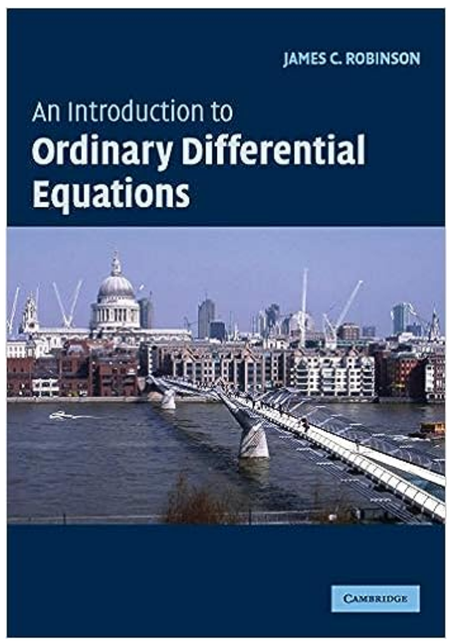
A Solution Manual For

AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004



Nasser M. Abbasi

May 15, 2024

Contents

1 Chapter 5, Trivial differential equations. Exercises page 33

1.1 problem 5.1 (i)

Internal problem ID [11968]
Internal file name [`OUTPUT/10620_Saturday_September_02_2023_02_48_34_PM_75656588/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 5, Trivial differential equations. Exercises page 33
Problem number: 5.1 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' = \cos(t) + \sin(t)}$$

1.1.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} x &= \int \cos(t) + \sin(t)\, dt \\ &= \sin(t) - \cos(t) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$x = \sin(t) - \cos(t) + c_1 \tag{1}$$

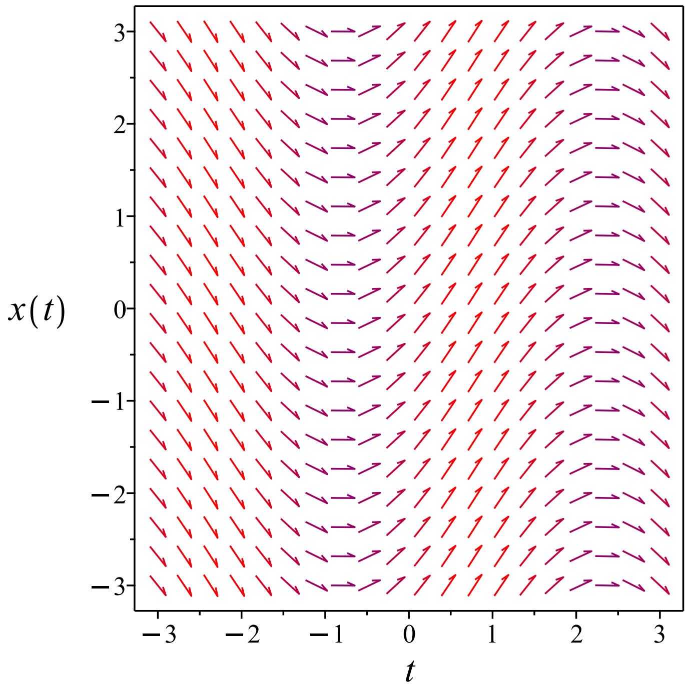


Figure 1: Slope field plot

Verification of solutions

$$x = \sin(t) - \cos(t) + c_1$$

Verified OK.

1.1.2 Maple step by step solution

Let's solve

$x' = \cos(t) + \sin(t)$

- Highest derivative means the order of the ODE is 1

  x'

- Integrate both sides with respect to t

  $\int x' dt = \int (\cos(t) + \sin(t))\, dt + c_1$

- Evaluate integral

  $x = \sin(t) - \cos(t) + c_1$

- Solve for x

  $x = \sin(t) - \cos(t) + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(x(t),t)=sin(t)+cos(t),x(t), singsol=all)
```

$$x(t) = -\cos(t) + \sin(t) + c_1$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 14

```
DSolve[x'[t]==Sin[t]+Cos[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \sin(t) - \cos(t) + c_1$$

1.2 problem 5.1 (ii)

Internal problem ID [11969]
Internal file name [OUTPUT/10621_Saturday_September_02_2023_02_48_35_PM_99659885/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 5, Trivial differential equations. Exercises page 33
Problem number: 5.1 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = \frac{1}{x^2 - 1}}$$

1.2.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x^2 - 1}\,\mathrm{d}x \\ &= -\operatorname{arctanh}(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\operatorname{arctanh}(x) + c_1 \tag{1}$$

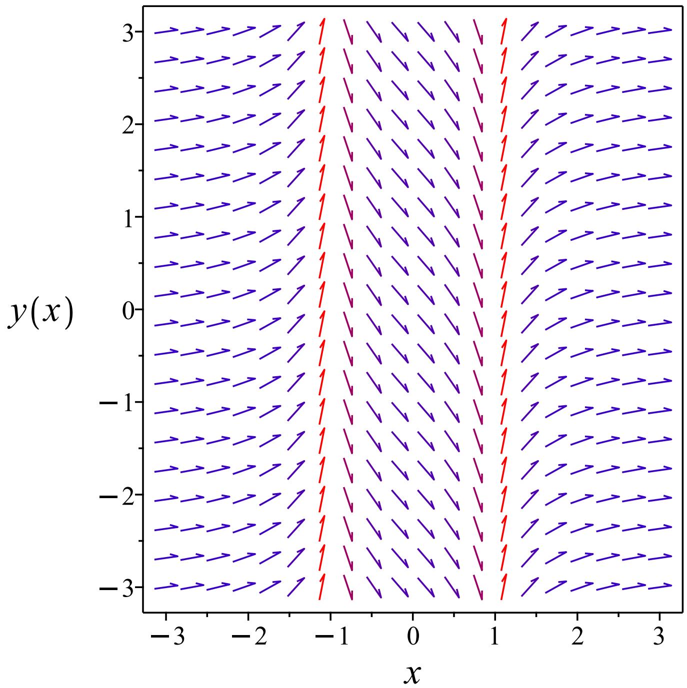


Figure 2: Slope field plot

Verification of solutions

$$y = -\operatorname{arctanh}(x) + c_1$$

Verified OK.

1.2.2 Maple step by step solution

Let's solve

$y' = \frac{1}{x^2-1}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{x^2-1} dx + c_1$

- Evaluate integral

  $y = -\operatorname{arctanh}(x) + c_1$

- Solve for y

$$y = -\operatorname{arctanh}(x) + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=1/(x^2-1),y(x), singsol=all)
```

$$y(x) = -\operatorname{arctanh}(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 26

```
DSolve[y'[x]==1/(x^2-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(\log(1-x) - \log(x+1) + 2c_1)$$

1.3 problem 5.1 (iii)

Internal problem ID [11970]
Internal file name [`OUTPUT/10622_Saturday_September_02_2023_02_48_35_PM_21740883/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 5, Trivial differential equations. Exercises page 33
Problem number: 5.1 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{u' = 4t \ln(t)}$$

1.3.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} u &= \int 4t \ln(t)\, \mathrm{d}t \\ &= 2t^2 \ln(t) - t^2 + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$u = 2t^2 \ln(t) - t^2 + c_1 \tag{1}$$

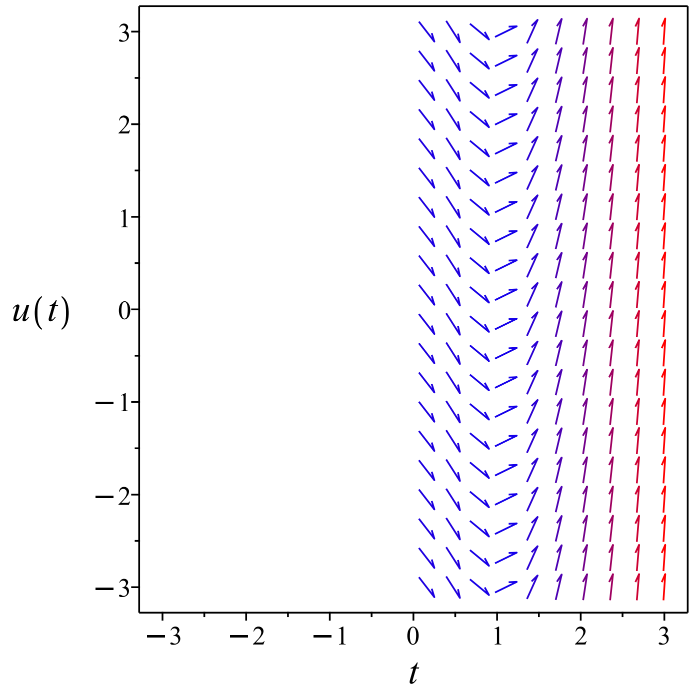


Figure 3: Slope field plot

Verification of solutions

$$u = 2t^2 \ln(t) - t^2 + c_1$$

Verified OK.

1.3.2 Maple step by step solution

Let's solve

$u' = 4t \ln(t)$

- Highest derivative means the order of the ODE is 1

  u'

- Integrate both sides with respect to t

  $\int u' dt = \int 4t \ln(t)\, dt + c_1$

- Evaluate integral

  $u = 2t^2 \ln(t) - t^2 + c_1$

- Solve for u

  $u = 2t^2 \ln(t) - t^2 + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(u(t),t)=4*t*ln(t),u(t), singsol=all)
```

$$u(t) = 2\ln(t)\, t^2 - t^2 + c_1$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 20

```
DSolve[u'[t]==4*t*Log[t],u[t],t,IncludeSingularSolutions -> True]
```

$$u(t) \to -t^2 + 2t^2 \log(t) + c_1$$

1.4 problem 5.1 (iv)

Internal problem ID [11971]
Internal file name [`OUTPUT/10623_Saturday_September_02_2023_02_48_36_PM_76338301/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 5, Trivial differential equations. Exercises page 33
Problem number: 5.1 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{z' = x\,\mathrm{e}^{-2x}}$$

1.4.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} z &= \int x\,\mathrm{e}^{-2x}\,\mathrm{d}x \\ &= -\frac{(2x+1)\,\mathrm{e}^{-2x}}{4} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$z = -\frac{(2x+1)\,\mathrm{e}^{-2x}}{4} + c_1 \tag{1}$$

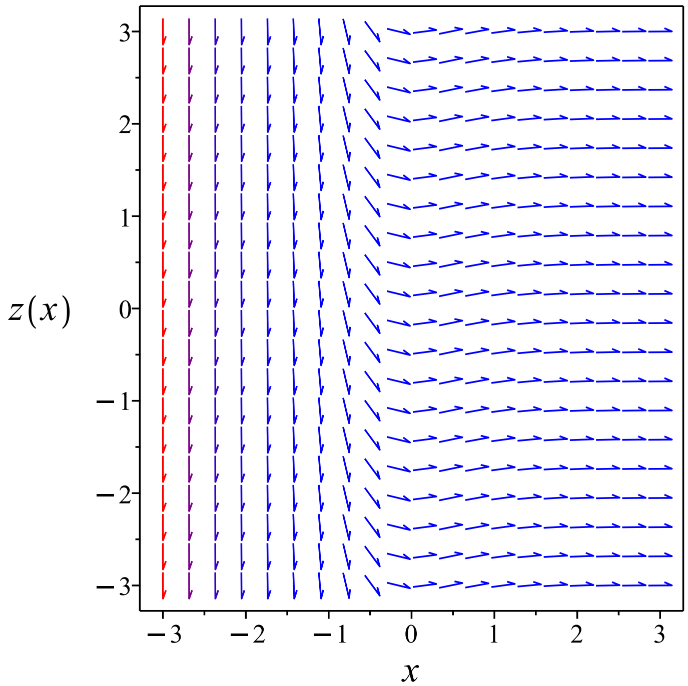


Figure 4: Slope field plot

Verification of solutions

$$z = -\frac{(2x+1)\,e^{-2x}}{4} + c_1$$

Verified OK.

1.4.2 Maple step by step solution

Let's solve

$z' = x\,e^{-2x}$

- Highest derivative means the order of the ODE is 1

  z'

- Integrate both sides with respect to x

  $\int z' dx = \int x\,e^{-2x} dx + c_1$

- Evaluate integral

  $z = -\frac{(2x+1)e^{-2x}}{4} + c_1$

- Solve for z

$$z = -\frac{x\,e^{-2x}}{2} - \frac{e^{-2x}}{4} + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(z(x),x)=x*exp(-2*x),z(x), singsol=all)
```

$$z(x) = \frac{(-2x-1)\,e^{-2x}}{4} + c_1$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 22

```
DSolve[z'[x]==x*Exp[-2*x],z[x],x,IncludeSingularSolutions -> True]
```

$$z(x) \to -\frac{1}{4}e^{-2x}(2x+1) + c_1$$

1.5 problem 5.1 (v)

Internal problem ID [11972]
Internal file name [`OUTPUT/10624_Saturday_September_02_2023_02_48_36_PM_68303959/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 5, Trivial differential equations. Exercises page 33
Problem number: 5.1 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{T' = \mathrm{e}^{-t} \sin(2t)}$$

1.5.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} T &= \int \mathrm{e}^{-t} \sin(2t)\, \mathrm{d}t \\ &= -\frac{2\,\mathrm{e}^{-t} \cos(2t)}{5} - \frac{\mathrm{e}^{-t} \sin(2t)}{5} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$T = -\frac{2\,\mathrm{e}^{-t} \cos(2t)}{5} - \frac{\mathrm{e}^{-t} \sin(2t)}{5} + c_1 \tag{1}$$

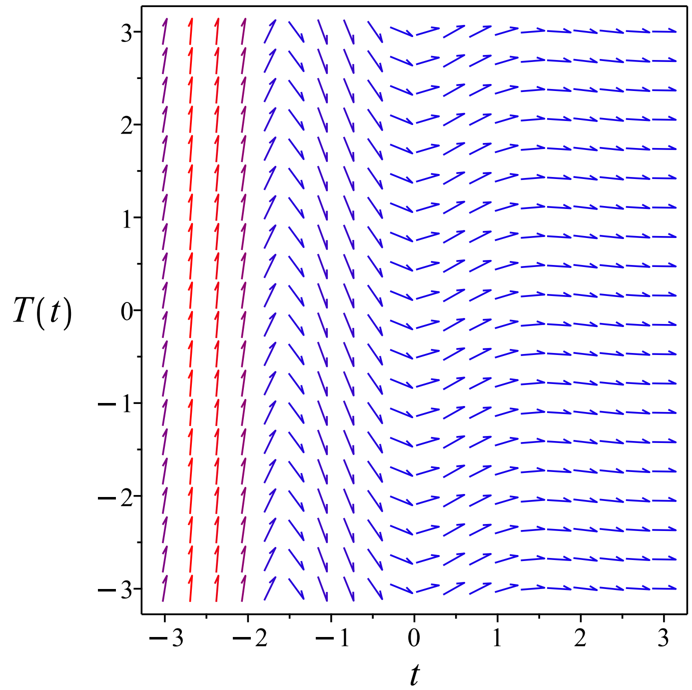


Figure 5: Slope field plot

Verification of solutions

$$T = -\frac{2\,e^{-t}\cos(2t)}{5} - \frac{e^{-t}\sin(2t)}{5} + c_1$$

Verified OK.

1.5.2 Maple step by step solution

Let's solve

$T' = e^{-t}\sin(2t)$

- Highest derivative means the order of the ODE is 1

  T'

- Integrate both sides with respect to t

  $\int T' dt = \int e^{-t}\sin(2t)\, dt + c_1$

- Evaluate integral

  $T = -\frac{2\,e^{-t}\cos(2t)}{5} - \frac{e^{-t}\sin(2t)}{5} + c_1$

- Solve for T

$$T = -\frac{2\,e^{-t}\cos(2t)}{5} - \frac{e^{-t}\sin(2t)}{5} + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(T(t),t)=exp(-t)*sin(2*t),T(t), singsol=all)
```

$$T(t) = \frac{e^{-t}(-2\cos(2t) - \sin(2t))}{5} + c_1$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 28

```
DSolve[T'[t]==Exp[-t]*Sin[2*t],T[t],t,IncludeSingularSolutions -> True]
```

$$T(t) \to -\frac{1}{5}e^{-t}(\sin(2t) + 2\cos(2t)) + c_1$$

1.6 problem 5.4 (i)

Internal problem ID [11973]
Internal file name [OUTPUT/10625_Saturday_September_02_2023_02_48_37_PM_85505114/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 5, Trivial differential equations. Exercises page 33
Problem number: 5.4 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' = \sec(t)^2}$$

With initial conditions

$$\left[x\left(\frac{\pi}{4}\right) = 0\right]$$

1.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = 0$$
$$q(t) = \sec(t)^2$$

Hence the ode is

$$x' = \sec(t)^2$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = \frac{\pi}{4}$ is inside this domain. The domain of $q(t) = \sec(t)^2$ is

$$\left\{t < \frac{1}{2}\pi + \pi\_Z116 \lor \frac{1}{2}\pi + \pi\_Z116 < t\right\}$$

And the point $t_0 = \frac{\pi}{4}$ is also inside this domain. Hence solution exists and is unique.

1.6.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} x &= \int \sec(t)^2 \, dt \\ &= \tan(t) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = \frac{\pi}{4}$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 1 + c_1$$

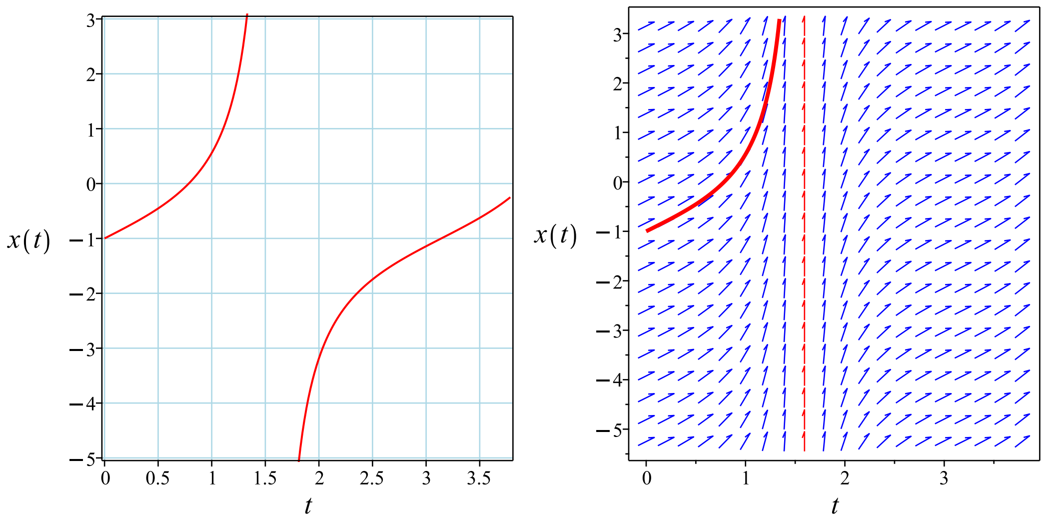
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$x = \tan(t) - 1$$

Summary
The solution(s) found are the following

$$x = \tan(t) - 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \tan(t) - 1$$

Verified OK.

1.6.3 Maple step by step solution

Let's solve

$\left[x' = \sec(t)^2, x\left(\frac{\pi}{4}\right) = 0\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Integrate both sides with respect to t

  $\int x' dt = \int \sec(t)^2 \, dt + c_1$

- Evaluate integral

  $x = \tan(t) + c_1$

- Solve for x

  $x = \tan(t) + c_1$

- Use initial condition $x\left(\frac{\pi}{4}\right) = 0$

  $0 = 1 + c_1$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $x = \tan(t) - 1$

- Solution to the IVP

  $x = \tan(t) - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 8

```
dsolve([diff(x(t),t)=sec(t)^2,x(1/4*Pi) = 0],x(t), singsol=all)
```

$$x(t) = \tan(t) - 1$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 9

```
DSolve[{x'[t]==Sec[t]^2,{x[Pi/4]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \tan(t) - 1$$

1.7 problem 5.4 (ii)

Internal problem ID [11974]
Internal file name [`OUTPUT/10626_Saturday_September_02_2023_02_48_37_PM_20629689/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 5, Trivial differential equations. Exercises page 33
Problem number: 5.4 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = x - \frac{1}{3}x^3}$$

With initial conditions

$$[y(-1) = 1]$$

1.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= x - \frac{1}{3}x^3 \end{aligned}$$

Hence the ode is

$$y' = x - \frac{1}{3}x^3$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = x - \frac{1}{3}x^3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

1.7.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int x - \frac{1}{3}x^3 \, \mathrm{d}x \\ &= -\frac{(x^2 - 3)^2}{12} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{3} + c_1$$

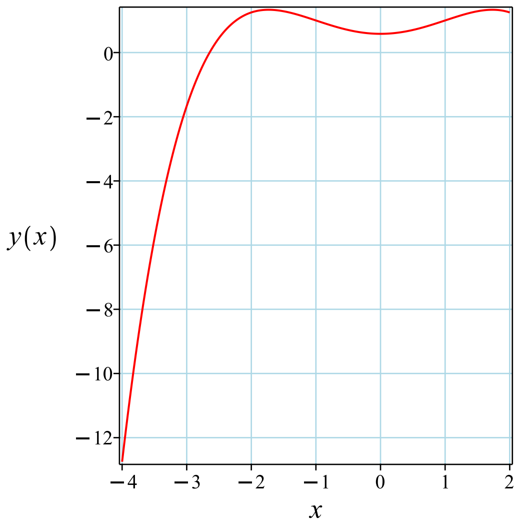
$$c_1 = \frac{4}{3}$$

Substituting c_1 found above in the general solution gives

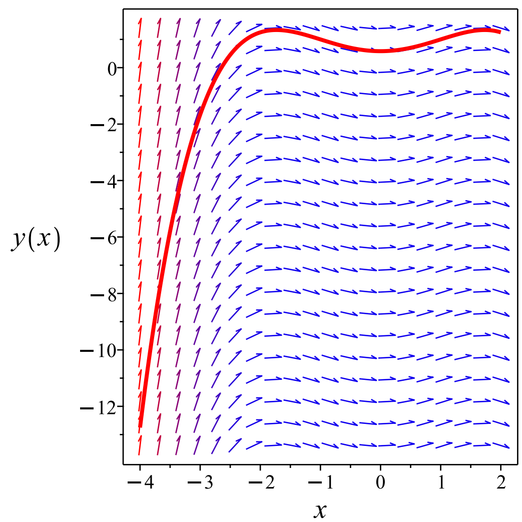
$$y = -\frac{1}{12}x^4 + \frac{1}{2}x^2 + \frac{7}{12}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{12}x^4 + \frac{1}{2}x^2 + \frac{7}{12} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{12}x^4 + \frac{1}{2}x^2 + \frac{7}{12}$$

Verified OK.

