)(r+2)(r+3)(r+4)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{2}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{2}{3}}\left(1 - \frac{6x}{5} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$-b_{n-1}(n+r-1)(n+r-2)+b_n(n+r)(n+r-1) - \frac{2b_{n-1}(n+r-1)}{3}+\frac{(n+r)\,b_n}{3}+\frac{20b_{n-1}}{9}=0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}(9n^2+18nr+9r^2-21n-21r-8)}{9n^2+18nr+9r^2-6n-6r} \tag{4}$$

Which for the root $r=0$ becomes

$$b_n = \frac{b_{n-1}(9n^2-21n-8)}{9n^2-6n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r=0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n=1$, using the above recursive equation gives

$$b_1 = \frac{9r^2-3r-20}{9r^2+12r+3}$$

Which for the root $r=0$ becomes

$$b_1 = -\frac{20}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9r^2-3r-20}{9r^2+12r+3}$	$-\frac{20}{3}$

For $n=2$, using the above recursive equation gives

$$b_2 = \frac{27r^3-117r+70}{27r^3+90r^2+81r+18}$$

Which for the root $r=0$ becomes

$$b_2 = \frac{35}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9r^2-3r-20}{9r^2+12r+3}$	$-\frac{20}{3}$
b_2	$\frac{27r^3-117r+70}{27r^3+90r^2+81r+18}$	$\frac{35}{9}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{27r^3 + 27r^2 - 180r + 100}{27\,(r+3)\,(r+2)\,(1+r)}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{50}{81}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9r^2-3r-20}{9r^2+12r+3}$	$-\frac{20}{3}$
b_2	$\frac{27r^3-117r+70}{27r^3+90r^2+81r+18}$	$\frac{35}{9}$
b_3	$\frac{27r^3+27r^2-180r+100}{27(r+3)(r+2)(1+r)}$	$\frac{50}{81}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81r^4 + 270r^3 - 513r^2 - 582r + 520}{81\,(1+r)\,(r+2)\,(r+3)\,(r+4)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{65}{243}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9r^2-3r-20}{9r^2+12r+3}$	$-\frac{20}{3}$
b_2	$\frac{27r^3-117r+70}{27r^3+90r^2+81r+18}$	$\frac{35}{9}$
b_3	$\frac{27r^3+27r^2-180r+100}{27(r+3)(r+2)(1+r)}$	$\frac{50}{81}$
b_4	$\frac{81r^4+270r^3-513r^2-582r+520}{81(1+r)(r+2)(r+3)(r+4)}$	$\frac{65}{243}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{243r^5 + 1620r^4 + 675r^3 - 6390r^2 - 3288r + 4480}{243\,(r+5)\,(1+r)\,(r+2)\,(r+3)\,(r+4)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{112}{729}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9r^2-3r-20}{9r^2+12r+3}$	$-\frac{20}{3}$
b_2	$\frac{27r^3-117r+70}{27r^3+90r^2+81r+18}$	$\frac{35}{9}$
b_3	$\frac{27r^3+27r^2-180r+100}{27(r+3)(r+2)(1+r)}$	$\frac{50}{81}$
b_4	$\frac{81r^4+270r^3-513r^2-582r+520}{81(1+r)(r+2)(r+3)(r+4)}$	$\frac{65}{243}$
b_5	$\frac{243r^5+1620r^4+675r^3-6390r^2-3288r+4480}{243(r+5)(1+r)(r+2)(r+3)(r+4)}$	$\frac{112}{729}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - \frac{20x}{3} + \frac{35x^2}{9} + \frac{50x^3}{81} + \frac{65x^4}{243} + \frac{112x^5}{729} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{2}{3}} \left(1 - \frac{6x}{5} + O(x^6)\right) + c_2 \left(1 - \frac{20x}{3} + \frac{35x^2}{9} + \frac{50x^3}{81} + \frac{65x^4}{243} + \frac{112x^5}{729} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{2}{3}} \left(1 - \frac{6x}{5} + O(x^6)\right) + c_2 \left(1 - \frac{20x}{3} + \frac{35x^2}{9} + \frac{50x^3}{81} + \frac{65x^4}{243} + \frac{112x^5}{729} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{2}{3}} \left(1 - \frac{6x}{5} + O(x^6)\right) + c_2 \left(1 - \frac{20x}{3} + \frac{35x^2}{9} + \frac{50x^3}{81} + \frac{65x^4}{243} + \frac{112x^5}{729} + O(x^6)\right) \quad (1)$$

Verification of solutions

$$y = c_1 x^{\frac{2}{3}} \left(1 - \frac{6x}{5} + O(x^6)\right) + c_2 \left(1 - \frac{20x}{3} + \frac{35x^2}{9} + \frac{50x^3}{81} + \frac{65x^4}{243} + \frac{112x^5}{729} + O(x^6)\right)$$

Verified OK.

5.4.1 Maple step by step solution

Let's solve

$-y''x(x-1) + \left(-\frac{2x}{3} + \frac{1}{3}\right) y' + \frac{20y}{9} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x-1)y'}{3x(x-1)} + \frac{20y}{9x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x-1)y'}{3x(x-1)} - \frac{20y}{9x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x-1}{3x(x-1)}, P_3(x) = -\frac{20}{9x(x-1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $9y''x(x-1) + (6x-3)\, y' - 20y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) \, x^{k+r-1+m}$$

- ◦ Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) \, x^{k+r}$$

- ◦ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1) \, x^{k+r-2+m}$$

- ◦ Shift index using $k- > k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r) \, x^{k+r}$$

Rewrite ODE with series expansions

$$-3a_0 r(-2+3r) \, x^{-1+r} + \left(\sum_{k=0}^{\infty} (-3a_{k+1}(k+1+r)\,(3k+1+3r) + a_k(3k+3r+4)\,(3k+3r-5$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-3r(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-9\left(k+\frac{1}{3}+r\right)(k+1+r)\,a_{k+1} + 9a_k\left(k+r-\frac{5}{3}\right)\left(k+r+\frac{4}{3}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(3k+3r-5)(3k+3r+4)}{3(3k+1+3r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k(3k-5)(3k+4)}{3(3k+1)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(3k-5)(3k+4)}{3(3k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{2}{3}$; series terminates at $k = 1$

$a_{k+1} = \frac{a_k(3k-3)(3k+6)}{3(3k+3)\left(k+\frac{5}{3}\right)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{6a_0}{5}$

- Terminating series solution of the ODE for $r = \frac{2}{3}$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{6x}{5}\right)$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + b_0 \cdot \left(1 - \frac{6x}{5}\right), a_{k+1} = \frac{a_k(3k-5)(3k+4)}{3(3k+1)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
Order:=6;
dsolve(x*(1-x)*diff(y(x),x$2)+1/3*(1-2*x)*diff(y(x),x)+20/9*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{2}{3}}\left(1 - \frac{6}{5}x + \mathrm{O}\left(x^6\right)\right) + c_2\left(1 - \frac{20}{3}x + \frac{35}{9}x^2 + \frac{50}{81}x^3 + \frac{65}{243}x^4 + \frac{112}{729}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 57

```
AsymptoticDSolveValue[x*(1-x)*y''[x]+1/3*(1-2*x)*y'[x]+20/9*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(1 - \frac{6x}{5}\right) x^{2/3} + c_2 \left(\frac{112x^5}{729} + \frac{65x^4}{243} + \frac{50x^3}{81} + \frac{35x^2}{9} - \frac{20x}{3} + 1\right)$$

5.5 problem 10

Internal problem ID [4729]
Internal file name [`OUTPUT/4222_Sunday_June_05_2022_12_43_19_PM_2179187/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XVI. page 220
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{2x(1-x)\,y''+y'+4y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^2+2x\right)y''+y'+4y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{1}{2x\,(x-1)}$$
$$q(x)=-\frac{2}{x\,(x-1)}$$

Table 88: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{2}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-2y''x(x-1) + y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x(x-1) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}2x^{n+r-1}a_n(n+r)(n+r-1)\right) \\ +\left(\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}4a_nx^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}4a_nx^{n+r}=\sum_{n=1}^{\infty}4a_{n-1}x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty}2x^{n+r-1}a_n(n+r)(n+r-1)\right) \\ +\left(\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}\right)+\left(\sum_{n=1}^{\infty}4a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1}a_n(n+r)(n+r-1)+(n+r)a_nx^{n+r-1}=0$$

When $n=0$ the above becomes

$$2x^{-1+r}a_0r(-1+r)+ra_0x^{-1+r}=0$$

Or

$$\left(2x^{-1+r}r(-1+r)+r\,x^{-1+r}\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}(2r-1) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{2} \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2r-1) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-2a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) + a_n(n+r) + 4a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2(n+r-3)\,a_{n-1}}{2n+2r-1} \quad (4)$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{\left(n - \frac{5}{2}\right) a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-4 + 2r}{1 + 2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{1+2r}$	$-\frac{3}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2 - 12r + 8}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{3}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{1+2r}$	$-\frac{3}{2}$
a_2	$\frac{4r^2-12r+8}{4r^2+8r+3}$	$\frac{3}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^3 - 24r^2 + 16r}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{1+2r}$	$-\frac{3}{2}$
a_2	$\frac{4r^2-12r+8}{4r^2+8r+3}$	$\frac{3}{8}$
a_3	$\frac{8r^3-24r^2+16r}{8r^3+36r^2+46r+15}$	$\frac{1}{16}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16(-1+r)(-2+r)r(1+r)}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{3}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{1+2r}$	$-\frac{3}{2}$
a_2	$\frac{4r^2-12r+8}{4r^2+8r+3}$	$\frac{3}{8}$
a_3	$\frac{8r^3-24r^2+16r}{8r^3+36r^2+46r+15}$	$\frac{1}{16}$
a_4	$\frac{16(-1+r)(-2+r)r(1+r)}{16r^4+128r^3+344r^2+352r+105}$	$\frac{3}{128}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32r^5 - 160r^3 + 128r}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{3}{256}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4+2r}{1+2r}$	$-\frac{3}{2}$
a_2	$\frac{4r^2-12r+8}{4r^2+8r+3}$	$\frac{3}{8}$
a_3	$\frac{8r^3-24r^2+16r}{8r^3+36r^2+46r+15}$	$\frac{1}{16}$
a_4	$\frac{16(-1+r)(-2+r)r(1+r)}{16r^4+128r^3+344r^2+352r+105}$	$\frac{3}{128}$
a_5	$\frac{32r^5-160r^3+128r}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{3}{256}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= \sqrt{x}\left(1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$-2b_{n-1}(n + r - 1)(n + r - 2) + 2b_n(n + r)(n + r - 1) + (n + r)b_n + 4b_{n-1} = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{2(n + r - 3)\, b_{n-1}}{2n + 2r - 1} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{2(n - 3)\, b_{n-1}}{2n - 1} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-4 + 2r}{1 + 2r}$$

Which for the root $r = 0$ becomes

$$b_1 = -4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{1+2r}$	-4

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^2 - 12r + 8}{4r^2 + 8r + 3}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{8}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{1+2r}$	-4
b_2	$\frac{4r^2-12r+8}{4r^2+8r+3}$	$\frac{8}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8r^3 - 24r^2 + 16r}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{1+2r}$	-4
b_2	$\frac{4r^2-12r+8}{4r^2+8r+3}$	$\frac{8}{3}$
b_3	$\frac{8r^3-24r^2+16r}{8r^3+36r^2+46r+15}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16(-1+r)(-2+r)r(1+r)}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{1+2r}$	-4
b_2	$\frac{4r^2-12r+8}{4r^2+8r+3}$	$\frac{8}{3}$
b_3	$\frac{8r^3-24r^2+16r}{8r^3+36r^2+46r+15}$	0
b_4	$\frac{16(-1+r)(-2+r)r(1+r)}{16r^4+128r^3+344r^2+352r+105}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32r^5 - 160r^3 + 128r}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = 0$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4+2r}{1+2r}$	-4
b_2	$\frac{4r^2-12r+8}{4r^2+8r+3}$	$\frac{8}{3}$
b_3	$\frac{8r^3-24r^2+16r}{8r^3+36r^2+46r+15}$	0
b_4	$\frac{16(-1+r)(-2+r)r(1+r)}{16r^4+128r^3+344r^2+352r+105}$	0
b_5	$\frac{32r^5-160r^3+128r}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 - 4x + \frac{8x^2}{3} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} + O(x^6)\right) + c_2\left(1 - 4x + \frac{8x^2}{3} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} + O(x^6)\right) + c_2\left(1 - 4x + \frac{8x^2}{3} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \sqrt{x}\left(1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} + O(x^6)\right) + c_2\left(1 - 4x + \frac{8x^2}{3} + O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 \sqrt{x}\left(1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} + O(x^6)\right) + c_2\left(1 - 4x + \frac{8x^2}{3} + O(x^6)\right)
$$

Verified OK.

5.5.1 Maple step by step solution

Let's solve

$-2y''x(x-1)+y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2y}{x(x-1)}+\frac{y'}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{y'}{2x(x-1)}-\frac{2y}{x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{1}{2x(x-1)}, P_3(x)=-\frac{2}{x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2y''x(x-1)-y'-4y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

    $y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1}$

- Shift index using $k->k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(2r-1)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(2k+1+2r) + 2a_k(k+1+r)(k+r-2)\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(2r-1) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(\left(-k-r-\frac{1}{2}\right) a_{k+1} + a_k(k+r-2)\right)(k+1+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r-2)}{2k+1+2r}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{2a_k(k-2)}{2k+1}$

- Apply recursion relation for $k = 0$

  $a_1 = -4a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{2a_1}{3}$

- Express in terms of a_0

  $a_2 = \frac{8a_0}{3}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - 4x + \frac{8}{3}x^2\right)$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{2a_k\left(k-\frac{3}{2}\right)}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{3}{2}\right)}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(1 - 4x + \frac{8}{3}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2b_k\left(k-\frac{3}{2}\right)}{2k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 38

