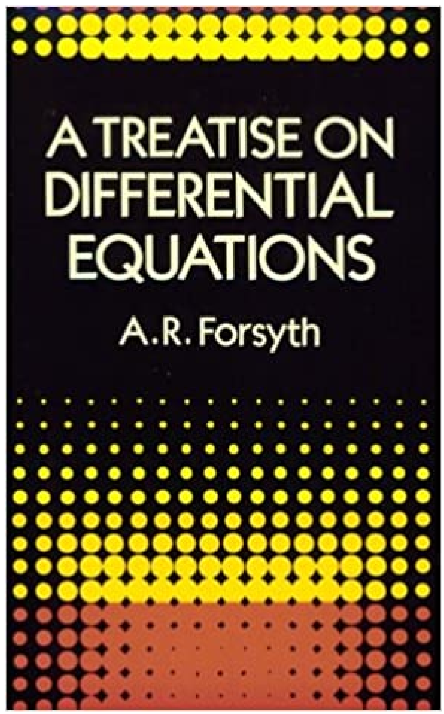
A Solution Manual For

A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956



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Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 5, page 256.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(-x^2+2)\,y''-(x^2+4x+2)\,((1-x)\,y'+y)=0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^3+2x)\,y''+(x^3+3x^2-2x-2)\,y'+(-x^2-4x-2)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{x^3+3x^2-2x-2}{(x^2-2)\,x}$$
$$q(x)=\frac{x^2+4x+2}{(x^2-2)\,x}$$

Table 1: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^3+3x^2-2x-2}{(x^2-2)x}$	
singularity	type
$x = 0$	"regular"
$x = \sqrt{2}$	"regular"
$x = -\sqrt{2}$	"regular"

$q(x) = \frac{x^2+4x+2}{(x^2-2)x}$	
singularity	type
$x = 0$	"regular"
$x = \sqrt{2}$	"regular"
$x = -\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \sqrt{2}, -\sqrt{2}]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''(x^2-2)x + (x^3+3x^2-2x-2)y' + (-x^2-4x-2)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned}
&-\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right)(x^2-2)x \\
&+ (x^3+3x^2-2x-2)\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) \\
&+ (-x^2-4x-2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2 x^{n+r-1} a_n(n+r)(n+r-1)\right) \\
&\quad+\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 3 x^{1+n+r} a_n(n+r)\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-2 x^{n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-2(n+r) a_n x^{n+r-1}\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-4 x^{1+n+r} a_n\right)+\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right) &= \sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n+r-3) x^{n+r-1}\right) \\
\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r) &= \sum_{n=3}^{\infty} a_{n-3}(n+r-3) x^{n+r-1} \\
\sum_{n=0}^{\infty} 3 x^{1+n+r} a_n(n+r) &= \sum_{n=2}^{\infty} 3 a_{n-2}(n+r-2) x^{n+r-1} \\
\sum_{n=0}^{\infty}\left(-2 x^{n+r} a_n(n+r)\right) &= \sum_{n=1}^{\infty}\left(-2 a_{n-1}(n+r-1) x^{n+r-1}\right) \\
\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n\right) &= \sum_{n=3}^{\infty}\left(-a_{n-3} x^{n+r-1}\right) \\
\sum_{n=0}^{\infty}\left(-4 x^{1+n+r} a_n\right) &= \sum_{n=2}^{\infty}\left(-4 a_{n-2} x^{n+r-1}\right) \\
\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right) &= \sum_{n=1}^{\infty}\left(-2 a_{n-1} x^{n+r-1}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned} &\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)\,(n+r-3)\,x^{n+r-1}\right) \\ &\quad+\left(\sum_{n=0}^{\infty} 2x^{n+r-1}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=3}^{\infty} a_{n-3}(n+r-3)\,x^{n+r-1}\right) \\ &\quad+\left(\sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)\,x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,x^{n+r-1}\right) \\ &\quad+\sum_{n=0}^{\infty}\left(-2(n+r)\,a_n x^{n+r-1}\right)+\sum_{n=3}^{\infty}\left(-a_{n-3}x^{n+r-1}\right) \\ &\quad+\sum_{n=2}^{\infty}\left(-4a_{n-2}x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1}x^{n+r-1}\right)=0 \end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1}a_n(n+r)\,(n+r-1)-2(n+r)\,a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$2x^{-1+r}a_0 r(-1+r)-2ra_0x^{-1+r}=0$$

Or

$$\left(2x^{-1+r}r(-1+r)-2r\,x^{-1+r}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$2r\,x^{-1+r}(-2+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$2r(-2+r)=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1&=2\\ r_2&=0 \end{aligned}$$

Since $a_0\neq 0$ then the indicial equation becomes

$$2r\,x^{-1+r}(-2+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{1}{-1 + r}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{r^2 - 5r + 10}{2(-1 + r)(2 + r)}$$

For $3 \leq n$ the recursive equation is

$$\begin{aligned} &-a_{n-2}(n + r - 2)(n + r - 3) + 2a_n(n + r)(n + r - 1) + a_{n-3}(n + r - 3) \\ &+ 3a_{n-2}(n + r - 2) - 2a_{n-1}(n + r - 1) - 2a_n(n + r) - a_{n-3} - 4a_{n-2} - 2a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n^2 a_{n-2} + 2nr a_{n-2} + r^2 a_{n-2} - n a_{n-3} - 8n a_{n-2} + 2n a_{n-1} - r a_{n-3} - 8r a_{n-2} + 2r a_{n-1} + 4a_{n-3} + 16a}{2n^2 + 4nr + 2r^2 - 4n - 4r} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{n^2 a_{n-2} + (-a_{n-3} - 4a_{n-2} + 2a_{n-1})\, n + 2a_{n-3} + 4a_{n-2} + 4a_{n-1}}{2n\,(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-1+r}$	1
a_2	$\frac{r^2-5r+10}{2(-1+r)(2+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r^2 - 5r + 10}{2r^3 + 4r^2 - 2r - 4}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-1+r}$	1
a_2	$\frac{r^2-5r+10}{2(-1+r)(2+r)}$	$\frac{1}{2}$
a_3	$\frac{r^2-5r+10}{2r^3+4r^2-2r-4}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 - 6r^3 + 17r^2 - 32r + 40}{4r^4 + 24r^3 + 28r^2 - 24r - 32}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-1+r}$	1
a_2	$\frac{r^2-5r+10}{2(-1+r)(2+r)}$	$\frac{1}{2}$
a_3	$\frac{r^2-5r+10}{2r^3+4r^2-2r-4}$	$\frac{1}{6}$
a_4	$\frac{r^4-6r^3+17r^2-32r+40}{4r^4+24r^3+28r^2-24r-32}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r^4 - 6r^3 + 17r^2 - 32r + 40}{4r^5 + 36r^4 + 100r^3 + 60r^2 - 104r - 96}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{-1+r}$	1
a_2	$\frac{r^2-5r+10}{2(-1+r)(2+r)}$	$\frac{1}{2}$
a_3	$\frac{r^2-5r+10}{2r^3+4r^2-2r-4}$	$\frac{1}{6}$
a_4	$\frac{r^4-6r^3+17r^2-32r+40}{4r^4+24r^3+28r^2-24r-32}$	$\frac{1}{24}$
a_5	$\frac{r^4-6r^3+17r^2-32r+40}{4r^5+36r^4+100r^3+60r^2-104r-96}$	$\frac{1}{120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_2 \\
&= \frac{r^2 - 5r + 10}{2(-1+r)(2+r)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\rightarrow r_2} \frac{r^2 - 5r + 10}{2(-1+r)(2+r)} &= \lim_{r\rightarrow 0} \frac{r^2 - 5r + 10}{2(-1+r)(2+r)} \\
&= -\frac{5}{2}
\end{aligned}$$

The limit is $-\frac{5}{2}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^n
\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = \frac{1}{-1+r}$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = \frac{r^2 - 5r + 10}{2(-1+r)(2+r)}$$

For $3 \leq n$ the recursive equation is

$$\begin{aligned}
&-b_{n-2}(n+r-2)(n+r-3) + 2b_n(n+r)(n+r-1) + b_{n-3}(n+r-3) \\
&+ 3b_{n-2}(n+r-2) - 2b_{n-1}(n+r-1) - 2(n+r)b_n - b_{n-3} - 4b_{n-2} - 2b_{n-1} = 0
\end{aligned} \tag{4}$$