1.7.3 Maple step by step solution

Let's solve

$\left[y' = x - \frac{1}{3}x^3, y(-1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'dx = \int \left(x - \frac{1}{3}x^3\right) dx + c_1$

- Evaluate integral

  $y = -\frac{(x^2-3)^2}{12} + c_1$

- Solve for y

  $y = -\frac{1}{12}x^4 + \frac{1}{2}x^2 - \frac{3}{4} + c_1$

- Use initial condition $y(-1) = 1$

  $1 = -\frac{1}{3} + c_1$

- Solve for c_1

  $c_1 = \frac{4}{3}$
- Substitute $c_1 = \frac{4}{3}$ into general solution and simplify

  $y = -\frac{1}{12}x^4 + \frac{1}{2}x^2 + \frac{7}{12}$
- Solution to the IVP

  $y = -\frac{1}{12}x^4 + \frac{1}{2}x^2 + \frac{7}{12}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(y(x),x)=x-1/3*x^3,y(-1) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{(x^2 - 3)^2}{12} + \frac{4}{3}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 21

```
DSolve[{y'[x]==x-1/3*x^3,{y[-1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{12}(-x^4 + 6x^2 + 7)$$

1.8 problem 5.4 (iii)

Internal problem ID [11975]
Internal file name [`OUTPUT/10627_Saturday_September_02_2023_02_48_38_PM_97933282/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 5, Trivial differential equations. Exercises page 33
Problem number: 5.4 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' = 2\sin(t)^2}$$

With initial conditions

$$\left[x\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\right]$$

1.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = 0$$
$$q(t) = 2\sin(t)^2$$

Hence the ode is

$$x' = 2\sin(t)^2$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = \frac{\pi}{4}$ is inside this domain. The domain of $q(t) = 2\sin(t)^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = \frac{\pi}{4}$ is also inside this domain. Hence solution exists and is unique.

1.8.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} x &= \int 2\sin(t)^2 \, \mathrm{d}t \\ &= -\cos(t)\sin(t) + t + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = \frac{\pi}{4}$ and $x = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = -\frac{1}{2} + \frac{\pi}{4} + c_1$$

$$c_1 = \frac{1}{2}$$

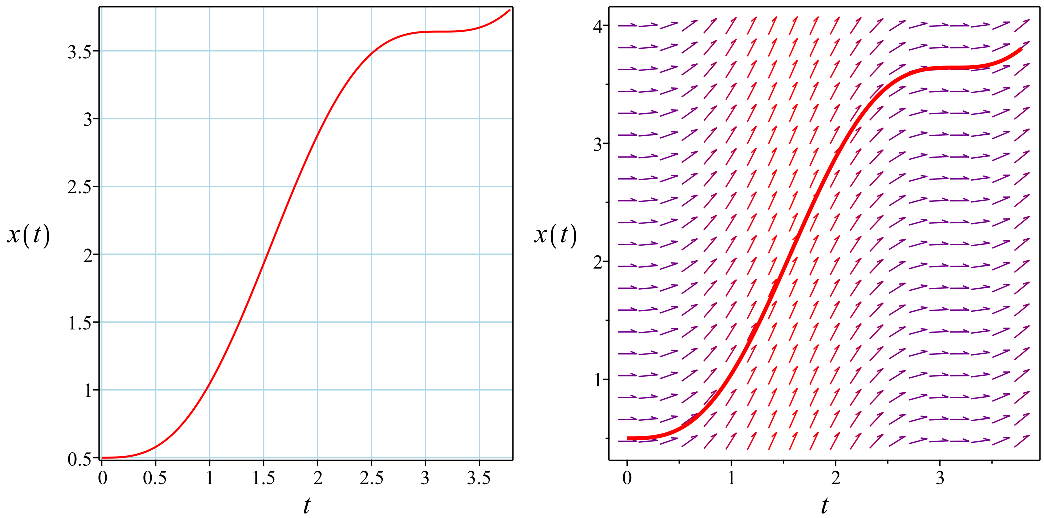
Substituting c_1 found above in the general solution gives

$$x = -\cos(t)\sin(t) + t + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$x = -\cos(t)\sin(t) + t + \frac{1}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$x = -\cos(t)\sin(t) + t + \frac{1}{2}$$

Verified OK.

1.8.3 Maple step by step solution

Let's solve

$\left[x' = 2\sin(t)^2, x\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Integrate both sides with respect to t

  $\int x' dt = \int 2\sin(t)^2 \, dt + c_1$

- Evaluate integral

  $x = -\cos(t)\sin(t) + t + c_1$

- Solve for x

  $x = -\cos(t)\sin(t) + t + c_1$

- Use initial condition $x\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$

  $\frac{\pi}{4} = -\frac{1}{2} + \frac{\pi}{4} + c_1$

- Solve for c_1

  $c_1 = \frac{1}{2}$
- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $x = -\frac{\sin(2t)}{2} + t + \frac{1}{2}$
- Solution to the IVP

  $x = -\frac{\sin(2t)}{2} + t + \frac{1}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([diff(x(t),t)=2*sin(t)^2,x(1/4*Pi) = 1/4*Pi],x(t), singsol=all)
```

$$x(t) = t + \frac{1}{2} - \frac{\sin(2t)}{2}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[{x'[t]==2*Sin[t]^2,{x[Pi/4]==Pi/4}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t - \sin(t)\cos(t) + \frac{1}{2}$$

1.9 problem 5.4 (iv)

Internal problem ID [11976]
Internal file name [`OUTPUT/10628_Saturday_September_02_2023_02_48_39_PM_65042289/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 5, Trivial differential equations. Exercises page 33
Problem number: 5.4 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{xV' = x^2 + 1}$$

With initial conditions

$$[V(1) = 1]$$

1.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$V' + p(x)V = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{x^2 + 1}{x}$$

Hence the ode is

$$V' = \frac{x^2 + 1}{x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{x^2+1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

1.9.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} V &= \int \frac{x^2+1}{x} \, dx \\ &= \frac{x^2}{2} + \ln(x) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $V = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + c_1$$

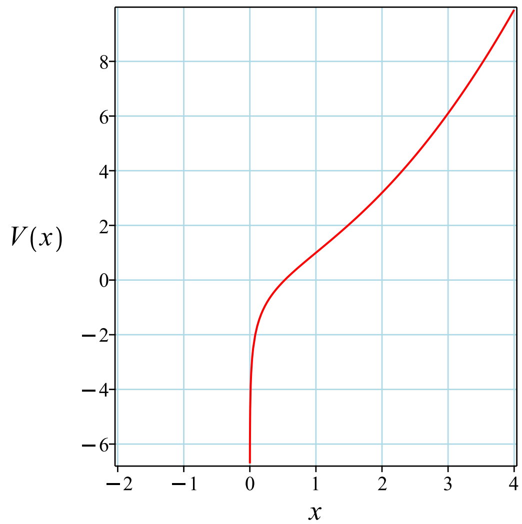
$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

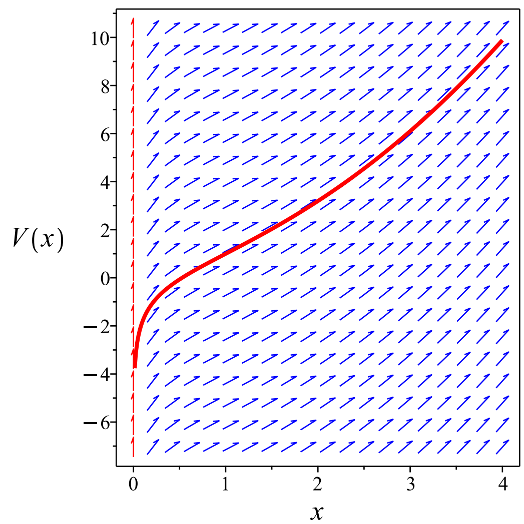
$$V = \frac{x^2}{2} + \ln(x) + \frac{1}{2}$$

Summary
The solution(s) found are the following

$$V = \frac{x^2}{2} + \ln(x) + \frac{1}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$V = \frac{x^2}{2} + \ln(x) + \frac{1}{2}$$

Verified OK.

1.9.3 Maple step by step solution

Let's solve

$[xV' = x^2 + 1, V(1) = 1]$

- Highest derivative means the order of the ODE is 1

  V'

- Separate variables

  $V' = \frac{x^2+1}{x}$

- Integrate both sides with respect to x

  $\int V'dx = \int \frac{x^2+1}{x}dx + c_1$

- Evaluate integral

  $V = \frac{x^2}{2} + \ln(x) + c_1$

- Solve for V

  $V = \frac{x^2}{2} + \ln(x) + c_1$

- Use initial condition $V(1) = 1$

  $1 = \frac{1}{2} + c_1$
- Solve for c_1

  $c_1 = \frac{1}{2}$
- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $V = \frac{x^2}{2} + \ln(x) + \frac{1}{2}$
- Solution to the IVP

  $V = \frac{x^2}{2} + \ln(x) + \frac{1}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve([x*diff(V(x),x)=1+x^2,V(1) = 1],V(x), singsol=all)
```

$$V(x) = \frac{x^2}{2} + \ln(x) + \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 18

```
DSolve[{x*V'[x]==1+x^2,{V[1]==1}},V[x],x,IncludeSingularSolutions -> True]
```

$$V(x) \to \frac{1}{2}\left(x^2 + 2\log(x) + 1\right)$$

1.10 problem 5.4 (v)

Internal problem ID [11977]
Internal file name [`OUTPUT/10629_Saturday_September_02_2023_02_48_40_PM_85883612/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 5, Trivial differential equations. Exercises page 33
Problem number: 5.4 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' e^{3t} + 3x\, e^{3t} = e^{-t}}$$

With initial conditions

$$[x(0) = 3]$$

1.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= e^{-4t} \end{aligned}$$

Hence the ode is

$$x' + 3x = e^{-4t}$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = e^{-4t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.10.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 3dt} \\ &= e^{3t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu x) &= (\mu)\left(e^{-4t}\right) \\ \frac{d}{dt}\left(e^{3t}x\right) &= \left(e^{3t}\right)\left(e^{-4t}\right) \\ d\left(e^{3t}x\right) &= e^{-t}\, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{3t}x &= \int e^{-t}\, dt \\ e^{3t}x &= -e^{-t} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{3t}$ results in

$$x = -e^{-3t}e^{-t} + e^{-3t}c_1$$

which simplifies to

$$x = -e^{-4t} + e^{-3t}c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -1 + c_1$$

$$c_1 = 4$$

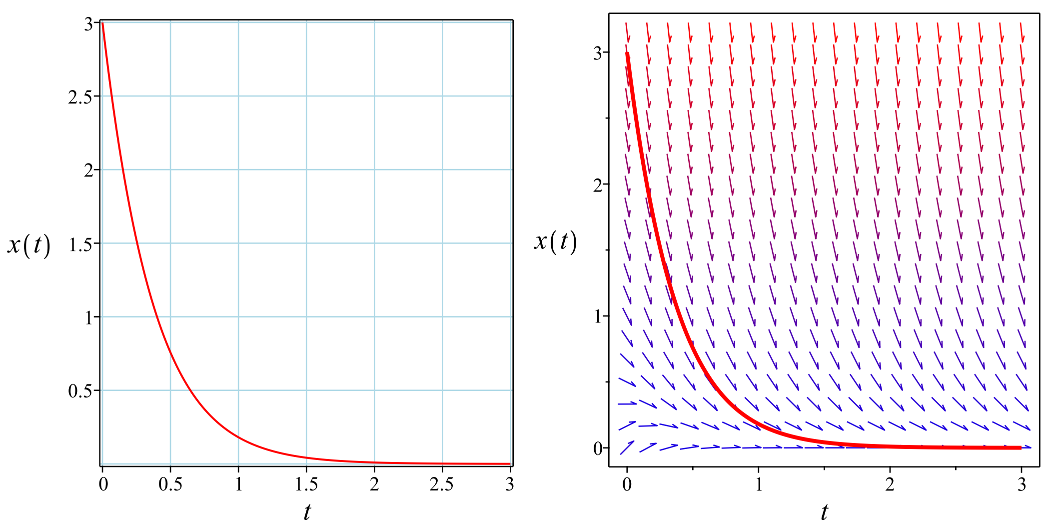
Substituting c_1 found above in the general solution gives

$$x = -\mathrm{e}^{-4t} + 4\,\mathrm{e}^{-3t}$$

Summary

The solution(s) found are the following

$$x = -\mathrm{e}^{-4t} + 4\,\mathrm{e}^{-3t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -\mathrm{e}^{-4t} + 4\,\mathrm{e}^{-3t}$$

Verified OK.

1.10.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= -\left(3\,\mathrm{e}^{3t}x - \mathrm{e}^{-t}\right)\mathrm{e}^{-3t} \\
x' &= \omega(t, x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 10: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,x) = 0$$
$$\eta(t,x) = e^{-3t} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-3t}} dy \end{aligned}$$

Which results in

$$S = e^{3t} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\left(3\, e^{3t} x - e^{-t}\right) e^{-3t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 3\, e^{3t} x \\ S_x &= e^{3t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$x\,\mathrm{e}^{3t} = -\mathrm{e}^{-t} + c_1$$

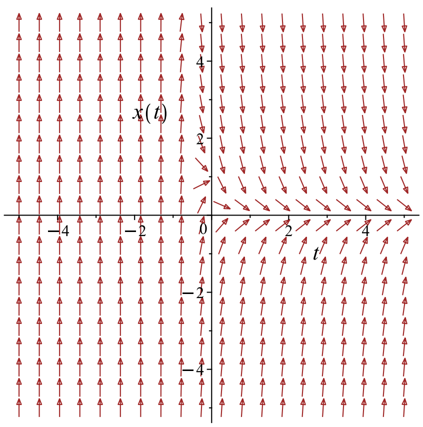
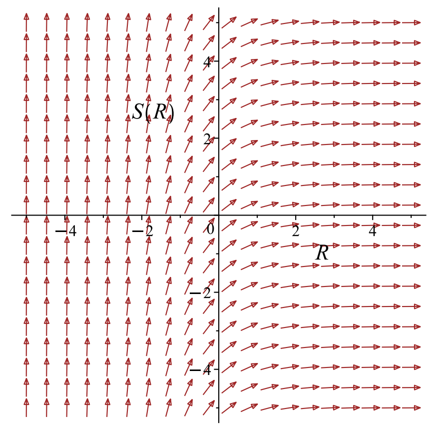
Which simplifies to

$$x\,\mathrm{e}^{3t} = -\mathrm{e}^{-t} + c_1$$

Which gives

$$x = -\left(\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{-3t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -(3\,\mathrm{e}^{3t}x - \mathrm{e}^{-t})\,\mathrm{e}^{-3t}$ 	$R = t$ $S = \mathrm{e}^{3t}x$	$\frac{dS}{dR} = \mathrm{e}^{-R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -1 + c_1$$

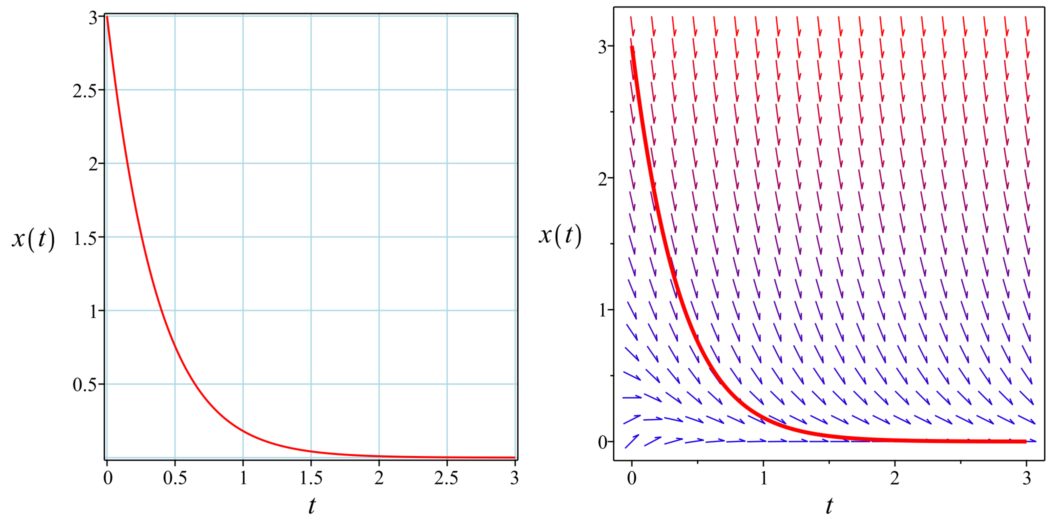
$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

$$x = -\mathrm{e}^{-4t} + 4\,\mathrm{e}^{-3t}$$

<u>Summary</u>
The solution(s) found are the following

$$x = -\mathrm{e}^{-4t} + 4\,\mathrm{e}^{-3t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -e^{-4t} + 4\,e^{-3t}$$

Verified OK.

1.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^{3t}\right)\mathrm{d}x &= \left(-3\,\mathrm{e}^{3t}x + \mathrm{e}^{-t}\right)\mathrm{d}t\\ \left(3\,\mathrm{e}^{3t}x - \mathrm{e}^{-t}\right)\mathrm{d}t + \left(\mathrm{e}^{3t}\right)\mathrm{d}x &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t,x) &= 3\,\mathrm{e}^{3t}x - \mathrm{e}^{-t}\\ N(t,x) &= \mathrm{e}^{3t}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(3\,\mathrm{e}^{3t}x - \mathrm{e}^{-t}\right)\\ &= 3\,\mathrm{e}^{3t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\mathrm{e}^{3t}\right)\\ &= 3\,\mathrm{e}^{3t}\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int 3\,\mathrm{e}^{3t}x - \mathrm{e}^{-t}\,\mathrm{d}t$$
$$\phi = \mathrm{e}^{3t}x + \mathrm{e}^{-t} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \mathrm{e}^{3t} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{3t}$. Therefore equation (4) becomes

$$\mathrm{e}^{3t} = \mathrm{e}^{3t} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{3t}x + \mathrm{e}^{-t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{3t}x + \mathrm{e}^{-t}$$

The solution becomes

$$x = -\left(\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{-3t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -1 + c_1$$

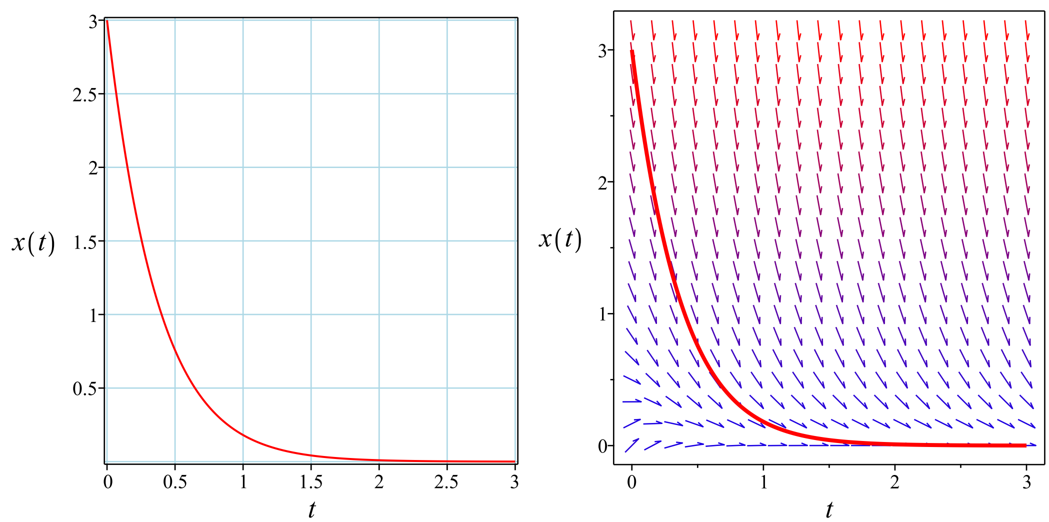
$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

$$x = -\mathrm{e}^{-4t} + 4\,\mathrm{e}^{-3t}$$

Summary
The solution(s) found are the following

$$x = -\mathrm{e}^{-4t} + 4\,\mathrm{e}^{-3t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -\mathrm{e}^{-4t} + 4\,\mathrm{e}^{-3t}$$

Verified OK.

1.10.5 Maple step by step solution

Let's solve

$[x'\mathrm{e}^{3t} + 3x\,\mathrm{e}^{3t} = \mathrm{e}^{-t}, x(0) = 3]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

$x' = -3x + \frac{e^{-t}}{e^{3t}}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + 3x = \frac{e^{-t}}{e^{3t}}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + 3x\right) = \frac{\mu(t)e^{-t}}{e^{3t}}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)\left(x' + 3x\right) = \mu'(t)\,x + \mu(t)\,x'$
- Isolate $\mu'(t)$

  $\mu'(t) = 3\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \left(e^{3t}\right)^2 e^{-3t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \frac{\mu(t)e^{-t}}{e^{3t}} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,x = \int \frac{\mu(t)e^{-t}}{e^{3t}} dt + c_1$
- Solve for x

  $x = \frac{\int \frac{\mu(t)e^{-t}}{e^{3t}} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \left(e^{3t}\right)^2 e^{-3t}$

  $x = \frac{\int e^{3t}e^{-3t}e^{-t} dt + c_1}{\left(e^{3t}\right)^2 e^{-3t}}$
- Evaluate the integrals on the rhs

  $x = \frac{-e^{-t} + c_1}{\left(e^{3t}\right)^2 e^{-3t}}$
- Simplify

  $x = e^{-3t}\left(-e^{-t} + c_1\right)$
- Use initial condition $x(0) = 3$

  $3 = -1 + c_1$
- Solve for c_1

  $c_1 = 4$
- Substitute $c_1 = 4$ into general solution and simplify

$$x = -(e^{-t} - 4)\, e^{-3t}$$

- Solution to the IVP

$$x = -(e^{-t} - 4)\, e^{-3t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(x(t)*exp(3*t),t)=exp(-t),x(0) = 3],x(t), singsol=all)
```

