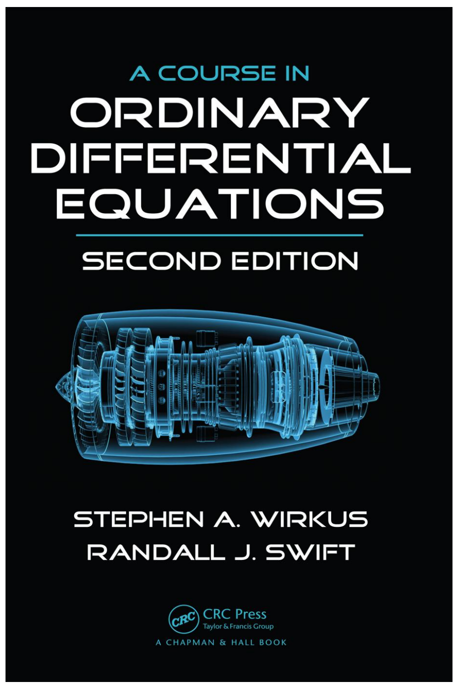
A Solution Manual For

A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition



Nasser M. Abbasi

May 15, 2024

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1.1 problem 1. Using series method

Internal problem ID [6544]
Internal file name [`OUTPUT/5792_Sunday_June_05_2022_03_54_44_PM_12664182/index.tex`]

Book: A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section: Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number: 1. Using series method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Riccati, _special]]
```

$$\boxed{y' - y^2 = -x}$$

With initial conditions

$$[y(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

1.1.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= y^2 - x \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 - x\right) \\ &= 2y\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.1.2 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\begin{aligned}
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned} \tag{2}$$

$$\begin{aligned}
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f
\end{aligned} \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$\begin{aligned}
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0
\end{aligned} \tag{5}$$

For example, for $n = 1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= -x + y^2 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= 2y^3 - 2xy - 1 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= 6y^4 - 8xy^2 + 2x^2 - 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= 24y^5 - 40y^3x + 16yx^2 - 10y^2 + 6x \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= 120y^6 - 240xy^4 + 136y^2x^2 - 60y^3 - 16x^3 + 52xy + 6 \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} F_4 \\
&= 720y^7 - 1680y^5x - 420y^4 + 1232y^3x^2 + 504xy^2 + \left(-272x^3 + 52\right) y - 100x^2 \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} F_5 \\
&= 5040y^8 - 13440xy^6 - 3360y^5 + 12096y^4x^2 + 5152y^3x + \left(-3968x^3 + 556\right) y^2 - 1824yx^2 + 272x^4 - 2
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 1 \\
F_2 &= 4 \\
F_3 &= 14 \\
F_4 &= 66 \\
F_5 &= 352 \\
F_6 &= 2236
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{7x^4}{12} + \frac{11x^5}{20} + \frac{22x^6}{45} + \frac{559x^7}{1260} + O(x^8)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y = 1 + x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{7}{12}x^4 + \frac{11}{20}x^5 + \frac{22}{45}x^6 + \frac{559}{1260}x^7$$

Hence the solution can be written as

$$y = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{7x^4}{12} + \frac{11x^5}{20} + \frac{22x^6}{45} + \frac{559x^7}{1260} + O(x^8)$$

which simplifies to

$$y = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{7x^4}{12} + \frac{11x^5}{20} + \frac{22x^6}{45} + \frac{559x^7}{1260} + O(x^8)$$

Unable to also solve using normal power series since not linear ode. Not currently supported.

Summary

The solution(s) found are the following

$$y = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{7x^4}{12} + \frac{11x^5}{20} + \frac{22x^6}{45} + \frac{559x^7}{1260} + O(x^8) \tag{1}$$

Verification of solutions

$$y = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{7x^4}{12} + \frac{11x^5}{20} + \frac{22x^6}{45} + \frac{559x^7}{1260} + O(x^8)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=8;
dsolve([diff(y(x),x)=y(x)^2-x,y(0) = 1],y(x),type='series',x=0);
```

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{7}{12}x^4 + \frac{11}{20}x^5 + \frac{22}{45}x^6 + \frac{559}{1260}x^7 + \mathrm{O}\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 48

```
AsymptoticDSolveValue[{y'[x]==y[x]^2-x,{y[0]==1}},y[x],{x,0,7}]
```

$$y(x) \to \frac{559x^7}{1260} + \frac{22x^6}{45} + \frac{11x^5}{20} + \frac{7x^4}{12} + \frac{2x^3}{3} + \frac{x^2}{2} + x + 1$$

1.2 problem 1. direct method

Internal problem ID [6545]
Internal file name [`OUTPUT/5793_Sunday_June_05_2022_03_54_47_PM_50830833/index.tex`]