Which for for the root $r = 0$ becomes

$$\begin{aligned}
&-b_{n-2}(n-2)(n-3) + 2b_n n(n-1) + b_{n-3}(n-3) + 3b_{n-2}(n-2) \\
&- 2b_{n-1}(n-1) - 2nb_n - b_{n-3} - 4b_{n-2} - 2b_{n-1} = 0
\end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{n^2 b_{n-2} + 2nrb_{n-2} + r^2 b_{n-2} - nb_{n-3} - 8nb_{n-2} + 2nb_{n-1} - rb_{n-3} - 8rb_{n-2} + 2rb_{n-1} + 4b_{n-3} + 16b_{n-2}}{2n^2 + 4nr + 2r^2 - 4n - 4r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{n^2 b_{n-2} - nb_{n-3} - 8nb_{n-2} + 2nb_{n-1} + 4b_{n-3} + 16b_{n-2}}{2n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{-1+r}$	-1
b_2	$\frac{r^2-5r+10}{2(-1+r)(2+r)}$	$-\frac{5}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{r^2 - 5r + 10}{2(1+r)(-1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{5}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{-1+r}$	-1
b_2	$\frac{r^2-5r+10}{2(-1+r)(2+r)}$	$-\frac{5}{2}$
b_3	$\frac{r^2-5r+10}{2r^3+4r^2-2r-4}$	$-\frac{5}{2}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^4 - 6r^3 + 17r^2 - 32r + 40}{4(1+r)(-1+r)(r^2+6r+8)}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{5}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{-1+r}$	-1
b_2	$\frac{r^2-5r+10}{2(-1+r)(2+r)}$	$-\frac{5}{2}$
b_3	$\frac{r^2-5r+10}{2r^3+4r^2-2r-4}$	$-\frac{5}{2}$
b_4	$\frac{r^4-6r^3+17r^2-32r+40}{4r^4+24r^3+28r^2-24r-32}$	$-\frac{5}{4}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{r^4 - 6r^3 + 17r^2 - 32r + 40}{4\,(r+3)\,(1+r)\,(-1+r)\,(r^2+6r+8)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{5}{12}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{-1+r}$	-1
b_2	$\frac{r^2-5r+10}{2(-1+r)(2+r)}$	$-\frac{5}{2}$
b_3	$\frac{r^2-5r+10}{2r^3+4r^2-2r-4}$	$-\frac{5}{2}$
b_4	$\frac{r^4-6r^3+17r^2-32r+40}{4r^4+24r^3+28r^2-24r-32}$	$-\frac{5}{4}$
b_5	$\frac{r^4-6r^3+17r^2-32r+40}{4r^5+36r^4+100r^3+60r^2-104r-96}$	$-\frac{5}{12}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - x - \frac{5x^2}{2} - \frac{5x^3}{2} - \frac{5x^4}{4} - \frac{5x^5}{12} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)+c_2\left(1-x-\frac{5x^2}{2}-\frac{5x^3}{2}-\frac{5x^4}{4}-\frac{5x^5}{12}\right. \\
&\qquad \left. +O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)+c_2\left(1-x-\frac{5x^2}{2}-\frac{5x^3}{2}-\frac{5x^4}{4}-\frac{5x^5}{12}\right. \\
&\qquad \left. +O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^2 &\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right) \\
&+ c_2\left(1-x-\frac{5x^2}{2}-\frac{5x^3}{2}-\frac{5x^4}{4}-\frac{5x^5}{12}+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^2 &\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right) \\
&+ c_2\left(1-x-\frac{5x^2}{2}-\frac{5x^3}{2}-\frac{5x^4}{4}-\frac{5x^5}{12}+O(x^6)\right)
\end{aligned}
$$

Verified OK.

1.1.1 Maple step by step solution

Let's solve

$-y''(x^2-2)\,x+(x^3+3x^2-2x-2)\,y'+(-x^2-4x-2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x^2+4x+2)y}{(x^2-2)x}+\frac{(x^3+3x^2-2x-2)y'}{(x^2-2)x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x^3+3x^2-2x-2)y'}{(x^2-2)x}+\frac{(x^2+4x+2)y}{(x^2-2)x}=0$

☐ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x)=-\frac{x^3+3x^2-2x-2}{(x^2-2)x}, P_3(x)=\frac{x^2+4x+2}{(x^2-2)x}\right]$

- ∘ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-1$

- ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

- ∘ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''(x^2-2)\,x+(-x^3-3x^2+2x+2)\,y'+(x^2+4x+2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ∘ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$$

- Convert $x^{m}\cdot y'$ to series expansion for $m=0..3$

  $$x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^{m}\cdot y''$ to series expansion for $m=1..3$

  $$x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_{0}r(-2+r)\,x^{-1+r}+(-2a_{1}(1+r)\,(-1+r)+2a_{0}(1+r))\,x^{r}+\big(-2a_{2}(2+r)\,r+2a_{1}(2+r)+$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- The coefficients of each power of x must be 0

  $\left[-2a_{1}(1+r)\,(-1+r)+2a_{0}(1+r)=0,-2a_{2}(2+r)\,r+2a_{1}(2+r)+a_{0}(-2+r)^{2}=0\right]$

- Solve for the dependent coefficient(s)

  $\left\{a_{1}=\frac{a_{0}}{-1+r},a_{2}=\frac{a_{0}(r^{2}-5r+10)}{2(r^{2}+r-2)}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(k-3+r)^{2}-2a_{k+1}(k+r+1)\,(k+r-1)+(2a_{k}-a_{k-2})\,k+(2a_{k}-a_{k-2})\,r+2a_{k}+3a_{k-}$

- Shift index using $k->k+2$

  $a_{k+1}(k+r-1)^{2}-2a_{k+3}(k+3+r)\,(k+r+1)+(2a_{k+2}-a_{k})\,(k+2)+(2a_{k+2}-a_{k})\,r+2a_{k+2}$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = \frac{k^2 a_{k+1}+2kr a_{k+1}+r^2 a_{k+1}-a_k k-2k a_{k+1}+2k a_{k+2}-a_k r-2r a_{k+1}+2r a_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3+r)(k+r+1)}$$

- Recursion relation for $r = 0$

$$a_{k+3} = \frac{k^2 a_{k+1}-a_k k-2k a_{k+1}+2k a_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k^2 a_{k+1}-a_k k-2k a_{k+1}+2k a_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}, a_1 = -a_0, a_2 = -\frac{5a_0}{2}\right]$$

- Recursion relation for $r = 2$

$$a_{k+3} = \frac{k^2 a_{k+1}-a_k k+2k a_{k+1}+2k a_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+3} = \frac{k^2 a_{k+1}-a_k k+2k a_{k+1}+2k a_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}, a_1 = a_0, a_2 = \frac{a_0}{2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+3} = \frac{k^2 a_{k+1}-k a_k-2k a_{k+1}+2k a_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}, a_1 = -a_0, a_2 =\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 44

```
Order:=6;
dsolve(x*(2-x^2)*diff(y(x),x$2)-(x^2+4*x+2)*((1-x)*diff(y(x),x)+y(x))=0,y(x),type='series',x
```

$$y(x) = c_1x^2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(-2 + 2x + 4x^2 + 4x^3 + 2x^4 + \frac{2}{3}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 64

```
AsymptoticDSolveValue[x*(2-x^2)*y''[x]-(x^2+4*x+2)*((1-x)*y'[x]+y[x])==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{5x^4}{4} - \frac{5x^3}{2} - \frac{5x^2}{2} - x + 1\right) + c_2\left(\frac{x^6}{24} + \frac{x^5}{6} + \frac{x^4}{2} + x^3 + x^2\right)$$

1.2 problem Ex. 6(i), page 257

Internal problem ID [5472]
Internal file name [`OUTPUT/4720_Sunday_June_05_2022_03_04_01_PM_46599014/index.tex`]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 6(i), page 257.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\,y''-(1+2x)\,(-y+xy')=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+\left(-2x^2-x\right)y'+(1+2x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{1+2x}{x\,(1+x)}$$
$$q(x)=\frac{1+2x}{x^2\,(1+x)}$$

Table 3: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1+2x}{x(1+x)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{1+2x}{x^2(1+x)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(1+x)\,y'' + \left(-2x^2 - x\right) y' + (1+2x)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x^2(1+x) + \left(-2x^2 - x\right)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (1+2x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} (n+r-1) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} (n+r-1) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(-1+r\right)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$\left(-1+r\right)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(-1+r\right)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\
y_2(x) &= y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n} \right)
\end{aligned}
$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\
- 2a_{n-1}(n+r-1) - a_n(n+r) + a_n + 2a_{n-1} = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 5n - 5r + 6)}{n^2 + 2nr + r^2 - 2n - 2r + 1} \tag{4}
$$

Which for the root $r = 1$ becomes

$$
a_n = -\frac{a_{n-1}(n^2 - 3n + 2)}{n^2} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = \frac{-r^2 + 3r - 2}{r^2}
$$

Which for the root $r = 1$ becomes

$$
a_1 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+3r-2}{r^2}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r-2)(-1+r)^2}{r(r+1)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+3r-2}{r^2}$	0
a_2	$\frac{(r-2)(-1+r)^2}{r(r+1)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r-2)(-1+r)^2}{(r+2)^2(r+1)}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+3r-2}{r^2}$	0
a_2	$\frac{(r-2)(-1+r)^2}{r(r+1)^2}$	0
a_3	$-\frac{(r-2)(-1+r)^2}{(r+2)^2(r+1)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r-2)(-1+r)^2}{(r+3)^2(r+2)}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+3r-2}{r^2}$	0
a_2	$\frac{(r-2)(-1+r)^2}{r(r+1)^2}$	0
a_3	$-\frac{(r-2)(-1+r)^2}{(r+2)^2(r+1)}$	0
a_4	$\frac{(r-2)(-1+r)^2}{(r+3)^2(r+2)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r-2)(-1+r)^2}{(r+4)^2(r+3)}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+3r-2}{r^2}$	0
a_2	$\frac{(r-2)(-1+r)^2}{r(r+1)^2}$	0
a_3	$-\frac{(r-2)(-1+r)^2}{(r+2)^2(r+1)}$	0
a_4	$\frac{(r-2)(-1+r)^2}{(r+3)^2(r+2)}$	0
a_5	$-\frac{(r-2)(-1+r)^2}{(r+4)^2(r+3)}$	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x\left(1 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{-r^2+3r-2}{r^2}$	0	$\frac{-3r+4}{r^3}$	1
b_2	$\frac{(r-2)(-1+r)^2}{r(r+1)^2}$	0	$\frac{6r^3-14r^2+6r+2}{r^2(r+1)^3}$	0
b_3	$-\frac{(r-2)(-1+r)^2}{(r+2)^2(r+1)}$	0	$-\frac{9(-1+r)(r^2-\frac{1}{3}r-2)}{(r+2)^3(r+1)^2}$	0
b_4	$\frac{(r-2)(-1+r)^2}{(r+3)^2(r+2)}$	0	$\frac{12r^3-4r^2-52r+44}{(r+3)^3(r+2)^2}$	0
b_5	$-\frac{(r-2)(-1+r)^2}{(r+4)^2(r+3)}$	0	$\frac{-15r^3-10r^2+105r-80}{(r+4)^3(r+3)^2}$	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= x\left(1 + O\left(x^6\right)\right)\ln(x) + x\left(x + O\left(x^6\right)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + O\left(x^6\right)\right) + c_2\left(x\left(1 + O\left(x^6\right)\right)\ln(x) + x\left(x + O\left(x^6\right)\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 + O\left(x^6\right)\right) + c_2\left(x\left(1 + O\left(x^6\right)\right)\ln(x) + x\left(x + O\left(x^6\right)\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x \left(1 + O\left(x^6\right)\right) + c_2 \left(x \left(1 + O\left(x^6\right)\right) \ln\left(x\right) + x \left(x + O\left(x^6\right)\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 + O\left(x^6\right)\right) + c_2 \left(x \left(1 + O\left(x^6\right)\right) \ln\left(x\right) + x \left(x + O\left(x^6\right)\right)\right)$$

Verified OK.

1.2.1 Maple step by step solution

Let's solve

$y''x^2(1+x) + (-2x^2 - x)\, y' + (1+2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(1+2x)y}{x^2(1+x)} + \frac{(1+2x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(1+2x)y'}{x(1+x)} + \frac{(1+2x)y}{x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1+2x}{x(1+x)}, P_3(x) = \frac{1+2x}{x^2(1+x)}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = -1$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $y''x^2(1+x) - x(1+2x)\, y' + (1+2x)\, y = 0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $\left(u^{3}-2u^{2}+u\right)\left(\frac{d^{2}}{du^{2}}y(u)\right)+\left(-2u^{2}+3u-1\right)\left(\frac{d}{du}y(u)\right)+(-1+2u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^{m}\cdot y(u)$ to series expansion for $m=0..1$

    $u^{m}\cdot y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^{m}\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^{m}\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..3$

    $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}r(-2+r)\,u^{-1+r}+\left(a_{1}(1+r)\,(-1+r)-a_{0}\left(2r^{2}-5r+1\right)\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- Each term must be 0

$a_1(1+r)(-1+r) - a_0(2r^2 - 5r + 1) = 0$

- Each term in the series must be 0, giving the recursion relation

$(-2a_k + a_{k-1} + a_{k+1}) k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1}) r + 5a_k - 5a_{k-1}) k + (-2a_k + a_{k-1} + a_{k+1})$

- Shift index using $k- > k+1$

$(-2a_{k+1} + a_k + a_{k+2}) (k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2}) r + 5a_{k+1} - 5a_k) (k+1) + (-2a_{k+1} +$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kra_k - 4kra_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 3ka_k + ka_{k+1} - 3ra_k + ra_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2kr + r^2 + 2k + 2r}$

- Recursion relation for $r = 0$

$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 3ka_k + ka_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2k}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 3ka_k + ka_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2k}$

- Recursion relation for $r = 2$

$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}$

- Solution for $r = 2$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}, 3a_1 + a_0 = 0\right]$

- Revert the change of variables $u = 1 + x$

$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}, 3a_1 + a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*(1+x)*diff(y(x),x$2)-(1+2*x)*(x*diff(y(x),x)-y(x))=0,y(x),type='series',x=0);
```