$$x(t) = -\left(e^{-t} - 4\right) e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 18

```
DSolve[{D[x[t]*Exp[3*t],t]==Exp[-t],{x[0]==3}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-4t}\left(4e^t - 1\right)$$

2 Chapter 7, Scalar autonomous ODEs. Exercises page 56

2.1 problem 7.1 (i)

Internal problem ID [11978]
Internal file name [`OUTPUT/10630_Saturday_September_02_2023_02_48_41_PM_84729893/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 7, Scalar autonomous ODEs. Exercises page 56
Problem number: 7.1 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x' + x = 1}$$

2.1.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{1-x} dx = \int dt$$
$$-\ln(1-x) = t + c_1$$

Raising both side to exponential gives

$$\frac{1}{1-x} = e^{t+c_1}$$

Which simplifies to

$$\frac{1}{1-x} = c_2 e^t$$

Summary
The solution(s) found are the following

$$x = -\frac{e^{-t}}{c_2} + 1 \tag{1}$$

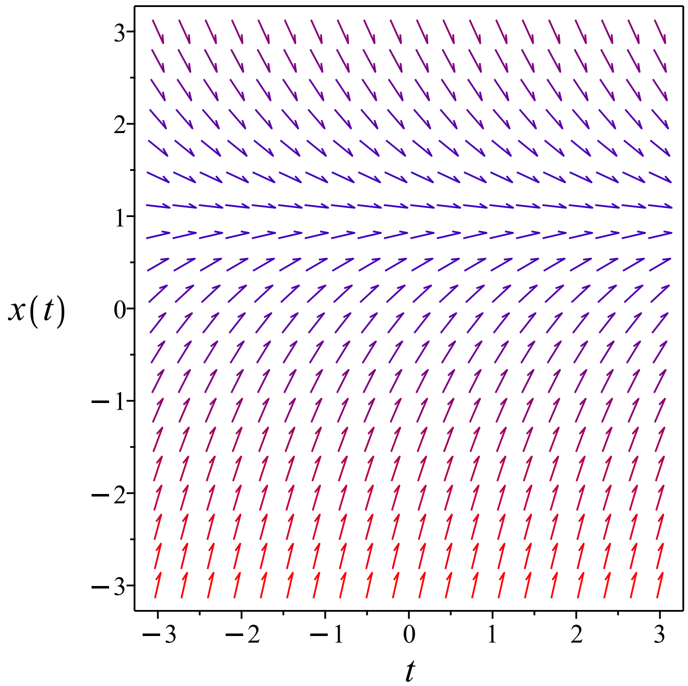


Figure 13: Slope field plot

Verification of solutions

$$x = -\frac{e^{-t}}{c_2} + 1$$

Verified OK.

2.1.2 Maple step by step solution

Let's solve

$x' + x = 1$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{1-x} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{1-x} dt = \int 1 dt + c_1$

- Evaluate integral

$$-\ln(1-x) = t + c_1$$

- Solve for x

$$x = -e^{-t-c_1} + 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(x(t),t)=-x(t)+1,x(t), singsol=all)
```

$$x(t) = 1 + e^{-t}c_1$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 20

```
DSolve[x'[t]==-x[t]+1,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 1 + c_1 e^{-t}$$
$$x(t) \to 1$$

2.2 problem 7.1 (ii)

Internal problem ID [11979]
Internal file name [OUTPUT/10631_Saturday_September_02_2023_02_48_42_PM_70744530/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 7, Scalar autonomous ODEs. Exercises page 56
Problem number: 7.1 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x' - x(-x+2) = 0}$$

2.2.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{x(x-2)}dx = \int dt$$
$$-\frac{\ln(x-2)}{2} + \frac{\ln(x)}{2} = t + c_1$$

The above can be written as

$$\left(-\frac{1}{2}\right)(\ln(x-2) - \ln(x)) = t + c_1$$
$$\ln(x-2) - \ln(x) = (-2)(t + c_1)$$
$$= -2t - 2c_1$$

Raising both side to exponential gives

$$e^{\ln(x-2)-\ln(x)} = -2c_1 e^{-2t}$$

Which simplifies to

$$\frac{x-2}{x} = c_2 \mathrm{e}^{-2t}$$

Summary
The solution(s) found are the following

$$x = -\frac{2}{-1 + c_2 \mathrm{e}^{-2t}} \tag{1}$$

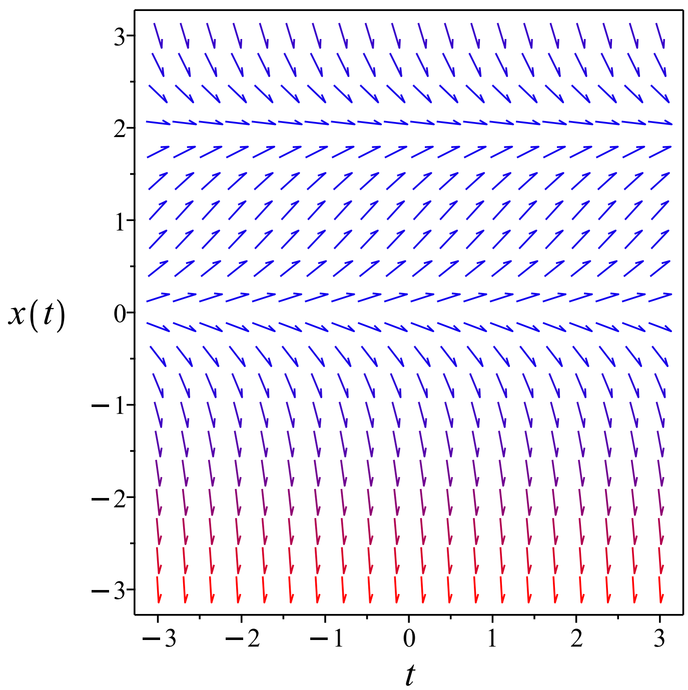


Figure 14: Slope field plot

Verification of solutions

$$x = -\frac{2}{-1 + c_2 \mathrm{e}^{-2t}}$$

Verified OK.

2.2.2 Maple step by step solution

Let's solve

$x' - x(-x+2) = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x(-x+2)} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{x(-x+2)}dt = \int 1dt + c_1$

- Evaluate integral

  $-\frac{\ln(x-2)}{2} + \frac{\ln(x)}{2} = t + c_1$

- Solve for x

  $x = \frac{2\,\mathrm{e}^{2t+2c_1}}{-1+\mathrm{e}^{2t+2c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(x(t),t)=x(t)*(2-x(t)),x(t), singsol=all)
```

$$x(t) = \frac{2}{1 + 2\,\mathrm{e}^{-2t} c_1}$$

✓ Solution by Mathematica

Time used: 0.503 (sec). Leaf size: 36

```
DSolve[x'[t]==x[t]*(2-x[t]),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{2e^{2t}}{e^{2t} + e^{2c_1}}$$
$$x(t) \to 0$$
$$x(t) \to 2$$

2.3 problem 7.1 (iii)

Internal problem ID [11980]
Internal file name [`OUTPUT/10632_Saturday_September_02_2023_02_48_42_PM_26439748/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 7, Scalar autonomous ODEs. Exercises page 56
Problem number: 7.1 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - (x+1)(-x+2)\sin(x) = 0}$$

2.3.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{\sin(x)(x+1)(x-2)}dx = \int dt$$
$$\int^{x} -\frac{1}{\sin(\_a)(\_a+1)(\_a-2)}d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^{x} -\frac{1}{\sin(\_a)(\_a+1)(\_a-2)}d\_a = t + c_1 \tag{1}$$

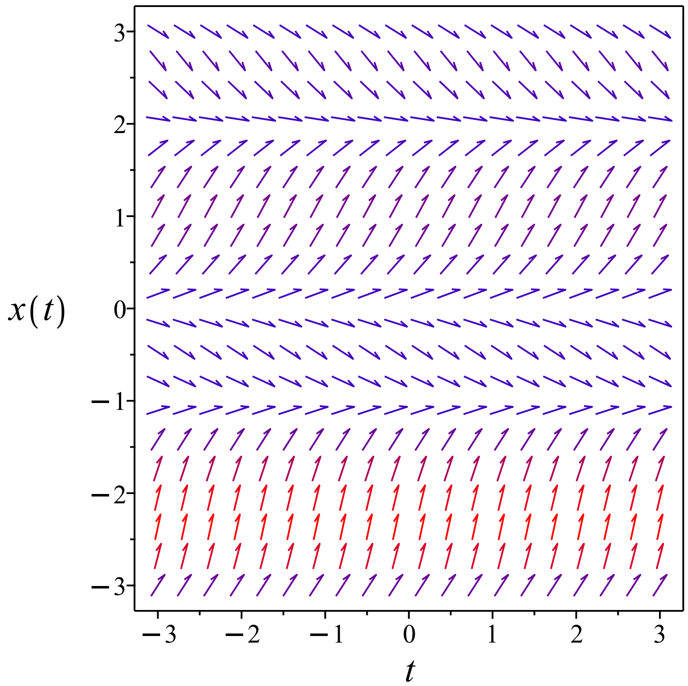


Figure 15: Slope field plot

Verification of solutions

$$\int^{x} -\frac{1}{\sin(\_a)(\_a+1)(\_a-2)} d\_a = t + c_1$$

Verified OK.

2.3.2 Maple step by step solution

Let's solve

$x' - (x+1)(-x+2)\sin(x) = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{(x+1)(-x+2)\sin(x)} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{(x+1)(-x+2)\sin(x)} dt = \int 1 dt + c_1$

- Cannot compute integral

$$\int \frac{x'}{(x+1)(-x+2)\sin(x)}dt = t + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(x(t),t)=(1+x(t))*(2-x(t))*sin(x(t)),x(t), singsol=all)
```

$$t + \int^{x(t)} \frac{\csc(\_a)}{(\_a+1)(\_a-2)}d\_a + c_1 = 0$$

✓ Solution by Mathematica

Time used: 15.593 (sec). Leaf size: 52

```
DSolve[x'[t]==(1+x[t])*(2-x[t])*Sin[x[t]],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\csc(K[1])}{(K[1]-2)(K[1]+1)}dK[1]\&\right][-t+c_1]$$
$$x(t) \to -1$$
$$x(t) \to 0$$
$$x(t) \to 2$$

2.4 problem 7.1 (iv)

Internal problem ID [11981]
Internal file name [`OUTPUT/10633_Saturday_September_02_2023_02_48_46_PM_3774711/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 7, Scalar autonomous ODEs. Exercises page 56
Problem number: 7.1 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' + x(1-x)(-x+2) = 0}$$

2.4.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{x(x-1)(x-2)} dx = \int dt$$
$$-\frac{\ln(x-2)}{2} + \ln(x-1) - \frac{\ln(x)}{2} = t + c_1$$

Raising both side to exponential gives

$$e^{-\frac{\ln(x-2)}{2} + \ln(x-1) - \frac{\ln(x)}{2}} = e^{t+c_1}$$

Which simplifies to

$$\frac{x-1}{\sqrt{x-2}\,\sqrt{x}} = c_2 e^t$$

Summary
The solution(s) found are the following

$$x = \frac{c_2^2 e^{2t} + \sqrt{e^{4t} c_2^4 - c_2^2 e^{2t}} - 1}{c_2^2 e^{2t} - 1} \tag{1}$$

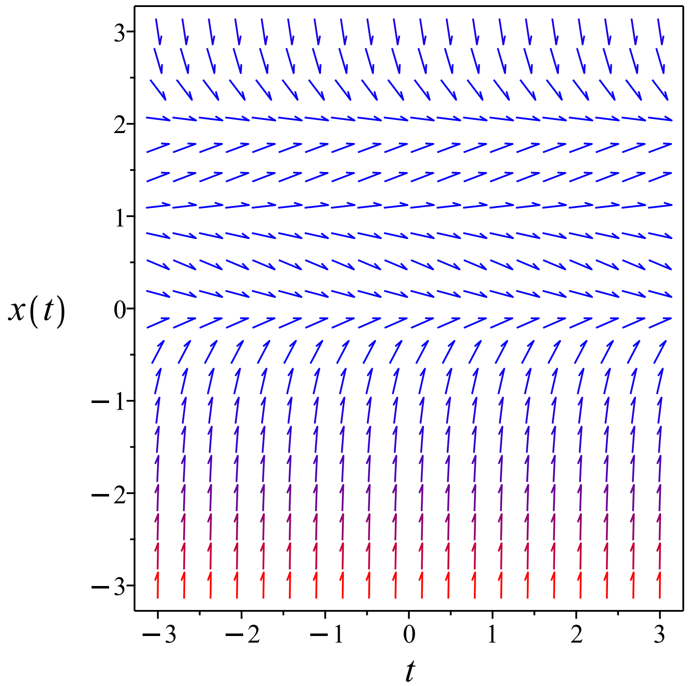


Figure 16: Slope field plot

Verification of solutions

$$x = \frac{c_2^2 e^{2t} + \sqrt{e^{4t} c_2^4 - c_2^2 e^{2t}} - 1}{c_2^2 e^{2t} - 1}$$

Verified OK.

2.4.2 Maple step by step solution

Let's solve

$x' + x(1-x)(-x+2) = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x(1-x)(-x+2)} = -1$

- Integrate both sides with respect to t

  $\int \frac{x'}{x(1-x)(-x+2)} dt = \int (-1) dt + c_1$

- Evaluate integral

$$\frac{\ln(x-2)}{2}-\ln(x-1)+\frac{\ln(x)}{2}=-t+c_1$$

- Solve for x

$$\left\{x=\frac{\sqrt{-\mathrm{e}^{-2t+2c_1}+1}-1}{\sqrt{-\mathrm{e}^{-2t+2c_1}+1}}, x=\frac{\sqrt{-\mathrm{e}^{-2t+2c_1}+1}+1}{\sqrt{-\mathrm{e}^{-2t+2c_1}+1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 34

```
dsolve(diff(x(t),t)=-x(t)*(1-x(t))*(2-x(t)),x(t), singsol=all)
```

$$x(t)=\frac{c_1\mathrm{e}^t+\sqrt{-1+\mathrm{e}^{2t}c_1^2}}{\sqrt{-1+\mathrm{e}^{2t}c_1^2}}$$

✓ Solution by Mathematica

Time used: 19.885 (sec). Leaf size: 159