```
Order:=6;
dsolve(2*x*(1-x)*diff(y(x),x$2)+diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \frac{3}{256}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(1 - 4x + \frac{8}{3}x^2 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 62

```
AsymptoticDSolveValue[2*x*(1-x)*y''[x]+y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{8x^2}{3} - 4x + 1\right) + c_1\sqrt{x}\left(\frac{3x^5}{256} + \frac{3x^4}{128} + \frac{x^3}{16} + \frac{3x^2}{8} - \frac{3x}{2} + 1\right)$$

5.6 problem 11

Internal problem ID [4730]
Internal file name [`OUTPUT/4223_Sunday_June_05_2022_12_43_27_PM_43790928/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter VII, Solutions in series. Examples XVI. page 220
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4y'' + \frac{3(-x^2+2)\,y}{(-x^2+1)^2} = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (88)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (89)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{3(x^2-2)\,y}{4\left(x^2-1\right)^2}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= \frac{(3x^4-9x^2+6)\,y'-6xy(x^2-3)}{4\left(x^2-1\right)^3}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= \frac{(-48x^5+192x^3-144x)\,y'+81y\left(x^4-4x^2-\frac{4}{9}\right)}{16\left(x^2-1\right)^4}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \frac{\frac{45(5x^6-25x^4+16x^2+4)y'}{16}-\frac{45xy(x^4-5x^2-2)}{2}}{\left(x^2-1\right)^5}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= \frac{(-5040x^7+30240x^5-9360x^3-15840x)\,y'+7875\left(x^6-6x^4-\frac{948}{175}x^2-\frac{8}{35}\right)y}{64\left(x^2-1\right)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{3y(0)}{2}\\
F_1 &= -\frac{3y'(0)}{2}\\
F_2 &= -\frac{9y(0)}{4}\\
F_3 &= -\frac{45y'(0)}{4}\\
F_4 &= -\frac{225y(0)}{8}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1-\frac{3}{4}x^2-\frac{3}{32}x^4-\frac{5}{128}x^6\right)y(0)+\left(x-\frac{1}{4}x^3-\frac{3}{32}x^5\right)y'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$4y''\left(x^2-1\right)^2+\left(-3x^2+6\right)y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty}n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$4\left(\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}\right)\left(x^2-1\right)^2+\left(-3x^2+6\right)\left(\sum_{n=0}^{\infty}a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}4n\,x^{n+2}a_n(n-1)\right)+\sum_{n=2}^{\infty}\left(-8x^n a_n n(n-1)\right)$$
$$+\left(\sum_{n=2}^{\infty}4n(n-1)\,a_n x^{n-2}\right)+\sum_{n=0}^{\infty}\left(-3x^{n+2}a_n\right)+\left(\sum_{n=0}^{\infty}6a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n\, x^{n+2} a_n (n-1) = \sum_{n=4}^{\infty} 4(n-2)\, a_{n-2} (n-3)\, x^n$$

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2} (n+1)\, x^n$$

$$\sum_{n=0}^{\infty} \left(-3x^{n+2} a_n\right) = \sum_{n=2}^{\infty} \left(-3a_{n-2} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{gathered}\left(\sum_{n=4}^{\infty} 4(n-2)\, a_{n-2} (n-3)\, x^n\right) + \sum_{n=2}^{\infty} \left(-8x^n a_n n(n-1)\right) \\ + \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2} (n+1)\, x^n\right) + \sum_{n=2}^{\infty} \left(-3a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} 6a_n x^n\right) = 0\end{gathered} \tag{3}$$

$n = 0$ gives

$$8a_2 + 6a_0 = 0$$

$$a_2 = -\frac{3a_0}{4}$$

$n = 1$ gives

$$24a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{4}$$

$n = 2$ gives

$$-10a_2 + 48a_4 - 3a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{3a_0}{32}$$

$n = 3$ gives

$$-42a_3 + 80a_5 - 3a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = -\frac{3a_1}{32}$$

For $4 \leq n$, the recurrence equation is

$$4(n-2)\, a_{n-2}(n-3) - 8na_n(n-1) + 4(n+2)\, a_{n+2}(n+1) - 3a_{n-2} + 6a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$(5) \qquad \begin{aligned} a_{n+2} &= \frac{8n^2a_n - 4n^2a_{n-2} - 8na_n + 20na_{n-2} - 6a_n - 21a_{n-2}}{4\,(n+2)\,(n+1)} \\ &= \frac{(8n^2 - 8n - 6)\, a_n}{4\,(n+2)\,(n+1)} + \frac{(-4n^2 + 20n - 21)\, a_{n-2}}{4\,(n+2)\,(n+1)} \end{aligned}$$

For $n = 4$ the recurrence equation gives

$$5a_2 - 90a_4 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{5a_0}{128}$$

For $n = 5$ the recurrence equation gives

$$21a_3 - 154a_5 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{7a_1}{128}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{3}{4} a_0 x^2 - \frac{1}{4} a_1 x^3 - \frac{3}{32} a_0 x^4 - \frac{3}{32} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{4}x^2 - \frac{3}{32}x^4\right) a_0 + \left(x - \frac{1}{4}x^3 - \frac{3}{32}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{4}x^2 - \frac{3}{32}x^4\right) c_1 + \left(x - \frac{1}{4}x^3 - \frac{3}{32}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{3}{4}x^2 - \frac{3}{32}x^4 - \frac{5}{128}x^6\right) y(0) + \left(x - \frac{1}{4}x^3 - \frac{3}{32}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{3}{4}x^2 - \frac{3}{32}x^4\right) c_1 + \left(x - \frac{1}{4}x^3 - \frac{3}{32}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{3}{4}x^2 - \frac{3}{32}x^4 - \frac{5}{128}x^6\right) y(0) + \left(x - \frac{1}{4}x^3 - \frac{3}{32}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{3}{4}x^2 - \frac{3}{32}x^4\right) c_1 + \left(x - \frac{1}{4}x^3 - \frac{3}{32}x^5\right) c_2 + O(x^6)$$

Verified OK.

5.6.1 Maple step by step solution

Let's solve

$4y''(x^2-1)^2 + (-3x^2+6)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{3(x^2-2)y}{4(x^2-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3(x^2-2)y}{4(x^2-1)^2} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{3(x^2-2)}{4(x^2-1)^2}\right]$

  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Bigg|_{x=-1} = 0$

  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Bigg|_{x=-1} = \frac{3}{16}$

  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $4y''(x^2-1)^2 + (-3x^2+6)\, y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^4 - 16u^3 + 16u^2)\left(\frac{d^2}{du^2}y(u)\right) + (-3u^2 + 6u + 3)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..4$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}} y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)(k+r-1) u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}} y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) u^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}(-1+4r)(-3+4r) u^{r}+\left(a_{1}(3+4r)(1+4r)-2a_{0}\left(8r^{2}-8r-3\right)\right) u^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(4k+4r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+4r)(-3+4r)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$$

- Each term must be 0

$$a_{1}(3+4r)(1+4r)-2a_{0}\left(8r^{2}-8r-3\right)=0$$

- Solve for the dependent coefficient(s)

$$a_{1}=\frac{2a_{0}\left(8r^{2}-8r-3\right)}{16r^{2}+16r+3}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(4a_{k}+a_{k-2}-4a_{k-1}\right) k^{2}+4\left(2\left(4a_{k}+a_{k-2}-4a_{k-1}\right) r-4a_{k}-5a_{k-2}+12a_{k-1}\right) k+4\left(4a_{k}+a_{k-2}\right.$$

- Shift index using $k->k+2$

$$4\left(4a_{k+2}+a_{k}-4a_{k+1}\right)(k+2)^{2}+4\left(2\left(4a_{k+2}+a_{k}-4a_{k+1}\right) r-4a_{k+2}-5a_{k}+12a_{k+1}\right)(k+2)+4$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{4k^{2}a_{k}-16k^{2}a_{k+1}+8kra_{k}-32kra_{k+1}+4r^{2}a_{k}-16r^{2}a_{k+1}-4ka_{k}-16ka_{k+1}-4ra_{k}-16ra_{k+1}-3a_{k}+6a_{k+1}}{16k^{2}+32kr+16r^{2}+48k+48r+35}$$

- Recursion relation for $r=\frac{1}{4}$

$$a_{k+2}=-\frac{4k^{2}a_{k}-16k^{2}a_{k+1}-2ka_{k}-24ka_{k+1}-\frac{15}{4}a_{k}+a_{k+1}}{16k^{2}+56k+48}$$

- Solution for $r=\frac{1}{4}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+\frac{1}{4}}, a_{k+2}=-\frac{4k^{2}a_{k}-16k^{2}a_{k+1}-2ka_{k}-24ka_{k+1}-\frac{15}{4}a_{k}+a_{k+1}}{16k^{2}+56k+48}, a_{1}=-\frac{9a_{0}}{8}\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_{k}(x+1)^{k+\frac{1}{4}}, a_{k+2}=-\frac{4k^{2}a_{k}-16k^{2}a_{k+1}-2ka_{k}-24ka_{k+1}-\frac{15}{4}a_{k}+a_{k+1}}{16k^{2}+56k+48}, a_{1}=-\frac{9a_{0}}{8}\right]$$

- Recursion relation for $r=\frac{3}{4}$

$a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} + 2k a_k - 40k a_{k+1} - \frac{15}{4} a_k - 15 a_{k+1}}{16k^2 + 72k + 80}$

- Solution for $r = \frac{3}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} + 2k a_k - 40k a_{k+1} - \frac{15}{4} a_k - 15 a_{k+1}}{16k^2 + 72k + 80}, a_1 = -\frac{3a_0}{8}\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} + 2k a_k - 40k a_{k+1} - \frac{15}{4} a_k - 15 a_{k+1}}{16k^2 + 72k + 80}, a_1 = -\frac{3a_0}{8}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} - 2k a_k - 24k a_{k+1} - \frac{15}{4} a_k + a_{k+1}}{16k^2 + 56k + 48},\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(4*diff(y(x),x$2)+3*(2-x^2)/(1-x^2)^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{4}x^2 - \frac{3}{32}x^4\right) y(0) + \left(x - \frac{1}{4}x^3 - \frac{3}{32}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[4*y''[x]+3*(2-x^2)/(1-x^2)^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{3x^5}{32}-\frac{x^3}{4}+x\right)+c_1\left(-\frac{3x^4}{32}-\frac{3x^2}{4}+1\right)$$

6 Chapter IX, Special forms of differential equations. Examples XVII. page 247

6.1 problem 1

Internal problem ID [4731]
Internal file name [`OUTPUT/4224_Sunday_June_05_2022_12_43_33_PM_739722/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{y' + y^2 = \frac{a^2}{x^4}}$$

6.1.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y^2x^4 - a^2}{x^4}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&2xb_4 + yb_5 + b_2 - \frac{(y^2x^4 - a^2)(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{x^4} \\ &- \frac{(y^2x^4 - a^2)^2 (xa_5 + 2ya_6 + a_3)}{x^8} \\ &- \left(-\frac{4y^2}{x} + \frac{4y^2x^4 - 4a^2}{x^5}\right)(x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1) \\ &+ 2y(x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1) = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^9y^4a_5 + 2x^8y^5a_6 + x^8y^4a_3 - 2x^{10}yb_4 - 2x^9y^2a_4 - x^9y^2b_5 - x^8y^3a_5 - 2x^9yb_2 - x^8y^2a_2 - x^8y^2b_3 - 2a^2x^5}{}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}&-x^9y^4a_5 - 2x^8y^5a_6 - x^8y^4a_3 + 2x^{10}yb_4 + 2x^9y^2a_4 + x^9y^2b_5 + x^8y^3a_5 + 2x^9yb_2 \\ &+ x^8y^2a_2 + x^8y^2b_3 + 2a^2x^5y^2a_5 + 4a^2x^4y^3a_6 + 2x^9b_4 + 2x^8yb_1 + yb_5x^8 \\ &+ 2a^2x^4y^2a_3 + b_2x^8 + 2a^2x^5a_4 + a^2x^5b_5 + 3a^2x^4ya_5 + 2a^2x^4yb_6 + 4a^2x^3y^2a_6 \\ &+ 3a^2x^4a_2 + a^2x^4b_3 + 4a^2x^3ya_3 - a^4xa_5 - 2a^4ya_6 + 4a^2x^3a_1 - a^4a_3 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-a_5v_1^9v_2^4 - 2a_6v_1^8v_2^5 - a_3v_1^8v_2^4 + 2a_4v_1^9v_2^2 + a_5v_1^8v_2^3 + 2b_4v_1^{10}v_2 + b_5v_1^9v_2^2 + a_2v_1^8v_2^2 \\ &+ 2b_2v_1^9v_2 + b_3v_1^8v_2^2 + 2a^2a_5v_1^5v_2^2 + 4a^2a_6v_1^4v_2^3 + 2b_1v_1^8v_2 + 2b_4v_1^9 + b_5v_1^8v_2 \\ &+ 2a^2a_3v_1^4v_2^2 + b_2v_1^8 + 2a^2a_4v_1^5 + 3a^2a_5v_1^4v_2 + 4a^2a_6v_1^3v_2^2 + a^2b_5v_1^5 + 2a^2b_6v_1^4v_2 \\ &+ 3a^2a_2v_1^4 + 4a^2a_3v_1^3v_2 + a^2b_3v_1^4 - a^4a_5v_1 - 2a^4a_6v_2 + 4a^2a_1v_1^3 - a^4a_3 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&2b_4v_1^{10}v_2 - a_5v_1^9v_2^4 + (2a_4 + b_5)\,v_1^9v_2^2 + 2b_2v_1^9v_2 + 2b_4v_1^9 - 2a_6v_1^8v_2^5 \\
&- a_3v_1^8v_2^4 + a_5v_1^8v_2^3 + (a_2 + b_3)\,v_1^8v_2^2 + (2b_1 + b_5)\,v_1^8v_2 + b_2v_1^8 \\
&+ 2a^2a_5v_1^5v_2^2 + \left(2a^2a_4 + a^2b_5\right)v_1^5 + 4a^2a_6v_1^4v_2^3 + 2a^2a_3v_1^4v_2^2 \\
&+ \left(3a^2a_5 + 2a^2b_6\right)v_1^4v_2 + \left(3a^2a_2 + a^2b_3\right)v_1^4 + 4a^2a_6v_1^3v_2^2 \\
&+ 4a^2a_3v_1^3v_2 + 4a^2a_1v_1^3 - a^4a_5v_1 - 2a^4a_6v_2 - a^4a_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_5 &= 0 \\
b_2 &= 0 \\
-a_3 &= 0 \\
-a_5 &= 0 \\
-2a_6 &= 0 \\
2b_2 &= 0 \\
2b_4 &= 0 \\
4a^2a_1 &= 0 \\
2a^2a_3 &= 0 \\
4a^2a_3 &= 0 \\
2a^2a_5 &= 0 \\
4a^2a_6 &= 0 \\
-a^4a_3 &= 0 \\
-a^4a_5 &= 0 \\
-2a^4a_6 &= 0 \\
a_2 + b_3 &= 0 \\
2a_4 + b_5 &= 0 \\
2b_1 + b_5 &= 0 \\
3a^2a_2 + a^2b_3 &= 0 \\
2a^2a_4 + a^2b_5 &= 0 \\
3a^2a_5 + 2a^2b_6 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= b_1 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= 0 \\
b_5 &= -2b_1 \\
b_6 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x^2 \\
\eta &= -2xy + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= -2xy + 1 - \left(-\frac{y^2 x^4 - a^2}{x^4}\right) (x^2) \\
&= \frac{y^2 x^4 - 2y\, x^3 - a^2 + x^2}{x^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2 x^4 - 2y\, x^3 - a^2 + x^2}{x^2}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(y\, x^2 - a - x\right)}{2a} - \frac{\ln\left(y\, x^2 + a - x\right)}{2a}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2 x^4 - a^2}{x^4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-2xy + 1}{\left(-y\, x^2 + a + x\right)\left(y\, x^2 + a - x\right)} \\ S_y &= -\frac{x^2}{\left(-y\, x^2 + a + x\right)\left(y\, x^2 + a - x\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(yx^2 - a - x) - \ln(yx^2 + a - x)}{2a} = c_1 + \frac{1}{x}$$

Which simplifies to

$$\frac{\ln(yx^2 - a - x) - \ln(yx^2 + a - x)}{2a} = c_1 + \frac{1}{x}$$

Summary

The solution(s) found are the following

$$\frac{\ln(yx^2 - a - x) - \ln(yx^2 + a - x)}{2a} = c_1 + \frac{1}{x} \tag{1}$$

Verification of solutions

$$\frac{\ln(yx^2 - a - x) - \ln(yx^2 + a - x)}{2a} = c_1 + \frac{1}{x}$$

Verified OK.

6.1.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^2x^4 - a^2}{x^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2 + \frac{a^2}{x^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a^2}{x^4}$, $f_1(x) = 0$ and $f_2(x) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{a^2}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + \frac{a^2 u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(c_1 \sinh\left(\frac{a}{x}\right) + c_2 \cosh\left(\frac{a}{x}\right)\right)$$

The above shows that

$$u'(x) = c_1 \sinh\left(\frac{a}{x}\right) + c_2 \cosh\left(\frac{a}{x}\right) - \frac{a\left(c_1 \cosh\left(\frac{a}{x}\right) + c_2 \sinh\left(\frac{a}{x}\right)\right)}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_1 \sinh\left(\frac{a}{x}\right) + c_2 \cosh\left(\frac{a}{x}\right) - \frac{a\left(c_1 \cosh\left(\frac{a}{x}\right) + c_2 \sinh\left(\frac{a}{x}\right)\right)}{x}}{x\left(c_1 \sinh\left(\frac{a}{x}\right) + c_2 \cosh\left(\frac{a}{x}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-a c_3 + x)\cosh\left(\frac{a}{x}\right) - \sinh\left(\frac{a}{x}\right)(-c_3 x + a)}{\left(c_3 \sinh\left(\frac{a}{x}\right) + \cosh\left(\frac{a}{x}\right)\right) x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{(-a c_3 + x)\cosh\left(\frac{a}{x}\right) - \sinh\left(\frac{a}{x}\right)(-c_3 x + a)}{\left(c_3 \sinh\left(\frac{a}{x}\right) + \cosh\left(\frac{a}{x}\right)\right) x^2} \tag{1}$$

Verification of solutions

$$y = \frac{(-a c_3 + x)\cosh\left(\frac{a}{x}\right) - \sinh\left(\frac{a}{x}\right)(-c_3 x + a)}{\left(c_3 \sinh\left(\frac{a}{x}\right) + \cosh\left(\frac{a}{x}\right)\right) x^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x)+y(x)^2=a^2/x^4,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{-a^2}\,\tan\left(\frac{\sqrt{-a^2}\,(c_1 x - 1)}{x}\right) + x}{x^2}$$

✓ Solution by Mathematica

Time used: 0.384 (sec). Leaf size: 71