Book: A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section: Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number: 1. direct method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_Riccati, _special]]
```

$$\boxed{y' - y^2 = -x}$$

With initial conditions

$$[y(0) = 1]$$

1.2.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^2 - x \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 - x)\\ &= 2y\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.2.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= y^2 - x\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - x$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -x$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{u} \qquad (1)\end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)$$

But

$$\begin{aligned}f_2' &= 0\\ f_1 f_2 &= 0\\ f_2^2 f_0 &= -x\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - xu(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{AiryAi}(x) + c_2 \operatorname{AiryBi}(x)$$

The above shows that

$$u'(x) = c_1 \operatorname{AiryAi}(1, x) + c_2 \operatorname{AiryBi}(1, x)$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 \operatorname{AiryAi}(1, x) + c_2 \operatorname{AiryBi}(1, x)}{c_1 \operatorname{AiryAi}(x) + c_2 \operatorname{AiryBi}(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 \operatorname{AiryAi}(1, x) - \operatorname{AiryBi}(1, x)}{c_3 \operatorname{AiryAi}(x) + \operatorname{AiryBi}(x)}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-3\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} + 3\Gamma\left(\frac{2}{3}\right)^2 c_3 3^{\frac{1}{6}}}{2\, 3^{\frac{5}{6}}\pi + 2\pi c_3 3^{\frac{1}{3}}}$$

$$c_3 = \frac{2\, 3^{\frac{5}{6}}\pi + 3\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}}}{3\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} - 2\pi 3^{\frac{1}{3}}}$$

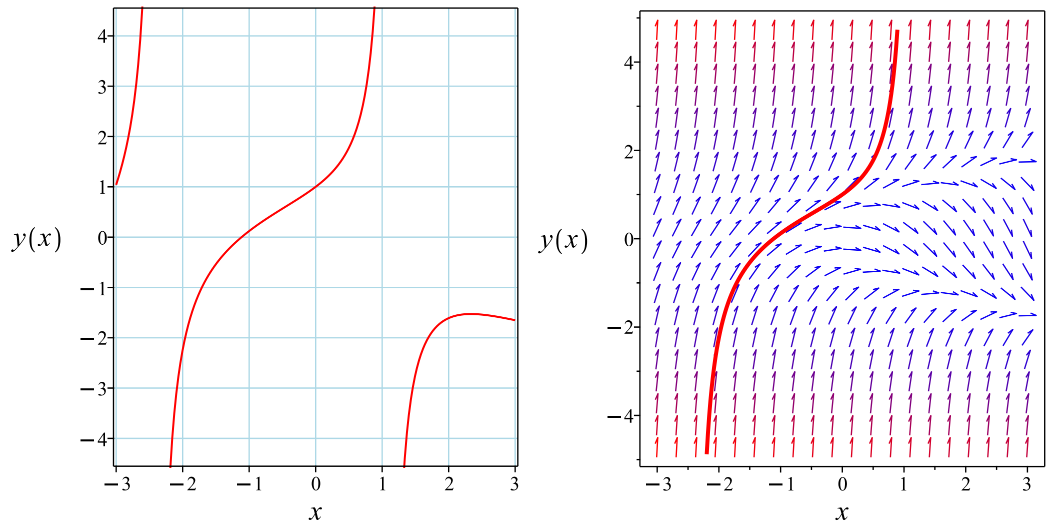
Substituting c_3 found above in the general solution gives

$$y = \frac{-2 \operatorname{AiryAi}(1, x)\, \pi 3^{\frac{5}{6}} - 3 \operatorname{AiryAi}(1, x)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} - 3 \operatorname{AiryBi}(1, x)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} + 2 \operatorname{AiryBi}(1, x)\, \pi 3^{\frac{1}{3}}}{2 \operatorname{AiryAi}(x)\, \pi 3^{\frac{5}{6}} + 3 \operatorname{AiryAi}(x)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} + 3 \operatorname{AiryBi}(x)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} - 2 \operatorname{AiryBi}(x)\, \pi 3^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{-2 \operatorname{AiryAi}(1, x)\, \pi 3^{\frac{5}{6}} - 3 \operatorname{AiryAi}(1, x)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} - 3 \operatorname{AiryBi}(1, x)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} + 2 \operatorname{AiryBi}(1, x)\, \pi 3^{\frac{1}{3}}}{2 \operatorname{AiryAi}(x)\, \pi 3^{\frac{5}{6}} + 3 \operatorname{AiryAi}(x)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} + 3 \operatorname{AiryBi}(x)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} - 2 \operatorname{AiryBi}(x)\, \pi 3^{\frac{1}{3}}} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-2\,\mathrm{AiryAi}(1,x)\,\pi 3^{\frac{5}{6}} - 3\,\mathrm{AiryAi}(1,x)\,\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} - 3\,\mathrm{AiryBi}(1,x)\,\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} + 2\,\mathrm{AiryBi}(1,x)\,\pi 3^{\frac{1}{3}}}{2\,\mathrm{AiryAi}(x)\,\pi 3^{\frac{5}{6}} + 3\,\mathrm{AiryAi}(x)\,\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} + 3\,\mathrm{AiryBi}(x)\,\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} - 2\,\mathrm{AiryBi}(x)\,\pi 3^{\frac{1}{3}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.219 (sec). Leaf size: 90