```
DSolve[x'[t]==-x[t]*(1-x[t])*(2-x[t]),x[t],t,IncludeSingularSolutions -> True]
```

$$\begin{aligned}
x(t)&\to\frac{e^{2t}-\sqrt{e^{4t}+e^{2(t+c_1)}}+e^{2c_1}}{e^{2t}+e^{2c_1}}\\
x(t)&\to\frac{e^{2t}+\sqrt{e^{4t}+e^{2(t+c_1)}}+e^{2c_1}}{e^{2t}+e^{2c_1}}\\
x(t)&\to 0\\
x(t)&\to 1\\
x(t)&\to 2\\
x(t)&\to 1-e^{-2t}\sqrt{e^{4t}}\\
x(t)&\to e^{-2t}\sqrt{e^{4t}}+1
\end{aligned}$$

2.5 problem 7.1 (v)

Internal problem ID [11982]
Internal file name [`OUTPUT/10634_Saturday_September_02_2023_02_48_47_PM_21281142/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 7, Scalar autonomous ODEs. Exercises page 56
Problem number: 7.1 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x' - x^2 + x^4 = 0}$$

2.5.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-x^4 + x^2} dx = \int dt$$
$$\int^x \frac{1}{-\_a^4 + \_a^2} d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^x \frac{1}{-\_a^4 + \_a^2} d\_a = t + c_1 \tag{1}$$

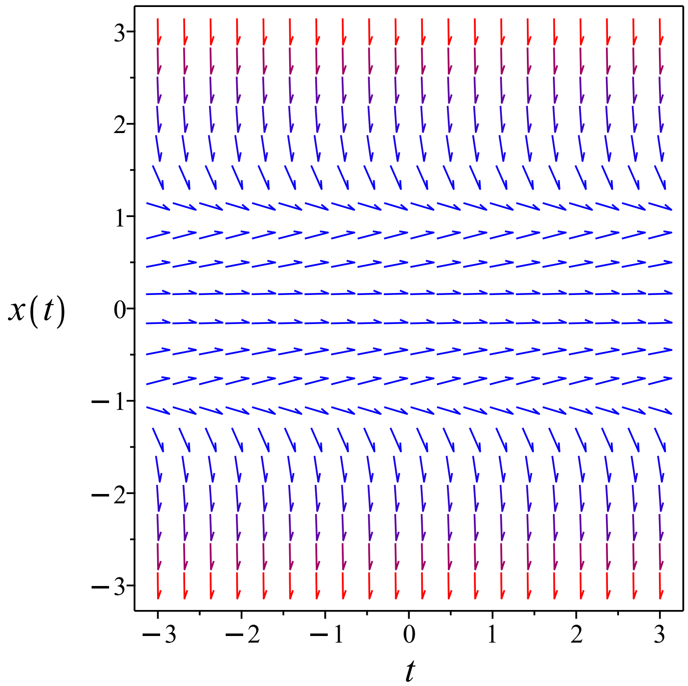


Figure 17: Slope field plot

Verification of solutions

$$\int^{x} \frac{1}{-\_a^{4}+\_a^{2}}d\_a = t + c_1$$

Verified OK.

2.5.2 Maple step by step solution

Let's solve

$x' - x^2 + x^4 = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x^2 - x^4} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{x^2 - x^4} dt = \int 1 dt + c_1$

- Evaluate integral

$$-\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}-\frac{1}{x}=t+c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 47

```
dsolve(diff(x(t),t)=x(t)^2-x(t)^4,x(t), singsol=all)
```

$$x(t)=\mathrm{e}^{\mathrm{RootOf}\left(\ln\left(\mathrm{e}^{\_Z}-2\right)\mathrm{e}^{\_Z}+2c_1\mathrm{e}^{\_Z}-\_Z\mathrm{e}^{\_Z}+2t\,\mathrm{e}^{\_Z}-\ln\left(\mathrm{e}^{\_Z}-2\right)-2c_1+\_Z-2t+2\right)}-1$$

✓ Solution by Mathematica

Time used: 0.414 (sec). Leaf size: 53

```
DSolve[x'[t]==x[t]^2-x[t]^4,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t)\to \text{InverseFunction}\left[\frac{1}{\#1}+\frac{1}{2}\log(1-\#1)-\frac{1}{2}\log(\#1+1)\&\right][-t+c_1]$$
$$x(t)\to -1$$
$$x(t)\to 0$$
$$x(t)\to 1$$

3 Chapter 8, Separable equations. Exercises page 72

3.1 problem 8.1 (i)

Internal problem ID [11983]
Internal file name [`OUTPUT/10635_Saturday_September_02_2023_02_48_48_PM_3805520/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.1 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - t^3(1 - x) = 0}$$

With initial conditions

$$[x(0) = 3]$$

3.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = t^3$$
$$q(t) = t^3$$

Hence the ode is

$$x' + t^3 x = t^3$$

The domain of $p(t) = t^3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = t^3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.1.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= f(t)g(x) \\ &= t^3(1 - x) \end{aligned}$$

Where $f(t) = t^3$ and $g(x) = 1 - x$. Integrating both sides gives

$$\begin{aligned} \frac{1}{1-x}\,dx &= t^3\,dt \\ \int \frac{1}{1-x}\,dx &= \int t^3\,dt \\ -\ln(x-1) &= \frac{t^4}{4} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{x-1} = e^{\frac{t^4}{4}+c_1}$$

Which simplifies to

$$\frac{1}{x-1} = c_2 e^{\frac{t^4}{4}}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{e^{c_1} e^{-c_1} c_2 + e^{-c_1}}{c_2}$$

$$c_1 = -\ln(2c_2)$$

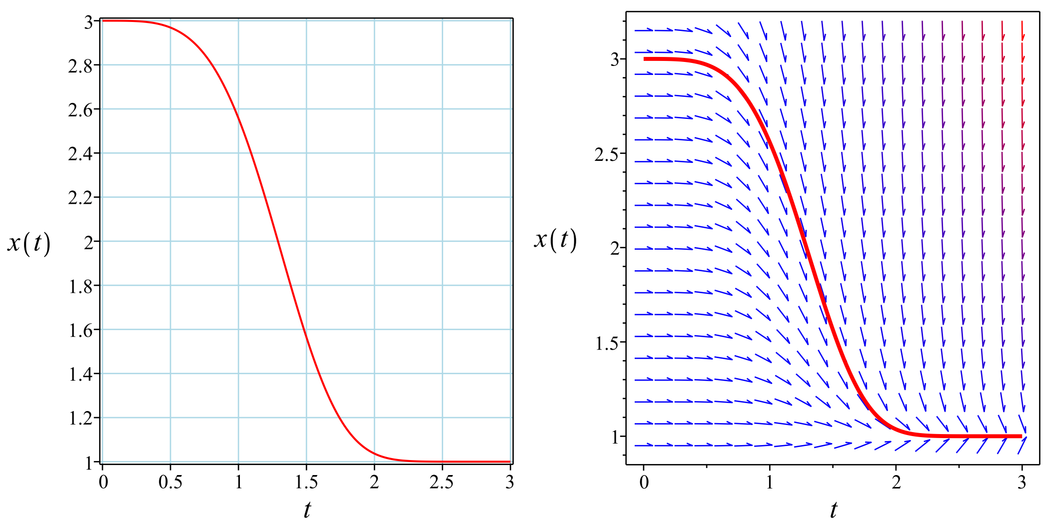
Substituting c_1 found above in the general solution gives

$$x = 1 + 2\,e^{-\frac{t^4}{4}}$$

Summary

The solution(s) found are the following

$$x = 1 + 2\,e^{-\frac{t^4}{4}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = 1 + 2\,e^{-\frac{t^4}{4}}$$

Verified OK.

3.1.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int t^3 dt} = e^{\frac{t^4}{4}}$$

The ode becomes

$$\frac{d}{dt}(\mu x) = (\mu)(t^3)$$
$$\frac{d}{dt}\left(e^{\frac{t^4}{4}} x\right) = \left(e^{\frac{t^4}{4}}\right)(t^3)$$
$$d\left(e^{\frac{t^4}{4}} x\right) = \left(t^3 e^{\frac{t^4}{4}}\right) dt$$

Integrating gives

$$e^{\frac{t^4}{4}} x = \int t^3 e^{\frac{t^4}{4}} dt$$
$$e^{\frac{t^4}{4}} x = e^{\frac{t^4}{4}} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{t^4}{4}}$ results in

$$x = e^{-\frac{t^4}{4}} e^{\frac{t^4}{4}} + c_1 e^{-\frac{t^4}{4}}$$

which simplifies to

$$x = 1 + c_1 e^{-\frac{t^4}{4}}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 1 + c_1$$

$$c_1 = 2$$

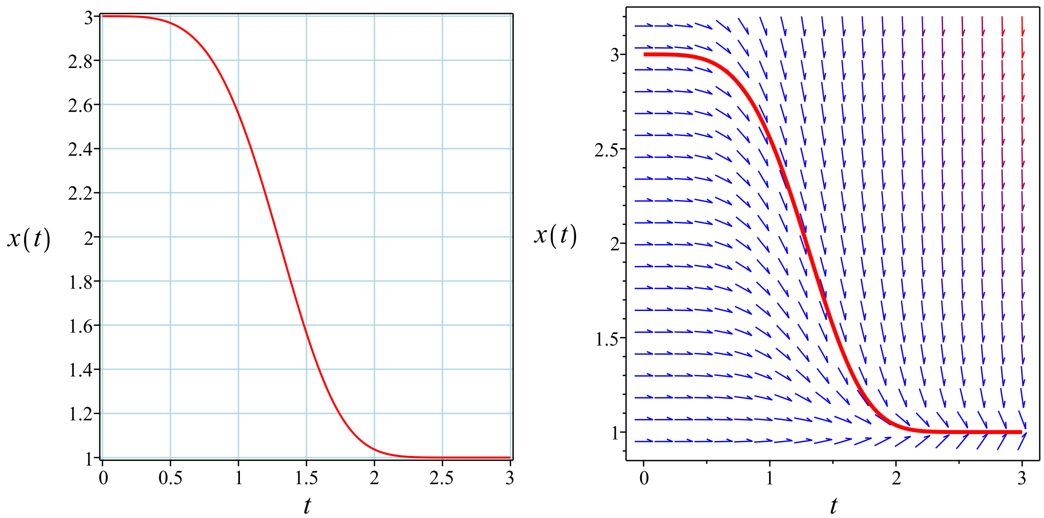
Substituting c_1 found above in the general solution gives

$$x = 1 + 2 e^{-\frac{t^4}{4}}$$

Summary

The solution(s) found are the following

$$x = 1 + 2 e^{-\frac{t^4}{4}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = 1 + 2\,\mathrm{e}^{-\frac{t^4}{4}}$$

Verified OK.

3.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -t^3(x-1)$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 18: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \mathrm{e}^{-\frac{t^4}{4}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{t^4}{4}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{t^4}{4}} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -t^3(x-1)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= t^3 e^{\frac{t^4}{4}} x \\ S_x &= e^{\frac{t^4}{4}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t^3 e^{\frac{t^4}{4}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3 e^{\frac{R^4}{4}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \mathrm{e}^{\frac{R^4}{4}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\mathrm{e}^{\frac{t^4}{4}} x = \mathrm{e}^{\frac{t^4}{4}} + c_1$$

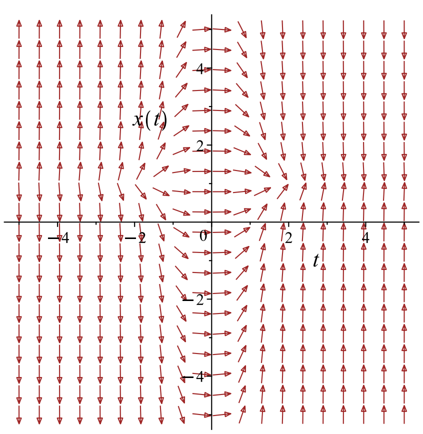
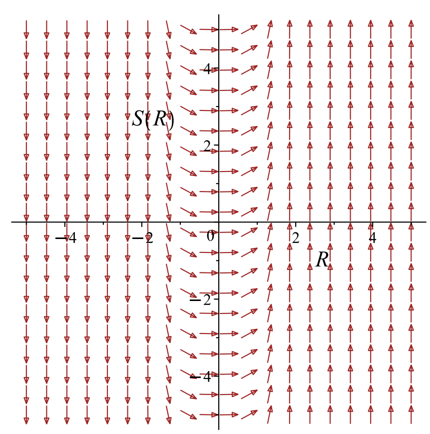
Which simplifies to

$$(x-1)\,\mathrm{e}^{\frac{t^4}{4}} - c_1 = 0$$

Which gives

$$x = \left(\mathrm{e}^{\frac{t^4}{4}} + c_1\right)\mathrm{e}^{-\frac{t^4}{4}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -t^3(x-1)$ 	$R = t$ $S = \mathrm{e}^{\frac{t^4}{4}} x$	$\frac{dS}{dR} = R^3 \mathrm{e}^{\frac{R^4}{4}}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 1 + c_1$$

$$c_1 = 2$$

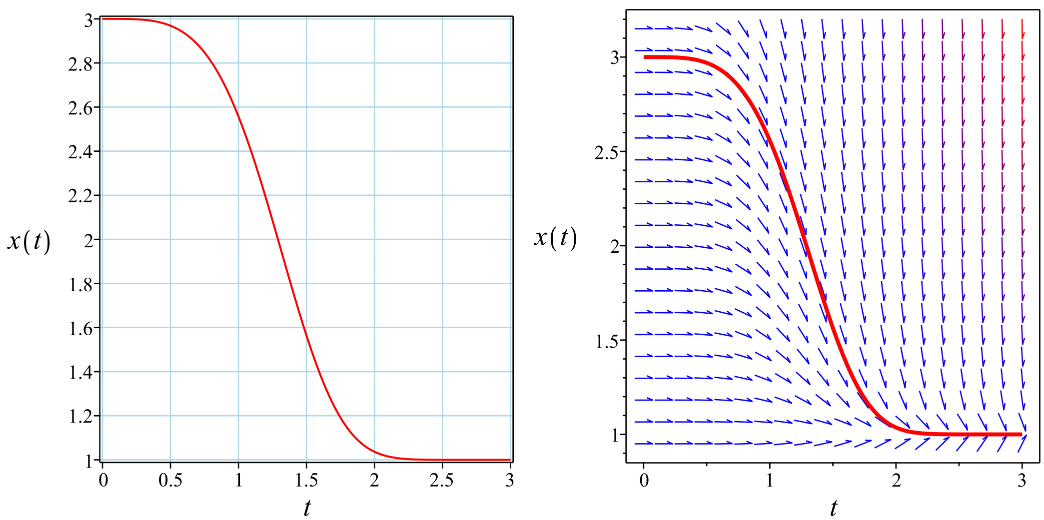
Substituting c_1 found above in the general solution gives

$$x = 1 + 2\,\mathrm{e}^{-\frac{t^4}{4}}$$

Summary

The solution(s) found are the following

$$x = 1 + 2\,\mathrm{e}^{-\frac{t^4}{4}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = 1 + 2\,\mathrm{e}^{-\frac{t^4}{4}}$$

Verified OK.

3.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{1-x}\right)\mathrm{d}x&=\left(t^3\right)\mathrm{d}t\\ \left(-t^3\right)\mathrm{d}t+\left(\frac{1}{1-x}\right)\mathrm{d}x&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-t^3\\ N(t,x)&=\frac{1}{1-x}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x}&=\frac{\partial}{\partial x}\left(-t^3\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{1-x}\right)\\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -t^3\,\mathrm{d}t
\end{aligned}$$

$$\phi = -\frac{t^4}{4} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{1-x}$. Therefore equation (4) becomes

$$\frac{1}{1-x} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{1}{x-1}$$

Integrating the above w.r.t x gives

$$\begin{aligned}
\int f'(x)\,\mathrm{d}x &= \int \left(-\frac{1}{x-1}\right)\mathrm{d}x\\
f(x) &= -\ln\left(x-1\right) + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^4}{4} - \ln(x-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^4}{4} - \ln(x-1)$$

The solution becomes

$$x = \mathrm{e}^{-\frac{t^4}{4} - c_1} + 1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \mathrm{e}^{-c_1} + 1$$

$$c_1 = -\ln(2)$$

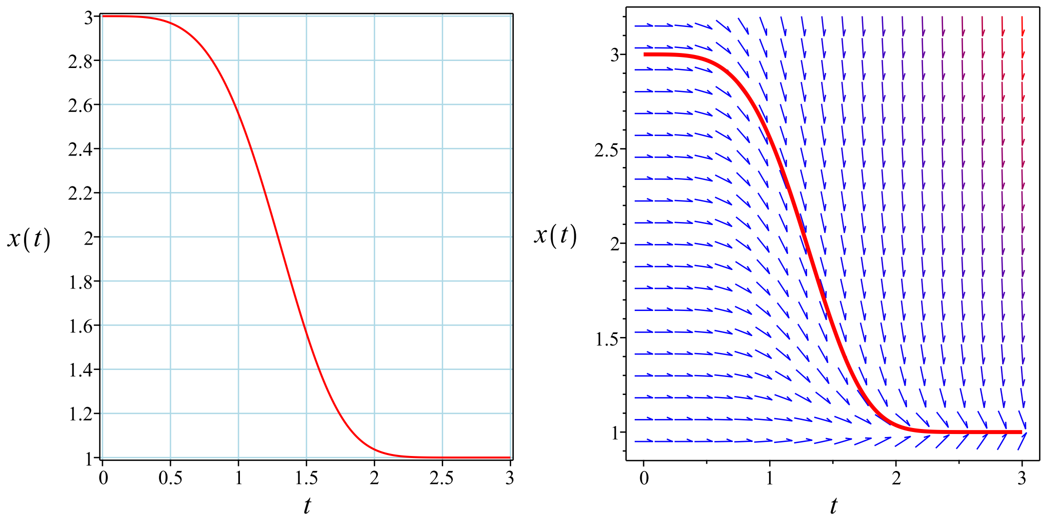
Substituting c_1 found above in the general solution gives

$$x = 1 + 2\,\mathrm{e}^{-\frac{t^4}{4}}$$

Summary

The solution(s) found are the following

$$x = 1 + 2\,\mathrm{e}^{-\frac{t^4}{4}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = 1 + 2\,e^{-\frac{t^4}{4}}$$

Verified OK.

3.1.6 Maple step by step solution

Let's solve

$[x' - t^3(1-x) = 0, x(0) = 3]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{1-x} = t^3$

- Integrate both sides with respect to t

  $\int \frac{x'}{1-x}dt = \int t^3 dt + c_1$

- Evaluate integral

  $-\ln(1-x) = \frac{t^4}{4} + c_1$

- Solve for x

  $x = -e^{-\frac{t^4}{4}-c_1} + 1$

- Use initial condition $x(0) = 3$

  $3 = -e^{-c_1} + 1$

- Solve for c_1

  $c_1 = -\ln(2) - I\pi$

- Substitute $c_1 = -\ln(2) - I\pi$ into general solution and simplify

  $x = 1 + 2\,e^{-\frac{t^4}{4}}$

- Solution to the IVP

  $x = 1 + 2\,e^{-\frac{t^4}{4}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(x(t),t)=t^3*(1-x(t)),x(0) = 3],x(t), singsol=all)
```

$$x(t) = 1 + 2\,e^{-\frac{t^4}{4}}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 18

```
DSolve[{x'[t]==t^3*(1-x[t]),{x[0]==3}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 2e^{-\frac{t^4}{4}} + 1$$

3.2 problem 8.1 (ii)

Internal problem ID [11984]
Internal file name [`OUTPUT/10636_Saturday_September_02_2023_02_48_49_PM_56060584/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.1 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - (1 + y^2)\tan(x) = 0}$$

With initial conditions

$$[y(0) = 1]$$

3.2.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= (y^2 + 1)\tan(x) \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z117 \vee \frac{1}{2}\pi + \pi\_Z117 < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left((y^2+1)\tan(x)\right)\\ &= 2y\tan(x)\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z117 \vee \frac{1}{2}\pi + \pi\_Z117 < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.2.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= (y^2+1)\tan(x)\end{aligned}$$

Where $f(x) = \tan(x)$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2+1}\,dy &= \tan(x)\,dx\\ \int\frac{1}{y^2+1}\,dy &= \int\tan(x)\,dx\\ \arctan(y) &= -\ln(\cos(x)) + c_1\end{aligned}$$

Which results in

$$y = \tan(-\ln(\cos(x)) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \tan(c_1)$$

$$c_1 = \frac{\pi}{4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{i\cos(x)^{2i} - 1 + i + \cos(x)^{2i}}{i\cos(x)^{2i} + 1 + i - \cos(x)^{2i}}$$

Summary

The solution(s) found are the following

$$y = \frac{i\cos(x)^{2i} - 1 + i + \cos(x)^{2i}}{i\cos(x)^{2i} + 1 + i - \cos(x)^{2i}} \tag{1}$$

Verification of solutions

$$y = \frac{i\cos(x)^{2i} - 1 + i + \cos(x)^{2i}}{i\cos(x)^{2i} + 1 + i - \cos(x)^{2i}}$$

Verified OK.