```
DSolve[y'[x]+y[x]^2==a^2/x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2a^2c_1e^{\frac{2a}{x}} + 2ac_1xe^{\frac{2a}{x}} + a + x}{x^2\left(1 + 2ac_1e^{\frac{2a}{x}}\right)}$$

$$y(x) \to \frac{x-a}{x^2}$$

6.2 problem 2

Internal problem ID [4732]
Internal file name [`OUTPUT/4225_Sunday_June_05_2022_12_43_44_PM_18743969/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{u'' - \frac{a^2 u}{x^{\frac{2}{3}}} = 0}$$

6.2.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 u'' - a^2 x^{\frac{4}{3}} u = 0 \tag{1}$$

Bessel ode has the form

$$x^2 u'' + u' x + \left(-n^2 + x^2\right) u = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 u'' + (1 - 2\alpha) x u' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) u = 0 \tag{3}$$

With the standard solution

$$u = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = \frac{3ia}{2}$$
$$n = \frac{3}{4}$$
$$\gamma = \frac{2}{3}$$

Substituting all the above into (4) gives the solution as

$$u = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{3}{4}, \frac{3ia\,x^{\frac{2}{3}}}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{3}{4}, \frac{3ia\,x^{\frac{2}{3}}}{2}\right)$$

Summary

The solution(s) found are the following

$$u = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{3}{4}, \frac{3ia\,x^{\frac{2}{3}}}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{3}{4}, \frac{3ia\,x^{\frac{2}{3}}}{2}\right) \tag{1}$$

Verification of solutions

$$u = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{3}{4}, \frac{3ia\,x^{\frac{2}{3}}}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{3}{4}, \frac{3ia\,x^{\frac{2}{3}}}{2}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(u(x),x$2)-a^2*x^(-2/3)*u(x)=0,u(x), singsol=all)
```

$$u(x) = \sqrt{x}\left(\text{BesselY}\left(\frac{3}{4}, \frac{3\sqrt{-a^2}\, x^{\frac{2}{3}}}{2}\right) c_2 + \text{BesselJ}\left(\frac{3}{4}, \frac{3\sqrt{-a^2}\, x^{\frac{2}{3}}}{2}\right) c_1\right)$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 79

```
DSolve[u''[x]-a^2*x^(-2/3)*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{3^{3/4}a^{3/4}\sqrt{x}\left(16c_1 \operatorname{Gamma}\left(\frac{5}{4}\right) \operatorname{BesselI}\left(-\frac{3}{4}, \frac{3}{2}ax^{2/3}\right) + 3(-1)^{3/4}c_2 \operatorname{Gamma}\left(\frac{3}{4}\right) \operatorname{BesselI}\left(\frac{3}{4}, \frac{3}{2}ax^{2/3}\right)\right)}{8\sqrt{2}}$$

6.3 problem 3

Internal problem ID [4733]
Internal file name [`OUTPUT/4226_Sunday_June_05_2022_12_43_50_PM_66870393/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' - \frac{2u'}{x} - a^2 u = 0}$$

6.3.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 u'' - 2u'x - a^2 x^2 u = 0 \tag{1}$$

Bessel ode has the form

$$x^2 u'' + u'x + \left(-n^2 + x^2\right) u = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 u'' + (1 - 2\alpha)\, x u' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) u = 0 \tag{3}$$

With the standard solution

$$u = x^{\alpha}(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{3}{2}$$
$$\beta = ia$$
$$n = \frac{3}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$u = -\frac{c_1 \sqrt{x}\, \sqrt{2}\, (\cosh(ax)\, ax - \sinh(ax))}{\sqrt{\pi}\, \sqrt{iax}\, a} + \frac{i c_2 \sqrt{x}\, \sqrt{2}\, (-\sinh(ax)\, ax + \cosh(ax))}{\sqrt{\pi}\, \sqrt{iax}\, a}$$

Summary

The solution(s) found are the following

$$u = -\frac{c_1 \sqrt{x}\, \sqrt{2}\, (\cosh(ax)\, ax - \sinh(ax))}{\sqrt{\pi}\, \sqrt{iax}\, a} + \frac{i c_2 \sqrt{x}\, \sqrt{2}\, (-\sinh(ax)\, ax + \cosh(ax))}{\sqrt{\pi}\, \sqrt{iax}\, a} \tag{1}$$

Verification of solutions

$$u = -\frac{c_1 \sqrt{x}\, \sqrt{2}\, (\cosh(ax)\, ax - \sinh(ax))}{\sqrt{\pi}\, \sqrt{iax}\, a} + \frac{i c_2 \sqrt{x}\, \sqrt{2}\, (-\sinh(ax)\, ax + \cosh(ax))}{\sqrt{\pi}\, \sqrt{iax}\, a}$$

Verified OK.