$$y(x) = x\left(\left(c_2 \ln\left(x\right) + c_1\right)\left(1 + \mathrm{O}\left(x^6\right)\right) + \left(x + \mathrm{O}\left(x^6\right)\right) c_2\right)$$

✓ Solution by Mathematica
Time used: 0.006 (sec). Leaf size: 2760

```
AsymptoticDSolveValue[x^2*(1+x)*y''[x]-(1+2*x)*(x*y'[x]+y[x])==0,y[x],{x,0,5}]
```

Too large to display

1.3 problem Ex. 6(ii), page 257

Internal problem ID [5473]
Internal file name [`OUTPUT/4721_Sunday_June_05_2022_03_04_02_PM_75632679/index.tex`]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 6(ii), page 257.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
trying Louvillian solutions for 3rd order ODEs, imprimitive case
Louvillian solutions for 3rd order ODEs, imprimitive case: input is reducible, switching to
checking if the LODE is of Euler type
expon. solutions partially successful. Result(s) =`, [x^2]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 79

```
Order:=6;
dsolve(x^3*(1+x)*diff(y(x),x$3)-(2+4*x)*x^2*diff(y(x),x$2)+(4+10*x)*x*diff(y(x),x)-(4+12*x)*
```

$$y(x) = x\left(\left(2x + \mathrm{O}\left(x^6\right)\right)\ln\left(x\right)^2 c_3 + \ln\left(x\right)\left(2 + \mathrm{O}\left(x^6\right)\right)c_2 x + 2\left(\left(-4\right)x + \mathrm{O}\left(x^6\right)\right)\ln\left(x\right)c_3 + \left(5 + \mathrm{O}\left(x^6\right)\right)c_2 x + c_1 x\left(1 + \mathrm{O}\left(x^6\right)\right) + \left(2 + 4x + 2x^2 + \mathrm{O}\left(x^6\right)\right)c_3\right)$$

✓ Solution by Mathematica

Time used: 0.514 (sec). Leaf size: 49

```
AsymptoticDSolveValue[x^3*(1+x)*y'''[x]-(2+4*x)*x^2*y''[x]+(4+10*x)*x*y'[x]-(4+12*x)*y[x]==0
```

$$y(x) \to c_2 x^2 + c_1\left(2\left(x^2 + 11x + 1\right)x + 2x^2\log^2(x) - 14x^2\log(x)\right) + c_3 x^2\log(x)$$

1.4 problem Ex. 6(iii), page 257

Internal problem ID [5474]
Internal file name [`OUTPUT/4722_Sunday_June_05_2022_03_04_03_PM_37151700/index.tex`]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 6(iii), page 257.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
trying Louvillian solutions for 3rd order ODEs, imprimitive case
Louvillian solutions for 3rd order ODEs, imprimitive case: input is reducible, switching to
checking if the LODE is of Euler type
expon. solutions partially successful. Result(s) =`, [x^2, x^3+x]
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 52

```
Order:=6;
dsolve(x^3*(1+x^2)*diff(y(x),x$3)-(2+4*x^2)*x^2*diff(y(x),x$2)+(4+10*x^2)*x*diff(y(x),x)-(4+
```

$$y(x) = \left(c_3\left(2+2x^2+\mathrm{O}\left(x^6\right)\right) + \left(\left(1+\mathrm{O}\left(x^6\right)\right)c_1 + c_2\left(\ln\left(x\right)\left(2+\mathrm{O}\left(x^6\right)\right)+\left(5+\mathrm{O}\left(x^6\right)\right)\right)\right)x\right)x$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 30

```
AsymptoticDSolveValue[x^3*(1+x^2)*y'''[x]-(2+4*x^2)*x^2*y''[x]+(4+10*x^2)*x*y'[x]-(4+12*x^2)
```

$$y(x) \to c_1\left(2x^3+2x\right) + c_2x^2 + c_3x^2\log(x)$$

1.5 problem Ex. 6(iv), page 257

Internal problem ID [5475]
Internal file name [OUTPUT/4723_Sunday_June_05_2022_03_04_04_PM_81533320/index.tex]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 6(iv), page 257.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2(2-x)\,x^2y''-(4-x)\,xy'+(-x+3)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^3+4x^2\right)y''+\left(x^2-4x\right)y'+(-x+3)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=-\frac{-4+x}{2x\,(-2+x)}\\ q(x)&=\frac{x-3}{2x^2\,(-2+x)}\end{aligned}$$

Table 5: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{-4+x}{2x(-2+x)}$	
singularity	type
$x = 0$	"regular"
$x = 2$	"regular"

$q(x) = \frac{x-3}{2x^2(-2+x)}$	
singularity	type
$x = 0$	"regular"
$x = 2$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 2, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-2y''x^2(-2+x) + \left(x^2 - 4x\right) y' + (-x+3)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(-2+x) + \left(x^2-4x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-x+3)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\
&\quad + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) \\
&\quad + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) \\
&\quad + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&\quad + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) - 4x^{n+r} a_n (n+r) + 3a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) - 4x^r a_0 r + 3a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) - 4x^r r + 3x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 8r + 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 8r + 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 8r + 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{3}{2}}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \sqrt{x}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -2a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) \\ + a_{n-1}(n+r-1) - 4a_n(n+r) - a_{n-1} + 3a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r-2)\, a_{n-1}}{2n+2r-1} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = \frac{(2n-1)\, a_{n-1}}{4+4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1+r}{1+2r}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+2r}$	$\frac{1}{8}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(-1+r)}{4r^2+8r+3}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{1}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+2r}$	$\frac{1}{8}$
a_2	$\frac{r(-1+r)}{4r^2+8r+3}$	$\frac{1}{32}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r^3 - r}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = \frac{5}{512}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+2r}$	$\frac{1}{8}$
a_2	$\frac{r(-1+r)}{4r^2+8r+3}$	$\frac{1}{32}$
a_3	$\frac{r^3-r}{8r^3+36r^2+46r+15}$	$\frac{5}{512}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(r^2 - 1)(2 + r)}{(4r^2 + 8r + 3)(5 + 2r)(7 + 2r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{7}{2048}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+2r}$	$\frac{1}{8}$
a_2	$\frac{r(-1+r)}{4r^2+8r+3}$	$\frac{1}{32}$
a_3	$\frac{r^3-r}{8r^3+36r^2+46r+15}$	$\frac{5}{512}$
a_4	$\frac{r(r^2-1)(2+r)}{(4r^2+8r+3)(5+2r)(7+2r)}$	$\frac{7}{2048}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r(r^2-1)(2+r)(3+r)}{(4r^2+8r+3)(5+2r)(7+2r)(9+2r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = \frac{21}{16384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{1+2r}$	$\frac{1}{8}$
a_2	$\frac{r(-1+r)}{4r^2+8r+3}$	$\frac{1}{32}$
a_3	$\frac{r^3-r}{8r^3+36r^2+46r+15}$	$\frac{5}{512}$
a_4	$\frac{r(r^2-1)(2+r)}{(4r^2+8r+3)(5+2r)(7+2r)}$	$\frac{7}{2048}$
a_5	$\frac{r(r^2-1)(2+r)(3+r)}{(4r^2+8r+3)(5+2r)(7+2r)(9+2r)}$	$\frac{21}{16384}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 + \frac{x}{8} + \frac{x^2}{32} + \frac{5x^3}{512} + \frac{7x^4}{2048} + \frac{21x^5}{16384} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-1+r}{1+2r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{-1+r}{1+2r} &= \lim_{r\to \frac{1}{2}} \frac{-1+r}{1+2r} \\ &= -\frac{1}{4} \end{aligned}$$