3.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= (y^2 + 1)\tan(x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 21: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{\tan(x)} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{\tan(x)}} dx \end{aligned}$$

Which results in

$$S = -\ln(\cos(x))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (y^2 + 1) \tan(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \tan(x) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(\cos(x)) = \arctan(y) + c_1$$

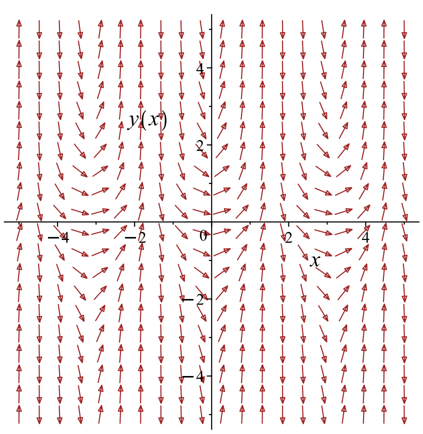
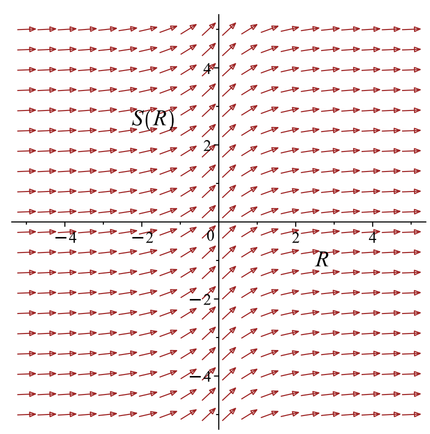
Which simplifies to

$$-\ln(\cos(x)) = \arctan(y) + c_1$$

Which gives

$$y = -\tan(\ln(\cos(x)) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (y^2+1)\tan(x)$ 	$R = y$ $S = -\ln(\cos(x))$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\tan(c_1)$$

$$c_1 = -\frac{\pi}{4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{i\cos(x)^{2i} - 1 + i + \cos(x)^{2i}}{i\cos(x)^{2i} + 1 + i - \cos(x)^{2i}}$$

Summary

The solution(s) found are the following

$$y = \frac{i\cos(x)^{2i} - 1 + i + \cos(x)^{2i}}{i\cos(x)^{2i} + 1 + i - \cos(x)^{2i}} \tag{1}$$

Verification of solutions

$$y = \frac{i\cos(x)^{2i} - 1 + i + \cos(x)^{2i}}{i\cos(x)^{2i} + 1 + i - \cos(x)^{2i}}$$

Verified OK.

3.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2+1}\right)\mathrm{d}y &= (\tan(x))\,\mathrm{d}x\\ (-\tan(x))\,\mathrm{d}x + \left(\frac{1}{y^2+1}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\tan(x)\\ N(x,y) &= \frac{1}{y^2+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\tan(x))\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathop{\mathrm{d}x} &= \int M \mathop{\mathrm{d}x} \\
\int \frac{\partial \phi}{\partial x} \mathop{\mathrm{d}x} &= \int -\tan\left(x\right) \mathop{\mathrm{d}x} \\
\phi &= \ln\left(\cos\left(x\right)\right) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$
\frac{1}{y^2+1} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathop{\mathrm{d}y} &= \int \left(\frac{1}{y^2+1}\right) \mathop{\mathrm{d}y} \\
f(y) &= \arctan\left(y\right) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \ln\left(\cos\left(x\right)\right) + \arctan\left(y\right) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \ln\left(\cos\left(x\right)\right) + \arctan\left(y\right)
$$

The solution becomes

$$y = \tan\left(-\ln\left(\cos\left(x\right)\right) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \tan\left(c_1\right)$$

$$c_1 = \frac{\pi}{4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{i\cos\left(x\right)^{2i} - 1 + i + \cos\left(x\right)^{2i}}{i\cos\left(x\right)^{2i} + 1 + i - \cos\left(x\right)^{2i}}$$

Summary

The solution(s) found are the following

$$y = \frac{i\cos\left(x\right)^{2i} - 1 + i + \cos\left(x\right)^{2i}}{i\cos\left(x\right)^{2i} + 1 + i - \cos\left(x\right)^{2i}} \tag{1}$$

Verification of solutions

$$y = \frac{i\cos\left(x\right)^{2i} - 1 + i + \cos\left(x\right)^{2i}}{i\cos\left(x\right)^{2i} + 1 + i - \cos\left(x\right)^{2i}}$$

Verified OK.

3.2.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \left(y^2 + 1\right)\tan\left(x\right) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \tan\left(x\right)y^2 + \tan\left(x\right)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \tan(x)$, $f_1(x) = 0$ and $f_2(x) = \tan(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\tan(x) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \tan(x)^2 + 1 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \tan(x)^3 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\tan(x) u''(x) - \left(\tan(x)^2 + 1\right) u'(x) + \tan(x)^3 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \cos(x)^{-i} + c_2 \cos(x)^{i}$$

The above shows that

$$u'(x) = i \tan(x) \left(c_1 \cos(x)^{-i} - c_2 \cos(x)^{i}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{i\left(c_1 \cos(x)^{-i} - c_2 \cos(x)^{i}\right)}{c_1 \cos(x)^{-i} + c_2 \cos(x)^{i}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{i\left(-c_3 + \cos(x)^{2i}\right)}{\cos(x)^{2i} + c_3}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-c_3 i + i}{1 + c_3}$$

$$c_3 = i$$

Substituting c_3 found above in the general solution gives

$$y = \frac{i \cos(x)^{2i} + 1}{\cos(x)^{2i} + i}$$

Summary

The solution(s) found are the following

$$y = \frac{i \cos(x)^{2i} + 1}{\cos(x)^{2i} + i} \tag{1}$$

Verification of solutions

$$y = \frac{i \cos(x)^{2i} + 1}{\cos(x)^{2i} + i}$$

Verified OK.

3.2.6 Maple step by step solution

Let's solve

$[y' - (1 + y^2) \tan(x) = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = \tan(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int \tan(x) \, dx + c_1$

- Evaluate integral

  $\arctan(y) = -\ln(\cos(x)) + c_1$

- Solve for y

  $y = \tan(-\ln(\cos(x)) + c_1)$
- Use initial condition $y(0) = 1$

  $1 = \tan(c_1)$
- Solve for c_1

  $c_1 = \frac{\pi}{4}$
- Substitute $c_1 = \frac{\pi}{4}$ into general solution and simplify

  $y = \cot\left(\ln(\cos(x)) + \frac{\pi}{4}\right)$
- Solution to the IVP

  $y = \cot\left(\ln(\cos(x)) + \frac{\pi}{4}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 12

```
dsolve([diff(y(x),x)=(1+y(x)^2)*tan(x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \cot\left(\frac{\pi}{4} + \ln(\cos(x))\right)$$

✓ Solution by Mathematica

Time used: 0.472 (sec). Leaf size: 15

```
DSolve[{y'[x]==(1+y[x]^2)*Tan[x],{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cot\left(\log(\cos(x)) + \frac{\pi}{4}\right)$$

3.3 problem 8.1 (iii)

Internal problem ID [11985]
Internal file name [`OUTPUT/10637_Saturday_September_02_2023_02_48_51_PM_89378805/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.1 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - xt^2 = 0}$$

3.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= f(t)g(x) \\ &= t^2 x \end{aligned}$$

Where $f(t) = t^2$ and $g(x) = x$. Integrating both sides gives

$$\frac{1}{x}\,dx = t^2\,dt$$
$$\int \frac{1}{x}\,dx = \int t^2\,dt$$
$$\ln(x) = \frac{t^3}{3} + c_1$$
$$x = e^{\frac{t^3}{3}+c_1}$$
$$= c_1 e^{\frac{t^3}{3}}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{\frac{t^3}{3}} \tag{1}$$

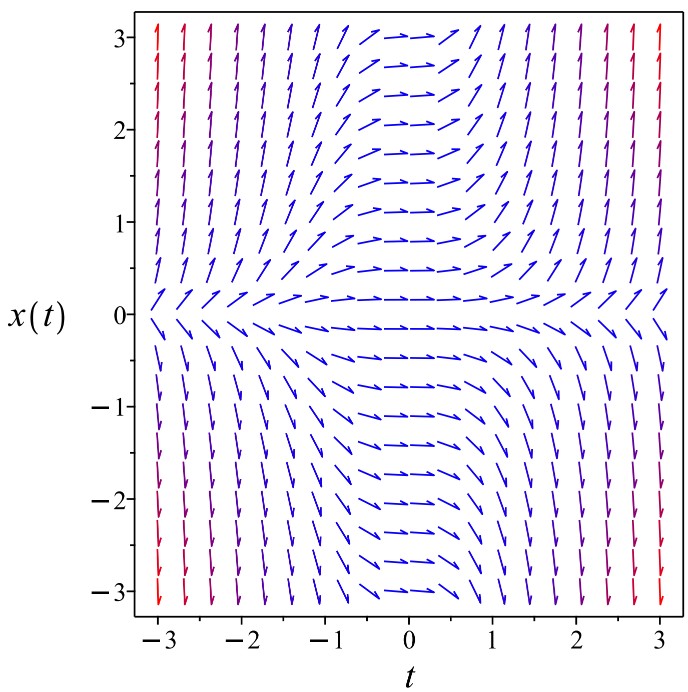


Figure 22: Slope field plot

Verification of solutions

$$x = c_1 e^{\frac{t^3}{3}}$$

Verified OK.

3.3.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -t^2 \\ q(t) &= 0 \end{aligned}$$

Hence the ode is

$$x' - x t^2 = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -t^2 dt} \\ &= \mathrm{e}^{-\frac{t^3}{3}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mu x &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{-\frac{t^3}{3}} x \right) &= 0 \end{aligned}$$

Integrating gives

$$\mathrm{e}^{-\frac{t^3}{3}} x = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{t^3}{3}}$ results in

$$x = c_1 \mathrm{e}^{\frac{t^3}{3}}$$

Summary
The solution(s) found are the following

$$x = c_1 \mathrm{e}^{\frac{t^3}{3}} \tag{1}$$

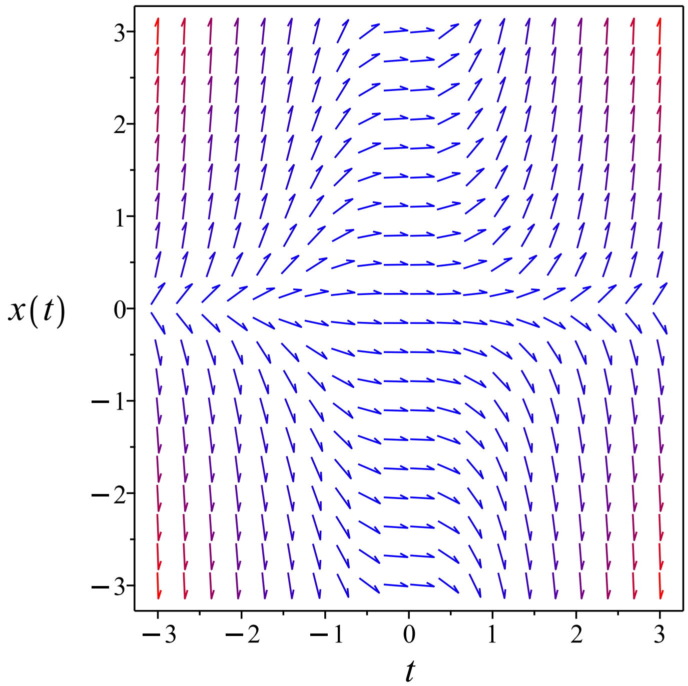


Figure 23: Slope field plot

Verification of solutions

$$x = c_1 e^{\frac{t^3}{3}}$$

Verified OK.

3.3.3 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u'(t)\, t + u(t) - u(t)\, t^3 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= \frac{u(t^3 - 1)}{t} \end{aligned}$$

Where $f(t) = \frac{t^3-1}{t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{t^3-1}{t}\, dt \\
\int \frac{1}{u}\, du &= \int \frac{t^3-1}{t}\, dt \\
\ln\left(u\right) &= \frac{t^3}{3} - \ln\left(t\right) + c_2 \\
u &= \mathrm{e}^{\frac{t^3}{3} - \ln(t) + c_2} \\
&= c_2 \mathrm{e}^{\frac{t^3}{3} - \ln(t)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = \frac{c_2 \mathrm{e}^{\frac{t^3}{3}}}{t}
$$

Therefore the solution x is

$$
\begin{aligned}
x &= tu \\
&= c_2 \mathrm{e}^{\frac{t^3}{3}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_2 \mathrm{e}^{\frac{t^3}{3}} \tag{1}
$$

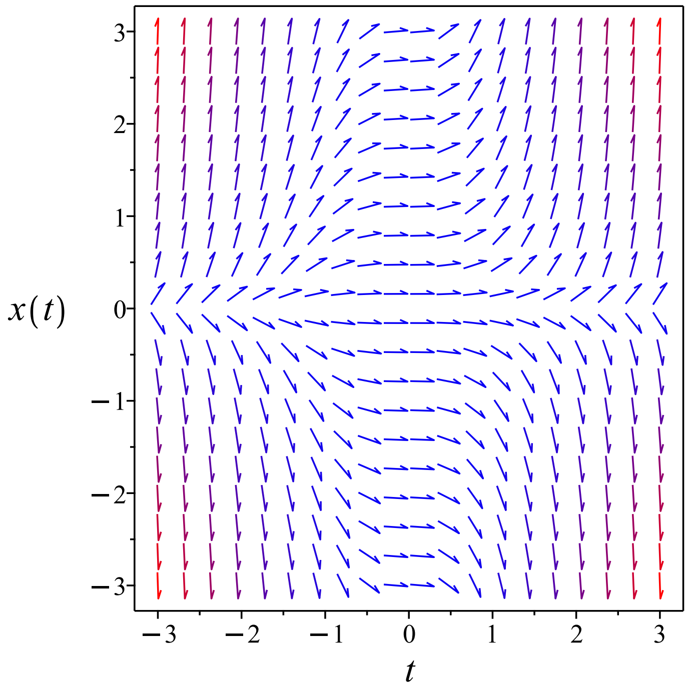


Figure 24: Slope field plot

Verification of solutions

$$x = c_2 e^{\frac{t^3}{3}}$$

Verified OK.

3.3.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = t^2 x$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 24: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, x) &= 0 \\ \eta(t, x) &= e^{\frac{t^3}{3}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{t^3}{3}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\frac{t^3}{3}} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = t^2 x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -t^2 \mathrm{e}^{-\frac{t^3}{3}} x \\ S_x &= \mathrm{e}^{-\frac{t^3}{3}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$e^{-\frac{t^3}{3}} x = c_1$$

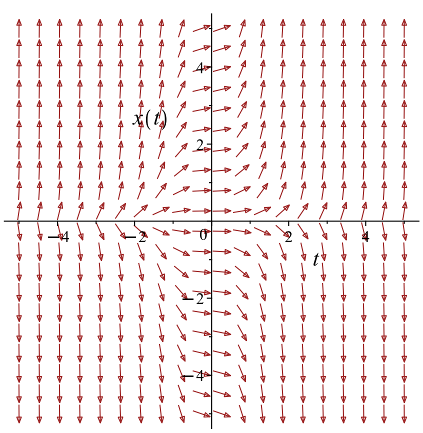
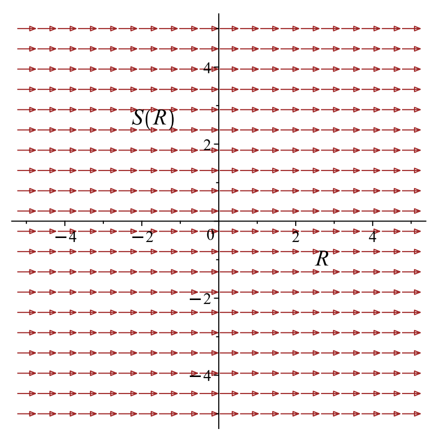
Which simplifies to

$$e^{-\frac{t^3}{3}} x = c_1$$

Which gives

$$x = c_1 e^{\frac{t^3}{3}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = t^2 x$ 	$R = t$ $S = e^{-\frac{t^3}{3}} x$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$x = c_1 e^{\frac{t^3}{3}} \tag{1}$$

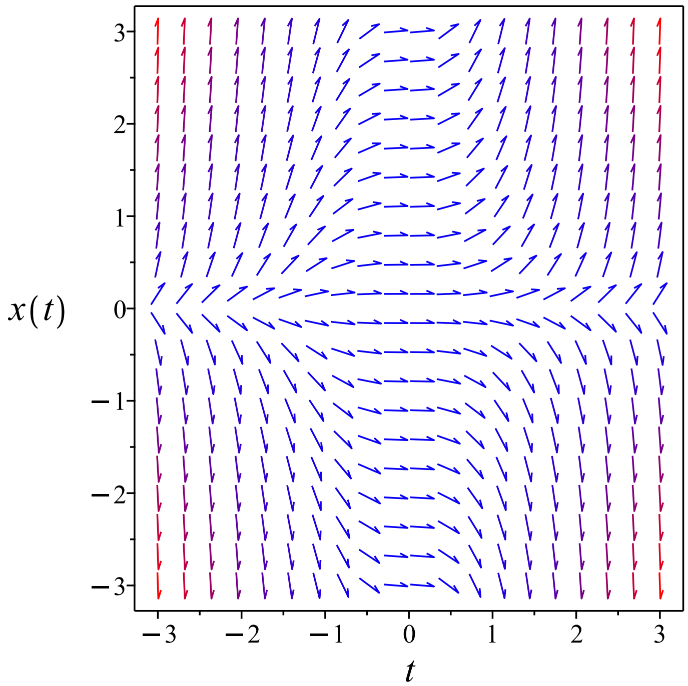


Figure 25: Slope field plot

Verification of solutions

$$x = c_1 e^{\frac{t^3}{3}}$$

Verified OK.

3.3.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x) \, \mathrm{d}t + N(t, x) \, \mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{x}\right) \mathrm{d}x = \left(t^2\right) \mathrm{d}t$$
$$\left(-t^2\right) \mathrm{d}t + \left(\frac{1}{x}\right) \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, x) = -t^2$$
$$N(t, x) = \frac{1}{x}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(-t^2\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{x}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t\\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t^2 \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{t^3}{3} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{x}$. Therefore equation (4) becomes

$$
\frac{1}{x} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = \frac{1}{x}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x) \, \mathrm{d}x &= \int \left(\frac{1}{x}\right) \mathrm{d}x\\
f(x) &= \ln(x) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^3}{3} + \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^3}{3} + \ln(x)$$

The solution becomes

$$x = e^{\frac{t^3}{3} + c_1}$$

Summary
The solution(s) found are the following

$$x = e^{\frac{t^3}{3} + c_1} \tag{1}$$

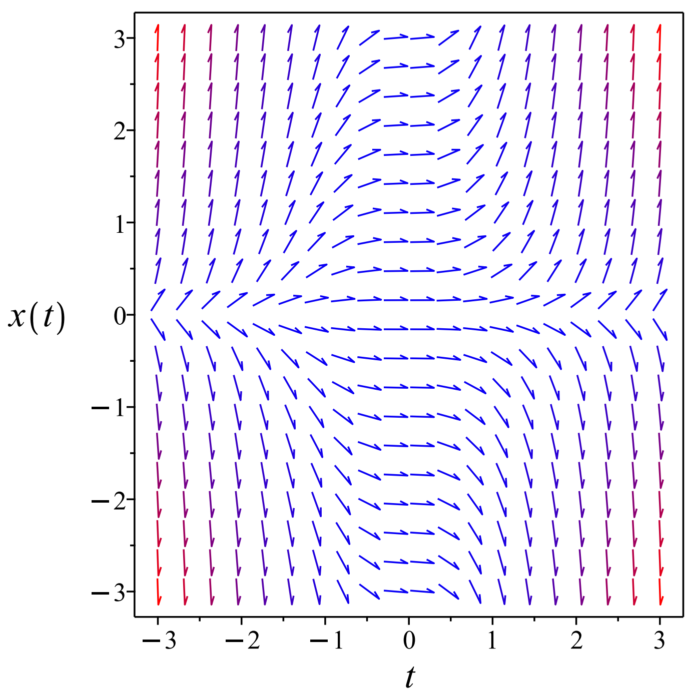


Figure 26: Slope field plot

Verification of solutions

$$x = e^{\frac{t^3}{3}+c_1}$$

Verified OK.