6.3.2 Solving using Kovacic algorithm

Writing the ode as

$$a^2 u x - u'' x + 2u' = 0 \tag{1}$$
$$A u'' + B u' + C u = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x \\ B &= 2 \\ C &= a^2 x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 91: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = a^2 + \frac{2}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \quad (10)$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= a^2 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{0}{a}-0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{0}{a}-0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2x^2+2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(a) \\
&= -\frac{1}{x} - a \\
&= \frac{-ax - 1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2aa_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{1}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{1}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{-ax - \ln(x)} \\
&= \frac{(ax + 1)\, e^{-ax}}{ax}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{-x}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(ax+1)\,\mathrm{e}^{-ax}}{a}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{2}{-x}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{2\ln(x)}}{(u_1)^2}\,dx \\
&= u_1 \left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2\,(ax+1)\,a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\right) + c_2 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2\,(ax+1)\,a}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
u = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a} + \frac{c_2(ax-1)\,\mathrm{e}^{ax}}{2a^2} \tag{1}
$$

Verification of solutions

$$
u = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a} + \frac{c_2(ax-1)\,\mathrm{e}^{ax}}{2a^2}
$$

Verified OK.

6.3.3 Maple step by step solution

Let's solve

$a^2ux - u''x + 2u' = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Isolate 2nd derivative

  $u'' = \frac{2u'}{x} + a^2u$

- Group terms with u on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $u'' - \frac{2u'}{x} - a^2u = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -a^2\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -2$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $-a^2ux + u''x - 2u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x \cdot u$ to series expansion

  $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- ◦ Shift index using $k- > k - 1$

$$x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert u' to series expansion

$$u' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$u' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

- Convert $x \cdot u''$ to series expansion

$$x \cdot u'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, x^{-1+r} + a_1(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r) - a^2 a_{k-1}\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term must be 0

$$a_1(1+r)\,(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+r+1)\,(k-2+r) - a^2 a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)\,(k+r-1) - a^2 a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+r-1)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}$$

- Solution for $r = 0$

$$\left[u=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a^2 a_k}{(k+2)(k-1)}, -2a_1=0\right]$$

- Recursion relation for $r=3$

$$a_{k+2}=\frac{a^2 a_k}{(k+5)(k+2)}$$

- Solution for $r=3$

$$\left[u=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=\frac{a^2 a_k}{(k+5)(k+2)}, 4a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[u=\left(\sum_{k=0}^{\infty} b_k x^k\right)+\left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2}=\frac{a^2 b_k}{(k+2)(k-1)}, -2b_1=0, c_{k+2}=\frac{a^2 c_k}{(k+5)(k+2)}, 4c_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(u(x),x$2)-2/x*diff(u(x),x)-a^2*u(x)=0,u(x), singsol=all)
```

$$u(x)=c_1 e^{ax}(ax-1)+c_2 e^{-ax}(ax+1)$$

✓ Solution by Mathematica

Time used: 0.13 (sec). Leaf size: 68

```
DSolve[u''[x]-2/x*u'[x]-a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{a\sqrt{-iax}}$$

6.4 problem 4

Internal problem ID [4734]
Internal file name [`OUTPUT/4227_Sunday_June_05_2022_12_44_00_PM_95572498/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' + \frac{2u'}{x} - a^2 u = 0}$$

6.4.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$u'' + p(x)\, u' + q(x)\, u = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -a^2 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= -a^2 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4} \\
&= -a^2 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= -a^2 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= -a^2
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
u = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{2}{x}}{2}} \\
&= \frac{1}{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
u = \frac{v(x)}{x} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
-a^2 v(x) + v''(x) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = -a^2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda x} - a^2 \mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$-a^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -a^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-a^2)}\\ &= \pm\sqrt{a^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{a^2}\\ \lambda_2 &= -\sqrt{a^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{a^2}\\ \lambda_2 &= -\sqrt{a^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ v(x) &= c_1 e^{\left(\sqrt{a^2}\right)x} + c_2 e^{\left(-\sqrt{a^2}\right)x}\end{aligned}$$

Or

$$v(x) = c_1 \mathrm{e}^{x\sqrt{a^2}} + c_2 \mathrm{e}^{-x\sqrt{a^2}}$$

Now that $v(x)$ is known, then

$$\begin{aligned}u &= v(x)\,z(x)\\ &= \left(c_1 \mathrm{e}^{x\sqrt{a^2}} + c_2 \mathrm{e}^{-x\sqrt{a^2}}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$u = \frac{c_1 e^{x\sqrt{a^2}} + c_2 e^{-x\sqrt{a^2}}}{x}$$

Summary

The solution(s) found are the following

$$u = \frac{c_1 e^{x\sqrt{a^2}} + c_2 e^{-x\sqrt{a^2}}}{x} \tag{1}$$

Verification of solutions

$$u = \frac{c_1 e^{x\sqrt{a^2}} + c_2 e^{-x\sqrt{a^2}}}{x}$$

Verified OK.

6.4.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 u'' + 2u'x - a^2 x^2 u = 0 \tag{1}$$

Bessel ode has the form

$$x^2 u'' + u'x + \left(-n^2 + x^2\right) u = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 u'' + (1 - 2\alpha)\, x u' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) u = 0 \tag{3}$$

With the standard solution

$$u = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = -\frac{1}{2}$$
$$\beta = ia$$
$$n = \frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$u = \frac{ic_1\sqrt{2}\,\sinh(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{iax}} - \frac{c_2\sqrt{2}\,\cosh(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{iax}}$$

Summary

The solution(s) found are the following

$$u = \frac{ic_1\sqrt{2}\,\sinh(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{iax}} - \frac{c_2\sqrt{2}\,\cosh(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{iax}} \tag{1}$$

Verification of solutions

$$u = \frac{ic_1\sqrt{2}\,\sinh(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{iax}} - \frac{c_2\sqrt{2}\,\cosh(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{iax}}$$

Verified OK.

6.4.3 Solving using Kovacic algorithm

Writing the ode as

$$-a^2ux + u''x + 2u' = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= -a^2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (a^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 93: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{x\sqrt{a^2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$u_1 = \frac{\mathrm{e}^{x\,\mathrm{csgn}(a)a}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2}\,dx \\
&= u_1\left(-\frac{\mathrm{csgn}\,(a)\,\mathrm{e}^{-2x\,\mathrm{csgn}(a)a}}{2a}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{\mathrm{e}^{x \operatorname{csgn}(a) a}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{x \operatorname{csgn}(a) a}}{x}\left(-\frac{\operatorname{csgn}(a)\, \mathrm{e}^{-2x \operatorname{csgn}(a) a}}{2a}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are th

Simplifying the solution $u = \frac{c_1 \mathrm{e}^{x \operatorname{csgn}(a) a}}{x} - \frac{c_2 \operatorname{csgn}(a) \mathrm{e}^{-x \operatorname{csgn}(a) a}}{2ax}$ to $u = \frac{c_1 \mathrm{e}^{ax}}{x} - \frac{c_2 \mathrm{e}^{-ax}}{2ax}$

Verification of solutions

$$
u = \frac{c_1 \mathrm{e}^{ax}}{x} - \frac{c_2 \mathrm{e}^{-ax}}{2ax}
$$

Verified OK.

6.4.4 Maple step by step solution

Let's solve

$-a^2 u x + u'' x + 2u' = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Isolate 2nd derivative

  $u'' = -\frac{2u'}{x} + a^2 u$

- Group terms with u on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $u'' + \frac{2u'}{x} - a^2 u = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -a^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$-a^2 u x + u'' x + 2u' = 0$

- Assume series solution for u

$u = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x \cdot u$ to series expansion

  $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k- > k-1$

  $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert u' to series expansion

  $u' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

  - Shift index using $k- > k+1$

  $u' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$

  - Convert $x \cdot u''$ to series expansion

  $x \cdot u'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- > k+1$

  $x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + a_1 (1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r)-a^2a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+3+r)-a^2a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a^2a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=\frac{a^2a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

  $\left[u=\sum_{k=0}^{\infty}a_kx^{k-1}, a_{k+2}=\frac{a^2a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{a^2a_k}{(k+2)(k+3)}$

- Solution for $r=0$

  $\left[u=\sum_{k=0}^{\infty}a_kx^k, a_{k+2}=\frac{a^2a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

  $\left[u=\left(\sum_{k=0}^{\infty}b_kx^{k-1}\right)+\left(\sum_{k=0}^{\infty}c_kx^k\right), b_{k+2}=\frac{a^2b_k}{(k+1)(k+2)}, 0=0, c_{k+2}=\frac{a^2c_k}{(k+2)(k+3)}, 2c_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(u(x),x$2)+2/x*diff(u(x),x)-a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{c_1 \sinh(ax) + c_2 \cosh(ax)}{x}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 35

```
DSolve[u''[x]+2/x*u'[x]-a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{2ac_1e^{-ax} + c_2e^{ax}}{2ax}$$

6.5 problem 5

Internal problem ID [4735]
Internal file name [`OUTPUT/4228_Sunday_June_05_2022_12_44_08_PM_82211255/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' + \frac{2u'}{x} + a^2 u = 0}$$

6.5.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$u'' + p(x)\, u' + q(x)\, u = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = a^2$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= a^2 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4} \\
&= a^2 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= a^2 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= a^2
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$u = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{2}{x}}{2}} \\
&= \frac{1}{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$u = \frac{v(x)}{x} \tag{4}$$

Applying this change of variable to the original ode results in

$$a^2 v(x) + v''(x) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = a^2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + a^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$a^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = a^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(a^2)} \\ &= \pm\sqrt{-a^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-a^2} \\ \lambda_2 &= -\sqrt{-a^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-a^2} \\ \lambda_2 &= -\sqrt{-a^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ v(x) &= c_1 e^{\left(\sqrt{-a^2}\right)x} + c_2 e^{\left(-\sqrt{-a^2}\right)x}\end{aligned}$$

Or

$$v(x) = c_1 \mathrm{e}^{\sqrt{-a^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-a^2}\, x}$$

Now that $v(x)$ is known, then

$$\begin{aligned}u &= v(x)\, z(x) \\ &= \left(c_1 \mathrm{e}^{\sqrt{-a^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-a^2}\, x}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$u = \frac{c_1 \mathrm{e}^{\sqrt{-a^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-a^2}\, x}}{x}$$

Summary

The solution(s) found are the following

$$u = \frac{c_1 \mathrm{e}^{\sqrt{-a^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-a^2}\, x}}{x} \tag{1}$$

Verification of solutions

$$u = \frac{c_1 \mathrm{e}^{\sqrt{-a^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-a^2}\, x}}{x}$$

Verified OK.

6.5.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 u'' + 2u'x + a^2 x^2 u = 0 \tag{1}$$

Bessel ode has the form

$$x^2 u'' + u'x + \left(-n^2 + x^2\right) u = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 u'' + (1 - 2\alpha)\, x u' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) u = 0 \tag{3}$$

With the standard solution

$$u = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= -\frac{1}{2} \\ \beta &= a \\ n &= \frac{1}{2} \\ \gamma &= 1 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$u = \frac{c_1\sqrt{2}\,\sin(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ax}} - \frac{c_2\sqrt{2}\,\cos(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ax}}$$

Summary

The solution(s) found are the following

$$u = \frac{c_1\sqrt{2}\,\sin(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ax}} - \frac{c_2\sqrt{2}\,\cos(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ax}} \tag{1}$$

Verification of solutions

$$u = \frac{c_1\sqrt{2}\,\sin(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ax}} - \frac{c_2\sqrt{2}\,\cos(ax)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ax}}$$

Verified OK.

6.5.3 Solving using Kovacic algorithm

Writing the ode as

$$a^2ux + u''x + 2u' = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= a^2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-a^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 95: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{-a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{2}{x}\, dx}}{(u_1)^2}\, dx \\ &= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2}\, dx \\ &= u_1 \left(\frac{\sqrt{-a^2}\, \mathrm{e}^{-2\sqrt{-a^2}\, x}}{2a^2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\,x}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\,x}}{x} \left(\frac{\sqrt{-a^2}\,\mathrm{e}^{-2\sqrt{-a^2}\,x}}{2a^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
u = \frac{c_1 \mathrm{e}^{\sqrt{-a^2}\,x}}{x} + \frac{c_2 \sqrt{-a^2}\,\mathrm{e}^{-\sqrt{-a^2}\,x}}{2a^2 x} \tag{1}
$$

Verification of solutions

$$
u = \frac{c_1 \mathrm{e}^{\sqrt{-a^2}\,x}}{x} + \frac{c_2 \sqrt{-a^2}\,\mathrm{e}^{-\sqrt{-a^2}\,x}}{2a^2 x}
$$

Verified OK.

6.5.4 Maple step by step solution

Let's solve

$a^2 u x + u'' x + 2u' = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Isolate 2nd derivative

  $u'' = -\frac{2u'}{x} - a^2 u$

- Group terms with u on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $u'' + \frac{2u'}{x} + a^2 u = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = a^2\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $a^2 u x + u'' x + 2u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x \cdot u$ to series expansion

  $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k- >k-1$

  $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert u' to series expansion

  $u' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $u' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$

- Convert $x \cdot u''$ to series expansion

  $x \cdot u'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r (1+r)\, x^{-1+r} + a_1 (1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+r+1)(k+2+r) + a^2 a_{k-1} = 0$

- Shift index using $k- > k+1$

$a_{k+2}(k+2+r)(k+3+r) + a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

$a_{k+2} = -\frac{a^2 a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

$\left[u = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a^2 a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

$a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = -\frac{a^2 b_k}{(k+1)(k+2)}, 0 = 0, c_{k+2} = -\frac{a^2 c_k}{(k+2)(k+3)}, 2c_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(u(x),x$2)+2/x*diff(u(x),x)+a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{c_1 \sin(ax) + c_2 \cos(ax)}{x}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 42