```
dsolve([diff(y(x),x)=y(x)^2-x,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{-2\,\mathrm{AiryAi}\,(1,x)\,3^{\frac{5}{6}}\pi - 3\,\mathrm{AiryAi}\,(1,x)\,\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} - 3\,\mathrm{AiryBi}\,(1,x)\,3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)^2 + 2\,\mathrm{AiryBi}\,(1,x)\,3^{\frac{1}{3}}\pi}{2\,\mathrm{AiryAi}\,(x)\,3^{\frac{5}{6}}\pi + 3\,\mathrm{AiryAi}\,(x)\,\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} + 3\,\mathrm{AiryBi}\,(x)\,3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)^2 - 2\,\mathrm{AiryBi}\,(x)\,3^{\frac{1}{3}}\pi}$$

✓ Solution by Mathematica

Time used: 7.282 (sec). Leaf size: 164

```
DSolve[{y'[x]==y[x]^2-x,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{-3}\,\mathrm{Gamma}\left(\frac{2}{3}\right)\left(ix^{3/2}\,\mathrm{BesselJ}\left(-\frac{4}{3},\frac{2}{3}ix^{3/2}\right) - ix^{3/2}\,\mathrm{BesselJ}\left(\frac{2}{3},\frac{2}{3}ix^{3/2}\right) + \mathrm{BesselJ}\left(-\frac{1}{3},\frac{2}{3}ix^{3/2}\right)\right) - 2ix^{3}}{2x\left(\mathrm{Gamma}\left(\frac{1}{3}\right)\mathrm{BesselJ}\left(\frac{1}{3},\frac{2}{3}ix^{3/2}\right) - \sqrt[3]{-3}\,\mathrm{Gamma}\left(\frac{2}{3}\right)\mathrm{BesselJ}\left(-\frac{1}{3},\frac{2}{3}ix\right.\right.}$$

1.3 problem 2. Using series method

Internal problem ID [6546]
Internal file name [`OUTPUT/5794_Sunday_June_05_2022_03_54_50_PM_87293275/index.tex`]

Book: A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section: Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number: 2. Using series method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = x^2}$$

With initial conditions

$$[y(1) = 1]$$

With the expansion point for the power series method at $x = 1$.

1.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -2$$
$$q(x) = x^2$$

Hence the ode is

$$y' - 2y = x^2$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

1.3.2 Solving as series ode

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d}{dt}y(t) - 2y(t) = (t+1)^2$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 1$$

The transformed ODE is now solved.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + x f + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y) && (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) F_0 \\
&= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) f \\
&= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$
y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0, y_0} \tag{6}
$$

Hence

$$
\begin{aligned}
F_0 &= t^2 + 2t + 2y(t) + 1 \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y} F_0 \\
&= 6t + 4 + 2t^2 + 4y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y} F_1 \\
&= 10 + 12t + 4t^2 + 8y(t) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y} F_2 \\
&= 20 + 24t + 8t^2 + 16y(t) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y} F_3 \\
&= 40 + 48t + 16t^2 + 32y(t) \\
F_5 &= \frac{dF_4}{dt} \\
&= \frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial y} F_4 \\
&= 80 + 96t + 32t^2 + 64y(t) \\
F_6 &= \frac{dF_5}{dt} \\
&= \frac{\partial F_5}{\partial t} + \frac{\partial F_5}{\partial y} F_5 \\
&= 160 + 192t + 64t^2 + 128y(t)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t(0) = 0$ and $y(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 3 \\
F_1 &= 8 \\
F_2 &= 18 \\
F_3 &= 36 \\
F_4 &= 72 \\
F_5 &= 144 \\
F_6 &= 288
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$y(t) = 3t^3 + 4t^2 + 3t + 1 + \frac{3t^4}{2} + \frac{3t^5}{5} + \frac{t^6}{5} + \frac{2t^7}{35} + O(t^8)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y(t) = 3t^3 + 4t^2 + 3t + 1 + \frac{3}{2}t^4 + \frac{3}{5}t^5 + \frac{1}{5}t^6 + \frac{2}{35}t^7$$

Hence the solution can be written as

$$y(t) = 3t^3 + 4t^2 + 3t + 1 + \frac{3t^4}{2} + \frac{3t^5}{5} + \frac{t^6}{5} + \frac{2t^7}{35} + O(t^8)$$

which simplifies to

$$y(t) = 3t^3 + 4t^2 + 3t + 1 + \frac{3t^4}{2} + \frac{3t^5}{5} + \frac{t^6}{5} + \frac{2t^7}{35} + O(t^8)$$

Since $t = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
\frac{d}{dt}y(t) + q(t)y(t) &= p(t) \\
\frac{d}{dt}y(t) - 2y(t) &= (t+1)^2
\end{aligned}
$$

Where

$$
\begin{aligned}
q(t) &= -2 \\
p(t) &= (t+1)^2
\end{aligned}
$$

Next, the type of the expansion point $t = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $t = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $t = 0$ is called an ordinary point $q(t)$ has a Taylor series expansion around the point $t = 0$. $t = 0$ is called a regular singular point if $q(t)$ is not not analytic at $t = 0$ but $tq(t)$ has Taylor series expansion. And finally, $t = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $t = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n t^n\right) = (t+1)^2 \tag{1}$$

Expanding $(t+1)^2$ as Taylor series around $t = 0$ and keeping only the first 8 terms gives