3.3.6 Maple step by step solution

Let's solve

$x' - x t^2 = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = t^2$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int t^2 dt + c_1$

- Evaluate integral

  $\ln(x) = \frac{t^3}{3} + c_1$

- Solve for x

  $x = e^{\frac{t^3}{3}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(x(t),t)=t^2*x(t),x(t), singsol=all)
```

$$x(t) = c_1 e^{\frac{t^3}{3}}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 22

```
DSolve[x'[t]==t^2*x[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{\frac{t^3}{3}}$$
$$x(t) \to 0$$

3.4 problem 8.1 (iv)

Internal problem ID [11986]
Internal file name [`OUTPUT/10638_Saturday_September_02_2023_02_48_51_PM_15979896/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.1 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x' + x^2 = 0}$$

3.4.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{x^2} dx = t + c_1$$
$$\frac{1}{x} = t + c_1$$

Solving for x gives these solutions

$$x_1 = \frac{1}{t + c_1}$$

Summary
The solution(s) found are the following

$$x = \frac{1}{t + c_1} \tag{1}$$

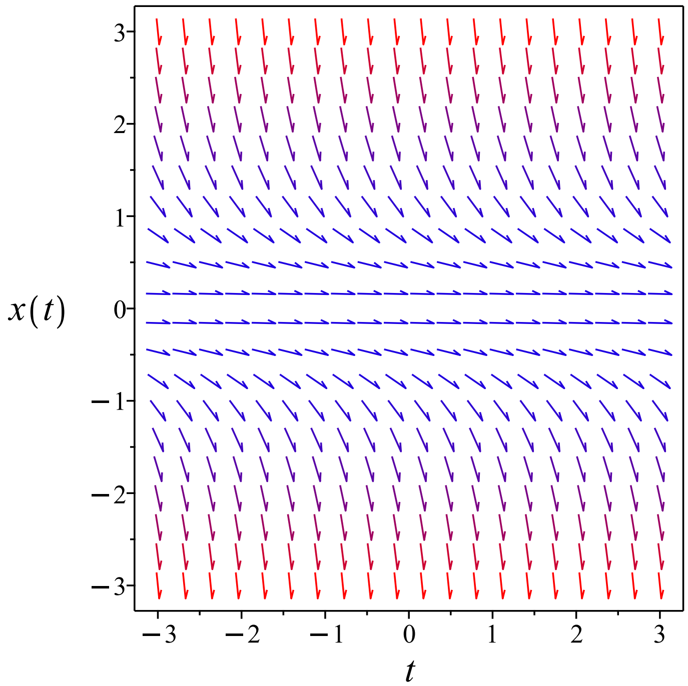


Figure 27: Slope field plot

Verification of solutions

$$x = \frac{1}{t + c_1}$$

Verified OK.

3.4.2 Maple step by step solution

Let's solve

$x' + x^2 = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x^2} = -1$

- Integrate both sides with respect to t

  $\int \frac{x'}{x^2} dt = \int (-1)\, dt + c_1$

- Evaluate integral

$$-\frac{1}{x} = -t + c_1$$

- Solve for x

$$x = -\frac{1}{-t + c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(x(t),t)=-x(t)^2,x(t), singsol=all)
```

$$x(t) = \frac{1}{t + c_1}$$

✓ Solution by Mathematica

Time used: 0.169 (sec). Leaf size: 18

```
DSolve[x'[t]==-x[t]^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{t - c_1}$$
$$x(t) \to 0$$

3.5 problem 8.1 (v)

Internal problem ID [11987]
Internal file name [`OUTPUT/10639_Saturday_September_02_2023_02_48_52_PM_81002036/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.1 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y^2 \mathrm{e}^{-t^2} = 0}$$

3.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t) g(y) \\ &= y^2 \mathrm{e}^{-t^2} \end{aligned}$$

Where $f(t) = \mathrm{e}^{-t^2}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\, dy = \mathrm{e}^{-t^2}\, dt$$

$$\int \frac{1}{y^2}\, dy = \int \mathrm{e}^{-t^2}\, dt$$

$$-\frac{1}{y} = \frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2} + c_1$$

Which results in

$$y = -\frac{2}{\sqrt{\pi}\, \operatorname{erf}(t) + 2c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{\sqrt{\pi}\, \operatorname{erf}(t) + 2c_1} \tag{1}$$

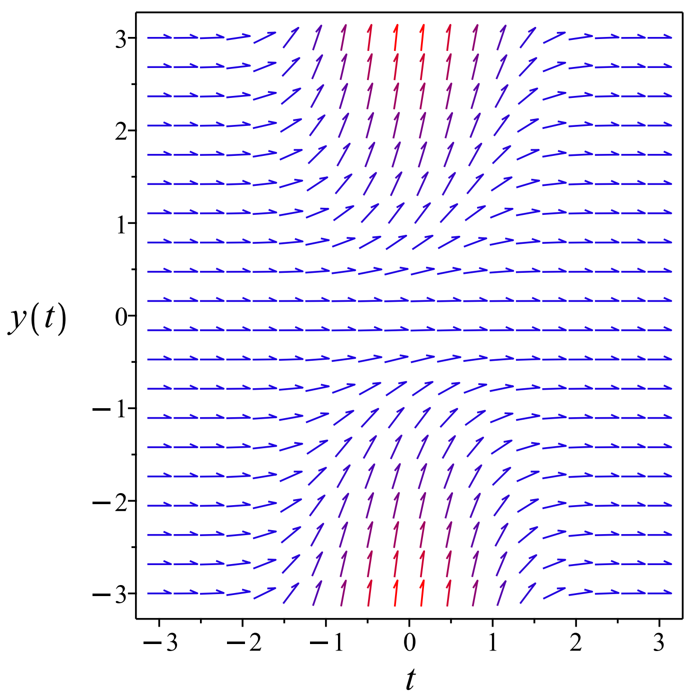


Figure 28: Slope field plot

Verification of solutions

$$y = -\frac{2}{\sqrt{\pi}\, \operatorname{erf}(t) + 2c_1}$$

Verified OK.

3.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y^2 e^{-t^2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 28: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= \mathrm{e}^{t^2} \\
\eta(t,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{\mathrm{e}^{t^2}} dt
\end{aligned}
$$

Which results in

$$
S = \frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = y^2 \mathrm{e}^{-t^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= \mathrm{e}^{-t^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2} = -\frac{1}{y} + c_1$$

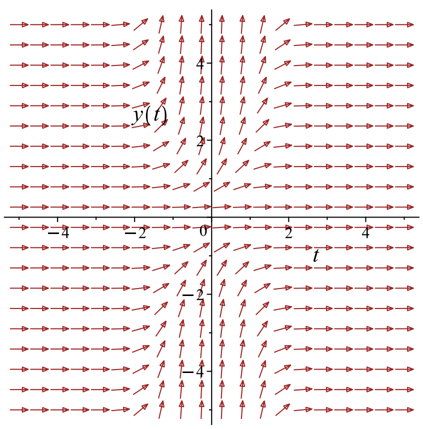
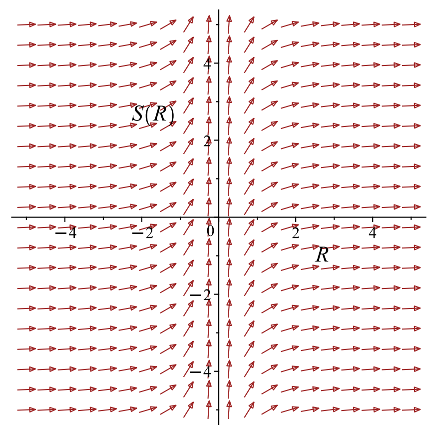
Which simplifies to

$$\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2} = -\frac{1}{y} + c_1$$

Which gives

$$y = -\frac{2}{\sqrt{\pi}\, \operatorname{erf}(t) - 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y^2 e^{-t^2}$ 	$R = y$ $S = \frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{2}{\sqrt{\pi}\, \mathrm{erf}(t) - 2c_1} \tag{1}$$

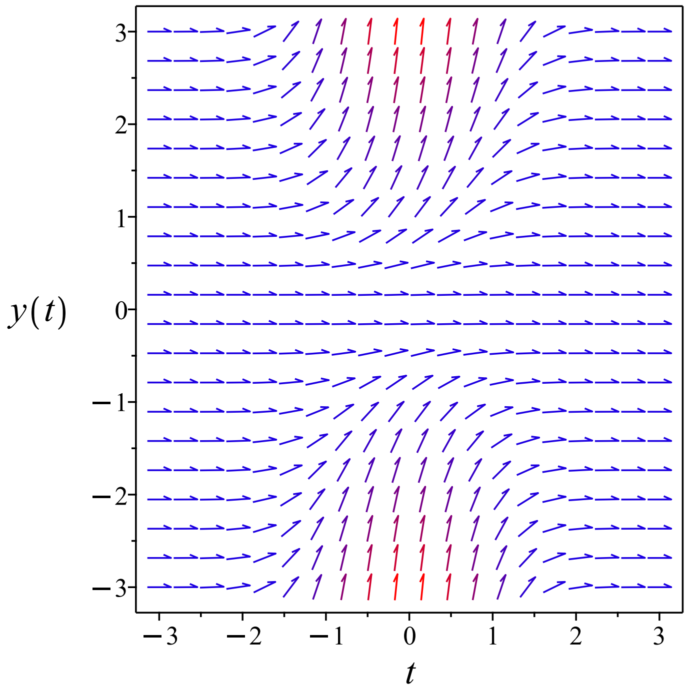


Figure 29: Slope field plot

Verification of solutions

$$y = -\frac{2}{\sqrt{\pi}\, \operatorname{erf}(t) - 2c_1}$$

Verified OK.

3.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2}\right)\mathrm{d}y &= \left(\mathrm{e}^{-t^2}\right)\mathrm{d}t\\ \left(-\mathrm{e}^{-t^2}\right)\mathrm{d}t + \left(\frac{1}{y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -\mathrm{e}^{-t^2}\\ N(t,y) &= \frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^{-t^2}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -\mathrm{e}^{-t^2} \,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{\sqrt{\pi}\,\operatorname{erf}(t)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$
\frac{1}{y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2}\right) \mathrm{d}y$$

$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sqrt{\pi}\ \mathrm{erf}\,(t)}{2} - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sqrt{\pi}\ \mathrm{erf}\,(t)}{2} - \frac{1}{y}$$

The solution becomes

$$y = -\frac{2}{\sqrt{\pi}\ \mathrm{erf}\,(t) + 2c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{\sqrt{\pi}\ \mathrm{erf}\,(t) + 2c_1} \tag{1}$$

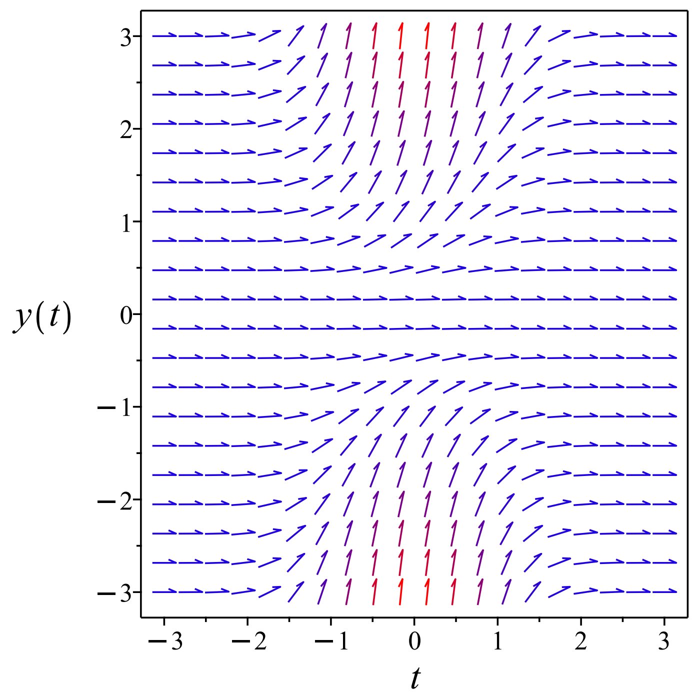


Figure 30: Slope field plot

Verification of solutions

$$y = -\frac{2}{\sqrt{\pi}\, \operatorname{erf}(t) + 2c_1}$$

Verified OK.

3.5.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= y^2 \mathrm{e}^{-t^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 \mathrm{e}^{-t^2}$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = e^{-t^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{e^{-t^2} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -2\,e^{-t^2} t \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$e^{-t^2} u''(t) + 2\,e^{-t^2} t u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + \operatorname{erf}(t)\, c_2$$

The above shows that

$$u'(t) = \frac{2\,e^{-t^2} c_2}{\sqrt{\pi}}$$

Using the above in (1) gives the solution

$$y = -\frac{2 c_2}{\sqrt{\pi}\, (c_1 + \operatorname{erf}(t)\, c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2}{\sqrt{\pi}\, (c_3 + \operatorname{erf}(t))}$$

Summary
The solution(s) found are the following

$$y = -\frac{2}{\sqrt{\pi}\,(c_3 + \operatorname{erf}(t))} \tag{1}$$

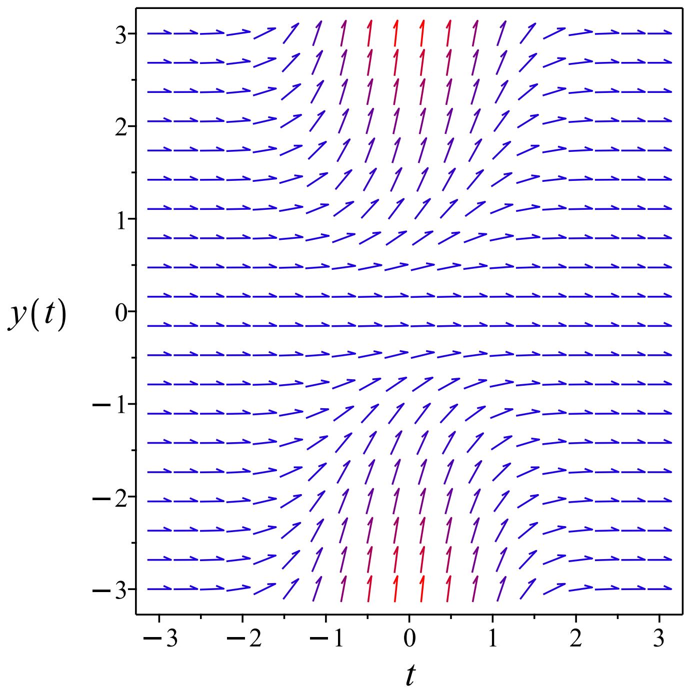


Figure 31: Slope field plot

Verification of solutions

$$y = -\frac{2}{\sqrt{\pi}\,(c_3 + \operatorname{erf}(t))}$$

Verified OK.

3.5.5 Maple step by step solution

Let's solve

$y' - y^2 e^{-t^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = e^{-t^2}$

- Integrate both sides with respect to t

$\int \frac{y'}{y^2} dt = \int e^{-t^2} dt + c_1$

- Evaluate integral

$-\frac{1}{y} = \frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2} + c_1$

- Solve for y

$y = -\frac{2}{\sqrt{\pi}\, \mathrm{erf}(t) + 2c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(t),t)=exp(-t^2)*y(t)^2,y(t), singsol=all)
```

$$y(t) = -\frac{2}{\sqrt{\pi}\, \mathrm{erf}(t) - 2c_1}$$

✓ Solution by Mathematica

Time used: 0.347 (sec). Leaf size: 27

```
DSolve[y'[t]==Exp[-t^2]*y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{2}{\sqrt{\pi}\mathrm{erf}(t) + 2c_1}$$
$$y(t) \to 0$$

3.6 problem 8.2

Internal problem ID [11988]
Internal file name [`OUTPUT/10640_Saturday_September_02_2023_02_48_52_PM_58373890/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.2 .
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x' + px = q}$$

3.6.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-px+q}dx = \int dt$$
$$-\frac{\ln(-px+q)}{p} = t + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln(-px+q)}{p}} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$(-px+q)^{-\frac{1}{p}} = c_2\mathrm{e}^t$$

Summary
The solution(s) found are the following

$$x = -\frac{(c_2\mathrm{e}^t)^{-p} - q}{p} \tag{1}$$

Verification of solutions

$$x = -\frac{\left(c_2 \mathrm{e}^{t}\right)^{-p} - q}{p}$$

Verified OK.

3.6.2 Maple step by step solution

Let's solve

$x' + px = q$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{-px+q} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{-px+q} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{\ln(-px+q)}{p} = t + c_1$

- Solve for x

  $x = -\frac{\mathrm{e}^{-c_1 p - tp} - q}{p}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(x(t),t)+p*x(t)=q,x(t), singsol=all)
```

$$x(t) = \frac{e^{-pt}c_1 p + q}{p}$$

✓ Solution by Mathematica

Time used: 0.063 (sec). Leaf size: 29

```
DSolve[x'[t]+p*x[t]==q,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{q}{p} + c_1 e^{-pt}$$
$$x(t) \to \frac{q}{p}$$

3.7 problem 8.3

Internal problem ID [11989]
Internal file name [`OUTPUT/10641_Saturday_September_02_2023_02_48_53_PM_12102685/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x - yk = 0}$$

3.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{yk}{x} \end{aligned}$$

Where $f(x) = \frac{k}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{k}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{k}{x}\,dx \\
\ln(y) &= k\ln(x) + c_1 \\
y &= \mathrm{e}^{k\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{k\ln(x)}
\end{aligned}$$

Which simplifies to

$$y = c_1 x^k$$

Summary
The solution(s) found are the following

$$y = c_1 x^k \tag{1}$$

Verification of solutions

$$y = c_1 x^k$$

Verified OK.

3.7.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{k}{x} \\
q(x) &= 0
\end{aligned}$$

Hence the ode is

$$y' - \frac{yk}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{k}{x}dx} \\
&= \mathrm{e}^{-k\ln(x)}
\end{aligned}$$

Which simplifies to

$$\mu = x^{-k}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-k}y\right) &= 0\end{aligned}$$

Integrating gives

$$x^{-k}y = c_1$$

Dividing both sides by the integrating factor $\mu = x^{-k}$ results in

$$y = c_1 x^k$$

Summary
The solution(s) found are the following

$$y = c_1 x^k \tag{1}$$

Verification of solutions

$$y = c_1 x^k$$

Verified OK.

3.7.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x - u(x)\,xk = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{u(k-1)}{x}\end{aligned}$$

Where $f(x) = \frac{k-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{k-1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{k-1}{x}\,dx \\
\ln(u) &= (k-1)\ln(x) + c_2 \\
u &= e^{(k-1)\ln(x)+c_2} \\
&= c_2 e^{(k-1)\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2 x^k}{x}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= c_2 x^k
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x^k \tag{1}$$

Verification of solutions

$$y = c_2 x^k$$

Verified OK.