```
DSolve[u''[x]+2/x*u'[x]+a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \rightarrow \frac{e^{-iax}\left(2c_1 - \frac{ic_2e^{2iax}}{a}\right)}{2x}$$

6.6 problem 6

Internal problem ID [4736]
Internal file name [`OUTPUT/4229_Sunday_June_05_2022_12_44_16_PM_83559563/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' + \frac{4u'}{x} - a^2 u = 0}$$

6.6.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 u'' + 4u'x - a^2 x^2 u = 0 \tag{1}$$

Bessel ode has the form

$$x^2 u'' + u'x + \left(-n^2 + x^2\right) u = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 u'' + (1 - 2\alpha) x u' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) u = 0 \tag{3}$$

With the standard solution

$$u = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= -\frac{3}{2} \\
\beta &= ia \\
n &= \frac{3}{2} \\
\gamma &= 1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$u = -\frac{c_1\sqrt{2}\left(\cosh\left(ax\right)ax - \sinh\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{iax}\,a} + \frac{ic_2\sqrt{2}\left(-\sinh\left(ax\right)ax + \cosh\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{iax}\,a}$$

Summary

The solution(s) found are the following

$$u = -\frac{c_1\sqrt{2}\left(\cosh\left(ax\right)ax - \sinh\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{iax}\,a} + \frac{ic_2\sqrt{2}\left(-\sinh\left(ax\right)ax + \cosh\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{iax}\,a} \tag{1}$$

Verification of solutions

$$u = -\frac{c_1\sqrt{2}\left(\cosh\left(ax\right)ax - \sinh\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{iax}\,a} + \frac{ic_2\sqrt{2}\left(-\sinh\left(ax\right)ax + \cosh\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{iax}\,a}$$

Verified OK.

6.6.2 Solving using Kovacic algorithm

Writing the ode as

$$-a^2ux + u''x + 4u' = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x \\
B &= 4 \\
C &= -a^2x
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 97: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= a^2 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(a) \\
&= -\frac{1}{x} - a \\
&= \frac{-ax - 1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2aa_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{1}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{1}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{-ax - \ln(x)} \\
&= \frac{(ax + 1)\, e^{-ax}}{ax}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{x}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1\left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(ax+1)\,\mathrm{e}^{-ax}}{a\,x^3}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{4}{x}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2}\,dx \\
&= u_1\left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2\,(ax+1)\,a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1\left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a\,x^3}\right) + c_2\left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a\,x^3}\left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2\,(ax+1)\,a}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
u = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a\,x^3} + \frac{c_2(ax-1)\,\mathrm{e}^{ax}}{2a^2x^3} \tag{1}
$$

Verification of solutions

$$u = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a\,x^3} + \frac{c_2(ax-1)\,\mathrm{e}^{ax}}{2a^2x^3}$$

Verified OK.

6.6.3 Maple step by step solution

Let's solve

$-a^2ux + u''x + 4u' = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Isolate 2nd derivative

  $u'' = -\frac{4u'}{x} + a^2u$

- Group terms with u on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $u'' + \frac{4u'}{x} - a^2u = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = -a^2\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $-a^2ux + u''x + 4u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot u$ to series expansion

    $$x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$
  - Shift index using $k- >k-1$

    $$x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$
  - Convert u' to series expansion

    $$u' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$
  - Shift index using $k- >k+1$

    $$u' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$
  - Convert $x \cdot u''$ to series expansion

    $$x \cdot u'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$
  - Shift index using $k- >k+1$

    $$x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(3+r)\, x^{-1+r} + a_1(1+r)\,(4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+4+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(3+r) = 0$$
- Values of r that satisfy the indicial equation

  $$r \in \{-3, 0\}$$
- Each term must be 0

  $$a_1(1+r)\,(4+r) = 0$$
- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+r+1)\,(k+4+r) - a^2 a_{k-1} = 0$$
- Shift index using $k- >k+1$

  $$a_{k+2}(k+2+r)\,(k+5+r) - a^2 a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+5+r)}$$

- Recursion relation for $r = -3$

$$a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}$$

- Solution for $r = -3$

$$\left[u = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}$$

- Solution for $r = 0$

$$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = \frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(k+5)(k+2)}, 4c_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(u(x),x$2)+4/x*diff(u(x),x)-a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{c_1 e^{ax}(ax - 1) + c_2 e^{-ax}(ax + 1)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 68

```
DSolve[u''[x]+4/x*u'[x]-a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{ax^{5/2}\sqrt{-iax}}$$

6.7 problem 7

Internal problem ID [4737]
Internal file name [`OUTPUT/4230_Sunday_June_05_2022_12_44_29_PM_92144711/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' + \frac{4u'}{x} + a^2 u = 0}$$

6.7.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 u'' + 4u'x + a^2 x^2 u = 0 \tag{1}$$

Bessel ode has the form

$$x^2 u'' + u'x + \left(-n^2 + x^2\right) u = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 u'' + (1 - 2\alpha)\, x u' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) u = 0 \tag{3}$$

With the standard solution

$$u = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= -\frac{3}{2} \\
\beta &= a \\
n &= \frac{3}{2} \\
\gamma &= 1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
u = -\frac{c_1\sqrt{2}\left(\cos\left(ax\right)ax - \sin\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{ax}\,a} - \frac{c_2\sqrt{2}\left(\sin\left(ax\right)ax + \cos\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{ax}\,a}
$$

Summary

The solution(s) found are the following

$$
u = -\frac{c_1\sqrt{2}\left(\cos\left(ax\right)ax - \sin\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{ax}\,a} - \frac{c_2\sqrt{2}\left(\sin\left(ax\right)ax + \cos\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{ax}\,a} \tag{1}
$$

Verification of solutions

$$
u = -\frac{c_1\sqrt{2}\left(\cos\left(ax\right)ax - \sin\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{ax}\,a} - \frac{c_2\sqrt{2}\left(\sin\left(ax\right)ax + \cos\left(ax\right)\right)}{x^{\frac{5}{2}}\sqrt{\pi}\,\sqrt{ax}\,a}
$$

Verified OK.

6.7.2 Solving using Kovacic algorithm

Writing the ode as

$$
a^2ux + u''x + 4u' = 0 \tag{1}
$$

$$
Au'' + Bu' + Cu = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x \\
B &= 4 \\
C &= a^2x
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ue^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a^2x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 99: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -a^2 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ia - \frac{i}{a\,x^2} - \frac{i}{2a^3x^4} - \frac{i}{2a^5x^6} - \frac{5i}{8a^7x^8} - \frac{7i}{8a^9x^{10}} - \frac{21i}{16a^{11}x^{12}} - \frac{33i}{16a^{13}x^{14}} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = ia$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= ia \end{aligned} \quad (10)$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-a^2) + \left(\frac{2}{x^2}\right) \\ &= -a^2 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= ia \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{ia} - 0\right) &&= 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{ia} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-a^2x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	ia	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(ia) \\
&= -\frac{1}{x} - ia \\
&= -\frac{1}{x} - ia
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - ia\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - ia\right)^2 - \left(\frac{-a^2x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2iaa_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{i}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{i}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{i}{a}\right) \mathrm{e}^{\int \left(-\frac{1}{x} - ia\right) dx} \\
&= \left(x - \frac{i}{a}\right) \mathrm{e}^{-iax - \ln(x)} \\
&= \frac{(ax - i)\,\mathrm{e}^{-iax}}{xa}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{x}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1 \left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(ax - i)\,\mathrm{e}^{-iax}}{x^3 a}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{4}{x}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2}\,dx \\
&= u_1 \left(\frac{(iax - 1)\,\mathrm{e}^{2iax}}{2a\,(-ax + i)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{(ax - i)\,\mathrm{e}^{-iax}}{x^3 a}\right) + c_2 \left(\frac{(ax - i)\,\mathrm{e}^{-iax}}{x^3 a}\left(\frac{(iax - 1)\,\mathrm{e}^{2iax}}{2a\,(-ax + i)}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
u = \frac{c_1 (ax - i)\,\mathrm{e}^{-iax}}{x^3 a} - \frac{c_2 (iax - 1)\,\mathrm{e}^{iax}}{2a^2 x^3} \tag{1}
$$

Verification of solutions

$$u = \frac{c_1(ax - i)\,\mathrm{e}^{-iax}}{x^3 a} - \frac{c_2(iax - 1)\,\mathrm{e}^{iax}}{2a^2x^3}$$

Verified OK.

6.7.3 Maple step by step solution

Let's solve

$a^2ux + u''x + 4u' = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Isolate 2nd derivative

  $u'' = -\frac{4u'}{x} - a^2u$

- Group terms with u on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $u'' + \frac{4u'}{x} + a^2u = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = a^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $a^2ux + u''x + 4u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert u' to series expansion

    $u' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $u' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  - Convert $x \cdot u''$ to series expansion

    $x \cdot u'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0 r(3+r)\, x^{-1+r} + a_1(1+r)\,(4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+4+r) + a^2 a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$
- Each term must be 0

  $a_1(1+r)\,(4+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+4+r) + a^2 a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k+5+r) + a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+5+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}$

- Solution for $r = -3$

  $\left[u = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}$

- Solution for $r = 0$

  $\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = -\frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = -\frac{a^2 c_k}{(k+5)(k+2)}, 4c_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(diff(u(x),x$2)+4/x*diff(u(x),x)+a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{(ac_1x + c_2)\cos(ax) + \sin(ax)(ac_2x - c_1)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 57

```
DSolve[u''[x]+4/x*u'[x]+a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to -\frac{\sqrt{\frac{2}{\pi}}((ac_1x + c_2)\cos(ax) + (ac_2x - c_1)\sin(ax))}{x^{3/2}(ax)^{3/2}}$$

6.8 problem 8

Internal problem ID [4738]
Internal file name [`OUTPUT/4231_Sunday_June_05_2022_12_44_39_PM_81251389/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - ya^2 - \frac{6y}{x^2} = 0}$$

6.8.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + (-a^2x^2 - 6)\,y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + (-n^2 + x^2)\,y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\,y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\,x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\,x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= ia \\
n &= -\frac{5}{2} \\
\gamma &= 1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = -\frac{c_1\sqrt{2}\left(\cosh\left(ax\right)a^2x^2 - 3\sinh\left(ax\right)ax + 3\cosh\left(ax\right)\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a^2} + \frac{ic_2\sqrt{2}\left(-\sinh\left(ax\right)a^2x^2 + 3\cosh\left(ax\right)ax - 3\right.}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a^2}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &-\frac{c_1\sqrt{2}\left(\cosh\left(ax\right)a^2x^2 - 3\sinh\left(ax\right)ax + 3\cosh\left(ax\right)\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a^2} \\
&+ \frac{ic_2\sqrt{2}\left(-\sinh\left(ax\right)a^2x^2 + 3\cosh\left(ax\right)ax - 3\sinh\left(ax\right)\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a^2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = &-\frac{c_1\sqrt{2}\left(\cosh\left(ax\right)a^2x^2 - 3\sinh\left(ax\right)ax + 3\cosh\left(ax\right)\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a^2} \\
&+ \frac{ic_2\sqrt{2}\left(-\sinh\left(ax\right)a^2x^2 + 3\cosh\left(ax\right)ax - 3\sinh\left(ax\right)\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a^2}
\end{aligned}
$$

Verified OK.

