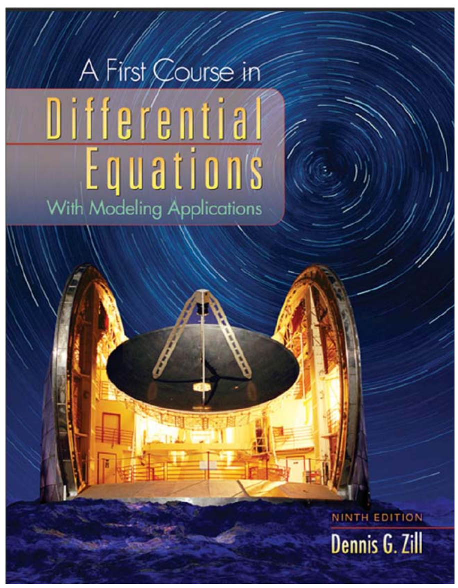
A Solution Manual For

A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.



Nasser M. Abbasi

May 15, 2024

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1.1 problem 15 (x=0)

Internal problem ID [5533]
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Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 15 (x=0).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^2-25\right)y''+2xy'+y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (2)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2xy' + y}{x^2 - 25} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{5x^2 y' + 4xy + 75y'}{(x^2 - 25)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-16y'x^3 - 17yx^2 - 800xy' - 175y}{(x^2 - 25)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(63x^4 + 7050x^2 + 24375) y' + (84x^3 + 2700x) y}{(x^2 - 25)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-294x^5 - 62100x^3 - 663750x) y' + (-483x^4 - 32250x^2 - 91875) y}{(x^2 - 25)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)}{25} \\
F_1 &= \frac{3y'(0)}{25} \\
F_2 &= \frac{7y(0)}{625} \\
F_3 &= \frac{39y'(0)}{625} \\
F_4 &= \frac{147y(0)}{15625}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1+\frac{1}{50}x^2+\frac{7}{15000}x^4+\frac{49}{3750000}x^6\right)y(0)+\left(x+\frac{1}{50}x^3+\frac{13}{25000}x^5\right)y'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2-25)\,y''+2xy'+y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_nx^n$$

Then

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1}$$
$$y''=\sum_{n=2}^{\infty}n(n-1)\,a_nx^{n-2}$$

Substituting the above back into the ode gives

$$(x^2-25)\left(\sum_{n=2}^{\infty}n(n-1)\,a_nx^{n-2}\right)+2x\left(\sum_{n=1}^{\infty}na_nx^{n-1}\right)+\left(\sum_{n=0}^{\infty}a_nx^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}x^na_nn(n-1)\right)+\sum_{n=2}^{\infty}\left(-25n(n-1)\,a_nx^{n-2}\right)+\left(\sum_{n=1}^{\infty}2na_nx^n\right)+\left(\sum_{n=0}^{\infty}a_nx^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-25n(n-1)\,a_nx^{n-2}\right)=\sum_{n=0}^{\infty}\left(-25(n+2)\,a_{n+2}(n+1)\,x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-25(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-50a_2 + a_0 = 0$$

$$a_2 = \frac{a_0}{50}$$

$n = 1$ gives

$$-150a_3 + 3a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_1}{50}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) - 25(n+2)\, a_{n+2}(n+1) + 2n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2+n+1)}{25\,(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$7a_2 - 300a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{7a_0}{15000}$$

For $n = 3$ the recurrence equation gives

$$13a_3 - 500a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{13a_1}{25000}$$

For $n = 4$ the recurrence equation gives

$$21a_4 - 750a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{49a_0}{3750000}$$

For $n = 5$ the recurrence equation gives

$$31a_5 - 1050a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{403a_1}{26250000}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \frac{1}{50}a_0x^2 + \frac{1}{50}a_1x^3 + \frac{7}{15000}a_0x^4 + \frac{13}{25000}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{50}x^2 + \frac{7}{15000}x^4\right)a_0 + \left(x + \frac{1}{50}x^3 + \frac{13}{25000}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x=0$ the solution above becomes

$$y = \left(1+\frac{1}{50}x^2+\frac{7}{15000}x^4\right)c_1+\left(x+\frac{1}{50}x^3+\frac{13}{25000}x^5\right)c_2+O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1+\frac{1}{50}x^2+\frac{7}{15000}x^4+\frac{49}{3750000}x^6\right)y(0)+\left(x+\frac{1}{50}x^3+\frac{13}{25000}x^5\right)y'(0)+O\left(x^6\right) \tag{1}$$

$$y = \left(1+\frac{1}{50}x^2+\frac{7}{15000}x^4\right)c_1+\left(x+\frac{1}{50}x^3+\frac{13}{25000}x^5\right)c_2+O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1+\frac{1}{50}x^2+\frac{7}{15000}x^4+\frac{49}{3750000}x^6\right)y(0)+\left(x+\frac{1}{50}x^3+\frac{13}{25000}x^5\right)y'(0)+O\left(x^6\right)$$

Verified OK.

$$y = \left(1+\frac{1}{50}x^2+\frac{7}{15000}x^4\right)c_1+\left(x+\frac{1}{50}x^3+\frac{13}{25000}x^5\right)c_2+O\left(x^6\right)$$

Verified OK.

1.1.1 Maple step by step solution

Let's solve

$(x^2-25)\,y''+2xy'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2xy'}{x^2-25}-\frac{y}{x^2-25}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^2-25}+\frac{y}{x^2-25}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x}{x^2-25}, P_3(x)=\frac{1}{x^2-25}\right]$

- $(5+x)\cdot P_2(x)$ is analytic at $x=-5$

  $$\left((5+x)\cdot P_2(x)\right)\bigg|_{x=-5}=1$$

- $(5+x)^2\cdot P_3(x)$ is analytic at $x=-5$

  $$\left((5+x)^2\cdot P_3(x)\right)\bigg|_{x=-5}=0$$

- $x=-5$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-5$

- Multiply by denominators

  $$\left(x^2-25\right)y''+2xy'+y=0$$

- Change variables using $x=u-5$ so that the regular singular point is at $u=0$

  $$\left(u^2-10u\right)\left(\frac{d^2}{du^2}y(u)\right)+(2u-10)\left(\frac{d}{du}y(u)\right)+y(u)=0$$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-10a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-10a_{k+1}(k+1+r)^2+a_k\left(k^2+2kr+r^2+k+r+1\right)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-10r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-10a_{k+1}(k+1)^2 + a_k(k^2+k+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k^2+k+1)}{10(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k^2+k+1)}{10(k+1)^2}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k^2+k+1)}{10(k+1)^2}\right]$

- Revert the change of variables $u = 5 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(5+x)^k, a_{k+1} = \frac{a_k(k^2+k+1)}{10(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((x^2-25)*diff(y(x),x$2)+2*x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{50}x^2 + \frac{7}{15000}x^4\right) y(0) + \left(x + \frac{1}{50}x^3 + \frac{13}{25000}x^5\right) D(y)\left(0\right) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[(x^2-25)*y''[x]+2*x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{13x^5}{25000} + \frac{x^3}{50} + x\right) + c_1\left(\frac{7x^4}{15000} + \frac{x^2}{50} + 1\right)$$

1.2 problem 15 (x=1)

Internal problem ID [5534]
Internal file name [OUTPUT/4782_Sunday_June_05_2022_03_05_46_PM_87687443/index.tex]

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Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 15 (x=1).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2-25)\,y''+2xy'+y=0}$$

With the expansion point for the power series method at $x=1$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=x-1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t+1)^2-25\right)\left(\frac{d^2}{dt^2}y(t)\right)+2(t+1)\left(\frac{d}{dt}y(t)\right)+y(t)=0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x,y,y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (4) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2t\left(\frac{d}{dt}y(t)\right) + 2\frac{d}{dt}y(t) + y(t)}{t^2 + 2t - 24} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{5(t^2 + 2t + 16)\left(\frac{d}{dt}y(t)\right) + 4(t+1)\,y(t)}{\left(t^2 + 2t - 24\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{(-16t^3 - 48t^2 - 848t - 816)\left(\frac{d}{dt}y(t)\right) - 17y(t)\left(t^2 + 2t + \frac{192}{17}\right)}{\left(t^2 + 2t - 24\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{(63t^4 + 252t^3 + 7428t^2 + 14352t + 31488)\left(\frac{d}{dt}y(t)\right) + 84(t+1)\,y(t)\left(t^2 + 2t + \frac{232}{7}\right)}{\left(t^2 + 2t - 24\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{(-294t^5 - 1470t^4 - 65040t^3 - 189240t^2 - 851520t - 726144)\left(\frac{d}{dt}y(t)\right) - 483y(t)\left(t^4 + 4t^3 + \frac{11716}{161}t^2}{\left(t^2 + 2t - 24\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)}{24} + \frac{y'(0)}{12} \\
F_1 &= \frac{y(0)}{144} + \frac{5y'(0)}{36} \\
F_2 &= \frac{y(0)}{72} + \frac{17y'(0)}{288} \\
F_3 &= \frac{29y(0)}{3456} + \frac{41y'(0)}{432} \\
F_4 &= \frac{649y(0)}{41472} + \frac{1891y'(0)}{20736}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = {} & \left(1 + \frac{1}{48}t^2 + \frac{1}{864}t^3 + \frac{1}{1728}t^4 + \frac{29}{414720}t^5 + \frac{649}{29859840}t^6\right) y(0) \\
& + \left(t + \frac{1}{24}t^2 + \frac{5}{216}t^3 + \frac{17}{6912}t^4 + \frac{41}{51840}t^5 + \frac{1891}{14929920}t^6\right) y'(0) + O(t^6)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(\frac{d^2}{dt^2}y(t)\right)\left(t^2 + 2t - 24\right) + (2t + 2)\left(\frac{d}{dt}y(t)\right) + y(t) = 0
$$

Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)\left(t^2 + 2t - 24\right) + (2t + 2)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} 2 n\, t^{n-1} a_{n}(n-1)\right)+\sum_{n=2}^{\infty}\left(-24 n(n-1) a_{n} t^{n-2}\right) \\ +\left(\sum_{n=1}^{\infty} 2 n a_{n} t^{n}\right)+\left(\sum_{n=1}^{\infty} 2 n a_{n} t^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_{n} t^{n}\right)=0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2 n\, t^{n-1} a_{n}(n-1)=\sum_{n=1}^{\infty} 2(n+1) a_{n+1} n\, t^{n}$$

$$\sum_{n=2}^{\infty}\left(-24 n(n-1) a_{n} t^{n-2}\right)=\sum_{n=0}^{\infty}\left(-24(n+2) a_{n+2}(n+1) t^{n}\right)$$

$$\sum_{n=1}^{\infty} 2 n a_{n} t^{n-1}=\sum_{n=0}^{\infty} 2(n+1) a_{n+1} t^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^{n} a_{n} n(n-1)\right)+\left(\sum_{n=1}^{\infty} 2(n+1) a_{n+1} n\, t^{n}\right) \\ +\sum_{n=0}^{\infty}\left(-24(n+2) a_{n+2}(n+1) t^{n}\right)+\left(\sum_{n=1}^{\infty} 2 n a_{n} t^{n}\right) \\ +\left(\sum_{n=0}^{\infty} 2(n+1) a_{n+1} t^{n}\right)+\left(\sum_{n=0}^{\infty} a_{n} t^{n}\right)=0 \tag{3}$$

$n=0$ gives

$$-48 a_{2}+2 a_{1}+a_{0}=0$$

$$a_{2}=\frac{a_{0}}{48}+\frac{a_{1}}{24}$$

$n = 1$ gives

$$8a_2 - 144a_3 + 3a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{864} + \frac{5a_1}{216}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + 2(n+1)\, a_{n+1}n - 24(n+2)\, a_{n+2}(n+1) + 2na_n + 2(n+1)\, a_{n+1} + a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2a_n + 2n^2a_{n+1} + na_n + 4na_{n+1} + a_n + 2a_{n+1}}{24\,(n+2)\,(n+1)} \\ &= \frac{(n^2+n+1)\, a_n}{24\,(n+2)\,(n+1)} + \frac{(2n^2+4n+2)\, a_{n+1}}{24\,(n+2)\,(n+1)} \end{aligned} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$7a_2 + 18a_3 - 288a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{1728} + \frac{17a_1}{6912}$$

For $n = 3$ the recurrence equation gives

$$13a_3 + 32a_4 - 480a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{29a_0}{414720} + \frac{41a_1}{51840}$$

For $n = 4$ the recurrence equation gives

$$21a_4 + 50a_5 - 720a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{649a_0}{29859840} + \frac{1891a_1}{14929920}$$

For $n = 5$ the recurrence equation gives

$$31a_5 + 72a_6 - 1008a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{43a_0}{11612160} + \frac{155a_1}{4644864}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t + \left(\frac{a_0}{48} + \frac{a_1}{24}\right) t^2 + \left(\frac{a_0}{864} + \frac{5a_1}{216}\right) t^3 \\ + \left(\frac{a_0}{1728} + \frac{17a_1}{6912}\right) t^4 + \left(\frac{29a_0}{414720} + \frac{41a_1}{51840}\right) t^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y(t) = &\left(1 + \frac{1}{48}t^2 + \frac{1}{864}t^3 + \frac{1}{1728}t^4 + \frac{29}{414720}t^5\right) a_0 \\ &+ \left(t + \frac{1}{24}t^2 + \frac{5}{216}t^3 + \frac{17}{6912}t^4 + \frac{41}{51840}t^5\right) a_1 + O(t^6) \end{aligned} \tag{3}$$

At $t = 0$ the solution above becomes

$$\begin{aligned} y(t) = &\left(1 + \frac{1}{48}t^2 + \frac{1}{864}t^3 + \frac{1}{1728}t^4 + \frac{29}{414720}t^5\right) c_1 \\ &+ \left(t + \frac{1}{24}t^2 + \frac{5}{216}t^3 + \frac{17}{6912}t^4 + \frac{41}{51840}t^5\right) c_2 + O(t^6) \end{aligned}$$

Replacing t in the above with the original independent variable $x$susing $t = x-1$ results in

$$y = \left(1+\frac{(x-1)^{2}}{48}+\frac{(x-1)^{3}}{864}+\frac{(x-1)^{4}}{1728}+\frac{29(x-1)^{5}}{414720}+\frac{649(x-1)^{6}}{29859840}\right)y(1) + \left(x-1+\frac{(x-1)^{2}}{24}+\frac{5(x-1)^{3}}{216}+\frac{17(x-1)^{4}}{6912}+\frac{41(x-1)^{5}}{51840}+\frac{1891(x-1)^{6}}{14929920}\right)y'(1) + O\left((x-1)^{6}\right)$$

Summary

The solution(s) found are the following

$$y = \left(1+\frac{(x-1)^{2}}{48}+\frac{(x-1)^{3}}{864}+\frac{(x-1)^{4}}{1728}+\frac{29(x-1)^{5}}{414720}+\frac{649(x-1)^{6}}{29859840}\right)y(1) + \left(x-1+\frac{(x-1)^{2}}{24}+\frac{5(x-1)^{3}}{216}+\frac{17(x-1)^{4}}{6912}+\frac{41(x-1)^{5}}{51840}+\frac{1891(x-1)^{6}}{14929920}\right)y'(1) + O\left((x-1)^{6}\right) \tag{1}$$

Verification of solutions

$$y = \left(1+\frac{(x-1)^{2}}{48}+\frac{(x-1)^{3}}{864}+\frac{(x-1)^{4}}{1728}+\frac{29(x-1)^{5}}{414720}+\frac{649(x-1)^{6}}{29859840}\right)y(1) + \left(x-1+\frac{(x-1)^{2}}{24}+\frac{5(x-1)^{3}}{216}+\frac{17(x-1)^{4}}{6912}+\frac{41(x-1)^{5}}{51840}+\frac{1891(x-1)^{6}}{14929920}\right)y'(1) + O\left((x-1)^{6}\right)$$

Verified OK.

1.2.1 Maple step by step solution

Let's solve

$(x^2-25)\,y''+2xy'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{2xy'}{x^2-25} - \frac{y}{x^2-25}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{2xy'}{x^2-25} + \frac{y}{x^2-25} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

$\left[P_2(x) = \frac{2x}{x^2-25}, P_3(x) = \frac{1}{x^2-25}\right]$

- $(5+x) \cdot P_2(x)$ is analytic at $x = -5$

$\left((5+x) \cdot P_2(x)\right)\bigg|_{x=-5} = 1$

- $(5+x)^2 \cdot P_3(x)$ is analytic at $x = -5$

$\left((5+x)^2 \cdot P_3(x)\right)\bigg|_{x=-5} = 0$

- $x = -5$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -5$

- Multiply by denominators

$(x^2-25)\, y'' + 2xy' + y = 0$

- Change variables using $x = u-5$ so that the regular singular point is at $u = 0$

$(u^2-10u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-10)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^{m}\cdot\left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k=0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k->k +2-m$

$$u^{m}\cdot\left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$-10a_{0} r^{2} u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-10a_{k +1} \left(k +1+r \right)^{2}+a_{k} \left(k^{2}+2 k r +r^{2}+k +r +1\right)\right) u^{k +r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$-10r^{2}=0$$

- Values of r that satisfy the indicial equation

$$r =0$$

- Each term in the series must be 0, giving the recursion relation

$$-10a_{k +1} \left(k +1\right)^{2}+a_{k} \left(k^{2}+k +1\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k +1}=\frac{a_{k} \left(k^{2}+k +1\right)}{10 \left(k +1\right)^{2}}$$

- Recursion relation for $r =0$

$$a_{k +1}=\frac{a_{k} \left(k^{2}+k +1\right)}{10 \left(k +1\right)^{2}}$$

- Solution for $r =0$

$$\left[y \left(u \right)=\sum_{k=0}^{\infty}a_{k} u^{k}, a_{k +1}=\frac{a_{k} \left(k^{2}+k +1\right)}{10 \left(k +1\right)^{2}}\right]$$

- Revert the change of variables $u =5+x$

$$\left[y=\sum_{k=0}^{\infty}a_{k} \left(5+x \right)^{k}, a_{k +1}=\frac{a_{k} \left(k^{2}+k +1\right)}{10 \left(k +1\right)^{2}}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve((x^2-25)*diff(y(x),x$2)+2*x*diff(y(x),x)+y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1 + \frac{(x-1)^2}{48} + \frac{(x-1)^3}{864} + \frac{(x-1)^4}{1728} + \frac{29(x-1)^5}{414720}\right) y(1) + \left(x - 1 + \frac{(x-1)^2}{24} + \frac{5(x-1)^3}{216} + \frac{17(x-1)^4}{6912} + \frac{41(x-1)^5}{51840}\right) D(y)(1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 87

```
AsymptoticDSolveValue[(x^2-25)*y''[x]+2*x*y'[x]+y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1\left(\frac{29(x-1)^5}{414720} + \frac{(x-1)^4}{1728} + \frac{1}{864}(x-1)^3 + \frac{1}{48}(x-1)^2 + 1\right) + c_2\left(\frac{41(x-1)^5}{51840} + \frac{17(x-1)^4}{6912} + \frac{5}{216}(x-1)^3 + \frac{1}{24}(x-1)^2 + x - 1\right)$$

1.3 problem 16 (x=0)

Internal problem ID [5535]
Internal file name [`OUTPUT/4783_Sunday_June_05_2022_03_05_47_PM_36303290/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 16 (x=0).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2-2x+10)\,y''+xy'-4y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (7)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (8)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{xy' - 4y}{x^2 - 2x + 10} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(6x^2 - 8x + 30)\, y' + (-12x + 8)\, y}{(x^2 - 2x + 10)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{30\left(\left(x^2 - \frac{4}{3}x + 5\right) y' - 2\left(x - \frac{2}{3}\right) y\right) \left(x - \frac{4}{5}\right)}{(x^2 - 2x + 10)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{180\left(x^2 - \frac{8}{5}x - \frac{13}{15}\right) \left(\left(x^2 - \frac{4}{3}x + 5\right) y' - 2\left(x - \frac{2}{3}\right) y\right)}{(x^2 - 2x + 10)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{1260\left(x^3 - \frac{12}{5}x^2 - \frac{13}{5}x + \frac{344}{105}\right) \left(\left(x^2 - \frac{4}{3}x + 5\right) y' - 2\left(x - \frac{2}{3}\right) y\right)}{(x^2 - 2x + 10)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{2y(0)}{5} \\
F_1 &= \frac{2y(0)}{25} + \frac{3y'(0)}{10} \\
F_2 &= \frac{4y(0)}{125} + \frac{3y'(0)}{25} \\
F_3 &= -\frac{13y(0)}{625} - \frac{39y'(0)}{500} \\
F_4 &= -\frac{172y(0)}{3125} - \frac{129y'(0)}{625}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1+\frac{1}{5}x^{2}+\frac{1}{75}x^{3}+\frac{1}{750}x^{4}-\frac{13}{75000}x^{5}-\frac{43}{562500}x^{6}\right)y(0) \\
&+\left(x+\frac{1}{20}x^{3}+\frac{1}{200}x^{4}-\frac{13}{20000}x^{5}-\frac{43}{150000}x^{6}\right)y'(0)+O(x^{6})
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x^{2}-2x+10)\,y''+xy'-4y=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x^{2}-2x+10)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-4\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \quad (1)
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} x^{n} a_n n(n-1)\right)+\sum_{n=2}^{\infty}\left(-2n\, x^{n-1} a_n (n-1)\right) \\
&+\left(\sum_{n=2}^{\infty} 10n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_n x^{n}\right)+\sum_{n=0}^{\infty}\left(-4 a_n x^{n}\right)=0
\end{aligned} \quad (2)
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-2n\, x^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-2(n+1)\, a_{n+1} n\, x^n\right)$$

$$\sum_{n=2}^{\infty} 10n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 10(n+2)\, a_{n+2} (n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=1}^{\infty} \left(-2(n+1)\, a_{n+1} n\, x^n\right) \\ &+ \left(\sum_{n=0}^{\infty} 10(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-4 a_n x^n\right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$20 a_2 - 4 a_0 = 0$$

$$a_2 = \frac{a_0}{5}$$

$n = 1$ gives

$$-4 a_2 + 60 a_3 - 3 a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{75} + \frac{a_1}{20}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) - 2(n+1)\, a_{n+1} n + 10(n+2)\, a_{n+2} (n+1) + n a_n - 4 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n - 2n^2 a_{n+1} - 2n a_{n+1} - 4 a_n}{10\,(n+2)\,(n+1)} \\ &= -\frac{(n^2-4)\, a_n}{10\,(n+2)\,(n+1)} - \frac{(-2n^2-2n)\, a_{n+1}}{10\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-12a_3 + 120a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{750} + \frac{a_1}{200}$$

For $n = 3$ the recurrence equation gives

$$5a_3 - 24a_4 + 200a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{13a_0}{75000} - \frac{13a_1}{20000}$$

For $n = 4$ the recurrence equation gives

$$12a_4 - 40a_5 + 300a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{43a_0}{562500} - \frac{43a_1}{150000}$$

For $n = 5$ the recurrence equation gives

$$21a_5 - 60a_6 + 420a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{71a_0}{31500000} - \frac{71a_1}{8400000}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{a_0 x^2}{5} + \left(\frac{a_0}{75} + \frac{a_1}{20}\right) x^3 + \left(\frac{a_0}{750} + \frac{a_1}{200}\right) x^4 + \left(-\frac{13 a_0}{75000} - \frac{13 a_1}{20000}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$\begin{aligned} y &= \left(1 + \frac{1}{5}x^2 + \frac{1}{75}x^3 + \frac{1}{750}x^4 - \frac{13}{75000}x^5\right) a_0 \\ &\quad + \left(x + \frac{1}{20}x^3 + \frac{1}{200}x^4 - \frac{13}{20000}x^5\right) a_1 + O(x^6) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{5}x^2 + \frac{1}{75}x^3 + \frac{1}{750}x^4 - \frac{13}{75000}x^5\right) c_1 + \left(x + \frac{1}{20}x^3 + \frac{1}{200}x^4 - \frac{13}{20000}x^5\right) c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned} y &= \left(1 + \frac{1}{5}x^2 + \frac{1}{75}x^3 + \frac{1}{750}x^4 - \frac{13}{75000}x^5 - \frac{43}{562500}x^6\right) y(0) \\ &\quad + \left(x + \frac{1}{20}x^3 + \frac{1}{200}x^4 - \frac{13}{20000}x^5 - \frac{43}{150000}x^6\right) y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$\begin{aligned} y &= \left(1 + \frac{1}{5}x^2 + \frac{1}{75}x^3 + \frac{1}{750}x^4 - \frac{13}{75000}x^5\right) c_1 \\ &\quad + \left(x + \frac{1}{20}x^3 + \frac{1}{200}x^4 - \frac{13}{20000}x^5\right) c_2 + O(x^6) \end{aligned} \tag{2}$$

<u>Verification of solutions</u>

$$\begin{aligned} y &= \left(1 + \frac{1}{5}x^2 + \frac{1}{75}x^3 + \frac{1}{750}x^4 - \frac{13}{75000}x^5 - \frac{43}{562500}x^6\right) y(0) \\ &\quad + \left(x + \frac{1}{20}x^3 + \frac{1}{200}x^4 - \frac{13}{20000}x^5 - \frac{43}{150000}x^6\right) y'(0) + O(x^6) \end{aligned}$$

Verified OK.

$$y = \left(1 + \frac{1}{5}x^2 + \frac{1}{75}x^3 + \frac{1}{750}x^4 - \frac{13}{75000}x^5\right) c_1 + \left(x + \frac{1}{20}x^3 + \frac{1}{200}x^4 - \frac{13}{20000}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve((x^2-2*x+10)*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{5}x^2 + \frac{1}{75}x^3 + \frac{1}{750}x^4 - \frac{13}{75000}x^5\right) y(0) + \left(x + \frac{1}{20}x^3 + \frac{1}{200}x^4 - \frac{13}{20000}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[(x^2-2*x+10)*y''[x]+x*y'[x]-4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{13x^5}{20000} + \frac{x^4}{200} + \frac{x^3}{20} + x\right) + c_1\left(-\frac{13x^5}{75000} + \frac{x^4}{750} + \frac{x^3}{75} + \frac{x^2}{5} + 1\right)$$

1.4 problem 16 (x=1)

Internal problem ID [5536]
Internal file name [`OUTPUT/4784_Sunday_June_05_2022_03_05_49_PM_50528159/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 16 (x=1).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2 - 2x + 10)\, y'' + xy' - 4y = 0}$$

With the expansion point for the power series method at $x = 1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t+1)^2 - 2t + 8\right)\left(\frac{d^2}{dt^2}y(t)\right) + (t+1)\left(\frac{d}{dt}y(t)\right) - 4y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (10) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (11) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned} F_0 &= f(x, y, y') && (4) \\ F_1 &= \frac{df}{dx} \\ &= \frac{dF_0}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5) \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\ F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\ &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\ &\vdots \\ F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6) \end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{t\left(\frac{d}{dt}y(t)\right) + \frac{d}{dt}y(t) - 4y(t)}{t^2+9} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(6t^2+4t+28\right)\left(\frac{d}{dt}y(t)\right) + \left(-12t-4\right)y(t)}{\left(t^2+9\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= -\frac{30\left(\left(t^2+\frac{2}{3}t+\frac{14}{3}\right)\left(\frac{d}{dt}y(t)\right) - 2y(t)\left(t+\frac{1}{3}\right)\right)\left(\frac{1}{5}+t\right)}{\left(t^2+9\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{180\left(\left(t^2+\frac{2}{3}t+\frac{14}{3}\right)\left(\frac{d}{dt}y(t)\right) - 2y(t)\left(t+\frac{1}{3}\right)\right)\left(t^2+\frac{2}{5}t-\frac{22}{15}\right)}{\left(t^2+9\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= -\frac{1260\left(\left(t^2+\frac{2}{3}t+\frac{14}{3}\right)\left(\frac{d}{dt}y(t)\right) - 2y(t)\left(t+\frac{1}{3}\right)\right)\left(t^3+\frac{3}{5}t^2-\frac{22}{5}t-\frac{76}{105}\right)}{\left(t^2+9\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{4y(0)}{9} - \frac{y'(0)}{9} \\
F_1 &= -\frac{4y(0)}{81} + \frac{28y'(0)}{81} \\
F_2 &= \frac{4y(0)}{729} - \frac{28y'(0)}{729} \\
F_3 &= \frac{176y(0)}{6561} - \frac{1232y'(0)}{6561} \\
F_4 &= -\frac{608y(0)}{59049} + \frac{4256y'(0)}{59049}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = &\left(1 + \frac{2}{9}t^2 - \frac{2}{243}t^3 + \frac{1}{4374}t^4 + \frac{22}{98415}t^5 - \frac{38}{2657205}t^6\right) y(0) \\
&+ \left(t - \frac{1}{18}t^2 + \frac{14}{243}t^3 - \frac{7}{4374}t^4 - \frac{154}{98415}t^5 + \frac{266}{2657205}t^6\right) y'(0) + O(t^6)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(\frac{d^2}{dt^2}y(t)\right)(t^2 + 9) + (t + 1)\left(\frac{d}{dt}y(t)\right) - 4y(t) = 0
$$

Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(t^2 + 9) + (t + 1)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (1)
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 9n(n-1) a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \sum_{n=0}^{\infty} (-4 a_n t^n) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 9n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} 9(n+2) a_{n+2}(n+1) t^n$$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 9(n+2) a_{n+2}(n+1) t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=0}^{\infty} (n+1) a_{n+1} t^n\right) + \sum_{n=0}^{\infty} (-4 a_n t^n) = 0 \tag{3}$$

$n = 0$ gives

$$18a_2 + a_1 - 4a_0 = 0$$

$$a_2 = \frac{2a_0}{9} - \frac{a_1}{18}$$

$n = 1$ gives

$$54a_3 - 3a_1 + 2a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{2a_0}{243} + \frac{14a_1}{243}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + 9(n+2)\, a_{n+2}(n+1) + n a_n + (n+1)\, a_{n+1} - 4 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n^2 a_n + n a_{n+1} - 4 a_n + a_{n+1}}{9\,(n+2)\,(n+1)} \\
&= -\frac{(n^2-4)\, a_n}{9\,(n+2)\,(n+1)} - \frac{a_{n+1}}{9\,(n+2)}
\end{aligned}
\tag{5}
$$

For $n = 2$ the recurrence equation gives

$$108 a_4 + 3 a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{4374} - \frac{7 a_1}{4374}$$

For $n = 3$ the recurrence equation gives

$$5 a_3 + 180 a_5 + 4 a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{22 a_0}{98415} - \frac{154 a_1}{98415}$$

For $n = 4$ the recurrence equation gives

$$12 a_4 + 270 a_6 + 5 a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{38 a_0}{2657205} + \frac{266 a_1}{2657205}$$

For $n = 5$ the recurrence equation gives

$$21 a_5 + 378 a_7 + 6 a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{2041a_0}{167403915} + \frac{2041a_1}{23914845}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t + \left(\frac{2a_0}{9} - \frac{a_1}{18}\right) t^2 + \left(-\frac{2a_0}{243} + \frac{14a_1}{243}\right) t^3 \\ + \left(\frac{a_0}{4374} - \frac{7a_1}{4374}\right) t^4 + \left(\frac{22a_0}{98415} - \frac{154a_1}{98415}\right) t^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y(t) = \left(1 + \frac{2}{9}t^2 - \frac{2}{243}t^3 + \frac{1}{4374}t^4 + \frac{22}{98415}t^5\right) a_0 \\ + \left(t - \frac{1}{18}t^2 + \frac{14}{243}t^3 - \frac{7}{4374}t^4 - \frac{154}{98415}t^5\right) a_1 + O(t^6) \end{aligned} \tag{3}$$

At $t = 0$ the solution above becomes

$$\begin{aligned} y(t) = \left(1 + \frac{2}{9}t^2 - \frac{2}{243}t^3 + \frac{1}{4374}t^4 + \frac{22}{98415}t^5\right) c_1 \\ + \left(t - \frac{1}{18}t^2 + \frac{14}{243}t^3 - \frac{7}{4374}t^4 - \frac{154}{98415}t^5\right) c_2 + O(t^6) \end{aligned}$$

Replacing t in the above with the original independent variable $x$using $t = x - 1$ results in

$$\begin{aligned} y = \left(1 + \frac{2(x-1)^2}{9} - \frac{2(x-1)^3}{243} + \frac{(x-1)^4}{4374} + \frac{22(x-1)^5}{98415} - \frac{38(x-1)^6}{2657205}\right) y(1) \\ + \left(x - 1 - \frac{(x-1)^2}{18} + \frac{14(x-1)^3}{243} - \frac{7(x-1)^4}{4374} - \frac{154(x-1)^5}{98415} + \frac{266(x-1)^6}{2657205}\right) y'(1) \\ + O\left((x-1)^6\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(1+\frac{2(x-1)^2}{9}-\frac{2(x-1)^3}{243}+\frac{(x-1)^4}{4374}+\frac{22(x-1)^5}{98415}-\frac{38(x-1)^6}{2657205}\right)y(1) + \left(x-1-\frac{(x-1)^2}{18}+\frac{14(x-1)^3}{243}-\frac{7(x-1)^4}{4374}-\frac{154(x-1)^5}{98415}+\frac{266(x-1)^6}{2657205}\right)y'(1)+O\left((x-1)^6\right) \tag{1}$$

Verification of solutions

$$y = \left(1+\frac{2(x-1)^2}{9}-\frac{2(x-1)^3}{243}+\frac{(x-1)^4}{4374}+\frac{22(x-1)^5}{98415}-\frac{38(x-1)^6}{2657205}\right)y(1) + \left(x-1-\frac{(x-1)^2}{18}+\frac{14(x-1)^3}{243}-\frac{7(x-1)^4}{4374}-\frac{154(x-1)^5}{98415}+\frac{266(x-1)^6}{2657205}\right)y'(1) + O\left((x-1)^6\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
Order:=6;
dsolve((x^2-2*x+10)*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1 + \frac{2(x-1)^2}{9} - \frac{2(x-1)^3}{243} + \frac{(x-1)^4}{4374} + \frac{22(x-1)^5}{98415}\right) y(1) + \left(x - 1 - \frac{(x-1)^2}{18} + \frac{14(x-1)^3}{243} - \frac{7(x-1)^4}{4374} - \frac{154(x-1)^5}{98415}\right) D(y)(1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 87

```
AsymptoticDSolveValue[(x^2-2*x+10)*y''[x]+x*y'[x]-4*y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1\left(\frac{22(x-1)^5}{98415} + \frac{(x-1)^4}{4374} - \frac{2}{243}(x-1)^3 + \frac{2}{9}(x-1)^2 + 1\right) + c_2\left(-\frac{154(x-1)^5}{98415} - \frac{7(x-1)^4}{4374} + \frac{14}{243}(x-1)^3 - \frac{1}{18}(x-1)^2 + x - 1\right)$$

1.5 problem 17

Internal problem ID [5537]
Internal file name [OUTPUT/4785_Sunday_June_05_2022_03_05_50_PM_79571363/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - xy = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (13)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (14)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= xy \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= xy' + y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= yx^2 + 2y' \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x(xy' + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= yx^3 + 6xy' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= y(0) \\
F_2 &= 2y'(0) \\
F_3 &= 0 \\
F_4 &= 4y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x + \frac{1}{12}x^4\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = x\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\sum_{n\,=0}^{\infty} \left(-x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n\,=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n\,=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right)+\sum_{n\,=1}^{\infty} \left(-a_{n-1} x^n\right) = 0 \tag{3}$$

For $1\le n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{6} a_0 x^3 + \frac{1}{12} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^3}{6}\right) a_0 + \left(x + \frac{1}{12} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O(x^6)$$

<ins>Summary</ins>
The solution(s) found are the following

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x + \frac{1}{12} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O(x^6) \tag{2}$$

<ins>Verification of solutions</ins>

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x + \frac{1}{12} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Verified OK.

1.5.1 Maple step by step solution

Let's solve

$y'' = xy$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- > k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k + 2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_{k-1} = 0$

- Shift index using $k- > k + 1$

$$\left((k+1)^2+3k+5\right)a_{k+3}-a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+3}=\frac{a_k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1+\frac{x^3}{6}\right)y(0)+\left(x+\frac{1}{12}x^4\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]-x*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(\frac{x^4}{12}+x\right)+c_1\left(\frac{x^3}{6}+1\right)$$

1.6 problem 18

Internal problem ID [5538]
Internal file name [OUTPUT/4786_Sunday_June_05_2022_03_05_51_PM_99063011/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + yx^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\ &= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (16) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (17) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned} F_0 &= f(x,y,y') \qquad (4)\\ F_1 &= \frac{df}{dx}\\ &= \frac{dF_0}{dx}\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\ F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\ &= \frac{d}{dx}(F_1)\\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\ &\vdots\\ F_n &= \frac{d}{dx}(F_{n-1})\\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6) \end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -yx^2 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -x(xy' + 2y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= yx^4 - 4xy' - 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y'x^4 + 8yx^3 - 6y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12y'x^3 - x^2 y\left(x^4 - 30\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -2y(0) \\
F_3 &= -6y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) + a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-2}}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{12} a_0 x^4 - \frac{1}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x - \frac{1}{20} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Verified OK.

1.6.1 Maple step by step solution

Let's solve

$y'' = -y x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + yx^2 = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- Convert $x^2 \cdot y$ to series expansion

$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

- Shift index using $k- > k - 2$

$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$

- Convert y'' to series expansion

$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k- > k + 2$

$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

$[2a_2 = 0, 6a_3 = 0]$

- Solve for the dependent coefficient(s)

$\{a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} + a_{k-2} = 0$

- Shift index using $k- > k + 2$

$\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_k = 0$

- Recursion relation that defines the series solution to the ODE

$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{x^5}{20}\right) + c_1\left(1 - \frac{x^4}{12}\right)$$

1.7 problem 19

Internal problem ID [5539]
Internal file name [`OUTPUT/4787_Sunday_June_05_2022_03_05_52_PM_25384544/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{y'' - 2xy' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{19}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{20}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= 2xy' - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= 4x^2y' - 2xy + y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= 8y'x^3 - 4yx^2 + 8xy' - 3y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(16x^4 + 36x^2 + 5\right) y' - 8yx\left(x^2+2\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(32x^5 + 128x^3 + 66x\right) y' + \left(-16x^4 - 60x^2 - 21\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= y'(0) \\
F_2 &= -3y(0) \\
F_3 &= 5y'(0) \\
F_4 &= -21y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2na_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(2n-1)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 7a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 9a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{112}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 - \frac{1}{8} a_0 x^4 + \frac{1}{24} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4\right) a_0 + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5\right) c_2 + O(x^6)$$

Verified OK.

1.7.1 Maple step by step solution

Let's solve

$y'' = 2xy' - y$

- Highest derivative means the order of the ODE is 2

y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - 2xy' + y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(2k-1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$(k^2+3k+2)\, a_{k+2} - 2a_k k + a_k = 0$

- Recursion relation that defines the series solution to the ODE

$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k(2k-1)}{k^2+3k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]-2*x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{24} + \frac{x^3}{6} + x\right) + c_1\left(-\frac{x^4}{8} - \frac{x^2}{2} + 1\right)$$

1.8 problem 20

Internal problem ID [5540]
Internal file name [`OUTPUT/4788_Sunday_June_05_2022_03_05_53_PM_82539984/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Hermite]`

$$\boxed{y'' - xy' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (22)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (23)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$\begin{aligned}
F_0 &= xy' - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x^2 y' - 2xy - y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 - x\right) y' - 2yx^2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y'x^4 - 2yx^3 - 2xy - y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= x\left(x^2+3\right)\left(\left(x^2-1\right) y' - 2xy\right)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -y'(0) \\
F_2 &= 0 \\
F_3 &= -y'(0) \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(-x^2+1\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right) y'(0) + O\left(x^6\right)$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-n\, x^n a_n\right)+\left(\sum_{n=0}^{\infty} 2a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^n\right)+\sum_{n=1}^{\infty}\left(-n\, x^n a_n\right)+\left(\sum_{n=0}^{\infty} 2a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$2a_2+2a_0=0$$

$$a_2=-a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) - n a_n + 2 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n-2)}{(n+2)(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{1680}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{6} a_1 x^3 - \frac{1}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(-x^2 + 1\right) a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(-x^2 + 1\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(-x^2 + 1\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(-x^2 + 1\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(-x^2 + 1\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right) y'(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(-x^2 + 1\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right) c_2 + O\left(x^6\right)$$

Verified OK.

1.8.1 Maple step by step solution

Let's solve

$y'' = xy' - 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - xy' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_k(k-2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - a_k(k-2) = 0$

- Recursion relation; series terminates at $k = 2$

  $a_{k+2} = \frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

  $a_2 = -a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y = A_2x^2 + A_1x - a_0$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
Order:=6;
dsolve(diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (-x^2+1)\,y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right) D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 33

```
AsymptoticDSolveValue[y''[x]-x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1(1-x^2) + c_2\left(-\frac{x^5}{120} - \frac{x^3}{6} + x\right)$$

1.9 problem 21

Internal problem ID [5541]
Internal file name [OUTPUT/4789_Sunday_June_05_2022_03_05_54_PM_2370240/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + x^2 y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (25)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (26)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -x^2y' - xy \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \left(x^4 - 3x\right) y' + y\left(x^3 - 1\right) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \left(-x^6 + 8x^3 - 4\right) y' - yx^2\left(x^3 - 6\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= x\left(\left(x^7 - 15x^4 + 34x\right) y' + y\left(x^6 - 13x^3 + 16\right)\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(-x^{10} + 24x^7 - 122x^4 + 84x\right) y' - y\left(x^9 - 22x^6 + 86x^3 - 16\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -4y'(0) \\
F_3 &= 0 \\
F_4 &= 16y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{45}x^6\right) y(0) + \left(x - \frac{1}{6}x^4\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}=-x^2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-x\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n\,x^{1+n} a_n\right)+\left(\sum_{n=0}^{\infty} x^{1+n} a_n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)\,a_{n+2}(1+n)\,x^n$$

$$\sum_{n=1}^{\infty} n\,x^{1+n} a_n=\sum_{n=2}^{\infty} (n-1)\,a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n=\sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\,a_{n+2}(1+n)\,x^n\right)+\left(\sum_{n=2}^{\infty} (n-1)\,a_{n-1} x^n\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^n\right)=0 \tag{3}$$

$n = 1$ gives

$$6a_3 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + (n-1)\, a_{n-1} + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-1} n}{(n+2)\,(1+n)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{5a_1}{252}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{6} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{6}\right) a_0 + \left(x - \frac{1}{6} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{6} x^4\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{45} x^6\right) y(0) + \left(x - \frac{1}{6} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{6} x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{45} x^6\right) y(0) + \left(x - \frac{1}{6} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{6} x^4\right) c_2 + O(x^6)$$

Verified OK.

1.9.1 Maple step by step solution

Let's solve

$y'' = -x^2y' - xy$

- Highest derivative means the order of the ODE is 2

y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + x^2y' + xy = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- > k - 1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  - Shift index using $k- > k - 1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  - Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- > k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-1}k\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} + a_{k-1}k = 0$

- Shift index using $k- >k + 1$

  $\left((k + 1)^2 + 3k + 5\right) a_{k+3} + a_k(k + 1) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k(k+1)}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x^2*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^3}{6}\right) y(0) + \left(x - \frac{1}{6}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x^2*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{x^4}{6}\right) + c_1\left(1 - \frac{x^3}{6}\right)$$

1.10 problem 22

Internal problem ID [5542]
Internal file name [OUTPUT/4790_Sunday_June_05_2022_03_05_55_PM_93081706/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + 2xy' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f\big|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'\big|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (28) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (29) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2xy' - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4x^2 y' + 4xy - 4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -8y' x^3 - 8yx^2 + 20xy' + 12y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^4 - 72x^2 + 32\right) y' + \left(16x^3 - 56x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-32x^5 + 224x^3 - 264x\right) y' - 32\left(x^4 - 6x^2 + \frac{15}{4}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -4y'(0) \\
F_2 &= 12y(0) \\
F_3 &= 32y'(0) \\
F_4 &= -120y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=-2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right)+\left(\sum_{n=0}^{\infty} 2a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right)+\left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right)+\left(\sum_{n=0}^{\infty} 2a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$2a_2+2a_0=0$$

$$a_2=-a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) + 2na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 8a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{4a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 10a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{6}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 12a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{8a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{2}{3} a_1 x^3 + \frac{1}{2} a_0 x^4 + \frac{4}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) a_0 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5\right) c_2 + O(x^6)$$

Verified OK.

1.10.1 Maple step by step solution

Let's solve

$y'' = -2xy' - 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2xy' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_k(k+1))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\,(a_{k+2}(k+2) + 2a_k) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{1}{2}x^4\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[y''[x]+2*x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{4x^5}{15} - \frac{2x^3}{3} + x\right) + c_1\left(\frac{x^4}{2} - x^2 + 1\right)$$

1.11 problem 23

Internal problem ID [5543]
Internal file name [`OUTPUT/4791_Sunday_June_05_2022_03_05_56_PM_13245501/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second order series method. Ordinary point", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(x-1)\,y''+y'=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (31) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (32) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y'}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{2y'}{(x-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{6y'}{(x-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{24y'}{(x-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{120y'}{(x-1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y'(0) \\
F_1 &= 2y'(0) \\
F_2 &= 6y'(0) \\
F_3 &= 24y'(0) \\
F_4 &= 120y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x-1)\, y'' + y' = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x-1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n$$
$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2} (n+1)\, x^n\right)$$
$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^{n}\right) + \sum_{n\,=0}^{\infty} \left(-(n+2)\, a_{n+2} (n+1)\, x^{n}\right) + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^{n}\right) = 0 \quad (3)$$

$n = 0$ gives

$$-2a_2 + a_1 = 0$$

$$a_2 = \frac{a_1}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n - (n+2)\, a_{n+2} (n+1) + (n+1)\, a_{n+1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{(n+1)\, a_{n+1}}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$4a_2 - 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$9a_3 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$16a_4 - 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$25a_5 - 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_1}{6}$$

For $n = 5$ the recurrence equation gives

$$36a_6 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{7}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{4} a_1 x^4 + \frac{1}{5} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = a_0 + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6\right) y'(0) + O(x^6) \tag{1}$$

$$y = c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6\right) y'(0) + O(x^6)$$

Verified OK.

$$y = c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5\right) c_2 + O(x^6)$$

Verified OK.

1.11.1 Maple step by step solution

Let's solve

$(x-1)\, y'' + y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x-1}, P_3(x) = 0\right]$

  ◦ $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

    $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = 1$

  ◦ $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

$$\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$$

- $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $(x-1)\, y'' + y' = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2} y(u)\right) + \frac{d}{du} y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite DE with series expansions

- Convert $\frac{d}{du} y(u)$ to series expansion

  $\frac{d}{du} y(u) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1}$

- Shift index using $k- > k+1$

  $\frac{d}{du} y(u) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$

- Shift index using $k- > k+1$

  $u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=-1}^{\infty} a_{k+1} (k+1+r)^2\, u^{k+r} = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+1)^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = 0$

- Recursion relation for $r = 0$

  $a_{k+1} = 0$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = 0\right]$$

- Revert the change of variables $u = x - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((x-1)*diff(y(x),x$2)+diff(y(x),x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 39

```
AsymptoticDSolveValue[(x-1)*y''[x]+y'[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{5}+\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x\right)+c_1$$

1.12 problem 24

Internal problem ID [5544]
Internal file name [OUTPUT/4792_Sunday_June_05_2022_03_05_57_PM_61031580/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+2)\, y'' + xy' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (34) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (35) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned} F_0 &= f(x, y, y') \qquad (4) \\ F_1 &= \frac{df}{dx} \\ &= \frac{dF_0}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5) \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\ F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\ &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\ &\vdots \\ F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6) \end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{-y + xy'}{x+2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(1+x)(-y+xy')}{(x+2)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{x(-y+xy')}{(x+2)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(-y+xy')(x-1)}{(x+2)^2} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{(-2+x)(-y+xy')}{(x+2)^2}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)}{2} \\
F_1 &= -\frac{y(0)}{4} \\
F_2 &= 0 \\
F_3 &= \frac{y(0)}{4} \\
F_4 &= -\frac{y(0)}{2}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1+\frac{1}{4}x^{2}-\frac{1}{24}x^{3}+\frac{1}{480}x^{5}-\frac{1}{1440}x^{6}\right) y(0)+x y'(0)+O\left(x^{6}\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x+2)\, y''+x y'-y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_{n} x^{n}$$

Then

$$y' = \sum_{n=1}^{\infty} n a_{n} x^{n -1}$$

$$y'' = \sum_{n=2}^{\infty} n \left(n -1\right) a_{n} x^{n -2}$$

Substituting the above back into the ode gives

$$(x+2) \left(\sum_{n=2}^{\infty} n \left(n -1\right) a_{n} x^{n -2}\right)+x \left(\sum_{n=1}^{\infty} n a_{n} x^{n -1}\right)-\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n -1} a_{n} \left(n -1\right)\right)+\left(\sum_{n=2}^{\infty} 2 n \left(n -1\right) a_{n} x^{n -2}\right)+\left(\sum_{n=1}^{\infty} n a_{n} x^{n}\right)+\sum_{n =0}^{\infty} \left(-a_{n} x^{n}\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the power and the corresponding index gives

$$\sum_{n =2}^{\infty} n\, x^{n -1} a_{n} \left(n -1\right) = \sum_{n=1}^{\infty} \left(n +1\right) a_{n +1} n\, x^{n}$$

$$\sum_{n =2}^{\infty} 2 n \left(n -1\right) a_{n} x^{n -2} = \sum_{n=0}^{\infty} 2 \left(n +2\right) a_{n +2} \left(n +1\right) x^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^{n}\right)+\left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^{n}\right) \\
&+\left(\sum_{n=1}^{\infty} n a_{n} x^{n}\right)+\sum_{n=0}^{\infty} \left(-a_{n} x^{n}\right)=0
\end{aligned}
\tag{3}
$$

$n=0$ gives

$$
4a_{2}-a_{0}=0
$$

$$
a_{2}=\frac{a_{0}}{4}
$$

For $1 \leq n$, the recurrence equation is

$$
(n+1)\, a_{n+1} n+2(n+2)\, a_{n+2}(n+1)+n a_{n}-a_{n}=0 \tag{4}
$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n^{2} a_{n+1}+n a_{n}+n a_{n+1}-a_{n}}{2\,(n+2)\,(n+1)} \\
&= -\frac{(n-1)\, a_{n}}{2\,(n+2)\,(n+1)}-\frac{(n^{2}+n)\, a_{n+1}}{2\,(n+2)\,(n+1)}
\end{aligned}
\tag{5}
$$

For $n=1$ the recurrence equation gives

$$
2a_{2}+12a_{3}=0
$$

Which after substituting the earlier terms found becomes

$$
a_{3}=-\frac{a_{0}}{24}
$$

For $n=2$ the recurrence equation gives

$$
6a_{3}+24a_{4}+a_{2}=0
$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$12a_4 + 40a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{480}$$

For $n = 4$ the recurrence equation gives

$$20a_5 + 60a_6 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{1440}$$

For $n = 5$ the recurrence equation gives

$$30a_6 + 84a_7 + 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{6720}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{4} a_0 x^2 - \frac{1}{24} a_0 x^3 + \frac{1}{480} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1+\frac{1}{4}x^{2}-\frac{1}{24}x^{3}+\frac{1}{480}x^{5}\right)a_{0}+a_{1}x+O\left(x^{6}\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1+\frac{1}{4}x^{2}-\frac{1}{24}x^{3}+\frac{1}{480}x^{5}\right)c_{1}+c_{2}x+O\left(x^{6}\right)$$

Summary
The solution(s) found are the following

$$y = \left(1+\frac{1}{4}x^{2}-\frac{1}{24}x^{3}+\frac{1}{480}x^{5}-\frac{1}{1440}x^{6}\right)y(0)+xy'(0)+O\left(x^{6}\right) \tag{1}$$

$$y = \left(1+\frac{1}{4}x^{2}-\frac{1}{24}x^{3}+\frac{1}{480}x^{5}\right)c_{1}+c_{2}x+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$y = \left(1+\frac{1}{4}x^{2}-\frac{1}{24}x^{3}+\frac{1}{480}x^{5}-\frac{1}{1440}x^{6}\right)y(0)+xy'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1+\frac{1}{4}x^{2}-\frac{1}{24}x^{3}+\frac{1}{480}x^{5}\right)c_{1}+c_{2}x+O\left(x^{6}\right)$$

Verified OK.

1.12.1 Maple step by step solution

Let's solve

$(x+2)\,y''+xy'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{x+2}-\frac{xy'}{x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{xy'}{x+2}-\frac{y}{x+2}=0$

☐ Check to see if $x_0 = -2$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{x}{x+2}, P_3(x) = -\frac{1}{x+2}\right]$

- $(x+2)\cdot P_2(x)$ is analytic at $x = -2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2} = -2$

- $(x+2)^2\cdot P_3(x)$ is analytic at $x = -2$

  $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2} = 0$

- $x = -2$is a regular singular point

  Check to see if $x_0 = -2$ is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $(x+2)\,y'' + xy' - y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (u-2)\left(\frac{d}{du}y(u)\right) - y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- Shift index using $k- >k+1$

$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-1)}{(k+1+r)(k-2+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = -\frac{a_k(k-1)}{(k+1)(k-2)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0}{2}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$

- Revert the change of variables $u = x + 2$

  $\left[y = -\frac{a_0 x}{2}\right]$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}$

- Solution for $r = 3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$

- Revert the change of variables $u = x + 2$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+3}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$

- Combine solutions and rename parameters

$$\left[y=-\frac{a_0x}{2}+\left(\sum_{k=0}^{\infty} b_k (x+2)^{k+3}\right), b_{k+1}=-\frac{b_k(k+2)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 29

```
Order:=6;
dsolve((x+2)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1+\frac{1}{4}x^2-\frac{1}{24}x^3+\frac{1}{480}x^5\right)y(0)+D(y)(0)\,x+O\left(x^6\right)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[(x+2)*y''[x]+x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^5}{480}-\frac{x^3}{24}+\frac{x^2}{4}+1\right)+c_2x$$

1.13 problem 25

Internal problem ID [5545]
Internal file name [`OUTPUT/4793_Sunday_June_05_2022_03_05_58_PM_40041956/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - (1+x)\,y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (37) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (38) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n\big|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= xy' + y' + y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^2 + 2x + 3\right) y' + (1 + x)\, y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 + 3x^2 + 8x + 6\right) y' + y\left(x^2 + 2x + 4\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 + 4x^3 + 15x^2 + 22x + 18\right) y' + y(1 + x)\left(x^2 + 2x + 8\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^5 + 5x^4 + 24x^3 + 52x^2 + 80x + 48\right) y' + y\left(x^4 + 4x^3 + 18x^2 + 28x + 28\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) + y'(0) \\
F_1 &= 3y'(0) + y(0) \\
F_2 &= 6y'(0) + 4y(0) \\
F_3 &= 18y'(0) + 8y(0) \\
F_4 &= 48y'(0) + 28y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{15}x^5 + \frac{7}{180}x^6\right) y(0) \\
& + \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{3}{20}x^5 + \frac{1}{15}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-(n+1) a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=0}^{\infty} \left(-(n+1) a_{n+1} x^n\right) + \sum_{n=1}^{\infty} \left(-n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_1 - a_0 = 0$$

$$a_2 = \frac{a_0}{2} + \frac{a_1}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - (n+1)\, a_{n+1} - n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{a_n + a_{n+1}}{n+2} \\ &= \frac{a_n}{n+2} + \frac{a_{n+1}}{n+2} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 2a_2 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6} + \frac{a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_3 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{6} + \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_4 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{15} + \frac{3a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_5 - 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{7a_0}{180} + \frac{a_1}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_6 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{19a_0}{1260} + \frac{13a_1}{420}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(\frac{a_0}{2} + \frac{a_1}{2}\right) x^2 + \left(\frac{a_0}{6} + \frac{a_1}{2}\right) x^3 + \left(\frac{a_0}{6} + \frac{a_1}{4}\right) x^4 + \left(\frac{a_0}{15} + \frac{3a_1}{20}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{15}x^5\right) a_0 + \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{3}{20}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{15}x^5\right) c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{3}{20}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{6}x^{4}+\frac{1}{15}x^{5}+\frac{7}{180}x^{6}\right) y(0) \\
& + \left(x+\frac{1}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{3}{20}x^{5}+\frac{1}{15}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned} \tag{1}
$$

$$
y = \left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{6}x^{4}+\frac{1}{15}x^{5}\right) c_{1}+\left(x+\frac{1}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{3}{20}x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{6}x^{4}+\frac{1}{15}x^{5}+\frac{7}{180}x^{6}\right) y(0) \\
& + \left(x+\frac{1}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{3}{20}x^{5}+\frac{1}{15}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

$$
y = \left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{6}x^{4}+\frac{1}{15}x^{5}\right) c_{1}+\left(x+\frac{1}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{3}{20}x^{5}\right) c_{2}+O\left(x^{6}\right)
$$

Verified OK.

1.13.1 Maple step by step solution

Let's solve

$y'' = xy' + y' + y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (-1 - x)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) - a_k(k+1)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+1)\left(a_{k+2}(k+2) - a_{k+1} - a_k\right) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_{k+1}+a_k}{k+2}\right]$$

<u>Maple trace</u>

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
Order:=6;
dsolve(diff(y(x),x$2)-(x+1)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{15}x^5\right) y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{3}{20}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 70

```
AsymptoticDSolveValue[y''[x]-(x+1)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^5}{15} + \frac{x^4}{6} + \frac{x^3}{6} + \frac{x^2}{2} + 1\right) + c_2\left(\frac{3x^5}{20} + \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{2} + x\right)$$

1.14 problem 26

Internal problem ID [5546]
Internal file name [`OUTPUT/4794_Sunday_June_05_2022_03_05_59_PM_14832022/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(x^2+1)\,y''-6y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{40}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{41}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{6y}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{6x^2 y' - 12xy + 6y'}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-24y'x^3 + 72yx^2 - 24xy' + 24y}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{144((x^3+x)\,y' + (-3x^2-1)\,y)\,x}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{1008((x^3+x)\,y' + (-3x^2-1)\,y)\left(x^2 - \frac{1}{7}\right)}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 6y(0) \\
F_1 &= 6y'(0) \\
F_2 &= 24y(0) \\
F_3 &= 0 \\
F_4 &= -144y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6\right) y(0) + \left(x^3 + x\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2+1)\, y'' - 6y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 6\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-6 a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} (-6 a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 6a_0 = 0$$

$$a_2 = 3a_0$$

$n = 1$ gives

$$6a_3 - 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = a_1$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + (n+2) a_{n+2} (n+1) - 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{(n-3) a_n}{n+1} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-4a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = a_0$$

For $n = 3$ the recurrence equation gives

$$20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$6a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{5}$$

For $n = 5$ the recurrence equation gives

$$14a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 x^4 + a_1 x^3 + 3a_0 x^2 + a_1 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = \left(x^4 + 3x^2 + 1\right) a_0 + \left(x^3 + x\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(x^4 + 3x^2 + 1\right) c_1 + \left(x^3 + x\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6\right) y(0) + \left(x^3 + x\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(x^4 + 3x^2 + 1\right) c_1 + \left(x^3 + x\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1+3x^2+x^4-\frac{1}{5}x^6\right) y(0)+\left(x^3+x\right) y'(0)+O\left(x^6\right)$$

Verified OK.

$$y = \left(x^4+3x^2+1\right) c_1+\left(x^3+x\right) c_2+O\left(x^6\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
Order:=6;
dsolve((x^2+1)*diff(y(x),x$2)-6*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(x^4+3x^2+1\right) y(0)+\left(x^3+x\right) D(y)\left(0\right)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 25

```
AsymptoticDSolveValue[(x^2+1)*y''[x]-6*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x^3+x\right)+c_1\left(x^4+3x^2+1\right)$$

1.15 problem 27

Internal problem ID [5547]
Internal file name [`OUTPUT/4795_Sunday_June_05_2022_03_06_00_PM_34013847/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+2)\, y''+3xy'-y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy_0'+\frac{x^2}{2}\, f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'}+\cdots \\ &= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (43)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (44)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$\begin{aligned}
F_0 &= -\frac{3xy'-y}{x^2+2}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= \frac{13x^2y'-5xy-4y'}{\left(x^2+2\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= \frac{-70y'x^3+28yx^2+70xy'-14y}{\left(x^2+2\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \frac{\left(448x^4-938x^2+112\right)y'+\left(-182x^3+266x\right)y}{\left(x^2+2\right)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= \frac{\left(-3318x^5+11928x^3-4452x\right)y'+\left(1358x^4-3892x^2+644\right)y}{\left(x^2+2\right)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$\begin{aligned}
F_0 &= \frac{y(0)}{2}\\
F_1 &= -y'(0)\\
F_2 &= -\frac{7y(0)}{4}\\
F_3 &= 7y'(0)\\
F_4 &= \frac{161y(0)}{8}
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1+\frac{1}{4}x^2-\frac{7}{96}x^4+\frac{161}{5760}x^6\right)y(0)+\left(x-\frac{1}{6}x^3+\frac{7}{120}x^5\right)y'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2+2)\,y''+3xy'-y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2+2)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+3x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} 2n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 3n a_n x^n\right)+\sum_{n=0}^{\infty}(-a_n x^n)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty} 2(n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=0}^{\infty} 2(n+2) a_{n+2}(n+1) x^{n}\right) \\ &+\left(\sum_{n=1}^{\infty} 3 n a_{n} x^{n}\right)+\sum_{n=0}^{\infty}\left(-a_{n} x^{n}\right)=0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$4a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{4}$$

$n = 1$ gives

$$12a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + 2(n+2) a_{n+2}(n+1) + 3n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 + 2n - 1)}{2(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$7a_2 + 24a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{7a_0}{96}$$

For $n = 3$ the recurrence equation gives

$$14a_3 + 40a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$23a_4 + 60a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{161a_0}{5760}$$

For $n = 5$ the recurrence equation gives

$$34a_5 + 84a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{17a_1}{720}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{4} a_0 x^2 - \frac{1}{6} a_1 x^3 - \frac{7}{96} a_0 x^4 + \frac{7}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4\right)c_1 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4 + \frac{161}{5760}x^6\right)y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right)y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4\right)c_1 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4 + \frac{161}{5760}x^6\right)y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right)y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4\right)c_1 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right)c_2 + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((x^2+2)*diff(y(x),x$2)+3*x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[(x^2+2)*y''[x]+3*x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{7x^5}{120} - \frac{x^3}{6} + x\right) + c_1\left(-\frac{7x^4}{96} + \frac{x^2}{4} + 1\right)$$

1.16 problem 28

Internal problem ID [5548]
Internal file name [`OUTPUT/4796_Sunday_June_05_2022_03_06_02_PM_21756012/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2-1)\,y''+xy'-y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \tag{46}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{47}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{-y+xy'}{x^2-1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= \frac{3(-y+xy')\,x}{(x^2-1)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= -\frac{3(-y+xy')\,(4x^2+1)}{(x^2-1)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \frac{60(-y+xy')\left(x^2+\frac{3}{4}\right)x}{(x^2-1)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= -\frac{45(-y+xy')\,(8x^4+12x^2+1)}{(x^2-1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)\\
F_1 &= 0\\
F_2 &= -3y(0)\\
F_3 &= 0\\
F_4 &= -45y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y=\left(1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6\right)y(0)+xy'(0)+O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2-1)\,y''+xy'-y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2-1)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\sum_{n=2}^{\infty}\left(-n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-n(n-1)\,a_n x^{n-2}\right)=\sum_{n=0}^{\infty}\left(-(n+2)\,a_{n+2}(n+1)\,x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\sum_{n=0}^{\infty}\left(-(n+2)\,a_{n+2}(n+1)\,x^n\right)+\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-a_n x^n\right)=0 \tag{3}$$

$n = 0$ gives

$$-2a_2 - a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) - (n+2)\, a_{n+2} (n+1) + n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{(n-1)\, a_n}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$3a_2 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$8a_3 - 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$15a_4 - 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{16}$$

For $n = 5$ the recurrence equation gives

$$24a_5 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{8} a_0 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4\right) a_0 + a_1 x + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4\right) c_1 + c_2 x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6\right) y(0) + x y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4\right) c_1 + c_2 x + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6\right) y(0) + x y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4\right) c_1 + c_2 x + O(x^6)$$

Verified OK.

1.16.1 Maple step by step solution

Let's solve

$(x^2-1)\,y''+xy'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{xy'}{x^2-1}+\frac{y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{xy'}{x^2-1}-\frac{y}{x^2-1}=0$

- Multiply by denominators of ODE

  $(-x^2+1)\,y''-xy'+y=0$

- Make a change of variables

  $\theta=\arccos(x)$

- Calculate y' with change of variables

  $y'=\left(\frac{d}{d\theta}y(\theta)\right)\theta'(x)$

- Compute 1st derivative y'

  $y'=-\frac{\frac{d}{d\theta}y(\theta)}{\sqrt{-x^2+1}}$

- Calculate y'' with change of variables

  $y''=\left(\frac{d^2}{d\theta^2}y(\theta)\right)\theta'(x)^2+\theta''(x)\left(\frac{d}{d\theta}y(\theta)\right)$

- Compute 2nd derivative y''

  $y''=\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1}-\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}$

- Apply the change of variables to the ODE

  $(-x^2+1)\left(\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1}-\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}\right)+\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\sqrt{-x^2+1}}+y=0$

- Multiply through

  $-\frac{\left(\frac{d^2}{d\theta^2}y(\theta)\right)x^2}{-x^2+1}+\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1}+\frac{x^3\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}-\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}+\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\sqrt{-x^2+1}}+y=0$

- Simplify ODE

$y + \frac{d^2}{d\theta^2} y(\theta) = 0$

- ODE is that of a harmonic oscillator with given general solution

  $y(\theta) = c_1 \sin(\theta) + c_2 \cos(\theta)$

- Revert back to x

  $y = c_1 \sin(\arccos(x)) + c_2 \cos(\arccos(x))$

- Use trig identity to simplify $\sin(\arccos(x))$

  $\sin(\arccos(x)) = \sqrt{-x^2 + 1}$

- Simplify solution to the ODE

  $y = c_1 \sqrt{-x^2 + 1} + c_2 x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve((x^2-1)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4\right) y(0) + D(y)(0)\, x + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 27

```
AsymptoticDSolveValue[(x^2-1)*y''[x]+x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{x^4}{8} - \frac{x^2}{2} + 1\right) + c_2 x$$

1.17 problem 29

Internal problem ID [5549]
Internal file name [`OUTPUT/4797_Sunday_June_05_2022_03_06_03_PM_6727615/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-xy'+y=0}$$

With initial conditions

$$[y(0)=-2, y'(0)=6]$$

With the expansion point for the power series method at $x=0$.