The limit is $-\frac{1}{4}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &-2b_{n-1}(n+r-1)(n+r-2) + 4b_n(n+r)(n+r-1) \\ &+ b_{n-1}(n+r-1) - 4b_n(n+r) - b_{n-1} + 3b_n = 0 \end{aligned} \tag{4}$$

Which for for the root $r = \frac{1}{2}$ becomes

$$\begin{aligned} &-2b_{n-1}\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + 4b_n\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) \\ &+ b_{n-1}\left(n-\frac{1}{2}\right) - 4b_n\left(n+\frac{1}{2}\right) - b_{n-1} + 3b_n = 0 \end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{(n+r-2)\, b_{n-1}}{2n+2r-1} \tag{5}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{\left(n - \frac{3}{2}\right) b_{n-1}}{2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-1 + r}{1 + 2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = -\frac{1}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+r}{1+2r}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r(-1 + r)}{(1 + 2r)(3 + 2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = -\frac{1}{32}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+r}{1+2r}$	$-\frac{1}{4}$
b_2	$\frac{r(-1+r)}{4r^2+8r+3}$	$-\frac{1}{32}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(-1+r)\, r(1+r)}{(1+2r)\,(3+2r)\,(5+2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = -\frac{1}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+r}{1+2r}$	$-\frac{1}{4}$
b_2	$\frac{r(-1+r)}{4r^2+8r+3}$	$-\frac{1}{32}$
b_3	$\frac{r^3-r}{8r^3+36r^2+46r+15}$	$-\frac{1}{128}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(-1+r)\, r(1+r)\,(2+r)}{(1+2r)\,(3+2r)\,(5+2r)\,(7+2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = -\frac{5}{2048}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+r}{1+2r}$	$-\frac{1}{4}$
b_2	$\frac{r(-1+r)}{4r^2+8r+3}$	$-\frac{1}{32}$
b_3	$\frac{r^3-r}{8r^3+36r^2+46r+15}$	$-\frac{1}{128}$
b_4	$\frac{r^4+2r^3-r^2-2r}{16r^4+128r^3+344r^2+352r+105}$	$-\frac{5}{2048}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{(-1+r)\, r(1+r)\,(2+r)\,(3+r)}{(1+2r)\,(3+2r)\,(5+2r)\,(7+2r)\,(9+2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = -\frac{7}{8192}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+r}{1+2r}$	$-\frac{1}{4}$
b_2	$\frac{r(-1+r)}{4r^2+8r+3}$	$-\frac{1}{32}$
b_3	$\frac{r^3-r}{8r^3+36r^2+46r+15}$	$-\frac{1}{128}$
b_4	$\frac{r^4+2r^3-r^2-2r}{16r^4+128r^3+344r^2+352r+105}$	$-\frac{5}{2048}$
b_5	$\frac{r^5+5r^4+5r^3-5r^2-6r}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$-\frac{7}{8192}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{3}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{x}{4} - \frac{x^2}{32} - \frac{x^3}{128} - \frac{5x^4}{2048} - \frac{7x^5}{8192} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{3}{2}}\left(1 + \frac{x}{8} + \frac{x^2}{32} + \frac{5x^3}{512} + \frac{7x^4}{2048} + \frac{21x^5}{16384} + O(x^6)\right) \\ &\quad + c_2\sqrt{x}\left(1 - \frac{x}{4} - \frac{x^2}{32} - \frac{x^3}{128} - \frac{5x^4}{2048} - \frac{7x^5}{8192} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{3}{2}}\left(1 + \frac{x}{8} + \frac{x^2}{32} + \frac{5x^3}{512} + \frac{7x^4}{2048} + \frac{21x^5}{16384} + O(x^6)\right) \\ &\quad + c_2\sqrt{x}\left(1 - \frac{x}{4} - \frac{x^2}{32} - \frac{x^3}{128} - \frac{5x^4}{2048} - \frac{7x^5}{8192} + O(x^6)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{3}{2}} \left(1+\frac{x}{8}+\frac{x^2}{32}+\frac{5x^3}{512}+\frac{7x^4}{2048}+\frac{21x^5}{16384}+O(x^6)\right) + c_2\sqrt{x}\left(1-\frac{x}{4}-\frac{x^2}{32}-\frac{x^3}{128}-\frac{5x^4}{2048}-\frac{7x^5}{8192}+O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} \left(1+\frac{x}{8}+\frac{x^2}{32}+\frac{5x^3}{512}+\frac{7x^4}{2048}+\frac{21x^5}{16384}+O(x^6)\right) + c_2\sqrt{x}\left(1-\frac{x}{4}-\frac{x^2}{32}-\frac{x^3}{128}-\frac{5x^4}{2048}-\frac{7x^5}{8192}+O(x^6)\right)$$

Verified OK.

1.5.1 Maple step by step solution

Let's solve

$-2y''x^2(-2+x)+(x^2-4x)\,y'+(-x+3)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(-4+x)y'}{2x(-2+x)}-\frac{(x-3)y}{2x^2(-2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(-4+x)y'}{2x(-2+x)}+\frac{(x-3)y}{2x^2(-2+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{-4+x}{2x(-2+x)}, P_3(x)=\frac{x-3}{2x^2(-2+x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x^2(-2+x) - x(-4+x)\,y' + (x-3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-1+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} (-a_k(2k+2r-1)\,(2k+2r-3) + a_{k-1}(2k+2r-3)\,(k-2$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$-4\left(\left(-\frac{k}{2}-\frac{r}{2}+1\right) a_{k-1}+a_k\left(k+r-\frac{1}{2}\right)\right)\left(k+r-\frac{3}{2}\right)=0$$

- Shift index using $k->k+1$

$$-4\left(\left(-\frac{k}{2}+\frac{1}{2}-\frac{r}{2}\right) a_k+a_{k+1}\left(k+\frac{1}{2}+r\right)\right)\left(k+r-\frac{1}{2}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{(k+r-1) a_k}{2k+1+2r}$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1}=\frac{\left(k-\frac{1}{2}\right) a_k}{2k+2}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{\left(k-\frac{1}{2}\right) a_k}{2k+2}\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+1}=\frac{\left(k+\frac{1}{2}\right) a_k}{2k+4}$$