6.8.2 Solving using Kovacic algorithm

Writing the ode as

$$
x^2y'' + \left(-a^2x^2 - 6\right)y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= 0 \\
C &= -a^2x^2 - 6
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 101: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{3}{a\,x^2} - \frac{9}{2a^3x^4} + \frac{27}{2a^5x^6} - \frac{405}{8a^7x^8} + \frac{1701}{8a^9x^{10}} - \frac{15309}{16a^{11}x^{12}} + \frac{72171}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2+6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{6}{x^2}\right) \\ &= a^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= a \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{a} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2x^2+6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\ &= 0 - (-2)\\ &= 2\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{2}{x} + (-)\,(a)\\ &= -\frac{2}{x} - a\\ &= \frac{-ax-2}{x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(2) + 2\left(-\frac{2}{x} - a\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - a\right)^2 - \left(\frac{a^2x^2+6}{x^2}\right)\right) &= 0\\ \frac{2axa_1 + 4aa_0 - 6x - 4a_1}{x} &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{3}{a^2}, a_1 = \frac{3}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + \frac{3x}{a} + \frac{3}{a^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{\int \left(-\frac{2}{x} - a\right) dx} \\
&= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{-ax - 2\ln(x)} \\
&= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2}
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \int \frac{1}{\frac{(a^2x^2+3ax+3)^2 e^{-2ax}}{a^4x^4}} \, dx \\
&= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \left(\frac{(a^2x^2 - 3ax + 3)\, e^{2ax}}{2a\,(a^2x^2 + 3ax + 3)}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(a^2x^2+3ax+3)\,\mathrm{e}^{-ax}}{a^2x^2}\right) + c_2 \left(\frac{(a^2x^2+3ax+3)\,\mathrm{e}^{-ax}}{a^2x^2}\left(\frac{(a^2x^2-3ax+3)\,\mathrm{e}^{2ax}}{2a\,(a^2x^2+3ax+3)}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1(a^2x^2+3ax+3)\,\mathrm{e}^{-ax}}{a^2x^2} + \frac{c_2\mathrm{e}^{ax}(a^2x^2-3ax+3)}{2a^3x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(a^2x^2+3ax+3)\,\mathrm{e}^{-ax}}{a^2x^2} + \frac{c_2\mathrm{e}^{ax}(a^2x^2-3ax+3)}{2a^3x^2}
$$

Verified OK.

6.8.3 Maple step by step solution

Let's solve

$x^2y'' + (-a^2x^2-6)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(a^2x^2+6)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(a^2x^2+6)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{a^2x^2+6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -6$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + (-a^2x^2 - 6)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2 + r)\,(-3 + r)\, x^r + a_1(3 + r)\,(-2 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k + r + 2)\,(k + r - 3) - a_{k-2}a^2\right) x^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2 + r)\,(-3 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 3\}$

- Each term must be 0

  $a_1(3 + r)\,(-2 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r+2)(k+r-3) - a_{k-2}a^2 = 0$

- Shift index using $k- >k+2$

$a_{k+2}(k+4+r)(k+r-1) - a_k a^2 = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = \frac{a_k a^2}{(k+4+r)(k+r-1)}$

- Recursion relation for $r = -2$

$a_{k+2} = \frac{a_k a^2}{(k+2)(k-3)}$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = \frac{a_k a^2}{(k+2)(k-3)}, a_1 = 0\right]$$

- Recursion relation for $r = 3$

$a_{k+2} = \frac{a_k a^2}{(k+7)(k+2)}$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a_k a^2}{(k+7)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2} = \frac{b_k a^2}{(k+2)(k-3)}, b_1 = 0, c_{k+2} = \frac{c_k a^2}{(k+7)(k+2)}, c_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$2)-a^2*y(x)=6*y(x)/x^2,y(x), singsol=all)
```

$$y(x) = \frac{c_2 e^{-ax}(x^2a^2 + 3ax + 3) + c_1 e^{ax}(x^2a^2 - 3ax + 3)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 90

```
DSolve[y''[x]-a^2*y[x]==6*y[x]/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}((a^2c_2x^2 - 3iac_1x + 3c_2)\cosh(ax) + i(c_1(a^2x^2 + 3) + 3iac_2x)\sinh(ax))}{a^2x^{3/2}\sqrt{-iax}}$$

6.9 problem 9

Internal problem ID [4739]
Internal file name [`OUTPUT/4232_Sunday_June_05_2022_12_44_49_PM_20146333/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y n^2 - \frac{6y}{x^2} = 0}$$

6.9.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + \left(n^2 x^2 - 6\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + x y' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= n \\
n &= -\frac{5}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = -\frac{c_1\sqrt{2}\,(\cos(nx)\,n^2x^2 - 3\sin(nx)\,nx - 3\cos(nx))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{nx}\,n^2} - \frac{c_2\sqrt{2}\,(\sin(nx)\,n^2x^2 + 3\cos(nx)\,nx - 3\sin(nx))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{nx}\,n^2}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = &-\frac{c_1\sqrt{2}\,(\cos(nx)\,n^2x^2 - 3\sin(nx)\,nx - 3\cos(nx))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{nx}\,n^2} \\
&- \frac{c_2\sqrt{2}\,(\sin(nx)\,n^2x^2 + 3\cos(nx)\,nx - 3\sin(nx))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{nx}\,n^2}
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y = &-\frac{c_1\sqrt{2}\,(\cos(nx)\,n^2x^2 - 3\sin(nx)\,nx - 3\cos(nx))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{nx}\,n^2} \\
&- \frac{c_2\sqrt{2}\,(\sin(nx)\,n^2x^2 + 3\cos(nx)\,nx - 3\sin(nx))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{nx}\,n^2}
\end{aligned}$$

Verified OK.

6.9.2 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + (n^2x^2 - 6)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= x^2 \\
B &= 0 \\
C &= n^2x^2 - 6
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-n^2x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -n^2x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-n^2x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 103: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -n^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx in - \frac{3i}{n\,x^2} - \frac{9i}{2n^3x^4} - \frac{27i}{2n^5x^6} - \frac{405i}{8n^7x^8} - \frac{1701i}{8n^9x^{10}} - \frac{15309i}{16n^{11}x^{12}} - \frac{72171i}{16n^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = in$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= in \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -n^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-n^2x^2+6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-n^2) + \left(\frac{6}{x^2}\right) \\ &= -n^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= in \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{in} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{in} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-n^2x^2+6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	in	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)\,(in) \\ &= -\frac{2}{x} - in \\ &= -\frac{2}{x} - in \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - in\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - in\right)^2 - \left(\frac{-n^2x^2 + 6}{x^2}\right) \right) = 0$$
$$\frac{(2ina_1 - 6)\,x + 4ina_0 - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = -\frac{3}{n^2}, a_1 = -\frac{3i}{n} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - \frac{3ix}{n} - \frac{3}{n^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) \mathrm{e}^{\int \left(-\frac{2}{x} - in\right) dx} \\
&= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) \mathrm{e}^{-inx - 2\ln(x)} \\
&= \frac{(n^2x^2 - 3inx - 3)\, \mathrm{e}^{-inx}}{x^2 n^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \frac{(n^2x^2 - 3inx - 3)\, \mathrm{e}^{-inx}}{x^2 n^2}
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(n^2x^2 - 3inx - 3)\, \mathrm{e}^{-inx}}{x^2 n^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \frac{(n^2x^2 - 3inx - 3)\, \mathrm{e}^{-inx}}{x^2 n^2} \int \frac{1}{\frac{(n^2x^2 - 3inx - 3)^2 \mathrm{e}^{-2inx}}{x^4 n^4}} \, dx \\
&= \frac{(n^2x^2 - 3inx - 3)\, \mathrm{e}^{-inx}}{x^2 n^2} \left(\frac{(in^2x^2 - 3nx - 3i)\, \mathrm{e}^{2inx}}{6\left(-\frac{1}{3}n^2x^2 + inx + 1\right) n}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \right) \\
&\quad + c_2 \left(\frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \left(\frac{(in^2x^2 - 3nx - 3i)\,\mathrm{e}^{2inx}}{6\left(-\frac{1}{3}n^2x^2 + inx + 1\right) n} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} - \frac{c_2\mathrm{e}^{inx}(in^2x^2 - 3nx - 3i)}{2n^3x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} - \frac{c_2\mathrm{e}^{inx}(in^2x^2 - 3nx - 3i)}{2n^3x^2}
$$

Verified OK.

6.9.3 Maple step by step solution

Let's solve

$x^2y'' + (n^2x^2 - 6)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(n^2x^2-6)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(n^2x^2-6)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{n^2x^2-6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -6$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + \left(n^2 x^2 - 6\right) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (2+r)(-3+r)\, x^r + a_1 (3+r)(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r+2)(k+r-3) + a_{k-2} n^2\right) x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 3\}$

- Each term must be 0

  $a_1 (3+r)(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-3) + a_{k-2}n^2 = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)(k+r-1) + a_k n^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k n^2}{(k+4+r)(k+r-1)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k n^2}{(k+2)(k-3)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k n^2}{(k+2)(k-3)}, a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k n^2}{(k+7)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k n^2}{(k+7)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k n^2}{(k+2)(k-3)}, a_1 = 0, b_{k+2} = -\frac{b_k n^2}{(k+7)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 53

```
dsolve(diff(y(x),x$2)+n^2*y(x)=6*y(x)/x^2,y(x), singsol=all)
```

$$y(x) = \frac{(c_1 n^2 x^2 + 3c_2 nx - 3c_1)\cos(nx) + \sin(nx)(c_2 n^2 x^2 - 3c_1 nx - 3c_2)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 79

```
DSolve[y''[x]+n^2*y[x]==6*y[x]/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((c_2(-n^2)x^2 + 3c_1 nx + 3c_2)\cos(nx) + (c_1(n^2x^2 - 3) + 3c_2 nx)\sin(nx))}{(nx)^{5/2}}$$

6.10 problem 10

Internal problem ID [4740]
Internal file name [`OUTPUT/4233_Sunday_June_05_2022_12_44_59_PM_85709635/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^{2} y'' + x y' - \left(x^{2}+\frac{1}{4}\right) y = 0}$$

6.10.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{-x^{2}-\frac{1}{4}}{x^{2}}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{-x^2 - \frac{1}{4}}{x^2} - \frac{\left(\frac{1}{x}\right)'}{2} - \frac{\left(\frac{1}{x}\right)^2}{4} \\
&= \frac{-x^2 - \frac{1}{4}}{x^2} - \frac{\left(-\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\
&= \frac{-x^2 - \frac{1}{4}}{x^2} - \left(-\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\
&= -1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{1}{x}}{2}} \\
&= \frac{1}{\sqrt{x}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = \frac{v(x)}{\sqrt{x}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^{\frac{3}{2}} (v''(x) - v(x)) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ v(x) &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$v(x) = c_1 e^{x} + c_2 e^{-x}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x)\\ &= \left(c_1 e^{x} + c_2 e^{-x}\right)\left(z(x)\right)\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{\sqrt{x}}$$

Hence (7) becomes

$$y = \frac{c_1 e^x + c_2 e^{-x}}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^x + c_2 e^{-x}}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x + c_2 e^{-x}}{\sqrt{x}}$$

Verified OK.

6.10.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + x y' + \left(-x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + x y' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = i$$
$$n = -\frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\,\cosh(x)}{\sqrt{\pi}\,\sqrt{ix}} + \frac{ic_2\sqrt{2}\,\sinh(x)}{\sqrt{\pi}\,\sqrt{ix}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,\cosh(x)}{\sqrt{\pi}\,\sqrt{ix}} + \frac{ic_2\sqrt{2}\,\sinh(x)}{\sqrt{\pi}\,\sqrt{ix}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\cosh(x)}{\sqrt{\pi}\,\sqrt{ix}} + \frac{ic_2\sqrt{2}\,\sinh(x)}{\sqrt{\pi}\,\sqrt{ix}}$$

Verified OK.

6.10.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + xy' + \left(-x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{e^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right) + c_2\left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \mathrm{e}^{-x}}{\sqrt{x}} + \frac{c_2 \mathrm{e}^{x}}{2\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \mathrm{e}^{-x}}{\sqrt{x}} + \frac{c_2 \mathrm{e}^{x}}{2\sqrt{x}}
$$

Verified OK.

6.10.4 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(-x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(4x^2+1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(4x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{4x^2+1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $4x^2y'' + 4xy' + (-4x^2 - 1)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$
- ○ Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\,x^r + a_1(3 + 2r)\,(1 + 2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) - 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1)-4a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2+8(k+2)\,r+4r^2-1\right)-4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=\frac{4a_k}{4k^2+12k+8}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}},a_{k+2}=\frac{4a_k}{4k^2+12k+8},a_1=0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{4a_k}{4k^2+20k+24}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}},a_{k+2}=\frac{4a_k}{4k^2+20k+24},a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}}\right),a_{k+2}=\frac{4a_k}{4k^2+12k+8},a_1=0,b_{k+2}=\frac{4b_k}{4k^2+20k+24},b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-(x^2+1/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sinh(x) + c_2 \cosh(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 32