1.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{x}{x-1} \\
q(x) &= \frac{1}{x-1} \\
F &= 0
\end{aligned}
$$

Hence the ode is

$$
y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0
$$

The domain of $p(x) = -\frac{x}{x-1}$ is

$$
\{x < 1 \lor 1 < x\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{x-1}$ is

$$
\{x < 1 \lor 1 < x\}
$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$
y'' = f(x, y, y')
$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (49)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (50)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{-y + xy'}{x - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-y + xy'}{x - 1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-y + xy'}{x - 1} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{-y + xy'}{x - 1} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{-y + xy'}{x - 1}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = -2$ and $y'(0) = 6$ gives

$$
\begin{aligned}
F_0 &= -2 \\
F_1 &= -2 \\
F_2 &= -2 \\
F_3 &= -2 \\
F_4 &= -2
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -x^2 + 6x - 2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360} + O(x^6)
$$

$$
y=-x^{2}+6x-2-\frac{x^{3}}{3}-\frac{x^{4}}{12}-\frac{x^{5}}{60}-\frac{x^{6}}{360}+O\left(x^{6}\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x-1\right) y''-x y'+y=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &=\sum_{n=1}^{\infty} n a_{n} x^{n-1}\\
y'' &=\sum_{n=2}^{\infty} n \left(n-1\right) a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(x-1\right) \left(\sum_{n=2}^{\infty} n \left(n-1\right) a_{n} x^{n-2}\right)-x \left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)=0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_{n} \left(n-1\right)\right)+\sum_{n =2}^{\infty} \left(-n \left(n-1\right) a_{n} x^{n-2}\right)+\sum_{n =1}^{\infty} \left(-n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)=0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n =2}^{\infty} n\, x^{n-1} a_{n} \left(n-1\right) &= \sum_{n=1}^{\infty} \left(n+1\right) a_{n+1} n\, x^{n}\\
\sum_{n =2}^{\infty} \left(-n \left(n-1\right) a_{n} x^{n-2}\right) &= \sum_{n=0}^{\infty} \left(-\left(n+2\right) a_{n+2} \left(n+1\right) x^{n}\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^{n}\right) + \sum_{n\, =0}^{\infty} \left(-(n+2)\, a_{n+2} (n+1)\, x^{n}\right) \\ + \sum_{n\, =1}^{\infty} \left(-n a_{n} x^{n}\right) + \left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 + a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n - (n+2)\, a_{n+2}(n+1) - n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_{n+1} - n a_n + n a_{n+1} + a_n}{(n+2)\,(n+1)} \\ &= \frac{(-n+1)\, a_n}{(n+2)\,(n+1)} + \frac{(n^2+n)\, a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 - 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$6a_3 - 12a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$12a_4 - 20a_5 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{120}$$

For $n = 4$ the recurrence equation gives

$$20a_5 - 30a_6 - 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$30a_6 - 42a_7 - 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_0 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) a_0 + a_1 x + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) c_1 + c_2 x + O(x^6)$$

$$y = -2 - x^2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} + 6x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -x^2 + 6x - 2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360} + O(x^6) \tag{1}$$

$$y = -2 - x^2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} + 6x + O(x^6) \tag{2}$$

Verification of solutions

$$y = -x^2 + 6x - 2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{360} + O(x^6)$$

Verified OK.

$$y = -2 - x^2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} + 6x + O(x^6)$$

Verified OK.

1.17.2 Maple step by step solution

Let's solve

$$\left[(x-1)\,y'' - xy' + y = 0, y(0) = -2, y'\Big|_{\{x=0\}} = 6\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- Define functions

$\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

$\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

$\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

- $x = 1$is a regular singular point

Check to see if $x_0 = 1$ is a regular singular point

$x_0 = 1$

- Multiply by denominators

$(x-1)\,y'' - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
Order:=6;
dsolve([(x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(0) = -2, D(y)(0) = 6],y(x),type='series
```

$$y(x)=-2+6x-x^2-\frac{1}{3}x^3-\frac{1}{12}x^4-\frac{1}{60}x^5+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{(x-1)*y''[x]-x*y'[x]+y[x]==0,{y[0]==-2,y'[0]==6}},y[x],{x,0,5}]
```

$$y(x)\to -\frac{x^5}{60}-\frac{x^4}{12}-\frac{x^3}{3}-x^2+6x-2$$

1.18 problem 30

Internal problem ID [5550]
Internal file name [`OUTPUT/4798_Sunday_June_05_2022_03_06_04_PM_70677692/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(1+x)\,y''-(2-x)\,y'+y=0}$$

With initial conditions

$$[y(0)=2, y'(0)=-1]$$

With the expansion point for the power series method at $x=0$.

1.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{-2+x}{1+x}\\q(x)&=\frac{1}{1+x}\\F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{(-2+x)\, y'}{1+x} + \frac{y}{1+x} = 0$$

The domain of $p(x) = \frac{-2+x}{1+x}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{1+x}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{52}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{53}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{xy' - 2y' + y}{1 + x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x^2 - 5x) y' + (x - 1) y}{(1 + x)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-x^3 + 8x^2 - 3x - 6) y' - y(x^2 - 4x - 3)}{(1 + x)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^3 - 12x^2 + 23x + 6) y' + y(x^2 - 8x + 1)}{(1 + x)^3} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-x^3 + 16x^2 - 55x + 18) y' - y(x^2 - 12x + 17)}{(1 + x)^3}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 2$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= -4 \\
F_1 &= -2 \\
F_2 &= 12 \\
F_3 &= -4 \\
F_4 &= -52
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -2x^2 - x + 2 - \frac{x^3}{3} + \frac{x^4}{2} - \frac{x^5}{30} - \frac{13x^6}{180} + O\left(x^6\right)
$$

$$
y = -2x^2 - x + 2 - \frac{x^3}{3} + \frac{x^4}{2} - \frac{x^5}{30} - \frac{13x^6}{180} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(1+x)\, y'' + (-2+x)\, y' + y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

$$
y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
$$

Substituting the above back into the ode gives

$$
(1+x)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + (-2+x)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{gathered}
\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \\
+ \sum_{n=1}^{\infty} \left(-2n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0
\end{gathered} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n
$$

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n
$$

$$
\sum_{n=1}^{\infty} \left(-2n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-2(n+1)\, a_{n+1} x^n\right)
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^{n}\right) \\ + \sum_{n\, =0}^{\infty} \left(-2(n+1)\, a_{n+1} x^{n}\right) + \left(\sum_{n=1}^{\infty} n a_{n} x^{n}\right) + \left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2} + a_1$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n + (n+2)\, a_{n+2}(n+1) - 2(n+1)\, a_{n+1} + n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n a_{n+1} + a_n - 2a_{n+1}}{n+2} \\ &= -\frac{a_n}{n+2} - \frac{(n-2)\, a_{n+1}}{n+2} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$-2a_2 + 6a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8} - \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$4a_4 + 20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{120} + \frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$10a_5 + 30a_6 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{17a_0}{720} + \frac{a_1}{40}$$

For $n = 5$ the recurrence equation gives

$$18a_6 + 42a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{112} - \frac{a_1}{56}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_0}{2} + a_1\right) x^2 - \frac{a_0 x^3}{6} + \left(\frac{a_0}{8} - \frac{a_1}{4}\right) x^4 + \left(\frac{a_0}{120} + \frac{a_1}{20}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y=\left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{8}x^{4}+\frac{1}{120}x^{5}\right)a_{0}+\left(x+x^{2}-\frac{1}{4}x^{4}+\frac{1}{20}x^{5}\right)a_{1}+O\left(x^{6}\right) \quad (3)$$

At $x=0$ the solution above becomes

$$y=\left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{8}x^{4}+\frac{1}{120}x^{5}\right)c_{1}+\left(x+x^{2}-\frac{1}{4}x^{4}+\frac{1}{20}x^{5}\right)c_{2}+O\left(x^{6}\right)$$

$$y=2-2x^{2}-\frac{x^{3}}{3}+\frac{x^{4}}{2}-\frac{x^{5}}{30}-x+O\left(x^{6}\right)$$

Summary

The solution(s) found are the following

$$y=-2x^{2}-x+2-\frac{x^{3}}{3}+\frac{x^{4}}{2}-\frac{x^{5}}{30}-\frac{13x^{6}}{180}+O\left(x^{6}\right) \quad (1)$$

$$y=2-2x^{2}-\frac{x^{3}}{3}+\frac{x^{4}}{2}-\frac{x^{5}}{30}-x+O\left(x^{6}\right) \quad (2)$$

Verification of solutions

$$y=-2x^{2}-x+2-\frac{x^{3}}{3}+\frac{x^{4}}{2}-\frac{x^{5}}{30}-\frac{13x^{6}}{180}+O\left(x^{6}\right)$$

Verified OK.

$$y=2-2x^{2}-\frac{x^{3}}{3}+\frac{x^{4}}{2}-\frac{x^{5}}{30}-x+O\left(x^{6}\right)$$

Verified OK.

1.18.2 Maple step by step solution

Let's solve

$$\left[(1+x)\,y''+(-2+x)\,y'+y=0, y(0)=2, y'\Big|_{\{x=0\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{1+x} - \frac{(-2+x)y'}{1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(-2+x)y'}{1+x} + \frac{y}{1+x} = 0$

□ Check to see if $x_0 = -1$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{-2+x}{1+x}, P_3(x) = \frac{1}{1+x}\right]$

- ○ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = -3$

- ○ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- ○ $x = -1$is a regular singular point

  Check to see if $x_0 = -1$ is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(1+x)\, y'' + (-2+x)\, y' + y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-3+u)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r-1}$$

- Shift index using $k- > k+1$

$$u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-4+r) u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k-3+r) + a_k(k+1+r)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-4+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 4\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+1+r)\left(a_{k+1}(k-3+r) + a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{k-3+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{k-3}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 3$

$$a_{k+1} = -\frac{a_k}{k-3}$$

- Recursion relation for $r = 4$

$$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = 4$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+4}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+4}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(x+1)*diff(y(x),x$2)-(2-x)*diff(y(x),x)+y(x)=0,y(0) = 2, D(y)(0) = -1],y(x),type='se
```

$$y(x) = 2 - x - 2x^2 - \frac{1}{3}x^3 + \frac{1}{2}x^4 - \frac{1}{30}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{(x+1)*y''[x]-(2-x)*y'[x]+y[x]==0,{y[0]==2,y'[0]==-1}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^5}{30} + \frac{x^4}{2} - \frac{x^3}{3} - 2x^2 - x + 2$$

1.19 problem 31

Internal problem ID [5551]
Internal file name [`OUTPUT/4799_Sunday_June_05_2022_03_06_06_PM_93892179/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2xy' + 8y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

1.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 8 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2xy' + 8y = 0$$

The domain of $p(x) = -2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 8$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{55}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{56}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= 2xy' - 8y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4x^2 y' - 16xy - 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 8y' x^3 - 32yx^2 - 20xy' + 32y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^4 - 48x^2 + 12\right) y' + \left(-64x^3 + 96x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 32x\left(\left(x^4 - 3x^2 + \frac{3}{4}\right) y' + \left(-4x^3 + 6x\right) y\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=3$ and $y'(0)=0$ gives

$$
\begin{aligned}
F_0 &= -24 \\
F_1 &= 0 \\
F_2 &= 96 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 4x^4 - 12x^2 + 3 + O\left(x^6\right)
$$

$$
y = 4x^4 - 12x^2 + 3 + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 8\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 8 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 8 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 8a_0 = 0$$

$$a_2 = -4a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2n a_n + 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2a_n(n-4)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 6a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -a_1$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{4a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{10}$$

For $n = 4$ the recurrence equation gives

$$30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{210}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 4a_0 x^2 - a_1 x^3 + \frac{4}{3} a_0 x^4 + \frac{1}{10} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - 4x^2 + \frac{4}{3}x^4\right) a_0 + \left(x - x^3 + \frac{1}{10}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - 4x^2 + \frac{4}{3}x^4\right) c_1 + \left(x - x^3 + \frac{1}{10}x^5\right) c_2 + O(x^6)$$

$$y = 4x^4 - 12x^2 + 3 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 4x^4 - 12x^2 + 3 + O(x^6) \tag{1}$$
$$y = 4x^4 - 12x^2 + 3 + O(x^6) \tag{2}$$

Verification of solutions

$$y = 4x^4 - 12x^2 + 3 + O(x^6)$$

Verified OK.

$$y = 4x^4 - 12x^2 + 3 + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)-2*x*diff(y(x),x)+8*y(x)=0,y(0) = 3, D(y)(0) = 0],y(x),type='series',x
```

$$y(x) = 3 - 12x^2 + 4x^4 + \mathrm{O}(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 22

```
AsymptoticDSolveValue[{y''[x]-2*x*y''[x]+8*y[x]==0,{y[0]==3,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to \frac{16x^5}{5} - 8x^3 - 12x^2 + 3$$

1.20 problem 32

Internal problem ID [5552]
Internal file name [`OUTPUT/4800_Sunday_June_05_2022_03_06_08_PM_59237499/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(x^2+1)\, y''+2xy'=0}$$

With initial conditions

$$[y(0)=0, y'(0)=1]$$

With the expansion point for the power series method at $x=0$.

1.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}
p(x) &= \frac{2x}{x^2+1} \\
q(x) &= 0 \\
F &= 0
\end{aligned}$$

Hence the ode is

$$y'' + \frac{2xy'}{x^2+1} = 0$$

The domain of $p(x) = \frac{2x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (58) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (59) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2xy'}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(6x^2-2)\, y'}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-24x^3+24x)\, y'}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{120\left(x^4-2x^2+\frac{1}{5}\right) y'}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-720x^5+2400x^3-720x)\, y'}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -2 \\
F_2 &= 0 \\
F_3 &= 24 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x - \frac{x^3}{3} + \frac{x^5}{5} + O\left(x^6\right)
$$

$$
y = x - \frac{x^{3}}{3} + \frac{x^{5}}{5} + O\left(x^{6}\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x^{2}+1\right) y'' + 2 x y' = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(x^{2}+1\right)\left(\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}\right) + 2x\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2 n a_{n} x^{n}\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^{n}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^{n}\right) + \left(\sum_{n=1}^{\infty} 2 n a_{n} x^{n}\right) = 0 \tag{3}
$$

$n=1$ gives

$$6a_3+2a_1=0$$

Which after substituting earlier equations, simplifies to

$$a_3=-\frac{a_1}{3}$$

For $2\leq n$, the recurrence equation is

$$na_n(n-1)+(n+2)\,a_{n+2}(n+1)+2na_n=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2}=-\frac{na_n}{n+2} \tag{5}$$

For $n=2$ the recurrence equation gives

$$6a_2+12a_4=0$$

Which after substituting the earlier terms found becomes

$$a_4=0$$

For $n=3$ the recurrence equation gives

$$12a_3+20a_5=0$$

Which after substituting the earlier terms found becomes

$$a_5=\frac{a_1}{5}$$

For $n=4$ the recurrence equation gives

$$20a_4+30a_6=0$$

Which after substituting the earlier terms found becomes

$$a_6=0$$

For $n = 5$ the recurrence equation gives

$$30a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{7}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{3} a_1 x^3 + \frac{1}{5} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = a_0 + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right) c_2 + O(x^6)$$

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} + O(x^6) \tag{1}$$

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} + O(x^6)$$

Verified OK.

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([(x^2+1)*diff(y(x),x$2)+2*x*diff(y(x),x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='series',
```

$$y(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{(x^2+1)*y''[x]+2*x*y'[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{5} - \frac{x^3}{3} + x$$

1.21 problem 33

Internal problem ID [5553]
Internal file name [`OUTPUT/4801_Sunday_June_05_2022_03_06_09_PM_91351806/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \sin(x) y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (61)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (62)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\sin(x)\, y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -y\cos(x) - \sin(x)\, y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2\cos(x)\, y' + y\sin(x)\,(1+\sin(x)) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\sin(x)^2 + 3\sin(x)\right) y' + 4\cos(x)\, y\left(\sin(x) + \frac{1}{4}\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= (6\sin(x)+4)\cos(x)\, y' + y\left((\sin(x)+11)\cos(x)^2 - 2\sin(x) - 7\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -2y'(0) \\
F_3 &= y(0) \\
F_4 &= 4y(0) + 4y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6\right) y(0) + \left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=-\sin(x)\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Expanding $\sin(x)$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}\sin(x)&=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\dots\\&=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7\right)\left(\sum_{n=0}^{\infty} a_n x^n\right)=0$$

Expanding the second term in (1) gives

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+x\cdot\left(\sum_{n=0}^{\infty} a_n x^n\right)-\frac{x^3}{6}\cdot\left(\sum_{n=0}^{\infty} a_n x^n\right)\\&+\frac{x^5}{120}\cdot\left(\sum_{n=0}^{\infty} a_n x^n\right)-\frac{x^7}{5040}\cdot\left(\sum_{n=0}^{\infty} a_n x^n\right)=0\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} x^{1+n} a_n\right)+\sum_{n=0}^{\infty}\left(-\frac{x^{n+3} a_n}{6}\right)\\&+\left(\sum_{n=0}^{\infty}\frac{x^{n+5} a_n}{120}\right)+\sum_{n=0}^{\infty}\left(-\frac{x^{n+7} a_n}{5040}\right)=0\end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) = \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+5} a_n}{120} = \sum_{n=5}^{\infty} \frac{a_{n-5} x^n}{120}$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+7} a_n}{5040}\right) = \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} x^n}{5040}\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right)\\ &+ \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right) + \left(\sum_{n=5}^{\infty} \frac{a_{n-5} x^n}{120}\right) + \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} x^n}{5040}\right) = 0\end{aligned} \tag{3}$$

$n = 1$ gives

$$6a_3 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6}$$

$n = 2$ gives

$$12a_4 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{a_1}{12}$$

$n = 3$ gives

$$20a_5 + a_2 - \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_0}{120}$$

$n = 4$ gives

$$30a_6 + a_3 - \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{a_0}{180} + \frac{a_1}{180}$$

$n = 5$ gives

$$42a_7 + a_4 - \frac{a_2}{6} + \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{a_0}{5040} + \frac{a_1}{504}$$

For $7 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} - \frac{a_{n-3}}{6} + \frac{a_{n-5}}{120} - \frac{a_{n-7}}{5040} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{-5040a_{n-1} + 840a_{n-3} - 42a_{n-5} + a_{n-7}}{5040\,(n+2)\,(1+n)} \\ &= \frac{a_{n-7}}{5040\,(n+2)\,(1+n)} - \frac{a_{n-5}}{120\,(n+2)\,(1+n)} + \frac{a_{n-3}}{6\,(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + \frac{1}{120} a_0 x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) a_0 + \left(x - \frac{1}{12}x^4\right) a_1 + O(x^6) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_1 + \left(x - \frac{1}{12}x^4\right) c_2 + O(x^6)
$$

Summary

The solution(s) found are the following

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6\right) y(0) + \left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6\right) y'(0) + O(x^6) \tag{1}
$$

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_1 + \left(x - \frac{1}{12}x^4\right) c_2 + O(x^6) \tag{2}
$$

Verification of solutions

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6\right) y(0) + \left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6\right) y'(0) + O(x^6)
$$

Verified OK.

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_1 + \left(x - \frac{1}{12}x^4\right) c_2 + O(x^6)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         Equivalence transformation and function parameters: {t = 1/2*t+1/2}, {kappa = -20,
         <- Equivalence to the rational form of Mathieu ODE successful
      <- Mathieu successful
   <- special function solution successful
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      (-t^2+1)^(1/2)*u(t)-t*diff(u(t),t)+(-t^2+1)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
Order:=6;
dsolve(diff(y(x),x$2)+sin(x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y(0) + \left(x - \frac{1}{12}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 35

```
AsymptoticDSolveValue[y''[x]+Sin[x]*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{x^4}{12}\right) + c_1\left(\frac{x^5}{120} - \frac{x^3}{6} + 1\right)$$

1.22 problem 34

Internal problem ID [5554]
Internal file name [`OUTPUT/4802_Sunday_June_05_2022_03_06_11_PM_16163762/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'e^x - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (64)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (65)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y' e^x + y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y' e^{2x} - y' e^x - e^x y + y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= (y + 3y') e^{2x} - y' e^{3x} - 3y' e^x + (-2 e^x + 1) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 5(y + 2y') e^{2x} + (-y - 6y') e^{3x} + y' e^{4x} + (-5 e^x + 1) y' - 5 e^x y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= (20y + 30y') e^{2x} + (-9y - 29y') e^{3x} + (y + 10y') e^{4x} - y' e^{5x} - 11y' e^x + (-10 e^x + 1) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) - y'(0) \\
F_1 &= -y(0) + y'(0) \\
F_2 &= -y'(0) \\
F_3 &= -y(0) + y'(0) \\
F_4 &= 3y(0) - y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 + \frac{1}{2} x^2 - \frac{1}{6} x^3 - \frac{1}{120} x^5 + \frac{1}{240} x^6\right) y(0) \\
&+ \left(x - \frac{1}{2} x^2 + \frac{1}{6} x^3 - \frac{1}{24} x^4 + \frac{1}{120} x^5 - \frac{1}{720} x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \mathrm{e}^{x} + \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Expanding e^{x} as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}
\mathrm{e}^{x} &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \dots \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6
\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \\
&+ \left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6\right)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0
\end{aligned}$$

Expanding the second term in (1) gives

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 1 \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + x \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \\
&+ \frac{x^2}{2} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \frac{x^3}{6} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \frac{x^4}{24} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \\
&+ \frac{x^5}{120} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \frac{x^6}{720} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0
\end{aligned}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\left(\sum_{n=1}^{\infty} n a_n x^{n}\right) \\ +\left(\sum_{n=1}^{\infty} \frac{n\, x^{1+n} a_n}{2}\right)+\left(\sum_{n=1}^{\infty} \frac{n\, x^{n+2} a_n}{6}\right)+\left(\sum_{n=1}^{\infty} \frac{n\, x^{n+3} a_n}{24}\right) \\ +\left(\sum_{n=1}^{\infty} \frac{n\, x^{n+4} a_n}{120}\right)+\left(\sum_{n=1}^{\infty} \frac{n\, x^{n+5} a_n}{720}\right)+\sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n)\, a_{1+n} x^n$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{1+n} a_n}{2} = \sum_{n=2}^{\infty} \frac{(n-1)\, a_{n-1} x^n}{2}$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+2} a_n}{6} = \sum_{n=3}^{\infty} \frac{(n-2)\, a_{n-2} x^n}{6}$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+3} a_n}{24} = \sum_{n=4}^{\infty} \frac{(n-3)\, a_{n-3} x^n}{24}$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+4} a_n}{120} = \sum_{n=5}^{\infty} \frac{(n-4)\, a_{n-4} x^n}{120}$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+5} a_n}{720} = \sum_{n=6}^{\infty} \frac{(n-5)\, a_{n-5} x^n}{720}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=2}^{\infty} \frac{(n-1)\, a_{n-1} x^n}{2}\right) + \left(\sum_{n=3}^{\infty} \frac{(n-2)\, a_{n-2} x^n}{6}\right) + \left(\sum_{n=4}^{\infty} \frac{(n-3)\, a_{n-3} x^n}{24}\right) + \left(\sum_{n=5}^{\infty} \frac{(n-4)\, a_{n-4} x^n}{120}\right) + \left(\sum_{n=6}^{\infty} \frac{(n-5)\, a_{n-5} x^n}{720}\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 - a_0 = 0$$

$$a_2 = \frac{a_0}{2} - \frac{a_1}{2}$$

$n = 1$ gives

$$6a_3 + 2a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$6a_3 + a_0 - a_1 = 0$$

Or

$$a_3 = -\frac{a_0}{6} + \frac{a_1}{6}$$

$n = 2$ gives

$$12a_4 + 3a_3 + a_2 + \frac{a_1}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{a_1}{24}$$

$n = 3$ gives

$$20a_5 + 4a_4 + 2a_3 + a_2 + \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = -\frac{a_0}{120} + \frac{a_1}{120}$$

$n = 4$ gives

$$30a_6 + 5a_5 + 3a_4 + \frac{3a_3}{2} + \frac{a_2}{3} + \frac{a_1}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{a_0}{240} - \frac{a_1}{720}$$

$n = 5$ gives

$$42a_7 + 6a_6 + 4a_5 + 2a_4 + \frac{a_3}{2} + \frac{a_2}{12} + \frac{a_1}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = \frac{a_0}{840} + \frac{a_1}{5040}$$

For $6 \leq n$, the recurrence equation is

$$\begin{aligned}(n+2)\,a_{n+2}(1+n) + (1+n)\,a_{1+n} + na_n + \frac{(n-1)\,a_{n-1}}{2} + \frac{(n-2)\,a_{n-2}}{6} \\ + \frac{(n-3)\,a_{n-3}}{24} + \frac{(n-4)\,a_{n-4}}{120} + \frac{(n-5)\,a_{n-5}}{720} - a_n = 0\end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

(5)

$$\begin{aligned}a_{n+2} &= \\ &-\frac{720na_n + 720na_{1+n} + na_{n-5} + 6na_{n-4} + 30na_{n-3} + 120na_{n-2} + 360na_{n-1} - 720a_n + 720a_{1+n} - 5a_n}{720\,(n+2)\,(1+n)} \\ &= -\frac{(720n-720)\,a_n}{720\,(n+2)\,(1+n)} - \frac{(720n+720)\,a_{1+n}}{720\,(n+2)\,(1+n)} - \frac{(n-5)\,a_{n-5}}{720\,(n+2)\,(1+n)} \\ &\quad - \frac{(6n-24)\,a_{n-4}}{720\,(n+2)\,(1+n)} - \frac{(30n-90)\,a_{n-3}}{720\,(n+2)\,(1+n)} \\ &\quad - \frac{(120n-240)\,a_{n-2}}{720\,(n+2)\,(1+n)} - \frac{(360n-360)\,a_{n-1}}{720\,(n+2)\,(1+n)}\end{aligned}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x + \left(\frac{a_0}{2} - \frac{a_1}{2}\right) x^2 + \left(-\frac{a_0}{6} + \frac{a_1}{6}\right) x^3 - \frac{a_1 x^4}{24} + \left(-\frac{a_0}{120} + \frac{a_1}{120}\right) x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 + \frac{1}{2} x^2 - \frac{1}{6} x^3 - \frac{1}{120} x^5\right) a_0 + \left(x - \frac{1}{2} x^2 + \frac{1}{6} x^3 - \frac{1}{24} x^4 + \frac{1}{120} x^5\right) a_1 + O\left(x^6\right) \quad (3)
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 + \frac{1}{2} x^2 - \frac{1}{6} x^3 - \frac{1}{120} x^5\right) c_1 + \left(x - \frac{1}{2} x^2 + \frac{1}{6} x^3 - \frac{1}{24} x^4 + \frac{1}{120} x^5\right) c_2 + O\left(x^6\right)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1 + \frac{1}{2} x^2 - \frac{1}{6} x^3 - \frac{1}{120} x^5 + \frac{1}{240} x^6\right) y(0) \\
& + \left(x - \frac{1}{2} x^2 + \frac{1}{6} x^3 - \frac{1}{24} x^4 + \frac{1}{120} x^5 - \frac{1}{720} x^6\right) y'(0) + O\left(x^6\right)
\end{aligned} \quad (1)
$$

$$
y = \left(1 + \frac{1}{2} x^2 - \frac{1}{6} x^3 - \frac{1}{120} x^5\right) c_1 + \left(x - \frac{1}{2} x^2 + \frac{1}{6} x^3 - \frac{1}{24} x^4 + \frac{1}{120} x^5\right) c_2 + O\left(x^6\right) \quad (2)
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1 + \frac{1}{2} x^2 - \frac{1}{6} x^3 - \frac{1}{120} x^5 + \frac{1}{240} x^6\right) y(0) \\
& + \left(x - \frac{1}{2} x^2 + \frac{1}{6} x^3 - \frac{1}{24} x^4 + \frac{1}{120} x^5 - \frac{1}{720} x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Verified OK.

$$
y = \left(1 + \frac{1}{2} x^2 - \frac{1}{6} x^3 - \frac{1}{120} x^5\right) c_1 + \left(x - \frac{1}{2} x^2 + \frac{1}{6} x^3 - \frac{1}{24} x^4 + \frac{1}{120} x^5\right) c_2 + O\left(x^6\right)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
   Change of variables used:
      [x = ln(t)]
   Linear ODE actually solved:
      -u(t)+(t^2+t)*diff(u(t),t)+t^2*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)+exp(x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{120}x^5\right)y(0) + \left(x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{120}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[y''[x]+Exp[x]*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{120} - \frac{x^3}{6} + \frac{x^2}{2} + 1\right) + c_2\left(\frac{x^5}{120} - \frac{x^4}{24} + \frac{x^3}{6} - \frac{x^2}{2} + x\right)$$

1.23 problem 39

Internal problem ID [5555]
Internal file name [`OUTPUT/4803_Sunday_June_05_2022_03_06_12_PM_86216556/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.1.2 page 230
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + xy' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (67)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (68)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -y - xy' \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= x^2y' + xy - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= -y'x^3 - yx^2 + 5xy' + 3y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(x^4 - 9x^2 + 8\right)y' + yx\left(x^2 - 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(-x^5 + 14x^3 - 33x\right)y' - y\left(x^4 - 12x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -2y'(0) \\
F_2 &= 3y(0) \\
F_3 &= 8y'(0) \\
F_4 &= -15y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right)y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right)y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) - x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6)$$

Verified OK.

1.23.1 Maple step by step solution

Let's solve

$y'' = -y - xy'$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + xy' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\,(a_{k+2}(k+2) + a_k) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{15} - \frac{x^3}{3} + x\right) + c_1\left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right)$$

2 Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239

2.1 problem 1

Internal problem ID [5556]
Internal file name [`OUTPUT/4804_Sunday_June_05_2022_03_06_13_PM_44411204/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + 4x^2y' + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^3y'' + 4x^2y' + 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{3}{x^3}$$

Table 16: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3}{x^3}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^3*diff(y(x),x$2)+4*x^2*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 282

```
AsymptoticDSolveValue[x^3*y''[x]+4*x^2*y'[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) &\to \frac{c_1 e^{-\frac{2i\sqrt{3}}{\sqrt{x}}}\left(-\frac{14315125825ix^{9/2}}{8796093022208\sqrt{3}}+\frac{8083075ix^{7/2}}{4294967296\sqrt{3}}-\frac{15015i\sqrt{3}x^{5/2}}{8388608}+\frac{385i\sqrt{3}x^{3/2}}{8192}+\frac{930483178625x^5}{844424930131968}-\frac{509233725x^4}{549755813888}+\frac{42542}{26843}\right.}{x^{5/4}} \\
&\quad+\frac{c_2 e^{\frac{2i\sqrt{3}}{\sqrt{x}}}\left(\frac{14315125825ix^{9/2}}{8796093022208\sqrt{3}}-\frac{8083075ix^{7/2}}{4294967296\sqrt{3}}+\frac{15015i\sqrt{3}x^{5/2}}{8388608}-\frac{385i\sqrt{3}x^{3/2}}{8192}+\frac{930483178625x^5}{844424930131968}-\frac{509233725x^4}{549755813888}+\frac{425425x}{2684354}\right.}{x^{5/4}}
\end{aligned}
$$

2.2 problem 2

Internal problem ID [5557]
Internal file name [OUTPUT/4805_Sunday_June_05_2022_03_06_14_PM_43330679/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x+3)^2 y'' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^3 + 6x^2 + 9x)\, y'' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = -\frac{1}{(x+3)^2 x}$$

Table 17: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{1}{(x+3)^2 x}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-3, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x^2 + 6x + 9) y'' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x^2 + 6x + 9) \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) \\ &+ \left(\sum_{n=0}^{\infty} 9x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 6a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r-1}\right) \\ &+ \left(\sum_{n=1}^{\infty} 6a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) \\ &+ \left(\sum_{n=0}^{\infty} 9x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$9x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$9x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$9x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$9x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-6r^2 + 6r + 1}{9r(1+r)}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-2}(n+r-2)(n-3+r) + 6a_{n-1}(n+r-1)(n+r-2) \\ &+ 9a_n(n+r)(n+r-1) - a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-2} + 6n^2 a_{n-1} + 2nra_{n-2} + 12nra_{n-1} + r^2 a_{n-2} + 6r^2 a_{n-1} - 5na_{n-2} - 18na_{n-1} - 5ra_{n-2} - 18}{9\left(n+r\right)\left(n+r-1\right)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{\left(-a_{n-2} - 6a_{n-1}\right) n^2 + \left(3a_{n-2} + 6a_{n-1}\right) n - 2a_{n-2} + a_{n-1}}{9n\left(1+n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$	$\frac{1}{18}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{27r^4 - 39r^2 + 1}{81r\left(1+r\right)^2\left(2+r\right)}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{11}{972}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$	$\frac{1}{18}$
a_2	$\frac{27r^4-39r^2+1}{81r(1+r)^2(2+r)}$	$-\frac{11}{972}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-108r^6 - 324r^5 - 18r^4 + 504r^3 + 270r^2 - 36r - 11}{729r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{277}{104976}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$	$\frac{1}{18}$
a_2	$\frac{27r^4-39r^2+1}{81r(1+r)^2(2+r)}$	$-\frac{11}{972}$
a_3	$\frac{-108r^6-324r^5-18r^4+504r^3+270r^2-36r-11}{729r(1+r)^2(2+r)^2(3+r)}$	$\frac{277}{104976}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{405r^8 + 3240r^7 + 8370r^6 + 4860r^5 - 10962r^4 - 15768r^3 - 4299r^2 + 1338r + 277}{6561r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{12539}{18895680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$	$\frac{1}{18}$
a_2	$\frac{27r^4-39r^2+1}{81r(1+r)^2(2+r)}$	$-\frac{11}{972}$
a_3	$\frac{-108r^6-324r^5-18r^4+504r^3+270r^2-36r-11}{729r(1+r)^2(2+r)^2(3+r)}$	$\frac{277}{104976}$
a_4	$\frac{405r^8+3240r^7+8370r^6+4860r^5-10962r^4-15768r^3-4299r^2+1338r+277}{6561r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{12539}{18895680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1458r^{10} - 21870r^9 - 128385r^8 - 359640r^7 - 413478r^6 + 167184r^5 + 890541r^4 + 735966r^3 + 1107}{59049r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)^2\left(5+r\right)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{893821}{5101833600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-6r^2+6r+1}{9r(1+r)}$	$\frac{1}{18}$
a_2	$\frac{27r^4-39r^2+1}{81r(1+r)^2(2+r)}$	$-\frac{11}{972}$
a_3	$\frac{-108r^6-324r^5-18r^4+504r^3+270r^2-36r-11}{729r(1+r)^2(2+r)^2(3+r)}$	$\frac{277}{104976}$
a_4	$\frac{405r^8+3240r^7+8370r^6+4860r^5-10962r^4-15768r^3-4299r^2+1338r+277}{6561r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{12539}{18895680}$
a_5	$\frac{-1458r^{10}-21870r^9-128385r^8-359640r^7-413478r^6+167184r^5+890541r^4+735966r^3+110706r^2-73206r-12539}{59049r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$\frac{893821}{5101833600}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 + \frac{x}{18} - \frac{11x^2}{972} + \frac{277x^3}{104976} - \frac{12539x^4}{18895680} + \frac{893821x^5}{5101833600} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= \frac{-6r^2 + 6r + 1}{9r(1 + r)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{-6r^2 + 6r + 1}{9r(1 + r)} &= \lim_{r\to 0} \frac{-6r^2 + 6r + 1}{9r(1 + r)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x(x^2+6x+9)\,y''-y=0$ gives

$$
\begin{aligned}
&x(x^2+6x+9)\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) - Cy_1(x)\ln(x) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x(x^2+6x+9)\,y_1''(x) - y_1(x)\right)\ln(x) + x(x^2+6x+9)\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right)C \\
&+ x(x^2+6x+9)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&- \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
x(x^2+6x+9)\,y_1''(x) - y_1(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&x\left(x^{2}+6x+9\right)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^{2}}\right)C\\
&+x\left(x^{2}+6x+9\right)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^{2}}{x^{2}}-\frac{b_n x^{n+r_2}(n+r_2)}{x^{2}}\right)\right)\\
&-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$
\frac{\left(2x(x+3)^{2}\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)-(x+3)^{2}\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}
+\frac{x^{2}(x+3)^{2}\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x}=0
\tag{9}
$$

Since $r_1=1$ and $r_2=0$ then the above becomes

$$
\frac{\left(2x(x+3)^{2}\left(\sum_{n=0}^{\infty}x^{n}a_n(1+n)\right)-(x+3)^{2}\left(\sum_{n=0}^{\infty}a_n x^{1+n}\right)\right)C}{x}
+\frac{x^{2}(x+3)^{2}\left(\sum_{n=0}^{\infty}x^{n-2}b_n n(n-1)\right)-\left(\sum_{n=0}^{\infty}b_n x^{n}\right)x}{x}=0
\tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}2C\,x^{n+2}a_n(1+n)\right)+\left(\sum_{n=0}^{\infty}12C\,x^{1+n}a_n(1+n)\right)\\
&+\left(\sum_{n=0}^{\infty}18C\,x^{n}a_n(1+n)\right)+\sum_{n=0}^{\infty}\left(-C\,x^{n+2}a_n\right)+\sum_{n=0}^{\infty}\left(-6C\,x^{1+n}a_n\right)\\
&+\sum_{n=0}^{\infty}\left(-9Ca_n x^{n}\right)+\left(\sum_{n=0}^{\infty}n\,x^{1+n}b_n(n-1)\right)+\left(\sum_{n=0}^{\infty}6x^{n}b_n n(n-1)\right)\\
&+\left(\sum_{n=0}^{\infty}9n\,x^{n-1}b_n(n-1)\right)+\sum_{n=0}^{\infty}\left(-b_n x^{n}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (1+n) &= \sum_{n=3}^{\infty} 2C a_{n-3} (n-2)\, x^{n-1} \\
\sum_{n=0}^{\infty} 12C\, x^{1+n} a_n (1+n) &= \sum_{n=2}^{\infty} 12C a_{n-2} (n-1)\, x^{n-1} \\
\sum_{n=0}^{\infty} 18C\, x^{n} a_n (1+n) &= \sum_{n=1}^{\infty} 18C a_{n-1} n\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-C\, x^{n+2} a_n\right) &= \sum_{n=3}^{\infty} \left(-C a_{n-3} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-6C\, x^{1+n} a_n\right) &= \sum_{n=2}^{\infty} \left(-6C a_{n-2} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-9C a_n x^{n}\right) &= \sum_{n=1}^{\infty} \left(-9C a_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} n\, x^{1+n} b_n (n-1) &= \sum_{n=2}^{\infty} (n-2)\, b_{n-2} (n-3)\, x^{n-1} \\
\sum_{n=0}^{\infty} 6 x^{n} b_n n (n-1) &= \sum_{n=1}^{\infty} 6(n-1)\, b_{n-1} (n-2)\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-b_n x^{n}\right) &= \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-1}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n-1$.

$$\left(\sum_{n=3}^{\infty} 2 C a_{n-3} (n-2)\, x^{n-1}\right) + \left(\sum_{n=2}^{\infty} 12 C a_{n-2} (n-1)\, x^{n-1}\right)$$
$$+ \left(\sum_{n=1}^{\infty} 18 C a_{n-1} n\, x^{n-1}\right) + \sum_{n=3}^{\infty} \left(-C a_{n-3} x^{n-1}\right)$$
$$+ \sum_{n=2}^{\infty} \left(-6 C a_{n-2} x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-9 C a_{n-1} x^{n-1}\right) \tag{2B}$$
$$+ \left(\sum_{n=2}^{\infty} (n-2)\, b_{n-2} (n-3)\, x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 6(n-1)\, b_{n-1} (n-2)\, x^{n-1}\right)$$
$$+ \left(\sum_{n=0}^{\infty} 9n\, x^{n-1} b_n (n-1)\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-1}\right) = 0$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$9C - 1 = 0$$

Which is solved for C. Solving for C gives

$$C = \frac{1}{9}$$

For $n = 2$, Eq (2B) gives

$$(6a_0 + 27a_1)\, C - b_1 + 18b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{5}{6} + 18b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{5}{108}$$

For $n = 3$, Eq (2B) gives

$$(a_0 + 18a_1 + 45a_2)\, C + 11b_2 + 54b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{167}{486} + 54b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{167}{26244}$$

For $n = 4$, Eq (2B) gives

$$(3a_1 + 30a_2 + 63a_3)\,C + 2b_2 + 35b_3 + 108b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{13583}{104976} + 108b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{13583}{11337408}$$

For $n = 5$, Eq (2B) gives

$$(5a_2 + 42a_3 + 81a_4)\,C + 6b_3 + 71b_4 + 180b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1327279}{28343520} + 180b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{1327279}{5101833600}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = \frac{1}{9}$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = {} & \frac{1}{9}\left(x\left(1 + \frac{x}{18} - \frac{11x^2}{972} + \frac{277x^3}{104976} - \frac{12539x^4}{18895680} + \frac{893821x^5}{5101833600} + O(x^6)\right)\right)\ln(x) \\ & + 1 - \frac{5x^2}{108} + \frac{167x^3}{26244} - \frac{13583x^4}{11337408} + \frac{1327279x^5}{5101833600} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x}{18} - \frac{11x^2}{972} + \frac{277x^3}{104976} - \frac{12539x^4}{18895680} + \frac{893821x^5}{5101833600} + O(x^6)\right) \\ &\quad + c_2\left(\frac{1}{9}\left(x\left(1 + \frac{x}{18} - \frac{11x^2}{972} + \frac{277x^3}{104976} - \frac{12539x^4}{18895680} + \frac{893821x^5}{5101833600} + O(x^6)\right)\right)\ln(x)\right. \\ &\qquad \left. + 1 - \frac{5x^2}{108} + \frac{167x^3}{26244} - \frac{13583x^4}{11337408} + \frac{1327279x^5}{5101833600} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}+O(x^6)\right) \\
&\quad + c_2 \left(\frac{x\left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}+O(x^6)\right)\ln(x)}{9}+1-\frac{5x^2}{108} \right. \\
&\quad \left. +\frac{167x^3}{26244}-\frac{13583x^4}{11337408}+\frac{1327279x^5}{5101833600}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}+O(x^6)\right) \\
&\quad + c_2 \left(\frac{x\left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}+O(x^6)\right)\ln(x)}{9}+1 \right. \\
&\quad \left. -\frac{5x^2}{108}+\frac{167x^3}{26244}-\frac{13583x^4}{11337408}+\frac{1327279x^5}{5101833600}+O(x^6)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}+O(x^6)\right) \\
&\quad + c_2 \left(\frac{x\left(1+\frac{x}{18}-\frac{11x^2}{972}+\frac{277x^3}{104976}-\frac{12539x^4}{18895680}+\frac{893821x^5}{5101833600}+O(x^6)\right)\ln(x)}{9}+1-\frac{5x^2}{108} \right. \\
&\quad \left. +\frac{167x^3}{26244}-\frac{13583x^4}{11337408}+\frac{1327279x^5}{5101833600}+O(x^6)\right)
\end{aligned}
$$

Verified OK.

2.2.1 Maple step by step solution

Let's solve

$x(x^2+6x+9)\,y''-y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=\frac{y}{x(x^2+6x+9)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''-\frac{y}{x(x^2+6x+9)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=0, P_3(x)=-\frac{1}{x(x^2+6x+9)}\right]$

  ◦ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3}=0$

  ◦ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3}=\frac{1}{3}$

  ◦ $x=-3$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-3$

- Multiply by denominators

$x(x^2+6x+9)\,y''-y=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

$(u^3-3u^2)\left(\frac{d^2}{du^2}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(3r^2-3r+1)\, u^r + \left(\sum_{k=1}^{\infty} \left(-a_k(3k^2+6kr+3r^2-3k-3r+1) + a_{k-1}(k+r-1)(k-2+r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-3r^2+3r-1=0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}, \frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k-1}(k+r-1)(k-2+r) - 3\left(k^2+(2r-1)\,k+r^2-r+\frac{1}{3}\right) a_k = 0$$

- Shift index using $k- >k+1$

  $$a_k(k+r)(k+r-1) - 3\left((k+1)^2+(2r-1)(k+1)+r^2-r+\frac{1}{3}\right) a_{k+1} = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r)(k+r-1)}{3k^2+6kr+3r^2+3k+3r+1}$$

- Recursion relation for $r = \frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}$

  $$a_{k+1} = \frac{a_k\left(k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)\left(k-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)+3\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}$$

- Solution for $r = \frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)\left(k-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)+3\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\right]$$

- Revert the change of variables $u = x+3$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)\left(k-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)+3\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\right]$$

- Recursion relation for $r = \frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}$

$$a_{k+1} = \frac{a_k \left(k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)\left(k-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)+3\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}$$

- Solution for $r = \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{6}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}}, a_{k+1} = \frac{a_k \left(k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)\left(k-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)+3\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}}, a_{k+1} = \frac{a_k \left(k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)\left(k-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)+3\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+3)^{k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}}\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{6}}\right), a_{k+1} = \frac{a_k \left(k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)\left(k-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)}{3k^2+6k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)+3\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{6}\right)^2+3k+\frac{5}{2}-}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
Order:=6;
dsolve(x*(x+3)^2*diff(y(x),x$2)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1x\left(1+\frac{1}{18}x-\frac{11}{972}x^2+\frac{277}{104976}x^3-\frac{12539}{18895680}x^4+\frac{893821}{5101833600}x^5+\mathrm{O}\left(x^6\right)\right) + c_2\left(\ln\left(x\right)\left(\frac{1}{9}x+\frac{1}{162}x^2-\frac{11}{8748}x^3+\frac{277}{944784}x^4-\frac{12539}{170061120}x^5+\mathrm{O}\left(x^6\right)\right) + \left(1-\frac{5}{108}x^2+\frac{167}{26244}x^3-\frac{13583}{11337408}x^4+\frac{1327279}{5101833600}x^5+\mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 87

```
AsymptoticDSolveValue[x*(x+3)^2*y''[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x(277x^3 - 1188x^2 + 5832x + 104976)\log(x)}{944784} + \frac{3037x^4 + 864x^3 - 174960x^2 + 6298560x + 11337408}{11337408} \right) + c_2 \left(-\frac{12539x^5}{18895680} + \frac{277x^4}{104976} - \frac{11x^3}{972} + \frac{x^2}{18} + x \right)$$

2.3 problem 3

Internal problem ID [5558]
Internal file name [OUTPUT/4806_Sunday_June_05_2022_03_06_16_PM_27990159/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2-9)^2 y''+(x+3)\,y'+2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots \\ &= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (71)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (72)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{xy'+3y'+2y}{x^4-18x^2+81} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{\left(x^3+9x^2-8x-78\right)y'+8y\left(x^2-3x+\frac{1}{4}\right)}{\left(x+3\right)^3\left(x-3\right)^4} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{\left(4x^5-84x^4+139x^3+843x^2-1576x-786\right)y'-40y\left(x^4-6x^3+\frac{56}{5}x^2-\frac{57}{5}x+\frac{289}{20}\right)}{\left(x-3\right)^6\left(x+3\right)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{\left(-60x^7+900x^6-2901x^5-3963x^4+27462x^3-29286x^2+31564x-71520\right)y'+240\left(x^6-9x^5+\cdots\right.}{\left(x+3\right)^5\left(x-3\right)^8} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{\left(600x^9-9720x^8+48444x^7-60612x^6-185109x^5+791229x^4-2239204x^3+4979016x^2-410561\cdots\right.}{\left(x-3\right)\cdots}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{2y(0)}{81}-\frac{y'(0)}{27} \\
F_1 &= \frac{2y(0)}{2187}-\frac{26y'(0)}{729} \\
F_2 &= -\frac{578y(0)}{59049}-\frac{262y'(0)}{19683} \\
F_3 &= \frac{2432y(0)}{1594323}-\frac{23840y'(0)}{531441} \\
F_4 &= -\frac{779924y(0)}{43046721}-\frac{313876y'(0)}{14348907}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{81}x^2+\frac{1}{6561}x^3-\frac{289}{708588}x^4+\frac{304}{23914845}x^5-\frac{194981}{7748409780}x^6\right)y(0) \\
&+\left(x-\frac{1}{54}x^2-\frac{13}{2187}x^3-\frac{131}{236196}x^4-\frac{596}{1594323}x^5-\frac{78469}{2582803260}x^6\right)y'(0)+O(x^6)
\end{aligned}
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
y''\left(x^4-18x^2+81\right)+(x+3)\,y'+2y=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)\left(x^4-18x^2+81\right)+(x+3)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+2\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n\,x^{n+2} a_n (n-1)\right)+\sum_{n\,=2}^{\infty}\left(-18x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} 81n(n-1)\,a_n x^{n-2}\right) \\
&+\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=1}^{\infty} 3n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} 2a_n x^n\right)=0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\,x^{n+2} a_n (n-1) = \sum_{n=4}^{\infty} (n-2)\, a_{n-2} (n-3)\, x^n$$

$$\sum_{n=2}^{\infty} 81 n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 81(n+2)\, a_{n+2} (n+1)\, x^n$$

$$\sum_{n=1}^{\infty} 3 n a_n x^{n-1} = \sum_{n=0}^{\infty} 3(n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=4}^{\infty} (n-2)\, a_{n-2}(n-3)\, x^n\right) + \sum_{n=2}^{\infty} \left(-18 x^n a_n n(n-1)\right) \\ &+ \left(\sum_{n=0}^{\infty} 81(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) \\ &+ \left(\sum_{n=0}^{\infty} 3(n+1)\, a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} 2 a_n x^n\right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$162 a_2 + 3 a_1 + 2 a_0 = 0$$

$$a_2 = -\frac{a_0}{81} - \frac{a_1}{54}$$

$n = 1$ gives

$$486 a_3 + 3 a_1 + 6 a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6561} - \frac{13 a_1}{2187}$$

$n = 2$ gives

$$-32 a_2 + 972 a_4 + 9 a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{289a_0}{729} + \frac{131a_1}{243} + 972a_4 = 0$$

Or

$$a_4 = -\frac{289a_0}{708588} - \frac{131a_1}{236196}$$

$n = 3$ gives

$$-103a_3 + 1620a_5 + 12a_4 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{1216a_0}{59049} + \frac{11920a_1}{19683} + 1620a_5 = 0$$

Or

$$a_5 = \frac{304a_0}{23914845} - \frac{596a_1}{1594323}$$

For $4 \leq n$, the recurrence equation is

$$(n-2)\,a_{n-2}(n-3) - 18na_n(n-1) + 81(n+2)\,a_{n+2}(n+1) \\ + na_n + 3(n+1)\,a_{n+1} + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{18n^2a_n - n^2a_{n-2} - 19na_n + 5na_{n-2} - 3na_{n+1} - 2a_n - 6a_{n-2} - 3a_{n+1}}{81\,(n+2)\,(n+1)} \\ &= \frac{(18n^2 - 19n - 2)\,a_n}{81\,(n+2)\,(n+1)} + \frac{(-n^2 + 5n - 6)\,a_{n-2}}{81\,(n+2)\,(n+1)} + \frac{(-3n-3)\,a_{n+1}}{81\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 4$ the recurrence equation gives

$$2a_2 - 210a_4 + 2430a_6 + 15a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{194981a_0}{7748409780} - \frac{78469a_1}{2582803260}$$

For $n = 5$ the recurrence equation gives

$$6a_3 - 353a_5 + 3402a_7 + 18a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{1732937a_0}{1464449448420} - \frac{13738871a_1}{488149816140}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1x + \left(-\frac{a_0}{81} - \frac{a_1}{54}\right)x^2 + \left(\frac{a_0}{6561} - \frac{13a_1}{2187}\right)x^3 \\ + \left(-\frac{289a_0}{708588} - \frac{131a_1}{236196}\right)x^4 + \left(\frac{304a_0}{23914845} - \frac{596a_1}{1594323}\right)x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y = \left(1 - \frac{1}{81}x^2 + \frac{1}{6561}x^3 - \frac{289}{708588}x^4 + \frac{304}{23914845}x^5\right)a_0 \\ + \left(x - \frac{1}{54}x^2 - \frac{13}{2187}x^3 - \frac{131}{236196}x^4 - \frac{596}{1594323}x^5\right)a_1 + O(x^6) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned} y = \left(1 - \frac{1}{81}x^2 + \frac{1}{6561}x^3 - \frac{289}{708588}x^4 + \frac{304}{23914845}x^5\right)c_1 \\ + \left(x - \frac{1}{54}x^2 - \frac{13}{2187}x^3 - \frac{131}{236196}x^4 - \frac{596}{1594323}x^5\right)c_2 + O(x^6) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= \left(1-\frac{1}{81}x^{2}+\frac{1}{6561}x^{3}-\frac{289}{708588}x^{4}+\frac{304}{23914845}x^{5}-\frac{194981}{7748409780}x^{6}\right) y(0) \\
&+\left(x-\frac{1}{54}x^{2}-\frac{13}{2187}x^{3}-\frac{131}{236196}x^{4}-\frac{596}{1594323}x^{5}-\frac{78469}{2582803260}x^{6}\right) y'(0)+O\left(x^{6}\right) \tag{1}
\end{align*}
$$

$$
\begin{align*}
y &= \left(1-\frac{1}{81}x^{2}+\frac{1}{6561}x^{3}-\frac{289}{708588}x^{4}+\frac{304}{23914845}x^{5}\right) c_{1} \\
&+\left(x-\frac{1}{54}x^{2}-\frac{13}{2187}x^{3}-\frac{131}{236196}x^{4}-\frac{596}{1594323}x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= \left(1-\frac{1}{81}x^{2}+\frac{1}{6561}x^{3}-\frac{289}{708588}x^{4}+\frac{304}{23914845}x^{5}-\frac{194981}{7748409780}x^{6}\right) y(0) \\
&+\left(x-\frac{1}{54}x^{2}-\frac{13}{2187}x^{3}-\frac{131}{236196}x^{4}-\frac{596}{1594323}x^{5}-\frac{78469}{2582803260}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{align*}
$$

Verified OK.

$$
\begin{align*}
y &= \left(1-\frac{1}{81}x^{2}+\frac{1}{6561}x^{3}-\frac{289}{708588}x^{4}+\frac{304}{23914845}x^{5}\right) c_{1} \\
&+\left(x-\frac{1}{54}x^{2}-\frac{13}{2187}x^{3}-\frac{131}{236196}x^{4}-\frac{596}{1594323}x^{5}\right) c_{2}+O\left(x^{6}\right)
\end{align*}
$$

Verified OK.

2.3.1 Maple step by step solution

Let's solve

$y''(x^4-18x^2+81)+(x+3)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{x^4-18x^2+81}-\frac{y'}{x^3-3x^2-9x+27}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x^3-3x^2-9x+27}+\frac{2y}{x^4-18x^2+81}=0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1}{x^3-3x^2-9x+27}, P_3(x) = \frac{2}{x^4-18x^2+81}\right]$

- $(x+3) \cdot P_2(x)$ is analytic at $x = -3$

  $\left((x+3) \cdot P_2(x)\right)\bigg|_{x=-3} = \frac{1}{36}$

- $(x+3)^2 \cdot P_3(x)$ is analytic at $x = -3$

  $\left((x+3)^2 \cdot P_3(x)\right)\bigg|_{x=-3} = \frac{1}{18}$

- $x = -3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -3$

• Multiply by denominators

  $y''(x^3-3x^2-9x+27)(x^4-18x^2+81) + y'(x^4-18x^2+81) + (2x^3-6x^2-18x+54)\, y = 0$

• Change variables using $x = u-3$ so that the regular singular point is at $u = 0$

  $(u^7-24u^6+216u^5-864u^4+1296u^3)\left(\frac{d^2}{du^2}y(u)\right) + (u^4-12u^3+36u^2)\left(\frac{d}{du}y(u)\right) + (2u^3-24u^2$

• Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 1..3$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..4$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 3..7$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$36a_0(36r^2-35r+2)\,u^{1+r} + (36a_1(36r^2+37r+3) - 12a_0(72r^2-71r+2))\,u^{2+r} + (36a_2(36r^2 +$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$1296r^2 - 1260r + 72 = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{35}{72} - \frac{\sqrt{937}}{72}, \frac{35}{72} + \frac{\sqrt{937}}{72}\right\}$$

- The coefficients of each power of u must be 0

$$[36a_1(36r^2+37r+3) - 12a_0(72r^2-71r+2) = 0, 36a_2(36r^2+109r+76) - 12a_1(72r^2+73r+$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = \frac{a_0(72r^2-71r+2)}{3(36r^2+37r+3)}, a_2 = \frac{a_0(4320r^4+108r^3-4019r^2+101r+6)}{12(1296r^4+5256r^3+6877r^2+3139r+228)}, a_3 = \frac{a_0(233280r^6+707616r^5+277488r^4-61842}{54(46656r^6+423792r^5+1485324r^4+2519317}\right.$$

- Each term in the series must be 0, giving the recursion relation

$$(a_{k-5} - 24a_{k-4} + 216a_{k-3} - 864a_{k-2} + 1296a_{k-1})\,k^2 + (2(a_{k-5} - 24a_{k-4} + 216a_{k-3} - 864a_{k-2} +$$

- Shift index using $k- >k+5$

$$(a_k - 24a_{k+1} + 216a_{k+2} - 864a_{k+3} + 1296a_{k+4})\,(k+5)^2 + (2(a_k - 24a_{k+1} + 216a_{k+2} - 864a_{k+3}$$

- Recursion relation that defines series solution to ODE

$$a_{k+4} = -\frac{k^2a_k - 24k^2a_{k+1} + 216k^2a_{k+2} - 864k^2a_{k+3} + 2kra_k - 48kra_{k+1} + 432kra_{k+2} - 1728kra_{k+3} + r^2a_k - 24r^2a_{k+1} + 216r^2a_{k+2}}{36(36k^2+72kr+36r^2+}$$

- Recursion relation for $r = \frac{35}{72} - \frac{\sqrt{937}}{72}$

$$a_{k+4} = -\frac{k^2a_k - 24k^2a_{k+1} + 216k^2a_{k+2} - 864k^2a_{k+3} + 2k\left(\frac{35}{72} - \frac{\sqrt{937}}{72}\right)a_k - 48k\left(\frac{35}{72} - \frac{\sqrt{937}}{72}\right)a_{k+1} + 432k\left(\frac{35}{72} - \frac{\sqrt{937}}{72}\right)a_{k+2} - 1728k\left(\right.}{}$$

- Solution for $r = \frac{35}{72} - \frac{\sqrt{937}}{72}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{35}{72}-\frac{\sqrt{937}}{72}}, a_{k+4} = -\frac{k^2a_k - 24k^2a_{k+1} + 216k^2a_{k+2} - 864k^2a_{k+3} + 2k\left(\frac{35}{72} - \frac{\sqrt{937}}{72}\right)a_k - 48k\left(\frac{35}{72} - \frac{\sqrt{937}}{72}\right)a_{k+}}{}\right.$$

- Revert the change of variables $u = x+3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k+\frac{35}{72}-\frac{\sqrt{937}}{72}}, a_{k+4} = -\frac{k^2 a_k - 24k^2 a_{k+1} + 216k^2 a_{k+2} - 864k^2 a_{k+3} + 2k\left(\frac{35}{72}-\frac{\sqrt{937}}{72}\right)a_k - 48k\left(\frac{35}{72}-\frac{\sqrt{937}}{72}\right.}{}\right.$$

- Recursion relation for $r = \frac{35}{72} + \frac{\sqrt{937}}{72}$

$$a_{k+4} = -\frac{k^2 a_k - 24k^2 a_{k+1} + 216k^2 a_{k+2} - 864k^2 a_{k+3} + 2k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_k - 48k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_{k+1} + 432k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_{k+2} - 1728k\left(\right.}{}$$

- Solution for $r = \frac{35}{72} + \frac{\sqrt{937}}{72}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{35}{72}+\frac{\sqrt{937}}{72}}, a_{k+4} = -\frac{k^2 a_k - 24k^2 a_{k+1} + 216k^2 a_{k+2} - 864k^2 a_{k+3} + 2k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_k - 48k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_{k+}}{}\right.$$

- Revert the change of variables $u = x+3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k+\frac{35}{72}+\frac{\sqrt{937}}{72}}, a_{k+4} = -\frac{k^2 a_k - 24k^2 a_{k+1} + 216k^2 a_{k+2} - 864k^2 a_{k+3} + 2k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right)a_k - 48k\left(\frac{35}{72}+\frac{\sqrt{937}}{72}\right.}{}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+3)^{k+\frac{35}{72}-\frac{\sqrt{937}}{72}}\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k+\frac{35}{72}+\frac{\sqrt{937}}{72}}\right), a_{k+4} = -\frac{k^2 a_k - 24k^2 a_{k+1} + 216k^2 a_{k+2} - 864}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
Order:=6;
dsolve((x^2-9)^2*diff(y(x),x$2)+(x+3)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{81}x^2 + \frac{1}{6561}x^3 - \frac{289}{708588}x^4 + \frac{304}{23914845}x^5\right) y(0) + \left(x - \frac{1}{54}x^2 - \frac{13}{2187}x^3 - \frac{131}{236196}x^4 - \frac{596}{1594323}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 70