3.7.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{yk}{x} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 32: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{k\ln(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int \frac{1}{\eta}dy\\&=\int \frac{1}{\mathrm{e}^{k\ln(x)}}dy\end{aligned}$$

Which results in

$$S=\mathrm{e}^{-k\ln(x)}y$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{yk}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=-ky\,x^{-1-k}\\S_y&=x^{-k}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^{-k} y = c_1$$

Which simplifies to

$$x^{-k} y = c_1$$

Which gives

$$y = c_1 x^k$$

Summary
The solution(s) found are the following

$$y = c_1 x^k \tag{1}$$

Verification of solutions

$$y = c_1 x^k$$

Verified OK.

3.7.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{yk}\right) \mathrm{d}y = \left(\frac{1}{x}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{1}{yk}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = \frac{1}{yk}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{yk}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{yk}$. Therefore equation (4) becomes

$$
\frac{1}{yk} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{yk}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{yk}\right) \mathrm{d}y \\
f(y) &= \frac{\ln(y)}{k} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{\ln(y)}{k} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{\ln(y)}{k}$$

The solution becomes

$$y = e^{k\ln(x)+c_1k}$$

Summary

The solution(s) found are the following

$$y = e^{k\ln(x)+c_1k} \tag{1}$$

Verification of solutions

$$y = e^{k\ln(x)+c_1k}$$

Verified OK.

3.7.6 Maple step by step solution

Let's solve

$y'x - yk = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{k}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{k}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = k\ln(x) + c_1$

- Solve for y

$$y = e^{k \ln(x)+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(x*diff(y(x),x)=k*y(x),y(x), singsol=all)
```

$$y(x) = c_1 x^k$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 16

```
DSolve[x*y'[x]==k*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x^k$$
$$y(x) \to 0$$

3.8 problem 8.4

Internal problem ID [11990]
Internal file name [`OUTPUT/10642_Saturday_September_02_2023_02_48_54_PM_51006278/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{i' - p(t)\, i = 0}$$

3.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} i' &= F(t, i) \\ &= f(t)g(i) \\ &= p(t)\, i \end{aligned}$$

Where $f(t) = p(t)$ and $g(i) = i$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{i}\, di &= p(t)\ dt \\
\int \frac{1}{i}\, di &= \int p(t)\ dt \\
\ln(i) &= \int p(t)\,dt + c_1 \\
i &= e^{\int p(t)dt + c_1} \\
&= c_1 e^{\int p(t)dt}
\end{aligned}$$

Summary

The solution(s) found are the following

$$i = c_1 e^{\int p(t)dt} \tag{1}$$

Verification of solutions

$$i = c_1 e^{\int p(t)dt}$$

Verified OK.

3.8.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$i' + p(t)i = q(t)$$

Where here

$$\begin{aligned}
p(t) &= -p(t) \\
q(t) &= 0
\end{aligned}$$

Hence the ode is

$$i' - p(t)\, i = 0$$

The integrating factor μ is

$$\mu = e^{\int -p(t)dt}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}\mu i &= 0 \\
\frac{d}{dt}\left(e^{\int -p(t)dt} i\right) &= 0
\end{aligned}$$

Integrating gives

$$e^{\int -p(t)dt} i = c_1$$

Dividing both sides by the integrating factor $\mu = e^{\int -p(t)dt}$ results in

$$i = c_1 e^{\int p(t)dt}$$

<u>Summary</u>
The solution(s) found are the following

$$i = c_1 e^{\int p(t)dt} \tag{1}$$

<u>Verification of solutions</u>

$$i = c_1 e^{\int p(t)dt}$$

Verified OK.

3.8.3 Solving as homogeneousTypeD2 ode

Using the change of variables $i = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u'(t)\, t + u(t) - p(t)\, u(t)\, t = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t) g(u) \\ &= \frac{u(p(t)\, t - 1)}{t} \end{aligned}$$

Where $f(t) = \frac{p(t)t-1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{p(t)\, t - 1}{t}\, dt \\ \int \frac{1}{u}\, du &= \int \frac{p(t)\, t - 1}{t}\, dt \\ \ln(u) &= \int \frac{p(t)\, t - 1}{t} dt + c_2 \\ u &= e^{\int \frac{p(t)t-1}{t} dt + c_2} \\ &= c_2 e^{\int \frac{p(t)t-1}{t} dt} \end{aligned}$$

Therefore the solution i is

$$
\begin{aligned}
i &= tu \\
&= t c_2 \mathrm{e}^{\int \frac{p(t)t-1}{t}dt}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
i = t c_2 \mathrm{e}^{\int \frac{p(t)t-1}{t}dt} \tag{1}
$$

Verification of solutions

$$
i = t c_2 \mathrm{e}^{\int \frac{p(t)t-1}{t}dt}
$$

Verified OK.

3.8.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
i' &= p(t)\, i \\
i' &= \omega(t, i)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_i - \xi_t) - \omega^2 \xi_i - \omega_t \xi - \omega_i \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 35: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, i) &= 0 \\
\eta(t, i) &= \mathrm{e}^{\int p(t) dt}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, i) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{di}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial i}\right) S(t, i) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\int p(t)dt}} dy \end{aligned}$$

3.8.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,i)\,\mathrm{d}t + N(t,i)\,\mathrm{d}i = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{i}\right)\mathrm{d}i &= (p(t))\,\mathrm{d}t\\ (-p(t))\,\mathrm{d}t + \left(\frac{1}{i}\right)\mathrm{d}i &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,i) &= -p(t)\\ N(t,i) &= \frac{1}{i}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial i} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial i} &= \frac{\partial}{\partial i}(-p(t))\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{i}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial i} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,i)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial i} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int M \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int -p(t) \, \mathrm{d}t$$

$$\phi = \int^{t} -p(\_a)\, d\_a + f(i) \tag{3}$$

Where $f(i)$ is used for the constant of integration since ϕ is a function of both t and i. Taking derivative of equation (3) w.r.t i gives

$$\frac{\partial \phi}{\partial i} = 0 + f'(i) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial i} = \frac{1}{i}$. Therefore equation (4) becomes

$$\frac{1}{i} = 0 + f'(i) \tag{5}$$

Solving equation (5) for $f'(i)$ gives

$$f'(i) = \frac{1}{i}$$

Integrating the above w.r.t i gives

$$\int f'(i)\, \mathrm{d}i = \int \left(\frac{1}{i}\right) \mathrm{d}i$$

$$f(i) = \ln(i) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(i)$ into equation (3) gives ϕ

$$\phi = \int^{t} -p(\_a)\, d\_a + \ln(i) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^{t} -p(\_a)\, d\_a + \ln(i)$$

The solution becomes

$$i = e^{-\left(\int^{t} -p(\_a)d\_a\right)+c_1}$$

Summary

The solution(s) found are the following

$$i = e^{-\left(\int^{t} -p(\_a)d\_a\right)+c_1} \tag{1}$$

Verification of solutions

$$i = e^{-\left(\int^{t} -p(\_a)d\_a\right)+c_1}$$

Verified OK.

3.8.6 Maple step by step solution

Let's solve

$i' - p(t)\, i = 0$

- Highest derivative means the order of the ODE is 1

  i'

- Separate variables

  $\frac{i'}{i} = p(t)$

- Integrate both sides with respect to t

  $\int \frac{i'}{i} dt = \int p(t)\, dt + c_1$

- Evaluate integral

  $\ln(i) = \int p(t)\, dt + c_1$

- Solve for i

  $i = e^{\int p(t)dt+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(i(t),t)=p(t)*i(t),i(t), singsol=all)
```

$$i(t) = c_1 e^{\int p(t)dt}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 25

```
DSolve[i'[t]==p[t]*i[t],i[t],t,IncludeSingularSolutions -> True]
```

$$i(t) \to c_1 \exp\left(\int_1^t p(K[1])dK[1]\right)$$
$$i(t) \to 0$$

3.9 problem 8.5

Internal problem ID [11991]
Internal file name [`OUTPUT/10643_Saturday_September_02_2023_02_48_54_PM_64009715/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - \lambda x = 0}$$

3.9.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\lambda x} dx = \int dt$$
$$\frac{\ln(x)}{\lambda} = t + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln(x)}{\lambda}} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$x^{\frac{1}{\lambda}} = c_2 \mathrm{e}^t$$

Summary
The solution(s) found are the following

$$x = \left(c_2 \mathrm{e}^t\right)^{\lambda} \tag{1}$$

Verification of solutions

$$x = (c_2 e^t)^\lambda$$

Verified OK.

3.9.2 Maple step by step solution

Let's solve

$x' - \lambda x = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = \lambda$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int \lambda dt + c_1$

- Evaluate integral

  $\ln(x) = \lambda t + c_1$

- Solve for x

  $x = e^{\lambda t + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(x(t),t)=lambda*x(t),x(t), singsol=all)
```

$$x(t) = c_1 e^{\lambda t}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 18

```
DSolve[x'[t]==\[Lambda]*x[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{\lambda t}$$
$$x(t) \to 0$$

3.10 problem 8.6

Internal problem ID [11992]
Internal file name [`OUTPUT/10644_Saturday_September_02_2023_02_48_55_PM_1776475/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{mv' - kv^2 = -mg}$$

3.10.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{m}{k\,v^2 - mg} dv = t + c_1$$
$$-\frac{m \operatorname{arctanh}\left(\frac{kv}{\sqrt{mgk}}\right)}{\sqrt{mgk}} = t + c_1$$

Solving for v gives these solutions

$$v_1 = -\frac{\tanh\left(\frac{\sqrt{mgk}\,(t+c_1)}{m}\right)\sqrt{mgk}}{k}$$

Summary
The solution(s) found are the following

$$v = -\frac{\tanh\left(\frac{\sqrt{mgk}\,(t+c_1)}{m}\right)\sqrt{mgk}}{k} \tag{1}$$

Verification of solutions

$$v = -\frac{\tanh\left(\frac{\sqrt{mgk}\,(t+c_1)}{m}\right)\sqrt{mgk}}{k}$$

Verified OK.

3.10.2 Maple step by step solution

Let's solve

$mv' - kv^2 = -mg$

- Highest derivative means the order of the ODE is 1

  v'

- Separate variables

  $\frac{v'}{-mg+kv^2} = \frac{1}{m}$

- Integrate both sides with respect to t

  $\int \frac{v'}{-mg+kv^2}dt = \int \frac{1}{m}dt + c_1$

- Evaluate integral

  $-\frac{\operatorname{arctanh}\left(\frac{vk}{\sqrt{mgk}}\right)}{\sqrt{mgk}} = \frac{t}{m} + c_1$

- Solve for v

  $v = -\frac{\tanh\left(\frac{\sqrt{mgk}\,(c_1 m+t)}{m}\right)\sqrt{mgk}}{k}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(m*diff(v(t),t)=-m*g+k*v(t)^2,v(t), singsol=all)
```

$$v(t) = -\frac{\tanh\left(\frac{\sqrt{mgk}\,(t+c_1)}{m}\right)\sqrt{mgk}}{k}$$

✓ Solution by Mathematica

Time used: 14.167 (sec). Leaf size: 87

```
DSolve[m*v'[t]==-m*g+k*v[t]^2,v[t],t,IncludeSingularSolutions -> True]
```

$$v(t) \to \frac{\sqrt{g}\sqrt{m}\tanh\left(\frac{\sqrt{g}\sqrt{k}(-t+c_1m)}{\sqrt{m}}\right)}{\sqrt{k}}$$
$$v(t) \to -\frac{\sqrt{g}\sqrt{m}}{\sqrt{k}}$$
$$v(t) \to \frac{\sqrt{g}\sqrt{m}}{\sqrt{k}}$$

3.11 problem 8.7

Internal problem ID [11993]
Internal file name [`OUTPUT/10645_Saturday_September_02_2023_02_48_56_PM_36373413/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - kx + x^2 = 0}$$

With initial conditions

$$[x(0) = x_0]$$

3.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t, x) \\ &= kx - x^2 \end{aligned}$$

The x domain of $f(t, x)$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

But the point $x_0 = x_0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

3.11.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{kx - x^2} dx = \int dt$$
$$-\frac{\ln(-k+x)}{k} + \frac{\ln(x)}{k} = t + c_1$$

The above can be written as

$$\left(-\frac{1}{k}\right)(\ln(-k+x) - \ln(x)) = t + c_1$$
$$\ln(-k+x) - \ln(x) = (-k)(t + c_1)$$
$$= -k(t + c_1)$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(-k+x)-\ln(x)} = -kc_1\mathrm{e}^{-kt}$$

Which simplifies to

$$\frac{-k+x}{x} = c_2\mathrm{e}^{-kt}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $x = x_0$ in the above solution gives an equation to solve for the constant of integration.

$$x_0 = -\frac{k}{-1 + c_2}$$

$$c_2 = -\frac{k - x_0}{x_0}$$

Substituting c_2 found above in the general solution gives

$$x = \frac{x_0 k}{\mathrm{e}^{-kt}k - \mathrm{e}^{-kt}x_0 + x_0}$$

Summary

The solution(s) found are the following

$$x = \frac{x_0 k}{\mathrm{e}^{-kt}k - \mathrm{e}^{-kt}x_0 + x_0} \tag{1}$$

Verification of solutions

$$x = \frac{x_0 k}{\mathrm{e}^{-kt}k - \mathrm{e}^{-kt}x_0 + x_0}$$

Verified OK.

3.11.3 Maple step by step solution

Let's solve

$[x' - kx + x^2 = 0, x(0) = x_0]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{kx - x^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{kx - x^2} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{\ln(x-k)}{k} + \frac{\ln(x)}{k} = t + c_1$

- Solve for x

  $x = \frac{k\, e^{c_1 k + kt}}{-1 + e^{c_1 k + kt}}$

- Use initial condition $x(0) = x_0$

  $x_0 = \frac{k\, e^{c_1 k}}{-1 + e^{c_1 k}}$

- Solve for c_1

  $c_1 = \frac{\ln\left(-\frac{x_0}{k - x_0}\right)}{k}$

- Substitute $c_1 = \frac{\ln\left(-\frac{x_0}{k - x_0}\right)}{k}$ into general solution and simplify

  $x = \frac{e^{kt} x_0 k}{x_0 e^{kt} + k - x_0}$

- Solution to the IVP

  $x = \frac{e^{kt} x_0 k}{x_0 e^{kt} + k - x_0}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 22

```
dsolve([diff(x(t),t)=k*x(t)-x(t)^2,x(0) = x__0],x(t), singsol=all)
```

$$x(t) = \frac{k x_0}{(-x_0 + k)\,\mathrm{e}^{-kt} + x_0}$$

✓ Solution by Mathematica

Time used: 1.052 (sec). Leaf size: 26

```
DSolve[{x'[t]==k*x[t]-x[t]^2,{x[0]==x0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{k\mathrm{x0}e^{kt}}{\mathrm{x0}\left(e^{kt} - 1\right) + k}$$

3.12 problem 8.8

Internal problem ID [11994]
Internal file name [`OUTPUT/10646_Saturday_September_02_2023_02_48_57_PM_58898071/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 8, Separable equations. Exercises page 72
Problem number: 8.8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

Unable to solve or complete the solution.

$$\boxed{x' + x(k^2 + x^2) = 0}$$

With initial conditions

$$[x(0) = x_0]$$

3.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t, x) \\ &= -x(k^2 + x^2) \end{aligned}$$

The x domain of $f(t, x)$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

But the point $x_0 = x_0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

3.12.2 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{x\left(k^2+x^2\right)}dx = \int dt$$
$$-\frac{\ln\left(x\right)}{k^2}+\frac{\ln\left(k^2+x^2\right)}{2k^2} = t+c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln(x)}{k^2}+\frac{\ln\left(k^2+x^2\right)}{2k^2}} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$x^{-\frac{1}{k^2}}\left(k^2+x^2\right)^{\frac{1}{2k^2}} = c_2\mathrm{e}^t$$

Initial conditions are used to solve for c_2. Substituting $t=0$ and $x=x_0$ in the above solution gives an equation to solve for the constant of integration.

$$x_0 = \frac{k}{\sqrt{\left(\frac{1}{c_2}\right)^{-2k^2}-1}}$$

$$c_2 = \mathrm{e}^{\frac{\ln\left(\sqrt{\frac{k^2+x_0^2}{x_0^2}}\right)}{k^2}}$$

Substituting c_2 found above in the general solution gives

$$x = \frac{k}{\sqrt{\left(\left(\frac{k^2+x_0^2}{x_0^2}\right)^{-\frac{1}{2k^2}}\right)^{-2k^2}\mathrm{e}^{2t\,k^2}-1}}$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

3.12.3 Maple step by step solution

Let's solve

$[x'+x(k^2+x^2)=0, x(0)=x_0]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x(k^2+x^2)} = -1$

- Integrate both sides with respect to t

  $\int \frac{x'}{x(k^2+x^2)}dt = \int (-1)\, dt + c_1$

- Evaluate integral

  $\frac{\ln(x)}{k^2} - \frac{\ln(k^2+x^2)}{2k^2} = -t + c_1$

- Solve for x

  $\left\{x = \frac{\sqrt{-\left(\mathrm{e}^{2c_1 k^2-2t\, k^2}-1\right)\mathrm{e}^{2c_1 k^2-2t\, k^2}}\, k}{\mathrm{e}^{2c_1 k^2-2t\, k^2}-1}, x = -\frac{\sqrt{-\left(\mathrm{e}^{2c_1 k^2-2t\, k^2}-1\right)\mathrm{e}^{2c_1 k^2-2t\, k^2}}\, k}{\mathrm{e}^{2c_1 k^2-2t\, k^2}-1}\right\}$

- Use initial condition $x(0) = x_0$

  $x_0 = \frac{\sqrt{-\left(\mathrm{e}^{2c_1 k^2}-1\right)\mathrm{e}^{2c_1 k^2}}\, k}{\mathrm{e}^{2c_1 k^2}-1}$

- Solve for c_1

  $c_1 = \frac{\ln\left(\frac{x_0^2}{k^2+x_0^2}\right)}{2k^2}$

- Substitute $c_1 = \frac{\ln\left(\frac{x_0^2}{k^2+x_0^2}\right)}{2k^2}$ into general solution and simplify

  $x = -\frac{\sqrt{\frac{\mathrm{e}^{-2t\, k^2} x_0^2\left(-x_0^2 \mathrm{e}^{-2t\, k^2}+k^2+x_0^2\right)}{\left(k^2+x_0^2\right)^2}}\, k\left(k^2+x_0^2\right)}{-x_0^2 \mathrm{e}^{-2t\, k^2}+k^2+x_0^2}$

- Use initial condition $x(0) = x_0$

  $x_0 = -\frac{\sqrt{-\left(\mathrm{e}^{2c_1 k^2}-1\right)\mathrm{e}^{2c_1 k^2}}\, k}{\mathrm{e}^{2c_1 k^2}-1}$

- Solve for c_1

  $c_1 = \frac{\ln\left(\frac{x_0^2}{k^2+x_0^2}\right)}{2k^2}$

- Substitute $c_1 = \frac{\ln\left(\frac{x_0^2}{k^2+x_0^2}\right)}{2k^2}$ into general solution and simplify

  $x = \frac{\sqrt{\frac{\mathrm{e}^{-2t\, k^2} x_0^2\left(-x_0^2 \mathrm{e}^{-2t\, k^2}+k^2+x_0^2\right)}{\left(k^2+x_0^2\right)^2}}\, k\left(k^2+x_0^2\right)}{-x_0^2 \mathrm{e}^{-2t\, k^2}+k^2+x_0^2}$

- Solutions to the IVP

$$\left\{x=\frac{\sqrt{\frac{e^{-2t\,k^2}x_0^2\left(-x_0^2e^{-2t\,k^2}+k^2+x_0^2\right)}{\left(k^2+x_0^2\right)^2}}\,k\left(k^2+x_0^2\right)}{-x_0^2e^{-2t\,k^2}+k^2+x_0^2},x=-\frac{\sqrt{\frac{e^{-2t\,k^2}x_0^2\left(-x_0^2e^{-2t\,k^2}+k^2+x_0^2\right)}{\left(k^2+x_0^2\right)^2}}\,k\left(k^2+x_0^2\right)}{-x_0^2e^{-2t\,k^2}+k^2+x_0^2}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✗ Solution by Maple