```
DSolve[x^2*y''[x]+x*y'[x]-(x^2+1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}(c_2 e^{2x} + 2c_1)}{2\sqrt{x}}$$

6.11 problem 11

Internal problem ID [4741]
Internal file name [`OUTPUT/4234_Sunday_June_05_2022_12_45_07_PM_19224807/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + xy' + \frac{(-9a^2 + 4x^2)\, y}{4a^2} = 0}$$

6.11.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + \left(-\frac{9}{4} + \frac{x^2}{a^2}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= \frac{1}{a} \\
n &= -\frac{3}{2} \\
\gamma &= 1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = -\frac{c_1 \sqrt{2}\,\left(\sin\left(\frac{x}{a}\right) x + \cos\left(\frac{x}{a}\right) a\right)}{\sqrt{\pi}\,\sqrt{\frac{x}{a}}\, x} - \frac{c_2 \sqrt{2}\,\left(\sin\left(\frac{x}{a}\right) a - \cos\left(\frac{x}{a}\right) x\right)}{\sqrt{\pi}\,\sqrt{\frac{x}{a}}\, x}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{c_1 \sqrt{2}\,\left(\sin\left(\frac{x}{a}\right) x + \cos\left(\frac{x}{a}\right) a\right)}{\sqrt{\pi}\,\sqrt{\frac{x}{a}}\, x} - \frac{c_2 \sqrt{2}\,\left(\sin\left(\frac{x}{a}\right) a - \cos\left(\frac{x}{a}\right) x\right)}{\sqrt{\pi}\,\sqrt{\frac{x}{a}}\, x} \tag{1}
$$

Verification of solutions

$$
y = -\frac{c_1 \sqrt{2}\,\left(\sin\left(\frac{x}{a}\right) x + \cos\left(\frac{x}{a}\right) a\right)}{\sqrt{\pi}\,\sqrt{\frac{x}{a}}\, x} - \frac{c_2 \sqrt{2}\,\left(\sin\left(\frac{x}{a}\right) a - \cos\left(\frac{x}{a}\right) x\right)}{\sqrt{\pi}\,\sqrt{\frac{x}{a}}\, x}
$$

Verified OK.

6.11.2 Solving using Kovacic algorithm

Writing the ode as

$$
4x^2 y'' a^2 + 4y' a^2 x + \left(-9a^2 + 4x^2\right) y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 4a^2 x^2 \\
B &= 4a^2 x \\
C &= -9a^2 + 4x^2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2a^2 - x^2}{a^2 x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2a^2 - x^2 \\ t &= a^2 x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2a^2 - x^2}{a^2 x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = a^2x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{a^2} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx -\frac{33ia^{13}}{16x^{14}} - \frac{21ia^{11}}{16x^{12}} - \frac{7ia^9}{8x^{10}} - \frac{5ia^7}{8x^8} - \frac{ia^5}{2x^6} - \frac{ia^3}{2x^4} - \frac{ia}{x^2} + \frac{i}{a} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{a}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{i}{a} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{1}{a^2}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2a^2 - x^2}{a^2x^2} \\ &= Q + \frac{R}{a^2x^2} \\ &= \left(-\frac{1}{a^2}\right) + \left(\frac{2}{x^2}\right) \\ &= -\frac{1}{a^2} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{i}{a} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i}{a}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i}{a}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2a^2 - x^2}{a^2 x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{i}{a}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{i}{a}\right) \\
&= -\frac{1}{x} - \frac{i}{a} \\
&= -\frac{ix+a}{ax}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{i}{a}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{i}{a}\right)^2 - \left(\frac{2a^2 - x^2}{a^2x^2}\right)\right) &= 0 \\
\frac{2ia_0 - 2a}{ax} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -ia\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = -ia + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-ia + x)\, \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{i}{a}\right) dx} \\
&= (-ia + x)\, \mathrm{e}^{-\frac{ix}{a} - \ln(x)} \\
&= \frac{(-ia + x)\, \mathrm{e}^{-\frac{ix}{a}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4a^2 x}{4a^2 x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-ia + x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4a^2 x}{4a^2 x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{a\,\mathrm{e}^{\frac{2ix}{a}}(ia + x)}{2ix + 2a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-ia + x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}}\right) + c_2 \left(\frac{(-ia + x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}} \left(\frac{a\,\mathrm{e}^{\frac{2ix}{a}}(ia + x)}{2ix + 2a}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1(-ia + x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}} - \frac{c_2(ix - a)\,a\,\mathrm{e}^{\frac{ix}{a}}}{2x^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(-ia+x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}} - \frac{c_2(ix-a)\,a\,\mathrm{e}^{\frac{ix}{a}}}{2x^{\frac{3}{2}}}$$

Verified OK.

6.11.3 Maple step by step solution

Let's solve

$4x^2y''a^2 + 4y'a^2x + (-9a^2 + 4x^2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(9a^2-4x^2)y}{4a^2x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(9a^2-4x^2)y}{4a^2x^2} = 0$

- Multiply by denominators of the ODE

  $4x^2y''a^2 + 4y'a^2x - (9a^2 - 4x^2)\,y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$4x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)a^2+4\left(\frac{d}{dt}y(t)\right)a^2-\left(9a^2-4x^2\right)y(t)=0$$

- Simplify

$$4a^2\left(\frac{d^2}{dt^2}y(t)\right)-9y(t)\,a^2+4y(t)\,x^2=0$$

- Isolate 2nd derivative

$$\frac{d^2}{dt^2}y(t)=\frac{(9a^2-4x^2)\,y(t)}{4a^2}$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dt^2}y(t)-\frac{(9a^2-4x^2)\,y(t)}{4a^2}=0$$

- Characteristic polynomial of ODE

$$r^2-\frac{9a^2-4x^2}{4a^2}=0$$

- Factor the characteristic polynomial

$$\frac{4r^2a^2-9a^2+4x^2}{4a^2}=0$$

- Roots of the characteristic polynomial

$$r=\left(\frac{\sqrt{9a^2-4x^2}}{2a},-\frac{\sqrt{9a^2-4x^2}}{2a}\right)$$

- 1st solution of the ODE

$$y_1(t)=\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- 2nd solution of the ODE

$$y_2(t)=\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}}+c_2\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- Change variables back using $t=\ln(x)$

$$y=c_1\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}}+c_2\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}}$$

- Simplify

$$y=c_1x^{\frac{\sqrt{9a^2-4x^2}}{2a}}+c_2x^{-\frac{\sqrt{9a^2-4x^2}}{2a}}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(4*x^2-9*a^2)/(4*a^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(ix + a)\, c_2 \mathrm{e}^{-\frac{ix}{a}} + (-ix + a)\, \mathrm{e}^{\frac{ix}{a}} c_1}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 62

```
DSolve[x^2*y''[x]+x*y'[x]+(4*x^2-9*a^2)/(4*a^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\left((ac_2 + c_1x) \cos\left(\frac{x}{a}\right) + (c_2x - ac_1) \sin\left(\frac{x}{a}\right)\right)}{x\sqrt{\frac{x}{a}}}$$

6.12 problem 12

Internal problem ID [4742]
Internal file name [`OUTPUT/4235_Sunday_June_05_2022_12_45_18_PM_82403323/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0$$

6.12.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = 1$$
$$n = -\frac{5}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = -\frac{c_1\sqrt{2}\,(\cos(x)\,x^2 - 3\sin(x)\,x - 3\cos(x))}{\sqrt{\pi}\,x^{\frac{5}{2}}} - \frac{c_2\sqrt{2}\,(x^2\sin(x) + 3\cos(x)\,x - 3\sin(x))}{\sqrt{\pi}\,x^{\frac{5}{2}}}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1\sqrt{2}\,(\cos(x)\,x^2 - 3\sin(x)\,x - 3\cos(x))}{\sqrt{\pi}\,x^{\frac{5}{2}}} - \frac{c_2\sqrt{2}\,(x^2\sin(x) + 3\cos(x)\,x - 3\sin(x))}{\sqrt{\pi}\,x^{\frac{5}{2}}} \tag{1}$$

Verification of solutions

$$y = -\frac{c_1\sqrt{2}\,(\cos(x)\,x^2 - 3\sin(x)\,x - 3\cos(x))}{\sqrt{\pi}\,x^{\frac{5}{2}}} - \frac{c_2\sqrt{2}\,(x^2\sin(x) + 3\cos(x)\,x - 3\sin(x))}{\sqrt{\pi}\,x^{\frac{5}{2}}}$$

Verified OK.

6.12.2 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = x^2$$
$$B = x \tag{3}$$
$$C = x^2 - \frac{25}{4}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= i
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2+6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{i} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)(i) \\ &= -\frac{2}{x} - i \\ &= -\frac{2}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2+6}{x^2}\right)\right) &= 0 \\ \frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3, a_1 = -3i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 3ix - 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 3ix - 3\right) \mathrm{e}^{\int \left(-\frac{2}{x} - i\right) dx} \\
&= \left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix - 2\ln(x)} \\
&= \frac{\left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix}}{x^{\frac{5}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{\left(i x^2 - 3x - 3i\right) \mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}\right) + c_2 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1 (x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2 \mathrm{e}^{ix} (ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1 (x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2 \mathrm{e}^{ix} (ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}}
$$

Verified OK.

6.12.3 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{25}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2 - 25) y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2 - 25) y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2 - 25}{4x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{25}{4}$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2y''+4xy'+(4x^2-25)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(5+2r)\,(-5+2r)\,x^r+a_1(7+2r)\,(-3+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-5)+4a_{}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)\,(-5+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{5}{2},\frac{5}{2}\right\}$

- Each term must be 0

  $a_1(7+2r)\,(-3+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+5)(2k+2r-5)+4a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(2k+9+2r)(2k-1+2r)+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$
- Recursion relation for $r=-\frac{5}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}$
- Solution for $r=-\frac{5}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0\right]$
- Recursion relation for $r=\frac{5}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r=\frac{5}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0, b_{k+2}=-\frac{4b_k}{(2k+14)(2k+4)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-25/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-3\left(ix - \frac{1}{3}x^2 + 1\right) c_2 \mathrm{e}^{-ix} + 3\left(ix + \frac{1}{3}x^2 - 1\right) c_1 \mathrm{e}^{ix}}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 59

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-25/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\left(\left(-c_2 x^2 + 3c_1 x + 3c_2\right)\cos(x) + \left(c_1\left(x^2 - 3\right) + 3c_2 x\right)\sin(x)\right)}{x^{5/2}}$$

6.13 problem 15

Internal problem ID [4743]
Internal file name [`OUTPUT/4236_Sunday_June_05_2022_12_45_29_PM_49732457/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + qy' - \frac{2y}{x^2} = 0}$$

6.13.1 Solving using Kovacic algorithm

Writing the ode as

$$qy'x^2 + x^2y'' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= q\,x^2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2q^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2q^2 + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2q^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 111: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{q^2}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{q}{2} + \frac{2}{q\,x^2} - \frac{4}{q^3 x^4} + \frac{16}{q^5 x^6} - \frac{80}{q^7 x^8} + \frac{448}{q^9 x^{10}} - \frac{2688}{q^{11} x^{12}} + \frac{16896}{q^{13} x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{q}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{q}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{q^2}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 q^2 + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{q^2}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{q^2}{4} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{q}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{q}{2}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{q}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2q^2 + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{q}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\left(\frac{q}{2}\right) \\
&= -\frac{1}{x} - \frac{q}{2} \\
&= -\frac{qx + 2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - \frac{q}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{q}{2}\right)^2 - \left(\frac{x^2q^2 + 8}{4x^2}\right)\right) = 0$$
$$\frac{qa_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{2}{q}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{2}{q}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{2}{q}\right) e^{\int \left(-\frac{1}{x} - \frac{q}{2}\right) dx} \\
&= \left(x + \frac{2}{q}\right) e^{-\frac{qx}{2} - \ln(x)} \\
&= \frac{(qx + 2)\, e^{-\frac{qx}{2}}}{qx}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{q x^2}{x^2}\,dx} \\
&= z_1 e^{-\frac{qx}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{qx}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(qx+2)\,\mathrm{e}^{-qx}}{qx}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{q x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-qx}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(qx-2)\,\mathrm{e}^{qx}}{q\,(qx+2)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(qx+2)\,\mathrm{e}^{-qx}}{qx}\right) + c_2 \left(\frac{(qx+2)\,\mathrm{e}^{-qx}}{qx}\left(\frac{(qx-2)\,\mathrm{e}^{qx}}{q\,(qx+2)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(qx+2)\,\mathrm{e}^{-qx}}{qx} + \frac{c_2(qx-2)}{q^2 x} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(qx+2)\,\mathrm{e}^{-qx}}{qx} + \frac{c_2(qx-2)}{q^2 x}$$

Verified OK.

6.13.2 Maple step by step solution

Let's solve

$qy'x^2 + x^2y'' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -qy' + \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + qy' - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = q, P_3(x) = -\frac{2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $qy'x^2 + x^2y'' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$
  - Shift index using $k->k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + q a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) + q a_{k-1}(k-1+r) = 0$
- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)\,(k-1+r) + q a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{q a_k(k+r)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r=-1$; series terminates at $k=1$

  $a_{k+1} = -\frac{q a_k(k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k=0$

  $a_1 = -\frac{q a_0}{2}$
- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second

  $y = a_0 \cdot \left(-\frac{qx}{2} + 1\right)$
- Recursion relation for $r=2$

  $a_{k+1} = -\frac{q a_k(k+2)}{(k+4)(k+1)}$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{q a_k (k+2)}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = a_0 \cdot \left(-\frac{qx}{2} + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{q b_k (k+2)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)+q*diff(y(x),x)=2*y(x)/x^2,y(x), singsol=all)
```