```
AsymptoticDSolveValue[(x^2-9)^2*y''[x]+(x+3)*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{304x^5}{23914845} - \frac{289x^4}{708588} + \frac{x^3}{6561} - \frac{x^2}{81} + 1 \right) + c_2 \left(-\frac{596x^5}{1594323} - \frac{131x^4}{236196} - \frac{13x^3}{2187} - \frac{x^2}{54} + x \right)$$

2.4 problem 4

Internal problem ID [5559]
Internal file name [`OUTPUT/4807_Sunday_June_05_2022_03_06_18_PM_7339101/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \frac{y'}{x} + \frac{y}{(x-1)^3} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' - \frac{y'}{x} + \frac{y}{(x-1)^3} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{(x-1)^3} \end{aligned}$$

Table 20: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{(x-1)^3}$	
singularity	type
$x = 1$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[1]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x(x-1)^3 - y'(x-1)^3 + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1)^3 - \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) (x-1)^3 + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-3 x^{1+n+r} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 3 x^{n+r} a_n(n+r)(n+r-1)\right) \\
&+\sum_{n=0}^{\infty}\left(-x^{n+r-1} a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 3 x^{1+n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-3 x^{n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty}(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)(n+r-1)=\sum_{n=3}^{\infty} a_{n-3}(n+r-3)(n-4+r) x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-3 x^{1+n+r} a_n(n+r)(n+r-1)\right)=\sum_{n=2}^{\infty}\left(-3 a_{n-2}(n+r-2)(n+r-3) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 3 x^{n+r} a_n(n+r)(n+r-1)=\sum_{n=1}^{\infty} 3 a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n(n+r)\right)=\sum_{n=3}^{\infty}\left(-a_{n-3}(n+r-3) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 3 x^{1+n+r} a_n(n+r)=\sum_{n=2}^{\infty} 3 a_{n-2}(n+r-2) x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-3 x^{n+r} a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-3 a_{n-1}(n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n=\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned}
&\left(\sum_{n=3}^{\infty} a_{n-3}(n+r-3)(n-4+r)\, x^{n+r-1}\right) \\
&+\sum_{n=2}^{\infty}\left(-3a_{n-2}(n+r-2)(n+r-3)\, x^{n+r-1}\right) \\
&\quad+\left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=3}^{\infty}\left(-a_{n-3}(n+r-3)\, x^{n+r-1}\right) \\
&\quad+\left(\sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)\, x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-3a_{n-1}(n+r-1)\, x^{n+r-1}\right) \\
&\quad+\left(\sum_{n=0}^{\infty}(n+r)\, a_n x^{n+r-1}\right)+\left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right)=0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-x^{n+r-1}a_n(n+r)(n+r-1)+(n+r)\, a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$-x^{-1+r}a_0 r(-1+r)+r a_0 x^{-1+r}=0$$

Or

$$\left(-x^{-1+r}r(-1+r)+r\, x^{-1+r}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2-r)=0$$

Since the above is true for all x then the indicial equation becomes

$$-r(-2+r)=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=2$$
$$r_2=0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(2-r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{3r(-2+r)}{r^2-1}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{6r^2 - 12r + 1}{r(2+r)}$$

For $3 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-3}(n+r-3)(n-4+r) - 3a_{n-2}(n+r-2)(n+r-3) \\ &+ 3a_{n-1}(n+r-1)(n+r-2) - a_n(n+r)(n+r-1) - a_{n-3}(n+r-3) \\ &+ 3a_{n-2}(n+r-2) - 3a_{n-1}(n+r-1) + a_n(n+r) + a_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n^2 a_{n-3} - 3n^2 a_{n-2} + 3n^2 a_{n-1} + 2nra_{n-3} - 6nra_{n-2} + 6nra_{n-1} + r^2 a_{n-3} - 3r^2 a_{n-2} + 3r^2 a_{n-1} - 8na}{n^2 + 2nr + r^2 - 2n} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{(a_{n-3} - 3a_{n-2} + 3a_{n-1}) n^2 + (-4a_{n-3} + 6a_{n-2}) n + 3a_{n-3} + a_{n-2} - 3a_{n-1}}{n (n + 2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r(-2+r)}{r^2-1}$	0
a_2	$\frac{6r^2-12r+1}{r(2+r)}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{10r^4 - 20r^3 - 4r^2 + 14r - 3}{(r + 3) (1 + r)^2 (-1 + r)}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r(-2+r)}{r^2-1}$	0
a_2	$\frac{6r^2-12r+1}{r(2+r)}$	$\frac{1}{8}$
a_3	$\frac{10r^4-20r^3-4r^2+14r-3}{(r+3)(1+r)^2(-1+r)}$	$\frac{1}{5}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{15r^6 - 54r^4 + 13r^2 - 1}{(4 + r) (2 + r)^2 r (-1 + r) (1 + r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{49}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r(-2+r)}{r^2-1}$	0
a_2	$\frac{6r^2-12r+1}{r(2+r)}$	$\frac{1}{8}$
a_3	$\frac{10r^4-20r^3-4r^2+14r-3}{(r+3)(1+r)^2(-1+r)}$	$\frac{1}{5}$
a_4	$\frac{15r^6-54r^4+13r^2-1}{(4+r)(2+r)^2r(-1+r)(1+r)}$	$\frac{49}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{21r^8 + 84r^7 + 14r^6 - 252r^5 - 298r^4 - 78r^3 + 47r^2 + 30r - 9}{(5+r)(r+3)^2 r(-1+r)(1+r)^2(2+r)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{423}{1400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r(-2+r)}{r^2-1}$	0
a_2	$\frac{6r^2-12r+1}{r(2+r)}$	$\frac{1}{8}$
a_3	$\frac{10r^4-20r^3-4r^2+14r-3}{(r+3)(1+r)^2(-1+r)}$	$\frac{1}{5}$
a_4	$\frac{15r^6-54r^4+13r^2-1}{(4+r)(2+r)^2r(-1+r)(1+r)}$	$\frac{49}{192}$
a_5	$\frac{21r^8+84r^7+14r^6-252r^5-298r^4-78r^3+47r^2+30r-9}{(5+r)(r+3)^2r(-1+r)(1+r)^2(2+r)}$	$\frac{423}{1400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= x^2\left(1 + \frac{x^2}{8} + \frac{x^3}{5} + \frac{49x^4}{192} + \frac{423x^5}{1400} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{6r^2 - 12r + 1}{r(2+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{6r^2 - 12r + 1}{r(2+r)} &= \lim_{r\to 0} \frac{6r^2 - 12r + 1}{r(2+r)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $y''x(x-1)^3 - y'(x-1)^3 + xy = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)^3 - \left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)(x-1)^3 + x\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(y_1''(x)\,x(x-1)^3 - y_1'(x)\,(x-1)^3 + y_1(x)\,x\right)\ln(x) + \left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x(x-1)^3 - \frac{y_1(x)\,(x-1)^3}{x}\right)C\right) + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)^3 - \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)(x-1)^3 + x\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x(x-1)^3 - y_1'(x)\,(x-1)^3 + y_1(x)\,x = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x(x-1)^3 - \frac{y_1(x)\,(x-1)^3}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x-1)^3 - \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)(x-1)^3 + x\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2x(x-1)^3 \left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)(x-1)^3\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 (x-1)^3 - \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right)(x-1)^3 x + x^2 \left(\sum_{n=0}^{\infty} b_n \right.}{x} = 0 \tag{9}$$

Since $r_1 = 2$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2x(x-1)^3 \left(\sum_{n=0}^{\infty} x^{1+n} a_n (n+2)\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+2}\right)(x-1)^3\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{n-2} b_n n(n-1)\right) x^2 (x-1)^3 - \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right)(x-1)^3 x + x^2 \left(\sum_{n=0}^{\infty} b_n x^n\right)}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+2)\right) + \sum_{n=0}^{\infty} \left(-6C\, x^{n+3} a_n (n+2)\right) \\
&+ \left(\sum_{n=0}^{\infty} 6C\, x^{n+2} a_n (n+2)\right) + \sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n (n+2)\right) \\
&+ \sum_{n=0}^{\infty} \left(-2C\, x^{n+4} a_n\right) + \left(\sum_{n=0}^{\infty} 6C\, x^{n+3} a_n\right) + \sum_{n=0}^{\infty} \left(-6C\, x^{n+2} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} n\, x^{n+2} b_n (n-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-3n\, x^{1+n} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 3x^n b_n n(n-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-n\, x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-n\, x^{n+2} b_n\right) + \left(\sum_{n=0}^{\infty} 3n\, x^{1+n} b_n\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^n b_n n\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) + \left(\sum_{n=0}^{\infty} x^{1+n} b_n\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+2) = \sum_{n=5}^{\infty} 2C a_{n-5} (n-3)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-6C\, x^{n+3} a_n (n+2)\right) = \sum_{n=4}^{\infty} \left(-6C a_{n-4} (n-2)\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 6C\, x^{n+2} a_n (n+2) = \sum_{n=3}^{\infty} 6C a_{n-3} (n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n (n+2)\right) = \sum_{n=2}^{\infty} \left(-2C a_{n-2} n\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2C\, x^{n+4} a_n\right) = \sum_{n=5}^{\infty} \left(-2C a_{n-5} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 6C\, x^{n+3} a_n = \sum_{n=4}^{\infty} 6C a_{n-4} x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-6C\, x^{n+2} a_n\right) = \sum_{n=3}^{\infty} \left(-6C a_{n-3} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n = \sum_{n=2}^{\infty} 2C a_{n-2} x^{n-1}$$

$$\sum_{n=0}^{\infty} n\, x^{n+2} b_n (n-1) = \sum_{n=3}^{\infty} (n-3)\, b_{n-3} (n-4)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-3n\, x^{1+n} b_n (n-1)\right) = \sum_{n=2}^{\infty} \left(-3(n-2)\, b_{n-2} (n-3)\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 3x^n b_n n (n-1) = \sum_{n=1}^{\infty} 3(n-1)\, b_{n-1} (n-2)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-n\, x^{n+2} b_n\right) = \sum_{n=3}^{\infty} \left(-(n-3)\, b_{n-3} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 3n\, x^{1+n} b_n = \sum_{n=2}^{\infty} 3(n-2)\, b_{n-2} x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-3x^n b_n n\right) = \sum_{n=1}^{\infty} \left(-3(n-1)\, b_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\left(\sum_{n=5}^{\infty} 2C a_{n-5}(n-3)\, x^{n-1}\right) + \sum_{n=4}^{\infty} \left(-6C a_{n-4}(n-2)\, x^{n-1}\right) \\
&+ \left(\sum_{n=3}^{\infty} 6C a_{n-3}(n-1)\, x^{n-1}\right) + \sum_{n=2}^{\infty} \left(-2C a_{n-2} n\, x^{n-1}\right) \\
&+ \sum_{n=5}^{\infty} \left(-2C a_{n-5} x^{n-1}\right) + \left(\sum_{n=4}^{\infty} 6C a_{n-4} x^{n-1}\right) + \sum_{n=3}^{\infty} \left(-6C a_{n-3} x^{n-1}\right) \\
&+ \left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{n-1}\right) + \left(\sum_{n=3}^{\infty} (n-3)\, b_{n-3}(n-4)\, x^{n-1}\right) \\
&+ \sum_{n=2}^{\infty} \left(-3(n-2)\, b_{n-2}(n-3)\, x^{n-1}\right) \\
&+ \left(\sum_{n=1}^{\infty} 3(n-1)\, b_{n-1}(n-2)\, x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-n\, x^{n-1} b_n (n-1)\right) \\
&+ \sum_{n=3}^{\infty} \left(-(n-3)\, b_{n-3} x^{n-1}\right) + \left(\sum_{n=2}^{\infty} 3(n-2)\, b_{n-2} x^{n-1}\right) \\
&+ \sum_{n=1}^{\infty} \left(-3(n-1)\, b_{n-1} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) + \left(\sum_{n=2}^{\infty} b_{n-2} x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$-2C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = \frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$(6a_0 - 4a_1)\, C + 4b_1 - 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3 - 3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 1$$

For $n = 4$, Eq (2B) gives

$$(-6a_0 + 12a_1 - 6a_2)\, C - b_1 + b_2 + 9b_3 - 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{45}{8} - 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{45}{64}$$

For $n = 5$, Eq (2B) gives

$$2(a_0 - 6a_1 + 9a_2 - 4a_3)\, C - 8b_3 + 24b_4 - 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{51}{5} - 15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{17}{25}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = \frac{1}{2}$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = \frac{1}{2}\left(x^2\left(1 + \frac{x^2}{8} + \frac{x^3}{5} + \frac{49x^4}{192} + \frac{423x^5}{1400} + O(x^6)\right)\right)\ln(x) \\ + 1 + x^3 + \frac{45x^4}{64} + \frac{17x^5}{25} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2\left(1 + \frac{x^2}{8} + \frac{x^3}{5} + \frac{49x^4}{192} + \frac{423x^5}{1400} + O(x^6)\right) \\ &\quad + c_2\left(\frac{1}{2}\left(x^2\left(1 + \frac{x^2}{8} + \frac{x^3}{5} + \frac{49x^4}{192} + \frac{423x^5}{1400} + O(x^6)\right)\right)\ln(x) + 1 + x^3 + \frac{45x^4}{64} \right. \\ &\quad \left. + \frac{17x^5}{25} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^2\left(1 + \frac{x^2}{8} + \frac{x^3}{5} + \frac{49x^4}{192} + \frac{423x^5}{1400} + O(x^6)\right) \\ &\quad + c_2\left(\frac{x^2\left(1 + \frac{x^2}{8} + \frac{x^3}{5} + \frac{49x^4}{192} + \frac{423x^5}{1400} + O(x^6)\right)\ln(x)}{2} + 1 + x^3 + \frac{45x^4}{64} + \frac{17x^5}{25} \right. \\ &\quad \left. + O(x^6)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^2 \left(1+\frac{x^2}{8}+\frac{x^3}{5}+\frac{49x^4}{192}+\frac{423x^5}{1400}+O\left(x^6\right)\right) \\
&+ c_2 \left(\frac{x^2 \left(1+\frac{x^2}{8}+\frac{x^3}{5}+\frac{49x^4}{192}+\frac{423x^5}{1400}+O\left(x^6\right)\right) \ln\left(x\right)}{2}+1+x^3+\frac{45x^4}{64}+\frac{17x^5}{25} \right. \\
&\left. + O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^2 \left(1+\frac{x^2}{8}+\frac{x^3}{5}+\frac{49x^4}{192}+\frac{423x^5}{1400}+O\left(x^6\right)\right) \\
&+ c_2 \left(\frac{x^2 \left(1+\frac{x^2}{8}+\frac{x^3}{5}+\frac{49x^4}{192}+\frac{423x^5}{1400}+O\left(x^6\right)\right) \ln\left(x\right)}{2}+1+x^3+\frac{45x^4}{64}+\frac{17x^5}{25} \right. \\
&\left. + O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a = 0, e <> 0, c <>
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 52

```
Order:=6;
dsolve(diff(y(x),x$2)-1/x*diff(y(x),x)+1/(x-1)^3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 + \frac{1}{8}x^2 + \frac{1}{5}x^3 + \frac{49}{192}x^4 + \frac{423}{1400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(-x^2 - \frac{1}{8}x^4 - \frac{1}{5}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(-2 - 2x^3 - \frac{45}{32}x^4 - \frac{34}{25}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 71

```
AsymptoticDSolveValue[y''[x]-1/x*y'[x]+1/(x-1)^3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{16}\left(x^2+8\right)x^2\log(x)+\frac{1}{64}\left(-5x^4+64x^3-400x^2+64\right)\right) + c_2\left(\frac{49x^6}{192}+\frac{x^5}{5}+\frac{x^4}{8}+x^2\right)$$

2.5 problem 5

Internal problem ID [5560]
Internal file name [`OUTPUT/4808_Sunday_June_05_2022_03_06_21_PM_37861751/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^{3}+4x\right)y''-2xy'+6y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^{3}+4x\right)y''-2xy'+6y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{2}{x^{2}+4}$$
$$q(x)=\frac{6}{x\left(x^{2}+4\right)}$$

Table 21: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2}{x^2+4}$	
singularity	type
$x = -2i$	"regular"
$x = 2i$	"regular"

$q(x) = \frac{6}{x(x^2+4)}$	
singularity	type
$x = 0$	"regular"
$x = -2i$	"regular"
$x = 2i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2i, 2i, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''(x^2+4)x - 2xy' + 6y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right)(x^2+4)x - 2x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 6\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) \\ + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 6a_n x^{n+r} = \sum_{n=1}^{\infty} 6a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) \\ + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 6a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$4x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$4x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$4x^{-1+r}r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$4x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{r-3}{2r(1+r)}$$

For $2 \leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r) + 4a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) + 6a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-2} + 2nr a_{n-2} + r^2 a_{n-2} - 5n a_{n-2} - 2n a_{n-1} - 5r a_{n-2} - 2r a_{n-1} + 6a_{n-2} + 8a_{n-1}}{4(n+r)(n+r-1)} \quad (4)$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-n^2 a_{n-2} + (3a_{n-2} + 2a_{n-1})n - 2a_{n-2} - 6a_{n-1}}{4n(1+n)} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-3}{2r(1+r)}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-r^4 + 2r^2 - 5r + 6}{4r(1+r)^2(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-3}{2r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+2r^2-5r+6}{4r(1+r)^2(2+r)}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-2r^5 + 9r^3 + 6r^2 + 17r - 6}{8r(1+r)^2(2+r)^2(3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-3}{2r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+2r^2-5r+6}{4r(1+r)^2(2+r)}$	$\frac{1}{24}$
a_3	$\frac{-2r^5+9r^3+6r^2+17r-6}{8r(1+r)^2(2+r)^2(3+r)}$	$\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^8 + 8r^7 + 19r^6 + 17r^5 + 9r^4 + 17r^3 - 5r^2 - 114r - 72}{16r(1+r)^2(2+r)^2(3+r)^2(4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{1}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-3}{2r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+2r^2-5r+6}{4r(1+r)^2(2+r)}$	$\frac{1}{24}$
a_3	$\frac{-2r^5+9r^3+6r^2+17r-6}{8r(1+r)^2(2+r)^2(3+r)}$	$\frac{1}{48}$
a_4	$\frac{r^8+8r^7+19r^6+17r^5+9r^4+17r^3-5r^2-114r-72}{16r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{3r^9 + 33r^8 + 124r^7 + 126r^6 - 396r^5 - 1408r^4 - 2077r^3 - 1967r^2 - 798r + 360}{32r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{5}{2304}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-3}{2r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+2r^2-5r+6}{4r(1+r)^2(2+r)}$	$\frac{1}{24}$
a_3	$\frac{-2r^5+9r^3+6r^2+17r-6}{8r(1+r)^2(2+r)^2(3+r)}$	$\frac{1}{48}$
a_4	$\frac{r^8+8r^7+19r^6+17r^5+9r^4+17r^3-5r^2-114r-72}{16r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{1}{384}$
a_5	$\frac{3r^9+33r^8+124r^7+126r^6-396r^5-1408r^4-2077r^3-1967r^2-798r+360}{32r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{5}{2304}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{r-3}{2r(1+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{r-3}{2r(1+r)} &= \lim_{r\to 0} \frac{r-3}{2r(1+r)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $y''(x^2+4)\,x - 2xy' + 6y = 0$ gives

$$\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)(x^2+4)\,x \\
&- 2x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ 6Cy_1(x)\ln(x) + 6\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(\left(y_1''(x)\left(x^2+4\right)x-2y_1'(x)\,x+6y_1(x)\right)\ln\left(x\right)+\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)\left(x^2+4\right)x\right.\right.$$
$$\left.-2y_1(x)\right)C+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\left(x^2+4\right)x \tag{7}$$
$$-2x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+6\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\left(x^2+4\right)x-2y_1'(x)\,x+6y_1(x)=0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)\left(x^2+4\right)x-2y_1(x)\right)C$$
$$+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\left(x^2+4\right)x \tag{8}$$
$$-2x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+6\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2x(x^2+4)\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)-(x^2+2x+4)\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} \tag{9}$$
$$+\frac{(x^4+4x^2)\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)\left(-1+n+r_2\right)\right)-2\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x^2+6\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)}{x}$$
$$=0$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2x(x^2+4)\left(\sum_{n=0}^{\infty} x^n a_n(1+n)\right) - (x^2+2x+4)\left(\sum_{n=0}^{\infty} a_n x^{1+n}\right)\right) C}{x} + \frac{(x^4+4x^2)\left(\sum_{n=0}^{\infty} x^{n-2} b_n n(n-1)\right) - 2\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x^2 + 6\left(\sum_{n=0}^{\infty} b_n x^n\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n(1+n)\right) + \left(\sum_{n=0}^{\infty} 8C\, x^n a_n(1+n)\right) + \sum_{n=0}^{\infty} \left(-C\, x^{n+2} a_n\right) \\
&+ \sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-4C a_n x^n\right) + \left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n(n-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 4n\, x^{n-1} b_n(n-1)\right) + \sum_{n=0}^{\infty} \left(-2x^n b_n n\right) + \left(\sum_{n=0}^{\infty} 6b_n x^n\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n(1+n) &= \sum_{n=3}^{\infty} 2C a_{n-3}(n-2)\, x^{n-1} \\
\sum_{n=0}^{\infty} 8C\, x^n a_n(1+n) &= \sum_{n=1}^{\infty} 8C a_{n-1} n\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-C\, x^{n+2} a_n\right) &= \sum_{n=3}^{\infty} \left(-C a_{n-3} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n\right) &= \sum_{n=2}^{\infty} \left(-2C a_{n-2} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-4C a_n x^n\right) &= \sum_{n=1}^{\infty} \left(-4C a_{n-1} x^{n-1}\right)
\end{aligned}$$

$$\sum_{n=0}^{\infty} n\,x^{1+n} b_n (n-1) = \sum_{n=2}^{\infty} (n-2)\, b_{n-2} (n-3)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-2x^n b_n n\right) = \sum_{n=1}^{\infty} \left(-2(n-1)\, b_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 6 b_n x^n = \sum_{n=1}^{\infty} 6 b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\left(\sum_{n=3}^{\infty} 2C a_{n-3} (n-2)\, x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 8C a_{n-1} n\, x^{n-1}\right) \\
&+ \sum_{n=3}^{\infty} \left(-C a_{n-3} x^{n-1}\right) + \sum_{n=2}^{\infty} \left(-2C a_{n-2} x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-4C a_{n-1} x^{n-1}\right) \\
&+ \left(\sum_{n=2}^{\infty} (n-2)\, b_{n-2} (n-3)\, x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 4n\, x^{n-1} b_n (n-1)\right) \\
&+ \sum_{n=1}^{\infty} \left(-2(n-1)\, b_{n-1} x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 6 b_{n-1} x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$4C + 6 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{3}{2}$$

For $n = 2$, Eq (2B) gives

$$(-2a_0 + 12a_1)\, C + 4b_1 + 8b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12 + 8b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{2}$$

For $n = 3$, Eq (2B) gives

$$(a_0 - 2a_1 + 20a_2)\, C + 2b_2 + 24b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{29}{4} + 24b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{29}{96}$$

For $n = 4$, Eq (2B) gives

$$(3a_1 - 2a_2 + 28a_3)\, C + 2b_2 + 48b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{3}{2} + 48b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{32}$$

For $n = 5$, Eq (2B) gives

$$(5a_2 - 2a_3 + 36a_4)\, C + 6b_3 - 2b_4 + 80b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{105}{64} + 80b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{21}{1024}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{3}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{3}{2}\left(x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + O(x^6)\right)\right)\ln(x) + 1 - \frac{3x^2}{2} + \frac{29x^3}{96} + \frac{x^4}{32} - \frac{21x^5}{1024} + O(x^6)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + O(x^6)\right) \\ &\quad + c_2\left(-\frac{3}{2}\left(x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + O(x^6)\right)\right)\ln(x) + 1 - \frac{3x^2}{2} + \frac{29x^3}{96} + \frac{x^4}{32} - \frac{21x^5}{1024} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + O(x^6)\right) \\ &\quad + c_2\left(-\frac{3x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + O(x^6)\right)\ln(x)}{2} + 1 - \frac{3x^2}{2} + \frac{29x^3}{96} + \frac{x^4}{32} - \frac{21x^5}{1024} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + O(x^6)\right) \\ &\quad + c_2\left(-\frac{3x\left(1 - \frac{x}{2} + \frac{x^2}{24} + \frac{x^3}{48} - \frac{x^4}{384} - \frac{5x^5}{2304} + O(x^6)\right)\ln(x)}{2} + 1 - \frac{3x^2}{2} + \frac{29x^3}{96} + \frac{x^4}{32} - \frac{21x^5}{1024} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1-\frac{x}{2}+\frac{x^2}{24}+\frac{x^3}{48}-\frac{x^4}{384}-\frac{5x^5}{2304}+O(x^6)\right) + c_2\left(-\frac{3x\left(1-\frac{x}{2}+\frac{x^2}{24}+\frac{x^3}{48}-\frac{x^4}{384}-\frac{5x^5}{2304}+O(x^6)\right)\ln(x)}{2}+1-\frac{3x^2}{2}+\frac{29x^3}{96}+\frac{x^4}{32} - \frac{21x^5}{1024}+O(x^6)\right)$$

Verified OK.

2.5.1 Maple step by step solution

Let's solve

$y''(x^2+4)\,x-2xy'+6y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{6y}{(x^2+4)x}+\frac{2y'}{x^2+4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2y'}{x^2+4}+\frac{6y}{(x^2+4)x}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{2}{x^2+4}, P_3(x)=\frac{6}{x(x^2+4)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''(x^2+4)\,x-2xy'+6y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  - Convert $x^m\cdot y''$ to series expansion for $m=1..3$

    $x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $4a_0r(-1+r)\,x^{-1+r}+(4a_1(1+r)\,r-2a_0(-3+r))\,x^r+\left(\sum_{k=1}^{\infty}(4a_{k+1}(k+1+r)\,(k+r)-2a_k(k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,1\}$

- Each term must be 0

  $4a_1(1+r)\,r-2a_0(-3+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $4a_{k+1}(k+1+r)\,(k+r)-2a_k(k+r-3)+a_{k-1}(k+r-1)\,(k-2+r)=0$

- Shift index using $k->k+1$

  $4a_{k+2}(k+2+r)\,(k+1+r)-2a_{k+1}(k-2+r)+a_k(k+r)\,(k+r-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{k^2a_k+2kra_k+r^2a_k-a_kk-2ka_{k+1}-a_kr-2ra_{k+1}+4a_{k+1}}{4(k+2+r)(k+1+r)}$

- Recursion relation for $r=0$

$$a_{k+2} = -\frac{k^2 a_k - a_k k - 2k a_{k+1} + 4a_{k+1}}{4(k+2)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{k^2 a_k - a_k k - 2k a_{k+1} + 4a_{k+1}}{4(k+2)(k+1)}, 6a_0 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{k^2 a_k + a_k k - 2k a_{k+1} + 2a_{k+1}}{4(k+3)(k+2)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k^2 a_k + a_k k - 2k a_{k+1} + 2a_{k+1}}{4(k+3)(k+2)}, 8a_1 + 4a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{k^2 a_k - k a_k - 2k a_{k+1} + 4a_{k+1}}{4(k+2)(k+1)}, 6a_0 = 0, b_{k+2} = -\frac{k^2 b_k + k b_k - 2k b_k}{4(k+3)(k}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 60

```
Order:=6;
dsolve((x^3+4*x)*diff(y(x),x$2)-2*x*diff(y(x),x)+6*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x\left(1-\frac{1}{2}x+\frac{1}{24}x^2+\frac{1}{48}x^3-\frac{1}{384}x^4-\frac{5}{2304}x^5+\mathrm{O}\left(x^6\right)\right) \\
& + c_2\left(\ln\left(x\right)\left(-\frac{3}{2}x+\frac{3}{4}x^2-\frac{1}{16}x^3-\frac{1}{32}x^4+\frac{1}{256}x^5+\mathrm{O}\left(x^6\right)\right)\right. \\
& \left.+\left(1+\frac{1}{2}x-\frac{7}{4}x^2+\frac{31}{96}x^3+\frac{1}{24}x^4-\frac{67}{3072}x^5+\mathrm{O}\left(x^6\right)\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 85

```
AsymptoticDSolveValue[(x^3+4*x)*y''[x]-2*x*y'[x]+6*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{96} \left(7x^4 + 37x^3 - 240x^2 + 192x + 96 \right) - \frac{1}{32} x \left(x^3 + 2x^2 - 24x + 48 \right) \log(x) \right) + c_2 \left(-\frac{x^5}{384} + \frac{x^4}{48} + \frac{x^3}{24} - \frac{x^2}{2} + x \right)$$

2.6 problem 6

Internal problem ID [5561]
Internal file name [`OUTPUT/4809_Sunday_June_05_2022_03_06_24_PM_93524911/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x-5)^2 y''+4xy'+\left(x^2-25\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^4-10x^3+25x^2\right)y''+4xy'+\left(x^2-25\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{4}{x\,(x-5)^2}$$
$$q(x)=\frac{5+x}{(x-5)\,x^2}$$

Table 23: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4}{x(x-5)^2}$	
singularity	type
$x = 0$	"regular"
$x = 5$	"irregular"

$q(x) = \frac{5+x}{(x-5)x^2}$	
singularity	type
$x = 0$	"regular"
$x = 5$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[5]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 (x^2 - 10x + 25) y'' + 4xy' + (x^2 - 25) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 (x^2 - 10x + 25) \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) + 4x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + (x^2 - 25) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-10 x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 25 x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-25 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-10 x^{1+n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-10 a_{n-1} (n+r-1)(n+r-2) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-10 a_{n-1} (n+r-1)(n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 25 x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-25 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$25 x^{n+r} a_n (n+r)(n+r-1) + 4 x^{n+r} a_n (n+r) - 25 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$25x^r a_0 r(-1+r) + 4x^r a_0 r - 25a_0 x^r = 0$$

Or

$$(25x^r r(-1+r) + 4x^r r - 25x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(25r^2 - 21r - 25\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$25r^2 - 21r - 25 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{21}{50} + \frac{\sqrt{2941}}{50}$$
$$r_2 = \frac{21}{50} - \frac{\sqrt{2941}}{50}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(25r^2 - 21r - 25\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{\sqrt{2941}}{25}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{21}{50}+\frac{\sqrt{2941}}{50}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{21}{50}-\frac{\sqrt{2941}}{50}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{10r(-1+r)}{25r^2 + 29r - 21}$$

For $2 \leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r) - 10a_{n-1}(n+r-1)(n+r-2) + 25a_n(n+r)(n+r-1) + 4a_n(n+r) + a_{n-2} - 25a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-2} - 10n^2 a_{n-1} + 2nra_{n-2} - 20nra_{n-1} + r^2 a_{n-2} - 10r^2 a_{n-1} - 5na_{n-2} + 30na_{n-1} - 5ra_{n-2} +}{25n^2 + 50nr + 25r^2 - 21n - 21r - 25} \tag{4}$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$a_n = \frac{((-25a_{n-2} + 250a_{n-1})\,n + 52a_{n-2} - 270a_{n-1})\sqrt{2941} + (-625a_{n-2} + 6250a_{n-1})\,n^2 + (2600a_{n-2} - 1}{625n\left(\sqrt{2941} + 25n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$\frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right)}{6250+250\sqrt{2941}}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{75r^4 - 4r^3 - 75r^2 - 50r + 21}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)}$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$a_2 = \frac{\frac{717381}{15625} + \frac{7911\sqrt{2941}}{15625}}{\left(25 + \sqrt{2941}\right)\left(50 + \sqrt{2941}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$\frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right)}{6250+250\sqrt{2941}}$
a_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}+\frac{7911\sqrt{2941}}{15625}}{\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{500r^6 + 1420r^5 + 300r^4 - 2580r^3 - 2000r^2 - 40r + 420}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r^2 + 129r + 137)}$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$a_3 = \frac{\frac{906742764}{1953125} + \frac{15291084\sqrt{2941}}{1953125}}{(25 + \sqrt{2941})(50 + \sqrt{2941})(75 + \sqrt{2941})}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$\frac{(21+\sqrt{2941})(-29+\sqrt{2941})}{6250+250\sqrt{2941}}$
a_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}+\frac{7911\sqrt{2941}}{15625}}{(25+\sqrt{2941})(50+\sqrt{2941})}$
a_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	$\frac{\frac{906742764}{1953125}+\frac{15291084\sqrt{2941}}{1953125}}{(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3125r^8 + 24000r^7 + 61766r^6 + 33496r^5 - 104033r^4 - 168040r^3 - 59654r^2 + 22392r + 16569}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r^2 + 129r + 137)(25r^2 + 179r + 291)}$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$a_4 = \frac{\frac{1473770634612}{244140625} + \frac{26407796172\sqrt{2941}}{244140625}}{(25 + \sqrt{2941})(50 + \sqrt{2941})(75 + \sqrt{2941})(100 + \sqrt{2941})}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$\frac{(21+\sqrt{2941})(-29+\sqrt{2941})}{6250+250\sqrt{2941}}$
a_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}+\frac{7911\sqrt{2941}}{15625}}{(25+\sqrt{2941})(50+\sqrt{2941})}$
a_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	$\frac{\frac{906742764}{1953125}+\frac{15291084\sqrt{2941}}{1953125}}{(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})}$
a_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$	$\frac{\frac{1473770634612}{244140625}+\frac{26407796172\sqrt{2941}}{244140625}}{(25+\sqrt{2941})(50+\sqrt{2941})(75+\sqrt{2941})(100+}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{18750r^{10} + 271250r^9 + 1552980r^8 + 4225260r^7 + 4278470r^6 - 4527750r^5 - 15744100r^4 - 1351844C}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r^2 + 129r + 137)(25r^2 + 179r + 291)(2}$$

Which for the root $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$ becomes

$$a_5 = \frac{\frac{10008934775328384}{152587890625} + \frac{181292058002304\sqrt{2941}}{152587890625}}{(25 + \sqrt{2941})(50 + \sqrt{2941})(75 + \sqrt{2941})(100 + \sqrt{2941})(125 + \sqrt{2941})}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$(2$
a_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$(2$
a_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	$(2$
a_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$	$(2$
a_5	$\frac{18750r^{10}+271250r^9+1552980r^8+4225260r^7+4278470r^6-4527750r^5-15744100r^4-13518440r^3-1814650r^2+2790990r+1132740}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)}$	$(2$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}}\left(1 + \frac{(21+\sqrt{2941})(-29+\sqrt{2941})\,x}{6250+250\sqrt{2941}} + \frac{9(79709+879\sqrt{2941})\,x^2}{15625\,(25+\sqrt{2941})(50+\sqrt{2941})} + \frac{}{195312\ldots}\right. \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{10r(-1+r)}{25r^2 + 29r - 21}$$

For $2 \le n$ the recursive equation is

$$\begin{aligned} b_{n-2}(n+r-2)(n-3+r) - 10b_{n-1}(n+r-1)(n+r-2) \\ + 25b_n(n+r)(n+r-1) + 4b_n(n+r) + b_{n-2} - 25b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{n^2 b_{n-2} - 10n^2 b_{n-1} + 2nrb_{n-2} - 20nrb_{n-1} + r^2 b_{n-2} - 10r^2 b_{n-1} - 5nb_{n-2} + 30nb_{n-1} - 5rb_{n-2} + 30}{25n^2 + 50nr + 25r^2 - 21n - 21r - 25} \tag{4}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_n = \frac{\left(\left(25b_{n-2} - 250b_{n-1}\right)n - 52b_{n-2} + 270b_{n-1}\right)\sqrt{2941} + \left(-625b_{n-2} + 6250b_{n-1}\right)n^2 + \left(2600b_{n-2} - 1350\right.}{625n\left(-\sqrt{2941} + 25n\right)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$-\frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)}{-6250+250\sqrt{2941}}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{75r^4 - 4r^3 - 75r^2 - 50r + 21}{\left(25r^2 + 29r - 21\right)\left(25r^2 + 79r + 33\right)}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_2 = \frac{\frac{717381}{15625} - \frac{7911\sqrt{2941}}{15625}}{\left(-25 + \sqrt{2941}\right)\left(-50 + \sqrt{2941}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$-\frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)}{-6250+250\sqrt{2941}}$
b_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}-\frac{7911\sqrt{2941}}{15625}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{500r^6 + 1420r^5 + 300r^4 - 2580r^3 - 2000r^2 - 40r + 420}{\left(25r^2 + 29r - 21\right)\left(25r^2 + 79r + 33\right)\left(25r^2 + 129r + 137\right)}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_3 = \frac{-\frac{906742764}{1953125} + \frac{15291084\sqrt{2941}}{1953125}}{\left(-25 + \sqrt{2941}\right)\left(-50 + \sqrt{2941}\right)\left(-75 + \sqrt{2941}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$-\frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)}{-6250+250\sqrt{2941}}$
b_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}-\frac{7911\sqrt{2941}}{15625}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)}$
b_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	$\frac{-\frac{906742764}{1953125}+\frac{15291084\sqrt{2941}}{1953125}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{3125r^8 + 24000r^7 + 61766r^6 + 33496r^5 - 104033r^4 - 168040r^3 - 59654r^2 + 22392r + 16569}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r^2 + 129r + 137)(25r^2 + 179r + 291)}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_4 = \frac{\frac{1473770634612}{244140625} - \frac{26407796172\sqrt{2941}}{244140625}}{\left(-25 + \sqrt{2941}\right)\left(-50 + \sqrt{2941}\right)\left(-75 + \sqrt{2941}\right)\left(-100 + \sqrt{2941}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$-\frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)}{-6250+250\sqrt{2941}}$
b_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{\frac{717381}{15625}-\frac{7911\sqrt{2941}}{15625}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)}$
b_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	$\frac{-\frac{906742764}{1953125}+\frac{15291084\sqrt{2941}}{1953125}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)}$
b_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$	$\frac{\frac{1473770634612}{244140625}-\frac{26407796172\sqrt{2941}}{244140625}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(\right.}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{18750r^{10} + 271250r^9 + 1552980r^8 + 4225260r^7 + 4278470r^6 - 4527750r^5 - 15744100r^4 - 13518440}{(25r^2 + 29r - 21)(25r^2 + 79r + 33)(25r^2 + 129r + 137)(25r^2 + 179r + 291)(2}$$

Which for the root $r = \frac{21}{50} - \frac{\sqrt{2941}}{50}$ becomes

$$b_5 = \frac{-\frac{10008934775328384}{152587890625} + \frac{181292058002304\sqrt{2941}}{152587890625}}{(-25 + \sqrt{2941})(-50 + \sqrt{2941})(-75 + \sqrt{2941})(-100 + \sqrt{2941})(-125 + \sqrt{2941})}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r(-1+r)}{25r^2+29r-21}$	$-\frac{(}{}$
b_2	$\frac{75r^4-4r^3-75r^2-50r+21}{(25r^2+29r-21)(25r^2+79r+33)}$	$\frac{}{(-}$
b_3	$\frac{500r^6+1420r^5+300r^4-2580r^3-2000r^2-40r+420}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)}$	$\frac{}{(-}$
b_4	$\frac{3125r^8+24000r^7+61766r^6+33496r^5-104033r^4-168040r^3-59654r^2+22392r+16569}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)}$	$\frac{}{(-}$
b_5	$\frac{18750r^{10}+271250r^9+1552980r^8+4225260r^7+4278470r^6-4527750r^5-15744100r^4-13518440r^3-1814650r^2+2790990r+1132740}{(25r^2+29r-21)(25r^2+79r+33)(25r^2+129r+137)(25r^2+179r+291)(25r^2+229r+495)}$	$\frac{}{(-}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}}(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots) \\ &= x^{\frac{21}{50}-\frac{\sqrt{2941}}{50}}\left(1 - \frac{(-21 + \sqrt{2941})(29 + \sqrt{2941})x}{-6250 + 250\sqrt{2941}} + \frac{9(79709 - 879\sqrt{2941})x^2}{15625(-25 + \sqrt{2941})(-50 + \sqrt{2941})} + \frac{}{195}\right. \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1y_1(x) + c_2y_2(x)$$

$$
\begin{aligned}
&= c_1 x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}} \left(1 + \frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right) x}{6250 + 250\sqrt{2941}} \right. \\
&+ \frac{9\left(79709 + 879\sqrt{2941}\right) x^2}{15625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)} \\
&+ \frac{12\left(75561897 + 1274257\sqrt{2941}\right) x^3}{1953125\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)} \\
&+ \frac{12\left(122814219551 + 2200649681\sqrt{2941}\right) x^4}{244140625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)} \\
&+ \frac{1152\left(8688311436917 + 157371578127\sqrt{2941}\right) x^5}{152587890625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)\left(125+\sqrt{2941}\right)} \\
&\left. + O\left(x^6\right)\right) + c_2 x^{\frac{21}{50}-\frac{\sqrt{2941}}{50}} \left(1 - \frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right) x}{-6250 + 250\sqrt{2941}} \right. \\
&+ \frac{9\left(79709 - 879\sqrt{2941}\right) x^2}{15625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)} \\
&+ \frac{12\left(-75561897 + 1274257\sqrt{2941}\right) x^3}{1953125\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)} \\
&+ \frac{12\left(122814219551 - 2200649681\sqrt{2941}\right) x^4}{244140625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)} \\
&+ \frac{1152\left(-8688311436917 + 157371578127\sqrt{2941}\right) x^5}{152587890625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)\left(-125+\sqrt{29\right.}} \\
&\left. + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$y = y_h$

$$= c_1 x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}} \left(1 + \frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right)x}{6250+250\sqrt{2941}} + \frac{9\left(79709+879\sqrt{2941}\right)x^2}{15625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)} + \frac{12\left(75561897+1274257\sqrt{2941}\right)x^3}{1953125\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)} + \frac{12\left(122814219551+2200649681\sqrt{2941}\right)x^4}{244140625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)} + \frac{1152\left(8688311436917+157371578127\sqrt{2941}\right)x^5}{152587890625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)\left(125+\sqrt{2941}\right)} + O\left(x^6\right)\right) + c_2 x^{\frac{21}{50}-\frac{\sqrt{2941}}{50}} \left(1 - \frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)x}{-6250+250\sqrt{2941}} + \frac{9\left(79709-879\sqrt{2941}\right)x^2}{15625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)} + \frac{12\left(-75561897+1274257\sqrt{2941}\right)x^3}{1953125\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)} + \frac{12\left(122814219551-2200649681\sqrt{2941}\right)x^4}{244140625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)} + \frac{1152\left(-8688311436917+157371578127\sqrt{2941}\right)x^5}{152587890625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)\left(-125+\sqrt{2941}\right)} + O\left(x^6\right)\right)$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}} \left(1+\frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right) x}{6250+250\sqrt{2941}} \right. \\
& + \frac{9\left(79709+879\sqrt{2941}\right) x^2}{15625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)} \\
& + \frac{12\left(75561897+1274257\sqrt{2941}\right) x^3}{1953125\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)} \\
& + \frac{12\left(122814219551+2200649681\sqrt{2941}\right) x^4}{244140625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)} \\
& + \frac{1152\left(8688311436917+157371578127\sqrt{2941}\right) x^5}{152587890625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)\left(125+\sqrt{2941}\right)} \\
& \left. + O\left(x^6\right)\right) + c_2 x^{\frac{21}{50}-\frac{\sqrt{2941}}{50}} \left(1-\frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right) x}{-6250+250\sqrt{2941}} \right. \\
& + \frac{9\left(79709-879\sqrt{2941}\right) x^2}{15625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)} \\
& + \frac{12\left(-75561897+1274257\sqrt{2941}\right) x^3}{1953125\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)} \\
& + \frac{12\left(122814219551-2200649681\sqrt{2941}\right) x^4}{244140625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)} \\
& + \frac{1152\left(-8688311436917+157371578127\sqrt{2941}\right) x^5}{152587890625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)\left(-125+\sqrt{2941}\right)} \\
& \left. + O\left(x^6\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{21}{50}+\frac{\sqrt{2941}}{50}} \left(1+\frac{\left(21+\sqrt{2941}\right)\left(-29+\sqrt{2941}\right)x}{6250+250\sqrt{2941}} + \frac{9\left(79709+879\sqrt{2941}\right)x^2}{15625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)} + \frac{12\left(75561897+1274257\sqrt{2941}\right)x^3}{1953125\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)} + \frac{12\left(122814219551+2200649681\sqrt{2941}\right)x^4}{244140625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)} + \frac{1152\left(8688311436917+157371578127\sqrt{2941}\right)x^5}{152587890625\left(25+\sqrt{2941}\right)\left(50+\sqrt{2941}\right)\left(75+\sqrt{2941}\right)\left(100+\sqrt{2941}\right)\left(125+\sqrt{2941}\right)} + O\left(x^6\right)\right) + c_2 x^{\frac{21}{50}-\frac{\sqrt{2941}}{50}} \left(1-\frac{\left(-21+\sqrt{2941}\right)\left(29+\sqrt{2941}\right)x}{-6250+250\sqrt{2941}} + \frac{9\left(79709-879\sqrt{2941}\right)x^2}{15625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)} + \frac{12\left(-75561897+1274257\sqrt{2941}\right)x^3}{1953125\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)} + \frac{12\left(122814219551-2200649681\sqrt{2941}\right)x^4}{244140625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)} + \frac{1152\left(-8688311436917+157371578127\sqrt{2941}\right)x^5}{152587890625\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)\left(-125+\sqrt{2941}\right)} + O\left(x^6\right)\right)$$

Verified OK.

2.6.1 Maple step by step solution

Let's solve

$x^2\left(x^2-10x+25\right)y''+4xy'+\left(x^2-25\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(5+x)y}{(x-5)x^2}-\frac{4y'}{x\left(x^2-10x+25\right)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4y'}{x(x^2-10x+25)} + \frac{(5+x)y}{(x-5)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{4}{x(x^2-10x+25)}, P_3(x) = \frac{5+x}{(x-5)x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{4}{25}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -1$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2(x^2-10x+25)(x-5) + 4y'x(x-5) + (x^2-10x+25)(5+x)y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..3$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..5$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-5a_0(25r^2-21r-25)\, x^r + (-5a_1(25r^2+29r-21) + a_0(75r^2-71r-25))\, x^{1+r} + (-5a_2(25r^2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-125r^2+105r+125=0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{21}{50}-\frac{\sqrt{2941}}{50}, \frac{21}{50}+\frac{\sqrt{2941}}{50}\right\}$$

- The coefficients of each power of x must be 0

  $$[-5a_1(25r^2+29r-21)+a_0(75r^2-71r-25)=0, -5a_2(25r^2+79r+33)+a_1(75r^2+79r-2$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{a_0(75r^2-71r-25)}{5(25r^2+29r-21)}, a_2 = \frac{6a_0(625r^4+50r^3-989r^2-464r+175)}{25(625r^4+2700r^3+2591r^2-702r-693)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(-125a_k+a_{k-3}-15a_{k-2}+75a_{k-1})\, k^2 + (2(-125a_k+a_{k-3}-15a_{k-2}+75a_{k-1})\, r + 105a_k - 7a_k$$

- Shift index using $k- >k+3$

  $$(-125a_{k+3}+a_k-15a_{k+1}+75a_{k+2})\,(k+3)^2 + (2(-125a_{k+3}+a_k-15a_{k+1}+75a_{k+2})\, r + 105a_{k}$$

- Recursion relation that defines series solution to ODE

  $$a_{k+3} = \frac{k^2a_k-15k^2a_{k+1}+75k^2a_{k+2}+2kra_k-30kra_{k+1}+150kra_{k+2}+r^2a_k-15r^2a_{k+1}+75r^2a_{k+2}-ka_k-15ka_{k+1}+229ka_{k+2}-ra}{5(25k^2+50kr+25r^2+129k+129r+137)}$$

- Recursion relation for $r = \frac{21}{50}-\frac{\sqrt{2941}}{50}$

  $$a_{k+3} = \frac{k^2a_k-15k^2a_{k+1}+75k^2a_{k+2}+2k\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)a_k-30k\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)a_{k+1}+150k\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)a_{k+2}+\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)^2a_k-15(}{5\left(25k^2+50k\left(\frac{21}{50}\right.\right.}$$

- Solution for $r = \frac{21}{50}-\frac{\sqrt{2941}}{50}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{21}{50}-\frac{\sqrt{2941}}{50}}, a_{k+3} = \frac{k^2a_k-15k^2a_{k+1}+75k^2a_{k+2}+2k\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)a_k-30k\left(\frac{21}{50}-\frac{\sqrt{2941}}{50}\right)a_{k+1}+150k\left(\frac{21}{50}-\frac{\sqrt{29}}{5}\right.}{}\right.$$

- Recursion relation for $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$

$$a_{k+3} = \frac{k^2 a_k - 15k^2 a_{k+1} + 75k^2 a_{k+2} + 2k\left(\frac{21}{50} + \frac{\sqrt{2941}}{50}\right) a_k - 30k\left(\frac{21}{50} + \frac{\sqrt{2941}}{50}\right) a_{k+1} + 150k\left(\frac{21}{50} + \frac{\sqrt{2941}}{50}\right) a_{k+2} + \left(\frac{21}{50} + \frac{\sqrt{2941}}{50}\right)^2 a_k - 15(}{5\left(25k^2 + 50k\left(\frac{21}{50}\right.\right.}$$

- Solution for $r = \frac{21}{50} + \frac{\sqrt{2941}}{50}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{21}{50}+\frac{\sqrt{2941}}{50}}, a_{k+3} = \frac{k^2 a_k - 15k^2 a_{k+1} + 75k^2 a_{k+2} + 2k\left(\frac{21}{50} + \frac{\sqrt{2941}}{50}\right) a_k - 30k\left(\frac{21}{50} + \frac{\sqrt{2941}}{50}\right) a_{k+1} + 150k\left(\frac{21}{50} + \frac{\sqrt{29}}{5}\right.}{}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{21}{50}-\frac{\sqrt{2941}}{50}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{21}{50}+\frac{\sqrt{2941}}{50}}\right), a_{k+3} = \frac{k^2 a_k - 15k^2 a_{k+1} + 75k^2 a_{k+2} + 2k\left(\frac{21}{50} - \frac{\sqrt{2941}}{50}\right) a_k - 3}{}\right.$$

<u>Maple trace</u>

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a = 0, e <> 0, c <>
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 1179

```
Order:=6;
dsolve(x^2*(x-5)^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(x^2-25)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^{\frac{21}{50}}\left(c_1 x^{-\frac{\sqrt{2941}}{50}}\left(1+\frac{-1166-4\sqrt{2941}}{-3125+125\sqrt{2941}}x-\frac{9}{15625}\frac{879\sqrt{2941}-79709}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)}x^2 + \frac{\frac{15291084\sqrt{2941}}{1953125}-\frac{906742764}{1953125}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)}x^3 - \frac{12}{244140625}\frac{2200649681\sqrt{2941}-122814219551}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)}x^4 + \frac{\frac{181292058002304\sqrt{2941}}{152587890625}-\frac{10008934775328384}{152587890625}}{\left(-25+\sqrt{2941}\right)\left(-50+\sqrt{2941}\right)\left(-75+\sqrt{2941}\right)\left(-100+\sqrt{2941}\right)\left(-125+\sqrt{2941}\right)}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\frac{\sqrt{2941}}{50}}\left(1+\frac{1166-4\sqrt{2941}}{125\sqrt{2941}+3125}x+\frac{\frac{7911\sqrt{2941}}{15625}+\frac{717381}{15625}}{\left(\sqrt{2941}+25\right)\left(50+\sqrt{2941}\right)}x^2 + \frac{\frac{15291084\sqrt{2941}}{1953125}+\frac{906742764}{1953125}}{\left(\sqrt{2941}+25\right)\left(50+\sqrt{2941}\right)\left(\sqrt{2941}+75\right)}x^3 + \frac{\frac{26407796172\sqrt{2941}}{244140625}+\frac{1473770634612}{244140625}}{\left(\sqrt{2941}+25\right)\left(50+\sqrt{2941}\right)\left(\sqrt{2941}+75\right)\left(100+\sqrt{2941}\right)}x^4 + \frac{\frac{181292058002304\sqrt{2941}}{152587890625}+\frac{10008934775328384}{152587890625}}{\left(\sqrt{2941}+25\right)\left(50+\sqrt{2941}\right)\left(\sqrt{2941}+75\right)\left(100+\sqrt{2941}\right)\left(125+\sqrt{2941}\right)}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 5384