- Solution for $r=\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1}=\frac{\left(k+\frac{1}{2}\right) a_k}{2k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1}=\frac{\left(k-\frac{1}{2}\right) a_k}{2k+2}, b_{k+1}=\frac{\left(k+\frac{1}{2}\right) b_k}{2k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(2*(2-x)*x^2*diff(y(x),x$2)-(4-x)*x*diff(y(x),x)+(3-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left(x\left(1+\frac{1}{8}x+\frac{1}{32}x^2+\frac{5}{512}x^3+\frac{7}{2048}x^4+\frac{21}{16384}x^5+\mathrm{O}\left(x^6\right)\right)c_1 + \left(1+\frac{1}{4}x+\frac{1}{32}x^2+\frac{1}{128}x^3+\frac{5}{2048}x^4+\frac{7}{8192}x^5+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 94

```
AsymptoticDSolveValue[2*(2-x)*x^2*y''[x]-(4-x)*x*y'[x]+(3-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{5x^{9/2}}{2048}-\frac{x^{7/2}}{128}-\frac{x^{5/2}}{32}-\frac{x^{3/2}}{4}+\sqrt{x}\right)+c_2\left(\frac{7x^{11/2}}{2048}+\frac{5x^{9/2}}{512}+\frac{x^{7/2}}{32}+\frac{x^{5/2}}{8}+x^{3/2}\right)$$

1.6 problem Ex. 6(v), page 257

Internal problem ID [5476]
Internal file name [`OUTPUT/4724_Sunday_June_05_2022_03_04_06_PM_73340437/index.tex`]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 6(v), page 257.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-x)\,x^2y''+(5x-4)\,xy'+(6-9x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^3+x^2)\,y''+(5x^2-4x)\,y'+(6-9x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=-\frac{5x-4}{x\,(x-1)}\\q(x)&=\frac{-6+9x}{(x-1)\,x^2}\end{aligned}$$

Table 7: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5x-4}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{-6+9x}{(x-1)x^2}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''(x-1)\,x^2 + \left(5x^2 - 4x\right) y' + (6 - 9x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right)(x-1)\,x^2 + \left(5x^2 - 4x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (6 - 9x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{1+n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-4x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-9x^{1+n+r} a_n\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 5x^{1+n+r} a_n(n+r)=\sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\,x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-9x^{1+n+r} a_n\right)=\sum_{n=1}^{\infty}\left(-9a_{n-1}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\,x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-4x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right)+\sum_{n=1}^{\infty}\left(-9a_{n-1}x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1)-4x^{n+r} a_n(n+r)+6a_n x^{n+r}=0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 4x^r a_0 r + 6a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - 4x^r r + 6x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 5r + 6\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 5r + 6 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 5r + 6\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^2 \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + 5a_{n-1}(n+r-1) - 4a_n(n+r) + 6a_n - 9a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n^2 + 2nr + r^2 - 8n - 8r + 16)}{n^2 + 2nr + r^2 - 5n - 5r + 6} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = \frac{a_{n-1}(n-1)^2}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{(r-3)^2}{r^2 - 3r + 2}$$

Which for the root $r = 3$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(r-3)^2}{r^2-3r+2}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r-2)(r-3)^2}{(-1+r)^2 r}$$

Which for the root $r = 3$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(r-3)^2}{r^2-3r+2}$	0
a_2	$\frac{(r-2)(r-3)^2}{(-1+r)^2 r}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(r-2)(r-3)^2}{(r+1)r^2}$$

Which for the root $r = 3$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(r-3)^2}{r^2-3r+2}$	0
a_2	$\frac{(r-2)(r-3)^2}{(-1+r)^2 r}$	0
a_3	$\frac{(r-2)(r-3)^2}{(r+1)r^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r-2)(r-3)^2}{(r+1)^2(r+2)}$$

Which for the root $r = 3$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(r-3)^2}{r^2-3r+2}$	0
a_2	$\frac{(r-2)(r-3)^2}{(-1+r)^2 r}$	0
a_3	$\frac{(r-2)(r-3)^2}{(r+1)r^2}$	0
a_4	$\frac{(r-2)(r-3)^2}{(r+1)^2(r+2)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(r-2)(r-3)^2}{(r+3)(r+2)^2}$$

Which for the root $r = 3$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(r-3)^2}{r^2-3r+2}$	0
a_2	$\frac{(r-2)(r-3)^2}{(-1+r)^2 r}$	0
a_3	$\frac{(r-2)(r-3)^2}{(r+1)r^2}$	0
a_4	$\frac{(r-2)(r-3)^2}{(r+1)^2(r+2)}$	0
a_5	$\frac{(r-2)(r-3)^2}{(r+3)(r+2)^2}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^3\left(1 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{(r-3)^2}{r^2 - 3r + 2} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{(r-3)^2}{r^2 - 3r + 2} &= \lim_{r \to 2} \frac{(r-3)^2}{r^2 - 3r + 2} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x} \right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2) \right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2} \right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2) (-1 + n + r_2) \right) \end{aligned}$$

Substituting these back into the given ode $-y''(x-1)\,x^2 + (5x^2 - 4x)\,y' + (6 - 9x)\,y = 0$

gives

$$
\begin{aligned}
&-\left(C y_1''(x)\ln(x)+\frac{2C y_1'(x)}{x}-\frac{C y_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)(x-1)\,x^2\\
&+\left(5x^2-4x\right)\left(C y_1'(x)\ln(x)+\frac{C y_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+(6-9x)\left(C y_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(-y_1''(x)(x-1)\,x^2+\left(5x^2-4x\right)y_1'(x)+(6-9x)\,y_1(x)\right)\ln(x)\right.\\
&\left.-\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)(x-1)\,x^2+\frac{\left(5x^2-4x\right)y_1(x)}{x}\right)C\\
&-\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)(x-1)\,x^2\\
&+\left(5x^2-4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(6-9x)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
-y_1''(x)(x-1)\,x^2+\left(5x^2-4x\right)y_1'(x)+(6-9x)\,y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(-\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)(x-1)\,x^2+\frac{\left(5x^2-4x\right)y_1(x)}{x}\right)C\\
&-\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)(x-1)\,x^2\\
&+\left(5x^2-4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(6-9x)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\left(-2x(x-1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) + (6x-5)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+ \left(-x^3+x^2\right)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\left(-1+n+r_2\right)\right) \\
&+ \left(5x^2-4x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) + 3(2-3x)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}$$

Since $r_1 = 3$ and $r_2 = 2$ then the above becomes

$$\begin{aligned}
&\left(-2x(x-1)\left(\sum_{n=0}^{\infty} x^{n+2} a_n(n+3)\right) + (6x-5)\left(\sum_{n=0}^{\infty} a_n x^{n+3}\right)\right) C \\
&+ \left(-x^3+x^2\right)\left(\sum_{n=0}^{\infty} x^{n} b_n(n+2)\left(1+n\right)\right) \\
&+ \left(5x^2-4x\right)\left(\sum_{n=0}^{\infty} x^{1+n} b_n(n+2)\right) + 3(2-3x)\left(\sum_{n=0}^{\infty} b_n x^{n+2}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(-2C\, x^{n+4} a_n(n+3)\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n(n+3)\right) + \left(\sum_{n=0}^{\infty} 6C\, x^{n+4} a_n\right) \\
&\quad + \sum_{n=0}^{\infty}\left(-5C\, x^{n+3} a_n\right) + \sum_{n=0}^{\infty}\left(-x^{n+3} b_n(n+2)\left(1+n\right)\right) \\
&\quad + \left(\sum_{n=0}^{\infty} x^{n+2} b_n\left(n^2+3n+2\right)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+3} b_n(n+2)\right) \\
&\quad + \sum_{n=0}^{\infty}\left(-4x^{n+2} b_n(n+2)\right) + \left(\sum_{n=0}^{\infty} 6b_n x^{n+2}\right) + \sum_{n=0}^{\infty}\left(-9x^{n+3} b_n\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+2} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2C\,x^{n+4}a_n(n+3)\right)=\sum_{n=2}^{\infty}\left(-2Ca_{n-2}(1+n)\,x^{n+2}\right)$$

$$\sum_{n=0}^{\infty}2C\,x^{n+3}a_n(n+3)=\sum_{n=1}^{\infty}2Ca_{n-1}(n+2)\,x^{n+2}$$

$$\sum_{n=0}^{\infty}6C\,x^{n+4}a_n=\sum_{n=2}^{\infty}6Ca_{n-2}x^{n+2}$$

$$\sum_{n=0}^{\infty}\left(-5C\,x^{n+3}a_n\right)=\sum_{n=1}^{\infty}\left(-5Ca_{n-1}x^{n+2}\right)$$

$$\sum_{n=0}^{\infty}\left(-x^{n+3}b_n(n+2)\,(1+n)\right)=\sum_{n=1}^{\infty}\left(-b_{n-1}n(1+n)\,x^{n+2}\right)$$

$$\sum_{n=0}^{\infty}5x^{n+3}b_n(n+2)=\sum_{n=1}^{\infty}5b_{n-1}(1+n)\,x^{n+2}$$

$$\sum_{n=0}^{\infty}\left(-9x^{n+3}b_n\right)=\sum_{n=1}^{\infty}\left(-9b_{n-1}x^{n+2}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+2$.

$$\begin{aligned}&\sum_{n=2}^{\infty}\left(-2Ca_{n-2}(1+n)\,x^{n+2}\right)+\left(\sum_{n=1}^{\infty}2Ca_{n-1}(n+2)\,x^{n+2}\right)\\&+\left(\sum_{n=2}^{\infty}6Ca_{n-2}x^{n+2}\right)+\sum_{n=1}^{\infty}\left(-5Ca_{n-1}x^{n+2}\right)\\&+\sum_{n=1}^{\infty}\left(-b_{n-1}n(1+n)\,x^{n+2}\right)+\left(\sum_{n=0}^{\infty}x^{n+2}b_n\left(n^2+3n+2\right)\right)\\&+\left(\sum_{n=1}^{\infty}5b_{n-1}(1+n)\,x^{n+2}\right)+\sum_{n=0}^{\infty}\left(-4x^{n+2}b_n(n+2)\right)\\&+\left(\sum_{n=0}^{\infty}6b_nx^{n+2}\right)+\sum_{n=1}^{\infty}\left(-9b_{n-1}x^{n+2}\right)=0\end{aligned}\tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C - 1 = 0$$

Which is solved for C. Solving for C gives

$$C = 1$$

For $n = 2$, Eq (2B) gives

$$3Ca_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = 0$$

For $n = 3$, Eq (2B) gives

$$(-2a_1 + 5a_2)\,C - b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$(-4a_2 + 7a_3)\,C - 4b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = 0$$

For $n = 5$, Eq (2B) gives

$$(-6a_3 + 9a_4)\,C - 9b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$y_2(x) = 1\left(x^3\left(1 + O\left(x^6\right)\right)\right)\ln(x) + x^2\left(1 + O\left(x^6\right)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^3\left(1 + O\left(x^6\right)\right) + c_2\left(1\left(x^3\left(1 + O\left(x^6\right)\right)\right)\ln(x) + x^2\left(1 + O\left(x^6\right)\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^3\left(1 + O\left(x^6\right)\right) + c_2\left(x^3\left(1 + O\left(x^6\right)\right)\ln(x) + x^2\left(1 + O\left(x^6\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^3\left(1 + O\left(x^6\right)\right) + c_2\left(x^3\left(1 + O\left(x^6\right)\right)\ln(x) + x^2\left(1 + O\left(x^6\right)\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^3\left(1 + O\left(x^6\right)\right) + c_2\left(x^3\left(1 + O\left(x^6\right)\right)\ln(x) + x^2\left(1 + O\left(x^6\right)\right)\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 43