$$\begin{aligned}(t+1)^2 &= t^2 + 2t + 1 + \dots \\ &= t^2 + 2t + 1\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n t^n\right) = t^2 + 2t + 1 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \sum_{n=0}^{\infty} \left(-2 a_n t^n\right) = t^2 + 2t + 1 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} t^n\right) + \sum_{n=0}^{\infty} (-2 a_n t^n) = t^2 + 2t + 1 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$((n+1) a_{n+1} - 2a_n) t^n = t^2 + 2t + 1 \tag{4}$$

For $n = 0$ the recurrence equation gives

$$\begin{aligned}(a_1 - 2a_0) 1 &= 1 \\ a_1 - 2a_0 &= 1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_1 = 1 + 2a_0$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned}(2a_2 - 2a_1) t &= 2t \\ 2a_2 - 2a_1 &= 2\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_2 = 2 + 2a_0$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(3a_3 - 2a_2) t^2 &= t^2 \\ 3a_3 - 2a_2 &= 1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{5}{3} + \frac{4a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$\left(4a_4 - 2a_3\right)t^3 = 0$$
$$4a_4 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5}{6} + \frac{2a_0}{3}$$

For $n = 4$ the recurrence equation gives

$$\left(5a_5 - 2a_4\right)t^4 = 0$$
$$5a_5 - 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{3} + \frac{4a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$\left(6a_6 - 2a_5\right)t^5 = 0$$
$$6a_6 - 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1}{9} + \frac{4a_0}{45}$$

For $n = 6$ the recurrence equation gives

$$\left(7a_7 - 2a_6\right)t^6 = 0$$
$$7a_7 - 2a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{2}{63} + \frac{8a_0}{315}$$

For $n = 7$ the recurrence equation gives

$$\begin{aligned}(8a_8 - 2a_7) t^7 &= 0 \\ 8a_8 - 2a_7 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{1}{126} + \frac{2a_0}{315}$$

And so on. Therefore the solution is

$$\begin{aligned}y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned}y(t) = a_0 + (1 + 2a_0) t + (2 + 2a_0) t^2 + \left(\frac{5}{3} + \frac{4a_0}{3}\right) t^3 + \left(\frac{5}{6} + \frac{2a_0}{3}\right) t^4 \\ + \left(\frac{1}{3} + \frac{4a_0}{15}\right) t^5 + \left(\frac{1}{9} + \frac{4a_0}{45}\right) t^6 + \left(\frac{2}{63} + \frac{8a_0}{315}\right) t^7 + \dots\end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned}y(t) = \left(1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4 + \frac{4}{15}t^5 + \frac{4}{45}t^6 + \frac{8}{315}t^7\right) a_0 \\ + t + 2t^2 + \frac{5t^3}{3} + \frac{5t^4}{6} + \frac{t^5}{3} + \frac{t^6}{9} + \frac{2t^7}{63} + O(t^8)\end{aligned} \tag{3}$$

At $t = 0$ the solution above becomes

$$y(0) = a_0$$

Therefore the solution in Eq(3) now can be written as

$$y(t) = \left(1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4 + \frac{4}{15}t^5 + \frac{4}{45}t^6 + \frac{8}{315}t^7\right) y(0) + t + 2t^2 + \frac{5t^3}{3} + \frac{5t^4}{6} + \frac{t^5}{3} + \frac{t^6}{9} + \frac{2t^7}{63} + O(t^8)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = 1$$

Therefore the solution becomes

$$y(t) = 3t^3 + 4t^2 + 3t + 1 + \frac{3}{2}t^4 + \frac{3}{5}t^5 + \frac{1}{5}t^6 + \frac{2}{35}t^7$$

Hence the solution can be written as

$$y(t) = 3t^3 + 4t^2 + 3t + 1 + \frac{3t^4}{2} + \frac{3t^5}{5} + \frac{t^6}{5} + \frac{2t^7}{35} + O(t^8)$$

which simplifies to

$$y(t) = 3t^3 + 4t^2 + 3t + 1 + \frac{3t^4}{2} + \frac{3t^5}{5} + \frac{t^6}{5} + \frac{2t^7}{35} + O(t^8)$$

Replacing t in the above with the original independent variable xs using $t = x - 1$ results in

$$y = 3(x-1)^3 + 4(x-1)^2 + 3x - 2 + \frac{3(x-1)^4}{2} + \frac{3(x-1)^5}{5} + \frac{(x-1)^6}{5} + \frac{2(x-1)^7}{35} + O\left((x-1)^8\right)$$

Summary

The solution(s) found are the following

$$y = 3(x-1)^3 + 4(x-1)^2 + 3x - 2 + \frac{3(x-1)^4}{2} + \frac{3(x-1)^5}{5} + \frac{(x-1)^6}{5} + \frac{2(x-1)^7}{35} + O\left((x-1)^8\right) \tag{1}$$

Verification of solutions

$$y = 3(x-1)^3 + 4(x-1)^2 + 3x - 2 + \frac{3(x-1)^4}{2} + \frac{3(x-1)^5}{5} + \frac{(x-1)^6}{5} + \frac{2(x-1)^7}{35} + O\left((x-1)^8\right)$$

Verified OK.

1.3.3 Maple step by step solution

Let's solve

$[y' - 2y = x^2, y(1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2y + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2y = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - 2y) = \mu(x)\,x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' - 2y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -2\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = e^{-2x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{-2x}$

  $y = \frac{\int x^2 e^{-2x} dx + c_1}{e^{-2x}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{(2x^2+2x+1)e^{-2x}}{4} + c_1}{e^{-2x}}$

- Simplify

$y = c_1 e^{2x} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$

- Use initial condition $y(1) = 1$

  $1 = c_1 e^2 - \frac{5}{4}$

- Solve for c_1

  $c_1 = \frac{9}{4\,e^2}$

- Substitute $c_1 = \frac{9}{4e^2}$ into general solution and simplify

  $y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$

- Solution to the IVP

  $y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=8;
dsolve([diff(y(x),x)-2*y(x)=x^2,y(1) = 1],y(x),type='series',x=1);
```