```
AsymptoticDSolveValue[x^2*(x-5)^2*y''[x]+4*x*y'[x]+(x^2-25)*y[x]==0,y[x],{x,0,5}]
```

Too large to display

2.7 problem 7

Internal problem ID [5562]
Internal file name [OUTPUT/4810_Sunday_June_05_2022_03_06_27_PM_7600750/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+x-6)\,y''+(x+3)\,y'+(-2+x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy_0'+\frac{x^2}{2}\,f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y_0'}+\cdots\\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (77)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (78)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{xy + xy' - 2y + 3y'}{x^2 + x - 6} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(-x^3 + 3x^2 + 20x + 6\right) y' + y(2x^2 - 3x - 2)}{\left(x^2 + x - 6\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\left(4x^4 - 8x^3 - 88x^2 - 130x - 138\right) y' + y(-2 + x)\left(x^3 - 7x^2 - 19x - 17\right)}{\left(x^2 + x - 6\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x+3)\left(x^5 - 23x^4 + 73x^3 + 219x^2 + 704x + 468\right) y' - 6\left(x^4 - \frac{16}{3}x^3 - 22x^2 - \frac{140}{3}x - 42\right) y(-2 + x)}{\left(x^2 + x - 6\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(-9x^7 + 117x^6 + 132x^5 - 2220x^4 - 12075x^3 - 36237x^2 - 43104x - 22284\right) y' - y(-2 + x)\left(x^6 - 38\right.}{\left(x^2 + x - 6\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{y(0)}{3} + \frac{y'(0)}{2} \\
F_1 &= -\frac{y(0)}{18} + \frac{y'(0)}{6} \\
F_2 &= -\frac{17y(0)}{108} + \frac{23y'(0)}{36} \\
F_3 &= -\frac{7y(0)}{18} + \frac{13y'(0)}{12} \\
F_4 &= -\frac{139y(0)}{162} + \frac{619y'(0)}{216}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{6}x^{2}-\frac{1}{108}x^{3}-\frac{17}{2592}x^{4}-\frac{7}{2160}x^{5}-\frac{139}{116640}x^{6}\right)y(0) \\
& + \left(x+\frac{1}{4}x^{2}+\frac{1}{36}x^{3}+\frac{23}{864}x^{4}+\frac{13}{1440}x^{5}+\frac{619}{155520}x^{6}\right)y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x^{2}+x-6\right)y''+(x+3)\,y'+(-2+x)\,y=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\,a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(x^{2}+x-6\right)\left(\sum_{n=2}^{\infty} n(n-1)\,a_{n} x^{n-2}\right)+(x+3)\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)+(-2+x)\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)=0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
& \left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} n\,x^{n-1} a_{n}(n-1)\right)+\sum_{n=2}^{\infty}\left(-6n(n-1)\,a_{n} x^{n-2}\right) \\
& + \left(\sum_{n=1}^{\infty} n a_{n} x^{n}\right)+\left(\sum_{n=1}^{\infty} 3n a_{n} x^{n-1}\right)+\sum_{n=0}^{\infty}\left(-2 a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} x^{1+n} a_{n}\right)=0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n$$

$$\sum_{n=2}^{\infty} \left(-6n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-6(n+2)\, a_{n+2}(1+n)\, x^n\right)$$

$$\sum_{n=1}^{\infty} 3n a_n x^{n-1} = \sum_{n=0}^{\infty} 3(1+n)\, a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{gathered}\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n\right) \\ + \sum_{n=0}^{\infty} \left(-6(n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) \\ + \left(\sum_{n=0}^{\infty} 3(1+n)\, a_{1+n} x^n\right) + \sum_{n=0}^{\infty} (-2 a_n x^n) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0\end{gathered} \tag{3}$$

$n = 0$ gives

$$-12a_2 + 3a_1 - 2a_0 = 0$$

$$a_2 = -\frac{a_0}{6} + \frac{a_1}{4}$$

$n = 1$ gives

$$8a_2 - 36a_3 - a_1 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{108} + \frac{a_1}{36}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + (1+n)\, a_{1+n} n - 6(n+2)\, a_{n+2} (1+n) + n a_n + 3(1+n)\, a_{1+n} - 2a_n + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_n + n^2 a_{1+n} + 4n a_{1+n} - 2a_n + 3a_{1+n} + a_{n-1}}{6\,(n+2)\,(1+n)} \\ &= \frac{(n^2-2)\, a_n}{6\,(n+2)\,(1+n)} + \frac{(n^2+4n+3)\, a_{1+n}}{6\,(n+2)\,(1+n)} + \frac{a_{n-1}}{6\,(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$2a_2 + 15a_3 - 72a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{17a_0}{2592} + \frac{23a_1}{864}$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 24a_4 - 120a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{7a_0}{2160} + \frac{13a_1}{1440}$$

For $n = 4$ the recurrence equation gives

$$14a_4 + 35a_5 - 180a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{139a_0}{116640} + \frac{619a_1}{155520}$$

For $n = 5$ the recurrence equation gives

$$23a_5 + 48a_6 - 252a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{5377a_0}{9797760} + \frac{689a_1}{408240}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1x + \left(-\frac{a_0}{6} + \frac{a_1}{4}\right)x^2 + \left(-\frac{a_0}{108} + \frac{a_1}{36}\right)x^3 \\ + \left(-\frac{17a_0}{2592} + \frac{23a_1}{864}\right)x^4 + \left(-\frac{7a_0}{2160} + \frac{13a_1}{1440}\right)x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y = \left(1 - \frac{1}{6}x^2 - \frac{1}{108}x^3 - \frac{17}{2592}x^4 - \frac{7}{2160}x^5\right)a_0 \\ + \left(x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{23}{864}x^4 + \frac{13}{1440}x^5\right)a_1 + O(x^6) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned} y = \left(1 - \frac{1}{6}x^2 - \frac{1}{108}x^3 - \frac{17}{2592}x^4 - \frac{7}{2160}x^5\right)c_1 \\ + \left(x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{23}{864}x^4 + \frac{13}{1440}x^5\right)c_2 + O(x^6) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = \left(1 - \frac{1}{6}x^2 - \frac{1}{108}x^3 - \frac{17}{2592}x^4 - \frac{7}{2160}x^5 - \frac{139}{116640}x^6\right)y(0) \\ + \left(x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{23}{864}x^4 + \frac{13}{1440}x^5 + \frac{619}{155520}x^6\right)y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$\begin{aligned} y = \left(1 - \frac{1}{6}x^2 - \frac{1}{108}x^3 - \frac{17}{2592}x^4 - \frac{7}{2160}x^5\right)c_1 \\ + \left(x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{23}{864}x^4 + \frac{13}{1440}x^5\right)c_2 + O(x^6) \end{aligned} \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{6}x^{2}-\frac{1}{108}x^{3}-\frac{17}{2592}x^{4}-\frac{7}{2160}x^{5}-\frac{139}{116640}x^{6}\right)y(0) + \left(x+\frac{1}{4}x^{2}+\frac{1}{36}x^{3}+\frac{23}{864}x^{4}+\frac{13}{1440}x^{5}+\frac{619}{155520}x^{6}\right)y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1-\frac{1}{6}x^{2}-\frac{1}{108}x^{3}-\frac{17}{2592}x^{4}-\frac{7}{2160}x^{5}\right)c_{1} + \left(x+\frac{1}{4}x^{2}+\frac{1}{36}x^{3}+\frac{23}{864}x^{4}+\frac{13}{1440}x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Verified OK.

2.7.1 Maple step by step solution

Let's solve

$(x^2+x-6)\,y''+(x+3)\,y'+(-2+x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x+3}-\frac{y'}{-2+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{-2+x}+\frac{y}{x+3}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1}{-2+x}, P_3(x)=\frac{1}{x+3}\right]$

  - $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3}=0$

  - $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

    $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3}=0$

  - $x=-3$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -3$

- Multiply by denominators

$y''(-2+x)(x+3) + (x+3)y' + (-2+x)y = 0$

- Change variables using $x = u-3$ so that the regular singular point is at $u = 0$

$(u^2-5u)\left(\frac{d^2}{du^2}y(u)\right) + u\left(\frac{d}{du}y(u)\right) + (-5+u)y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)u^{k+r}$

  Rewrite ODE with series expansions

  $-5a_0r(-1+r)u^{-1+r} + (-5a_1(1+r)r + a_0(r^2-5))u^r + \left(\sum_{k=1}^{\infty} (-5a_{k+1}(k+1+r)(k+r) + a_k($

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-5r(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 1\}$

- Each term must be 0

$$-5a_1(1+r)\,r + a_0(r^2-5) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-5a_{k+1}(k+1+r)\,(k+r) + k^2a_k + 2kra_k + r^2a_k - 5a_k + a_{k-1} = 0$$

- Shift index using $k->k+1$

$$-5a_{k+2}(k+2+r)\,(k+1+r) + (k+1)^2\,a_{k+1} + 2(k+1)\,ra_{k+1} + r^2a_{k+1} - 5a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}+2ka_{k+1}+2ra_{k+1}+a_k-4a_{k+1}}{5(k+2+r)(k+1+r)}$$

- Recursion relation for $r=0$

$$a_{k+2} = \frac{k^2a_{k+1}+2ka_{k+1}+a_k-4a_{k+1}}{5(k+2)(k+1)}$$

- Solution for $r=0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = \frac{k^2a_{k+1}+2ka_{k+1}+a_k-4a_{k+1}}{5(k+2)(k+1)}, -5a_0 = 0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+3)^k, a_{k+2} = \frac{k^2a_{k+1}+2ka_{k+1}+a_k-4a_{k+1}}{5(k+2)(k+1)}, -5a_0 = 0\right]$$

- Recursion relation for $r=1$

$$a_{k+2} = \frac{k^2a_{k+1}+4ka_{k+1}+a_k-a_{k+1}}{5(k+3)(k+2)}$$

- Solution for $r=1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = \frac{k^2a_{k+1}+4ka_{k+1}+a_k-a_{k+1}}{5(k+3)(k+2)}, -10a_1 - 4a_0 = 0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+3)^{k+1}, a_{k+2} = \frac{k^2a_{k+1}+4ka_{k+1}+a_k-a_{k+1}}{5(k+3)(k+2)}, -10a_1 - 4a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k(x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+3)^{k+1}\right), a_{k+2} = \frac{k^2a_{k+1}+2ka_{k+1}+a_k-4a_{k+1}}{5(k+2)(k+1)}, -5a_0 = 0, b_{k+2} = \frac{k}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
Order:=6;
dsolve((x^2+x-6)*diff(y(x),x$2)+(x+3)*diff(y(x),x)+(x-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{6}x^2 - \frac{1}{108}x^3 - \frac{17}{2592}x^4 - \frac{7}{2160}x^5\right)y(0) + \left(x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{23}{864}x^4 + \frac{13}{1440}x^5\right)D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 70

```
AsymptoticDSolveValue[(x^2+x-6)*y''[x]+(x+3)*y'[x]+(x-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{7x^5}{2160} - \frac{17x^4}{2592} - \frac{x^3}{108} - \frac{x^2}{6} + 1\right) + c_2 \left(\frac{13x^5}{1440} + \frac{23x^4}{864} + \frac{x^3}{36} + \frac{x^2}{4} + x\right)$$

2.8 problem 8

Internal problem ID [5563]
Internal file name [`OUTPUT/4811_Sunday_June_05_2022_03_06_28_PM_79720233/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x^2+1)^2 y'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^5 + 2x^3 + x)\, y'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{1}{x\,(x^2+1)^2}$$

Table 26: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x(x^2+1)^2}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x^4 + 2x^2 + 1) y'' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x^4 + 2x^2 + 1) \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \qquad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+3} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+3} a_n (n+r)(n+r-1) = \sum_{n=4}^{\infty} a_{n-4} (n+r-4)(n-5+r)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 2a_{n-2} (n+r-2)(n-3+r)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=4}^{\infty} a_{n-4} (n+r-4)(n-5+r)\, x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} (n+r-2)(n-3+r)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{r(1+r)}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{-2r^4 + 2r^2 + 1}{r(1+r)^2 (2+r)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = \frac{4r^4 + 8r^3 + 8r^2 + 4r - 1}{r(1+r)^2 (2+r)^2 (3+r)}$$

For $4 \le n$ the recursive equation is

$$\begin{aligned} &a_{n-4}(n+r-4)(n-5+r) + 2a_{n-2}(n+r-2)(n-3+r) \\ &+ a_n(n+r)(n+r-1) + a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-4} + 2n^2 a_{n-2} + 2nra_{n-4} + 4nra_{n-2} + r^2 a_{n-4} + 2r^2 a_{n-2} - 9na_{n-4} - 10na_{n-2} - 9ra_{n-4} - 10r}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(-a_{n-4} - 2a_{n-2})\, n^2 + (7a_{n-4} + 6a_{n-2})\, n - 12a_{n-4} - 4a_{n-2} - a_{n-1}}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-2r^4+2r^2+1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{4r^4+8r^3+8r^2+4r-1}{r(1+r)^2(2+r)^2(3+r)}$	$\frac{23}{144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3r^8 + 24r^7 + 66r^6 + 60r^5 - 39r^4 - 108r^3 - 90r^2 - 60r - 23}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{167}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-2r^4+2r^2+1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{4r^4+8r^3+8r^2+4r-1}{r(1+r)^2(2+r)^2(3+r)}$	$\frac{23}{144}$
a_4	$\frac{3r^8+24r^7+66r^6+60r^5-39r^4-108r^3-90r^2-60r-23}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{167}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-10r^8 - 120r^7 - 592r^6 - 1548r^5 - 2342r^4 - 2172r^3 - 1208r^2 - 168r + 167}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{7993}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-2r^4+2r^2+1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{4r^4+8r^3+8r^2+4r-1}{r(1+r)^2(2+r)^2(3+r)}$	$\frac{23}{144}$
a_4	$\frac{3r^8+24r^7+66r^6+60r^5-39r^4-108r^3-90r^2-60r-23}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{167}{2880}$
a_5	$\frac{-10r^8-120r^7-592r^6-1548r^5-2342r^4-2172r^3-1208r^2-168r+167}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{7993}{86400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{23x^3}{144} - \frac{167x^4}{2880} - \frac{7993x^5}{86400} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= -\frac{1}{r(1+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -\frac{1}{r(1+r)} &= \lim_{r \to 0} -\frac{1}{r(1+r)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n + r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2)\right) \end{aligned}$$

$$\begin{aligned} \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n + r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2)(-1 + n + r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x(x^4+2x^2+1)\,y''+y=0$ gives

$$x\left(x^4+2x^2+1\right)\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.$$
$$\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0$$

Which can be written as

$$\begin{aligned}&\left(\left(x\left(x^4+2x^2+1\right)y_1''(x)+y_1(x)\right)\ln(x)\right.\\&\left.+x\left(x^4+2x^2+1\right)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)\right)C\\&+x\left(x^4+2x^2+1\right)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\&+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x\left(x^4+2x^2+1\right)y_1''(x)+y_1(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&x\left(x^4+2x^2+1\right)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)C\\&+x\left(x^4+2x^2+1\right)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\&+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2x(x^2+1)^2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)-(x^2+1)^2\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}$$
$$+\frac{x^2(x^2+1)^2\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x}=0\tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2x(x^2+1)^2\left(\sum_{n=0}^{\infty} x^n a_n(1+n)\right) - (x^2+1)^2\left(\sum_{n=0}^{\infty} a_n x^{1+n}\right)\right)C}{x} + \frac{x^2(x^2+1)^2\left(\sum_{n=0}^{\infty} x^{n-2} b_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right)x}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n(1+n)\right) + \left(\sum_{n=0}^{\infty} 4C\, x^{n+2} a_n(1+n)\right)\\&+\left(\sum_{n=0}^{\infty} 2C\, x^n a_n(1+n)\right) + \sum_{n=0}^{\infty}\left(-C\, x^{n+4} a_n\right) + \sum_{n=0}^{\infty}\left(-2C\, x^{n+2} a_n\right)\\&+\sum_{n=0}^{\infty}\left(-C a_n x^n\right) + \left(\sum_{n=0}^{\infty} n\, x^{n+3} b_n(n-1)\right) + \left(\sum_{n=0}^{\infty} 2n\, x^{1+n} b_n(n-1)\right)\\&+\left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n(n-1)\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n(1+n) = \sum_{n=5}^{\infty} 2C a_{n-5}(n-4)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} 4C\, x^{n+2} a_n(1+n) = \sum_{n=3}^{\infty} 4C a_{-3+n}(n-2)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} 2C\, x^n a_n(1+n) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty}\left(-C\, x^{n+4} a_n\right) = \sum_{n=5}^{\infty}\left(-C a_{n-5} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-2C\, x^{n+2} a_n\right) = \sum_{n=3}^{\infty}\left(-2C a_{-3+n} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-C a_n x^n\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} n\, x^{n+3} b_n (n-1) = \sum_{n=4}^{\infty} (n-4)\, b_{n-4} (n-5)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} 2n\, x^{1+n} b_n (n-1) = \sum_{n=2}^{\infty} 2(n-2)\, b_{n-2} (-3+n)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\left(\sum_{n=5}^{\infty} 2C a_{n-5} (n-4)\, x^{n-1}\right) + \left(\sum_{n=3}^{\infty} 4C a_{-3+n} (n-2)\, x^{n-1}\right) \\
&+ \left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right) + \sum_{n=5}^{\infty} \left(-C a_{n-5} x^{n-1}\right) + \sum_{n=3}^{\infty} \left(-2C a_{-3+n} x^{n-1}\right) \\
&+ \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) + \left(\sum_{n=4}^{\infty} (n-4)\, b_{n-4} (n-5)\, x^{n-1}\right) \\
&+ \left(\sum_{n=2}^{\infty} 2(n-2)\, b_{n-2} (-3+n)\, x^{n-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 2$, Eq (2B) gives

$$3C a_1 + b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2 + \frac{3}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$(2a_0 + 5a_2) C + b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{19}{6} + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{19}{36}$$

For $n = 4$, Eq (2B) gives

$$(6a_1 + 7a_3) C + 4b_2 + b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{85}{144} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{85}{1728}$$

For $n = 5$, Eq (2B) gives

$$(a_0 + 10a_2 + 9a_4) C + 12b_3 + b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{21907}{4320} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{21907}{86400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1)\left(x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+O(x^6)\right)\right)\ln(x) + 1-\frac{3x^2}{4}+\frac{19x^3}{36}+\frac{85x^4}{1728}-\frac{21907x^5}{86400}+O(x^6)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x)+c_2y_2(x)\\ &= c_1x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+O(x^6)\right)\\ &\quad + c_2\left((-1)\left(x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+O(x^6)\right)\right)\ln(x)+1\right.\\ &\qquad \left.-\frac{3x^2}{4}+\frac{19x^3}{36}+\frac{85x^4}{1728}-\frac{21907x^5}{86400}+O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h\\ &= c_1x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+O(x^6)\right)\\ &\quad + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}\right.\\ &\qquad \left.+\frac{19x^3}{36}+\frac{85x^4}{1728}-\frac{21907x^5}{86400}+O(x^6)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$\begin{aligned} y &= c_1x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+O(x^6)\right)\\ &\quad + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}\right.(1)\\ &\qquad \left.+\frac{19x^3}{36}+\frac{85x^4}{1728}-\frac{21907x^5}{86400}+O(x^6)\right) \end{aligned}$$

Verification of solutions

$$y = c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+O(x^6)\right) + c_2 \left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{23x^3}{144}-\frac{167x^4}{2880}-\frac{7993x^5}{86400}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}+\frac{19x^3}{36}+\frac{85x^4}{1728}-\frac{21907x^5}{86400}+O(x^6)\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
Order:=6;
dsolve(x*(x^2+1)^2*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 + \frac{23}{144}x^3 - \frac{167}{2880}x^4 - \frac{7993}{86400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(-x + \frac{1}{2}x^2 - \frac{1}{12}x^3 - \frac{23}{144}x^4 + \frac{167}{2880}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - \frac{3}{4}x^2 + \frac{19}{36}x^3 + \frac{85}{1728}x^4 - \frac{21907}{86400}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 87

```
AsymptoticDSolveValue[x*(x^2+1)^2*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{361x^4 + 1056x^3 - 2160x^2 + 1728x + 1728}{1728} - \frac{1}{144}x\left(23x^3 + 12x^2 - 72x + 144\right)\log(x)\right) + c_2\left(-\frac{167x^5}{2880} + \frac{23x^4}{144} + \frac{x^3}{12} - \frac{x^2}{2} + x\right)$$

2.9 problem 9

Internal problem ID [5564]
Internal file name [`OUTPUT/4812_Sunday_June_05_2022_03_06_31_PM_60913564/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^3(x^2-25)(-2+x)^2y''+3x(-2+x)y'+7(5+x)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^7-4x^6-21x^5+100x^4-100x^3)y''+(3x^2-6x)y'+(7x+35)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{x^2(-2+x)(x-5)(5+x)}$$
$$q(x)=\frac{7}{(x-5)(-2+x)^2x^3}$$

Table 27: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x^2(-2+x)(x-5)(5+x)}$	
singularity	type
$x = -5$	"regular"
$x = 0$	"irregular"
$x = 2$	"regular"
$x = 5$	"regular"

$q(x) = \frac{7}{(x-5)(-2+x)^2 x^3}$	
singularity	type
$x = 0$	"irregular"
$x = 2$	"regular"
$x = 5$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-5, 2, 5, \infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

2.9.1 Maple step by step solution

Let's solve

$y''x^3(x-5)(5+x)(-2+x)^2 + (3x^2 - 6x)y' + (7x+35)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x^2(-2+x)(x-5)(5+x)} - \frac{7y}{x^3(x-5)(-2+x)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x^2(-2+x)(x-5)(5+x)} + \frac{7y}{x^3(x-5)(-2+x)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3}{x^2(-2+x)(x-5)(5+x)}, P_3(x) = \frac{7}{(x-5)(-2+x)^2 x^3}\right]$

  ◦ $(5+x) \cdot P_2(x)$ is analytic at $x = -5$

$\left((5+x)\cdot P_2(x)\right)\Big|_{x=-5} = \frac{3}{1750}$

- $(5+x)^2\cdot P_3(x)$ is analytic at $x=-5$

  $\left((5+x)^2\cdot P_3(x)\right)\Big|_{x=-5} = 0$

- $x=-5$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-5$

- Multiply by denominators

  $y''x^3(x-5)(5+x)(-2+x)^2+3x(-2+x)y'+(7x+35)y=0$

- Change variables using $x=u-5$ so that the regular singular point is at $u=0$

  $(u^7-39u^6+624u^5-5250u^4+24525u^3-60375u^2+61250u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^2-36u+105)\,($

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u\cdot y(u)$ to series expansion

  $u\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+1}$

- Shift index using $k->k-1$

  $u\cdot y(u)=\sum_{k=1}^{\infty} a_{k-1} u^{k+r}$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..7$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$35a_0r(-1747+1750r)\,u^{-1+r} + (35a_1(1+r)(3+1750r) - 3a_0r(-20113+20125r))\,u^r + (35a_2($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $35r(-1747+1750r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1747}{1750}\right\}$

- The coefficients of each power of u must be 0

  $[35a_1(1+r)(3+1750r) - 3a_0r(-20113+20125r) = 0, 35a_2(2+r)(1753+1750r) - 3a_1(1+r)$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = \frac{3a_0r(-20113+20125r)}{35(1750r^2+1753r+3)}, a_2 = \frac{a_0(2142984375r^3-2141396250r^2-26144r-735)}{1225(3062500r^3+9198000r^2+6151259r+10518)}, a_3 = \frac{3a_0(19461900390625r^5+194685}{42875(5359375000r^5+3754}\right.$

- Each term in the series must be 0, giving the recursion relation

  $(-60375a_k + a_{k-5} - 39a_{k-4} + 624a_{k-3} - 5250a_{k-2} + 24525a_{k-1} + 61250a_{k+1})\,k^2 + (2(-60375a_k$

- Shift index using $k->k+5$

  $(-60375a_{k+5} + a_k - 39a_{k+1} + 624a_{k+2} - 5250a_{k+3} + 24525a_{k+4} + 61250a_{k+6})\,(k+5)^2 + (2(-60$

- Recursion relation that defines series solution to ODE

  $a_{k+6} = -\frac{k^2a_k - 39k^2a_{k+1} + 624k^2a_{k+2} - 5250k^2a_{k+3} + 24525k^2a_{k+4} - 60375k^2a_{k+5} + 2kra_k - 78kra_{k+1} + 1248kra_{k+2} - 10500kra_k}{}$

- Recursion relation for $r = 0$

  $a_{k+6} = -\frac{k^2a_k - 39k^2a_{k+1} + 624k^2a_{k+2} - 5250k^2a_{k+3} + 24525k^2a_{k+4} - 60375k^2a_{k+5} - ka_k - 39ka_{k+1} + 1872ka_{k+2} - 26250ka_{k+3} + 1}{35(1750k^2+19253k+52518)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+6} = -\frac{k^2a_k - 39k^2a_{k+1} + 624k^2a_{k+2} - 5250k^2a_{k+3} + 24525k^2a_{k+4} - 60375k^2a_{k+5} - ka_k - 39ka_{k+1} + 187}{35(1750}\right.$

- Revert the change of variables $u = 5 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(5+x)^k, a_{k+6} = -\frac{k^2a_k - 39k^2a_{k+1} + 624k^2a_{k+2} - 5250k^2a_{k+3} + 24525k^2a_{k+4} - 60375k^2a_{k+5} - ka_k - 39ka_{k+1}}{35(1}\right.$

- Recursion relation for $r = \frac{1747}{1750}$

  $a_{k+6} = -\frac{k^2a_k - 39k^2a_{k+1} + 624k^2a_{k+2} - 5250k^2a_{k+3} + 24525k^2a_{k+4} - 60375k^2a_{k+5} + \frac{872}{875}ka_k - \frac{102258}{875}ka_{k+1} + \frac{2728128}{875}ka_{k+2} - 367}{35(1}$

- Solution for $r = \frac{1747}{1750}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1747}{1750}}, a_{k+6} = -\frac{k^2 a_k - 39k^2 a_{k+1} + 624k^2 a_{k+2} - 5250k^2 a_{k+3} + 24525k^2 a_{k+4} - 60375k^2 a_{k+5} + \frac{872}{875} k a_k - \frac{102}{87}}{}\right.$$

- Revert the change of variables $u = 5 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (5+x)^{k+\frac{1747}{1750}}, a_{k+6} = -\frac{k^2 a_k - 39k^2 a_{k+1} + 624k^2 a_{k+2} - 5250k^2 a_{k+3} + 24525k^2 a_{k+4} - 60375k^2 a_{k+5} + \frac{872}{875} k a_k}{}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (5+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (5+x)^{k+\frac{1747}{1750}}\right), a_{k+6} = -\frac{k^2 a_k - 39k^2 a_{k+1} + 624k^2 a_{k+2} - 5250k^2 a_{k+3} + 24525}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^3*(x^2-25)*(x-2)^2*diff(y(x),x$2)+3*x*(x-2)*diff(y(x),x)+7*(x+5)*y(x)=0,y(x),type='
```

No solution found

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 99

```
AsymptoticDSolveValue[x^3*(x^2-25)*(x-2)^2*y''[x]+3*x*(x-2)*y'[x]+7*(x+5)*y[x]==0,y[x],{x,0,
```

$$y(x) \to c_2\left(-\frac{133769872016978219061888 1x^5}{352638738432}+\frac{4284030153765326450 5x^4}{3265173504} - \frac{344729362309955x^3}{7558272}+\frac{3590248795x^2}{23328}-\frac{50309x}{108}+1\right)x^{35/6} + \frac{c_1 e^{\frac{3}{50}/x}\left(-\frac{37907198008560463448473952765642999x^5}{538084012500000000000000000}+\frac{27497874350326089989823180601x^4}{79716150000000000000000}+\frac{10649898771731482781701x^3}{14762250000000000}+ 9\right.}{x^{1159/300}}$$

2.10 problem 10

Internal problem ID [5565]
Internal file name [OUTPUT/4813_Sunday_June_05_2022_03_06_32_PM_30756957/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^3-2x^2+3x\right)^2 y''+x(x-3)^2 y'-(1+x)\, y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^6-4x^5+10x^4-12x^3+9x^2\right) y''+\left(x^3-6x^2+9x\right) y'+(-1-x)\, y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{(x-3)^2}{x\left(x^2-2x+3\right)^2}$$
$$q(x)=-\frac{1+x}{x^2\left(x^2-2x+3\right)^2}$$

Table 29: Table $p(x), q(x)$ singularites.

$p(x) = \frac{(x-3)^2}{x(x^2-2x+3)^2}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}+1$	"irregular"
$x = i\sqrt{2}+1$	"irregular"

$q(x) = -\frac{1+x}{x^2(x^2-2x+3)^2}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}+1$	"regular"
$x = i\sqrt{2}+1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $\left[-i\sqrt{2}+1, i\sqrt{2}+1\right]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^4 - 4x^3 + 10x^2 - 12x + 9)\, y'' + (x^3 - 6x^2 + 9x)\, y' + (-1 - x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^4 - 4x^3 + 10x^2 - 12x + 9)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (x^3 - 6x^2 + 9x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-1-x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+4} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r+3} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 10x^{n+r+2} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-12x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+4} a_n (n+r)(n+r-1) = \sum_{n=4}^{\infty} a_{n-4} (n-4+r)(n-5+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-4x^{n+r+3} a_n (n+r)(n+r-1)\right) = \sum_{n=3}^{\infty} \left(-4a_{n-3} (-3+n+r)(n-4+r)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 10x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 10a_{n-2} (n+r-2)(-3+n+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-12x^{1+n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-12a_{n-1} (n+r-1)(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-6a_{n-1} (n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=4}^{\infty} a_{n-4}(n-4+r)\,(n-5+r)\,x^{n+r}\right) \\
&+ \sum_{n=3}^{\infty} \left(-4a_{n-3}(-3+n+r)\,(n-4+r)\,x^{n+r}\right) \\
&+ \left(\sum_{n=2}^{\infty} 10a_{n-2}(n+r-2)\,(-3+n+r)\,x^{n+r}\right) \\
&+ \sum_{n=1}^{\infty} \left(-12a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\,x^{n+r}\right) \\
&+ \sum_{n=1}^{\infty} \left(-6a_{n-1}(n+r-1)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n(n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n(n+r)\,(n+r-1) + 9x^{n+r} a_n(n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 9x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 9x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -\frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{12r^2 - 6r + 1}{9r^2 + 18r + 8}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{54r^4 + 45r^3 + 70r^2 + 48r + 7}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = \frac{-108r^6 - 540r^5 + 528r^4 + 3732r^3 + 3489r^2 + 739r + 224}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)}$$

For $4 \leq n$ the recursive equation is

$$
\begin{aligned}
&a_{n-4}(n-4+r)(n-5+r) - 4a_{n-3}(-3+n+r)(n-4+r) \\
&+ 10a_{n-2}(n+r-2)(-3+n+r) - 12a_{n-1}(n+r-1)(n+r-2) \\
&+ 9a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) \\
&- 6a_{n-1}(n+r-1) + 9a_n(n+r) - a_n - a_{n-1} = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{n^2 a_{n-4} - 4n^2 a_{n-3} + 10n^2 a_{n-2} - 12n^2 a_{n-1} + 2nra_{n-4} - 8nra_{n-3} + 20nra_{n-2} - 24nra_{n-1} + r^2 a_{n-\dots}}{} \tag{4}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
a_n = \frac{(-9a_{n-4} + 36a_{n-3} - 90a_{n-2} + 108a_{n-1})\, n^2 + (75a_{n-4} - 228a_{n-3} + 381a_{n-2} - 198a_{n-1})\, n - 154a_{n-\dots}}{81n^2 + 54n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	$\frac{1}{45}$
a_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{149}{3240}$
a_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{2701}{192456}$

For $n = 4$, using the above recursive equation gives

$$
a_4 = \frac{-2997r^8 - 29079r^7 - 99054r^6 - 125883r^5 + 9826r^4 + 138853r^3 + 80233r^2 + 5477r + 8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
a_4 = \frac{236933}{121247280}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	$\frac{1}{45}$
a_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{149}{3240}$
a_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{2701}{192456}$
a_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{236933}{121247280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-17496r^{10} - 292572r^9 - 2032830r^8 - 7628094r^7 - 16915518r^6 - 23128560r^5 - 20473273r^4 - 1293}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)(9r^2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{67092967}{92754169200}$$

And the table now becomes

n	$a_{n,r}$	a
a_0	1	1
a_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	$\frac{1}{4}$
a_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{1}{3}$
a_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	1
a_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	1
a_5	$\frac{-17496r^{10}-292572r^9-2032830r^8-7628094r^7-16915518r^6-23128560r^5-20473273r^4-12932200r^3-6220667r^2-1947462r-14560}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$	$-$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 + \frac{x}{45} + \frac{149x^2}{3240} + \frac{2701x^3}{192456} + \frac{236933x^4}{121247280} - \frac{67092967x^5}{92754169200} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{12r^2 - 6r + 1}{9r^2 + 18r + 8}$$

Substituting $n = 2$ in Eq. (2B) gives

$$b_2 = \frac{54r^4 + 45r^3 + 70r^2 + 48r + 7}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$b_3 = \frac{-108r^6 - 540r^5 + 528r^4 + 3732r^3 + 3489r^2 + 739r + 224}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)}$$

For $4 \leq n$ the recursive equation is

$$\begin{aligned} &b_{n-4}(n-4+r)(n-5+r) - 4b_{n-3}(-3+n+r)(n-4+r) \\ &+ 10b_{n-2}(n+r-2)(-3+n+r) - 12b_{n-1}(n+r-1)(n+r-2) \\ &+ 9b_n(n+r)(n+r-1) + b_{n-2}(n+r-2) \\ &- 6b_{n-1}(n+r-1) + 9b_n(n+r) - b_n - b_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{n^2b_{n-4} - 4n^2b_{n-3} + 10n^2b_{n-2} - 12n^2b_{n-1} + 2nrb_{n-4} - 8nrb_{n-3} + 20nrb_{n-2} - 24nrb_{n-1} + r^2b_{n-4} - \cdots}{\cdots} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_n = \frac{(-9b_{n-4} + 36b_{n-3} - 90b_{n-2} + 108b_{n-1})n^2 + (87b_{n-4} - 276b_{n-3} + 501b_{n-2} - 342b_{n-1})n - 208b_{n-4} + \cdots}{81n^2 - 54n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	$\frac{13}{9}$
b_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{5}{162}$
b_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1591}{30618}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{-2997r^8 - 29079r^7 - 99054r^6 - 125883r^5 + 9826r^4 + 138853r^3 + 80233r^2 + 5477r + 8064}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = \frac{106583}{5511240}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	$\frac{13}{9}$
b_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{5}{162}$
b_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1591}{30618}$
b_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{106583}{5511240}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-17496r^{10} - 292572r^9 - 2032830r^8 - 7628094r^7 - 16915518r^6 - 23128560r^5 - 20473273r^4 - 1293}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)(9r^2}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_5 = \frac{7435523}{3224075400}$$

And the table now becomes

n	$b_{n,r}$	b
b_0	1	1
b_1	$\frac{12r^2-6r+1}{9r^2+18r+8}$	1 9
b_2	$\frac{54r^4+45r^3+70r^2+48r+7}{(9r^2+18r+8)(9r^2+36r+35)}$	−
b_3	$\frac{-108r^6-540r^5+528r^4+3732r^3+3489r^2+739r+224}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	1 3
b_4	$\frac{-2997r^8-29079r^7-99054r^6-125883r^5+9826r^4+138853r^3+80233r^2+5477r+8064}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	1 5
b_5	$\frac{-17496r^{10}-292572r^9-2032830r^8-7628094r^7-16915518r^6-23128560r^5-20473273r^4-12932200r^3-6220667r^2-1947462r-14560}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$	3

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right)\\ &= \frac{1 + \frac{13x}{9} - \frac{5x^2}{162} + \frac{1591x^3}{30618} + \frac{106583x^4}{5511240} + \frac{7435523x^5}{3224075400} + O(x^6)}{x^{\frac{1}{3}}}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\ &= c_1 x^{\frac{1}{3}}\left(1 + \frac{x}{45} + \frac{149x^2}{3240} + \frac{2701x^3}{192456} + \frac{236933x^4}{121247280} - \frac{67092967x^5}{92754169200} + O(x^6)\right)\\ &\quad + \frac{c_2\left(1 + \frac{13x}{9} - \frac{5x^2}{162} + \frac{1591x^3}{30618} + \frac{106583x^4}{5511240} + \frac{7435523x^5}{3224075400} + O(x^6)\right)}{x^{\frac{1}{3}}}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1 x^{\frac{1}{3}}\left(1 + \frac{x}{45} + \frac{149x^2}{3240} + \frac{2701x^3}{192456} + \frac{236933x^4}{121247280} - \frac{67092967x^5}{92754169200} + O(x^6)\right)\\ &\quad + \frac{c_2\left(1 + \frac{13x}{9} - \frac{5x^2}{162} + \frac{1591x^3}{30618} + \frac{106583x^4}{5511240} + \frac{7435523x^5}{3224075400} + O(x^6)\right)}{x^{\frac{1}{3}}}\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned}y &= c_1 x^{\frac{1}{3}}\left(1 + \frac{x}{45} + \frac{149x^2}{3240} + \frac{2701x^3}{192456} + \frac{236933x^4}{121247280} - \frac{67092967x^5}{92754169200} + O(x^6)\right)\\ &\quad + \frac{c_2\left(1 + \frac{13x}{9} - \frac{5x^2}{162} + \frac{1591x^3}{30618} + \frac{106583x^4}{5511240} + \frac{7435523x^5}{3224075400} + O(x^6)\right)}{x^{\frac{1}{3}}}\end{aligned} \tag{1}$$

<u>Verification of solutions</u>

$$\begin{aligned}y &= c_1 x^{\frac{1}{3}}\left(1 + \frac{x}{45} + \frac{149x^2}{3240} + \frac{2701x^3}{192456} + \frac{236933x^4}{121247280} - \frac{67092967x^5}{92754169200} + O(x^6)\right)\\ &\quad + \frac{c_2\left(1 + \frac{13x}{9} - \frac{5x^2}{162} + \frac{1591x^3}{30618} + \frac{106583x^4}{5511240} + \frac{7435523x^5}{3224075400} + O(x^6)\right)}{x^{\frac{1}{3}}}\end{aligned}$$

Verified OK.

2.10.1 Maple step by step solution

Let's solve

$x^2(x^4-4x^3+10x^2-12x+9)\,y''+(x^3-6x^2+9x)\,y'+(-1-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(1+x)y}{x^2(x^4-4x^3+10x^2-12x+9)}-\frac{(x^2-6x+9)y'}{x(x^4-4x^3+10x^2-12x+9)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x^2-6x+9)y'}{x(x^4-4x^3+10x^2-12x+9)}-\frac{(1+x)y}{x^2(x^4-4x^3+10x^2-12x+9)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x^2-6x+9}{x(x^4-4x^3+10x^2-12x+9)},P_3(x)=-\frac{1+x}{x^2(x^4-4x^3+10x^2-12x+9)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{9}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^4-4x^3+10x^2-12x+9)\,y''+x(x^2-6x+9)\,y'+(-1-x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..6$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+3r)\,(-1+3r)\,x^{r}+\left(a_{1}(4+3r)\,(2+3r)-a_{0}(12r^{2}-6r+1)\right)x^{1+r}+\left(a_{2}(7+3r)\,(5+3r)\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+3r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{1}{3},\frac{1}{3}\right\}$

- The coefficients of each power of x must be 0

  $[a_{1}(4+3r)\,(2+3r)-a_{0}(12r^{2}-6r+1)=0,a_{2}(7+3r)\,(5+3r)-a_{1}(12r^{2}+18r+7)+a_{0}r(-9$

- Solve for the dependent coefficient(s)

  $\left\{a_{1}=\frac{a_{0}(12r^{2}-6r+1)}{9r^{2}+18r+8},a_{2}=\frac{a_{0}(54r^{4}+45r^{3}+70r^{2}+48r+7)}{81r^{4}+486r^{3}+1035r^{2}+918r+280},a_{3}=-\frac{a_{0}(108r^{6}+540r^{5}-528r^{4}-3732r^{3}-3489r^{2}-739r}{729r^{6}+8748r^{5}+42039r^{4}+103032r^{3}+134892r^{2}+8850}\right.$

- Each term in the series must be 0, giving the recursion relation

  $(9a_{k}+a_{k-4}-4a_{k-3}+10a_{k-2}-12a_{k-1})\,k^{2}+(2(9a_{k}+a_{k-4}-4a_{k-3}+10a_{k-2}-12a_{k-1})\,r-9a_{k}$

- Shift index using $k->k+4$

  $(9a_{k+4}+a_{k}-4a_{k+1}+10a_{k+2}-12a_{k+3})\,(k+4)^{2}+(2(9a_{k+4}+a_{k}-4a_{k+1}+10a_{k+2}-12a_{k+3})\,r$

- Recursion relation that defines series solution to ODE

$$a_{k+4} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 10k^2 a_{k+2} - 12k^2 a_{k+3} + 2kr a_k - 8kr a_{k+1} + 20kr a_{k+2} - 24kr a_{k+3} + r^2 a_k - 4r^2 a_{k+1} + 10r^2 a_{k+2} - 12r^2 a_{k+3}}{9k^2 + 18kr + 9r^2 + 72k + 72r + 14}$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+4} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 10k^2 a_{k+2} - 12k^2 a_{k+3} - \frac{5}{3} k a_k - \frac{4}{3} k a_{k+1} + \frac{73}{3} k a_{k+2} - 58k a_{k+3} + \frac{4}{9} a_k + \frac{8}{9} a_{k+1} + \frac{115}{9} a_{k+2} - \frac{211}{3} a_{k+3}}{9k^2 + 66k + 120}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+4} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 10k^2 a_{k+2} - 12k^2 a_{k+3} - \frac{5}{3} k a_k - \frac{4}{3} k a_{k+1} + \frac{73}{3} k a_{k+2} - 58k a_{k+3} + \frac{4}{9} a_k + \frac{8}{9} a_{k+1} + \frac{11}{9}}{9k^2 + 66k + 120}\right.$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+4} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 10k^2 a_{k+2} - 12k^2 a_{k+3} - \frac{1}{3} k a_k - \frac{20}{3} k a_{k+1} + \frac{113}{3} k a_{k+2} - 74k a_{k+3} - \frac{2}{9} a_k - \frac{16}{9} a_{k+1} + \frac{301}{9} a_{k+2} - \frac{343}{3} a_{k+3}}{9k^2 + 78k + 168}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+4} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 10k^2 a_{k+2} - 12k^2 a_{k+3} - \frac{1}{3} k a_k - \frac{20}{3} k a_{k+1} + \frac{113}{3} k a_{k+2} - 74k a_{k+3} - \frac{2}{9} a_k - \frac{16}{9} a_{k+1}}{9k^2 + 78k + 168}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+4} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 10k^2 a_{k+2} - 12k^2 a_{k+3} - \frac{5}{3} k a_k - \frac{4}{3} k a_{k+1} + \frac{73}{3} k a_{k+2}}{9k^2 + 66k + 120}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve((x^3-2*x^2+3*x)^2*diff(y(x),x$2)+x*(x-3)^2*diff(y(x),x)-(x+1)*y(x)=0,y(x),type='serie
```

$$y(x) = \frac{c_2 x^{\frac{2}{3}}\left(1+\frac{1}{45}x+\frac{149}{3240}x^2+\frac{2701}{192456}x^3+\frac{236933}{121247280}x^4-\frac{67092967}{92754169200}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1+\frac{13}{9}x-\frac{5}{162}x^2+\frac{1591}{30618}x^3\right.}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 90

```
AsymptoticDSolveValue[(x^3-2*x^2+3*x)^2*y''[x]+x*(x-3)^2*y'[x]-(x+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt[3]{x}\left(-\frac{67092967x^5}{92754169200}+\frac{236933x^4}{121247280}+\frac{2701x^3}{192456}+\frac{149x^2}{3240}+\frac{x}{45}+1\right) + \frac{c_2\left(\frac{7435523x^5}{3224075400}+\frac{106583x^4}{5511240}+\frac{1591x^3}{30618}-\frac{5x^2}{162}+\frac{13x}{9}+1\right)}{\sqrt[3]{x}}$$

2.11 problem 11

Internal problem ID [5566]
Internal file name [`OUTPUT/4814_Sunday_June_05_2022_03_06_36_PM_63783283/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^{2}-1\right) y''+5(1+x) y'+\left(x^{2}-x\right) y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{82}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{83}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{yx^2 - xy + 5xy' + 5y'}{x^2-1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
&= \frac{\left(-x^4 + x^3 + 31x^2 + 59x + 30\right) y' + 5y\left(x^2 + \frac{4}{5}x + \frac{1}{5}\right)(x-1)}{\left(x^2-1\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1\\
&= \frac{\left(10x^5 - 2x^4 - 226x^3 - 630x^2 - 624x - 208\right) y' + y(x-1)\left(x^5 - x^4 - 36x^3 - 62x^2 - 39x - 3\right)}{\left(x^2-1\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2\\
&= \frac{\left((1+x)\left(x^7 - 3x^6 - 93x^5 + 33x^4 + 1793x^3 + 5077x^2 + 4971x + 1661\right) y' - 10\left(x^6 - \frac{2}{5}x^5 - \frac{146}{5}x^4 - \right.\right.}{\left(x^2-1\right)^5}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3\\
&= \frac{\left(-15x^9 + 21x^8 + 972x^7 + 1452x^6 - 15804x^5 - 78540x^4 - 152496x^3 - 149952x^2 - 74577x - 14901\right)}{\left(x^2 - \right.}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 5y'(0)\\
F_1 &= -y(0) + 30y'(0)\\
F_2 &= -3y(0) + 208y'(0)\\
F_3 &= -36y(0) + 1661y'(0)\\
F_4 &= -272y(0) + 14901y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{6}x^{3}-\frac{1}{8}x^{4}-\frac{3}{10}x^{5}-\frac{17}{45}x^{6}\right) y(0) \\
&+\left(x+\frac{5}{2}x^{2}+5x^{3}+\frac{26}{3}x^{4}+\frac{1661}{120}x^{5}+\frac{4967}{240}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x^{2}-1\right) y''+\left(5x+5\right) y'+\left(x^{2}-x\right) y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(x^{2}-1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+\left(5x+5\right)\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)+\left(x^{2}-x\right)\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\sum_{n=2}^{\infty}\left(-n(n-1)\, a_{n} x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 5n a_{n} x^{n}\right) \\
&+\left(\sum_{n=1}^{\infty} 5n a_{n} x^{n-1}\right)+\left(\sum_{n=0}^{\infty} x^{n+2} a_{n}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n} a_{n}\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty}\left(-(n+2)\, a_{n+2}(1+n)\, x^n\right)$$

$$\sum_{n=1}^{\infty} 5n a_n x^{n-1} = \sum_{n=0}^{\infty} 5(1+n)\, a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$\sum_{n=0}^{\infty}\left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty}\left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=0}^{\infty}\left(-(n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} 5n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 5(1+n)\, a_{1+n} x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) + \sum_{n=1}^{\infty}\left(-a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 + 5a_1 = 0$$

$$a_2 = \frac{5a_1}{2}$$

$n = 1$ gives

$$-6a_3 + 5a_1 + 10a_2 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} + 5a_1$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) - (n+2)\, a_{n+2}(1+n) + 5n a_n + 5(1+n)\, a_{1+n} + a_{n-2} - a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{n^2 a_n + 4n a_n + 5n a_{1+n} + 5a_{1+n} + a_{n-2} - a_{n-1}}{(n+2)(1+n)} \\
&= \frac{(n^2+4n) a_n}{(n+2)(1+n)} + \frac{(5n+5) a_{1+n}}{(n+2)(1+n)} + \frac{a_{n-2}}{(n+2)(1+n)} - \frac{a_{n-1}}{(n+2)(1+n)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$
12a_2 - 12a_4 + 15a_3 + a_0 - a_1 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_4 = \frac{26a_1}{3} - \frac{a_0}{8}
$$

For $n = 3$ the recurrence equation gives

$$
21a_3 - 20a_5 + 20a_4 + a_1 - a_2 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_5 = -\frac{3a_0}{10} + \frac{1661a_1}{120}
$$

For $n = 4$ the recurrence equation gives

$$
32a_4 - 30a_6 + 25a_5 + a_2 - a_3 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_6 = \frac{4967a_1}{240} - \frac{17a_0}{45}
$$

For $n = 5$ the recurrence equation gives

$$
45a_5 - 42a_7 + 30a_6 + a_3 - a_4 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_7 = -\frac{199a_0}{336} + \frac{14881a_1}{504}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x + \frac{5 a_1 x^2}{2} + \left(-\frac{a_0}{6} + 5 a_1\right) x^3 + \left(\frac{26 a_1}{3} - \frac{a_0}{8}\right) x^4 + \left(-\frac{3 a_0}{10} + \frac{1661 a_1}{120}\right) x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{3}{10}x^5\right) a_0 + \left(x + \frac{5}{2}x^2 + 5x^3 + \frac{26}{3}x^4 + \frac{1661}{120}x^5\right) a_1 + O(x^6) \quad (3)
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{3}{10}x^5\right) c_1 + \left(x + \frac{5}{2}x^2 + 5x^3 + \frac{26}{3}x^4 + \frac{1661}{120}x^5\right) c_2 + O(x^6)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\left(1 - \frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{3}{10}x^5 - \frac{17}{45}x^6\right) y(0) \\
&+ \left(x + \frac{5}{2}x^2 + 5x^3 + \frac{26}{3}x^4 + \frac{1661}{120}x^5 + \frac{4967}{240}x^6\right) y'(0) + O(x^6)
\end{aligned} \quad (1)
$$

$$
y = \left(1 - \frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{3}{10}x^5\right) c_1 + \left(x + \frac{5}{2}x^2 + 5x^3 + \frac{26}{3}x^4 + \frac{1661}{120}x^5\right) c_2 + O(x^6) \quad (2)
$$

Verification of solutions

$$
\begin{aligned}
y = &\left(1 - \frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{3}{10}x^5 - \frac{17}{45}x^6\right) y(0) \\
&+ \left(x + \frac{5}{2}x^2 + 5x^3 + \frac{26}{3}x^4 + \frac{1661}{120}x^5 + \frac{4967}{240}x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Verified OK.

$$
y = \left(1 - \frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{3}{10}x^5\right) c_1 + \left(x + \frac{5}{2}x^2 + 5x^3 + \frac{26}{3}x^4 + \frac{1661}{120}x^5\right) c_2 + O(x^6)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve((x^2-1)*diff(y(x),x$2)+5*(x+1)*diff(y(x),x)+(x^2-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{6}x^{3}-\frac{1}{8}x^{4}-\frac{3}{10}x^{5}\right)y(0) + \left(x+\frac{5}{2}x^{2}+5x^{3}+\frac{26}{3}x^{4}+\frac{1661}{120}x^{5}\right)D(y)(0)+O\left(x^{6}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 61

```
AsymptoticDSolveValue[(x^2-1)*y''[x]+5*(x+1)*y'[x]+(x^2-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{3x^5}{10} - \frac{x^4}{8} - \frac{x^3}{6} + 1\right) + c_2\left(\frac{1661x^5}{120} + \frac{26x^4}{3} + 5x^3 + \frac{5x^2}{2} + x\right)$$

2.12 problem 12

Internal problem ID [5567]
Internal file name [`OUTPUT/4815_Sunday_June_05_2022_03_06_37_PM_64655398/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (x + 3)\, y' + 7yx^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (x + 3)\, y' + 7yx^2 = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x + 3}{x}$$
$$q(x) = 7x$$

Table 31: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 7x$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (x+3)\,y' + 7yx^2 = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x + (x+3)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + 7\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x^2 = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 7x^{2+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 7x^{2+n+r} a_n = \sum_{n=3}^{\infty} 7a_{n-3} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=3}^{\infty} 7a_{n-3} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{r}{r^2 + 4r + 3}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{r}{r^3 + 9r^2 + 26r + 24}$$

For $3 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + 3a_n(n+r) + 7a_{n-3} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n a_{n-1} + r a_{n-1} + 7a_{n-3} - a_{n-1}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{-n a_{n-1} - 7a_{n-3} + a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+4r+3}$	0
a_2	$\frac{r}{r^3+9r^2+26r+24}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-7r^2 - 50r - 84}{(5+r)(r+3)^2(4+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{7}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+4r+3}$	0
a_2	$\frac{r}{r^3+9r^2+26r+24}$	0
a_3	$\frac{-7r^2-50r-84}{(5+r)(r+3)^2(4+r)}$	$-\frac{7}{15}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{14r^3 + 120r^2 + 274r + 84}{(r+6)(4+r)^2(r+3)(1+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{7}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+4r+3}$	0
a_2	$\frac{r}{r^3+9r^2+26r+24}$	0
a_3	$\frac{-7r^2-50r-84}{(5+r)(r+3)^2(4+r)}$	$-\frac{7}{15}$
a_4	$\frac{14r^3+120r^2+274r+84}{(r+6)(4+r)^2(r+3)(1+r)(5+r)}$	$\frac{7}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-21r^4 - 232r^3 - 801r^2 - 842r - 168}{(r+7)(5+r)^2(r+3)(4+r)(2+r)(1+r)(r+6)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{150}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+4r+3}$	0
a_2	$\frac{r}{r^3+9r^2+26r+24}$	0
a_3	$\frac{-7r^2-50r-84}{(5+r)(r+3)^2(4+r)}$	$-\frac{7}{15}$
a_4	$\frac{14r^3+120r^2+274r+84}{(r+6)(4+r)^2(r+3)(1+r)(5+r)}$	$\frac{7}{120}$
a_5	$\frac{-21r^4-232r^3-801r^2-842r-168}{(r+7)(5+r)^2(r+3)(4+r)(2+r)(1+r)(r+6)}$	$-\frac{1}{150}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + O(x^6)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_2 \\ &= \frac{r}{r^3 + 9r^2 + 26r + 24}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} \frac{r}{r^3 + 9r^2 + 26r + 24} &= \lim_{r\to -2} \frac{r}{r^3 + 9r^2 + 26r + 24} \\ &= \textit{undefined}\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2) (-1 + n + r_2)\right)\end{aligned}$$

Substituting these back into the given ode $xy''+(x+3)\,y'+7y x^2=0$ gives

$$\begin{aligned}&\left(C y_1''(x)\ln(x)+\frac{2C y_1'(x)}{x}-\frac{C y_1(x)}{x^2}\right.\\&+\left.\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\&+(x+3)\left(C y_1'(x)\ln(x)+\frac{C y_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\&+7\left(C y_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)x^2=0\end{aligned}$$

Which can be written as

$$\begin{aligned}&\left(\left(y_1''(x)\,x+(x+3)\,y_1'(x)+7y_1(x)\,x^2\right)\ln(x)+\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x\right.\\&+\left.\frac{(x+3)\,y_1(x)}{x}\right)C+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\&+(x+3)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+7\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x^2=0\end{aligned}\tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x+(x+3)\,y_1'(x)+7y_1(x)\,x^2=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x+\frac{(x+3)\,y_1(x)}{x}\right)C\\&+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\&+(x+3)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+7\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x^2=0\end{aligned}\tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + (x+2)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} \\
&+ \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + (x^2+3x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) + 7\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^3}{x} \\
&= 0
\end{aligned}
\tag{9}
$$

Since $r_1 = 0$ and $r_2 = -2$ then the above becomes

$$
\begin{aligned}
&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{n-1} a_n n\right) x + (x+2)\left(\sum_{n=0}^{\infty} a_n x^{n}\right)\right) C}{x} \\
&+ \frac{\left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(n-3)\right) x^2 + (x^2+3x)\left(\sum_{n=0}^{\infty} x^{n-3} b_n (n-2)\right) + 7\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) x^3}{x} \\
&= 0
\end{aligned}
\tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n n\right) + \left(\sum_{n=0}^{\infty} C a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-3} b_n (n^2 - 5n + 6)\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{n-3} b_n (n-2)\right) + \left(\sum_{n=0}^{\infty} 7 b_n x^n\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-3$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-3} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n n = \sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{n-3}
$$

$$
\sum_{n=0}^{\infty} C a_n x^n = \sum_{n=3}^{\infty} C a_{n-3} x^{n-3}
$$

$$\sum_{n=0}^{\infty} 2C\,x^{n-1} a_n = \sum_{n=2}^{\infty} 2C a_{n-2} x^{n-3}$$

$$\sum_{n=0}^{\infty} x^{n-2} b_n (n-2) = \sum_{n=1}^{\infty} b_{n-1} (n-3)\, x^{n-3}$$

$$\sum_{n=0}^{\infty} 7 b_n x^n = \sum_{n=3}^{\infty} 7 b_{n-3} x^{n-3}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-3$.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{n-3}\right) + \left(\sum_{n=3}^{\infty} C a_{n-3} x^{n-3}\right) + \left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{n-3}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-3} b_n \left(n^2 - 5n + 6\right)\right) + \left(\sum_{n=1}^{\infty} b_{n-1} (n-3)\, x^{n-3}\right) \\ &+ \left(\sum_{n=0}^{\infty} 3 x^{n-3} b_n (n-2)\right) + \left(\sum_{n=3}^{\infty} 7 b_{n-3} x^{n-3}\right) = 0 \end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-b_1 - 2b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 - 2 = 0$$

Solving the above for b_1 gives

$$b_1 = -2$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$2C + 2 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 3$, Eq (2B) gives

$$(a_0 + 4a_1)\, C + 7b_0 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6+3b_3=0$$

Solving the above for b_3 gives

$$b_3=-2$$

For $n=4$, Eq (2B) gives

$$(a_1+6a_2)\,C+7b_1+b_3+8b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-16+8b_4=0$$

Solving the above for b_4 gives

$$b_4=2$$

For $n=5$, Eq (2B) gives

$$(a_2+8a_3)\,C+7b_2+2b_4+15b_5=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{116}{15}+15b_5=0$$

Solving the above for b_5 gives

$$b_5=-\frac{116}{225}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)$$

Using the above value found for $C=-1$ and all b_n, then the second solution becomes

$$y_2(x)=(-1)\left(1-\frac{7x^3}{15}+\frac{7x^4}{120}-\frac{x^5}{150}+O(x^6)\right)\ln(x)+\frac{1-2x-2x^3+2x^4-\frac{116x^5}{225}+O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + O(x^6)\right) \\
&\quad + c_2 \left((-1)\left(1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + O(x^6)\right)\ln(x) \right. \\
&\quad \left. + \frac{1 - 2x - 2x^3 + 2x^4 - \frac{116x^5}{225} + O(x^6)}{x^2}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + O(x^6)\right) + c_2 \left(\left(-1 + \frac{7x^3}{15} - \frac{7x^4}{120} + \frac{x^5}{150} - O(x^6)\right)\ln(x) \right. \\
&\quad \left. + \frac{1 - 2x - 2x^3 + 2x^4 - \frac{116x^5}{225} + O(x^6)}{x^2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + O(x^6)\right) + c_2 \left(\left(-1 + \frac{7x^3}{15} - \frac{7x^4}{120} + \frac{x^5}{150} - O(x^6)\right)\ln(x) \right. \\
&\quad \left. + \frac{1 - 2x - 2x^3 + 2x^4 - \frac{116x^5}{225} + O(x^6)}{x^2}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \left(1 - \frac{7x^3}{15} + \frac{7x^4}{120} - \frac{x^5}{150} + O(x^6)\right) + c_2 \left(\left(-1 + \frac{7x^3}{15} - \frac{7x^4}{120} + \frac{x^5}{150} - O(x^6)\right)\ln(x) \right. \\
&\quad \left. + \frac{1 - 2x - 2x^3 + 2x^4 - \frac{116x^5}{225} + O(x^6)}{x^2}\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 56

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(x+3)*diff(y(x),x)+7*x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1-\frac{7}{15}x^3+\frac{7}{120}x^4-\frac{1}{150}x^5+\mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(\ln\left(x\right)\left(2x^2-\frac{14}{15}x^5+\mathrm{O}\left(x^6\right)\right)+\left(-2+4x-3x^2+4x^3-4x^4+\frac{547}{225}x^5+\mathrm{O}\left(x^6\right)\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 55

```
AsymptoticDSolveValue[x*y''[x]+(x+3)*y'[x]+7*x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{7x^4}{120}-\frac{7x^3}{15}+1\right)+c_1\left(\frac{2x^4-2x^3+2x^2-2x+1}{x^2}-\log(x)\right)$$

2.13 problem 13

Internal problem ID [5568]
Internal file name [OUTPUT/4816_Sunday_June_05_2022_03_06_40_PM_69573029/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3x + 5}{3x}$$
$$q(x) = -\frac{1}{3x^2}$$

Table 32: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x+5}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{3x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + \left(\frac{5}{3}x + x^2\right) y' - \frac{y}{3} = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(\frac{5}{3}x + x^2\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \frac{\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)}{3} = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} \frac{5x^{n+r} a_n (n+r)}{3}\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{3}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} \frac{5x^{n+r} a_n (n+r)}{3}\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{3}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + \frac{5x^{n+r} a_n (n+r)}{3} - \frac{a_n x^{n+r}}{3} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + \frac{5x^r a_0 r}{3} - \frac{a_0 x^r}{3} = 0$$

Or

$$\left(x^r r(-1+r) + \frac{5x^r r}{3} - \frac{x^r}{3}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(3r^2 + 2r - 1)\, x^r}{3} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + \frac{2}{3}r - \frac{1}{3} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(3r^2 + 2r - 1)\, x^r}{3} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + \frac{5a_n(n+r)}{3} - \frac{a_n}{3} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}(n+r-1)}{3n^2 + 6nr + 3r^2 + 2n + 2r - 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{a_{n-1}(2-3n)}{3n^2 + 4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{3r}{3r^2 + 8r + 4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = -\frac{1}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9r(1+r)}{(3r^2 + 8r + 4)(3r^2 + 14r + 15)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{1}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$
a_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{1}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27r(1+r)}{(3r^2 + 20r + 32)(3r^2 + 14r + 15)(3r + 2)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{1}{195}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$
a_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{1}{35}$
a_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	$-\frac{1}{195}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{1}{1248}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$
a_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{1}{35}$
a_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	$-\frac{1}{195}$
a_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	$\frac{1}{1248}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{1}{9120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3r}{3r^2+8r+4}$	$-\frac{1}{7}$
a_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{1}{35}$
a_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	$-\frac{1}{195}$
a_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	$\frac{1}{1248}$
a_5	$-\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$	$-\frac{1}{9120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{x}{7} + \frac{x^2}{35} - \frac{x^3}{195} + \frac{x^4}{1248} - \frac{x^5}{9120} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) + \frac{5b_n(n+r)}{3} - \frac{b_n}{3} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{3b_{n-1}(n+r-1)}{3n^2+6nr+3r^2+2n+2r-1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{3b_{n-1}(n-2)}{n(3n-4)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{3r}{3r^2 + 8r + 4}$$

Which for the root $r = -1$ becomes

$$b_1 = -3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9r(1 + r)}{(3r^2 + 8r + 4)(3r^2 + 14r + 15)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3
b_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{27r(1 + r)}{(3r^2 + 20r + 32)(3r^2 + 14r + 15)(3r + 2)}$$

Which for the root $r = -1$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3
b_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	0
b_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$$

Which for the root $r = -1$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3
b_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	0
b_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	0
b_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$$

Which for the root $r = -1$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3r}{3r^2+8r+4}$	-3
b_2	$\frac{9r(1+r)}{(3r^2+8r+4)(3r^2+14r+15)}$	0
b_3	$-\frac{27r(1+r)}{(3r^2+20r+32)(3r^2+14r+15)(3r+2)}$	0
b_4	$\frac{81r(1+r)}{(3r^2+26r+55)(3r+2)(3r+5)(3r^2+20r+32)}$	0
b_5	$-\frac{243r(1+r)}{(3r^2+32r+84)(3r+8)(3r+5)(3r+2)(3r^2+26r+55)}$	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - 3x + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{x}{7} + \frac{x^2}{35} - \frac{x^3}{195} + \frac{x^4}{1248} - \frac{x^5}{9120} + O(x^6)\right) + \frac{c_2(1 - 3x + O(x^6))}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{x}{7} + \frac{x^2}{35} - \frac{x^3}{195} + \frac{x^4}{1248} - \frac{x^5}{9120} + O(x^6)\right) + \frac{c_2(1 - 3x + O(x^6))}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{3}}\left(1 - \frac{x}{7} + \frac{x^2}{35} - \frac{x^3}{195} + \frac{x^4}{1248} - \frac{x^5}{9120} + O(x^6)\right) + \frac{c_2(1 - 3x + O(x^6))}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{3}}\left(1 - \frac{x}{7} + \frac{x^2}{35} - \frac{x^3}{195} + \frac{x^4}{1248} - \frac{x^5}{9120} + O(x^6)\right) + \frac{c_2(1 - 3x + O(x^6))}{x}
$$

Verified OK.