```
Order:=6;
dsolve((1-x)*x^2*diff(y(x),x$2)+(5*x-4)*x*diff(y(x),x)+(6-9*x)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = x^2\left(\ln(x)\left(x + \mathrm{O}\left(x^6\right)\right)c_2 + c_1x\left(1 + \mathrm{O}\left(x^6\right)\right) + \left(1 - x + \mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 30

```
AsymptoticDSolveValue[(1-x)*x^2*y''[x]+(5*x-4)*x*y'[x]+(6-9*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2x^3 + c_1\left(x^3\log(x) - x^2(3x - 1)\right)$$

1.7 problem Ex. 6(vi), page 257

Internal problem ID [5477]
Internal file name [OUTPUT/4725_Sunday_June_05_2022_03_04_08_PM_84588508/index.tex]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 6(vi), page 257.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (4x^2 + 1)\,y' + 4xy(x^2 + 1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (4x^2 + 1)\,y' + (4x^3 + 4x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{4x^2 + 1}{x}$$
$$q(x) = 4x^2 + 4$$

Table 8: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4x^2+1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = 4x^2 + 4$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (4x^2 + 1)\, y' + (4x^3 + 4x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + (4x^2+1) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (4x^3+4x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r+3} a_n\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r) = \sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 4x^{n+r+3} a_n = \sum_{n=4}^{\infty} 4a_{n-4} x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=2}^{\infty} 4a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=4}^{\infty} 4a_{n-4} x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{-4 - 4r}{(2 + r)^2}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = 0$$

For $4 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 4a_{n-2}(n+r-2) + a_n(n+r) + 4a_{n-4} + 4a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4(na_{n-2} + ra_{n-2} + a_{n-4} - a_{n-2})}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(-4n+4)\, a_{n-2} - 4a_{n-4}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-4r}{(2+r)^2}$	-1
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{12r^2 + 48r + 32}{(2+r)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-4r}{(2+r)^2}$	-1
a_3	0	0
a_4	$\frac{12r^2+48r+32}{(2+r)^2(r+4)^2}$	$\frac{1}{2}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-4r}{(2+r)^2}$	-1
a_3	0	0
a_4	$\frac{12r^2+48r+32}{(2+r)^2(r+4)^2}$	$\frac{1}{2}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= -x^2 + 1 + \frac{x^4}{2} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{-4-4r}{(2+r)^2}$	-1	$\frac{4r}{(2+r)^3}$	0
b_3	0	0	0	0
b_4	$\frac{12r^2+48r+32}{(2+r)^2(r+4)^2}$	$\frac{1}{2}$	$\frac{-24r^3-144r^2-224r}{(2+r)^3(r+4)^3}$	0
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(-x^2 + 1 + \frac{x^4}{2} + O(x^6)\right)\ln(x) + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(-x^2 + 1 + \frac{x^4}{2} + O(x^6)\right) + c_2\left(\left(-x^2 + 1 + \frac{x^4}{2} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\left(-x^2 + 1 + \frac{x^4}{2} + O(x^6)\right) + c_2\left(\left(-x^2 + 1 + \frac{x^4}{2} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\left(-x^2 + 1 + \frac{x^4}{2} + O(x^6)\right) + c_2\left(\left(-x^2 + 1 + \frac{x^4}{2} + O(x^6)\right)\ln(x) + O(x^6)\right) \quad (1)$$

Verification of solutions

$$y = c_1\left(-x^2 + 1 + \frac{x^4}{2} + O(x^6)\right) + c_2\left(\left(-x^2 + 1 + \frac{x^4}{2} + O(x^6)\right)\ln(x) + O(x^6)\right)$$

Verified OK.

1.7.1 Maple step by step solution

Let's solve

$y''x + (4x^2 + 1)\,y' + (4x^3 + 4x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = (-4x^2 - 4)\, y - \frac{(4x^2+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(4x^2+1)y'}{x} + (4x^2+4)\, y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{4x^2+1}{x}, P_3(x) = 4x^2+4\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (4x^2+1)\, y' + 4xy(x^2+1) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 1..3$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + a_1(1+r)^2 x^r + \left(a_2(2+r)^2 + 4a_0(1+r)\right) x^{1+r} + \left(a_3(3+r)^2 + 4a_1(2+r)\right) x^{2+r} + \Big($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- The coefficients of each power of x must be 0

  $\left[a_1(1+r)^2 = 0, a_2(2+r)^2 + 4a_0(1+r) = 0, a_3(3+r)^2 + 4a_1(2+r) = 0\right]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{4a_0(1+r)}{r^2+4r+4}, a_3 = 0\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + 4a_{k-1}k + 4a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4)^2 + 4a_{k+2}(k+3) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}, a_1 = 0, a_2 = -a_0, a_3 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(4*x^2+1)*diff(y(x),x)+4*x*(x^2+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{1}{2}x^4\right)(c_2 \ln (x) + c_1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 40

```
AsymptoticDSolveValue[x*y''[x]+(4*x^2+1)*y'[x]+4*x*(x^2+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{2} - x^2 + 1\right) + c_2\left(\frac{x^4}{2} - x^2 + 1\right)\log(x)$$

1.8 problem Ex. 8(i), page 258

Internal problem ID [5478]
Internal file name [`OUTPUT/4726_Sunday_June_05_2022_03_04_10_PM_91628320/index.tex`]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 8(i), page 258.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 4(a+x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (4a + 4x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{4a + 4x}{x^2}$$

Table 10: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{4a+4x}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (4a + 4x)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (4a+4x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4a\,x^{n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4a\, x^{n+r} a_n\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 4a\, x^{n+r} a_n = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 4a\, x^r a_0 = 0$$

Or

$$(x^r r(-1+r) + 4a\, x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + 4a - r\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 4a - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2} + \frac{\sqrt{1-16a}}{2}$$
$$r_2 = \frac{1}{2} - \frac{\sqrt{1-16a}}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 4a - r\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Assuming the roots differ by non-integer Since $r_1 - r_2 = \sqrt{1-16a}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}+\frac{\sqrt{1-16a}}{2}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}-\frac{\sqrt{1-16a}}{2}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \le n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + 4a a_n + 4a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{4a_{n-1}}{n^2 + 2nr + r^2 + 4a - n - r} \tag{4}
$$

Which for the root $r = \frac{1}{2} + \frac{\sqrt{1-16a}}{2}$ becomes

$$
a_n = -\frac{4a_{n-1}}{n\left(\sqrt{1-16a} + n\right)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2} + \frac{\sqrt{1-16a}}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = -\frac{4}{r^2 + 4a + r}
$$

Which for the root $r = \frac{1}{2} + \frac{\sqrt{1-16a}}{2}$ becomes

$$a_1 = -\frac{4}{1 + \sqrt{1 - 16a}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r^2+4a+r}$	$-\frac{4}{1+\sqrt{1-16a}}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{(r^2 + 4a + r)(r^2 + 4a + 3r + 2)}$$

Which for the root $r = \frac{1}{2} + \frac{\sqrt{1-16a}}{2}$ becomes

$$a_2 = \frac{8}{\left(1 + \sqrt{1 - 16a}\right)\left(2 + \sqrt{1 - 16a}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r^2+4a+r}$	$-\frac{4}{1+\sqrt{1-16a}}$
a_2	$\frac{16}{(r^2+4a+r)(r^2+4a+3r+2)}$	$\frac{8}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{(r^2 + 4a + r)(r^2 + 4a + 3r + 2)(r^2 + 4a + 5r + 6)}$$

Which for the root $r = \frac{1}{2} + \frac{\sqrt{1-16a}}{2}$ becomes

$$a_3 = -\frac{32}{3\left(1 + \sqrt{1 - 16a}\right)\left(2 + \sqrt{1 - 16a}\right)\left(3 + \sqrt{1 - 16a}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r^2+4a+r}$	$-\frac{4}{1+\sqrt{1-16a}}$
a_2	$\frac{16}{(r^2+4a+r)(r^2+4a+3r+2)}$	$\frac{8}{(1+\sqrt{1-16a})(2+\sqrt{1-16a})}$
a_3	$-\frac{64}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)}$	$-\frac{32}{3(1+\sqrt{1-16a})(2+\sqrt{1-16a})(3+\sqrt{1-16a})}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)(r^2+4a+7r+12)}$$

Which for the root $r = \frac{1}{2} + \frac{\sqrt{1-16a}}{2}$ becomes

$$a_4 = \frac{32}{3\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r^2+4a+r}$	$-\frac{4}{1+\sqrt{1-16a}}$
a_2	$\frac{16}{(r^2+4a+r)(r^2+4a+3r+2)}$	$\frac{8}{(1+\sqrt{1-16a})(2+\sqrt{1-16a})}$
a_3	$-\frac{64}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)}$	$-\frac{32}{3(1+\sqrt{1-16a})(2+\sqrt{1-16a})(3+\sqrt{1-16a})}$
a_4	$\frac{256}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)(r^2+4a+7r+12)}$	$\frac{32}{3(1+\sqrt{1-16a})(2+\sqrt{1-16a})(3+\sqrt{1-16a})(4+\sqrt{1-16a})}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)(r^2+4a+7r+12)(r^2+4a+9r+20)}$$

Which for the root $r = \frac{1}{2} + \frac{\sqrt{1-16a}}{2}$ becomes

$$a_5 = -\frac{128}{15\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)\left(5+\sqrt{1-16a}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r^2+4a+r}$	$-\frac{4}{1+\sqrt{1-16a}}$
a_2	$\frac{16}{(r^2+4a+r)(r^2+4a+3r+2)}$	$\frac{8}{(1+\sqrt{1-16a})(2+\sqrt{1-16a})}$
a_3	$-\frac{64}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)}$	$-\frac{32}{3(1+\sqrt{1-16a})(2+\sqrt{1-16a})(3+\sqrt{1-16a})}$
a_4	$\frac{256}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)(r^2+4a+7r+12)}$	$\frac{32}{3(1+\sqrt{1-16a})(2+\sqrt{1-16a})(3+\sqrt{1-16a})(4+\sqrt{1-16a}}$
a_5	$-\frac{1024}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)(r^2+4a+7r+12)(r^2+4a+9r+20)}$	$-\frac{128}{15(1+\sqrt{1-16a})(2+\sqrt{1-16a})(3+\sqrt{1-16a})(4+\sqrt{1-}}$

Using the above table, then the solution $y_1(x)$ is

$$y_1(x) = x^{\frac{1}{2}+\frac{\sqrt{1-16a}}{2}} (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots)$$
$$= x^{\frac{1}{2}+\frac{\sqrt{1-16a}}{2}} \left(1 - \frac{4x}{1+\sqrt{1-16a}} + \frac{8x^2}{(1+\sqrt{1-16a})(2+\sqrt{1-16a})} - \frac{32x^3}{3(1+\sqrt{1-16a})(2+\sqrt{1-}\right.$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 4ab_n + 4b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{4b_{n-1}}{n^2 + 2nr + r^2 + 4a - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2} - \frac{\sqrt{1-16a}}{2}$ becomes

$$b_n = \frac{4b_{n-1}}{n\left(\sqrt{1-16a} - n\right)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2} - \frac{\sqrt{1-16a}}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{4}{r^2 + 4a + r}$$

Which for the root $r = \frac{1}{2} - \frac{\sqrt{1-16a}}{2}$ becomes

$$b_1 = \frac{4}{-1 + \sqrt{1 - 16a}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{r^2+4a+r}$	$\frac{4}{-1+\sqrt{1-16a}}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{16}{(r^2 + 4a + r)(r^2 + 4a + 3r + 2)}$$