$$y(x) = 1 + 3(x-1) + 4(x-1)^2 + 3(x-1)^3 + \frac{3}{2}(x-1)^4 + \frac{3}{5}(x-1)^5 + \frac{1}{5}(x-1)^6 + \frac{2}{35}(x-1)^7 + \mathrm{O}\left((x-1)^8\right)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 60

```
AsymptoticDSolveValue[{y'[x]-2*y[x]==x^2,{y[1]==1}},y[x],{x,1,7}]
```

$$y(x) \to \frac{2}{35}(x-1)^7 + \frac{1}{5}(x-1)^6 + \frac{3}{5}(x-1)^5 + \frac{3}{2}(x-1)^4 + 3(x-1)^3 + 4(x-1)^2 + 3(x-1) + 1$$

1.4 problem 2. direct method

Internal problem ID [6547]
Internal file name [`OUTPUT/5795_Sunday_June_05_2022_03_54_54_PM_5663266/index.tex`]

Book: A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section: Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number: 2. direct method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = x^2}$$

With initial conditions

$$[y(1) = 1]$$

1.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -2$$
$$q(x) = x^2$$

Hence the ode is

$$y' - 2y = x^2$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

1.4.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-2)dx} \\ &= e^{-2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(x^2\right) \\ \frac{d}{dx}\left(y\,e^{-2x}\right) &= \left(e^{-2x}\right)\left(x^2\right) \\ d\left(y\,e^{-2x}\right) &= \left(x^2 e^{-2x}\right)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,e^{-2x} &= \int x^2 e^{-2x}\,dx \\ y\,e^{-2x} &= -\frac{(2x^2 + 2x + 1)\,e^{-2x}}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-2x}$ results in

$$y = -\frac{e^{2x}(2x^2 + 2x + 1)\,e^{-2x}}{4} + c_1 e^{2x}$$

which simplifies to

$$y = c_1 e^{2x} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 e^2 - \frac{5}{4}$$