2.13.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(\frac{5}{3}x + x^2\right) y' - \frac{y}{3} = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{y}{3x^2} - \frac{(3x+5)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(3x+5)y'}{3x} - \frac{y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

  $\left[P_2(x) = \frac{3x+5}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{5}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{3}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$3x^2 y'' + x(3x+5) y' - y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

$$x^{m} \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{2} \cdot y''$ to series expansion

$$x^{2} \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+3r)\, x^{r} + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(3k+3r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-1, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3(k+r+1)\left(k+r-\frac{1}{3}\right) a_k + 3a_{k-1}(k+r-1) = 0$$

- Shift index using $k- >k+1$

$$3(k+2+r)\left(k+\frac{2}{3}+r\right) a_{k+1} + 3a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{3a_k(k+r)}{(k+2+r)(3k+2+3r)}$$

- Recursion relation for $r=-1$; series terminates at $k=1$

$$a_{k+1} = -\frac{3a_k(k-1)}{(k+1)(3k-1)}$$

- Apply recursion relation for $k=0$

$$a_1 = -3a_0$$

- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second

$$y = a_0 \cdot (1-3x)$$

- Recursion relation for $r=\frac{1}{3}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}$$

- Solution for $r=\frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot (1-3x) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{k+1} = -\frac{3b_k \left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 39

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+(5/3*x+x^2)*diff(y(x),x)-1/3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}} \left(1 - \frac{1}{7}x + \frac{1}{35}x^2 - \frac{1}{195}x^3 + \frac{1}{1248}x^4 - \frac{1}{9120}x^5 + \mathrm{O}\left(x^6\right)\right) + c_1\left(1 - 3x + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 58

```
AsymptoticDSolveValue[x^2*y''[x]+(5/3*x+x^2)*y'[x]-1/3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1 \sqrt[3]{x}\left(-\frac{x^5}{9120}+\frac{x^4}{1248}-\frac{x^3}{195}+\frac{x^2}{35}-\frac{x}{7}+1\right)+\frac{c_2(1-3x)}{x}$$

2.14 problem 14

Internal problem ID [5569]
Internal file name [OUTPUT/4817_Sunday_June_05_2022_03_06_42_PM_44446926/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' + 10y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + 10y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{10}{x}$$

Table 34: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{10}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + 10y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 10\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 10a_n x^{n+r} = \sum_{n=1}^{\infty} 10a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 10a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + 10a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{10a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{10a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{10}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -10$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{(r+1)^2}$	-10

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{100}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 25$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{(r+1)^2}$	-10
a_2	$\frac{100}{(r+1)^2(r+2)^2}$	25

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1000}{(r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{250}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{(r+1)^2}$	-10
a_2	$\frac{100}{(r+1)^2(r+2)^2}$	25
a_3	$-\frac{1000}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{250}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{10000}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{625}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{(r+1)^2}$	-10
a_2	$\frac{100}{(r+1)^2(r+2)^2}$	25
a_3	$-\frac{1000}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{250}{9}$
a_4	$\frac{10000}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{625}{36}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{100000}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2 (5+r)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{125}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{(r+1)^2}$	-10
a_2	$\frac{100}{(r+1)^2(r+2)^2}$	25
a_3	$-\frac{1000}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{250}{9}$
a_4	$\frac{10000}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{625}{36}$
a_5	$-\frac{100000}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{125}{18}$

Using the above table, then the first solution $y_1(x)$ becomes

$$y_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots$$
$$= 25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + O(x^6)$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{10}{(r+1)^2}$	-10	$\frac{20}{(r+1)^3}$	20
b_2	$\frac{100}{(r+1)^2 (r+2)^2}$	25	$\frac{-400r-600}{(r+1)^3 (r+2)^3}$	-75
b_3	$-\frac{1000}{(r+1)^2 (r+2)^2 (r+3)^2}$	$-\frac{250}{9}$	$\frac{6000r^2+24000r+22000}{(r+1)^3 (r+2)^3 (r+3)^3}$	$\frac{2750}{27}$
b_4	$\frac{10000}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2}$	$\frac{625}{36}$	$-\frac{80000\left(r+\frac{5}{2}\right)(r^2+5r+5)}{(r+1)^3 (r+2)^3 (r+3)^3 (4+r)^3}$	$-\frac{15625}{216}$
b_5	$-\frac{100000}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2 (5+r)^2}$	$-\frac{125}{18}$	$\frac{1000000r^4+12000000r^3+51000000r^2+90000000r+54800000}{(r+1)^3 (r+2)^3 (r+3)^3 (4+r)^3 (5+r)^3}$	$\frac{3425}{108}$

The above table gives all values of b_n needed. Hence the second solution is

$$y_2(x) = y_1(x) \ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots$$
$$= \left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + O(x^6)\right) \ln(x)$$
$$- 75x^2 + 20x + \frac{2750x^3}{27} - \frac{15625x^4}{216} + \frac{3425x^5}{108} + O(x^6)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + O(x^6)\right) \\
&\quad + c_2\left(\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + O(x^6)\right)\ln(x) - 75x^2 \right.\\
&\quad \left. + 20x + \frac{2750x^3}{27} - \frac{15625x^4}{216} + \frac{3425x^5}{108} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + O(x^6)\right) \\
&\quad + c_2\left(\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + O(x^6)\right)\ln(x) - 75x^2 + 20x \right.\\
&\quad \left. + \frac{2750x^3}{27} - \frac{15625x^4}{216} + \frac{3425x^5}{108} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + O(x^6)\right) \\
&\quad + c_2\left(\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + O(x^6)\right)\ln(x) - 75x^2 \right.\\
&\quad \left. + 20x + \frac{2750x^3}{27} - \frac{15625x^4}{216} + \frac{3425x^5}{108} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + O(x^6)\right) \\
&\quad + c_2\left(\left(25x^2 - 10x + 1 - \frac{250x^3}{9} + \frac{625x^4}{36} - \frac{125x^5}{18} + O(x^6)\right)\ln(x) - 75x^2 + 20x \right.\\
&\quad \left. + \frac{2750x^3}{27} - \frac{15625x^4}{216} + \frac{3425x^5}{108} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

2.14.1 Maple step by step solution

Let's solve

$y''x + y' + 10y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{10y}{x} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{10y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{10}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y' + 10y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- > k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + 10a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + 10a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{10a_k}{(k+1)^2}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{10a_k}{(k+1)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{10a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+10*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln (x) + c_1) \left(1 - 10x + 25x^2 - \frac{250}{9}x^3 + \frac{625}{36}x^4 - \frac{125}{18}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(20x - 75x^2 + \frac{2750}{27}x^3 - \frac{15625}{216}x^4 + \frac{3425}{108}x^5 + \mathrm{O}\left(x^6\right)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 105

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+10*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{125x^5}{18} + \frac{625x^4}{36} - \frac{250x^3}{9} + 25x^2 - 10x + 1\right) + c_2 \left(\frac{3425x^5}{108} - \frac{15625x^4}{216} + \frac{2750x^3}{27} - 75x^2 + \left(-\frac{125x^5}{18} + \frac{625x^4}{36} - \frac{250x^3}{9} + 25x^2 - 10x + 1\right) \log(x) + 20x\right)$$

2.15 problem 15

Internal problem ID [5570]
Internal file name [`OUTPUT/4818_Sunday_June_05_2022_03_06_43_PM_77920638/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2xy'' - y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' - y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = \frac{1}{x}$$

Table 36: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' - y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x - \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) - (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) - r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) - r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-3+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-3+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}}{2n^2 + 4nr + 2r^2 - 3n - 3r} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{2a_{n-1}}{n(2n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{2r^2 + r - 1}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = -\frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+r-1}$	$-\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{4r^4 + 12r^3 + 7r^2 - 3r - 2}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{2}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	$\frac{2}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{8r^6 + 60r^5 + 158r^4 + 165r^3 + 32r^2 - 45r - 18}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = -\frac{4}{945}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	$\frac{2}{35}$
a_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$-\frac{4}{945}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{16r^8 + 224r^7 + 1256r^6 + 3584r^5 + 5369r^4 + 3626r^3 + 19r^2 - 1134r - 360}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{2}{10395}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	$\frac{2}{35}$
a_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$-\frac{4}{945}$
a_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$\frac{2}{10395}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{32r^{10} + 720r^9 + 6880r^8 + 36360r^7 + 115626r^6 + 223965r^5 + 249595r^4 + 124965r^3 - 19333r^2 - 45}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = -\frac{4}{675675}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+r-1}$	$-\frac{2}{5}$
a_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	$\frac{2}{35}$
a_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$-\frac{4}{945}$
a_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$\frac{2}{10395}$
a_5	$-\frac{32}{32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600}$	$-\frac{4}{675675}$

Using the above table, then the solution $y_1(x)$ is

$$y_1(x) = x^{\frac{3}{2}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right)$$
$$= x^{\frac{3}{2}}\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + O(x^6)\right)$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \le n$ the recursive equation is

$$2b_n(n + r)(n + r - 1) - (n + r)b_n + 2b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{2b_{n-1}}{2n^2 + 4nr + 2r^2 - 3n - 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{2b_{n-1}}{n(2n - 3)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{2}{2r^2 + r - 1}$$

Which for the root $r = 0$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+r-1}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{4r^4 + 12r^3 + 7r^2 - 3r - 2}$$

Which for the root $r = 0$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+r-1}$	2
b_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{8r^6 + 60r^5 + 158r^4 + 165r^3 + 32r^2 - 45r - 18}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{4}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+r-1}$	2
b_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	-2
b_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{4}{9}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{16r^8 + 224r^7 + 1256r^6 + 3584r^5 + 5369r^4 + 3626r^3 + 19r^2 - 1134r - 360}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{2}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+r-1}$	2
b_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	-2
b_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{4}{9}$
b_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$-\frac{2}{45}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{32r^{10} + 720r^9 + 6880r^8 + 36360r^7 + 115626r^6 + 223965r^5 + 249595r^4 + 124965r^3 - 19333r^2 - 45}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{4}{1575}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+r-1}$	2
b_2	$\frac{4}{4r^4+12r^3+7r^2-3r-2}$	-2
b_3	$-\frac{8}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{4}{9}$
b_4	$\frac{16}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$-\frac{2}{45}$
b_5	$-\frac{32}{32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600}$	$\frac{4}{1575}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{3}{2}} \left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + O(x^6)\right) \\ &\quad + c_2 \left(1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}}\left(1-\frac{2x}{5}+\frac{2x^2}{35}-\frac{4x^3}{945}+\frac{2x^4}{10395}-\frac{4x^5}{675675}+O(x^6)\right) \\
&\quad + c_2\left(1+2x-2x^2+\frac{4x^3}{9}-\frac{2x^4}{45}+\frac{4x^5}{1575}+O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = c_1 x^{\frac{3}{2}}&\left(1-\frac{2x}{5}+\frac{2x^2}{35}-\frac{4x^3}{945}+\frac{2x^4}{10395}-\frac{4x^5}{675675}+O(x^6)\right) \\
&+ c_2\left(1+2x-2x^2+\frac{4x^3}{9}-\frac{2x^4}{45}+\frac{4x^5}{1575}+O(x^6)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y = c_1 x^{\frac{3}{2}}&\left(1-\frac{2x}{5}+\frac{2x^2}{35}-\frac{4x^3}{945}+\frac{2x^4}{10395}-\frac{4x^5}{675675}+O(x^6)\right) \\
&+ c_2\left(1+2x-2x^2+\frac{4x^3}{9}-\frac{2x^4}{45}+\frac{4x^5}{1575}+O(x^6)\right)
\end{aligned}$$

Verified OK.

2.15.1 Maple step by step solution

Let's solve

$2y''x - y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{2x} - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $$x_0 = 0$$

- Multiply by denominators

  $$2y''x - y' + 2y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k-\frac{1}{2}+r\right)a_{k+1}+2a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{2a_k}{(k+1+r)(2k-1+2r)}$

- Recursion relation for $r=0$

  $a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}\right]$

- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$

- Solution for $r=\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}, b_{k+1}=-\frac{2b_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 44

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)-diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{3}{2}} \left(1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{945}x^3 + \frac{2}{10395}x^4 - \frac{4}{675675}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(1 + 2x - 2x^2 + \frac{4}{9}x^3 - \frac{2}{45}x^4 + \frac{4}{1575}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 81

```
AsymptoticDSolveValue[2*x*y''[x]-y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{4x^5}{1575} - \frac{2x^4}{45} + \frac{4x^3}{9} - 2x^2 + 2x + 1\right) + c_1 \left(-\frac{4x^5}{675675} + \frac{2x^4}{10395} - \frac{4x^3}{945} + \frac{2x^2}{35} - \frac{2x}{5} + 1\right) x^{3/2}$$

2.16 problem 16

Internal problem ID [5571]
Internal file name [OUTPUT/4819_Sunday_June_05_2022_03_06_45_PM_43329254/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + 5y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + 5y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{5}{2x}$$
$$q(x) = \frac{1}{2}$$

Table 38: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{2}$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + 5y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + 5\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + 5(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + 5r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + 5r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(3+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -\frac{3}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(3+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 5a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{2n^2 + 4nr + 2r^2 + 3n + 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n(2n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{2r^2 + 11r + 14}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 60r^3 + 325r^2 + 750r + 616}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{616}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{616}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{616}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - \frac{x^2}{14} + \frac{x^4}{616} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + 5(n+r)b_n + b_{n-2} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{2n^2 + 4nr + 2r^2 + 3n + 3r} \tag{4}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_n = -\frac{b_{n-2}}{n(2n-3)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{2r^2 + 11r + 14}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 60r^3 + 325r^2 + 750r + 616}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_4 = \frac{1}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{40}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{40}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$y_2(x) = 1\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right)$$
$$= \frac{1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)}{x^{\frac{3}{2}}}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1y_1(x) + c_2y_2(x)$$
$$= c_1\left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{x^{\frac{3}{2}}}$$

Hence the final solution is

$$y = y_h$$
$$= c_1\left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = c_1\left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = c_1\left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{x^{\frac{3}{2}}}$$

Verified OK.

2.16.1 Maple step by step solution

Let's solve

$2y''x + 5y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{5y'}{2x} - \frac{y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{2x} + \frac{y}{2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{5}{2x}, P_3(x) = \frac{1}{2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Bigg|_{x=0} = \frac{5}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + 5y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(3+2r)\, x^{-1+r} + a_1(1+r)\,(5+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(2k+5+2r) + a_{k-1}\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(1+r)\,(5+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+r+1)\left(k+\frac{5}{2}+r\right) a_{k+1} + a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$2(k+2+r)\left(k+\frac{7}{2}+r\right) a_{k+2} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+2+r)(2k+7+2r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k}{(k+2)(2k+7)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(2k+7)}, 5a_1 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+4)}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2}=-\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+4)}, -a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2}=-\frac{a_k}{(k+2)(2k+7)}, 5a_1=0, b_{k+2}=-\frac{b_k}{\left(k+\frac{1}{2}\right)(2k+4)}, -b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)+5*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1\left(1-\frac{1}{2}x^2+\frac{1}{40}x^4+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{3}{2}}}+c_2\left(1-\frac{1}{14}x^2+\frac{1}{616}x^4+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 47

```
AsymptoticDSolveValue[2*x*y''[x]+5*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^4}{616}-\frac{x^2}{14}+1\right)+\frac{c_2\left(\frac{x^4}{40}-\frac{x^2}{2}+1\right)}{x^{3/2}}$$

2.17 problem 17

Internal problem ID [5572]
Internal file name [`OUTPUT/4820_Sunday_June_05_2022_03_06_47_PM_44334659/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{4xy'' + \frac{y'}{2} + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4xy'' + \frac{y'}{2} + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{8x}$$
$$q(x) = \frac{1}{4x}$$

Table 40: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{8x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{4x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' + \frac{y'}{2} + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \frac{\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right)}{2} + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} \frac{(n+r)\, a_n x^{n+r-1}}{2}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} \frac{(n+r)\, a_n x^{n+r-1}}{2}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r)(n+r-1) + \frac{(n+r)\, a_n x^{n+r-1}}{2} = 0$$

When $n=0$ the above becomes

$$4x^{-1+r} a_0 r(-1+r) + \frac{r a_0 x^{-1+r}}{2} = 0$$

Or

$$\left(4x^{-1+r} r(-1+r) + \frac{r\, x^{-1+r}}{2}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}\left(-\frac{7}{2} + 4r\right) = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - \frac{7}{2} r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{7}{8}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}\left(-\frac{7}{2}+4r\right) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{7}{8}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{7}{8}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + \frac{a_n(n+r)}{2} + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}}{8n^2 + 16nr + 8r^2 - 7n - 7r} \tag{4}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_n = -\frac{2a_{n-1}}{n(8n+7)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{7}{8}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{8r^2 + 9r + 1}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_1 = -\frac{2}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{(8r^2 + 9r + 1)(8r^2 + 25r + 18)}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_2 = \frac{2}{345}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$
a_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{345}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{(8r^2 + 9r + 1)(8r^2 + 25r + 18)(8r^2 + 41r + 51)}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_3 = -\frac{4}{32085}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$
a_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{345}$
a_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{32085}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(8r^2 + 9r + 1)(8r^2 + 25r + 18)(8r^2 + 41r + 51)(8r^2 + 57r + 100)}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_4 = \frac{2}{1251315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$
a_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{345}$
a_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{32085}$
a_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{1251315}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{(8r^2 + 9r + 1)(8r^2 + 25r + 18)(8r^2 + 41r + 51)(8r^2 + 57r + 100)(8r^2 + 73r + 165)}$$

Which for the root $r = \frac{7}{8}$ becomes

$$a_5 = -\frac{4}{294059025}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{8r^2+9r+1}$	$-\frac{2}{15}$
a_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{345}$
a_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{32085}$
a_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{1251315}$
a_5	$-\frac{32}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)}$	$-\frac{4}{294059025}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{7}{8}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{7}{8}}\left(1 - \frac{2x}{15} + \frac{2x^2}{345} - \frac{4x^3}{32085} + \frac{2x^4}{1251315} - \frac{4x^5}{294059025} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
4b_n(n+r)(n+r-1) + \frac{(n+r)\, b_n}{2} + b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{2b_{n-1}}{8n^2 + 16nr + 8r^2 - 7n - 7r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{2b_{n-1}}{n(8n-7)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{2}{8r^2 + 9r + 1}
$$

Which for the root $r = 0$ becomes

$$b_1 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{(8r^2 + 9r + 1)(8r^2 + 25r + 18)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{2}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2
b_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{9}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{(8r^2 + 9r + 1)(8r^2 + 25r + 18)(8r^2 + 41r + 51)}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{4}{459}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2
b_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{9}$
b_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{459}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{2}{11475}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2
b_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{9}$
b_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{459}$
b_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{11475}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{4}{1893375}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{8r^2+9r+1}$	-2
b_2	$\frac{4}{(8r^2+9r+1)(8r^2+25r+18)}$	$\frac{2}{9}$
b_3	$-\frac{8}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)}$	$-\frac{4}{459}$
b_4	$\frac{16}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)}$	$\frac{2}{11475}$
b_5	$-\frac{32}{(8r^2+9r+1)(8r^2+25r+18)(8r^2+41r+51)(8r^2+57r+100)(8r^2+73r+165)}$	$-\frac{4}{1893375}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - 2x + \frac{2x^2}{9} - \frac{4x^3}{459} + \frac{2x^4}{11475} - \frac{4x^5}{1893375} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{7}{8}} \left(1 - \frac{2x}{15} + \frac{2x^2}{345} - \frac{4x^3}{32085} + \frac{2x^4}{1251315} - \frac{4x^5}{294059025} + O(x^6)\right) \\ &\quad + c_2 \left(1 - 2x + \frac{2x^2}{9} - \frac{4x^3}{459} + \frac{2x^4}{11475} - \frac{4x^5}{1893375} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{7}{8}} \left(1 - \frac{2x}{15} + \frac{2x^2}{345} - \frac{4x^3}{32085} + \frac{2x^4}{1251315} - \frac{4x^5}{294059025} + O(x^6)\right) \\ &\quad + c_2 \left(1 - 2x + \frac{2x^2}{9} - \frac{4x^3}{459} + \frac{2x^4}{11475} - \frac{4x^5}{1893375} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_1 x^{\frac{7}{8}} \left(1 - \frac{2x}{15} + \frac{2x^2}{345} - \frac{4x^3}{32085} + \frac{2x^4}{1251315} - \frac{4x^5}{294059025} + O(x^6)\right) \\ & + c_2 \left(1 - 2x + \frac{2x^2}{9} - \frac{4x^3}{459} + \frac{2x^4}{11475} - \frac{4x^5}{1893375} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & c_1 x^{\frac{7}{8}} \left(1 - \frac{2x}{15} + \frac{2x^2}{345} - \frac{4x^3}{32085} + \frac{2x^4}{1251315} - \frac{4x^5}{294059025} + O(x^6)\right) \\ & + c_2 \left(1 - 2x + \frac{2x^2}{9} - \frac{4x^3}{459} + \frac{2x^4}{11475} - \frac{4x^5}{1893375} + O(x^6)\right) \end{aligned}$$

Verified OK.

2.17.1 Maple step by step solution

Let's solve

$4y''x+\frac{y'}{2}+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{4x}-\frac{y'}{8x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{8x}+\frac{y}{4x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1}{8x}, P_3(x)=\frac{1}{4x}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{1}{8}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $8y''x+2y+y'=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$

  - Shift index using $k->k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-7+8r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(8k+1+8r) + 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-7+8r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{7}{8}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$8\left(k+\frac{1}{8}+r\right)(k+1+r)\, a_{k+1} + 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2a_k}{(8k+1+8r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{2a_k}{(8k+1)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(8k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{7}{8}$

$$a_{k+1} = -\frac{2a_k}{(8k+8)\left(k+\frac{15}{8}\right)}$$

- Solution for $r = \frac{7}{8}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{7}{8}}, a_{k+1} = -\frac{2a_k}{(8k+8)\left(k+\frac{15}{8}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{7}{8}}\right), a_{k+1}=-\frac{2a_k}{(8k+1)(k+1)}, b_{k+1}=-\frac{2b_k}{(8k+8)\left(k+\frac{15}{8}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
Order:=6;
dsolve(4*x*diff(y(x),x$2)+1/2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1 x^{\frac{7}{8}}\left(1-\frac{2}{15}x+\frac{2}{345}x^2-\frac{4}{32085}x^3+\frac{2}{1251315}x^4-\frac{4}{294059025}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(1-2x+\frac{2}{9}x^2-\frac{4}{459}x^3+\frac{2}{11475}x^4-\frac{4}{1893375}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 83

```
AsymptoticDSolveValue[4*x*y''[x]+1/2*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(-\frac{4x^5}{1893375}+\frac{2x^4}{11475}-\frac{4x^3}{459}+\frac{2x^2}{9}-2x+1\right)+c_1 x^{7/8}\left(-\frac{4x^5}{294059025}+\frac{2x^4}{1251315}-\frac{4x^3}{32085}+\frac{2x^2}{345}-\frac{2x}{15}+1\right)$$

2.18 problem 18

Internal problem ID [5573]
Internal file name [`OUTPUT/4821_Sunday_June_05_2022_03_06_50_PM_9834430/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - xy' + \left(x^2+1\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = \frac{x^2+1}{2x^2}$$

Table 42: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2+1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + (x^2 + 1) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2} \right) - x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \right) + (x^2 + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1) - x^{n+r} a_n(n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) - a_n(n + r) + a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) + b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = -\frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} + O\left(x^6\right)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} + O\left(x^6\right)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} + O\left(x^6\right)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} + O\left(x^6\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} + O\left(x^6\right)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} + O\left(x^6\right)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} + O\left(x^6\right)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} + O\left(x^6\right)\right)
$$

Verified OK.

2.18.1 Maple step by step solution

Let's solve

$2x^2 y'' - x y' + \left(x^2 + 1\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+1)y}{2x^2} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} + \frac{(x^2+1)y}{2x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{x^2+1}{2x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{2}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2y'' - xy' + (x^2 + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+2r)(-1+r)x^r + a_1(1+2r)r\,x^{1+r} + \left(\sum_{k=2}^{\infty}\left(a_k(2k+2r-1)(k+r-1)+a_{k-2}\right)x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)(-1+r)=0$
- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(1+2r)r=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $2(k+r-1)\left(k-\frac{1}{2}+r\right)a_k + a_{k-2} = 0$
- Shift index using $k->k+2$

  $2(k+1+r)\left(k+\frac{3}{2}+r\right)a_{k+2} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(2k+3+2r)}$
- Recursion relation for $r=1$

  $a_{k+2} = -\frac{a_k}{(k+2)(2k+5)}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(2k+5)}, a_1 = 0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+4)}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+4)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k}{(k+2)(2k+5)}, a_1 = 0, b_{k+2} = -\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+4)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+(x^2+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2x\left(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 48

```
AsymptoticDSolveValue[2*x^2*y''[x]-x*y'[x]+(x^2+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1x\left(\frac{x^4}{360} - \frac{x^2}{10} + 1\right) + c_2\sqrt{x}\left(\frac{x^4}{168} - \frac{x^2}{6} + 1\right)$$

2.19 problem 19

Internal problem ID [5574]
Internal file name [`OUTPUT/4822_Sunday_June_05_2022_03_06_52_PM_61396382/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{3xy'' + (2-x)\,y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3xy'' + (2-x)\,y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{-2+x}{3x}$$
$$q(x) = -\frac{1}{3x}$$

Table 44: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{-2+x}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{3x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3xy'' + (2 - x)\,y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x + (2-x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 3x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$3x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(3x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+3r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1 + 3r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$3a_n(n + r)(n + r - 1) - a_{n-1}(n + r - 1) + 2a_n(n + r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{3n - 1 + 3r} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{a_{n-1}}{3n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2 + 3r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{9r^2 + 21r + 10}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{1}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$
a_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{18}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = \frac{1}{162}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$
a_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{18}$
a_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{162}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{1}{1944}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$
a_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{18}$
a_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{162}$
a_4	$\frac{1}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1}{1944}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = \frac{1}{29160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+3r}$	$\frac{1}{3}$
a_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{18}$
a_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{162}$
a_4	$\frac{1}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1}{1944}$
a_5	$\frac{1}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$\frac{1}{29160}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 + \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{162} + \frac{x^4}{1944} + \frac{x^5}{29160} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$3b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + 2(n+r)b_n - b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{3n - 1 + 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}}{3n - 1} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{2 + 3r}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{9r^2 + 21r + 10}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{10}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$
b_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{80}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$
b_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{10}$
b_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{80}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{880}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$
b_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{10}$
b_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{80}$
b_4	$\frac{1}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1}{880}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{12320}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+3r}$	$\frac{1}{2}$
b_2	$\frac{1}{9r^2+21r+10}$	$\frac{1}{10}$
b_3	$\frac{1}{27r^3+135r^2+198r+80}$	$\frac{1}{80}$
b_4	$\frac{1}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1}{880}$
b_5	$\frac{1}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$\frac{1}{12320}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + \frac{x}{2} + \frac{x^2}{10} + \frac{x^3}{80} + \frac{x^4}{880} + \frac{x^5}{12320} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}} \left(1 + \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{162} + \frac{x^4}{1944} + \frac{x^5}{29160} + O(x^6)\right) \\
&\quad + c_2 \left(1 + \frac{x}{2} + \frac{x^2}{10} + \frac{x^3}{80} + \frac{x^4}{880} + \frac{x^5}{12320} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}} \left(1 + \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{162} + \frac{x^4}{1944} + \frac{x^5}{29160} + O(x^6)\right) \\
&\quad + c_2 \left(1 + \frac{x}{2} + \frac{x^2}{10} + \frac{x^3}{80} + \frac{x^4}{880} + \frac{x^5}{12320} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{1}{3}} &\left(1 + \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{162} + \frac{x^4}{1944} + \frac{x^5}{29160} + O(x^6)\right) \\
&+ c_2 \left(1 + \frac{x}{2} + \frac{x^2}{10} + \frac{x^3}{80} + \frac{x^4}{880} + \frac{x^5}{12320} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^{\frac{1}{3}} &\left(1 + \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{162} + \frac{x^4}{1944} + \frac{x^5}{29160} + O(x^6)\right) \\
&+ c_2 \left(1 + \frac{x}{2} + \frac{x^2}{10} + \frac{x^3}{80} + \frac{x^4}{880} + \frac{x^5}{12320} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

2.19.1 Maple step by step solution

Let's solve

$3y''x+(2-x)\,y'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{3x}+\frac{(-2+x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(-2+x)y'}{3x}-\frac{y}{3x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{-2+x}{3x},P_3(x)=-\frac{1}{3x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{2}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3y''x+(2-x)\,y'-y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(3k+2+3r) - a_k(k+1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3(k+1+r)\left(\left(k+r+\frac{2}{3}\right) a_{k+1} - \frac{a_k}{3}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{3k+2+3r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{3k+2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{3k+2}\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = \frac{a_k}{3k+3}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{a_k}{3k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = \frac{a_k}{3k+2}, b_{k+1} = \frac{b_k}{3k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form is not straightforward to achieve - returning hypergeometric solution
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(3*x*diff(y(x),x$2)+(2-x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4 + \frac{1}{29160}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 + \frac{1}{880}x^4 + \frac{1}{12320}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 85

```
AsymptoticDSolveValue[3*x*y''[x]+(2-x)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x} \left(\frac{x^5}{29160} + \frac{x^4}{1944} + \frac{x^3}{162} + \frac{x^2}{18} + \frac{x}{3} + 1\right) + c_2 \left(\frac{x^5}{12320} + \frac{x^4}{880} + \frac{x^3}{80} + \frac{x^2}{10} + \frac{x}{2} + 1\right)$$

2.20 problem 20

Internal problem ID [5575]
Internal file name [`OUTPUT/4823_Sunday_June_05_2022_03_06_55_PM_58278675/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(x - \frac{2}{9}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x + \frac{2}{9}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{9x - 2}{9x^2} \end{aligned}$$

Table 46: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{9x-2}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-x + \frac{2}{9}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(-x + \frac{2}{9}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n + r) (n + r - 1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{2a_n x^{n+r}}{9}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} \frac{2 a_n x^{n+r}}{9}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + \frac{2 a_n x^{n+r}}{9} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + \frac{2 a_0 x^r}{9} = 0$$

Or

$$\left(x^r r(-1+r) + \frac{2x^r}{9}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(9r^2 - 9r + 2)\, x^r}{9} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r + \frac{2}{9} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{2}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(9r^2 - 9r + 2)\, x^r}{9} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1} + \frac{2a_n}{9} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{9a_{n-1}}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_n = \frac{3a_{n-1}}{3n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{9}{9r^2 + 9r + 2}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_1 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{81}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_2 = \frac{9}{56}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$
a_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{56}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{729}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_3 = \frac{9}{560}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$
a_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{56}$
a_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{560}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_4 = \frac{27}{29120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$
a_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{56}$
a_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{560}$
a_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{29120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{59049}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_5 = \frac{81}{2329600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{4}$
a_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{56}$
a_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{560}$
a_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{29120}$
a_5	$\frac{59049}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$\frac{81}{2329600}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{2}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x^{\frac{2}{3}}\left(1 + \frac{3x}{4} + \frac{9x^2}{56} + \frac{9x^3}{560} + \frac{27x^4}{29120} + \frac{81x^5}{2329600} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n + r)(n + r - 1) - b_{n-1} + \frac{2b_n}{9} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{9b_{n-1}}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
b_n = \frac{3b_{n-1}}{n(3n - 1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{9}{9r^2 + 9r + 2}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
b_1 = \frac{3}{2}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{81}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = \frac{9}{20}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$
b_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{20}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{729}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_3 = \frac{9}{160}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$
b_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{20}$
b_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{160}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{6561}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)(9r^2 + 63r + 110)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = \frac{27}{7040}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$
b_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{20}$
b_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{160}$
b_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{7040}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{59049}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_5 = \frac{81}{492800}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{9}{9r^2+9r+2}$	$\frac{3}{2}$
b_2	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{9}{20}$
b_3	$\frac{729}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$\frac{9}{160}$
b_4	$\frac{6561}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{27}{7040}$
b_5	$\frac{59049}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$\frac{81}{492800}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{2}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 + \frac{3x}{2} + \frac{9x^2}{20} + \frac{9x^3}{160} + \frac{27x^4}{7040} + \frac{81x^5}{492800} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{2}{3}} \left(1 + \frac{3x}{4} + \frac{9x^2}{56} + \frac{9x^3}{560} + \frac{27x^4}{29120} + \frac{81x^5}{2329600} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}} \left(1 + \frac{3x}{2} + \frac{9x^2}{20} + \frac{9x^3}{160} + \frac{27x^4}{7040} + \frac{81x^5}{492800} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{2}{3}} \left(1 + \frac{3x}{4} + \frac{9x^2}{56} + \frac{9x^3}{560} + \frac{27x^4}{29120} + \frac{81x^5}{2329600} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}} \left(1 + \frac{3x}{2} + \frac{9x^2}{20} + \frac{9x^3}{160} + \frac{27x^4}{7040} + \frac{81x^5}{492800} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{2}{3}} &\left(1 + \frac{3x}{4} + \frac{9x^2}{56} + \frac{9x^3}{560} + \frac{27x^4}{29120} + \frac{81x^5}{2329600} + O(x^6)\right) \\
&+ c_2 x^{\frac{1}{3}} \left(1 + \frac{3x}{2} + \frac{9x^2}{20} + \frac{9x^3}{160} + \frac{27x^4}{7040} + \frac{81x^5}{492800} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^{\frac{2}{3}} &\left(1 + \frac{3x}{4} + \frac{9x^2}{56} + \frac{9x^3}{560} + \frac{27x^4}{29120} + \frac{81x^5}{2329600} + O(x^6)\right) \\
&+ c_2 x^{\frac{1}{3}} \left(1 + \frac{3x}{2} + \frac{9x^2}{20} + \frac{9x^3}{160} + \frac{27x^4}{7040} + \frac{81x^5}{492800} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

2.20.1 Maple step by step solution

Let's solve

$x^2y'' + \left(-x + \frac{2}{9}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(9x-2)y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(9x-2)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{9x-2}{9x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{2}{9}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2y'' + (-9x + 2) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

$$x^{m} \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^{2} \cdot y''$ to series expansion

$$x^{2} \cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}(-1+3r)(-2+3r) x^{r} + \left(\sum_{k=1}^{\infty} \left(a_{k}(3k+3r-1)(3k+3r-2) - 9a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+3r)(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$9\left(k+r-\frac{2}{3}\right)\left(k+r-\frac{1}{3}\right) a_{k} - 9a_{k-1} = 0$$

- Shift index using $k->k+1$

$$9\left(k+\frac{1}{3}+r\right)\left(k+\frac{2}{3}+r\right) a_{k+1} - 9a_{k} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{9a_{k}}{(3k+1+3r)(3k+2+3r)}$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = \frac{9a_{k}}{(3k+2)(3k+3)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{3}}, a_{k+1} = \frac{9a_{k}}{(3k+2)(3k+3)}\right]$$

- Recursion relation for $r = \frac{2}{3}$

$$a_{k+1} = \frac{9a_{k}}{(3k+3)(3k+4)}$$

- Solution for $r = \frac{2}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_{k} x^{k+\frac{2}{3}}, a_{k+1} = \frac{9a_{k}}{(3k+3)(3k+4)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_{k} x^{k+\frac{2}{3}}\right), a_{k+1} = \frac{9a_{k}}{(3k+2)(3k+3)}, b_{k+1} = \frac{9b_{k}}{(3k+3)(3k+4)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-(x-2/9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \frac{27}{7040}x^4 + \frac{81}{492800}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\frac{2}{3}} \left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \frac{27}{29120}x^4 + \frac{81}{2329600}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 90

```
AsymptoticDSolveValue[x^2*y''[x]-(x-2/9)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \sqrt[3]{x} \left(\frac{81x^5}{492800} + \frac{27x^4}{7040} + \frac{9x^3}{160} + \frac{9x^2}{20} + \frac{3x}{2} + 1\right) + c_1 x^{2/3} \left(\frac{81x^5}{2329600} + \frac{27x^4}{29120} + \frac{9x^3}{560} + \frac{9x^2}{56} + \frac{3x}{4} + 1\right)$$

2.21 problem 21

Internal problem ID [5576]
Internal file name [`OUTPUT/4824_Sunday_June_05_2022_03_06_57_PM_24876484/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[_Laguerre]
```

$$\boxed{2xy'' - (2x+3)\,y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + (-2x-3)\,y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x+3}{2x}$$
$$q(x) = \frac{1}{2x}$$

Table 48: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x+3}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (-2x - 3) y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) x + (-2x - 3) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) - 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) - 3r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) - 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-5+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 5r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{5}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-5 + 2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) - 2a_{n-1}(n + r - 1) - 3a_n(n + r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(2n + 2r - 3)}{2n^2 + 4nr + 2r^2 - 5n - 5r} \tag{4}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_n = \frac{2a_{n-1}(n + 1)}{n\,(2n + 5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1+2r}{2r^2 - r - 3}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_1 = \frac{4}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1+2r}{2r^3 + 3r^2 - 5r - 6}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_2 = \frac{4}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$
a_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$\frac{4}{21}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{3+2r}{2r^4 + 9r^3 + 4r^2 - 21r - 18}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_3 = \frac{32}{693}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$
a_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$\frac{4}{21}$
a_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$\frac{32}{693}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5 + 2r}{2r^5 + 17r^4 + 40r^3 - 5r^2 - 102r - 72}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_4 = \frac{80}{9009}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$
a_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$\frac{4}{21}$
a_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$\frac{32}{693}$
a_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$\frac{80}{9009}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{7 + 2r}{(r + 5)(2r^5 + 17r^4 + 40r^3 - 5r^2 - 102r - 72)}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_5 = \frac{64}{45045}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{4}{7}$
a_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$\frac{4}{21}$
a_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$\frac{32}{693}$
a_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$\frac{80}{9009}$
a_5	$\frac{7+2r}{(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$\frac{64}{45045}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{5}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{5}{2}}\left(1 + \frac{4x}{7} + \frac{4x^2}{21} + \frac{32x^3}{693} + \frac{80x^4}{9009} + \frac{64x^5}{45045} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) - 2b_{n-1}(n+r-1) - 3(n+r)b_n + b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}(2n+2r-3)}{2n^2+4nr+2r^2-5n-5r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{b_{n-1}(2n-3)}{n(2n-5)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-1 + 2r}{2r^2 - r - 3}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1 + 2r}{2r^3 + 3r^2 - 5r - 6}$$

Which for the root $r = 0$ becomes

$$b_2 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$
b_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{3 + 2r}{2r^4 + 9r^3 + 4r^2 - 21r - 18}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$
b_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$-\frac{1}{6}$
b_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{5 + 2r}{2r^5 + 17r^4 + 40r^3 - 5r^2 - 102r - 72}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{5}{72}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$
b_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$-\frac{1}{6}$
b_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$-\frac{1}{6}$
b_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$-\frac{5}{72}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{7 + 2r}{(r + 5)(2r^5 + 17r^4 + 40r^3 - 5r^2 - 102r - 72)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{7}{360}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+2r}{2r^2-r-3}$	$\frac{1}{3}$
b_2	$\frac{1+2r}{2r^3+3r^2-5r-6}$	$-\frac{1}{6}$
b_3	$\frac{3+2r}{2r^4+9r^3+4r^2-21r-18}$	$-\frac{1}{6}$
b_4	$\frac{5+2r}{2r^5+17r^4+40r^3-5r^2-102r-72}$	$-\frac{5}{72}$
b_5	$\frac{7+2r}{(r+5)(2r^5+17r^4+40r^3-5r^2-102r-72)}$	$-\frac{7}{360}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + \frac{x}{3} - \frac{x^2}{6} - \frac{x^3}{6} - \frac{5x^4}{72} - \frac{7x^5}{360} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{5}{2}} \left(1 + \frac{4x}{7} + \frac{4x^2}{21} + \frac{32x^3}{693} + \frac{80x^4}{9009} + \frac{64x^5}{45045} + O(x^6)\right) \\ &+ c_2 \left(1 + \frac{x}{3} - \frac{x^2}{6} - \frac{x^3}{6} - \frac{5x^4}{72} - \frac{7x^5}{360} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{5}{2}} \left(1 + \frac{4x}{7} + \frac{4x^2}{21} + \frac{32x^3}{693} + \frac{80x^4}{9009} + \frac{64x^5}{45045} + O(x^6)\right) \\ &+ c_2 \left(1 + \frac{x}{3} - \frac{x^2}{6} - \frac{x^3}{6} - \frac{5x^4}{72} - \frac{7x^5}{360} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 x^{\frac{5}{2}} &\left(1 + \frac{4x}{7} + \frac{4x^2}{21} + \frac{32x^3}{693} + \frac{80x^4}{9009} + \frac{64x^5}{45045} + O(x^6)\right) \\ &+ c_2 \left(1 + \frac{x}{3} - \frac{x^2}{6} - \frac{x^3}{6} - \frac{5x^4}{72} - \frac{7x^5}{360} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = c_1 x^{\frac{5}{2}} &\left(1 + \frac{4x}{7} + \frac{4x^2}{21} + \frac{32x^3}{693} + \frac{80x^4}{9009} + \frac{64x^5}{45045} + O(x^6)\right) \\ &+ c_2 \left(1 + \frac{x}{3} - \frac{x^2}{6} - \frac{x^3}{6} - \frac{5x^4}{72} - \frac{7x^5}{360} + O(x^6)\right) \end{aligned}$$

Verified OK.

2.21.1 Maple step by step solution

Let's solve

$2y''x + (-2x-3)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x} + \frac{(2x+3)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x+3)y'}{2x} + \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x+3}{2x}, P_3(x) = \frac{1}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{3}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + (-2x-3)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x\cdot y''$ to series expansion

  $$x\cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x\cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-5+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-3+2r) - a_k(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-5+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{5}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2(k+1+r)\left(k-\frac{3}{2}+r\right) a_{k+1} - 2a_k\left(k-\frac{1}{2}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(2k+2r-1)}{(k+1+r)(2k-3+2r)}$$

- Recursion relation for $r=0$

  $$a_{k+1} = \frac{a_k(2k-1)}{(k+1)(2k-3)}$$

- Solution for $r=0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+1} = \frac{a_k(2k-1)}{(k+1)(2k-3)}\right]$$

- Recursion relation for $r=\frac{5}{2}$

  $$a_{k+1} = \frac{a_k(2k+4)}{\left(k+\frac{7}{2}\right)(2k+2)}$$

- Solution for $r=\frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+1} = \frac{a_k(2k+4)}{\left(k+\frac{7}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+1}=\frac{a_k(2k-1)}{(k+1)(2k-3)}, b_{k+1}=\frac{b_k(2k+4)}{(k+\frac{7}{2})(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)-(3+2*x)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1x^{\frac{5}{2}}\left(1+\frac{4}{7}x+\frac{4}{21}x^2+\frac{32}{693}x^3+\frac{80}{9009}x^4+\frac{64}{45045}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(1+\frac{1}{3}x-\frac{1}{6}x^2-\frac{1}{6}x^3-\frac{5}{72}x^4-\frac{7}{360}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 85

```
AsymptoticDSolveValue[2*x*y''[x]-(3+2*x)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{7x^5}{360}-\frac{5x^4}{72}-\frac{x^3}{6}-\frac{x^2}{6}+\frac{x}{3}+1\right) + c_1\left(\frac{64x^5}{45045}+\frac{80x^4}{9009}+\frac{32x^3}{693}+\frac{4x^2}{21}+\frac{4x}{7}+1\right)x^{5/2}$$

2.22 problem 22

Internal problem ID [5577]
Internal file name [`OUTPUT/4825_Sunday_June_05_2022_03_07_00_PM_24081568/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{9x^2 - 4}{9x^2}$$

Table 50: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{9x^2-4}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(x^2 - \frac{4}{9}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{4 a_n x^{n+r}}{9}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{4 a_n x^{n+r}}{9}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - \frac{4 a_n x^{n+r}}{9} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - \frac{4 a_0 x^r}{9} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - \frac{4 x^r}{9}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(9r^2 - 4)\, x^r}{9} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{4}{9} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{2}{3}$$
$$r_2 = -\frac{2}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(9r^2 - 4)\, x^r}{9} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{2}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - \frac{4a_n}{9} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{9a_{n-2}}{9n^2 + 18nr + 9r^2 - 4} \tag{4}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_n = -\frac{3a_{n-2}}{n(3n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{9}{9r^2 + 36r + 32}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_2 = -\frac{3}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{20}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(9r^2 + 36r + 32)(9r^2 + 72r + 140)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_4 = \frac{9}{1280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{20}$
a_3	0	0
a_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{1280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{20}$
a_3	0	0
a_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{1280}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{2}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{2}{3}}\left(1 - \frac{3x^2}{20} + \frac{9x^4}{1280} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - \frac{4b_n}{9} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{9b_{n-2}}{9n^2 + 18nr + 9r^2 - 4} \tag{4}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_n = -\frac{3b_{n-2}}{n(3n-4)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{9}{9r^2 + 36r + 32}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_2 = -\frac{3}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{4}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(9r^2 + 36r + 32)(9r^2 + 72r + 140)}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_4 = \frac{9}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{4}$
b_3	0	0
b_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{128}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+32}$	$-\frac{3}{4}$
b_3	0	0
b_4	$\frac{81}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{9}{128}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{2}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{3x^2}{4} + \frac{9x^4}{128} + O(x^6)}{x^{\frac{2}{3}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{2}{3}}\left(1 - \frac{3x^2}{20} + \frac{9x^4}{1280} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^2}{4} + \frac{9x^4}{128} + O(x^6)\right)}{x^{\frac{2}{3}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{2}{3}}\left(1 - \frac{3x^2}{20} + \frac{9x^4}{1280} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^2}{4} + \frac{9x^4}{128} + O(x^6)\right)}{x^{\frac{2}{3}}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{2}{3}}\left(1 - \frac{3x^2}{20} + \frac{9x^4}{1280} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^2}{4} + \frac{9x^4}{128} + O(x^6)\right)}{x^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{2}{3}}\left(1 - \frac{3x^2}{20} + \frac{9x^4}{1280} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^2}{4} + \frac{9x^4}{128} + O(x^6)\right)}{x^{\frac{2}{3}}}$$

Verified OK.

2.22.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2-4)y}{9x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(9x^2-4)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{9x^2-4}{9x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{4}{9}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $9x^2y'' + 9xy' + (9x^2 - 4) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+3r)\,(-2+3r)\, x^{r}+a_1(5+3r)\,(1+3r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(3k+3r+2)\,(3k+3r-2)+9a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+3r)\,(-2+3r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{2}{3},\frac{2}{3}\right\}$

- Each term must be 0

  $a_1(5+3r)\,(1+3r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r+2)\,(3k+3r-2)+9a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(3k+8+3r)\,(3k+4+3r)+9a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{9a_k}{(3k+8+3r)(3k+4+3r)}$

- Recursion relation for $r=-\frac{2}{3}$

  $a_{k+2}=-\frac{9a_k}{(3k+6)(3k+2)}$

- Solution for $r=-\frac{2}{3}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{2}{3}}, a_{k+2}=-\frac{9a_k}{(3k+6)(3k+2)}, a_1=0\right]$

- Recursion relation for $r = \frac{2}{3}$

  $a_{k+2} = -\frac{9a_k}{(3k+10)(3k+6)}$

- Solution for $r = \frac{2}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+2} = -\frac{9a_k}{(3k+10)(3k+6)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{2}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{k+2} = -\frac{9a_k}{(3k+6)(3k+2)}, a_1 = 0, b_{k+2} = -\frac{9b_k}{(3k+10)(3k+6)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-4/9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}} \left(1 - \frac{3}{20}x^2 + \frac{9}{1280}x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 - \frac{3}{4}x^2 + \frac{9}{128}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-4/9)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x^{2/3}\left(\frac{9x^4}{1280} - \frac{3x^2}{20} + 1\right) + \frac{c_2\left(\frac{9x^4}{128} - \frac{3x^2}{4} + 1\right)}{x^{2/3}}$$

2.23 problem 23

Internal problem ID [5578]
Internal file name [OUTPUT/4826_Sunday_June_05_2022_03_07_03_PM_7002621/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' + 9x^2y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + 9x^2y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 1$$
$$q(x) = \frac{2}{9x^2}$$

Table 52: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = \frac{2}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + 9x^2y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 9x^2 \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{1+n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 9x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n(n+r)\,(n+r-1) + 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 2a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 9r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 9r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{2}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2 - 9r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) + 9a_{n-1}(n+r-1) + 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{9a_{n-1}(n+r-1)}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_n = \frac{a_{n-1}(1-3n)}{3n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{9r}{9r^2 + 9r + 2}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_2 = \frac{5}{28}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
a_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{5}{28}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_3 = -\frac{1}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
a_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{5}{28}$
a_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{21}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_4 = \frac{11}{1092}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
a_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{5}{28}$
a_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{21}$
a_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{11}{1092}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_5 = -\frac{11}{6240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
a_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{5}{28}$
a_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{21}$
a_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{11}{1092}$
a_5	$-\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{11}{6240}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{2}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{2}{3}}\left(1 - \frac{x}{2} + \frac{5x^2}{28} - \frac{x^3}{21} + \frac{11x^4}{1092} - \frac{11x^5}{6240} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
9b_n(n+r)(n+r-1) + 9b_{n-1}(n+r-1) + 2b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{9b_{n-1}(n+r-1)}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
b_n = \frac{b_{n-1}(2-3n)}{3n^2 - n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{9r}{9r^2 + 9r + 2}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
b_1 = -\frac{1}{2}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = \frac{1}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{5}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_3 = -\frac{7}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{5}$
b_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{7}{120}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = \frac{7}{528}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{5}$
b_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{7}{120}$
b_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{7}{528}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_5 = -\frac{13}{5280}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{9r}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{81r(1+r)}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{5}$
b_3	$-\frac{729r(1+r)(2+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{7}{120}$
b_4	$\frac{6561r(1+r)(2+r)(3+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{7}{528}$
b_5	$-\frac{59049r(1+r)(2+r)(3+r)(4+r)}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{13}{5280}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{2}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{7x^3}{120} + \frac{7x^4}{528} - \frac{13x^5}{5280} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{2}{3}} \left(1 - \frac{x}{2} + \frac{5x^2}{28} - \frac{x^3}{21} + \frac{11x^4}{1092} - \frac{11x^5}{6240} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{7x^3}{120} + \frac{7x^4}{528} - \frac{13x^5}{5280} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{2}{3}} \left(1 - \frac{x}{2} + \frac{5x^2}{28} - \frac{x^3}{21} + \frac{11x^4}{1092} - \frac{11x^5}{6240} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{7x^3}{120} + \frac{7x^4}{528} - \frac{13x^5}{5280} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{2}{3}} &\left(1 - \frac{x}{2} + \frac{5x^2}{28} - \frac{x^3}{21} + \frac{11x^4}{1092} - \frac{11x^5}{6240} + O(x^6)\right) \\
&+ c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{7x^3}{120} + \frac{7x^4}{528} - \frac{13x^5}{5280} + O(x^6)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^{\frac{2}{3}} &\left(1 - \frac{x}{2} + \frac{5x^2}{28} - \frac{x^3}{21} + \frac{11x^4}{1092} - \frac{11x^5}{6240} + O(x^6)\right) \\
&+ c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{5} - \frac{7x^3}{120} + \frac{7x^4}{528} - \frac{13x^5}{5280} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

2.23.1 Maple step by step solution

Let's solve

$9x^2y'' + 9x^2y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{2y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{2y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 1, P_3(x) = \frac{2}{9x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{2}{9}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2y'' + 9x^2y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

  - Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+3r)\,(-2+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-1)\,(3k+3r-2) + 9a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+3r)\,(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$9\left(k+r-\frac{2}{3}\right)\left(k+r-\frac{1}{3}\right) a_k + 9a_{k-1}(k-1+r) = 0$$

- Shift index using $k->k+1$

$$9\left(k+\frac{1}{3}+r\right)\left(k+\frac{2}{3}+r\right) a_{k+1} + 9a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{9a_k(k+r)}{(3k+1+3r)(3k+2+3r)}$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = -\frac{9a_k\left(k+\frac{1}{3}\right)}{(3k+2)(3k+3)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{9a_k\left(k+\frac{1}{3}\right)}{(3k+2)(3k+3)}\right]$$

- Recursion relation for $r = \frac{2}{3}$

$$a_{k+1} = -\frac{9a_k\left(k+\frac{2}{3}\right)}{(3k+3)(3k+4)}$$

- Solution for $r = \frac{2}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+1} = -\frac{9a_k\left(k+\frac{2}{3}\right)}{(3k+3)(3k+4)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{k+1} = -\frac{9a_k\left(k+\frac{1}{3}\right)}{(3k+2)(3k+3)}, b_{k+1} = -\frac{9b_k\left(k+\frac{2}{3}\right)}{(3k+3)(3k+4)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)+9*x^2*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 - \frac{1}{2}x + \frac{1}{5}x^2 - \frac{7}{120}x^3 + \frac{7}{528}x^4 - \frac{13}{5280}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\frac{2}{3}} \left(1 - \frac{1}{2}x + \frac{5}{28}x^2 - \frac{1}{21}x^3 + \frac{11}{1092}x^4 - \frac{11}{6240}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 90

```
AsymptoticDSolveValue[9*x^2*y''[x]+9*x^2*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \sqrt[3]{x} \left(-\frac{13x^5}{5280} + \frac{7x^4}{528} - \frac{7x^3}{120} + \frac{x^2}{5} - \frac{x}{2} + 1\right) + c_1 x^{2/3} \left(-\frac{11x^5}{6240} + \frac{11x^4}{1092} - \frac{x^3}{21} + \frac{5x^2}{28} - \frac{x}{2} + 1\right)$$

2.24 problem 24

Internal problem ID [5579]
Internal file name [`OUTPUT/4827_Sunday_June_05_2022_03_07_05_PM_89504628/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + 3xy' + (2x-1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 3xy' + (2x-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{3}{2x} \\ q(x) &= \frac{2x-1}{2x^2} \end{aligned}$$

Table 54: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2x-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 3xy' + (2x-1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + 3x\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (2x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + 3x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) + 3x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 3a_n(n+r) + 2a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}}{2n^2 + 4nr + 2r^2 + n + r - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{2a_{n-1}}{n(2n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{2r^2 + 5r + 2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{2}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$
a_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	$\frac{2}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)(2r^2 + 13r + 20)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{4}{945}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$
a_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	$\frac{2}{35}$
a_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$-\frac{4}{945}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)(2r^2 + 13r + 20)(2r^2 + 17r + 35)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{2}{10395}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$
a_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	$\frac{2}{35}$
a_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$-\frac{4}{945}$
a_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$\frac{2}{10395}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)(2r^2 + 13r + 20)(2r^2 + 17r + 35)(2r^2 + 21r + 54)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{4}{675675}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+5r+2}$	$-\frac{2}{5}$
a_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	$\frac{2}{35}$
a_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$-\frac{4}{945}$
a_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$\frac{2}{10395}$
a_5	$-\frac{32}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)}$	$-\frac{4}{675675}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + 3b_n(n+r) + 2b_{n-1} - b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{2b_{n-1}}{2n^2 + 4nr + 2r^2 + n + r - 1} \tag{4}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{2b_{n-1}}{n(2n-3)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{2}{2r^2 + 5r + 2}
$$

Which for the root $r = -1$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)}$$

Which for the root $r = -1$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2
b_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{(2r^2 + 5r + 2)(2r^2 + 9r + 9)(2r^2 + 13r + 20)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{4}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2
b_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	-2
b_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$\frac{4}{9}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{2}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2
b_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	-2
b_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$\frac{4}{9}$
b_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$-\frac{2}{45}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{4}{1575}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+5r+2}$	2
b_2	$\frac{4}{(2r^2+5r+2)(2r^2+9r+9)}$	-2
b_3	$-\frac{8}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)}$	$\frac{4}{9}$
b_4	$\frac{16}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)}$	$-\frac{2}{45}$
b_5	$-\frac{32}{(2r^2+5r+2)(2r^2+9r+9)(2r^2+13r+20)(2r^2+17r+35)(2r^2+21r+54)}$	$\frac{4}{1575}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} + O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} + O(x^6)\right)}{x}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} + O(x^6)\right)}{x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 \sqrt{x}\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} + O(x^6)\right)}{x}
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 \sqrt{x}\left(1 - \frac{2x}{5} + \frac{2x^2}{35} - \frac{4x^3}{945} + \frac{2x^4}{10395} - \frac{4x^5}{675675} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + 2x - 2x^2 + \frac{4x^3}{9} - \frac{2x^4}{45} + \frac{4x^5}{1575} + O(x^6)\right)}{x}
\end{aligned}$$

Verified OK.

2.24.1 Maple step by step solution

Let's solve

$2x^2y'' + 3xy' + (2x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{2x} - \frac{(2x-1)y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2x} + \frac{(2x-1)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3}{2x}, P_3(x) = \frac{2x-1}{2x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{2}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + 3xy' + (2x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r-1) + 2a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-1, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2(k+r+1)\left(k - \frac{1}{2} + r\right) a_k + 2a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$2(k+2+r)\left(k + \frac{1}{2} + r\right) a_{k+1} + 2a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{2a_k}{(k+2+r)(2k+1+2r)}$$

- Recursion relation for $r = -1$

  $$a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}, b_{k+1}=-\frac{2b_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 47

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+(2*x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{3}{2}}\left(1-\frac{2}{5}x+\frac{2}{35}x^2-\frac{4}{945}x^3+\frac{2}{10395}x^4-\frac{4}{675675}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1+2x-2x^2+\frac{4}{9}x^3-\frac{2}{45}x^4+\frac{4}{1575}x^5+\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 84

```
AsymptoticDSolveValue[2*x^2*y''[x]+3*x*y'[x]+(2*x-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\sqrt{x}\left(-\frac{4x^5}{675675}+\frac{2x^4}{10395}-\frac{4x^3}{945}+\frac{2x^2}{35}-\frac{2x}{5}+1\right)+\frac{c_2\left(\frac{4x^5}{1575}-\frac{2x^4}{45}+\frac{4x^3}{9}-2x^2+2x+1\right)}{x}$$

2.25 problem 25

Internal problem ID [5580]
Internal file name [`OUTPUT/4828_Sunday_June_05_2022_03_07_09_PM_87162009/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + 2y' - xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = -1$$

Table 56: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y' - xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}2(n+r)a_nx^{n+r-1}\right)+\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)(n+r-1)+2(n+r)a_nx^{n+r-1}=0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0r(-1+r)+2ra_0x^{-1+r}=0$$

Or

$$\left(x^{-1+r}r(-1+r)+2r\,x^{-1+r}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$r\,x^{-1+r}(1+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r)=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=0$$
$$r_2=-1$$

Since $a_0\neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(1+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_n(n+r) - a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-2}}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r^2 + 5r + 6}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 14r^3 + 71r^2 + 154r + 120}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 + \frac{x^2}{6} + \frac{x^4}{120} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -1} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 2(n+r)\, b_n - b_{n-2} = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + 2(n-1)\, b_n - b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{b_{n-2}}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+5r+6}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 + \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1+\frac{x^2}{6}+\frac{x^4}{120}+O\left(x^6\right)\right)+\frac{c_2\left(1+\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\left(1+\frac{x^2}{6}+\frac{x^4}{120}+O\left(x^6\right)\right)+\frac{c_2\left(1+\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1+\frac{x^2}{6}+\frac{x^4}{120}+O\left(x^6\right)\right)+\frac{c_2\left(1+\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x}
$$

Verified OK.

2.25.1 Maple step by step solution

Let's solve

$y''x + 2y' - xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^{2} \cdot P_{3}(x)\right)\bigg|_{x=0} = 0$$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + 2y' - xy = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y$ to series expansion

  $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- ◦ Shift index using $k- >k - 1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- ◦ Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- ◦ Shift index using $k- >k + 1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

- ◦ Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- ◦ Shift index using $k- >k + 1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^{r} + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$a_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+r+1)(k+2+r)-a_{k-1}=0$

- Shift index using $k->k+1$

$a_{k+2}(k+2+r)(k+3+r)-a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

$a_{k+2}=\frac{a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=\frac{a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

$a_{k+2}=\frac{a_k}{(k+2)(k+3)}$

- Solution for $r=0$

$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2}=\frac{a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=\frac{b_k}{(k+2)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1+\frac{1}{6}x^2+\frac{1}{120}x^4+\mathrm{O}\left(x^6\right)\right)+\frac{c_2\left(1+\frac{1}{2}x^2+\frac{1}{24}x^4+\mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 42

```
AsymptoticDSolveValue[x*y''[x]+2*y'[x]-x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^3}{24}+\frac{x}{2}+\frac{1}{x}\right)+c_2\left(\frac{x^4}{120}+\frac{x^2}{6}+1\right)$$

2.26 problem 26

Internal problem ID [5581]
Internal file name [`OUTPUT/4829_Sunday_June_05_2022_03_07_11_PM_30919380/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{4x^2 - 1}{4x^2} \end{aligned}$$

Table 58: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2-1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(x^2 - \frac{1}{4}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - \frac{a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - \frac{a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - \frac{x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2-1)\, x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 1)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - \frac{a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\rightarrow r_2} 0 &= \lim_{r\rightarrow -\frac{1}{2}} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - \frac{b_n}{4} = 0 \tag{4}
$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$b_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)+b_n\left(n-\frac{1}{2}\right)+b_{n-2}-\frac{b_n}{4}=0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-2}}{4n^2+8nr+4r^2-1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{4b_{n-2}}{4n^2-4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4}{4r^2+16r+15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{\sqrt{x}}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}}$$

Verified OK.