Which for the root $r = \frac{1}{2} - \frac{\sqrt{1-16a}}{2}$ becomes

$$b_2 = \frac{8}{\left(-1 + \sqrt{1 - 16a}\right)\left(-2 + \sqrt{1 - 16a}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{r^2+4a+r}$	$\frac{4}{-1+\sqrt{1-16a}}$
b_2	$\frac{16}{(r^2+4a+r)(r^2+4a+3r+2)}$	$\frac{8}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{64}{(r^2 + 4a + r)(r^2 + 4a + 3r + 2)(r^2 + 4a + 5r + 6)}$$

Which for the root $r = \frac{1}{2} - \frac{\sqrt{1-16a}}{2}$ becomes

$$b_3 = \frac{32}{3\left(-1 + \sqrt{1 - 16a}\right)\left(-2 + \sqrt{1 - 16a}\right)\left(-3 + \sqrt{1 - 16a}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{r^2+4a+r}$	$\frac{4}{-1+\sqrt{1-16a}}$
b_2	$\frac{16}{(r^2+4a+r)(r^2+4a+3r+2)}$	$\frac{8}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)}$
b_3	$-\frac{64}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)}$	$\frac{32}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{256}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)(r^2+4a+7r+12)}$$

Which for the root $r = \frac{1}{2} - \frac{\sqrt{1-16a}}{2}$ becomes

$$b_4 = \frac{32}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{r^2+4a+r}$	$\frac{4}{-1+\sqrt{1-16a}}$
b_2	$\frac{16}{(r^2+4a+r)(r^2+4a+3r+2)}$	$\frac{8}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)}$
b_3	$-\frac{64}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)}$	$\frac{32}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)}$
b_4	$\frac{256}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)(r^2+4a+7r+12)}$	$\frac{32}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1024}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)(r^2+4a+7r+12)(r^2+4a+9r+20)}$$

Which for the root $r = \frac{1}{2} - \frac{\sqrt{1-16a}}{2}$ becomes

$$b_5 = \frac{128}{15\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)\left(-5+\sqrt{1-16a}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{r^2+4a+r}$	$\frac{4}{-1+\sqrt{1-16a}}$
b_2	$\frac{16}{(r^2+4a+r)(r^2+4a+3r+2)}$	$\frac{8}{(-1+\sqrt{1-16a})(-2+\sqrt{1-16a})}$
b_3	$-\frac{64}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)}$	$\frac{32}{3(-1+\sqrt{1-16a})(-2+\sqrt{1-16a})(-3+\sqrt{1-16a})}$
b_4	$\frac{256}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)(r^2+4a+7r+12)}$	$\frac{32}{3(-1+\sqrt{1-16a})(-2+\sqrt{1-16a})(-3+\sqrt{1-16a})(-4+\sqrt{}}$
b_5	$-\frac{1024}{(r^2+4a+r)(r^2+4a+3r+2)(r^2+4a+5r+6)(r^2+4a+7r+12)(r^2+4a+9r+20)}$	$\frac{128}{15(-1+\sqrt{1-16a})(-2+\sqrt{1-16a})(-3+\sqrt{1-16a})(-4+}$

Using the above table, then the solution $y_2(x)$ is

$$y_2(x) = x^{\frac{1}{2}+\frac{\sqrt{1-16a}}{2}}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right)$$
$$= x^{\frac{1}{2}-\frac{\sqrt{1-16a}}{2}}\left(1 + \frac{4x}{-1+\sqrt{1-16a}} + \frac{8x^2}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)} + \frac{}{3\left(-1+\sqrt{1-16a}\right)\left(-\right.}\right.$$

Therefore the homogeneous solution is

$$y_h(x) = c_1y_1(x) + c_2y_2(x)$$

$$
\begin{aligned}
&= c_1 x^{\frac{1}{2}+\frac{\sqrt{1-16a}}{2}} \left(1 - \frac{4x}{1+\sqrt{1-16a}} + \frac{8x^2}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)} \right. \\
&\quad - \frac{32x^3}{3\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)} \\
&\quad + \frac{32x^4}{3\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)} \\
&\quad - \frac{128x^5}{15\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)\left(5+\sqrt{1-16a}\right)} \\
&\quad \left. + O\!\left(x^6\right)\right) \\
&\quad + c_2 x^{\frac{1}{2}-\frac{\sqrt{1-16a}}{2}} \left(1 + \frac{4x}{-1+\sqrt{1-16a}} + \frac{8x^2}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)} \right. \\
&\quad + \frac{32x^3}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)} \\
&\quad + \frac{32x^4}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)} \\
&\quad + \frac{128x^5}{15\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)\left(-5+\sqrt{1-16a}\right)} \\
&\quad \left. + O\!\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$y = y_h$

$$
\begin{aligned}
&= c_1 x^{\frac{1}{2}+\frac{\sqrt{1-16a}}{2}} \left(1 - \frac{4x}{1+\sqrt{1-16a}} + \frac{8x^2}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)} \right. \\
&\quad - \frac{32x^3}{3\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)} \\
&\quad + \frac{32x^4}{3\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)} \\
&\quad - \frac{128x^5}{15\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)\left(5+\sqrt{1-16a}\right)} \\
&\quad \left. + O\left(x^6\right)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{1-16a}}{2}} \left(1 + \frac{4x}{-1+\sqrt{1-16a}} + \frac{8x^2}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)} \right. \\
&\quad + \frac{32x^3}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)} \\
&\quad + \frac{32x^4}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)} \\
&\quad + \frac{128x^5}{15\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)\left(-5+\sqrt{1-16a}\right)} \\
&\quad \left. + O\left(x^6\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^{\frac{1}{2}+\frac{\sqrt{1-16a}}{2}} \left(1-\frac{4x}{1+\sqrt{1-16a}}+\frac{8x^2}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)}\right. \\
& -\frac{32x^3}{3\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)} \\
& +\frac{32x^4}{3\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)} \\
& -\frac{128x^5}{15\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)\left(5+\sqrt{1-16a}\right)} \\
& \left.+O\left(x^6\right)\right)+c_2 x^{\frac{1}{2}-\frac{\sqrt{1-16a}}{2}}\left(1+\frac{4x}{-1+\sqrt{1-16a}}+\frac{8x^2}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)}\right. \\
& +\frac{32x^3}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)} \\
& +\frac{32x^4}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)} \\
& +\frac{128x^5}{15\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)\left(-5+\sqrt{1-16a}\right)} \\
& \left.+O\left(x^6\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}+\frac{\sqrt{1-16a}}{2}} \left(1 - \frac{4x}{1+\sqrt{1-16a}} + \frac{8x^2}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)} - \frac{32x^3}{3\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)} + \frac{32x^4}{3\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)} - \frac{128x^5}{15\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)\left(5+\sqrt{1-16a}\right)} + O\left(x^6\right)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{1-16a}}{2}} \left(1 + \frac{4x}{-1+\sqrt{1-16a}} + \frac{8x^2}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)} + \frac{32x^3}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)} + \frac{32x^4}{3\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)} + \frac{128x^5}{15\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)\left(-5+\sqrt{1-16a}\right)} + O\left(x^6\right)\right)$$

Verified OK.

1.8.1 Maple step by step solution

Let's solve

$x^2 y'' + (4a + 4x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4(a+x)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4(a+x)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x) = 0, P_3(x) = \frac{4(a+x)}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4a$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + (4a + 4x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(r^2 + 4a - r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k^2 + 2kr + r^2 + 4a - k - r) + 4a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 + 4a - r = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2} - \frac{\sqrt{1-16a}}{2}, \frac{1}{2} + \frac{\sqrt{1-16a}}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $$\left(k^2+(2r-1)\,k+r^2+4a-r\right)a_k+4a_{k-1}=0$$

- Shift index using $k->k+1$

  $$\left((k+1)^2+(2r-1)\,(k+1)+r^2+4a-r\right)a_{k+1}+4a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=-\frac{4a_k}{k^2+2kr+r^2+4a+k+r}$$

- Recursion relation for $r=\frac{1}{2}-\frac{\sqrt{1-16a}}{2}$

  $$a_{k+1}=-\frac{4a_k}{k^2+2k\left(\frac{1}{2}-\frac{\sqrt{1-16a}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{1-16a}}{2}\right)^2+4a+k+\frac{1}{2}-\frac{\sqrt{1-16a}}{2}}$$

- Solution for $r=\frac{1}{2}-\frac{\sqrt{1-16a}}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}-\frac{\sqrt{1-16a}}{2}},a_{k+1}=-\frac{4a_k}{k^2+2k\left(\frac{1}{2}-\frac{\sqrt{1-16a}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{1-16a}}{2}\right)^2+4a+k+\frac{1}{2}-\frac{\sqrt{1-16a}}{2}}\right]$$

- Recursion relation for $r=\frac{1}{2}+\frac{\sqrt{1-16a}}{2}$

  $$a_{k+1}=-\frac{4a_k}{k^2+2k\left(\frac{1}{2}+\frac{\sqrt{1-16a}}{2}\right)+\left(\frac{1}{2}+\frac{\sqrt{1-16a}}{2}\right)^2+4a+k+\frac{1}{2}+\frac{\sqrt{1-16a}}{2}}$$

- Solution for $r=\frac{1}{2}+\frac{\sqrt{1-16a}}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}+\frac{\sqrt{1-16a}}{2}},a_{k+1}=-\frac{4a_k}{k^2+2k\left(\frac{1}{2}+\frac{\sqrt{1-16a}}{2}\right)+\left(\frac{1}{2}+\frac{\sqrt{1-16a}}{2}\right)^2+4a+k+\frac{1}{2}+\frac{\sqrt{1-16a}}{2}}\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}-\frac{\sqrt{1-16a}}{2}}\right)+\left(\sum_{k=0}^{\infty}c_k x^{k+\frac{1}{2}+\frac{\sqrt{1-16a}}{2}}\right),b_{k+1}=-\frac{4b_k}{k^2+2k\left(\frac{1}{2}-\frac{\sqrt{1-16a}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{1-16a}}{2}\right)^2+4a+k+}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 947