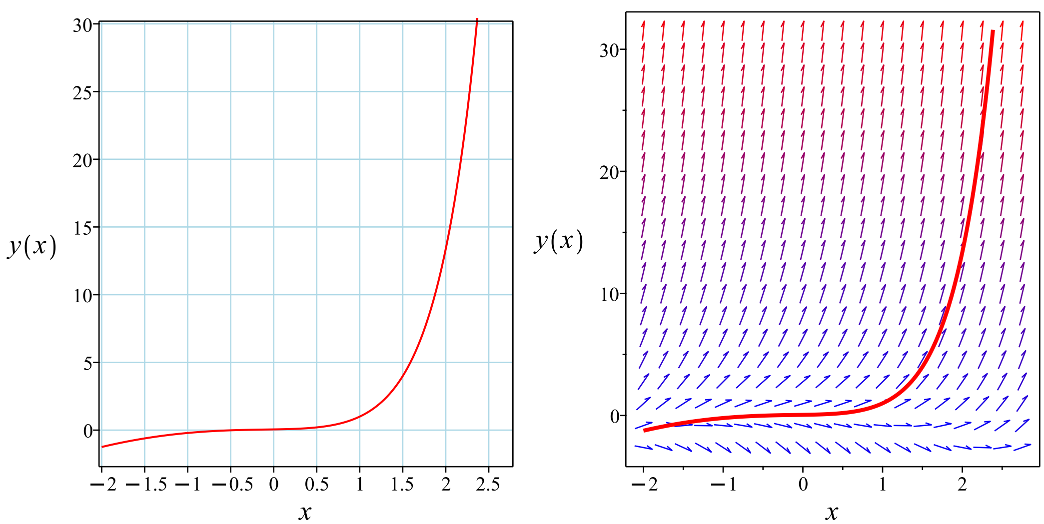
$$c_1 = \frac{9\,e^{-2}}{4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$

Summary
The solution(s) found are the following

$$y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$

Verified OK.

1.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= x^2 + 2y \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 2: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= e^{2x}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{e^{2x}} dy\end{aligned}$$

Which results in

$$S = y\,e^{-2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x^2 + 2y$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= -2y\,e^{-2x}\\ S_y &= e^{-2x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^2 \mathrm{e}^{-2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 \mathrm{e}^{-2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{(2R^2 + 2R + 1)\,\mathrm{e}^{-2R}}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-2x} y = -\frac{(2x^2 + 2x + 1)\,\mathrm{e}^{-2x}}{4} + c_1$$

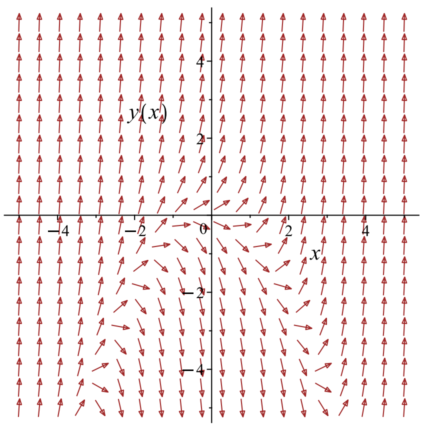
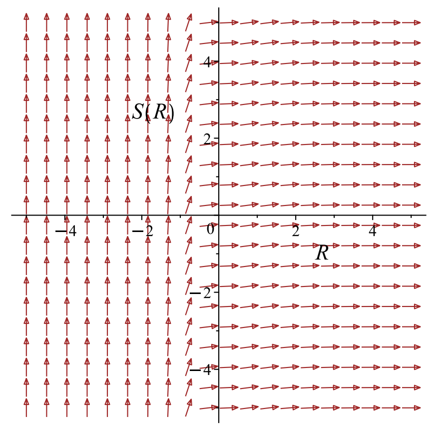
Which simplifies to

$$\mathrm{e}^{-2x} y = -\frac{(2x^2 + 2x + 1)\,\mathrm{e}^{-2x}}{4} + c_1$$

Which gives

$$y = -\frac{(2x^2 \mathrm{e}^{-2x} + 2x\,\mathrm{e}^{-2x} + \mathrm{e}^{-2x} - 4c_1)\,\mathrm{e}^{2x}}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 + 2y$ 	$R = x$ $S = y\,e^{-2x}$	$\frac{dS}{dR} = R^2 e^{-2R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 e^2 - \frac{5}{4}$$