2.26.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)(-1+2r)x^r + a_1(3+2r)(1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1x\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 58

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^{7/2}}{24}-\frac{x^{3/2}}{2}+\frac{1}{\sqrt{x}}\right)+c_2\left(\frac{x^{9/2}}{120}-\frac{x^{5/2}}{6}+\sqrt{x}\right)$$

2.27 problem 27

Internal problem ID [5582]
Internal file name [`OUTPUT/4830_Sunday_June_05_2022_03_07_13_PM_36920882/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Laguerre, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{xy'' - xy' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - xy' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -1$$
$$q(x) = \frac{1}{x}$$

Table 60: Table $p(x), q(x)$ singularites.

$p(x) = -1$	
singularity	type

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - xy' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x - x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$
$$\sum_{n=0}^{\infty} a_n x^{n+r}=\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)\,(n+r-1)\right) + \sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)\,(n+r-1)=0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r)=0$$

Or

$$x^{-1+r} a_0 r(-1+r)=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r)=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=1$$
$$r_2=0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-2)}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}(n-1)}{(n+1)n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1 + r}{(1 + r) r}$$

Which for the root $r = 1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1 + r}{(1 + r)^2 (2 + r)}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-1 + r}{(1 + r) (2 + r)^2 (3 + r)}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0
a_4	$\frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1+r}{(1+r)(2+r)(3+r)(4+r)^2(5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0
a_4	$\frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$	0
a_5	$\frac{-1+r}{(1+r)(2+r)(3+r)(4+r)^2(5+r)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\right) \\ &= x\left(1 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-1+r}{(1+r)\, r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{-1+r}{(1+r)\, r} &= \lim_{r \to 0} \frac{-1+r}{(1+r)\, r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $xy'' - xy' + y = 0$ gives

$$
\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x \\
&- x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(y_1''(x)\,x - y_1'(x)\,x + y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - y_1(x)\right)C \\
&+ \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&- x\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\, x - y_1'(x)\, x + y_1(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x - y_1(x)\right) C + \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x - x\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - (1+x)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \frac{-\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^n a_n (n+1)\right) x - (1+x)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} + \frac{-\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^n\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1)\right) + \sum_{n=0}^{\infty} (-C a_n x^n) + \sum_{n=0}^{\infty} (-C\, x^{n+1} a_n) + \sum_{n=0}^{\infty} (-x^n b_n n) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n} a_{n} (n+1) &= \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-C a_{n} x^{n}\right) &= \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-C\, x^{n+1} a_{n}\right) &= \sum_{n=2}^{\infty} \left(-C a_{-2+n} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-x^{n} b_{n} n\right) &= \sum_{n=1}^{\infty} \left(-(n-1)\, b_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} b_{n} x^{n} &= \sum_{n=1}^{\infty} b_{n-1} x^{n-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) + \sum_{n=2}^{\infty} \left(-C a_{-2+n} x^{n-1}\right) \\
&+ \sum_{n=1}^{\infty} \left(-(n-1)\, b_{n-1} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_{n} (n-1)\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$
C + 1 = 0
$$

Which is solved for C. Solving for C gives

$$
C = -1
$$

For $n = 2$, Eq (2B) gives

$$
(-a_0 + 3a_1)\, C + 2b_2 = 0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 + 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$(-a_1 + 5a_2) C - b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{2} + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{12}$$

For $n = 4$, Eq (2B) gives

$$(-a_2 + 7a_3) C - 2b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{6} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1}{72}$$

For $n = 5$, Eq (2B) gives

$$(-a_3 + 9a_4) C - 3b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{24} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{1}{480}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1)\left(x\left(1+O\left(x^6\right)\right)\right)\ln(x) + 1 - \frac{x^2}{2} - \frac{x^3}{12} - \frac{x^4}{72} - \frac{x^5}{480} + O\left(x^6\right)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1+O\left(x^6\right)\right) + c_2\left((-1)\left(x\left(1+O\left(x^6\right)\right)\right)\ln(x) + 1 - \frac{x^2}{2} - \frac{x^3}{12} - \frac{x^4}{72} - \frac{x^5}{480} + O\left(x^6\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1+O\left(x^6\right)\right) + c_2\left(-x\left(1+O\left(x^6\right)\right)\ln(x) + 1 - \frac{x^2}{2} - \frac{x^3}{12} - \frac{x^4}{72} - \frac{x^5}{480} + O\left(x^6\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\left(1+O\left(x^6\right)\right) + c_2\left(-x\left(1+O\left(x^6\right)\right)\ln(x) + 1 - \frac{x^2}{2} - \frac{x^3}{12} - \frac{x^4}{72} - \frac{x^5}{480} + O\left(x^6\right)\right) \quad (1)$$

Verification of solutions

$$y = c_1 x\left(1+O\left(x^6\right)\right) + c_2\left(-x\left(1+O\left(x^6\right)\right)\ln(x) + 1 - \frac{x^2}{2} - \frac{x^3}{12} - \frac{x^4}{72} - \frac{x^5}{480} + O\left(x^6\right)\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
Order:=6;
dsolve(x*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \ln(x)\left(-x + \mathrm{O}\left(x^6\right)\right)c_2 + c_1 x\left(1 + \mathrm{O}\left(x^6\right)\right) + \left(1 + x - \frac{1}{2}x^2 - \frac{1}{12}x^3 - \frac{1}{72}x^4 - \frac{1}{480}x^5 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 41

```
AsymptoticDSolveValue[x*y''[x]-x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{72}\left(-x^4 - 6x^3 - 36x^2 + 144x + 72\right) - x\log(x)\right) + c_2 x$$

2.28 problem 28

Internal problem ID [5583]
Internal file name [`OUTPUT/4831_Sunday_June_05_2022_03_07_17_PM_1035518/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \frac{3y'}{x} - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + \frac{3y'}{x} - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = -2$$

Table 61: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -2$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-2xy + xy'' + 3y' = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-2x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + 3\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n + r)(n + r - 1) + 3a_n(n + r) - 2a_{n-2} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{2a_{n-2}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
a_n = \frac{2a_{n-2}}{n(n + 2)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2}{r^2 + 6r + 8}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r^2+6r+8}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r^2+6r+8}$	$\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4}{(4 + r)^2 (2 + r) (r + 6)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r^2+6r+8}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{4}{(4+r)^2(2+r)(r+6)}$	$\frac{1}{48}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r^2+6r+8}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{4}{(4+r)^2(2+r)(r+6)}$	$\frac{1}{48}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 + \frac{x^2}{4} + \frac{x^4}{48} + O(x^6)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= \frac{2}{r^2 + 6r + 8}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} \frac{2}{r^2 + 6r + 8} &= \lim_{r \to -2} \frac{2}{r^2 + 6r + 8} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x} \right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2) \right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2} \right) \\
&= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2) (-1 + n + r_2) \right)
\end{aligned}
$$

Substituting these back into the given ode $-2xy + xy'' + 3y' = 0$ gives

$$
\begin{aligned}
&-2x \left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) \right) + \left(C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} \right. \\
&\left. - \frac{C y_1(x)}{x^2} + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2} \right) \right) x \\
&+ 3 C y_1'(x) \ln(x) + \frac{3 C y_1(x)}{x} + 3 \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x} \right) = 0
\end{aligned}
$$

Which can be written as

$$\begin{aligned}&\left(\left(y_1''(x)\,x-2y_1(x)\,x+3y_1'(x)\right)\ln(x)+\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x+\frac{3y_1(x)}{x}\right)C\\&-2x\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\&+3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)=0\end{aligned}\tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x-2y_1(x)\,x+3y_1'(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x+\frac{3y_1(x)}{x}\right)C-2x\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\\&+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\&+3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)=0\end{aligned}\tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x+2\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}\\&+\frac{\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2-2x^2\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)+3\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x}{x}\\&=0\end{aligned}\tag{9}$$

Since $r_1 = 0$ and $r_2 = -2$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n} a_n n\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 - 2x^2 \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) + 3\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{gathered}\left(\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n\right) + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n \left(n^2 - 5n + 6\right)\right) \\ + \sum_{n=0}^{\infty} \left(-2x^{-1+n} b_n\right) + \left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n (n-2)\right) = 0\end{gathered} \tag{2A}$$

The next step is to make all powers of x be $-3+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-3+n} and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n n &= \sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n} \\ \sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n &= \sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n} \\ \sum_{n=0}^{\infty} \left(-2x^{-1+n} b_n\right) &= \sum_{n=2}^{\infty} \left(-2b_{n-2} x^{-3+n}\right)\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $-3+n$.

$$\left(\sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}\right)+\left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}\right)$$
$$+\left(\sum_{n=0}^{\infty} x^{-3+n} b_n\left(n^2-5n+6\right)\right)+\sum_{n=2}^{\infty}\left(-2b_{n-2} x^{-3+n}\right) \tag{2B}$$
$$+\left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n(n-2)\right)=0$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1=0$$

Solving the above for b_1 gives

$$b_1=0$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C-2=0$$

Which is solved for C. Solving for C gives

$$C=1$$

For $n=3$, Eq (2B) gives

$$4Ca_1-2b_1+3b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3=0$$

Solving the above for b_3 gives

$$b_3=0$$

For $n=4$, Eq (2B) gives

$$6Ca_2-2b_2+8b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 + \frac{3}{2} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{3}{16}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 - 2b_3 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$y_2(x) = 1\left(1 + \frac{x^2}{4} + \frac{x^4}{48} + O(x^6)\right)\ln(x) + \frac{1 - \frac{3x^4}{16} + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1+\frac{x^2}{4}+\frac{x^4}{48}+O(x^6)\right)+c_2\left(1\left(1+\frac{x^2}{4}+\frac{x^4}{48}+O(x^6)\right)\ln(x)+\frac{1-\frac{3x^4}{16}+O(x^6)}{x^2}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\left(1+\frac{x^2}{4}+\frac{x^4}{48}+O(x^6)\right)+c_2\left(\left(1+\frac{x^2}{4}+\frac{x^4}{48}+O(x^6)\right)\ln(x)+\frac{1-\frac{3x^4}{16}+O(x^6)}{x^2}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{align*}
y &= c_1 \left(1+\frac{x^2}{4}+\frac{x^4}{48}+O\left(x^6\right)\right) \\
&+ c_2 \left(\left(1+\frac{x^2}{4}+\frac{x^4}{48}+O\left(x^6\right)\right) \ln\left(x\right)+\frac{1-\frac{3x^4}{16}+O\left(x^6\right)}{x^2}\right) \tag{1}
\end{align*}
$$

Verification of solutions

$$
y = c_1 \left(1+\frac{x^2}{4}+\frac{x^4}{48}+O\left(x^6\right)\right)+c_2 \left(\left(1+\frac{x^2}{4}+\frac{x^4}{48}+O\left(x^6\right)\right) \ln\left(x\right)+\frac{1-\frac{3x^4}{16}+O\left(x^6\right)}{x^2}\right)
$$

Verified OK.

2.28.1 Maple step by step solution

Let's solve

$-2xy+y''x+3y'=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{x}+2y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{x}-2y=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3}{x}, P_3(x)=-2\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=3$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$-2xy + y''x + 3y' = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y$ to series expansion

$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- ◦ Shift index using $k- > k - 1$

$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- ◦ Convert y' to series expansion

$y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$

- ◦ Shift index using $k- > k + 1$

$y' = \sum_{k=-1}^{\infty} a_{k+1}(k + r + 1)\, x^{k+r}$

- ◦ Convert $x \cdot y''$ to series expansion

$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

- ◦ Shift index using $k- > k + 1$

$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + r + 1)\,(k + r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(2 + r)\, x^{-1+r} + a_1(1 + r)\,(3 + r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k + r + 1)\,(k + 3 + r) - 2a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(2 + r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-2, 0\}$

- Each term must be 0

  $a_1(1+r)(3+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+3+r) - 2a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+r+2)(k+4+r) - 2a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2a_k}{(k+r+2)(k+4+r)}$

- Recursion relation for $r = -2$

  $a_{k+2} = \frac{2a_k}{k(k+2)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = \frac{2a_k}{k(k+2)}, -a_1 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2a_k}{(k+2)(k+4)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k}{(k+2)(k+4)}, 3a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{2a_k}{k(k+2)}, -a_1 = 0, b_{k+2} = \frac{2b_k}{(k+2)(k+4)}, 3b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
Order:=6;
dsolve(diff(y(x),x$2)+3/x*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1+\frac{1}{4}x^2+\frac{1}{48}x^4+\mathrm{O}\left(x^6\right)\right)x^2+c_2\left(\ln\left(x\right)\left(\left(-2\right)x^2-\frac{1}{2}x^4+\mathrm{O}\left(x^6\right)\right)+\left(-2+\frac{3}{8}x^4+\mathrm{O}\left(x^6\right)\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 57

```
AsymptoticDSolveValue[y''[x]+3/x*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^4}{48}+\frac{x^2}{4}+1\right)+c_1\left(\frac{1}{4}\left(x^2+4\right)\log(x)-\frac{5x^4+8x^2-16}{16x^2}\right)$$

2.29 problem 29

Internal problem ID [5584]
Internal file name [`OUTPUT/4832_Sunday_June_05_2022_03_07_20_PM_38239968/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{xy'' + (1-x)\,y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (1-x)\,y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{x}$$
$$q(x) = -\frac{1}{x}$$

Table 63: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (1 - x) y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) x + (1 - x) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{1+r}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(2+r)(1+r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(2+r)(1+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(2+r)(1+r)(3+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(4+r)(2+r)(1+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(2+r)(1+r)(3+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(4+r)(2+r)(1+r)(3+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(5+r)(4+r)(2+r)(1+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(2+r)(1+r)(3+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(4+r)(2+r)(1+r)(3+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(5+r)(4+r)(2+r)(1+r)(3+r)}$	$\frac{1}{120}$

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned}y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{1+r}$	1	$-\frac{1}{(1+r)^2}$	-1
b_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$	$\frac{-3-2r}{(2+r)^2(1+r)^2}$	$-\frac{3}{4}$
b_3	$\frac{1}{(2+r)(1+r)(3+r)}$	$\frac{1}{6}$	$\frac{-3r^2-12r-11}{(2+r)^2(1+r)^2(3+r)^2}$	$-\frac{11}{36}$
b_4	$\frac{1}{(4+r)(2+r)(1+r)(3+r)}$	$\frac{1}{24}$	$\frac{-4r^3-30r^2-70r-50}{(4+r)^2(2+r)^2(1+r)^2(3+r)^2}$	$-\frac{25}{288}$
b_5	$\frac{1}{(5+r)(4+r)(2+r)(1+r)(3+r)}$	$\frac{1}{120}$	$\frac{-5r^4-60r^3-255r^2-450r-274}{(5+r)^2(4+r)^2(2+r)^2(1+r)^2(3+r)^2}$	$-\frac{137}{7200}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) - x - \frac{3x^2}{4} - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} \\
&\quad + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) - x - \frac{3x^2}{4} - \frac{11x^3}{36} \right. \\
&\quad \left. - \frac{25x^4}{288} - \frac{137x^5}{7200} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) - x - \frac{3x^2}{4} - \frac{11x^3}{36} - \frac{25x^4}{288} \right. \\
&\quad \left. - \frac{137x^5}{7200} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) - x - \frac{3x^2}{4} - \frac{11x^3}{36} \right. \\
&\quad \left. - \frac{25x^4}{288} - \frac{137x^5}{7200} + O(x^6)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right) + c_2\left(\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)\ln(x) - x - \frac{3x^2}{4} - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} + O(x^6)\right)$$

Verified OK.

2.29.1 Maple step by step solution

Let's solve

$y''x + (1-x)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-1)y'}{x} + \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} - \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$$y''x+(1-x)\,y'-y=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m=0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

  - Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0r^2x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)^2-a_k(k+1+r)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2=0$$

- Values of r that satisfy the indicial equation

$$r=0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+1)\,(a_{k+1}(k+1)-a_k)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k}{k+1}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{a_k}{k+1}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(1-x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x)=(c_2\ln(x)+c_1)\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\mathrm{O}\left(x^6\right)\right)+\left(-x-\frac{3}{4}x^2-\frac{11}{36}x^3-\frac{25}{288}x^4-\frac{137}{7200}x^5+\mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 107

```
AsymptoticDSolveValue[x*y''[x]+(1-x)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1\right)+c_2\left(-\frac{137x^5}{7200}-\frac{25x^4}{288}-\frac{11x^3}{36}-\frac{3x^2}{4}+\left(\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1\right)\log(x)-x\right)$$

2.30 problem 30

Internal problem ID [5585]
Internal file name [OUTPUT/4833_Sunday_June_05_2022_03_07_22_PM_64756893/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{1}{x}$$

Table 65: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad (2A)$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2 (5+r)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{14400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(r+1)^2}$	-1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{576}$
a_5	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{1}{14400}$

Using the above table, then the first solution $y_1(x)$ becomes

$$y_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots$$
$$= 1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{1}{(r+1)^2}$	-1	$\frac{2}{(r+1)^3}$	2
b_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$	$\frac{-4r-6}{(r+1)^3(r+2)^3}$	$-\frac{3}{4}$
b_3	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{36}$	$\frac{6r^2+24r+22}{(r+1)^3(r+2)^3(r+3)^3}$	$\frac{11}{108}$
b_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{576}$	$\frac{-8r^3-60r^2-140r-100}{(r+1)^3(r+2)^3(r+3)^3(4+r)^3}$	$-\frac{25}{3456}$
b_5	$-\frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(5+r)^2}$	$-\frac{1}{14400}$	$\frac{10r^4+120r^3+510r^2+900r+548}{(r+1)^3(r+2)^3(r+3)^3(4+r)^3(5+r)^3}$	$\frac{137}{432000}$

The above table gives all values of b_n needed. Hence the second solution is

$$y_2(x) = y_1(x) \ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots$$
$$= \left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \ln(x)$$
$$+ 2x - \frac{3x^2}{4} + \frac{11x^3}{108} - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right)\ln(x) + 2x - \frac{3x^2}{4} + \frac{11x^3}{108} \right. \\
&\quad \left. - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right)\ln(x) + 2x - \frac{3x^2}{4} + \frac{11x^3}{108} \right. \\
&\quad \left. - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right)\ln(x) + 2x - \frac{3x^2}{4} + \frac{11x^3}{108} \right. \\
&\quad \left. - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right)
\end{aligned}
\tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y &= c_1\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right)\ln(x) + 2x - \frac{3x^2}{4} + \frac{11x^3}{108} \right. \\
&\quad \left. - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

2.30.1 Maple step by step solution

Let's solve

$y''x + y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1)^2 + a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{(k+1)^2}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = -\frac{a_k}{(k+1)^2}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \frac{1}{576}x^4 - \frac{1}{14400}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(2x - \frac{3}{4}x^2 + \frac{11}{108}x^3 - \frac{25}{3456}x^4 + \frac{137}{432000}x^5 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 111

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{14400} + \frac{x^4}{576} - \frac{x^3}{36} + \frac{x^2}{4} - x + 1\right) + c_2\left(\frac{137x^5}{432000} - \frac{25x^4}{3456} + \frac{11x^3}{108} - \frac{3x^2}{4} + \left(-\frac{x^5}{14400} + \frac{x^4}{576} - \frac{x^3}{36} + \frac{x^2}{4} - x + 1\right)\log(x) + 2x\right)$$

2.31 problem 31

Internal problem ID [5586]
Internal file name [OUTPUT/4834_Sunday_June_05_2022_03_07_24_PM_77989509/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (x-6)\,y' - 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (x-6)\,y' - 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x-6}{x}$$
$$q(x) = -\frac{3}{x}$$

Table 67: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-6}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{3}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (x-6)\,y' - 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x + (x-6)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-6(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-6(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 6(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 6r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 6r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-7+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-7+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 7$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-7+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 7$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{7}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+7}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - 6a_n(n+r) - 3a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-4)}{n^2+2nr+r^2-7n-7r} \tag{4}$$

Which for the root $r = 7$ becomes

$$a_n = -\frac{a_{n-1}(n+3)}{n\,(n+7)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 7$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r+3}{r^2-5r-6}$$

Which for the root $r = 7$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(-2+r)\,(r-3)}{r^4-8r^3-r^2+68r+60}$$

Which for the root $r = 7$ becomes

$$a_2 = \frac{5}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(-2+r)(r-3)(-1+r)}{r^6 - 9r^5 - 5r^4 + 165r^3 + 4r^2 - 876r - 720}$$

Which for the root $r = 7$ becomes

$$a_3 = -\frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$
a_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-2+r)\, r(-1+r)}{(r+4)\,(r^6 - 9r^5 - 5r^4 + 165r^3 + 4r^2 - 876r - 720)}$$

Which for the root $r = 7$ becomes

$$a_4 = \frac{7}{1584}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$
a_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{36}$
a_4	$\frac{(-2+r)r(-1+r)}{(r+4)(r^6-9r^5-5r^4+165r^3+4r^2-876r-720)}$	$\frac{7}{1584}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{r(-1+r)}{(r+5)\,(r^5 - 10r^4 + 5r^3 + 160r^2 - 156r - 720)\,(r+4)}$$

Which for the root $r = 7$ becomes

$$a_5 = -\frac{7}{11880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$
a_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{36}$
a_4	$\frac{(-2+r)r(-1+r)}{(r+4)(r^6-9r^5-5r^4+165r^3+4r^2-876r-720)}$	$\frac{7}{1584}$
a_5	$-\frac{r(-1+r)}{(r+5)(r^5-10r^4+5r^3+160r^2-156r-720)(r+4)}$	$-\frac{7}{11880}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r}{(r+6)(r+4)(r-5)(r-6)(r+3)(r-4)(r+5)}$$

Which for the root $r = 7$ becomes

$$a_6 = \frac{7}{102960}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$
a_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{36}$
a_4	$\frac{(-2+r)r(-1+r)}{(r+4)(r^6-9r^5-5r^4+165r^3+4r^2-876r-720)}$	$\frac{7}{1584}$
a_5	$-\frac{r(-1+r)}{(r+5)(r^5-10r^4+5r^3+160r^2-156r-720)(r+4)}$	$-\frac{7}{11880}$
a_6	$\frac{r}{(r+6)(r+4)(r-5)(r-6)(r+3)(r-4)(r+5)}$	$\frac{7}{102960}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)}$$

Which for the root $r = 7$ becomes

$$a_7 = -\frac{1}{144144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
a_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{5}{36}$
a_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{36}$
a_4	$\frac{(-2+r)r(-1+r)}{(r+4)(r^6-9r^5-5r^4+165r^3+4r^2-876r-720)}$	$\frac{7}{1584}$
a_5	$-\frac{r(-1+r)}{(r+5)(r^5-10r^4+5r^3+160r^2-156r-720)(r+4)}$	$-\frac{7}{11880}$
a_6	$\frac{r}{(r+6)(r+4)(r-5)(r-6)(r+3)(r-4)(r+5)}$	$\frac{7}{102960}$
a_7	$-\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)}$	$-\frac{1}{144144}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^7\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= x^7\left(1 - \frac{x}{2} + \frac{5x^2}{36} - \frac{x^3}{36} + \frac{7x^4}{1584} - \frac{7x^5}{11880} + \frac{7x^6}{102960} - \frac{x^7}{144144} + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 7$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_7(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_7 \\ &= -\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)} &= \lim_{r \to 0} -\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)} \\ &= \frac{1}{100800} \end{aligned}$$

The limit is $\frac{1}{100800}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) - 6(n+r)b_n - 3b_{n-1} = 0 \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_n n(n-1) + b_{n-1}(n-1) - 6nb_n - 3b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n+r-4)}{n^2 + 2nr + r^2 - 7n - 7r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{b_{n-1}(n-4)}{n^2 - 7n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r-3}{r^2 - 5r - 6}$$

Which for the root $r = 0$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(r-3)(-2+r)}{(r^2-5r-6)(r^2-3r-10)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{10}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(-2+r)(r-3)(-1+r)}{(r^2-5r-6)(r^2-3r-10)(r^2-r-12)}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$
b_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{120}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(-2+r)\,r(-1+r)}{(r+4)(r^2-5r-6)(r^2-3r-10)(r^2-r-12)}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$
b_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{120}$
b_4	$\frac{(-2+r)r(-1+r)}{r^7-5r^6-41r^5+145r^4+664r^3-860r^2-4224r-2880}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{r(-1+r)}{(r+5)(r^2-r-12)(r^2-3r-10)(r-6)(r+4)}$$

Which for the root $r = 0$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$
b_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{120}$
b_4	$\frac{(-2+r)r(-1+r)}{r^7-5r^6-41r^5+145r^4+664r^3-860r^2-4224r-2880}$	0
b_5	$-\frac{r(-1+r)}{(r+5)(r-5)(r-6)(r+3)(2+r)(r-4)(r+4)}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{r}{(r+6)(r+4)(r-6)(r-5)(r^2-r-12)(r+5)}$$

Which for the root $r = 0$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$
b_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{120}$
b_4	$\frac{(-2+r)r(-1+r)}{r^7-5r^6-41r^5+145r^4+664r^3-860r^2-4224r-2880}$	0
b_5	$-\frac{r(-1+r)}{(r+5)(r-5)(r-6)(r+3)(2+r)(r-4)(r+4)}$	0
b_6	$\frac{r}{(r+4)(r^2-r-12)(r^2-36)(r^2-25)}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{1}{(r+6)(r+4)(r-5)(r-6)(r-4)(r+5)(r+7)}$$

Which for the root $r = 0$ becomes

$$b_7 = \frac{1}{100800}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r+3}{r^2-5r-6}$	$-\frac{1}{2}$
b_2	$\frac{(-2+r)(r-3)}{r^4-8r^3-r^2+68r+60}$	$\frac{1}{10}$
b_3	$-\frac{(-2+r)(r-3)(-1+r)}{r^6-9r^5-5r^4+165r^3+4r^2-876r-720}$	$-\frac{1}{120}$
b_4	$\frac{(-2+r)r(-1+r)}{r^7-5r^6-41r^5+145r^4+664r^3-860r^2-4224r-2880}$	0
b_5	$-\frac{r(-1+r)}{(r+5)(r-5)(r-6)(r+3)(2+r)(r-4)(r+4)}$	0
b_6	$\frac{r}{(r+4)(r^2-r-12)(r^2-36)(r^2-25)}$	0
b_7	$-\frac{1}{(r+7)(r^2-36)(r^2-16)(r^2-25)}$	$\frac{1}{100800}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= 1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \frac{x^7}{100800} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^7 \left(1 - \frac{x}{2} + \frac{5x^2}{36} - \frac{x^3}{36} + \frac{7x^4}{1584} - \frac{7x^5}{11880} + \frac{7x^6}{102960} - \frac{x^7}{144144} + O(x^8)\right) \\
&+ c_2 \left(1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \frac{x^7}{100800} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^7 \left(1 - \frac{x}{2} + \frac{5x^2}{36} - \frac{x^3}{36} + \frac{7x^4}{1584} - \frac{7x^5}{11880} + \frac{7x^6}{102960} - \frac{x^7}{144144} + O(x^8)\right) \\
&+ c_2 \left(1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \frac{x^7}{100800} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^7 \left(1 - \frac{x}{2} + \frac{5x^2}{36} - \frac{x^3}{36} + \frac{7x^4}{1584} - \frac{7x^5}{11880} + \frac{7x^6}{102960} - \frac{x^7}{144144} + O(x^8)\right) \\
+ c_2 \left(1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \frac{x^7}{100800} + O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^7 \left(1 - \frac{x}{2} + \frac{5x^2}{36} - \frac{x^3}{36} + \frac{7x^4}{1584} - \frac{7x^5}{11880} + \frac{7x^6}{102960} - \frac{x^7}{144144} + O(x^8)\right) \\
+ c_2 \left(1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \frac{x^7}{100800} + O(x^8)\right)
\end{aligned}
$$

Verified OK.

2.31.1 Maple step by step solution

Let's solve

$y''x + (x-6)\, y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{3y}{x} - \frac{(x-6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(x-6)y'}{x} - \frac{3y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x-6}{x}, P_3(x) = -\frac{3}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -6$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$y''x + (x-6)\, y' - 3y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x\cdot y''$ to series expansion

$$x\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k->k+1$

$$x\cdot y''=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}r(-7+r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-6+r)+a_{k}(k+r-3)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-7+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{0,7\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k-6+r)+a_{k}(k+r-3)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{a_{k}(k+r-3)}{(k+1+r)(k-6+r)}$$

- Recursion relation for $r=0$; series terminates at $k=3$

$$a_{k+1}=-\frac{a_{k}(k-3)}{(k+1)(k-6)}$$

- Apply recursion relation for $k=0$

$$a_{1}=-\frac{a_{0}}{2}$$

- Apply recursion relation for $k=1$

$$a_{2}=-\frac{a_{1}}{5}$$

- Express in terms of a_0

$$a_{2}=\frac{a_{0}}{10}$$

- Apply recursion relation for $k=2$

$$a_{3}=-\frac{a_{2}}{12}$$

- Express in terms of a_0

$$a_{3}=-\frac{a_{0}}{120}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right)$

- Recursion relation for $r = 7$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}$

- Solution for $r = 7$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), b_{k+1} = -\frac{b_k(k+4)}{(k+8)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 40

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(x-6)*diff(y(x),x)-3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^7 \left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1584}x^4 - \frac{7}{11880}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(3628800 - 1814400x + 362880x^2 - 30240x^3 + \mathrm{O}\left(x^6\right)\right)$$

✓ <u>Solution by Mathematica</u>

Time used: 0.03 (sec). Leaf size: 63

```
AsymptoticDSolveValue[x*y''[x]+(x-6)*y'[x]-3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^3}{120}+\frac{x^2}{10}-\frac{x}{2}+1\right)+c_2\left(\frac{7x^{11}}{1584}-\frac{x^{10}}{36}+\frac{5x^9}{36}-\frac{x^8}{2}+x^7\right)$$

2.32 problem 32

Internal problem ID [5587]
Internal file name [OUTPUT/4835_Sunday_June_05_2022_03_07_28_PM_18175668/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(x-1)\,y''+3y'-2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2-x\right)y''-2y+3y'=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{x\,(x-1)}$$
$$q(x)=-\frac{2}{x\,(x-1)}$$

Table 69: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{2}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x-1)\, y'' + 3y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) + 3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-2 a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2 a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2 a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$-x^{-1+r} a_0 r(-1+r) + 3 r a_0 x^{-1+r} = 0$$

Or

$$\left(-x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(4-r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$-r(-4+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1 &= 4\\ r_2 &= 0\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(4-r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= x^{4}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+4}\\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) - a_n(n+r)(n+r-1) + 3a_n(n+r) - 2a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n + r - 3)\, a_{n-1}}{n - 4 + r} \tag{4}$$

Which for the root $r = 4$ becomes

$$a_n = \frac{(n + 1)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2 + r}{r - 3}$$

Which for the root $r = 4$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1 + r}{r - 3}$$

Which for the root $r = 4$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_3 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_4 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{2+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_5 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5
a_5	$\frac{2+r}{r-3}$	6

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^4\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^4\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{1+r}{r-3} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1+r}{r-3} &= \lim_{r \to 0} \frac{1+r}{r-3} \\ &= -\frac{1}{3} \end{aligned}$$

The limit is $-\frac{1}{3}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_{n-1}(n+r-1)(n+r-2) - b_n(n+r)(n+r-1) + 3(n+r)b_n - 2b_{n-1} = 0 \quad (4)$$

Which for for the root $r = 0$ becomes

$$b_{n-1}(n-1)(n-2) - b_n n(n-1) + 3nb_n - 2b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{(n+r-3)\,b_{n-1}}{n-4+r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{(n-3)\,b_{n-1}}{n-4} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-2+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-1+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{2+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$
b_5	$\frac{2+r}{r-3}$	$-\frac{2}{3}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^4 \left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right)+c_2 \left(1+\frac{2x}{3}+\frac{x^2}{3}-\frac{x^4}{3}-\frac{2x^5}{3}+O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$y = y_h$$
$$= c_1 x^4 \left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right)+c_2\left(1+\frac{2x}{3}+\frac{x^2}{3}-\frac{x^4}{3}-\frac{2x^5}{3}+O(x^6)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^4 \left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right) + c_2\left(1+\frac{2x}{3}+\frac{x^2}{3}-\frac{x^4}{3}-\frac{2x^5}{3}+O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^4 \left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right)+c_2\left(1+\frac{2x}{3}+\frac{x^2}{3}-\frac{x^4}{3}-\frac{2x^5}{3}+O(x^6)\right)$$

Verified OK.

2.32.1 Maple step by step solution

Let's solve

$y''x(x-1)+3y'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2y}{x(x-1)}-\frac{3y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{x(x-1)}-\frac{2y}{x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3}{x(x-1)}, P_3(x)=-\frac{2}{x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x(x-1) + 3y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-4+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-3+r) + a_k(k+1+r)\,(k+r-2)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 4\}$

- Each term in the series must be 0, giving the recursion relation

$$((-k-r+3)\, a_{k+1}+a_{k}(k+r-2))\, (k+1+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_{k}(k+r-2)}{k-3+r}$$

- Recursion relation for $r=0$; series terminates at $k=2$

$$a_{k+1}=\frac{a_{k}(k-2)}{k-3}$$

- Apply recursion relation for $k=0$

$$a_{1}=\frac{2a_{0}}{3}$$

- Apply recursion relation for $k=1$

$$a_{2}=\frac{a_{1}}{2}$$

- Express in terms of a_0

$$a_{2}=\frac{a_{0}}{3}$$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

$$y=a_{0}\cdot \left(1+\frac{2}{3}x+\frac{1}{3}x^{2}\right)$$

- Recursion relation for $r=4$

$$a_{k+1}=\frac{a_{k}(k+2)}{k+1}$$

- Solution for $r=4$

$$\left[y=\sum_{k=0}^{\infty}a_{k}x^{k+4}, a_{k+1}=\frac{a_{k}(k+2)}{k+1}\right]$$

- Combine solutions and rename parameters

$$\left[y=a_{0}\cdot \left(1+\frac{2}{3}x+\frac{1}{3}x^{2}\right)+\left(\sum_{k=0}^{\infty}b_{k}x^{k+4}\right), b_{k+1}=\frac{b_{k}(k+2)}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
Order:=6;
dsolve(x*(x-1)*diff(y(x),x$2)+3*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + c_2 \left(-144 - 96x - 48x^2 + 48x^4 + 96x^5 + \mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 55

```
AsymptoticDSolveValue[x*(x-1)*y''[x]+3*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$
y(x) \to c_1 \left(-\frac{x^4}{3} + \frac{x^2}{3} + \frac{2x}{3} + 1\right) + c_2 \left(5x^8 + 4x^7 + 3x^6 + 2x^5 + x^4\right)
$$

2.33 problem 33

Internal problem ID [5588]
Internal file name [OUTPUT/4836_Sunday_June_05_2022_03_07_31_PM_28516887/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

Unable to solve or complete the solution.

$$\boxed{x^4y'' + \lambda y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^4y'' + \lambda y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{\lambda}{x^4}$$

Table 71: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{\lambda}{x^4}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^4*diff(y(x),x$2)+lambda*y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 50

```
AsymptoticDSolveValue[x^4*y''[x]+\[Lambda]*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x e^{\frac{i\sqrt{\lambda}}{x}} - \frac{i c_2 x e^{-\frac{i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}$$

2.34 problem 36 (a)

Internal problem ID [5589]
Internal file name [`OUTPUT/4837_Sunday_June_05_2022_03_07_31_PM_81374011/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 36 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^3y'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{1}{x^3} \end{aligned}$$

Table 72: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x^3}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^3*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 222

```
AsymptoticDSolveValue[x^3*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \rightarrow\, & c_1 e^{-\frac{2i}{\sqrt{x}}} x^{3/4} \left(-\frac{468131288625 i x^{9/2}}{8796093022208} + \frac{66891825 i x^{7/2}}{4294967296} - \frac{72765 i x^{5/2}}{8388608} + \frac{105 i x^{3/2}}{8192} \right. \\
& + \frac{33424574007825 x^5}{281474976710656} - \frac{14783093325 x^4}{549755813888} + \frac{2837835 x^3}{268435456} - \frac{4725 x^2}{524288} + \frac{15 x}{512} - \frac{3 i \sqrt{x}}{16} \\
& \left. + 1 \right) + c_2 e^{\frac{2i}{\sqrt{x}}} x^{3/4} \left(\frac{468131288625 i x^{9/2}}{8796093022208} - \frac{66891825 i x^{7/2}}{4294967296} + \frac{72765 i x^{5/2}}{8388608} - \frac{105 i x^{3/2}}{8192} + \frac{33424574007825 x^5}{281474976710656} - \right.
\end{aligned}
$$

2.35 problem 36 (b)

Internal problem ID [5590]
Internal file name [`OUTPUT/4838_Sunday_June_05_2022_03_07_32_PM_45094205/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.2 page 239
Problem number: 36 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' + (3x-1)\,y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (3x-1)\,y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3x-1}{x^2}$$
$$q(x) = \frac{1}{x^2}$$

Table 73: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x-1}{x^2}$	
singularity	type
$x = 0$	"irregular"

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+(3*x-1)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 43

```
AsymptoticDSolveValue[x^2*y''[x]+(3*x-1)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(120x^5 + 24x^4 + 6x^3 + 2x^2 + x + 1\right) + \frac{c_2 e^{-1/x}}{x}$$

3 Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250

3.1 problem 1

Internal problem ID [5591]
Internal file name [OUTPUT/4839_Sunday_June_05_2022_03_07_33_PM_52973265/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{9x^2 - 1}{9x^2}$$

Table 74: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{9x^2-1}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(x^2 - \frac{1}{9}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{9}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{9}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - \frac{a_n x^{n+r}}{9} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - \frac{a_0 x^r}{9} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - \frac{x^r}{9}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(9r^2-1)\, x^r}{9} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{9} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -\frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(9r^2 - 1)\, x^r}{9} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - \frac{a_n}{9} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{9a_{n-2}}{9n^2 + 18nr + 9r^2 - 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{3a_{n-2}}{n(3n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{9}{9r^2 + 36r + 35}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = -\frac{3}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{16}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(9r^2 + 36r + 35)(9r^2 + 72r + 143)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{9}{896}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{16}$
a_3	0	0
a_4	$\frac{81}{(9r^2+36r+35)(9r^2+72r+143)}$	$\frac{9}{896}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{16}$
a_3	0	0
a_4	$\frac{81}{(9r^2+36r+35)(9r^2+72r+143)}$	$\frac{9}{896}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - \frac{b_n}{9} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{9b_{n-2}}{9n^2 + 18nr + 9r^2 - 1} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_n = -\frac{3b_{n-2}}{n(3n-2)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{9}{9r^2 + 36r + 35}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_2 = -\frac{3}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{8}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(9r^2 + 36r + 35)(9r^2 + 72r + 143)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = \frac{9}{320}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{8}$
b_3	0	0
b_4	$\frac{81}{(9r^2+36r+35)(9r^2+72r+143)}$	$\frac{9}{320}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{8}$
b_3	0	0
b_4	$\frac{81}{(9r^2+36r+35)(9r^2+72r+143)}$	$\frac{9}{320}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{3x^2}{8} + \frac{9x^4}{320} + O(x^6)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^2}{8} + \frac{9x^4}{320} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^2}{8} + \frac{9x^4}{320} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^2}{8} + \frac{9x^4}{320} + O(x^6)\right)}{x^{\frac{1}{3}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^2}{8} + \frac{9x^4}{320} + O(x^6)\right)}{x^{\frac{1}{3}}}
$$

Verified OK.

3.1.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2-1)y}{9x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(9x^2-1)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{9x^2-1}{9x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{9}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2y'' + 9xy' + (9x^2 - 1) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)\,(-1+3r)\, x^r+a_1(4+3r)\,(2+3r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(3k+3r+1)\,(3k+3r-1)+9a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+3r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(4+3r)\,(2+3r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r+1)\,(3k+3r-1)+9a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(3k+7+3r)\,(3k+5+3r)+9a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{9a_k}{(3k+7+3r)(3k+5+3r)}$

- Recursion relation for $r=-\frac{1}{3}$

  $a_{k+2}=-\frac{9a_k}{(3k+6)(3k+4)}$

- Solution for $r=-\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2}=-\frac{9a_k}{(3k+6)(3k+4)}, a_1=0\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{9a_k}{(3k+8)(3k+6)}$
- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{9a_k}{(3k+8)(3k+6)}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{9a_k}{(3k+6)(3k+4)}, a_1 = 0, b_{k+2} = -\frac{9b_k}{(3k+8)(3k+6)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{2}{3}} \left(1 - \frac{3}{16}x^2 + \frac{9}{896}x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 - \frac{3}{8}x^2 + \frac{9}{320}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-1/9)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x} \left(\frac{9x^4}{896} - \frac{3x^2}{16} + 1\right) + \frac{c_2 \left(\frac{9x^4}{320} - \frac{3x^2}{8} + 1\right)}{\sqrt[3]{x}}$$

3.2 problem 2

Internal problem ID [5592]
Internal file name [`OUTPUT/4840_Sunday_June_05_2022_03_07_35_PM_41596924/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Bessel]
```

$$\boxed{x^2y'' + xy' + (x^2 - 1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (x^2 - 1)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 - 1}{x^2}$$

Table 76: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + (x^2 - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (x^2 - 1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 4r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+3)^2 (1+r)(5+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{(r+3)^2(1+r)(5+r)}$	$\frac{1}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{(r+3)^2(1+r)(5+r)}$	$\frac{1}{192}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= -\frac{1}{r^2 + 4r + 3} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{1}{r^2 + 4r + 3} &= \lim_{r\to -1} -\frac{1}{r^2 + 4r + 3} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n + r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n + r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2)(-1 + n + r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x^2y'' + xy' + (x^2 - 1)y = 0$ gives

$$x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) + (x^2-1)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(x^2y_1''(x)+y_1'(x)\,x+(x^2-1)\,y_1(x)\right)\ln(x)+x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+y_1(x)\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (x^2-1)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2y_1''(x) + y_1'(x)\,x + (x^2-1)\,y_1(x) = 0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + y_1(x)\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (x^2-1)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
2x\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) C + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
+ \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{9}
$$

Since $r_1 = 1$ and $r_2 = -1$ then the above becomes

$$
\begin{aligned}
2x\left(\sum_{n=0}^{\infty} x^{n} a_n (n+1)\right) C + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-1)(n-2)\right) x^2 \\
+ \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) x^2 + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-1)\right) x - \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0
\end{aligned}
\tag{10}
$$

Which simplifies to

$$
\begin{aligned}
\left(\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1)\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2 - 3n + 2\right)\right) \\
+ \left(\sum_{n=0}^{\infty} x^{n+1} b_n\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1) &= \sum_{n=2}^{\infty} 2C a_{n-2} (n-1)\, x^{n-1} \\
\sum_{n=0}^{\infty} x^{n+1} b_n &= \sum_{n=2}^{\infty} b_{n-2} x^{n-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n-1$.

$$\left(\sum_{n=2}^{\infty} 2Ca_{n-2}(n-1)\, x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2-3n+2\right)\right) + \left(\sum_{n=2}^{\infty} b_{n-2} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n=3$, Eq (2B) gives

$$4Ca_1 + b_1 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n=4$, Eq (2B) gives

$$6Ca_2 + b_2 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 + \frac{3}{8} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{3}{64}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + b_3 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\right)\ln(x) + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{1}{2}\left(x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\right)\ln(x) + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\ln(x)}{2} + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x \left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) + c_2 \left(-\frac{x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\ln(x)}{2} + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) + c_2 \left(-\frac{x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\ln(x)}{2} + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x}\right)$$

Verified OK.

3.2.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + (x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-1)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -1$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x y' + (x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (1 + r)\, (-1 + r)\, x^r + a_1 (2 + r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k + r + 1)\, (k + r - 1) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + r)\, (-1 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term must be 0

  $a_1 (2 + r)\, r = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-1) + a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+3+r)(k+r+1) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+3+r)(k+r+1)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+2)k}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+2)k}, a_1 = 0\right]$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+4)(k+2)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+4)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{a_k}{k(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+4)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{1}{8}x^2 + \frac{1}{192}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(x^2 - \frac{1}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-2 + \frac{3}{32}x^4 + \mathrm{O}\left(x^6\right)\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 58

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{x^5}{192} - \frac{x^3}{8} + x\right) + c_1 \left(\frac{1}{16}x\left(x^2 - 8\right)\log(x) - \frac{5x^4 - 16x^2 - 64}{64x}\right)$$

3.3 problem 3

Internal problem ID [5593]
Internal file name [OUTPUT/4841_Sunday_June_05_2022_03_07_38_PM_905349/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4xy' + (4x^2 - 25)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + 4xy' + (4x^2 - 25)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{4x^2 - 25}{4x^2}$$

Table 78: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2-25}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + 4xy' + (4x^2 - 25)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 4x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (4x^2 - 25)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-25 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{n+r+2} a_n = \sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-25 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) - 25 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - 25 a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 4x^r r - 25x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 25\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 25 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{5}{2}$$
$$r_2 = -\frac{5}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 25\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{5}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{5}{2}}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{5}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 4a_n(n+r) + 4a_{n-2} - 25a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{4n^2 + 8nr + 4r^2 - 25} \tag{4}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{4r^2 + 16r - 9}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_2 = -\frac{1}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r-9}$	$-\frac{1}{14}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r-9}$	$-\frac{1}{14}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r - 9)(4r^2 + 32r + 39)}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_4 = \frac{1}{504}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r-9}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r-9)(4r^2+32r+39)}$	$\frac{1}{504}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r-9}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r-9)(4r^2+32r+39)}$	$\frac{1}{504}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{5}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{5}{2}}\left(1 - \frac{x^2}{14} + \frac{x^4}{504} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_5(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_5 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} 0 &= \lim_{r \to -\frac{5}{2}} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-\frac{5}{2}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
4b_n(n+r)(n+r-1) + 4b_n(n+r) + 4b_{n-2} - 25b_n = 0 \tag{4}
$$

Which for for the root $r = -\frac{5}{2}$ becomes

$$
4b_n\left(n - \frac{5}{2}\right)\left(n - \frac{7}{2}\right) + 4b_n\left(n - \frac{5}{2}\right) + 4b_{n-2} - 25b_n = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-2}}{4n^2 + 8nr + 4r^2 - 25} \tag{5}$$

Which for the root $r = -\frac{5}{2}$ becomes

$$b_n = -\frac{4b_{n-2}}{4n^2 - 20n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{5}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4}{4r^2 + 16r - 9}$$

Which for the root $r = -\frac{5}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r-9}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r-9}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(4r^2 + 16r - 9)(4r^2 + 32r + 39)}$$

Which for the root $r = -\frac{5}{2}$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r-9}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r-9)(4r^2+32r+39)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r-9}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r-9)(4r^2+32r+39)}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^{\frac{5}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + \frac{x^2}{6} + \frac{x^4}{24} + O(x^6)}{x^{\frac{5}{2}}}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{5}{2}}\left(1 - \frac{x^2}{14} + \frac{x^4}{504} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{6} + \frac{x^4}{24} + O(x^6)\right)}{x^{\frac{5}{2}}}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{5}{2}}\left(1 - \frac{x^2}{14} + \frac{x^4}{504} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{6} + \frac{x^4}{24} + O(x^6)\right)}{x^{\frac{5}{2}}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{5}{2}}\left(1 - \frac{x^2}{14} + \frac{x^4}{504} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{6} + \frac{x^4}{24} + O(x^6)\right)}{x^{\frac{5}{2}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{5}{2}}\left(1 - \frac{x^2}{14} + \frac{x^4}{504} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{6} + \frac{x^4}{24} + O(x^6)\right)}{x^{\frac{5}{2}}}$$

Verified OK.

3.3.1 Maple step by step solution

Let's solve

$4x^2 y'' + 4xy' + (4x^2 - 25)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(4x^2-25)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-25)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-25}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{25}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 25)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^{k+r}$

Rewrite ODE with series expansions

$a_0(5+2r)(-5+2r)x^r + a_1(7+2r)(-3+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)(2k+2r-5)+4a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)(-5+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$

- Each term must be 0

  $a_1(7+2r)(-3+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+5)(2k+2r-5) + 4a_{k-2} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}(2k+9+2r)(2k-1+2r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$

- Recursion relation for $r = -\frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}$

- Solution for $r = -\frac{5}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}$

- Solution for $r = \frac{5}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{5}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{5}{2}}\right), a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0, b_{k+2}=-\frac{4b_k}{(2k+14)(2k+4)}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2-25)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1x^5\left(1-\frac{1}{14}x^2+\frac{1}{504}x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(2880+480x^2+120x^4+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 58

```
AsymptoticDSolveValue[4*x^2*y''[x]+4*x*y'[x]+(4*x^2-25)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^{3/2}}{24}+\frac{1}{x^{5/2}}+\frac{1}{6\sqrt{x}}\right)+c_2\left(\frac{x^{13/2}}{504}-\frac{x^{9/2}}{14}+x^{5/2}\right)$$

3.4 problem 4

Internal problem ID [5594]
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Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{16x^2y'' + 16xy' + (16x^2 - 1) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$16x^2y'' + 16xy' + (16x^2 - 1) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{16x^2 - 1}{16x^2}$$

Table 80: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{16x^2-1}{16x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$16x^2y'' + 16xy' + \left(16x^2 - 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$16x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 16x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(16x^2 - 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r)\right) \\
+ \left(\sum_{n=0}^{\infty} 16x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 16x^{n+r+2} a_n = \sum_{n=2}^{\infty} 16a_{n-2} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r)\right) \\
+ \left(\sum_{n=2}^{\infty} 16a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
16x^{n+r} a_n (n+r) (n+r-1) + 16x^{n+r} a_n (n+r) - a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
16x^r a_0 r(-1+r) + 16x^r a_0 r - a_0 x^r = 0
$$

Or

$$
\left(16x^r r(-1+r) + 16x^r r - x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(16r^2 - 1\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
16r^2 - 1 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = -\frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(16r^2 - 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$16a_n(n+r)(n+r-1) + 16a_n(n+r) + 16a_{n-2} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{16a_{n-2}}{16n^2 + 32nr + 16r^2 - 1} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_n = -\frac{2a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{16}{16r^2 + 64r + 63}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_2 = -\frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16}{16r^2+64r+63}$	$-\frac{1}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16}{16r^2+64r+63}$	$-\frac{1}{5}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(16r^2 + 64r + 63)(16r^2 + 128r + 255)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = \frac{1}{90}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16}{16r^2+64r+63}$	$-\frac{1}{5}$
a_3	0	0
a_4	$\frac{256}{(16r^2+64r+63)(16r^2+128r+255)}$	$\frac{1}{90}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16}{16r^2+64r+63}$	$-\frac{1}{5}$
a_3	0	0
a_4	$\frac{256}{(16r^2+64r+63)(16r^2+128r+255)}$	$\frac{1}{90}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{4}}\left(1 - \frac{x^2}{5} + \frac{x^4}{90} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$16b_n(n+r)(n+r-1) + 16b_n(n+r) + 16b_{n-2} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{16b_{n-2}}{16n^2 + 32nr + 16r^2 - 1} \tag{4}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_n = -\frac{2b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{16}{16r^2 + 64r + 63}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_2 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{16}{16r^2+64r+63}$	$-\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{16}{16r^2+64r+63}$	$-\frac{1}{3}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{256}{(16r^2 + 64r + 63)(16r^2 + 128r + 255)}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_4 = \frac{1}{42}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{16}{16r^2+64r+63}$	$-\frac{1}{3}$
b_3	0	0
b_4	$\frac{256}{(16r^2+64r+63)(16r^2+128r+255)}$	$\frac{1}{42}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{16}{16r^2+64r+63}$	$-\frac{1}{3}$
b_3	0	0
b_4	$\frac{256}{(16r^2+64r+63)(16r^2+128r+255)}$	$\frac{1}{42}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{4}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x^2}{3} + \frac{x^4}{42} + O(x^6)}{x^{\frac{1}{4}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{4}}\left(1 - \frac{x^2}{5} + \frac{x^4}{90} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{3} + \frac{x^4}{42} + O(x^6)\right)}{x^{\frac{1}{4}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{4}}\left(1 - \frac{x^2}{5} + \frac{x^4}{90} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{3} + \frac{x^4}{42} + O(x^6)\right)}{x^{\frac{1}{4}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{4}}\left(1 - \frac{x^2}{5} + \frac{x^4}{90} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{3} + \frac{x^4}{42} + O(x^6)\right)}{x^{\frac{1}{4}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{4}}\left(1 - \frac{x^2}{5} + \frac{x^4}{90} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{3} + \frac{x^4}{42} + O(x^6)\right)}{x^{\frac{1}{4}}}
$$

Verified OK.

3.4.1 Maple step by step solution

Let's solve

$16x^2 y'' + 16xy' + (16x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(16x^2-1)y}{16x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(16x^2-1)y}{16x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{16x^2-1}{16x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{16}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2y'' + 16xy' + (16x^2 - 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- ◦ Convert $x^2 \cdot y''$ to series expansion

$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+4r)(-1+4r) x^r + a_1(5+4r)(3+4r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(4k+4r+1)(4k+4r-1) + 16a_k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+4r)(-1+4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{4}, \frac{1}{4}\right\}$

- Each term must be 0

  $a_1(5+4r)(3+4r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k+4r+1)(4k+4r-1) + 16a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(4k+9+4r)(4k+7+4r) + 16a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{16a_k}{(4k+9+4r)(4k+7+4r)}$

- Recursion relation for $r = -\frac{1}{4}$

  $a_{k+2} = -\frac{16a_k}{(4k+8)(4k+6)}$

- Solution for $r = -\frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{16a_k}{(4k+8)(4k+6)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = -\frac{16a_k}{(4k+10)(4k+8)}$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{16a_k}{(4k+10)(4k+8)}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2}=-\frac{16a_k}{(4k+8)(4k+6)}, a_1=0, b_{k+2}=-\frac{16b_k}{(4k+10)(4k+8)}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
Order:=6;
dsolve(16*x^2*diff(y(x),x$2)+16*x*diff(y(x),x)+(16*x^2-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_2\sqrt{x}\left(1-\frac{1}{5}x^2+\frac{1}{90}x^4+\mathrm{O}\left(x^6\right)\right)+c_1\left(1-\frac{1}{3}x^2+\frac{1}{42}x^4+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 52

```
AsymptoticDSolveValue[16*x^2*y''[x]+16*x*y'[x]+(16*x^2-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\sqrt[4]{x}\left(\frac{x^4}{90}-\frac{x^2}{5}+1\right)+\frac{c_2\left(\frac{x^4}{42}-\frac{x^2}{3}+1\right)}{\sqrt[4]{x}}$$

3.5 problem 5

Internal problem ID [5595]
Internal file name [OUTPUT/4843_Sunday_June_05_2022_03_07_43_PM_84878401/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{xy'' + y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = 1$$

Table 82: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + x \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+2)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(4+r)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(4+r)^2}$	$\frac{1}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$	$\frac{2}{(r+2)^3}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(r+2)^2(4+r)^2}$	$\frac{1}{64}$	$\frac{-12-4r}{(r+2)^3(4+r)^3}$	$-\frac{3}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) + c_2 \left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) + c_2 \left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 &\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \\
&+ c_2 \left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)+c_2\left(\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)\ln(x)+\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)$$

Verified OK.

3.5.1 Maple step by step solution

Let's solve

$y''x + y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = 1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$
  - Shift index using $k- >k-1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$
  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$
  - Shift index using $k- >k+1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$
  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$
  - Shift index using $k- >k+1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + a_1(1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term must be 0

  $$a_1(1+r)^2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1)^2 + a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$a_{k+2}(k+2)^2 + a_k = 0$$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \mathrm{O}(x^6)\right) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \mathrm{O}(x^6)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{64}-\frac{x^2}{4}+1\right)+c_2\left(-\frac{3x^4}{128}+\frac{x^2}{4}+\left(\frac{x^4}{64}-\frac{x^2}{4}+1\right)\log(x)\right)$$

3.6 problem 6

Internal problem ID [5596]
Internal file name [OUTPUT/4844_Sunday_June_05_2022_03_07_45_PM_74008201/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Bessel]
```

$$\boxed{xy'' + y' + \left(x - \frac{4}{x}\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + \left(x - \frac{4}{x}\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 - 4}{x^2}$$

Table 84: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + (x^2 - 4)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (x^2 - 4)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \\
+ \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-4 a_n x^{n+r}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \\
+ \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4 a_n x^{n+r}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 4 a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) + x^r a_0 r - 4 a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) + x^r r - 4 x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2 - 4\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
r^2 - 4 = 0
$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 2 \\
r_2 &= -2
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(r^2 - 4\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2} \right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n + r)(n + r - 1) + a_n(n + r) + a_{n-2} - 4a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-2}}{n (n + 4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r (r + 4)}$$

Which for the root $r = 2$ becomes

$$a_2 = -\frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+4)}$	$-\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+4)}$	$-\frac{1}{12}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r(r+4)(r+6)(r+2)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+4)}$	$-\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{r(r+4)(r+6)(r+2)}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+4)}$	$-\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{r(r+4)(r+6)(r+2)}$	$\frac{1}{384}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{1}{r(r+4)(r+6)(r+2)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1}{r(r+4)(r+6)(r+2)} &= \lim_{r \to -2} \frac{1}{r(r+4)(r+6)(r+2)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x^2y'' + xy' + (x^2 - 4)\,y = 0$ gives

$$
\begin{aligned}
&x^2\left(C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2}\right.\\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+ x\left(C y_1'(x)\ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+ \left(x^2-4\right)\left(C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x^2 y_1''(x) + y_1'(x)\,x + \left(x^2-4\right)y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + y_1(x)\right)C\\
&+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+ x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(x^2-4\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
x^2 y_1''(x) + y_1'(x)\,x + \left(x^2-4\right)y_1(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + y_1(x)\right)C\\
&+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+ x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(x^2-4\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$2x\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) C + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) x - 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 2$ and $r_2 = -2$ then the above becomes

$$2x\left(\sum_{n=0}^{\infty} x^{1+n} a_n(n+2)\right) C + \left(\sum_{n=0}^{\infty} x^{-4+n} b_n(n-2)(-3+n)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n(n-2)\right) x - 4\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n(n+2)\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n(n^2-5n+6)\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n(n-2)\right) + \sum_{n=0}^{\infty} \left(-4b_n x^{n-2}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n(n+2) = \sum_{n=4}^{\infty} 2C a_{-4+n}(n-2)\, x^{n-2}$$

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=2}^{\infty} b_{n-2} x^{n-2}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n-2$.

$$\left(\sum_{n=4}^{\infty} 2Ca_{-4+n}(n-2)\, x^{n-2}\right)+\left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right) \\ +\left(\sum_{n=2}^{\infty} b_{n-2} x^{n-2}\right)+\left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right)+\sum_{n=0}^{\infty} \left(-4 b_n x^{n-2}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-3b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1=0$$

Solving the above for b_1 gives

$$b_1=0$$

For $n=2$, Eq (2B) gives

$$b_0-4b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1-4b_2=0$$

Solving the above for b_2 gives

$$b_2=\frac{1}{4}$$

For $n=3$, Eq (2B) gives

$$b_1-3b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3=0$$

Solving the above for b_3 gives

$$b_3=0$$

For $n=N$, where $N=4$ which is the difference between the two roots, we are free to choose $b_4=0$. Hence for $n=4$, Eq (2B) gives

$$4C+\frac{1}{4}=0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{16}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{16}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{16}\left(x^2\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right)\right) \ln(x) + \frac{1 + \frac{x^2}{4} + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right) \\ &\quad + c_2\left(-\frac{1}{16}\left(x^2\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right)\right) \ln(x) + \frac{1 + \frac{x^2}{4} + O(x^6)}{x^2}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^2\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right) \\ &\quad + c_2\left(-\frac{x^2\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right) \ln(x)}{16} + \frac{1 + \frac{x^2}{4} + O(x^6)}{x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{align*}
y &= c_1 x^2 \left(1-\frac{x^2}{12}+\frac{x^4}{384}+O\left(x^6\right)\right) \\
&+ c_2\left(-\frac{x^2\left(1-\frac{x^2}{12}+\frac{x^4}{384}+O(x^6)\right)\ln\left(x\right)}{16}+\frac{1+\frac{x^2}{4}+O(x^6)}{x^2}\right) \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= c_1 x^2 \left(1-\frac{x^2}{12}+\frac{x^4}{384}+O\left(x^6\right)\right) \\
&+ c_2\left(-\frac{x^2\left(1-\frac{x^2}{12}+\frac{x^4}{384}+O(x^6)\right)\ln\left(x\right)}{16}+\frac{1+\frac{x^2}{4}+O(x^6)}{x^2}\right)
\end{align*}
$$

Verified OK.