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+4*(x+a)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left(c_1 x^{-\frac{\sqrt{1-16a}}{2}}\left(1+4\frac{1}{-1+\sqrt{1-16a}}x+8\frac{1}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)}x^2 + \frac{32}{3}\frac{1}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)}x^3 + \frac{32}{3}\frac{1}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)}x^4 + \frac{128}{15}\frac{1}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)\left(-5+\sqrt{1-16a}\right)}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\frac{\sqrt{1-16a}}{2}}\left(1-4\frac{1}{1+\sqrt{1-16a}}x+8\frac{1}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)}x^2 - \frac{32}{3}\frac{1}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)}x^3 + \frac{32}{3}\frac{1}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)}x^4 - \frac{128}{15}\frac{1}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)\left(5+\sqrt{1-16a}\right)}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 1356

```
AsymptoticDSolveValue[x^2*y''[x]+4*(x+a)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
&y(x) \\
&\to \left(-\frac{}{\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right.\right.\right.} \right. \\
&+\frac{}{\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)\right.\right.} \\
&-\frac{64x^3}{\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)\right.\right.} \\
&+\frac{16x^2}{\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)+4a\right)\left(\frac{1}{2}\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(1-\sqrt{1-16a}\right)\right.} \\
&\left.-\frac{4x}{\frac{1}{2}\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(1-\sqrt{1-16a}\right)+4a}+1\right)c_2x^{\frac{1}{2}\left(1-\sqrt{1-16a}\right)} \\
&+\left(-\frac{}{\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+\right.\right.\right.} \right. \\
&+\frac{}{\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)\right.\right.} \\
&-\frac{64x^3}{\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)\right.\right.} \\
&+\frac{16x^2}{\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)+4a\right)\left(\frac{1}{2}\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\sqrt{1-16a}+1\right)\right.} \\
&\left.-\frac{4x}{\frac{1}{2}\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\sqrt{1-16a}+1\right)+4a}+1\right)c_1x^{\frac{1}{2}\left(\sqrt{1-16a}+1\right)}
\end{aligned}
$$

1.9 problem Ex. 8(ii), page 258

Internal problem ID [5479]
Internal file name [OUTPUT/4727_Sunday_June_05_2022_03_04_11_PM_94591782/index.tex]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 8(ii), page 258.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (x^3+1)\,y' + bxy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (x^3+1)\,y' + bxy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^3+1}{x}$$
$$q(x) = b$$

Table 12: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^3+1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = b$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (x^3 + 1)\, y' + bxy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + (x^3+1) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + bx \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} b\, x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=3}^{\infty} a_{n-3} (n+r-3)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} b\, x^{1+n+r} a_n = \sum_{n=2}^{\infty} b a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=3}^{\infty} a_{n-3} (n+r-3)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} b a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = -\frac{b}{(2+r)^2}$$

For $3 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-3}(n+r-3) + a_n(n+r) + b a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{b a_{n-2} + n a_{n-3} + r a_{n-3} - 3 a_{n-3}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(-n+3) a_{n-3} - b a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{b}{(2+r)^2}$	$-\frac{b}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{r}{(r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{b}{(2+r)^2}$	$-\frac{b}{4}$
a_3	$-\frac{r}{(r+3)^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{b^2}{(2+r)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{b^2}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{b}{(2+r)^2}$	$-\frac{b}{4}$
a_3	$-\frac{r}{(r+3)^2}$	0
a_4	$\frac{b^2}{(2+r)^2(4+r)^2}$	$\frac{b^2}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{b(2r^2 + 8r + 9)}{(2 + r)(r + 3)^2 (5 + r)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{b}{50}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{b}{(2+r)^2}$	$-\frac{b}{4}$
a_3	$-\frac{r}{(r+3)^2}$	0
a_4	$\frac{b^2}{(2+r)^2(4+r)^2}$	$\frac{b^2}{64}$
a_5	$\frac{b(2r^2+8r+9)}{(2+r)(r+3)^2(5+r)^2}$	$\frac{b}{50}$

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - \frac{b\,x^2}{4} + \frac{b^2 x^4}{64} + \frac{b\,x^5}{50} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{b}{(2+r)^2}$	$-\frac{b}{4}$	$\frac{2b}{(2+r)^3}$	$\frac{b}{4}$
b_3	$-\frac{r}{(r+3)^2}$	0	$\frac{r-3}{(r+3)^3}$	$-\frac{1}{9}$
b_4	$\frac{b^2}{(2+r)^2 (4+r)^2}$	$\frac{b^2}{64}$	$-\frac{4b^2(r+3)}{(2+r)^3 (4+r)^3}$	$-\frac{3b^2}{128}$
b_5	$\frac{b(2r^2+8r+9)}{(2+r)(r+3)^2 (5+r)^2}$	$\frac{b}{50}$	$-\frac{b(6r^4+56r^3+191r^2+296r+183)}{(5+r)^3 (2+r)^2 (r+3)^3}$	$-\frac{61b}{4500}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\ &= \left(1 - \frac{b\,x^2}{4} + \frac{b^2 x^4}{64} + \frac{b\,x^5}{50} + O(x^6)\right) \ln(x) + \frac{b\,x^2}{4} - \frac{x^3}{9} - \frac{3b^2 x^4}{128} - \frac{61b\,x^5}{4500} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 - \frac{b\,x^2}{4} + \frac{b^2 x^4}{64} + \frac{b\,x^5}{50} + O(x^6)\right) + c_2 \left(\left(1 - \frac{b\,x^2}{4} + \frac{b^2 x^4}{64} + \frac{b\,x^5}{50} + O(x^6)\right) \ln(x) \right. \\ &\quad \left. + \frac{b\,x^2}{4} - \frac{x^3}{9} - \frac{3b^2 x^4}{128} - \frac{61b\,x^5}{4500} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \left(1 - \frac{b\,x^2}{4} + \frac{b^2 x^4}{64} + \frac{b\,x^5}{50} + O(x^6)\right) \\ &\quad + c_2 \left(\left(1 - \frac{b\,x^2}{4} + \frac{b^2 x^4}{64} + \frac{b\,x^5}{50} + O(x^6)\right) \ln(x) + \frac{b\,x^2}{4} - \frac{x^3}{9} - \frac{3b^2 x^4}{128} - \frac{61b\,x^5}{4500} + O(x^6)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\left(1-\frac{b\,x^2}{4}+\frac{b^2x^4}{64}+\frac{b\,x^5}{50}+O\left(x^6\right)\right)+c_2\left(\left(1-\frac{b\,x^2}{4}+\frac{b^2x^4}{64}+\frac{b\,x^5}{50}+O\left(x^6\right)\right)\ln\left(x\right)+\frac{b\,x^2}{4}-\frac{x^3}{9}-\frac{3b^2x^4}{128}-\frac{61b\,x^5}{4500}+O\left(x^6\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(1-\frac{b\,x^2}{4}+\frac{b^2x^4}{64}+\frac{b\,x^5}{50}+O\left(x^6\right)\right)+c_2\left(\left(1-\frac{b\,x^2}{4}+\frac{b^2x^4}{64}+\frac{b\,x^5}{50}+O\left(x^6\right)\right)\ln\left(x\right)+\frac{b\,x^2}{4}-\frac{x^3}{9}-\frac{3b^2x^4}{128}-\frac{61b\,x^5}{4500}+O\left(x^6\right)\right)$$

Verified OK.

1.9.1 Maple step by step solution

Let's solve

$y''x + (x^3 + 1)\,y' + bxy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^3+1)y'}{x} - by$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^3+1)y'}{x} + by = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{x^3+1}{x}, P_3(x) = b\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x+(x^3+1)\,y'+bxy=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x\cdot y$ to series expansion

  $x\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x\cdot y=\sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $x^m\cdot y'$ to series expansion for $m=0..3$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x\cdot y''$ to series expansion

  $x\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- Shift index using $k->k+1$

  $x\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r^2x^{-1+r}+a_1(1+r)^2\,x^r+\left(a_2(2+r)^2+a_0b\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k+1}(k+r+1)^2+ba_{k-1}+a_{k-2}(k\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2=0$

- Values of r that satisfy the indicial equation

$r = 0$

- The coefficients of each power of x must be 0

  $\left[a_1(1+r)^2 = 0, a_2(2+r)^2 + a_0 b = 0\right]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{a_0 b}{r^2+4r+4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + b a_{k-1} + a_{k-2}(k-2) = 0$

- Shift index using $k- >k+2$

  $a_{k+3}(k+3)^2 + b a_{k+1} + a_k k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+3} = -\frac{b a_{k+1} + a_k k}{(k+3)^2}$

- Recursion relation for $r = 0$

  $a_{k+3} = -\frac{b a_{k+1} + a_k k}{(k+3)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{b a_{k+1} + a_k k}{(k+3)^2}, a_1 = 0, a_2 = -\frac{a_0 b}{4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 73

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(1+x*x^2)*diff(y(x),x)+b*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1) \left(1 - \frac{1}{4} b\, x^2 + \frac{1}{64} b^2 x^4 + \frac{1}{50} b\, x^5 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{b}{4} x^2 - \frac{1}{9} x^3 - \frac{3}{128} b^2 x^4 - \frac{61}{4500} b\, x^5 + \mathrm{O}\left(x^6\right)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 103

```
AsymptoticDSolveValue[x*y''[x]+(1+x*x^2)*y'[x]+b*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{b^2 x^4}{64} + \frac{b x^5}{50} - \frac{b x^2}{4} + 1\right) + c_2 \left(-\frac{3 b^2 x^4}{128} + \left(\frac{b^2 x^4}{64} + \frac{b x^5}{50} - \frac{b x^2}{4} + 1\right) \log(x) - \frac{61 b x^5}{4500} + \frac{b x^2}{4} - \frac{x^3}{9}\right)$$