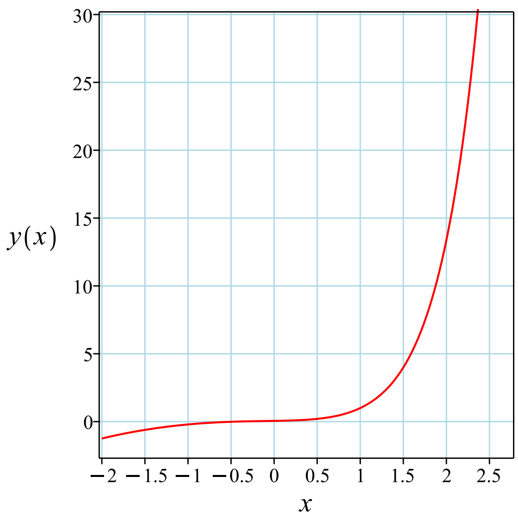
$$c_1 = \frac{9\,e^{-2}}{4}$$

Substituting c_1 found above in the general solution gives

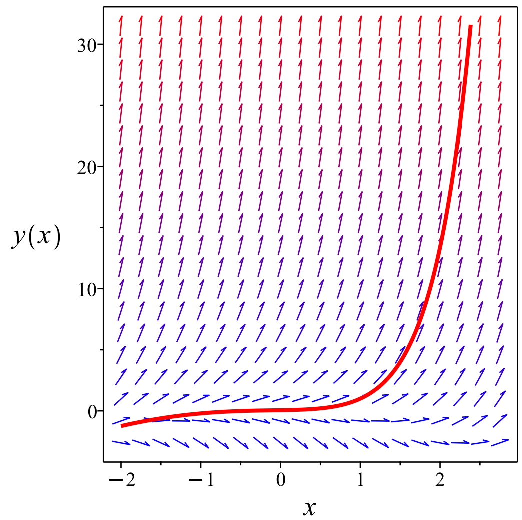
$$y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$

Verified OK.

1.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(x^2 + 2y\right)\mathrm{d}x\\ \left(-x^2 - 2y\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2 - 2y\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - 2y\right)\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-2) - (0))\\ &= -2\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -2\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2x}\\ &= \mathrm{e}^{-2x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-2x}\left(-x^2-2y\right)\\ &= -\mathrm{e}^{-2x}\left(x^2+2y\right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-2x}(1)\\ &= \mathrm{e}^{-2x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\mathrm{e}^{-2x}\left(x^2+2y\right)\right)+\left(\mathrm{e}^{-2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{-2x}\left(x^2+2y\right)\mathrm{d}x\end{aligned}$$

$$\phi = \frac{(2x^2+2x+4y+1)\,\mathrm{e}^{-2x}}{4}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-2x} = \mathrm{e}^{-2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(2x^2 + 2x + 4y + 1)\,\mathrm{e}^{-2x}}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(2x^2 + 2x + 4y + 1)\,\mathrm{e}^{-2x}}{4}$$

The solution becomes

$$y = -\frac{(2x^2\mathrm{e}^{-2x} + 2x\,\mathrm{e}^{-2x} + \mathrm{e}^{-2x} - 4c_1)\,\mathrm{e}^{2x}}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1\mathrm{e}^2 - \frac{5}{4}$$

$$c_1 = \frac{9\,\mathrm{e}^{-2}}{4}$$

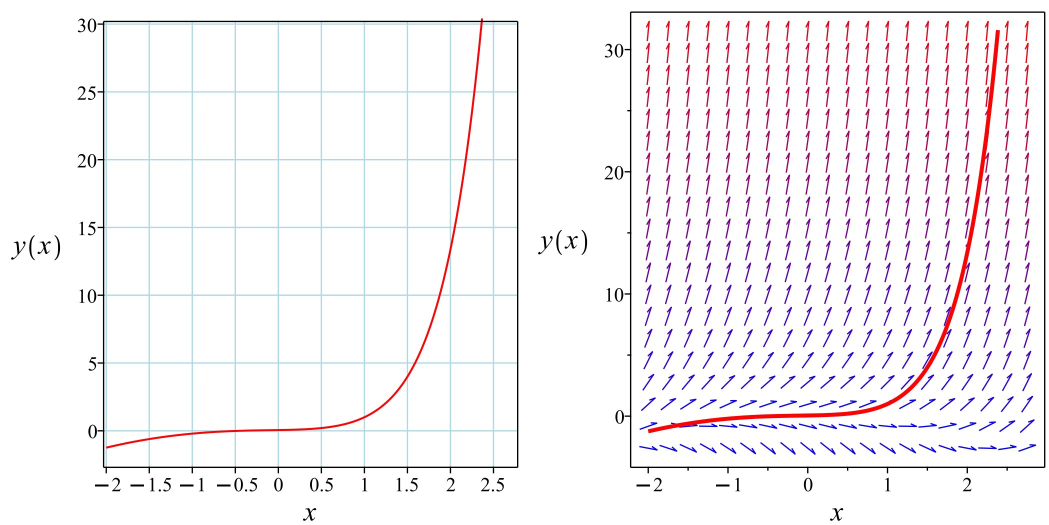
Substituting c_1 found above in the general solution gives

$$y = \frac{9\,\mathrm{e}^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{9\,\mathrm{e}^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{9\,\mathrm{e}^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$

Verified OK.