3.6.1 Maple step by step solution

Let's solve

$x^2y''+xy'+\left(x^2-4\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x^2-4)y}{x^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}+\frac{(x^2-4)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{x^2-4}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -4$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x y' + (x^2 - 4)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 (2 + r)\,(-2 + r)\, x^r + a_1 (3 + r)\,(-1 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k + r + 2)\,(k + r - 2) + a_{k-2}\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2 + r)\,(-2 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 2\}$

- Each term must be 0

  $a_1 (3 + r)\,(-1 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-2) + a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+4+r)(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+r)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-2)}$
- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-2)}$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+6)(k+2)}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+6)(k+2)}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
Order:=6;
dsolve(diff(x*diff(y(x),x),x)+(x-4/x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^4 \left(1 - \frac{1}{12}x^2 + \frac{1}{384}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 (\ln\left(x\right)\left(9x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-144 - 36x^2 + \mathrm{O}\left(x^6\right)\right))}{x^2}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 52

```
AsymptoticDSolveValue[D[x*D[y[x],x],x]+(x-4/x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{\left(x^2+8\right)^2}{64x^2} - \frac{1}{16}x^2 \log(x)\right) + c_2 \left(\frac{x^6}{384} - \frac{x^4}{12} + x^2\right)$$

3.7 problem 7

Internal problem ID [5597]
Internal file name [OUTPUT/4845_Sunday_June_05_2022_03_07_48_PM_13206810/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + (9x^2 - 4)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (9x^2 - 4)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{9x^2 - 4}{x^2}$$

Table 86: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{9x^2-4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + (9x^2 - 4)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (9x^2 - 4) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 9x^{n+r+2} a_n = \sum_{n=2}^{\infty} 9a_{n-2} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 9a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 4a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) + x^r a_0 r - 4a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) + x^r r - 4x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2 - 4\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
r^2 - 4 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 4)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_n(n + r) + 9a_{n-2} - 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{9a_{n-2}}{n^2 + 2nr + r^2 - 4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{9a_{n-2}}{n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{9}{r(r+4)}$$

Which for the root $r = 2$ becomes

$$a_2 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{r(r+4)}$	$-\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{r(r+4)}$	$-\frac{3}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{r(r+4)(r+6)(r+2)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{27}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{r(r+4)}$	$-\frac{3}{4}$
a_3	0	0
a_4	$\frac{81}{r(r+4)(r+6)(r+2)}$	$\frac{27}{128}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{r(r+4)}$	$-\frac{3}{4}$
a_3	0	0
a_4	$\frac{81}{r(r+4)(r+6)(r+2)}$	$\frac{27}{128}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= x^2\left(1 - \frac{3x^2}{4} + \frac{27x^4}{128} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{81}{r\left(r+4\right)\left(r+6\right)\left(r+2\right)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{81}{r\left(r+4\right)\left(r+6\right)\left(r+2\right)} &= \lim_{r\to -2} \frac{81}{r\left(r+4\right)\left(r+6\right)\left(r+2\right)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln\left(x\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln\left(x\right) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) \\ &= C y_1'(x) \ln\left(x\right) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln\left(x\right) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln\left(x\right) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2) \left(-1+n+r_2\right)\right) \end{aligned}$$

Substituting these back into the given ode $x^2y''+xy'+(9x^2-4)\,y=0$ gives

$$x^2\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+x\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)+(9x^2-4)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0$$

Which can be written as

$$\left(\left(x^2y_1''(x)+y_1'(x)\,x+\left(9x^2-4\right)y_1(x)\right)\ln(x)+x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+y_1(x)\right)C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(9x^2-4)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2y_1''(x)+y_1'(x)\,x+\left(9x^2-4\right)y_1(x)=0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+y_1(x)\right)C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(9x^2-4)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&2x\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) C + 9\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x - 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}
$$

Since $r_1 = 2$ and $r_2 = -2$ then the above becomes

$$
\begin{aligned}
&2x\left(\sum_{n=0}^{\infty} x^{1+n} a_n (n+2)\right) C + 9\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) x - 4\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0
\end{aligned} \tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2)\right) + \left(\sum_{n=0}^{\infty} 9 b_n x^n\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n^2 - 5n + 6)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right) + \sum_{n=0}^{\infty} \left(-4 b_n x^{n-2}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2) = \sum_{n=4}^{\infty} 2C a_{-4+n} (n-2)\, x^{n-2}
$$

$$
\sum_{n=0}^{\infty} 9 b_n x^n = \sum_{n=2}^{\infty} 9 b_{n-2} x^{n-2}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\left(\sum_{n=4}^{\infty} 2Ca_{-4+n}(n-2)\,x^{n-2}\right)+\left(\sum_{n=2}^{\infty} 9b_{n-2}x^{n-2}\right) \\ +\left(\sum_{n=0}^{\infty} x^{n-2}b_n\left(n^2-5n+6\right)\right)+\left(\sum_{n=0}^{\infty} x^{n-2}b_n(n-2)\right)+\sum_{n=0}^{\infty}\left(-4b_n x^{n-2}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-3b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1=0$$

Solving the above for b_1 gives

$$b_1=0$$

For $n=2$, Eq (2B) gives

$$9b_0-4b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$9-4b_2=0$$

Solving the above for b_2 gives

$$b_2=\frac{9}{4}$$

For $n=3$, Eq (2B) gives

$$9b_1-3b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3=0$$

Solving the above for b_3 gives

$$b_3=0$$

For $n=N$, where $N=4$ which is the difference between the two roots, we are free to choose $b_4=0$. Hence for $n=4$, Eq (2B) gives

$$4C+\frac{81}{4}=0$$

Which is solved for C. Solving for C gives

$$C = -\frac{81}{16}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + 9b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{81}{16}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{81}{16}\left(x^2\left(1 - \frac{3x^2}{4} + \frac{27x^4}{128} + O(x^6)\right)\right)\ln(x) + \frac{1 + \frac{9x^2}{4} + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x^2\left(1 - \frac{3x^2}{4} + \frac{27x^4}{128} + O(x^6)\right) \\ &\quad + c_2\left(-\frac{81}{16}\left(x^2\left(1 - \frac{3x^2}{4} + \frac{27x^4}{128} + O(x^6)\right)\right)\ln(x) + \frac{1 + \frac{9x^2}{4} + O(x^6)}{x^2}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x^2\left(1 - \frac{3x^2}{4} + \frac{27x^4}{128} + O(x^6)\right) \\ &\quad + c_2\left(-\frac{81x^2\left(1 - \frac{3x^2}{4} + \frac{27x^4}{128} + O(x^6)\right)\ln(x)}{16} + \frac{1 + \frac{9x^2}{4} + O(x^6)}{x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^2 \left(1-\frac{3x^2}{4}+\frac{27x^4}{128}+O\left(x^6\right)\right) \\
&+ c_2 \left(-\frac{81x^2 \left(1-\frac{3x^2}{4}+\frac{27x^4}{128}+O(x^6)\right)\ln\left(x\right)}{16}+\frac{1+\frac{9x^2}{4}+O(x^6)}{x^2}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^2 \left(1-\frac{3x^2}{4}+\frac{27x^4}{128}+O\left(x^6\right)\right) \\
&+ c_2 \left(-\frac{81x^2 \left(1-\frac{3x^2}{4}+\frac{27x^4}{128}+O(x^6)\right)\ln\left(x\right)}{16}+\frac{1+\frac{9x^2}{4}+O(x^6)}{x^2}\right)
\end{aligned}
$$

Verified OK.

3.7.1 Maple step by step solution

Let's solve

$x^2y''+xy'+\left(9x^2-4\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(9x^2-4)y}{x^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}+\frac{(9x^2-4)y}{x^2}=0$

- Simplify ODE

  $9yx^2+x^2y''+xy'-4y=0$

- Make a change of variables

  $t=3x$

- Compute y'

  $y'=3\frac{d}{dt}y(t)$

- Compute second derivative

$y'' = 9\frac{d^2}{dt^2}y(t)$

- Apply change of variables to the ODE

  $y(t)\,t^2 + t^2\left(\frac{d^2}{dt^2}y(t)\right) + t\left(\frac{d}{dt}y(t)\right) - 4y(t) = 0$

- ODE is now of the Bessel form
- Solution to Bessel ODE

  $y(t) = c_1 \mathit{BesselJ}(2, t) + c_2 \mathit{BesselY}(2, t)$

- Make the change from t back to x

  $y = c_1 \mathit{BesselJ}(2, 3x) + c_2 \mathit{BesselY}(2, 3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(9*x^2-4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^4 \left(1 - \frac{3}{4}x^2 + \frac{27}{128}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(\ln\left(x\right)\left(729x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-144 - 324x^2 + \mathrm{O}\left(x^6\right)\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 54

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(9*x^2-4)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{(9x^2+8)^2}{64x^2} - \frac{81}{16} x^2 \log(x) \right) + c_2 \left(\frac{27x^6}{128} - \frac{3x^4}{4} + x^2 \right)$$

3.8 problem 8

Internal problem ID [5598]
Internal file name [OUTPUT/4846_Sunday_June_05_2022_03_07_52_PM_30257725/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{144x^2 - 1}{4x^2}$$

Table 88: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{144x^2-1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(36x^2 - \frac{1}{4}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 36 x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 36 x^{n+r+2} a_n = \sum_{n=2}^{\infty} 36 a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 36 a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - \frac{a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - \frac{a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - \frac{x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2-1)\, x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1 &= \frac{1}{2}\\ r_2 &= -\frac{1}{2}\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 1)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}\\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)\end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1)+a_n(n+r)+36a_{n-2}-\frac{a_n}{4}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{144a_{n-2}}{4n^2+8nr+4r^2-1} \tag{4}$$

Which for the root $r=\frac{1}{2}$ becomes

$$a_n = -\frac{36a_{n-2}}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$a_2 = -\frac{144}{4r^2+16r+15}$$

Which for the root $r=\frac{1}{2}$ becomes

$$a_2 = -6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{144}{4r^2+16r+15}$	-6

For $n=3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{144}{4r^2+16r+15}$	-6
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{20736}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{54}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{144}{4r^2+16r+15}$	-6
a_3	0	0
a_4	$\frac{20736}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{54}{5}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{144}{4r^2+16r+15}$	-6
a_3	0	0
a_4	$\frac{20736}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{54}{5}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\
&= \sqrt{x}\left(1 - 6x^2 + \frac{54x^4}{5} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} 0 &= \lim_{r\to -\frac{1}{2}} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
b_n(n+r)\,(n+r-1) + b_n(n+r) + 36b_{n-2} - \frac{b_n}{4} = 0 \tag{4}
$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$b_n \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) + b_n \left(n - \frac{1}{2}\right) + 36 b_{n-2} - \frac{b_n}{4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{144 b_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{144 b_{n-2}}{4n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{144}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -18$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{144}{4r^2+16r+15}$	-18

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{144}{4r^2+16r+15}$	-18
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{20736}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = 54$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{144}{4r^2+16r+15}$	-18
b_3	0	0
b_4	$\frac{20736}{(4r^2+16r+15)(4r^2+32r+63)}$	54

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{144}{4r^2+16r+15}$	-18
b_3	0	0
b_4	$\frac{20736}{(4r^2+16r+15)(4r^2+32r+63)}$	54
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - 18x^2 + 54x^4 + O(x^6)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1 - 6x^2 + \frac{54x^4}{5} + O\left(x^6\right)\right) + \frac{c_2\left(1 - 18x^2 + 54x^4 + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1 - 6x^2 + \frac{54x^4}{5} + O\left(x^6\right)\right) + \frac{c_2\left(1 - 18x^2 + 54x^4 + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \sqrt{x}\left(1 - 6x^2 + \frac{54x^4}{5} + O\left(x^6\right)\right) + \frac{c_2\left(1 - 18x^2 + 54x^4 + O(x^6)\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \sqrt{x}\left(1 - 6x^2 + \frac{54x^4}{5} + O\left(x^6\right)\right) + \frac{c_2\left(1 - 18x^2 + 54x^4 + O(x^6)\right)}{\sqrt{x}}
$$

Verified OK.

3.8.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(36x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(144x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}+\frac{(144x^2-1)y}{4x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{144x^2-1}{4x^2}\right]$

  ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=1$

  ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{1}{4}$

  ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2y''+4xy'+(144x^2-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ○ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ○ Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  ○ Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)x^r + a_1(3+2r)(1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)(2k+2r-1) + 144a\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(3+2r)(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k^2+8kr+4r^2-1) + 144a_{k-2} = 0$$

- Shift index using $k->k+2$

$$a_{k+2}\left(4(k+2)^2 + 8(k+2)r + 4r^2 - 1\right) + 144a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{144a_k}{4k^2+8kr+4r^2+16k+16r+15}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+2} = -\frac{144a_k}{4k^2+12k+8}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{144a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{144b_k}{4k^2+20k+24}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(36*x^2-1/4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x\left(1 - 6x^2 + \frac{54}{5}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(1 - 18x^2 + 54x^4 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(36*x^2-1/4)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(54x^{7/2} - 18x^{3/2} + \frac{1}{\sqrt{x}}\right) + c_2\left(\frac{54x^{9/2}}{5} - 6x^{5/2} + \sqrt{x}\right)$$

3.9 problem 9

Internal problem ID [5599]
Internal file name [OUTPUT/4847_Sunday_June_05_2022_03_07_54_PM_87154698/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{225x^2 - 4}{9x^2}$$

Table 90: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{225x^2-4}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) + x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + \left(25x^2 - \frac{4}{9}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 25 x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{4 a_n x^{n+r}}{9}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 25 x^{n+r+2} a_n = \sum_{n=2}^{\infty} 25 a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 25 a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{4 a_n x^{n+r}}{9}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - \frac{4 a_n x^{n+r}}{9} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - \frac{4 a_0 x^r}{9} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - \frac{4 x^r}{9}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(9r^2-4)\, x^r}{9} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{4}{9} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{2}{3}$$
$$r_2 = -\frac{2}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(9r^2 - 4)\,x^r}{9} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{2}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + 25a_{n-2} - \frac{4a_n}{9} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{225a_{n-2}}{9n^2 + 18nr + 9r^2 - 4} \tag{4}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_n = -\frac{75 a_{n-2}}{n\,(3n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{225}{9r^2 + 36r + 32}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_2 = -\frac{15}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{225}{9r^2+36r+32}$	$-\frac{15}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{225}{9r^2+36r+32}$	$-\frac{15}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{50625}{(9r^2 + 36r + 32)(9r^2 + 72r + 140)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_4 = \frac{1125}{256}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{225}{9r^2+36r+32}$	$-\frac{15}{4}$
a_3	0	0
a_4	$\frac{50625}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{1125}{256}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{225}{9r^2+36r+32}$	$-\frac{15}{4}$
a_3	0	0
a_4	$\frac{50625}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{1125}{256}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{2}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{2}{3}}\left(1 - \frac{15x^2}{4} + \frac{1125x^4}{256} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + 25b_{n-2} - \frac{4b_n}{9} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{225b_{n-2}}{9n^2 + 18nr + 9r^2 - 4} \tag{4}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_n = -\frac{75b_{n-2}}{n(3n-4)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{225}{9r^2 + 36r + 32}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_2 = -\frac{75}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{225}{9r^2+36r+32}$	$-\frac{75}{4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{225}{9r^2+36r+32}$	$-\frac{75}{4}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{50625}{(9r^2 + 36r + 32)(9r^2 + 72r + 140)}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_4 = \frac{5625}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{225}{9r^2+36r+32}$	$-\frac{75}{4}$
b_3	0	0
b_4	$\frac{50625}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{5625}{128}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{225}{9r^2+36r+32}$	$-\frac{75}{4}$
b_3	0	0
b_4	$\frac{50625}{(9r^2+36r+32)(9r^2+72r+140)}$	$\frac{5625}{128}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^{\frac{2}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{75x^2}{4} + \frac{5625x^4}{128} + O(x^6)}{x^{\frac{2}{3}}}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{2}{3}}\left(1 - \frac{15x^2}{4} + \frac{1125x^4}{256} + O(x^6)\right) + \frac{c_2\left(1 - \frac{75x^2}{4} + \frac{5625x^4}{128} + O(x^6)\right)}{x^{\frac{2}{3}}}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{2}{3}}\left(1 - \frac{15x^2}{4} + \frac{1125x^4}{256} + O(x^6)\right) + \frac{c_2\left(1 - \frac{75x^2}{4} + \frac{5625x^4}{128} + O(x^6)\right)}{x^{\frac{2}{3}}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{2}{3}}\left(1 - \frac{15x^2}{4} + \frac{1125x^4}{256} + O(x^6)\right) + \frac{c_2\left(1 - \frac{75x^2}{4} + \frac{5625x^4}{128} + O(x^6)\right)}{x^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{2}{3}}\left(1 - \frac{15x^2}{4} + \frac{1125x^4}{256} + O(x^6)\right) + \frac{c_2\left(1 - \frac{75x^2}{4} + \frac{5625x^4}{128} + O(x^6)\right)}{x^{\frac{2}{3}}}$$

Verified OK.

3.9.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(225x^2-4)y}{9x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(225x^2-4)y}{9x^2} = 0$

- Simplify ODE

  $25yx^2 + x^2y'' + xy' - \frac{4y}{9} = 0$

- Make a change of variables

  $t = 5x$

- Compute y'

  $y' = 5\frac{d}{dt}y(t)$

- Compute second derivative

  $y'' = 25\frac{d^2}{dt^2}y(t)$

- Apply change of variables to the ODE

  $y(t)\,t^2 + t^2\left(\frac{d^2}{dt^2}y(t)\right) + t\left(\frac{d}{dt}y(t)\right) - \frac{4y(t)}{9} = 0$

- ODE is now of the Bessel form
- Solution to Bessel ODE

  $y(t) = c_1 \mathit{BesselJ}\left(\frac{2}{3}, t\right) + c_2 \mathit{BesselY}\left(\frac{2}{3}, t\right)$

- Make the change from t back to x

  $y = c_1 \mathit{BesselJ}\left(\frac{2}{3}, 5x\right) + c_2 \mathit{BesselY}\left(\frac{2}{3}, 5x\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(25*x^2-4/9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}} \left(1 - \frac{15}{4} x^2 + \frac{1125}{256} x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 - \frac{75}{4} x^2 + \frac{5625}{128} x^4 + \mathrm{O}\left(x^6\right)\right)}{x^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(25*x^2-4/9)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x^{2/3} \left(\frac{1125x^4}{256} - \frac{15x^2}{4} + 1\right) + \frac{c_2 \left(\frac{5625x^4}{128} - \frac{75x^2}{4} + 1\right)}{x^{2/3}}$$

3.10 problem 10

Internal problem ID [5600]
Internal file name [`OUTPUT/4848_Sunday_June_05_2022_03_07_56_PM_89338792/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + (2x^2 - 64)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (2x^2 - 64)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{2x^2 - 64}{x^2}$$

Table 92: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2x^2-64}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + (2x^2 - 64)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (2x^2 - 64) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-64 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-64 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 64 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 64 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 64 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 64\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 64 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 8 \\
r_2 &= -8
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(r^2 - 64\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 16$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^8 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^8}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+8} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-8}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_n(n+r) + 2a_{n-2} - 64a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-2}}{n^2 + 2nr + r^2 - 64} \tag{4}$$

Which for the root $r = 8$ becomes

$$a_n = -\frac{2a_{n-2}}{n(n+16)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 8$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{2}{r^2 + 4r - 60}$$

Which for the root $r = 8$ becomes

$$a_2 = -\frac{1}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4}{(r^2 + 4r - 60)(r^2 + 8r - 48)}$$

Which for the root $r = 8$ becomes

$$a_4 = \frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{8}{(r^2 + 4r - 60)(r^2 + 8r - 48)(r^2 + 12r - 28)}$$

Which for the root $r = 8$ becomes

$$a_6 = -\frac{1}{47520}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0
a_6	$-\frac{8}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)}$	$-\frac{1}{47520}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0
a_6	$-\frac{8}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)}$	$-\frac{1}{47520}$
a_7	0	0

For $n = 8$, using the above recursive equation gives

$$a_8 = \frac{16}{(r^2 + 4r - 60)(r^2 + 8r - 48)(r^2 + 12r - 28) r (r + 16)}$$

Which for the root $r = 8$ becomes

$$a_8 = \frac{1}{4561920}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0
a_6	$-\frac{8}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)}$	$-\frac{1}{47520}$
a_7	0	0
a_8	$\frac{16}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)}$	$\frac{1}{4561920}$

For $n = 9$, using the above recursive equation gives

$$a_9 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0
a_6	$-\frac{8}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)}$	$-\frac{1}{47520}$
a_7	0	0
a_8	$\frac{16}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)}$	$\frac{1}{4561920}$
a_9	0	0

For $n = 10$, using the above recursive equation gives

$$a_{10} = -\frac{32}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)}$$

Which for the root $r = 8$ becomes

$$a_{10} = -\frac{1}{593049600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0
a_6	$-\frac{8}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)}$	$-\frac{1}{47520}$
a_7	0	0
a_8	$\frac{16}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)}$	$\frac{1}{4561920}$
a_9	0	0
a_{10}	$-\frac{32}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)}$	$-\frac{1}{593049600}$

For $n = 11$, using the above recursive equation gives

$$a_{11} = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0
a_6	$-\frac{8}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)}$	$-\frac{1}{47520}$
a_7	0	0
a_8	$\frac{16}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)}$	$\frac{1}{4561920}$
a_9	0	0
a_{10}	$-\frac{32}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)}$	$-\frac{1}{593049600}$
a_{11}	0	0

For $n = 12$, using the above recursive equation gives

$$a_{12} = \frac{64}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)\, r\, (r+16)(r^2+20r+36)(r^2+24r+80)}$$

Which for the root $r = 8$ becomes

$$a_{12} = \frac{1}{99632332800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0
a_6	$-\frac{8}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)}$	$-\frac{1}{47520}$
a_7	0	0
a_8	$\frac{16}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)}$	$\frac{1}{4561920}$
a_9	0	0
a_{10}	$-\frac{32}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)}$	$-\frac{1}{593049600}$
a_{11}	0	0
a_{12}	$\frac{64}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)(r^2+24r+80)}$	$\frac{1}{99632332800}$

For $n = 13$, using the above recursive equation gives

$$a_{13} = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0
a_6	$-\frac{8}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)}$	$-\frac{1}{47520}$
a_7	0	0
a_8	$\frac{16}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)}$	$\frac{1}{4561920}$
a_9	0	0
a_{10}	$-\frac{32}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)}$	$-\frac{1}{593049600}$
a_{11}	0	0
a_{12}	$\frac{64}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)(r^2+24r+80)}$	$\frac{1}{99632332800}$
a_{13}	0	0

For $n = 14$, using the above recursive equation gives

$$a_{14} = -\frac{128}{\left(r^2+4r-60\right)\left(r^2+8r-48\right)\left(r^2+12r-28\right)r\left(r+16\right)\left(r^2+20r+36\right)\left(r^2+24r+80\right)\left(r^2+28r\right.}$$

Which for the root $r = 8$ becomes

$$a_{14} = -\frac{1}{20922789888000}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0
a_6	$-\frac{8}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)}$	$-\frac{1}{47520}$
a_7	0	0
a_8	$\frac{16}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)}$	$\frac{1}{4561920}$
a_9	0	0
a_{10}	$-\frac{32}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)}$	$-\frac{1}{593049600}$
a_{11}	0	0
a_{12}	$\frac{64}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)(r^2+24r+80)}$	$\frac{1}{99632332800}$
a_{13}	0	0
a_{14}	$-\frac{128}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)(r^2+24r+80)(r^2+28r+132)}$	$-\frac{1}{20922789888000}$

For $n = 15$, using the above recursive equation gives

$$a_{15} = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0
a_6	$-\frac{8}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)}$	$-\frac{1}{47520}$
a_7	0	0
a_8	$\frac{16}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)}$	$\frac{1}{4561920}$
a_9	0	0
a_{10}	$-\frac{32}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)}$	$-\frac{1}{593049600}$
a_{11}	0	0
a_{12}	$\frac{64}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)(r^2+24r+80)}$	$\frac{1}{99632332800}$
a_{13}	0	0
a_{14}	$-\frac{128}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)(r^2+24r+80)(r^2+28r+132)}$	$-\frac{1}{20922789888000}$
a_{15}	0	0

For $n = 16$, using the above recursive equation gives

$$a_{16} = \frac{256}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)\, r\, (r+16)(r^2+20r+36)(r^2+24r+80)(r^2+28r+$$

Which for the root $r = 8$ becomes

$$a_{16} = \frac{1}{5356234211328000}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{r^2+4r-60}$	$-\frac{1}{18}$
a_3	0	0
a_4	$\frac{4}{(r^2+4r-60)(r^2+8r-48)}$	$\frac{1}{720}$
a_5	0	0
a_6	$-\frac{8}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)}$	$-\frac{1}{47520}$
a_7	0	0
a_8	$\frac{16}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)}$	$\frac{1}{4561920}$
a_9	0	0
a_{10}	$-\frac{32}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)}$	$-\frac{1}{593049600}$
a_{11}	0	0
a_{12}	$\frac{64}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)(r^2+24r+80)}$	$\frac{1}{99632332800}$
a_{13}	0	0
a_{14}	$-\frac{128}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)(r^2+24r+80)(r^2+28r+132)}$	$-\frac{1}{20922789888000}$
a_{15}	0	0
a_{16}	$\frac{256}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)r(r+16)(r^2+20r+36)(r^2+24r+80)(r^2+28r+132)(r^2+32r+192)}$	$\frac{1}{5356234211328000}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^8 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + a_{10} x^{10} + a_{11} x^{11} + a_{12} x^{1} \right. \\
&= x^8 \left(1 - \frac{x^2}{18} + \frac{x^4}{720} - \frac{x^6}{47520} + \frac{x^8}{4561920} - \frac{x^{10}}{593049600} + \frac{x^{12}}{99632332800} - \frac{x^{14}}{20922789888000} + \frac{}{53562} \right.
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 16$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_{16}(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$a_N = a_{16}$$
$$= \frac{256}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)\, r\,(r+16)(r^2+20r+36)(r^2+24r+80)(r^2+28r+}$$

Therefore

$$\lim_{r\to r_2} \frac{256}{(r^2+4r-60)(r^2+8r-48)(r^2+12r-28)\, r\,(r+16)(r^2+20r+36)(r^2+24r+80)(r^2+28r+1}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y''+xy'+(2x^2-64)\,y=0$ gives

$$
\begin{aligned}
&x^2\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+x\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+(2x^2-64)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x^2y_1''(x)+y_1'(x)\,x+(2x^2-64)\,y_1(x)\right)\ln(x)+x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)\right.\\
&\left.+y_1(x)\right)C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(2x^2-64)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
x^2y_1''(x)+y_1'(x)\,x+(2x^2-64)\,y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+y_1(x)\right)C\\
&+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(2x^2-64)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&2x\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) C + 2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x - 64\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}
$$

Since $r_1 = 8$ and $r_2 = -8$ then the above becomes

$$
\begin{aligned}
&2x\left(\sum_{n=0}^{\infty} x^{7+n} a_n (n+8)\right) C + 2\left(\sum_{n=0}^{\infty} b_n x^{n-8}\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-10+n} b_n (n-8)(-9+n)\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-9+n} b_n (n-8)\right) x - 64\left(\sum_{n=0}^{\infty} b_n x^{n-8}\right) = 0
\end{aligned} \tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+8} a_n (n+8)\right) + \left(\sum_{n=0}^{\infty} 2x^{-6+n} b_n\right) + \left(\sum_{n=0}^{\infty} x^{n-8} b_n (-9+n)(n-8)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-8} b_n (n-8)\right) + \sum_{n=0}^{\infty} \left(-64 b_n x^{n-8}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-8$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-8} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2C\, x^{n+8} a_n (n+8) = \sum_{n=16}^{\infty} 2C a_{n-16} (n-8)\, x^{n-8}
$$

$$
\sum_{n=0}^{\infty} 2x^{-6+n} b_n = \sum_{n=2}^{\infty} 2 b_{n-2} x^{n-8}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-8$.

$$\left(\sum_{n=16}^{\infty} 2C a_{n-16}(n-8)\, x^{n-8}\right)+\left(\sum_{n=2}^{\infty} 2 b_{n-2} x^{n-8}\right) + \left(\sum_{n=0}^{\infty} x^{n-8} b_n(-9+n)\,(n-8)\right)+\left(\sum_{n=0}^{\infty} x^{n-8} b_n(n-8)\right)+\sum_{n=0}^{\infty}\left(-64 b_n x^{n-8}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-15 b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-15 b_1=0$$

Solving the above for b_1 gives

$$b_1=0$$

For $n=2$, Eq (2B) gives

$$2 b_0-28 b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2-28 b_2=0$$

Solving the above for b_2 gives

$$b_2=\frac{1}{14}$$

For $n=3$, Eq (2B) gives

$$2 b_1-39 b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-39 b_3=0$$

Solving the above for b_3 gives

$$b_3=0$$

For $n=4$, Eq (2B) gives

$$2 b_2-48 b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{7} - 48b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{336}$$

For $n = 5$, Eq (2B) gives

$$2b_3 - 55b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-55b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

For $n = 6$, Eq (2B) gives

$$2b_4 - 60b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{168} - 60b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{1}{10080}$$

For $n = 7$, Eq (2B) gives

$$2b_5 - 63b_7 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-63b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = 0$$

For $n = 8$, Eq (2B) gives

$$2b_6 - 64b_8 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{5040} - 64b_8 = 0$$

Solving the above for b_8 gives

$$b_8 = \frac{1}{322560}$$

For $n = 9$, Eq (2B) gives

$$2b_7 - 63b_9 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-63b_9 = 0$$

Solving the above for b_9 gives

$$b_9 = 0$$

For $n = 10$, Eq (2B) gives

$$2b_8 - 60b_{10} = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{161280} - 60b_{10} = 0$$

Solving the above for b_{10} gives

$$b_{10} = \frac{1}{9676800}$$

For $n = 11$, Eq (2B) gives

$$2b_9 - 55b_{11} = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-55b_{11} = 0$$

Solving the above for b_{11} gives

$$b_{11} = 0$$

For $n = 12$, Eq (2B) gives

$$2b_{10} - 48b_{12} = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{4838400} - 48b_{12} = 0$$

Solving the above for b_{12} gives

$$b_{12} = \frac{1}{232243200}$$

For $n = 13$, Eq (2B) gives

$$2b_{11} - 39b_{13} = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-39b_{13} = 0$$

Solving the above for b_{13} gives

$$b_{13} = 0$$

For $n = 14$, Eq (2B) gives

$$2b_{12} - 28b_{14} = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{116121600} - 28b_{14} = 0$$

Solving the above for b_{14} gives

$$b_{14} = \frac{1}{3251404800}$$

For $n = 15$, Eq (2B) gives

$$2b_{13} - 15b_{15} = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-15b_{15} = 0$$

Solving the above for b_{15} gives

$$b_{15} = 0$$

For $n = N$, where $N = 16$ which is the difference between the two roots, we are free to choose $b_{16} = 0$. Hence for $n = 16$, Eq (2B) gives

$$16C + \frac{1}{1625702400} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{26011238400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{26011238400}$ and all b_n, then the second solution becomes

$$
\begin{aligned}
y_2(x) = &-\frac{1}{26011238400}\left(x^8\left(1-\frac{x^2}{18}+\frac{x^4}{720}-\frac{x^6}{47520}+\frac{x^8}{4561920}-\frac{x^{10}}{593049600}\right.\right.\\
&\left.\left.+\frac{x^{12}}{99632332800}-\frac{x^{14}}{20922789888000}+\frac{x^{16}}{5356234211328000}+O\left(x^{17}\right)\right)\right)\ln\left(x\right)\\
&+\frac{1+\frac{x^2}{14}+\frac{x^4}{336}+\frac{x^6}{10080}+\frac{x^8}{322560}+\frac{x^{10}}{9676800}+\frac{x^{12}}{232243200}+\frac{x^{14}}{3251404800}+O(x^{17})}{x^8}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\
&= c_1 x^8\left(1-\frac{x^2}{18}+\frac{x^4}{720}-\frac{x^6}{47520}+\frac{x^8}{4561920}-\frac{x^{10}}{593049600}+\frac{x^{12}}{99632332800}\right.\\
&\qquad\left.-\frac{x^{14}}{20922789888000}+\frac{x^{16}}{5356234211328000}+O\left(x^{17}\right)\right)\\
&\quad+c_2\left(-\frac{1}{26011238400}\left(x^8\left(1-\frac{x^2}{18}+\frac{x^4}{720}-\frac{x^6}{47520}+\frac{x^8}{4561920}-\frac{x^{10}}{593049600}+\frac{x^{12}}{99632332800}-\frac{x}{2092278}\right.\right.\right.\\
&\qquad\left.+\frac{1+\frac{x^2}{14}+\frac{x^4}{336}+\frac{x^6}{10080}+\frac{x^8}{322560}+\frac{x^{10}}{9676800}+\frac{x^{12}}{232243200}+\frac{x^{14}}{3251404800}+O(x^{17})}{x^8}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h\\
&= c_1 x^8\left(1-\frac{x^2}{18}+\frac{x^4}{720}-\frac{x^6}{47520}+\frac{x^8}{4561920}-\frac{x^{10}}{593049600}+\frac{x^{12}}{99632332800}\right.\\
&\qquad\left.-\frac{x^{14}}{20922789888000}+\frac{x^{16}}{5356234211328000}+O\left(x^{17}\right)\right)\\
&\quad+c_2\left(-\frac{x^8\left(1-\frac{x^2}{18}+\frac{x^4}{720}-\frac{x^6}{47520}+\frac{x^8}{4561920}-\frac{x^{10}}{593049600}+\frac{x^{12}}{99632332800}-\frac{x^{14}}{20922789888000}+\frac{x^{16}}{5356234211328000}+O\right.}{26011238400}\right.\\
&\qquad\left.+\frac{1+\frac{x^2}{14}+\frac{x^4}{336}+\frac{x^6}{10080}+\frac{x^8}{322560}+\frac{x^{10}}{9676800}+\frac{x^{12}}{232243200}+\frac{x^{14}}{3251404800}+O(x^{17})}{x^8}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x^8 \left(1 - \frac{x^2}{18} + \frac{x^4}{720} - \frac{x^6}{47520} + \frac{x^8}{4561920} - \frac{x^{10}}{593049600} + \frac{x^{12}}{99632332800} - \frac{x^{14}}{20922789888000} + \frac{x^{16}}{5356234211328000} + O\left(x^{17}\right)\right) + c_2 \left(-\frac{x^8 \left(1 - \frac{x^2}{18} + \frac{x^4}{720} - \frac{x^6}{47520} + \frac{x^8}{4561920} - \frac{x^{10}}{593049600} + \frac{x^{12}}{99632332800} - \frac{x^{14}}{20922789888000} + \frac{x^{16}}{5356234211328000} + O(\right.}{26011238400} + \frac{1 + \frac{x^2}{14} + \frac{x^4}{336} + \frac{x^6}{10080} + \frac{x^8}{322560} + \frac{x^{10}}{9676800} + \frac{x^{12}}{232243200} + \frac{x^{14}}{3251404800} + O(x^{17})}{x^8}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^8 \left(1 - \frac{x^2}{18} + \frac{x^4}{720} - \frac{x^6}{47520} + \frac{x^8}{4561920} - \frac{x^{10}}{593049600} + \frac{x^{12}}{99632332800} - \frac{x^{14}}{20922789888000} + \frac{x^{16}}{5356234211328000} + O\left(x^{17}\right)\right) + c_2 \left(-\frac{x^8 \left(1 - \frac{x^2}{18} + \frac{x^4}{720} - \frac{x^6}{47520} + \frac{x^8}{4561920} - \frac{x^{10}}{593049600} + \frac{x^{12}}{99632332800} - \frac{x^{14}}{20922789888000} + \frac{x^{16}}{5356234211328000} + O(\right.}{26011238400} + \frac{1 + \frac{x^2}{14} + \frac{x^4}{336} + \frac{x^6}{10080} + \frac{x^8}{322560} + \frac{x^{10}}{9676800} + \frac{x^{12}}{232243200} + \frac{x^{14}}{3251404800} + O(x^{17})}{x^8}\right)$$

Verified OK.

3.10.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + (2x^2 - 64)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(x^2-32)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{2(x^2-32)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{2(x^2-32)}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -64$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x y' + (2x^2 - 64) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(8+r)(-8+r)x^r + a_1(9+r)(-7+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+8)(k+r-8) + 2a_{k-2})x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(8+r)(-8+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-8, 8\}$
- Each term must be 0

  $a_1(9+r)(-7+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+8)(k+r-8) + 2a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+10+r)(k-6+r) + 2a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k}{(k+10+r)(k-6+r)}$
- Recursion relation for $r = -8$

  $a_{k+2} = -\frac{2a_k}{(k+2)(k-14)}$
- Series not valid for $r = -8$, division by 0 in the recursion relation at $k = 14$

  $a_{k+2} = -\frac{2a_k}{(k+2)(k-14)}$
- Recursion relation for $r = 8$

  $a_{k+2} = -\frac{2a_k}{(k+18)(k+2)}$
- Solution for $r = 8$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+8}, a_{k+2} = -\frac{2a_k}{(k+18)(k+2)}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(2*x^2-64)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1x^8\left(1 - \frac{1}{18}x^2 + \frac{1}{720}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2(-2736019604358719078400000 - 195429971739908505600000x^2 - 8142915489162854400000x}{x^8}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 46

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(2*x^2-64)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^{12}}{720} - \frac{x^{10}}{18} + x^8\right) + c_1\left(\frac{1}{x^8} + \frac{1}{14x^6} + \frac{1}{336x^4}\right)$$

3.11 problem 13

Internal problem ID [5601]
Internal file name [OUTPUT/4849_Sunday_June_05_2022_03_08_02_PM_69386176/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + 2y' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y' + 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{4}{x}$$

Table 94: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r) (n+r-1) a_n x^{n+r-2}\right) x + 2\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4a_n x^{n+r} = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + 2a_n(n+r) + 4a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{4a_{n-1}}{n^2 + 2nr + r^2 + n + r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
a_n = -\frac{4a_{n-1}}{n(n+1)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{4}{r^2 + 3r + 2}$$

Which for the root $r = 0$ becomes

$$a_1 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r^2+3r+2}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{(r + 2)^2 (1 + r) (r + 3)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{4}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r^2+3r+2}$	-2
a_2	$\frac{16}{(r+2)^2(1+r)(r+3)}$	$\frac{4}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{(r + 2)^2 (1 + r) (r + 3)^2 (4 + r)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{4}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r^2+3r+2}$	-2
a_2	$\frac{16}{(r+2)^2(1+r)(r+3)}$	$\frac{4}{3}$
a_3	$-\frac{64}{(r+2)^2(1+r)(r+3)^2(4+r)}$	$-\frac{4}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(r+2)^2 (1+r) (r+3)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{4}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r^2+3r+2}$	-2
a_2	$\frac{16}{(r+2)^2(1+r)(r+3)}$	$\frac{4}{3}$
a_3	$-\frac{64}{(r+2)^2(1+r)(r+3)^2(4+r)}$	$-\frac{4}{9}$
a_4	$\frac{256}{(r+2)^2(1+r)(r+3)^2(4+r)^2(5+r)}$	$\frac{4}{45}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{(r+2)^2 (1+r) (r+3)^2 (4+r)^2 (5+r)^2 (r+6)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{8}{675}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r^2+3r+2}$	-2
a_2	$\frac{16}{(r+2)^2(1+r)(r+3)}$	$\frac{4}{3}$
a_3	$-\frac{64}{(r+2)^2(1+r)(r+3)^2(4+r)}$	$-\frac{4}{9}$
a_4	$\frac{256}{(r+2)^2(1+r)(r+3)^2(4+r)^2(5+r)}$	$\frac{4}{45}$
a_5	$-\frac{1024}{(r+2)^2(1+r)(r+3)^2(4+r)^2(5+r)^2(r+6)}$	$-\frac{8}{675}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 - 2x + \frac{4x^2}{3} - \frac{4x^3}{9} + \frac{4x^4}{45} - \frac{8x^5}{675} + O(x^6)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= -\frac{4}{r^2 + 3r + 2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} -\frac{4}{r^2 + 3r + 2} &= \lim_{r\to -1} -\frac{4}{r^2 + 3r + 2} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right)
$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$
$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$
$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$
$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $xy'' + 2y' + 4y = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + 2Cy_1'(x)\ln(x) + \frac{2Cy_1(x)}{x} + 2\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 4Cy_1(x)\ln(x) + 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left(\left(y_1''(x)\,x + 4y_1(x) + 2y_1'(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + \frac{2y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + 2\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x + 4y_1(x) + 2y_1'(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x+\frac{2y_1(x)}{x}\right)C+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x+2\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+4\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x+\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}+\frac{\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2+2\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x+4\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x}=0 \tag{9}$$

Since $r_1=0$ and $r_2=-1$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{n-1}a_n n\right)x+\left(\sum_{n=0}^{\infty}a_n x^{n}\right)\right)C}{x}+\frac{\left(\sum_{n=0}^{\infty}x^{-3+n}b_n(n-1)(-2+n)\right)x^2+2\left(\sum_{n=0}^{\infty}x^{-2+n}b_n(n-1)\right)x+4\left(\sum_{n=0}^{\infty}b_n x^{n-1}\right)x}{x}=0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}2C\,x^{n-1}a_n n\right)+\left(\sum_{n=0}^{\infty}C\,x^{n-1}a_n\right)+\left(\sum_{n=0}^{\infty}x^{-2+n}b_n(n^2-3n+2)\right)+\left(\sum_{n=0}^{\infty}2x^{-2+n}b_n(n-1)\right)+\left(\sum_{n=0}^{\infty}4b_n x^{n-1}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $-2+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-2+n} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n n &= \sum_{n=1}^{\infty} 2C(n-1)\, a_{n-1} x^{-2+n} \\
\sum_{n=0}^{\infty} C\, x^{n-1} a_n &= \sum_{n=1}^{\infty} C a_{n-1} x^{-2+n} \\
\sum_{n=0}^{\infty} 4 b_n x^{n-1} &= \sum_{n=1}^{\infty} 4 b_{n-1} x^{-2+n}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-2+n$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 2C(n-1)\, a_{n-1} x^{-2+n}\right) + \left(\sum_{n=1}^{\infty} C a_{n-1} x^{-2+n}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n} b_n \left(n^2-3n+2\right)\right) \\
&+ \left(\sum_{n=0}^{\infty} 2x^{-2+n} b_n (n-1)\right) + \left(\sum_{n=1}^{\infty} 4 b_{n-1} x^{-2+n}\right) = 0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$
C+4=0
$$

Which is solved for C. Solving for C gives

$$
C=-4
$$

For $n=2$, Eq (2B) gives

$$
3Ca_1+4b_1+2b_2=0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$
2b_2+24=0
$$

Solving the above for b_2 gives

$$b_2 = -12$$

For $n = 3$, Eq (2B) gives

$$5Ca_2 + 4b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3 - \frac{224}{3} = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{112}{9}$$

For $n = 4$, Eq (2B) gives

$$7Ca_3 + 4b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_4 + \frac{560}{9} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{140}{27}$$

For $n = 5$, Eq (2B) gives

$$9Ca_4 + 4b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$20b_5 - \frac{3232}{135} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{808}{675}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -4$ and all b_n, then the second solution becomes

$$y_2(x) = (-4)\left(1 - 2x + \frac{4x^2}{3} - \frac{4x^3}{9} + \frac{4x^4}{45} - \frac{8x^5}{675} + O(x^6)\right)\ln(x) + \frac{1 - 12x^2 + \frac{112x^3}{9} - \frac{140x^4}{27} + \frac{808x^5}{675} + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 - 2x + \frac{4x^2}{3} - \frac{4x^3}{9} + \frac{4x^4}{45} - \frac{8x^5}{675} + O(x^6)\right) \\
&\quad + c_2 \left((-4)\left(1 - 2x + \frac{4x^2}{3} - \frac{4x^3}{9} + \frac{4x^4}{45} - \frac{8x^5}{675} + O(x^6)\right)\ln(x) \right. \\
&\qquad \left. + \frac{1 - 12x^2 + \frac{112x^3}{9} - \frac{140x^4}{27} + \frac{808x^5}{675} + O(x^6)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - 2x + \frac{4x^2}{3} - \frac{4x^3}{9} + \frac{4x^4}{45} - \frac{8x^5}{675} + O(x^6)\right) \\
&\quad + c_2 \left(\left(-4 + 8x - \frac{16x^2}{3} + \frac{16x^3}{9} - \frac{16x^4}{45} + \frac{32x^5}{675} - 4O(x^6)\right)\ln(x) \right. \\
&\qquad \left. + \frac{1 - 12x^2 + \frac{112x^3}{9} - \frac{140x^4}{27} + \frac{808x^5}{675} + O(x^6)}{x}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(1 - 2x + \frac{4x^2}{3} - \frac{4x^3}{9} + \frac{4x^4}{45} - \frac{8x^5}{675} + O(x^6)\right) \\
&\quad + c_2 \left(\left(-4 + 8x - \frac{16x^2}{3} + \frac{16x^3}{9} - \frac{16x^4}{45} + \frac{32x^5}{675} - 4O(x^6)\right)\ln(x) \right. \\
&\qquad \left. + \frac{1 - 12x^2 + \frac{112x^3}{9} - \frac{140x^4}{27} + \frac{808x^5}{675} + O(x^6)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\left(1-2x+\frac{4x^2}{3}-\frac{4x^3}{9}+\frac{4x^4}{45}-\frac{8x^5}{675}+O(x^6)\right)$$
$$+c_2\left(\left(-4+8x-\frac{16x^2}{3}+\frac{16x^3}{9}-\frac{16x^4}{45}+\frac{32x^5}{675}-4O(x^6)\right)\ln(x)\right.$$
$$\left.+\frac{1-12x^2+\frac{112x^3}{9}-\frac{140x^4}{27}+\frac{808x^5}{675}+O(x^6)}{x}\right)$$

Verified OK.

3.11.1 Maple step by step solution

Let's solve

$y''x+2y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x}-\frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2y'}{x}+\frac{4y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2}{x}, P_3(x)=\frac{4}{x}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=2$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$y''x + 2y' + 4y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$

  - Shift index using $k->k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r) x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$

  - Shift index using $k->k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r) x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+2+r) + 4a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+1+r)(k+2+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{4a_k}{(k+1+r)(k+2+r)}$

- Recursion relation for $r = -1$

$a_{k+1} = -\frac{4a_k}{k(k+1)}$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{4a_k}{k(k+1)}\right]$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{4a_k}{(k+1)(k+2)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{4a_k}{(k+1)(k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+1} = -\frac{4a_k}{k(k+1)}, b_{k+1} = -\frac{4b_k}{(k+1)(k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 64

```
Order:=6;
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1 - 2x + \frac{4}{3}x^2 - \frac{4}{9}x^3 + \frac{4}{45}x^4 - \frac{8}{675}x^5 + \mathrm{O}\left(x^6\right)\right)x + c_2\left(\ln\left(x\right)\left(\left(-4\right)x + 8x^2 - \frac{16}{3}x^3 + \frac{16}{9}x^4 - \frac{16}{45}x^5 + \mathrm{O}\right.\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x*y''[x]+2*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{4x^4}{45} - \frac{4x^3}{9} + \frac{4x^2}{3} - 2x + 1\right) + c_1\left(\frac{4}{9}\left(4x^3 - 12x^2 + 18x - 9\right)\log(x) - \frac{188x^4 - 480x^3 + 540x^2 - 108x - 27}{27x}\right)$$

3.12 problem 14

Internal problem ID [5602]
Internal file name [OUTPUT/4850_Sunday_June_05_2022_03_08_05_PM_86311054/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{xy'' + 3y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 3y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = 1$$

Table 96: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 3y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + 3\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n + r - 1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n + r - 1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r)\,a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)\,(n+r-1) + 3(n+r)\,a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\,x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2+6r+8}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+6r+8}$	$-\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+6r+8}$	$-\frac{1}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(4+r)^2 (2+r) (r+6)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+6r+8}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{(4+r)^2(2+r)(r+6)}$	$\frac{1}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+6r+8}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{(4+r)^2(2+r)(r+6)}$	$\frac{1}{192}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= -\frac{1}{r^2+6r+8} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{1}{r^2+6r+8} &= \lim_{r\to -2} -\frac{1}{r^2+6r+8} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $xy'' + 3y' + xy = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + 3Cy_1'(x)\ln(x) + \frac{3Cy_1(x)}{x} + 3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + x\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left((y_1(x)\,x + y_1''(x)\,x + 3y_1'(x))\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + \frac{3y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) + 3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1(x)\,x + y_1''(x)\,x + 3y_1'(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + \frac{3y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) + 3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \frac{x^2 \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + 3\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x}{x} = 0 \tag{9}$$

Since $r_1 = 0$ and $r_2 = -2$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n} a_n n\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^{n}\right)\right) C}{x} + \frac{x^2 \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 + 3\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n\right) + \left(\sum_{n=0}^{\infty} x^{-1+n} b_n\right) + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n^2 - 5n + 6)\right) + \left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n (n-2)\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $-3+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-3+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n n = \sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n = \sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} x^{-1+n} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{-3+n}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-3+n$.

$$\left(\sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}\right) + \left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}\right) + \left(\sum_{n=2}^{\infty} b_{n-2} x^{-3+n}\right) + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n \left(n^2-5n+6\right)\right) + \left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n (n-2)\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C+1=0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n=3$, Eq (2B) gives

$$4Ca_1 + b_1 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n=4$, Eq (2B) gives

$$6Ca_2 + b_2 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 + \frac{3}{8} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{3}{64}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + b_3 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\ln(x) + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\ &= c_1\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) + c_2\left(-\frac{1}{2}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\ln(x) + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x^2}\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) + c_2\left(\left(-\frac{1}{2} + \frac{x^2}{16} - \frac{x^4}{384} - \frac{O(x^6)}{2}\right)\ln(x) + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x^2}\right)\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned}y = {}& c_1\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\\ &+ c_2\left(\left(-\frac{1}{2} + \frac{x^2}{16} - \frac{x^4}{384} - \frac{O(x^6)}{2}\right)\ln(x) + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x^2}\right)\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right) + c_2\left(\left(-\frac{1}{2}+\frac{x^2}{16}-\frac{x^4}{384}-\frac{O(x^6)}{2}\right)\ln(x)+\frac{1-\frac{3x^4}{64}+O(x^6)}{x^2}\right)$$

Verified OK.

3.12.1 Maple step by step solution

Let's solve

$y''x + 3y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} + y = 0$

- Simplify ODE

  $x^2y'' + 3xy' + yx^2 = 0$

- Make a change of variables

  $y = \frac{u(x)}{x}$

- Compute y'

  $y' = -\frac{u(x)}{x^2} + \frac{u'(x)}{x}$

- Compute y''

  $y'' = \frac{2u(x)}{x^3} - \frac{2u'(x)}{x^2} + \frac{u''(x)}{x}$

- Apply change of variables to the ODE

  $x^2u(x) + u''(x)\,x^2 + u'(x)\,x - u(x) = 0$

- ODE is now of the Bessel form
- Solution to Bessel ODE

  $u(x) = c_1 BesselJ(1, x) + c_2 BesselY(1, x)$

- Make the change from y back to y

$$y = \frac{c_1 \mathit{BesselJ}(1,x) + c_2 \mathit{BesselY}(1,x)}{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 46

```
Order:=6;
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{1}{8}x^2 + \frac{1}{192}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(x^2 - \frac{1}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-2 + \frac{3}{32}x^4 + \mathrm{O}\left(x^6\right)\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 57

```
AsymptoticDSolveValue[x*y''[x]+3*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{x^4}{192} - \frac{x^2}{8} + 1\right) + c_1 \left(\frac{1}{16}\left(x^2 - 8\right)\log(x) - \frac{5x^4 - 16x^2 - 64}{64x^2}\right)$$

3.13 problem 15

Internal problem ID [5603]
Internal file name [`OUTPUT/4851_Sunday_June_05_2022_03_08_09_PM_12831850/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' - y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = 1$$

Table 98: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x - \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 2n - 2r} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r(r+2)}$$

Which for the root $r = 2$ becomes

$$a_2 = -\frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+2)}$	$-\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+2)}$	$-\frac{1}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r(r+2)^2(4+r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+2)}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{r(r+2)^2(4+r)}$	$\frac{1}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+2)}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{r(r+2)^2(4+r)}$	$\frac{1}{192}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= x^2\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= -\frac{1}{r(r+2)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} -\frac{1}{r(r+2)} &= \lim_{r\to 0} -\frac{1}{r(r+2)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $xy'' - y' + xy = 0$ gives

$$
\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x - Cy_1'(x)\ln(x) - \frac{Cy_1(x)}{x} \\
&- \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + x\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$\begin{aligned}&\left(\left(y_1''(x)\,x+y_1(x)\,x-y_1'(x)\right)\ln(x)+\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x-\frac{y_1(x)}{x}\right)C\right.\\&+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\&\left.+x\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)-\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)=0\end{aligned}\tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x+y_1(x)\,x-y_1'(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x-\frac{y_1(x)}{x}\right)C\right.\\&+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\&\left.+x\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)-\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)=0\end{aligned}\tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x-2\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}\\&+\frac{\left(\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)+\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)\left(-1+n+r_2\right)\right)\right)x^2-\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x}{x}\\&=0\end{aligned}\tag{9}$$

Since $r_1=2$ and $r_2=0$ then the above becomes

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty}x^{1+n}a_n(n+2)\right)x-2\left(\sum_{n=0}^{\infty}a_n x^{n+2}\right)\right)C}{x}\\&+\frac{\left(\left(\sum_{n=0}^{\infty}b_n x^{n}\right)+\left(\sum_{n=0}^{\infty}x^{n-2}b_n n(-1+n)\right)\right)x^2-\left(\sum_{n=0}^{\infty}x^{-1+n}b_n n\right)x}{x}=0\end{aligned}\tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (n+2)\right) + \sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} x^{1+n} b_n\right) + \left(\sum_{n=0}^{\infty} n\, x^{-1+n} b_n (-1+n)\right) + \sum_{n=0}^{\infty} \left(-x^{-1+n} b_n n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $-1+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-1+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (n+2) = \sum_{n=2}^{\infty} 2C a_{n-2} n\, x^{-1+n}$$

$$\sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n\right) = \sum_{n=2}^{\infty} \left(-2C a_{n-2} x^{-1+n}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{-1+n}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-1+n$.

$$\left(\sum_{n=2}^{\infty} 2C a_{n-2} n\, x^{-1+n}\right) + \sum_{n=2}^{\infty} \left(-2C a_{n-2} x^{-1+n}\right) + \left(\sum_{n=2}^{\infty} b_{n-2} x^{-1+n}\right) + \left(\sum_{n=0}^{\infty} n\, x^{-1+n} b_n (-1+n)\right) + \sum_{n=0}^{\infty} \left(-x^{-1+n} b_n n\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$2C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 + b_1 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$6Ca_2 + b_2 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 + \frac{3}{8} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{3}{64}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + b_3 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right)\right)\ln(x)+1-\frac{3x^4}{64}+O(x^6)$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{1}{2}\left(x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right)\right)\ln(x)+1-\frac{3x^4}{64}+O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right)\ln(x)}{2}+1-\frac{3x^4}{64}+O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right)\ln(x)}{2}+1-\frac{3x^4}{64}+O(x^6)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right)\ln(x)}{2}+1-\frac{3x^4}{64}+O(x^6)\right)
\end{aligned}$$

Verified OK.

3.13.1 Maple step by step solution

Let's solve

$y''x - y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + y = 0$

- Simplify ODE

  $x^2y'' - xy' + yx^2 = 0$

- Make a change of variables

  $y = xu(x)$

- Compute y'

  $y' = u(x) + xu'(x)$

- Compute y''

  $y'' = 2u'(x) + xu''(x)$

- Apply change of variables to the ODE

  $u''(x)\,x^2 + x^2u(x) + xu'(x) - u(x) = 0$

- ODE is now of the Bessel form
- Solution to Bessel ODE

  $u(x) = c_1 \mathit{BesselJ}(1, x) + c_2 \mathit{BesselY}(1, x)$

- Make the change from y back to y

  $y = (c_1 \mathit{BesselJ}(1, x) + c_2 \mathit{BesselY}(1, x))\,x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 42

```
Order:=6;
dsolve(x*diff(y(x),x$2)-diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 - \frac{1}{8}x^2 + \frac{1}{192}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(x^2 - \frac{1}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-2 + \frac{3}{32}x^4 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 59

```
AsymptoticDSolveValue[x*y''[x]-y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{16}\left(x^2 - 8\right)x^2 \log(x) + \frac{1}{64}\left(-5x^4 + 16x^2 + 64\right)\right) + c_2 \left(\frac{x^6}{192} - \frac{x^4}{8} + x^2\right)$$

3.14 problem 16

Internal problem ID [5604]
Internal file name [OUTPUT/4852_Sunday_June_05_2022_03_08_12_PM_48301624/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{xy'' - 5y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - 5y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{5}{x}$$
$$q(x) = 1$$

Table 100: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - 5y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x - 5\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-5(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-5(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 5(n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 5r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 5r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-6+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-6+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 6$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-6+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^{6}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+6} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - 5a_n(n+r) + a_{n-2} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 6n - 6r} \tag{4}
$$

Which for the root $r = 6$ becomes

$$
a_n = -\frac{a_{n-2}}{n(n+6)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 6$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 - 2r - 8}$$

Which for the root $r = 6$ becomes

$$a_2 = -\frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 - 20r^2 + 64}$$

Which for the root $r = 6$ becomes

$$a_4 = \frac{1}{640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{r^4-20r^2+64}$	$\frac{1}{640}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{r^4-20r^2+64}$	$\frac{1}{640}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)}$$

Which for the root $r = 6$ becomes

$$a_6 = -\frac{1}{46080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{r^4-20r^2+64}$	$\frac{1}{640}$
a_5	0	0
a_6	$-\frac{1}{(r^4-20r^2+64)r(r+6)}$	$-\frac{1}{46080}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^6\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 \dots\right) \\
&= x^6\left(1 - \frac{x^2}{16} + \frac{x^4}{640} - \frac{x^6}{46080} + O(x^7)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_6 \\
&= -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)} &= \lim_{r\to 0} -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $xy'' - 5y' + xy = 0$ gives

$$
\begin{aligned}
&\left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \right.\\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x - 5C y_1'(x) \ln(x) - \frac{5C y_1(x)}{x} \\
&- 5\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) + x\left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(y_1''(x)\, x + y_1(x)\, x - 5 y_1'(x)\right) \ln(x) + \left(\frac{2 y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x - \frac{5 y_1(x)}{x}\right) C \\
&+ \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x \\
&+ x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) - 5\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
y_1''(x)\, x + y_1(x)\, x - 5 y_1'(x) = 0
$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - \frac{5y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) - 5\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right)x - 6\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)x^2 + x^2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) - 5\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right)x}{x} = 0 \tag{9}$$

Since $r_1 = 6$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{5+n} a_n (n+6)\right)x - 6\left(\sum_{n=0}^{\infty} a_n x^{n+6}\right)\right)C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{n-2} b_n n(-1+n)\right)x^2 + x^2\left(\sum_{n=0}^{\infty} b_n x^{n}\right) - 5\left(\sum_{n=0}^{\infty} x^{-1+n} b_n n\right)x}{x} = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{5+n} a_n (n+6)\right) + \sum_{n=0}^{\infty}\left(-6C\, x^{5+n} a_n\right) + \left(\sum_{n=0}^{\infty} n\, x^{-1+n} b_n (-1+n)\right) + \left(\sum_{n=0}^{\infty} x^{1+n} b_n\right) + \sum_{n=0}^{\infty}\left(-5x^{-1+n} b_n n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $-1+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-1+n} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{5+n} a_n (n+6) &= \sum_{n=6}^{\infty} 2C a_{-6+n} n\, x^{-1+n} \\
\sum_{n=0}^{\infty} \left(-6C\, x^{5+n} a_n\right) &= \sum_{n=6}^{\infty} \left(-6C a_{-6+n} x^{-1+n}\right) \\
\sum_{n=0}^{\infty} x^{1+n} b_n &= \sum_{n=2}^{\infty} b_{n-2} x^{-1+n}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-1+n$.

$$
\begin{aligned}
&\left(\sum_{n=6}^{\infty} 2C a_{-6+n} n\, x^{-1+n}\right) + \sum_{n=6}^{\infty} \left(-6C a_{-6+n} x^{-1+n}\right) \\
&\quad + \left(\sum_{n=0}^{\infty} n\, x^{-1+n} b_n (-1+n)\right) + \left(\sum_{n=2}^{\infty} b_{n-2} x^{-1+n}\right) + \sum_{n=0}^{\infty} \left(-5x^{-1+n} b_n n\right) = 0
\end{aligned}
\tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$
-5b_1 = 0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$
-5b_1 = 0
$$

Solving the above for b_1 gives

$$
b_1 = 0
$$

For $n=2$, Eq (2B) gives

$$
-8b_2 + b_0 = 0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$
-8b_2 + 1 = 0
$$

Solving the above for b_2 gives

$$
b_2 = \frac{1}{8}
$$

For $n = 3$, Eq (2B) gives

$$-9b_3 + b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-9b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$-8b_4 + b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-8b_4 + \frac{1}{8} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{64}$$

For $n = 5$, Eq (2B) gives

$$-5b_5 + b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

For $n = N$, where $N = 6$ which is the difference between the two roots, we are free to choose $b_6 = 0$. Hence for $n = 6$, Eq (2B) gives

$$6C + \frac{1}{64} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{384}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{384}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{384}\left(x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\right)\ln(x)+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x)\\
&= c_1x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\\
&\quad+c_2\left(-\frac{1}{384}\left(x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\right)\ln(x)+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h\\
&= c_1x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\\
&\quad+c_2\left(-\frac{x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\ln(x)}{384}+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\\
&\quad+c_2\left(-\frac{x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\ln(x)}{384}+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\\
&\quad+c_2\left(-\frac{x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\ln(x)}{384}+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned}$$

Verified OK.