$$y(x) = \frac{c_2 e^{-qx}(qx+2) + c_1(qx-2)}{x}$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 80

```
DSolve[y''[x]+q*y'[x]==2*y[x]/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{q x^{3/2} e^{-\frac{qx}{2}} \left(2(i c_2 q x + 2c_1)\sinh\left(\frac{qx}{2}\right) - 2(c_1 q x + 2 i c_2)\cosh\left(\frac{qx}{2}\right)\right)}{\sqrt{\pi}(-iqx)^{5/2}}$$

6.14 problem 18

Internal problem ID [4744]
Internal file name [`OUTPUT/4237_Sunday_June_05_2022_12_45_39_PM_285195/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode_form_A"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \mathrm{e}^{2x} y - y n^2 = 0}$$

6.14.1 Solving as second order bessel ode form A ode

Writing the ode as

$$y'' + \left(-n^2 + \mathrm{e}^{2x}\right) y = 0 \tag{1}$$

An ode of the form

$$a y'' + b y' + (c e^{rx} + m) y = 0 \tag{1}$$

can be transformed to Bessel ode using the transformation

$$\begin{aligned} x &= \ln(t) \\ e^x &= t \end{aligned}$$

Where a, b, c, m are not functions of x and where b and m are allowed to be be zero. Using this transformation gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \\ &= \frac{dy}{dt} e^x \\ &= t\frac{dy}{dt} \end{aligned} \tag{2}$$

And

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{dx}\left(t\frac{dy}{dt}\right) \\
&= \frac{d}{dt}\frac{dt}{dx}\left(t\frac{dy}{dt}\right) \\
&= \frac{dt}{dx}\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\
&= t\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\
&= t\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right) \qquad (3)
\end{aligned}
$$

Substituting (2,3) into (1) gives

$$
\begin{aligned}
at\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right) + bt\frac{dy}{dt} + (ce^{rx} + m)y &= 0 \\
(aty' + at^2y'') + bty' + (ct^r + m)y &= 0 \\
at^2y'' + (b+a)\, ty' + (ct^r + m)y &= 0 \\
t^2y'' + \frac{b+a}{a}ty' + \left(\frac{c}{a}t^r + \frac{m}{a}\right) y &= 0 \qquad (4)
\end{aligned}
$$

Which is Bessel ODE. Comparing the above to the general known Bowman form of Bessel ode which is

$$
t^2y'' + (1-2\alpha)\, ty' + \left(\beta^2\gamma^2 t^{2\gamma} - \left(n^2\gamma^2 - \alpha^2\right)\right) y = 0 \qquad \text{(C)}
$$

And now comparing (4) and (C) shows that

$$
\begin{aligned}
(1-2\alpha) &= \frac{b+a}{a} \qquad (5) \\
\beta^2\gamma^2 &= \frac{c}{a} \qquad (6) \\
2\gamma &= r \qquad (7) \\
\left(n^2\gamma^2 - \alpha^2\right) &= -\frac{m}{a} \qquad (8)
\end{aligned}
$$

(5) gives $\alpha = \frac{1}{2} - \frac{b+a}{2a}$. (7) gives $\gamma = \frac{r}{2}$. (8) now becomes $\left(n^2\left(\frac{r}{2}\right)^2 - \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2\right) = -\frac{m}{a}$ or $n^2 = \frac{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}{\left(\frac{r}{2}\right)^2}$. Hence $n = \frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}$ by taking the positive root.

And finally (6) gives $\beta^2 = \frac{c}{a\gamma^2}$ or $\beta = \sqrt{\frac{c}{a}\frac{1}{\gamma}} = \sqrt{\frac{c}{a}\frac{2}{r}}$ (also taking the positive root). Hence

$$
\begin{aligned}
\alpha &= \frac{1}{2} - \frac{b+a}{2a} \\
n &= \frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2} \\
\beta &= \sqrt{\frac{c}{a}\frac{2}{r}} \\
\gamma &= \frac{r}{2}
\end{aligned}
$$

But the solution to (C) which is general form of Bessel ode is known and given by

$$
y(t) = t^\alpha (c_1 J_n(\beta t^\gamma) + c_2 Y_n(\beta t^\gamma))
$$

Substituting the above values found into this solution gives

$$
y(t) = t^{\frac{1}{2}-\frac{b+a}{2a}}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}t^{\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}t^{\frac{r}{2}}\right)\right)
$$

Since $e^x = t$ then the above becomes

$$
\begin{aligned}
y(x) &= e^{x\left(\frac{1}{2}-\frac{b+a}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{-b}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{-b}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\frac{b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\frac{b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \qquad (9)
\end{aligned}
$$

Equation (9) above is the solution to $ay'' + by' + (ce^{rx} + m)y = 0$. Therefore we just need now to compare this form to the ode given and use (9) to obtain the final solution.

Comparing form (1) to the ode we are solving shows that

$$
\begin{aligned}
a &= 1 \\
b &= 0 \\
c &= 1 \\
r &= 2 \\
m &= -n^2
\end{aligned}
$$

Substituting these in (9) gives the solution as

$$y = c_1 \operatorname{BesselJ}(n, e^x) + c_2 \operatorname{BesselY}(n, e^x)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}(n, e^x) + c_2 \operatorname{BesselY}(n, e^x) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}(n, e^x) + c_2 \operatorname{BesselY}(n, e^x)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)]
   Linear ODE actually solved:
      (-n^2+t^2)*u(t)+t*diff(u(t),t)+t^2*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+exp(2*x)*y(x)=n^2*y(x),y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}(n, e^x) + c_2 \operatorname{BesselY}(n, e^x)$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 46

```
DSolve[y''[x]+Exp[2*x]*y[x]==n^2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{Gamma}(1-n) \operatorname{BesselJ}\left(-n, \sqrt{e^{2x}}\right) + c_2 \operatorname{Gamma}(n+1) \operatorname{BesselJ}\left(n, \sqrt{e^{2x}}\right)$$

6.15 problem 19

Internal problem ID [4745]
Internal file name [`OUTPUT/4238_Sunday_June_05_2022_12_45_45_PM_12574817/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{y'' + \frac{y}{4x} = 0}$$

6.15.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + \frac{xy}{4} = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 1$$
$$n = 1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}\left(1, \sqrt{x}\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(1, \sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}\left(1, \sqrt{x}\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(1, \sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}\left(1, \sqrt{x}\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(1, \sqrt{x}\right)$$

Verified OK.

6.15.2 Maple step by step solution

Let's solve

$4y''x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y}{4x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{1}{4x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4y''x + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

  - Shift index using $k->k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $4a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(4a_{k+1}(k+1+r)(k+r) + a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $4a_{k+1}(k+1+r)(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{4(k+1+r)(k+r)}$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{4(k+1)k}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{4(k+1)k}\right]$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{4(k+2)(k+1)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{4(k+2)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{a_k}{4(k+1)k}, b_{k+1} = -\frac{b_k}{4(k+2)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)/(4*x)=0,y(x), singsol=all)
```

$$y(x) = \left(\operatorname{BesselY}\left(1, \sqrt{x}\right) c_2 + \operatorname{BesselJ}\left(1, \sqrt{x}\right) c_1\right) \sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 38

```
DSolve[y''[x]+y[x]/(4*x)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sqrt{x}\left(c_1 \operatorname{BesselJ}\left(1, \sqrt{x}\right) + 2ic_2 \operatorname{BesselY}\left(1, \sqrt{x}\right)\right)$$

6.16 problem 20

Internal problem ID [4746]
Internal file name [`OUTPUT/4239_Sunday_June_05_2022_12_45_52_PM_51741631/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{y''x + y' + y = 0}$$

6.16.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + xy' + xy = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 0\\ \beta &= 2\\ n &= 0\\ \gamma &= \frac{1}{2}\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right)$$

Summary
The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right)$$

Verified OK.

6.16.2 Maple step by step solution

Let's solve

$y''x + y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{1}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{(k+1)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 31

```
DSolve[x*y''[x]+y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + 2c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right)$$

6.17 problem 21

Internal problem ID [4747]
Internal file name [`OUTPUT/4240_Sunday_June_05_2022_12_45_59_PM_64848712/index.tex`]

Book: A treatise on ordinary and partial differential equations by William Woolsey Johnson. 1913
Section: Chapter IX, Special forms of differential equations. Examples XVII. page 247
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y''x + 3y' + 4yx^3 = 0}$$

6.17.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 3xy' + 4yx^4 = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -1 \\
\beta &= 1 \\
n &= \frac{1}{2} \\
\gamma &= 2
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\,\sin\left(x^2\right)}{x\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2\sqrt{2}\,\cos\left(x^2\right)}{x\sqrt{\pi}\,\sqrt{x^2}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,\sin\left(x^2\right)}{x\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2\sqrt{2}\,\cos\left(x^2\right)}{x\sqrt{\pi}\,\sqrt{x^2}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\sin\left(x^2\right)}{x\sqrt{\pi}\,\sqrt{x^2}} - \frac{c_2\sqrt{2}\,\cos\left(x^2\right)}{x\sqrt{\pi}\,\sqrt{x^2}}$$

Verified OK.

6.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y''x + 3y' + 4yx^3 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= x \\
B &= 3 \\
C &= 4x^3
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 115: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 4\\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(-4x^2\right) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-16x^4 + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\left(2ix\right) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{\mathrm{e}^{-ix^2}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{x} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-i x^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{i e^{2 i x^2}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-i x^2}}{x^2}\right) + c_2 \left(\frac{e^{-i x^2}}{x^2}\left(-\frac{i e^{2 i x^2}}{4}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{-i x^2}}{x^2} - \frac{i c_2 e^{i x^2}}{4 x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{-i x^2}}{x^2} - \frac{i c_2 e^{i x^2}}{4 x^2}$$

Verified OK.

6.17.3 Maple step by step solution

Let's solve

$y''x + 3y' + 4y x^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - 4y x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} + 4y x^2 = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + 3y' + 4y x^3 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^3 \cdot y$ to series expansion

  $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

  ◦ Shift index using $k- >k - 3$

$$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 0\}$$

- The coefficients of each power of x must be 0

  $$[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_1 = 0, a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$$

- Shift index using $k->k+3$

  $$a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$$

- Recursion relation for $r = -2$

$a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Recursion relation for $r = 0$

$a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{4b_k}{(k+4)(k+6)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+4*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x^2) + c_2 \cos(x^2)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 41

```
DSolve[x*y''[x]+3*y'[x]+4*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_1e^{-ix^2} - ic_2e^{ix^2}}{4x^2}$$