1.4.5 Maple step by step solution

Let's solve

$[y' - 2y = x^2, y(1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = 2y + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2y = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - 2y\right) = \mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - 2y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -2\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-2x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{-2x}$

  $y = \frac{\int x^2 \mathrm{e}^{-2x} dx + c_1}{\mathrm{e}^{-2x}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{\left(2x^2 + 2x + 1\right)\mathrm{e}^{-2x}}{4} + c_1}{\mathrm{e}^{-2x}}$

- Simplify

  $y = c_1 \mathrm{e}^{2x} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$

- Use initial condition $y(1) = 1$

  $1 = c_1 \mathrm{e}^2 - \frac{5}{4}$

- Solve for c_1

  $c_1 = \frac{9}{4\,\mathrm{e}^2}$

- Substitute $c_1 = \frac{9}{4\mathrm{e}^2}$ into general solution and simplify

  $y = \frac{9\,\mathrm{e}^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$

- Solution to the IVP

  $y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 22

```
dsolve([diff(y(x),x)-2*y(x)=x^2,y(1) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} + \frac{9\,e^{2x-2}}{4}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 28

```
DSolve[{y'[x]-2*y[x]==x^2,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(-2x^2 - 2x + 9e^{2x-2} - 1\right)$$

1.5 problem 3. series method

Internal problem ID [6548]
Internal file name [`OUTPUT/5796_Sunday_June_05_2022_03_54_56_PM_99488822/index.tex`]

Book: A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section: Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number: 3. series method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{y' - y - x\,\mathrm{e}^{y} = 0}$$

With initial conditions

$$[y(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

1.5.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + x f + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) && (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$
y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n\big|_{x_0,y_0} \tag{6}
$$

Hence

$$
\begin{aligned}
F_0 &= y + x\,\mathrm{e}^y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \mathrm{e}^{2y} x^2 + (xy + x + 1)\,\mathrm{e}^y + y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= \left(3yx^2 + 2x^2 + 3x\right)\mathrm{e}^{2y} + 2x^3\mathrm{e}^{3y} + \left(xy^2 + (2 + 2x)\,y + x + 1\right)\mathrm{e}^y + y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \left(7y^2x^2 + \left(11x^2 + 14x\right)y + 3x^2 + 7x + 3\right)\mathrm{e}^{2y} + \left(12yx^3 + 7x^3 + 12x^2\right)\mathrm{e}^{3y} + 6x^4\mathrm{e}^{4y} + \left(y^3x + (4x + 3
\\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= \left(15y^3x^2 + \left(43x^2 + 45x\right)y^2 + \left(28x^2 + 61x + 20\right)y + 4x^2 + 12x + 7\right)\mathrm{e}^{2y} + \left(50x^3y^2 + \left(69x^3 + 100x^2\right)
\\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} F_4 \\
&= \left(31y^4x^2 + \left(142x^2 + 124x\right)y^3 + \left(174x^2 + 325x + 85\right)y^2 + \left(62x^2 + 180x + 95\right)y + 5x^2 + 18x + 12\right)
\\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} F_5 \\
&= \left(63y^5x^2 + \left(424x^2 + 315x\right)y^4 + \left(850x^2 + 1360x + 294\right)y^3 + \left(594x^2 + 1510x + 685\right)y^2 + \left(129x^2 +
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 0$ gives

$$F_0 = 0$$
$$F_1 = 1$$
$$F_2 = 1$$
$$F_3 = 4$$
$$F_4 = 8$$
$$F_5 = 43$$
$$F_6 = 151$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{6} + \frac{x^5}{15} + \frac{43x^6}{720} + \frac{151x^7}{5040} + O(x^8)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{15}x^5 + \frac{43}{720}x^6 + \frac{151}{5040}x^7$$

Hence the solution can be written as

$$y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{6} + \frac{x^5}{15} + \frac{43x^6}{720} + \frac{151x^7}{5040} + O(x^8)$$

which simplifies to

$$y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{6} + \frac{x^5}{15} + \frac{43x^6}{720} + \frac{151x^7}{5040} + O(x^8)$$

Unable to also solve using normal power series since not linear ode. Not currently supported.

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{6} + \frac{x^5}{15} + \frac{43x^6}{720} + \frac{151x^7}{5040} + O(x^8) \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{6} + \frac{x^5}{15} + \frac{43x^6}{720} + \frac{151x^7}{5040} + O(x^8)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
Order:=8;
dsolve([diff(y(x),x)=y(x)+x*exp(y(x)),y(0) = 0],y(x),type='series',x=0);
```

$$y(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{15}x^5 + \frac{43}{720}x^6 + \frac{151}{5040}x^7 + O\left(x^8\right)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 46

```
AsymptoticDSolveValue[{y'[x]==y[x]+x*Exp[y[x]],{y[0]==0}},y[x],{x,0,7}]
```

$$y(x) \to \frac{151x^7}{5040} + \frac{43x^6}{720} + \frac{x^5}{15} + \frac{x^4}{6} + \frac{x^3}{6} + \frac{x^2}{2}$$

1.6 problem 3. direct method

Internal problem ID [6549]
Internal file name [`OUTPUT/5797_Sunday_June_05_2022_03_54_59_PM_50844051/index.tex`]

Book: A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section: Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number: 3. direct method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' - y - x\,\mathrm{e}^{y} = 0}$$

With initial conditions

$$[y(0) = 0]$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve([diff(y(x),x)=y(x)+x*exp(y(x)),y(0) = 0],y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y'[x]==y[x]+x*Exp[y[x]],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

Not solved