3.14.1 Maple step by step solution

Let's solve

$y''x - 5y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{x} + y = 0$

- Simplify ODE

  $x^2 y'' - 5xy' + yx^2 = 0$

- Make a change of variables

  $y = x^3 u(x)$

- Compute y'

  $y' = 3x^2 u(x) + x^3 u'(x)$

- Compute y''

  $y'' = 6xu(x) + 6x^2 u'(x) + x^3 u''(x)$

- Apply change of variables to the ODE

  $x^2 u(x) + u''(x)\, x^2 + u'(x)\, x - 9u(x) = 0$

- ODE is now of the Bessel form

- Solution to Bessel ODE

  $u(x) = c_1 \mathit{BesselJ}(3, x) + c_2 \mathit{BesselY}(3, x)$

- Make the change from y back to y

  $y = (c_1 \mathit{BesselJ}(3, x) + c_2 \mathit{BesselY}(3, x))\, x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*diff(y(x),x$2)-5*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^6 \left(1 - \frac{1}{16}x^2 + \frac{1}{640}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(-86400 - 10800x^2 - 1350x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 44

```
AsymptoticDSolveValue[x*y''[x]-5*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{64} + \frac{x^2}{8} + 1\right) + c_2\left(\frac{x^{10}}{640} - \frac{x^8}{16} + x^6\right)$$

3.15 problem 17

Internal problem ID [5605]
Internal file name [`OUTPUT/4853_Sunday_June_05_2022_03_08_15_PM_51195457/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(x^2-2\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2-2\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{x^2-2}{x^2} \end{aligned}$$

Table 102: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{x^2-2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^2 - 2)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (x^2 - 2) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n + r)\, (n + r - 1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n\, =0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y_2(x) = Cy_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-2} - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - n - r - 2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r(r+3)}$$

Which for the root $r = 2$ becomes

$$a_2 = -\frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+3)}$	$-\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+3)}$	$-\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r(r+3)(5+r)(r+2)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+3)}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{r(r+3)(5+r)(r+2)}$	$\frac{1}{280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+3)}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{r(r+3)(5+r)(r+2)}$	$\frac{1}{280}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2 \left(1 - \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} 0 &= \lim_{r\rightarrow -1} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-2} - 2b_n = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + b_{n-2} - 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 - n - r - 2} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-2}}{n^2 - 3n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r(r+3)}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r(r+3)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r(r+3)}$	$\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{r(r+3)(r^2+7r+10)}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r(r+3)}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r(r+3)(5+r)(r+2)}$	$-\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r(r+3)}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r(r+3)(5+r)(r+2)}$	$-\frac{1}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^2 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \right) \\ &= \frac{1 + \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2 \left(1 - \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 \left(1 - \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2 \left(1 - \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)\right)}{x}
$$

Verified OK.

3.15.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 - 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{x^2-2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2 y'' + \left(x^2 - 2\right) y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)(-2+r)\, x^r + a_1(2+r)(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)(k+r-2) + a_{k-2}\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 2\}$$

- Each term must be 0

  $$a_1(2+r)(-1+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-2)+a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+3+r)(k+r)+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+3+r)(k+r)}$
- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_k}{(k+2)(k-1)}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, a_1=0\right]$
- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+5)(k+2)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+5)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+5)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+(x^2-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 - \frac{1}{10}x^2 + \frac{1}{280}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2 \left(12 + 6x^2 - \frac{3}{2}x^4 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 44

```
AsymptoticDSolveValue[x^2*y''[x]+(x^2-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{x^3}{8} + \frac{x}{2} + \frac{1}{x}\right) + c_2 \left(\frac{x^6}{280} - \frac{x^4}{10} + x^2\right)$$

3.16 problem 18

Internal problem ID [5606]
Internal file name [`OUTPUT/4854_Sunday_June_05_2022_03_08_17_PM_60331657/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + \left(16x^2+1\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + \left(16x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{16x^2+1}{4x^2}$$

Table 104: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{16x^2+1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + \left(16x^2 + 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + \left(16x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \qquad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n + r)\,(n + r - 1)\right) + \left(\sum_{n=0}^{\infty} 16x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad (2A)$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 16x^{n+r+2}a_n = \sum_{n=2}^{\infty} 16a_{n-2}x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 16a_{n-2}x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r(2r-1)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$4a_n(n + r)(n + r - 1) + 16a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{16a_{n-2}}{4n^2 + 8nr + 4r^2 - 4n - 4r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{4a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{16}{(2r+3)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16}{(2r+3)^2}$	-1

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16}{(2r+3)^2}$	-1
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(2r+3)^2 (2r+7)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16}{(2r+3)^2}$	-1
a_3	0	0
a_4	$\frac{256}{(2r+3)^2(2r+7)^2}$	$\frac{1}{4}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16}{(2r+3)^2}$	-1
a_3	0	0
a_4	$\frac{256}{(2r+3)^2(2r+7)^2}$	$\frac{1}{4}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(-x^2 + 1 + \frac{x^4}{4} + O(x^6)\right)
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = \frac{1}{2})$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{16}{(2r+3)^2}$	-1	$\frac{64}{(2r+3)^3}$	1
b_3	0	0	0	0
b_4	$\frac{256}{(2r+3)^2(2r+7)^2}$	$\frac{1}{4}$	$\frac{-4096r-10240}{(2r+3)^3(2r+7)^3}$	$-\frac{3}{8}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\ &= \sqrt{x}\left(-x^2 + 1 + \frac{x^4}{4} + O(x^6)\right) \ln(x) + \sqrt{x}\left(x^2 - \frac{3x^4}{8} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x}\left(-x^2 + 1 + \frac{x^4}{4} + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(-x^2 + 1 + \frac{x^4}{4} + O(x^6)\right) \ln(x) + \sqrt{x}\left(x^2 - \frac{3x^4}{8} + O(x^6)\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x}\left(-x^2 + 1 + \frac{x^4}{4} + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(-x^2 + 1 + \frac{x^4}{4} + O(x^6)\right) \ln(x) + \sqrt{x}\left(x^2 - \frac{3x^4}{8} + O(x^6)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\left(-x^2+1+\frac{x^4}{4}+O\left(x^6\right)\right) \\
& + c_2\left(\sqrt{x}\left(-x^2+1+\frac{x^4}{4}+O\left(x^6\right)\right)\ln\left(x\right)+\sqrt{x}\left(x^2-\frac{3x^4}{8}+O\left(x^6\right)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\left(-x^2+1+\frac{x^4}{4}+O\left(x^6\right)\right) \\
& + c_2\left(\sqrt{x}\left(-x^2+1+\frac{x^4}{4}+O\left(x^6\right)\right)\ln\left(x\right)+\sqrt{x}\left(x^2-\frac{3x^4}{8}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Verified OK.

3.16.1 Maple step by step solution

Let's solve

$4x^2y''+\left(16x^2+1\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(16x^2+1)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(16x^2+1)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=0, P_3(x)=\frac{16x^2+1}{4x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + (16x^2 + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + 2r)^2\, x^r + a_1(1 + 2r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k + 2r - 1)^2 + 16a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term must be 0

  $a_1(1 + 2r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k + r - \frac{1}{2}\right)^2 a_k + 16a_{k-2} = 0$

- Shift index using $k->k+2$

  $4\left(k+\frac{3}{2}+r\right)^{2} a_{k+2}+16 a_{k}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{16 a_{k}}{(2k+3+2r)^{2}}$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{16 a_{k}}{(2k+4)^{2}}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{2}}, a_{k+2}=-\frac{16 a_{k}}{(2k+4)^{2}}, a_{1}=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+(16*x^2+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\sqrt{x}\left(\left(c_{2} \ln (x)+c_{1}\right)\left(1-x^{2}+\frac{1}{4} x^{4}+\mathrm{O}\left(x^{6}\right)\right)+\left(x^{2}-\frac{3}{8} x^{4}+\mathrm{O}\left(x^{6}\right)\right) c_{2}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 69

```
AsymptoticDSolveValue[4*x^2*y''[x]+(16*x^2+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^4}{4}-x^2+1\right)+c_2\left(\sqrt{x}\left(x^2-\frac{3x^4}{8}\right)+\sqrt{x}\left(\frac{x^4}{4}-x^2+1\right)\log(x)\right)$$

3.17 problem 19

Internal problem ID [5607]
Internal file name [`OUTPUT/4855_Sunday_June_05_2022_03_08_20_PM_31829690/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + 3y' + yx^3 = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 3y' + yx^3 = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = x^2$$

Table 106: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = x^2$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 3y' + yx^3 = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + 3\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x^3 = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{3+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{3+n+r} a_n = \sum_{n=4}^{\infty} a_{n-4} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=4}^{\infty} a_{n-4} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r) (n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r} (2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r} (2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = 0$$

For $4 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_n(n+r) + a_{n-4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-4}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-4}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = -\frac{1}{r^2 + 10r + 24}$$

Which for the root $r = 0$ becomes

$$a_4 = -\frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{1}{r^2+10r+24}$	$-\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{1}{r^2+10r+24}$	$-\frac{1}{24}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 - \frac{x^4}{24} + O(x^6)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} 0 &= \lim_{r \to -2} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-2}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = 0$$

Substituting $n = 3$ in Eq(3) gives

$$b_3 = 0$$

For $4 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 3(n+r) b_n + b_{n-4} = 0 \tag{4}$$

Which for for the root $r = -2$ becomes

$$b_n(n-2)(n-3) + 3(n-2) b_n + b_{n-4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-4}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{5}$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{b_{n-4}}{n^2 - 2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = -\frac{1}{r^2 + 10r + 24}$$

Which for the root $r = -2$ becomes

$$b_4 = -\frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{1}{r^2+10r+24}$	$-\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{1}{r^2+10r+24}$	$-\frac{1}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^4}{8} + O(x^6)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 - \frac{x^4}{24} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^4}{8} + O(x^6)\right)}{x^2} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-\frac{x^4}{24}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^4}{8}+O(x^6)\right)}{x^2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\left(1-\frac{x^4}{24}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^4}{8}+O(x^6)\right)}{x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1-\frac{x^4}{24}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^4}{8}+O(x^6)\right)}{x^2}
$$

Verified OK.

3.17.1 Maple step by step solution

Let's solve

$y''x + 3y' + yx^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - yx^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} + yx^2 = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3}{x}, P_3(x) = x^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + 3y' + yx^3 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^3 \cdot y$ to series expansion

    $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

  - Shift index using $k- >k - 3$

    $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)(3+r)=0, a_2(2+r)(4+r)=0, a_3(3+r)(5+r)=0]$

- Solve for the dependent coefficient(s)

  $\{a_1=0, a_2=0, a_3=0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r+3)+a_{k-3}=0$

- Shift index using $k->k+3$

  $a_{k+4}(k+4+r)(k+6+r)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+4}=-\frac{a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r=-2$

  $a_{k+4}=-\frac{a_k}{(k+2)(k+4)}$

- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4}=-\frac{a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0\right]$

- Recursion relation for $r=0$

  $a_{k+4}=-\frac{a_k}{(k+4)(k+6)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4}=-\frac{a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{b_k}{(k+4)(k+6)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
Order:=6;
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+x^3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{1}{24}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(-2 + \frac{1}{4}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 30

```
AsymptoticDSolveValue[x*y''[x]+3*y'[x]+x^3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(1 - \frac{x^4}{24}\right) + c_1\left(\frac{1}{x^2} - \frac{x^2}{8}\right)$$

3.18 problem 20

Internal problem ID [5608]
Internal file name [OUTPUT/4856_Sunday_June_05_2022_03_08_22_PM_7104875/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' + 9xy' + (x^6 - 36)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + 9xy' + (x^6 - 36)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^6 - 36}{9x^2}$$

Table 108: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^6-36}{9x^2}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + 9xy' + (x^6 - 36)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 9x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (x^6 - 36) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+6} a_n\right) + \sum_{n=0}^{\infty} \left(-36 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+6} a_n = \sum_{n=6}^{\infty} a_{n-6} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=6}^{\infty} a_{n-6} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-36 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) + 9x^{n+r} a_n (n+r) - 36 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 9x^r a_0 r - 36 a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 9x^r r - 36x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 36\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 36 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(9r^2 - 36)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = 0$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = 0$$

Substituting $n = 5$ in Eq. (2B) gives

$$a_5 = 0$$

For $6 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) + 9a_n(n+r) + a_{n-6} - 36a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-6}}{9(n^2 + 2nr + r^2 - 4)} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-6}}{9n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= x^2(1 + O(x^6)) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -2} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-2} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = 0$$

Substituting $n = 3$ in Eq(3) gives

$$b_3 = 0$$

Substituting $n = 4$ in Eq(3) gives

$$b_4 = 0$$

Substituting $n = 5$ in Eq(3) gives

$$b_5 = 0$$

For $6 \leq n$ the recursive equation is

$$9b_n(n+r)(n+r-1) + 9b_n(n+r) + b_{n-6} - 36b_n = 0 \tag{4}$$

Which for for the root $r = -2$ becomes

$$9b_n(n-2)(n-3) + 9b_n(n-2) + b_{n-6} - 36b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-6}}{9(n^2 + 2nr + r^2 - 4)} \tag{5}$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{b_{n-6}}{9(n^2 - 4n)} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^2\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \ldots\right) \\ &= \frac{1 + O(x^6)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x^2\left(1 + O\left(x^6\right)\right) + \frac{c_2(1 + O(x^6))}{x^2} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x^2\left(1 + O\left(x^6\right)\right) + \frac{c_2(1 + O(x^6))}{x^2} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^2 \left(1 + O\left(x^6\right)\right) + \frac{c_2 (1 + O(x^6))}{x^2} \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(1 + O\left(x^6\right)\right) + \frac{c_2 (1 + O(x^6))}{x^2}$$

Verified OK.

3.18.1 Maple step by step solution

Let's solve

$9x^2 y'' + 9xy' + (x^6 - 36)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{(x^6-36)y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^6-36)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^6-36}{9x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -4$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$9x^2y'' + 9xy' + (x^6 - 36)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..6$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $9a_0(2 + r)\, (-2 + r)\, x^r + 9a_1(3 + r)\, (-1 + r)\, x^{1+r} + 9a_2(4 + r)\, r\, x^{2+r} + 9a_3(5 + r)\, (1 + r)\, x^{3+r} +$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $9(2 + r)\, (-2 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 2\}$

- The coefficients of each power of x must be 0

  $[9a_1(3 + r)\, (-1 + r) = 0, 9a_2(4 + r)\, r = 0, 9a_3(5 + r)\, (1 + r) = 0, 9a_4(6 + r)\, (2 + r) = 0, 9a_5(7 +$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $9a_k(k + r + 2)\, (k + r - 2) + a_{k-6} = 0$

- Shift index using $k- >k + 6$

  $9a_{k+6}(k + 8 + r)\, (k + 4 + r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $$a_{k+6} = -\frac{a_k}{9(k+8+r)(k+4+r)}$$

- Recursion relation for $r = -2$

  $$a_{k+6} = -\frac{a_k}{9(k+6)(k+2)}$$

- Solution for $r = -2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+6} = -\frac{a_k}{9(k+6)(k+2)}, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$$

- Recursion relation for $r = 2$

  $$a_{k+6} = -\frac{a_k}{9(k+10)(k+6)}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+6} = -\frac{a_k}{9(k+10)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+6} = -\frac{a_k}{9(k+6)(k+2)}, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, b\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)+9*x*diff(y(x),x)+(x^6-36)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2 \left(-144 + \mathrm{O}\left(x^6\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 16

```
AsymptoticDSolveValue[9*x^2*y''[x]+9*x*y'[x]+(x^6-36)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_2 x^2 + \frac{c_1}{x^2}$$

3.19 problem 22(a)

Internal problem ID [5609]
Internal file name [`OUTPUT/4857_Sunday_June_05_2022_03_08_24_PM_61357096/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 22(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - yx^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (124) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (125) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= yx^2 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x(xy' + 2y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= yx^4 + 4xy' + 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y'x^4 + 8yx^3 + 6y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12y'x^3 + x^2 y\left(x^4 + 30\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 2y(0) \\
F_3 &= 6y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20} x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-2}}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{12} a_0 x^4 + \frac{1}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^4}{12}\right) a_0 + \left(x + \frac{1}{20} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20} x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{20} x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20} x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Verified OK.

3.19.1 Maple step by step solution

Let's solve

$y'' = y x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - yx^2 = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

□ Rewrite ODE with series expansions

- ○ Convert $x^2 \cdot y$ to series expansion

$$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

- ○ Shift index using $k- >k-2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

- ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- ○ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)-x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]-x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{20} + x\right) + c_1\left(\frac{x^4}{12} + 1\right)$$

3.20 problem 22(b)

Internal problem ID [5610]
Internal file name [`OUTPUT/4858_Sunday_June_05_2022_03_08_25_PM_75345777/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 22(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' - 7yx^3 = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' - 7yx^3 = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -7x^2$$

Table 111: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -7x^2$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' - 7yx^3 = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - 7\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x^3 = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-7x^{3+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-7x^{3+n+r} a_n\right) = \sum_{n=4}^{\infty} \left(-7a_{n-4} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=4}^{\infty} \left(-7a_{n-4} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = 0$$

For $4 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) - 7a_{n-4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{7a_{n-4}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{7a_{n-4}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{7}{(4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{7}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$\frac{7}{(4+r)^2}$	$\frac{7}{16}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$\frac{7}{(4+r)^2}$	$\frac{7}{16}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 + \frac{7x^4}{16} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	0	0	0	0
b_3	0	0	0	0
b_4	$\frac{7}{(4+r)^2}$	$\frac{7}{16}$	$-\frac{14}{(4+r)^3}$	$-\frac{7}{32}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= \left(1 + \frac{7x^4}{16} + O(x^6)\right) \ln(x) - \frac{7x^4}{32} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 + \frac{7x^4}{16} + O(x^6)\right) + c_2 \left(\left(1 + \frac{7x^4}{16} + O(x^6)\right) \ln(x) - \frac{7x^4}{32} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1+\frac{7x^4}{16}+O\left(x^6\right)\right)+c_2\left(\left(1+\frac{7x^4}{16}+O\left(x^6\right)\right)\ln\left(x\right)-\frac{7x^4}{32}+O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1+\frac{7x^4}{16}+O\left(x^6\right)\right)+c_2\left(\left(1+\frac{7x^4}{16}+O\left(x^6\right)\right)\ln\left(x\right)-\frac{7x^4}{32}+O\left(x^6\right)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1+\frac{7x^4}{16}+O\left(x^6\right)\right)+c_2\left(\left(1+\frac{7x^4}{16}+O\left(x^6\right)\right)\ln\left(x\right)-\frac{7x^4}{32}+O\left(x^6\right)\right)
$$

Verified OK.

3.20.1 Maple step by step solution

Let's solve

$y''x+y'-7yx^3=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y'}{x}+7yx^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}-7yx^2=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{1}{x}, P_3(x)=-7x^2\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y' - 7yx^3 = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions

  - Convert $x^3 \cdot y$ to series expansion

    $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

  - Shift index using $k- >k - 3$

    $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r) x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r) (k + r) x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + a_1(1 + r)^2 x^r + a_2(2 + r)^2 x^{1+r} + a_3(3 + r)^2 x^{2+r} + \left(\sum_{k=3}^{\infty} \left(a_{k+1}(k + 1 + r)^2 - 7a_{k-3}\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- The coefficients of each power of x must be 0

  $\left[a_1(1+r)^2 = 0, a_2(2+r)^2 = 0, a_3(3+r)^2 = 0\right]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - 7a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4)^2 - 7a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = \frac{7a_k}{(k+4)^2}$

- Recursion relation for $r = 0$

  $a_{k+4} = \frac{7a_k}{(k+4)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{7a_k}{(k+4)^2}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)-7*x^3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1) \left(1 + \frac{7}{16}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-\frac{7}{32}x^4 + \mathrm{O}\left(x^6\right)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 39

```
AsymptoticDSolveValue[x*y''[x]+y'[x]-7*x^3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{7x^4}{16} + 1\right) + c_2 \left(\left(\frac{7x^4}{16} + 1\right) \log(x) - \frac{7x^4}{32}\right)$$

3.21 problem 23

Internal problem ID [5611]
Internal file name [`OUTPUT/4859_Sunday_June_05_2022_03_08_27_PM_30922448/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (128) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (129) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= -y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{6} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O\left(x^6\right) \tag{2}$$

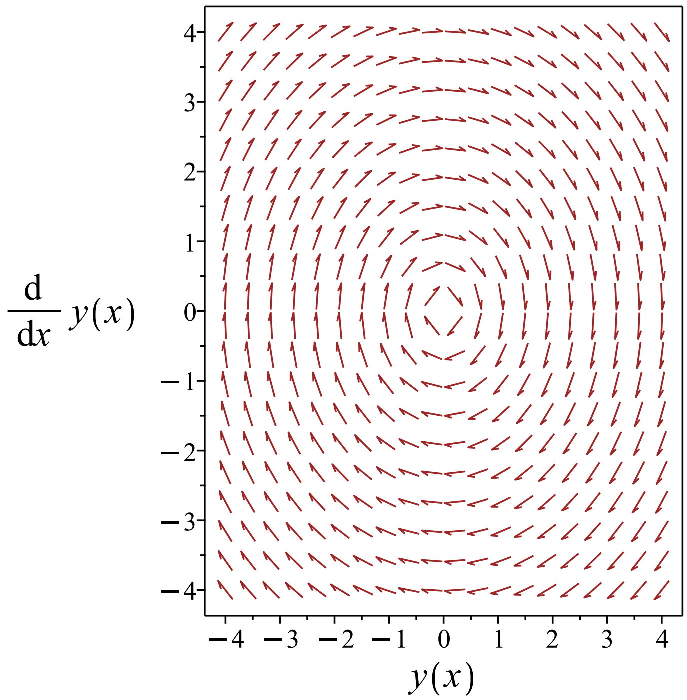

Figure 1: Slope field plot

<u>Verification of solutions</u>

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Verified OK.

3.21.1 Maple step by step solution

Let's solve

$y'' = -y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y = 0$

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = \cos(x) c_1 + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120} - \frac{x^3}{6} + x\right) + c_1\left(\frac{x^4}{24} - \frac{x^2}{2} + 1\right)$$

3.22 problem 24

Internal problem ID [5612]
Internal file name [`OUTPUT/4860_Sunday_June_05_2022_03_08_27_PM_20093763/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 4xy' + y(x^2 + 2) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 4xy' + y(x^2 + 2) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{x^2 + 2}{x^2}$$

Table 114: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2+2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + 4xy' + y(x^2 + 2) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r-2} \right) + 4x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) (x^2 + 2) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) + 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 4x^r a_0 r + 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 4x^r r + 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + 3r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 3r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 3r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{x}$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + 4a_n(n + r) + a_{n-2} + 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 + 3n + 3r + 2} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 7r + 12}$$

Which for the root $r = -1$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(4+r)(r+3)(r+6)(5+r)}$$

Which for the root $r = -1$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{(4+r)(r+3)(r+6)(5+r)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{(4+r)(r+3)(r+6)(5+r)}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)}{x} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -2} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-2} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 4b_n(n+r) + b_{n-2} + 2b_n = 0 \tag{4}$$

Which for for the root $r = -2$ becomes

$$b_n(n-2)(n-3) + 4b_n(n-2) + b_{n-2} + 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 + 3n + 3r + 2} \tag{5}$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{b_{n-2}}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 7r + 12}$$

Which for the root $r = -2$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 7r + 12)(r^2 + 11r + 30)}$$

Which for the root $r = -2$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(4+r)(r+3)(r+6)(5+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(4+r)(r+3)(r+6)(5+r)}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \frac{1}{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right)}{x} + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x^2} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= \frac{c_1\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right)}{x} + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x^2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right)}{x} + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right)}{x} + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x^2}$$

Verified OK.

3.22.1 Maple step by step solution

Let's solve

$x^2 y'' + 4xy' + y(x^2 + 2) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} - \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x)=\frac{4}{x}, P_3(x)=\frac{x^2+2}{x^2}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=4$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=2$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''+4xy'+y(x^2+2)=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(1+r)\,x^r+a_1(3+r)\,(2+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r+1)+a_{k-2}\right) x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$

- Each term must be 0

  $a_1(3+r)(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r+1) + a_{k-2} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}(k+4+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x \left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 40

```
AsymptoticDSolveValue[x^2*y''[x]+4*x*y'[x]+(x^2+2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{x^3}{120} - \frac{x}{6} + \frac{1}{x}\right) + c_1 \left(\frac{x^2}{24} + \frac{1}{x^2} - \frac{1}{2}\right)$$

3.23 problem 25

Internal problem ID [5613]
Internal file name [`OUTPUT/4861_Sunday_June_05_2022_03_08_30_PM_93011495/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{16x^2y'' + 32xy' + (x^4 - 12)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$16x^2y'' + 32xy' + (x^4 - 12)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{x^4 - 12}{16x^2}$$

Table 116: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^4-12}{16x^2}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$16x^2y'' + 32xy' + (x^4 - 12)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$16x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + 32x\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (x^4 - 12)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 32x^{n+r} a_n (n+r)\right) \\ + \left(\sum_{n=0}^{\infty} x^{n+r+4} a_n\right) + \sum_{n=0}^{\infty} \left(-12 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+4} a_n = \sum_{n=4}^{\infty} a_{n-4} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 32x^{n+r} a_n (n+r)\right) \\ + \left(\sum_{n=4}^{\infty} a_{n-4} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-12 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$16x^{n+r} a_n (n+r) (n+r-1) + 32x^{n+r} a_n (n+r) - 12 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$16x^r a_0 r(-1+r) + 32x^r a_0 r - 12 a_0 x^r = 0$$

Or

$$\left(16x^r r(-1+r) + 32x^r r - 12x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(16r^2 + 16r - 12\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$16r^2 + 16r - 12 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{3}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(16r^2 + 16r - 12\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{3}{2}}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = 0$$

For $4 \leq n$ the recursive equation is

$$16a_n(n+r)(n+r-1) + 32a_n(n+r) + a_{n-4} - 12a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-4}}{4(4n^2 + 8nr + 4r^2 + 4n + 4r - 3)} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-4}}{16n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = -\frac{1}{16r^2 + 144r + 308}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = -\frac{1}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{1}{16r^2+144r+308}$	$-\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{1}{16r^2+144r+308}$	$-\frac{1}{384}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= \sqrt{x}\left(1 - \frac{x^4}{384} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} 0 &= \lim_{r \to -\frac{3}{2}} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = 0$$

Substituting $n = 3$ in Eq(3) gives

$$b_3 = 0$$

For $4 \leq n$ the recursive equation is

$$16b_n(n+r)(n+r-1) + 32b_n(n+r) + b_{n-4} - 12b_n = 0 \tag{4}$$

Which for for the root $r = -\frac{3}{2}$ becomes

$$16b_n\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right) + 32b_n\left(n-\frac{3}{2}\right) + b_{n-4} - 12b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-4}}{4(4n^2+8nr+4r^2+4n+4r-3)} \tag{5}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_n = -\frac{b_{n-4}}{4(4n^2-8n)} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = -\frac{1}{4\left(4r^2 + 36r + 77\right)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_4 = -\frac{1}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{1}{16r^2+144r+308}$	$-\frac{1}{128}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{1}{16r^2+144r+308}$	$-\frac{1}{128}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\
&= \frac{1 - \frac{x^4}{128} + O(x^6)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1 - \frac{x^4}{384} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^4}{128} + O(x^6)\right)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1 - \frac{x^4}{384} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^4}{128} + O(x^6)\right)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x}\left(1 - \frac{x^4}{384} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^4}{128} + O(x^6)\right)}{x^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\left(1 - \frac{x^4}{384} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^4}{128} + O(x^6)\right)}{x^{\frac{3}{2}}}
$$

Verified OK.

3.23.1 Maple step by step solution

Let's solve

$16x^2y'' + 32xy' + (x^4 - 12)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{2y'}{x} - \frac{(x^4-12)y}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + \frac{(x^4-12)y}{16x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = \frac{x^4-12}{16x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{3}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2y'' + 32xy' + (x^4 - 12)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- ○ Convert $x^2 \cdot y''$ to series expansion

$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

Rewrite ODE with series expansions

$4a_0(3+2r)(-1+2r) x^r + 4a_1(5+2r)(1+2r) x^{1+r} + 4a_2(7+2r)(3+2r) x^{2+r} + 4a_3(9+2r)($

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4(3+2r)(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{1}{2}\right\}$

- The coefficients of each power of x must be 0

  $[4a_1(5+2r)(1+2r) = 0, 4a_2(7+2r)(3+2r) = 0, 4a_3(9+2r)(5+2r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $16\left(k+r+\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right) a_k + a_{k-4} = 0$

- Shift index using $k- > k+4$

  $16\left(k+\frac{11}{2}+r\right)\left(k+\frac{7}{2}+r\right) a_{k+4} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{a_k}{4(2k+11+2r)(2k+7+2r)}$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+4} = -\frac{a_k}{4(2k+8)(2k+4)}$

- Solution for $r = -\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+4} = -\frac{a_k}{4(2k+8)(2k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+4} = -\frac{a_k}{4(2k+12)(2k+8)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+4} = -\frac{a_k}{4(2k+12)(2k+8)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+4} = -\frac{a_k}{4(2k+8)(2k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{}{4(2k\cdots}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
Order:=6;
dsolve(16*x^2*diff(y(x),x$2)+32*x*diff(y(x),x)+(x^4-12)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{1}{384} x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(-2 + \frac{1}{64} x^4 + \mathrm{O}\left(x^6\right)\right)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 40

```
AsymptoticDSolveValue[16*x^2*y''[x]+32*x*y'[x]+(x^4-12)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{x^{3/2}} - \frac{x^{5/2}}{128}\right) + c_2 \left(\sqrt{x} - \frac{x^{9/2}}{384}\right)$$

3.24 problem 26

Internal problem ID [5614]
Internal file name [`OUTPUT/4862_Sunday_June_05_2022_03_08_32_PM_1535898/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Exercises. 6.3.1 page 250
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4xy' + \left(16x^4 + 3\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' - 4xy' + \left(16x^4 + 3\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{16x^4 + 3}{4x^2}$$

Table 118: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{16x^4+3}{4x^2}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' - 4xy' + \left(16x^4 + 3\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - 4x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(16x^4 + 3\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r}a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r}a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 16x^{n+r+4}a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 16x^{n+r+4}a_n = \sum_{n=4}^{\infty} 16a_{n-4}x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r}a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r}a_n(n+r)\right) + \left(\sum_{n=4}^{\infty} 16a_{n-4}x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)(n+r-1) - 4x^{n+r}a_n(n+r) + 3a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) - 4x^r a_0 r + 3a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) - 4x^r r + 3x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 8r + 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 8r + 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{3}{2} \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(4r^2 - 8r + 3)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^{\frac{3}{2}} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + \sqrt{x} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} \right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

Substituting $n = 2$ in Eq. (2B) gives

$$
a_2 = 0
$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = 0$$

For $4 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - 4a_n(n+r) + 16a_{n-4} + 3a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{16a_{n-4}}{4n^2 + 8nr + 4r^2 - 8n - 8r + 3} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{4a_{n-4}}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = -\frac{16}{4r^2 + 24r + 35}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = -\frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{16}{4r^2+24r+35}$	$-\frac{1}{5}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{16}{4r^2+24r+35}$	$-\frac{1}{5}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 - \frac{x^4}{5} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} 0 &= \lim_{r\rightarrow \frac{1}{2}} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = 0$$

Substituting $n = 3$ in Eq(3) gives

$$b_3 = 0$$

For $4 \leq n$ the recursive equation is

$$4b_n(n+r)(n+r-1) - 4b_n(n+r) + 16b_{n-4} + 3b_n = 0 \tag{4}$$

Which for for the root $r = \frac{1}{2}$ becomes

$$4b_n\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) - 4b_n\left(n+\frac{1}{2}\right) + 16b_{n-4} + 3b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{16b_{n-4}}{4n^2 + 8nr + 4r^2 - 8n - 8r + 3} \tag{5}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = -\frac{16b_{n-4}}{4n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = -\frac{16}{4r^2 + 24r + 35}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{16}{4r^2+24r+35}$	$-\frac{1}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{16}{4r^2+24r+35}$	$-\frac{1}{3}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{3}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{x^4}{3} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{3}{2}}\left(1 - \frac{x^4}{5} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^4}{3} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{3}{2}}\left(1 - \frac{x^4}{5} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^4}{3} + O(x^6)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{3}{2}}\left(1 - \frac{x^4}{5} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^4}{3} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}}\left(1 - \frac{x^4}{5} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^4}{3} + O(x^6)\right)$$

Verified OK.

3.24.1 Maple step by step solution

Let's solve

$4x^2y'' - 4xy' + (16x^4 + 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - \frac{(16x^4+3)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{(16x^4+3)y}{4x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{16x^4+3}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{3}{4}$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4xy' + (16x^4 + 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..4$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+2r)(-3+2r)x^r + a_1(1+2r)(-1+2r)x^{1+r} + a_2(3+2r)(1+2r)x^{2+r} + a_3(5+2r)(3$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- The coefficients of each power of x must be 0

  $[a_1(1+2r)(-1+2r) = 0, a_2(3+2r)(1+2r) = 0, a_3(5+2r)(3+2r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right)a_k + 16a_{k-4} = 0$

- Shift index using $k- >k+4$

  $4\left(k+\frac{7}{2}+r\right)\left(k+\frac{5}{2}+r\right)a_{k+4} + 16a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{16a_k}{(2k+7+2r)(2k+5+2r)}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+4} = -\frac{16a_k}{(2k+8)(2k+6)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+4} = -\frac{16a_k}{(2k+8)(2k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+4} = -\frac{16a_k}{(2k+10)(2k+8)}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+4} = -\frac{16a_k}{(2k+10)(2k+8)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+4} = -\frac{16a_k}{(2k+8)(2k+6)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{1}{(2k+1}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(16*x^4+3)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left(x\left(1-\frac{1}{5}x^4+\mathrm{O}\left(x^6\right)\right)c_1+\left(1-\frac{1}{3}x^4+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 40

```
AsymptoticDSolveValue[4*x^2*y''[x]-4*x*y'[x]+(16*x^4+3)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\sqrt{x}-\frac{x^{9/2}}{3}\right)+c_2\left(x^{3/2}-\frac{x^{11/2}}{5}\right)$$

4 Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Chapter 6 review exercises. page 253

4.1 problem 9

Internal problem ID [5615]
Internal file name [`OUTPUT/4863_Sunday_June_05_2022_03_08_35_PM_845170/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Chapter 6 review exercises. page 253
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{2xy'' + y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{1}{2x}$$

Table 120: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) + a_n(n + r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{1}{1247400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r+1}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$
a_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{1247400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \le n$ the recursive equation is

$$
2b_n(n + r)(n + r - 1) + (n + r)b_n + b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{b_{n-1}}{n(2n - 1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{1}{2r^2 + 3r + 1}
$$

Which for the root $r = 0$ becomes

$$
b_1 = -1
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{1}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r+1}$	-1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$
b_5	$-\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{1}{113400}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+O(x^6)\right) \\
&\quad + c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+O(x^6)\right) \\
&\quad + c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+O(x^6)\right) \\
&\quad + c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+O(x^6)\right) \\
&\quad + c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+O(x^6)\right)
\end{aligned}
$$

Verified OK.

4.1.1 Maple step by step solution

Let's solve

$2y''x + y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x} - \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} + \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{1}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1}(k+1+r)\,(2k+1+2r) + a_k)\, x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{(2k+1)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(2k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{a_k}{(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{a_k}{(2k+1)(k+1)}, b_{k+1}=-\frac{b_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1\sqrt{x}\left(1-\frac{1}{3}x+\frac{1}{30}x^2-\frac{1}{630}x^3+\frac{1}{22680}x^4-\frac{1}{1247400}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(1-x+\frac{1}{6}x^2-\frac{1}{90}x^3+\frac{1}{2520}x^4-\frac{1}{113400}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 83

```
AsymptoticDSolveValue[2*x*y''[x]+y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\sqrt{x}\left(-\frac{x^5}{1247400}+\frac{x^4}{22680}-\frac{x^3}{630}+\frac{x^2}{30}-\frac{x}{3}+1\right)+c_2\left(-\frac{x^5}{113400}+\frac{x^4}{2520}-\frac{x^3}{90}+\frac{x^2}{6}-x+1\right)$$

4.2 problem 10

Internal problem ID [5616]
Internal file name [OUTPUT/4864_Sunday_June_05_2022_03_08_37_PM_40457729/index.tex]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Chapter 6 review exercises. page 253
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second__order__integrable__as__is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - xy' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (135) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (136) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') &(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f &(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 &(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= xy' + y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x^2 y' + xy + 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 + 5x\right) y' + y\left(x^2 + 3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 + 9x^2 + 8\right) y' + yx\left(x^2 + 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^5 + 14x^3 + 33x\right) y' + y\left(x^4 + 12x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= 2y'(0) \\
F_2 &= 3y(0) \\
F_3 &= 8y'(0) \\
F_4 &= 15y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6\right) y(0) + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6\right) y(0) + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6\right) y(0) + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6)$$

Verified OK.

4.2.1 Maple step by step solution

Let's solve

$y'' = xy' + y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - xy' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$

  - Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) - a_k(k+1)) \, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)(a_{k+2}(k+2) - a_k) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{2}x^2+\frac{1}{8}x^4\right)y(0)+\left(x+\frac{1}{3}x^3+\frac{1}{15}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]-x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{15}+\frac{x^3}{3}+x\right)+c_1\left(\frac{x^4}{8}+\frac{x^2}{2}+1\right)$$

4.3 problem 11

Internal problem ID [5617]
Internal file name [`OUTPUT/4865_Sunday_June_05_2022_03_08_39_PM_72663456/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Chapter 6 review exercises. page 253
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(x-1)\,y''+3y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{138}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{139}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned} F_0 &= f(x,y,y') \qquad (4)\\ F_1 &= \frac{df}{dx}\\ &= \frac{dF_0}{dx}\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\ F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\ &= \frac{d}{dx}(F_1)\\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\ &\vdots\\ F_n &= \frac{d}{dx}(F_{n-1})\\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6) \end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3y}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(-3x+3)\, y' + 3y}{(x-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(6x-6)\, y' + (9x-15)\, y}{(x-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(9x^2-36x+27)\, y' + (-36x+54)\, y}{(x-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-54x^2+180x-126)\, y' - 27y\left(x^2-8x+\frac{29}{3}\right)}{(x-1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 3y(0) \\
F_1 &= 3y(0) + 3y'(0) \\
F_2 &= 15y(0) + 6y'(0) \\
F_3 &= 54y(0) + 27y'(0) \\
F_4 &= 261y(0) + 126y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1+\frac{3}{2}x^{2}+\frac{1}{2}x^{3}+\frac{5}{8}x^{4}+\frac{9}{20}x^{5}+\frac{29}{80}x^{6}\right) y(0) \\
&+\left(x+\frac{1}{2}x^{3}+\frac{1}{4}x^{4}+\frac{9}{40}x^{5}+\frac{7}{40}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x-1)\, y''+3y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x-1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+3\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right)+\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} 3 a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) &= \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n \\
\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) &= \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2} (n+1)\, x^n\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^{n}\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(n+1)\, x^{n}\right) + \left(\sum_{n=0}^{\infty} 3 a_{n} x^{n}\right) = 0 \quad (3)$$

$n = 0$ gives

$$-2a_2 + 3a_0 = 0$$

$$a_2 = \frac{3a_0}{2}$$

For $1 \le n$, the recurrence equation is

$$(n+1)\, a_{n+1} n - (n+2)\, a_{n+2}(n+1) + 3a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_{n+1} + n a_{n+1} + 3a_n}{(n+2)\,(n+1)} \\ &= \frac{3a_n}{(n+2)\,(n+1)} + \frac{(n^2+n)\, a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \quad (5)$$

For $n = 1$ the recurrence equation gives

$$2a_2 - 6a_3 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{2} + \frac{a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$6a_3 - 12a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{8} + \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$12a_4 - 20a_5 + 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{9a_0}{20} + \frac{9a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$20a_5 - 30a_6 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{29a_0}{80} + \frac{7a_1}{40}$$

For $n = 5$ the recurrence equation gives

$$30a_6 - 42a_7 + 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{163a_0}{560} + \frac{79a_1}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{3a_0 x^2}{2} + \left(\frac{a_0}{2} + \frac{a_1}{2}\right) x^3 + \left(\frac{5a_0}{8} + \frac{a_1}{4}\right) x^4 + \left(\frac{9a_0}{20} + \frac{9a_1}{40}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \frac{9}{20}x^5\right) a_0 + \left(x + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{9}{40}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \frac{9}{20}x^5\right)c_1 + \left(x + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{9}{40}x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \frac{9}{20}x^5 + \frac{29}{80}x^6\right)y(0) \\ &+ \left(x + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{9}{40}x^5 + \frac{7}{40}x^6\right)y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$y = \left(1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \frac{9}{20}x^5\right)c_1 + \left(x + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{9}{40}x^5\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$\begin{aligned} y = &\left(1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \frac{9}{20}x^5 + \frac{29}{80}x^6\right)y(0) \\ &+ \left(x + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{9}{40}x^5 + \frac{7}{40}x^6\right)y'(0) + O(x^6) \end{aligned}$$

Verified OK.

$$y = \left(1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \frac{9}{20}x^5\right)c_1 + \left(x + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \frac{9}{40}x^5\right)c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve((x-1)*diff(y(x),x$2)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{3}{2}x^2+\frac{1}{2}x^3+\frac{5}{8}x^4+\frac{9}{20}x^5\right)y(0)+\left(x+\frac{1}{2}x^3+\frac{1}{4}x^4+\frac{9}{40}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 63

```
AsymptoticDSolveValue[(x-1)*y''[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{9x^5}{40}+\frac{x^4}{4}+\frac{x^3}{2}+x\right)+c_1\left(\frac{9x^5}{20}+\frac{5x^4}{8}+\frac{x^3}{2}+\frac{3x^2}{2}+1\right)$$

4.4 problem 12

Internal problem ID [5618]
Internal file name [`OUTPUT/4866_Sunday_June_05_2022_03_08_40_PM_97857263/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Chapter 6 review exercises. page 253
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - x^2y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (141) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (142) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= x^2 y' - xy \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^4 + x\right) y' + \left(-x^3 - 1\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= x^2 \left(x^3 + 4\right) \left(-y + xy'\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^8 + 9x^5 + 8x^2\right) y' + \left(-x^7 - 9x^4 - 8x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^9 + 16x^6 + 44x^3 + 8\right) \left(-y + xy'\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= -8y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6\right) y(0) + xy'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=x^2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-x\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-n\, x^{1+n} a_n\right)+\left(\sum_{n=0}^{\infty} x^{1+n} a_n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=1}^{\infty}\left(-n\, x^{1+n} a_n\right)=\sum_{n=2}^{\infty}\left(-(n-1)\, a_{n-1} x^n\right)$$
$$\sum_{n=0}^{\infty} x^{1+n} a_n=\sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right)+\sum_{n=2}^{\infty}\left(-(n-1)\, a_{n-1} x^n\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^n\right)=0 \tag{3}$$

$n = 1$ gives

$$6a_3 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - (n-1)\, a_{n-1} + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-1}(n-2)}{(n+2)\,(1+n)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{90}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{6}\right) a_0 + a_1 x + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + c_2 x + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6\right) y(0) + x y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + c_2 x + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6\right) y(0) + x y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + c_2 x + O(x^6)$$

Verified OK.

4.4.1 Maple step by step solution

Let's solve

$y'' = x^2 y' - xy$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - x^2 y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  - Shift index using $k- >k - 1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_{k-1}(k-2)\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2} - a_{k-1}(k-2) = 0$

- Shift index using $k->k+1$

  $\left((k+1)^2+3k+5\right) a_{k+3} - a_k(k-1) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k(k-1)}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
Order:=6;
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{x^3}{6}\right) y(0) + D(y)\,(0)\,x + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 20

```
AsymptoticDSolveValue[y''[x]-x^2*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(1 - \frac{x^3}{6}\right) + c_2 x$$

4.5 problem 13

Internal problem ID [5619]
Internal file name [`OUTPUT/4867_Sunday_June_05_2022_03_08_41_PM_89602476/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Chapter 6 review exercises. page 253
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{xy'' - (x+2)\,y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (-x-2)\,y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x+2}{x}$$
$$q(x) = \frac{2}{x}$$

Table 124: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-x - 2)\, y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + (-x-2)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-3+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 2a_n(n+r) + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n+r} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = \frac{a_{n-1}}{n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{1+r}$$

Which for the root $r = 3$ becomes

$$a_1 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)(2+r)}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{1}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(1+r)(2+r)(3+r)}$$

Which for the root $r = 3$ becomes

$$a_3 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{20}$
a_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{120}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(2+r)(3+r)(4+r)(1+r)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{20}$
a_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{120}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{840}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$$

Which for the root $r = 3$ becomes

$$a_5 = \frac{1}{6720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{4}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{20}$
a_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{120}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{840}$
a_5	$\frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$	$\frac{1}{6720}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^3 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= \frac{1}{(1+r)(2+r)(3+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1}{(1+r)(2+r)(3+r)} &= \lim_{r \to 0} \frac{1}{(1+r)(2+r)(3+r)} \\ &= \frac{1}{6} \end{aligned}$$

The limit is $\frac{1}{6}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^n
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) - 2(n+r)b_n + 2b_{n-1} = 0 \tag{4}
$$

Which for for the root $r = 0$ becomes

$$
b_n n(n-1) - b_{n-1}(n-1) - 2nb_n + 2b_{n-1} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = \frac{b_{n-1}}{n+r} \tag{5}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{b_{n-1}}{n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{1}{1+r}
$$

Which for the root $r = 0$ becomes

$$
b_1 = 1
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{(1+r)(2+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(2+r)(3+r)(4+r)(1+r)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(1+r)(2+r)(3+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(1+r)}$	$\frac{1}{24}$
b_5	$\frac{1}{(3+r)(4+r)(1+r)(2+r)(5+r)}$	$\frac{1}{120}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^3 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O\left(x^6\right)\right) \\ &+ c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O\left(x^6\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^3 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O\left(x^6\right)\right) \\ &+ c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O\left(x^6\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 x^3 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O\left(x^6\right)\right) \\ + c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O\left(x^6\right)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = c_1 x^3 \left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O\left(x^6\right)\right) \\ + c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O\left(x^6\right)\right) \end{aligned}$$

Verified OK.

4.5.1 Maple step by step solution

Let's solve

$y''x + (-x-2)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x} + \frac{(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+2)y'}{x} + \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x+2}{x}, P_3(x) = \frac{2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (-x-2)\, y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

$$x^{m} \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_{0} r(-3+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-2) - a_{k}(k+r-2)\right) x^{k+r}\right) = 0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)\,(a_{k+1}(k+1+r) - a_{k}) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_{k}}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_{k}}{k+1}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_{k} x^{k}, a_{k+1} = \frac{a_{k}}{k+1}\right]$$

- Recursion relation for $r = 3$

$$a_{k+1} = \frac{a_{k}}{k+4}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_{k} x^{k+3}, a_{k+1} = \frac{a_{k}}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_{k} x^{k}\right) + \left(\sum_{k=0}^{\infty} b_{k} x^{k+3}\right), a_{k+1} = \frac{a_{k}}{k+1}, b_{k+1} = \frac{b_{k}}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 44

```
Order:=6;
dsolve(x*diff(y(x),x$2)-(x+2)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 \left(1 + \frac{1}{4}x + \frac{1}{20}x^2 + \frac{1}{120}x^3 + \frac{1}{840}x^4 + \frac{1}{6720}x^5 + O\left(x^6\right)\right) + c_2\left(12 + 12x + 6x^2 + 2x^3 + \frac{1}{2}x^4 + \frac{1}{10}x^5 + O\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 66

```
AsymptoticDSolveValue[x*y''[x]-(x+2)*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right) + c_2\left(\frac{x^7}{840} + \frac{x^6}{120} + \frac{x^5}{20} + \frac{x^4}{4} + x^3\right)$$

4.6 problem 14

Internal problem ID [5620]
Internal file name [`OUTPUT/4868_Sunday_June_05_2022_03_08_44_PM_35553539/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Chapter 6 review exercises. page 253
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\cos(x)\, y'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (145)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (146)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y}{\cos(x)} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -\sec(x)\,(y \tan(x) + y') \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2\sec(x)\left(\tan(x)\, y' + \left(\sec(x) + \frac{1}{2}\right) y(\sec(x) - 1)\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\left(-6\sec(x)^2 + \sec(x) + 3\right) y' + \sec(x)^2 y \tan(x)\left(\cos(x)^2 + 4\cos(x) - 6\right)\right)\sec(x) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -24\left(-\frac{\sec(x)^2 \tan(x)\left(\cos(x)^2 + \frac{3\cos(x)}{2} - 6\right) y'}{6} + y\left(\sec(x)^4 - \frac{3\sec(x)^3}{4} - \frac{19\sec(x)^2}{24} + \frac{11\sec(}{24}\right.\right.
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= 0 \\
F_3 &= -2y'(0) \\
F_4 &= y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\frac{\sum_{n=0}^{\infty} a_n x^n}{\cos(x)} \tag{1}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the first term in (1) gives

$$\begin{aligned}&1 \cdot \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - \frac{x^2}{2} \cdot \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \frac{x^4}{24} \\ &\cdot \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - \frac{x^6}{720} \cdot \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\sum_{n=2}^{\infty}\left(-\frac{n\, x^{n+4} a_n (n-1)}{720}\right) + \left(\sum_{n=2}^{\infty} \frac{n\, x^{n+2} a_n (n-1)}{24}\right) \\ &+ \sum_{n=2}^{\infty}\left(-\frac{n a_n x^n (n-1)}{2}\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0\end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-\frac{n\, x^{n+4} a_n (n-1)}{720}\right) = \sum_{n=6}^{\infty} \left(-\frac{(n-4)\, a_{n-4} (n-5)\, x^n}{720}\right)$$
$$\sum_{n=2}^{\infty} \frac{n\, x^{n+2} a_n (n-1)}{24} = \sum_{n=4}^{\infty} \frac{(n-2)\, a_{n-2} (n-3)\, x^n}{24}$$
$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=6}^{\infty} \left(-\frac{(n-4)\, a_{n-4} (n-5)\, x^n}{720}\right) + \left(\sum_{n=4}^{\infty} \frac{(n-2)\, a_{n-2} (n-3)\, x^n}{24}\right) + \sum_{n=2}^{\infty} \left(-\frac{n a_n x^n (n-1)}{2}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

$n = 3$ gives

$$-2a_3 + 20a_5 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{a_1}{3} + 20a_5 = 0$$

Or

$$a_5 = -\frac{a_1}{60}$$

$n = 4$ gives

$$\frac{a_2}{12} - 5a_4 + 30a_6 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{a_0}{24} + 30a_6 = 0$$

Or

$$a_6 = \frac{a_0}{720}$$

$n = 5$ gives

$$\frac{a_3}{4} - 9a_5 + 42a_7 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{13a_1}{120} + 42a_7 = 0$$

Or

$$a_7 = -\frac{13a_1}{5040}$$

For $6 \leq n$, the recurrence equation is

$$-\frac{(n-4)\,a_{n-4}(n-5)}{720} + \frac{(n-2)\,a_{n-2}(n-3)}{24} - \frac{na_n(n-1)}{2} + (n+2)\,a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{360n^2a_n + n^2a_{n-4} - 30n^2a_{n-2} - 360na_n - 9na_{n-4} + 150na_{n-2} - 720a_n + 20a_{n-4} - 180a_{n-2}}{720(n+2)(n+1)}$$

$$= \frac{(360n^2 - 360n - 720)a_n}{720(n+2)(n+1)} + \frac{(n^2 - 9n + 20)a_{n-4}}{720(n+2)(n+1)} + \frac{(-30n^2 + 150n - 180)a_{n-2}}{720(n+2)(n+1)} \quad (5)$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{6}a_1x^3 - \frac{1}{60}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(-\frac{x^2}{2} + 1\right)a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(-\frac{x^2}{2} + 1\right)c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6\right)y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)y'(0) + O(x^6) \quad (1)$$

$$y = \left(-\frac{x^2}{2} + 1\right)c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)c_2 + O(x^6) \quad (2)$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6\right)y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)y'(0) + O(x^6)$$

Verified OK.

$$y = \left(-\frac{x^2}{2} + 1\right)c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0,
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      u(t)-t^2*diff(u(t),t)+(-t^3+t)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
Order:=6;
dsolve(cos(x)*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^2}{2}\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 35

```
AsymptoticDSolveValue[Cos[x]*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(1 - \frac{x^2}{2}\right) + c_2\left(-\frac{x^5}{60} - \frac{x^3}{6} + x\right)$$

4.7 problem 15

Internal problem ID [5621]
Internal file name [`OUTPUT/4869_Sunday_June_05_2022_03_08_46_PM_69524103/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Chapter 6 review exercises. page 253
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + xy' + 2y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = -2]$$

With the expansion point for the power series method at $x = 0$.

4.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + xy' + 2y = 0$$

The domain of $p(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (148) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (149) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') &(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f &(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 &(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -xy' - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x^2 y' + 2xy - 3y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -y'x^3 - 2yx^2 + 7xy' + 8y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 12x^2 + 15\right) y' + 2\left(x^3 - 9x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 18x^3 - 57x\right) y' - 2y\left(x^4 - 15x^2 + 24\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=3$ and $y'(0)=-2$ gives

$$
\begin{aligned}
F_0 &= -6 \\
F_1 &= 6 \\
F_2 &= 24 \\
F_3 &= -30 \\
F_4 &= -144
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x^4 + x^3 - 3x^2 - 2x + 3 - \frac{x^5}{4} - \frac{x^6}{5} + O\left(x^6\right)
$$

$$
y = x^4 + x^3 - 3x^2 - 2x + 3 - \frac{x^5}{4} - \frac{x^6}{5} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n\,=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + 2 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+1} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 7a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{48}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{2} a_1 x^3 + \frac{1}{3} a_0 x^4 + \frac{1}{8} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) a_0 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) c_2 + O(x^6)$$

$$y = x^4 - 3x^2 + 3 - 2x + x^3 - \frac{x^5}{4} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x^4 + x^3 - 3x^2 - 2x + 3 - \frac{x^5}{4} - \frac{x^6}{5} + O(x^6) \tag{1}$$

$$y = x^4 - 3x^2 + 3 - 2x + x^3 - \frac{x^5}{4} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x^4 + x^3 - 3x^2 - 2x + 3 - \frac{x^5}{4} - \frac{x^6}{5} + O(x^6)$$

Verified OK.

$$y = x^4 - 3x^2 + 3 - 2x + x^3 - \frac{x^5}{4} + O(x^6)$$

Verified OK.

4.7.2 Maple step by step solution

Let's solve

$\left[y'' = -xy' - 2y, y(0) = 3, y'\Big|_{\{x=0\}} = -2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + xy' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_k(k+2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)(ka_{k+2}+a_k+a_{k+2})=0$

- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(0) = 3, D(y)(0) = -2],y(x),type='series',x=
```

$$y(x)=3-2x-3x^2+x^3+x^4-\frac{1}{4}x^5+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 26

```
AsymptoticDSolveValue[{y''[x]+x*y'[x]+2*y[x]==0,{y[0]==3,y'[0]==-2}},y[x],{x,0,5}]
```

$$y(x)\to -\frac{x^5}{4}+x^4+x^3-3x^2-2x+3$$

4.8 problem 16

Internal problem ID [5622]
Internal file name [`OUTPUT/4870_Sunday_June_05_2022_03_08_48_PM_8462272/index.tex`]

Book: A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications. Dennis G. Zill. 9th edition. Brooks/Cole. CA, USA.
Section: Chapter 6. SERIES SOLUTIONS OF LINEAR EQUATIONS. Chapter 6 review exercises. page 253
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(x+2)\, y'' + 3y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

4.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{3}{x+2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{3y}{x+2} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{3}{x+2}$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (151) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (152) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3y}{x+2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(-3x-6)\, y' + 3y}{(x+2)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(6x+12)\, y' + (9x+12)\, y}{(x+2)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{9x^2 y' + 18xy' - 36xy - 54y}{(x+2)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-54x^2 - 144x - 72)\, y' - 27y\left(x^2 - 2x - \frac{16}{3}\right)}{(x+2)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -\frac{3}{2} \\
F_2 &= \frac{3}{2} \\
F_3 &= 0 \\
F_4 &= -\frac{9}{4}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = x - \frac{x^3}{4} + \frac{x^4}{16} - \frac{x^6}{320} + O(x^6)$$

$$y = x - \frac{x^3}{4} + \frac{x^4}{16} - \frac{x^6}{320} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x+2)\, y'' + 3y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x+2)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2} (n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2} (n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} 3 a_n x^n\right) = 0 \quad (3)$$

$n = 0$ gives

$$4a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{4}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n + 2(n+2)\, a_{n+2}(n+1) + 3a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{n+1} + n a_{n+1} + 3a_n}{2\,(n+2)\,(n+1)} \\ &= -\frac{3a_n}{2\,(n+2)\,(n+1)} - \frac{(n^2+n)\, a_{n+1}}{2\,(n+2)\,(n+1)} \end{aligned} \quad (5)$$

For $n = 1$ the recurrence equation gives

$$2a_2 + 12a_3 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{8} - \frac{a_1}{4}$$

For $n = 2$ the recurrence equation gives

$$6a_3 + 24a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{16} + \frac{a_1}{16}$$

For $n = 3$ the recurrence equation gives

$$12a_4 + 40a_5 + 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{9a_0}{320}$$

For $n = 4$ the recurrence equation gives

$$20a_5 + 60a_6 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{160} - \frac{a_1}{320}$$

For $n = 5$ the recurrence equation gives

$$30a_6 + 84a_7 + 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{11a_0}{8960} + \frac{a_1}{896}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{3a_0x^2}{4} + \left(\frac{a_0}{8} - \frac{a_1}{4}\right)x^3 + \left(\frac{a_0}{16} + \frac{a_1}{16}\right)x^4 - \frac{9a_0x^5}{320} + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 - \frac{9}{320}x^5\right)a_0 + \left(x - \frac{1}{4}x^3 + \frac{1}{16}x^4\right)a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 - \frac{9}{320}x^5\right) c_1 + \left(x - \frac{1}{4}x^3 + \frac{1}{16}x^4\right) c_2 + O(x^6)$$

$$y = x - \frac{x^3}{4} + \frac{x^4}{16} + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = x - \frac{x^3}{4} + \frac{x^4}{16} - \frac{x^6}{320} + O(x^6) \tag{1}$$

$$y = x - \frac{x^3}{4} + \frac{x^4}{16} + O(x^6) \tag{2}$$

<u>Verification of solutions</u>

$$y = x - \frac{x^3}{4} + \frac{x^4}{16} - \frac{x^6}{320} + O(x^6)$$

Verified OK.

$$y = x - \frac{x^3}{4} + \frac{x^4}{16} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([(x+2)*diff(y(x),x$2)+3*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='series',x=0);
```

$$y(x) = x - \frac{1}{4}x^3 + \frac{1}{16}x^4 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{(x+2)*y''[x]+3*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^4}{16} - \frac{x^3}{4} + x$$