```
dsolve([diff(x(t),t)=-x(t)*(k^2+x(t)^2),x(0) = x__0],x(t), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 1.848 (sec). Leaf size: 62

```
DSolve[{x'[t]==-x[t]*(k^2+x[t]^2),{x[0]==x0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t)\to -\frac{k}{\sqrt{e^{2k^2t}\left(\frac{k^2}{\text{x0}^2}+1\right)-1}}$$
$$x(t)\to \frac{k}{\sqrt{e^{2k^2t}\left(\frac{k^2}{\text{x0}^2}+1\right)-1}}$$

4 Chapter 9, First order linear equations and the integrating factor. Exercises page 86

4.1 problem 9.1 (i)

Internal problem ID [11995]
Internal file name [`OUTPUT/10647_Saturday_September_02_2023_02_48_59_PM_41369347/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 9, First order linear equations and the integrating factor. Exercises page 86
Problem number: 9.1 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

Unable to solve or complete the solution.

$$\boxed{y' + \frac{y}{x} = x^2}$$

With initial conditions

$$[y(0) = y_0]$$

4.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= x^2 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = x^2$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

4.1.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(x^2\right) \\ \frac{d}{dx}(xy) &= (x)\left(x^2\right) \\ d(xy) &= x^3\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} xy &= \int x^3\, dx \\ xy &= \frac{x^4}{4} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{x^3}{4} + \frac{c_1}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = y_0$ in the above solution gives an equation to solve for the constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

4.1.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{y}{x} + x^2 \tag{1}$$

Which becomes

$$0 = (-x)\,dy + \left(x^3 - y\right)dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + \left(x^3 - y\right)dx = d\left(\frac{1}{4}x^4 - xy\right)$$

Hence (2) becomes

$$0 = d\left(\frac{1}{4}x^4 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^4 + 4c_1}{4x} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = y_0$ in the above solution gives an equation to solve for the constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

4.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^3 + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 42: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x}} dy
\end{aligned}
$$

Which results in

$$
S = xy
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{-x^3 + y}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \\
S_y &= x
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = x^3 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R^3
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^4}{4} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
yx = \frac{x^4}{4} + c_1
$$

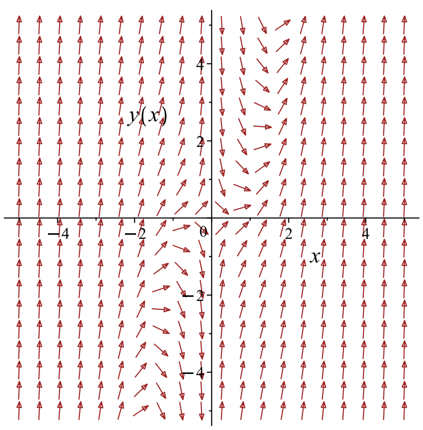
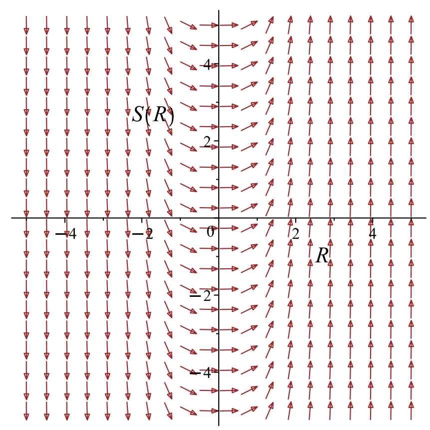
Which simplifies to

$$
yx = \frac{x^4}{4} + c_1
$$

Which gives

$$
y = \frac{x^4 + 4c_1}{4x}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^3+y}{x}$ 	$R = x$ $S = xy$	$\frac{dS}{dR} = R^3$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = y_0$ in the above solution gives an equation to solve for the constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

4.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x) \,\mathrm{d}y &= \left(x^3 - y\right) \mathrm{d}x \\
\left(-x^3 + y\right) \mathrm{d}x + (x) \,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^3 + y \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3 + y\right) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int -x^3 + y \, dx$$

$$\phi = -\frac{1}{4}x^4 + xy + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{4}x^4 + xy + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{4}x^4 + xy$$

The solution becomes

$$y = \frac{x^4 + 4c_1}{4x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = y_0$ in the above solution gives an equation to solve for the constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✗ Solution by Maple

```
dsolve([diff(y(x),x)+y(x)/x=x^2,y(0) = y__0],y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y'[x]+y[x]/x==x^2,{y[0]==y0}},y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.2 problem 9.1 (ii)

Internal problem ID [11996]
Internal file name [`OUTPUT/10648_Saturday_September_02_2023_02_49_01_PM_55576453/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 9, First order linear equations and the integrating factor. Exercises page 86
Problem number: 9.1 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x' + xt = 4t}$$

With initial conditions

$$[x(0) = 2]$$

4.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = t$$
$$q(t) = 4t$$

Hence the ode is

$$x' + xt = 4t$$

The domain of $p(t) = t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.2.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= f(t)g(x) \\ &= t(4 - x) \end{aligned}$$

Where $f(t) = t$ and $g(x) = 4 - x$. Integrating both sides gives

$$\begin{aligned} \frac{1}{4 - x}\, dx &= t\, dt \\ \int \frac{1}{4 - x}\, dx &= \int t\, dt \\ -\ln(-4 + x) &= \frac{t^2}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{-4 + x} = e^{\frac{t^2}{2} + c_1}$$

Which simplifies to

$$\frac{1}{-4 + x} = e^{\frac{t^2}{2}} c_2$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{4\,\mathrm{e}^{c_1}\mathrm{e}^{-c_1}c_2 + \mathrm{e}^{-c_1}}{c_2}$$

$$c_1 = -\ln\left(-2c_2\right)$$

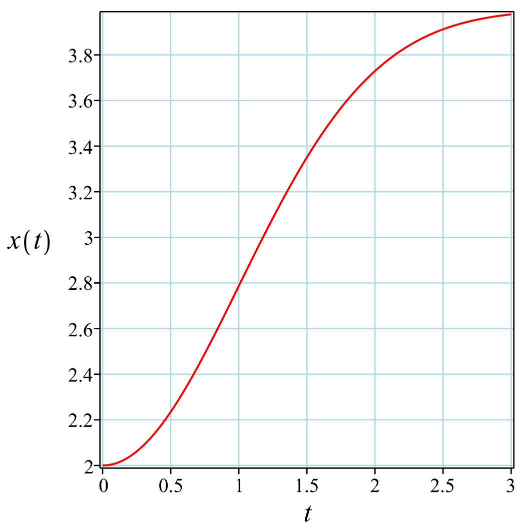
Substituting c_1 found above in the general solution gives

$$x = 4 - 2\,\mathrm{e}^{-\frac{t^2}{2}}$$

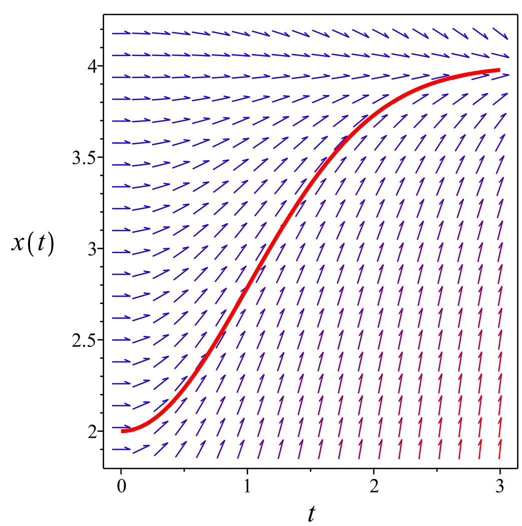
Summary

The solution(s) found are the following

$$x = 4 - 2\,\mathrm{e}^{-\frac{t^2}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 4 - 2\,\mathrm{e}^{-\frac{t^2}{2}}$$

Verified OK.

4.2.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int t dt}\\ &= \mathrm{e}^{\frac{t^2}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)(4t)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{t^2}{2}} x\right) &= \left(\mathrm{e}^{\frac{t^2}{2}}\right)(4t)\\ \mathrm{d}\left(\mathrm{e}^{\frac{t^2}{2}} x\right) &= \left(4t\,\mathrm{e}^{\frac{t^2}{2}}\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{\frac{t^2}{2}} x &= \int 4t\,\mathrm{e}^{\frac{t^2}{2}}\,\mathrm{d}t\\ \mathrm{e}^{\frac{t^2}{2}} x &= 4\,\mathrm{e}^{\frac{t^2}{2}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{t^2}{2}}$ results in

$$x = 4\,\mathrm{e}^{-\frac{t^2}{2}}\mathrm{e}^{\frac{t^2}{2}} + \mathrm{e}^{-\frac{t^2}{2}} c_1$$

which simplifies to

$$x = 4 + \mathrm{e}^{-\frac{t^2}{2}} c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 4 + c_1$$

$$c_1 = -2$$

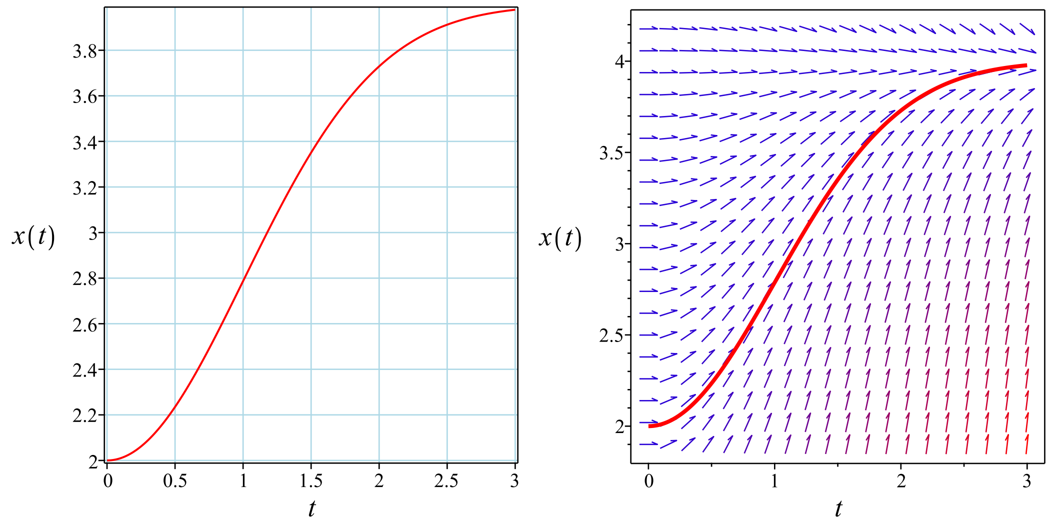
Substituting c_1 found above in the general solution gives

$$x = 4 - 2\,\mathrm{e}^{-\frac{t^2}{2}}$$

Summary

The solution(s) found are the following

$$x = 4 - 2\,\mathrm{e}^{-\frac{t^2}{2}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$x = 4 - 2\,\mathrm{e}^{-\frac{t^2}{2}}$$

Verified OK.

4.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} x' &= -tx + 4t \\ x' &= \omega(t, x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 44: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,x) = 0$$
$$\eta(t,x) = \mathrm{e}^{-\frac{t^2}{2}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{t^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{t^2}{2}} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = -tx + 4t$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= t\, e^{\frac{t^2}{2}} x \\ S_x &= e^{\frac{t^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4t\, e^{\frac{t^2}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4R\, e^{\frac{R^2}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 4\,\mathrm{e}^{\frac{R^2}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\mathrm{e}^{\frac{t^2}{2}} x = 4\,\mathrm{e}^{\frac{t^2}{2}} + c_1$$

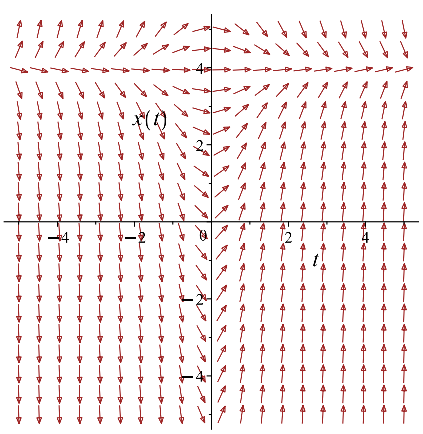
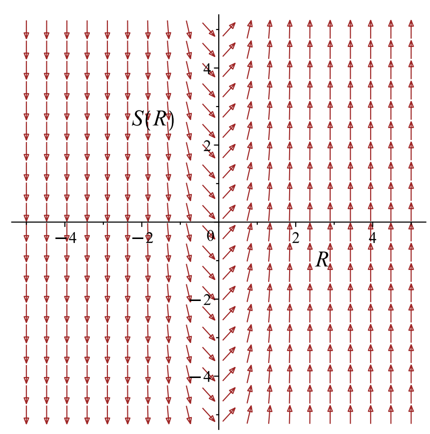
Which simplifies to

$$(-4 + x)\,\mathrm{e}^{\frac{t^2}{2}} - c_1 = 0$$

Which gives

$$x = \left(4\,\mathrm{e}^{\frac{t^2}{2}} + c_1\right)\mathrm{e}^{-\frac{t^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -tx + 4t$ 	$R = t$ $S = \mathrm{e}^{\frac{t^2}{2}} x$	$\frac{dS}{dR} = 4R\,\mathrm{e}^{\frac{R^2}{2}}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 4 + c_1$$

$$c_1 = -2$$

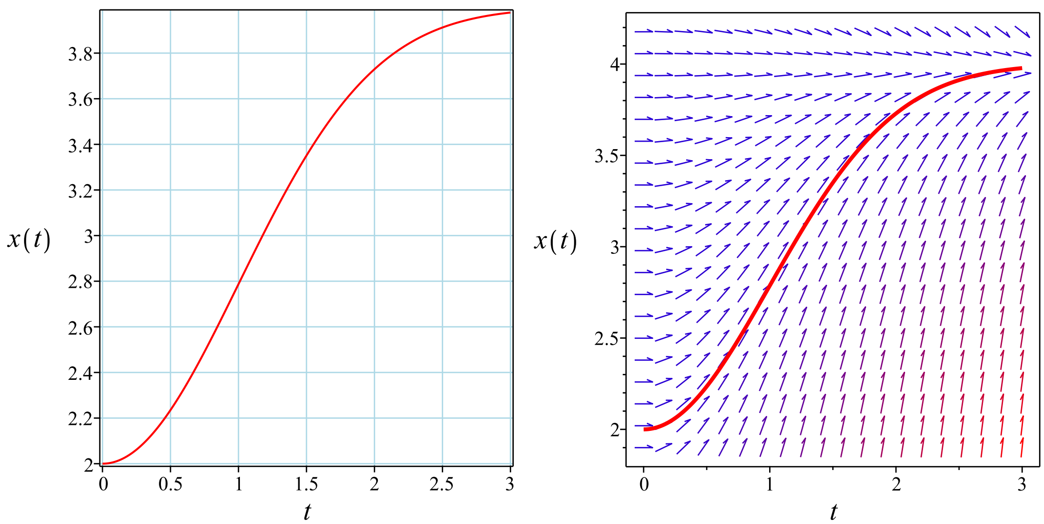
Substituting c_1 found above in the general solution gives

$$x = 4 - 2\,\mathrm{e}^{-\frac{t^2}{2}}$$

Summary

The solution(s) found are the following

$$x = 4 - 2\,\mathrm{e}^{-\frac{t^2}{2}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = 4 - 2\,\mathrm{e}^{-\frac{t^2}{2}}$$

Verified OK.

4.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{4-x}\right)\mathrm{d}x&=(t)\,\mathrm{d}t\\ (-t)\,\mathrm{d}t+\left(\frac{1}{4-x}\right)\mathrm{d}x&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-t\\ N(t,x)&=\frac{1}{4-x}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x}&=\frac{\partial}{\partial x}(-t)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{4-x}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{t^2}{2} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{4-x}$. Therefore equation (4) becomes

$$
\frac{1}{4-x} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = -\frac{1}{-4+x}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x) \, \mathrm{d}x &= \int \left(-\frac{1}{-4+x}\right) \mathrm{d}x \\
f(x) &= -\ln(-4+x) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} - \ln(-4+x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} - \ln(-4+x)$$

The solution becomes

$$x = e^{-\frac{t^2}{2}-c_1} + 4$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = e^{-c_1} + 4$$

$$c_1 = -\ln(2) - i\pi$$

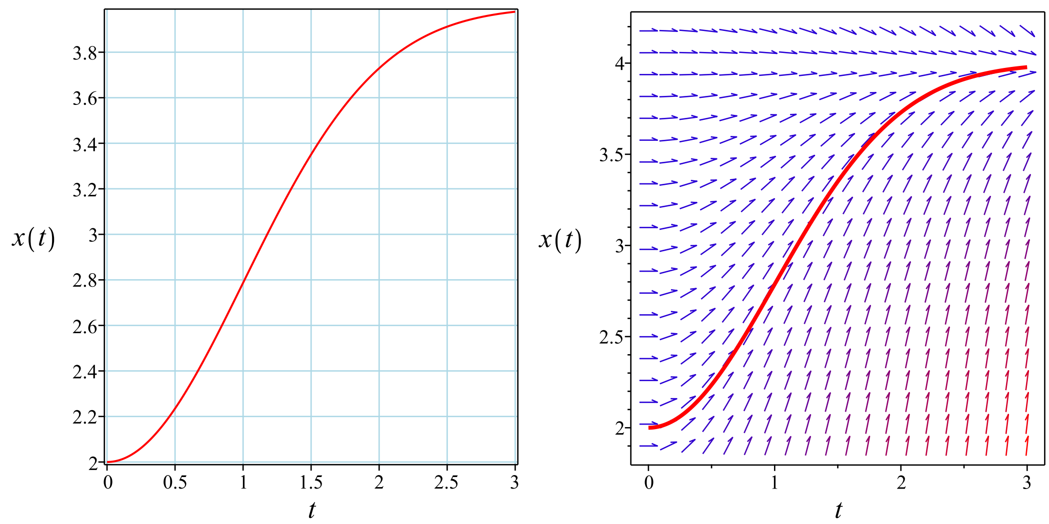
Substituting c_1 found above in the general solution gives

$$x = 4 - 2\,e^{-\frac{t^2}{2}}$$

Summary

The solution(s) found are the following

$$x = 4 - 2\,e^{-\frac{t^2}{2}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = 4 - 2\,\mathrm{e}^{-\frac{t^2}{2}}$$

Verified OK.

4.2.6 Maple step by step solution

Let's solve

$[x' + xt = 4t, x(0) = 2]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{-4+x} = -t$

- Integrate both sides with respect to t

  $\int \frac{x'}{-4+x} dt = \int -t dt + c_1$

- Evaluate integral

  $\ln(-4 + x) = -\frac{t^2}{2} + c_1$

- Solve for x

  $x = \mathrm{e}^{-\frac{t^2}{2}+c_1} + 4$

- Use initial condition $x(0) = 2$

  $2 = e^{c_1} + 4$
- Solve for c_1

  $c_1 = \ln(2) + I\pi$
- Substitute $c_1 = \ln(2) + I\pi$ into general solution and simplify

  $x = 4 - 2\,e^{-\frac{t^2}{2}}$
- Solution to the IVP

  $x = 4 - 2\,e^{-\frac{t^2}{2}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(x(t),t)+t*x(t)=4*t,x(0) = 2],x(t), singsol=all)
```

$$x(t) = 4 - 2\,e^{-\frac{t^2}{2}}$$

✓ Solution by Mathematica
Time used: 0.066 (sec). Leaf size: 18

```
DSolve[{x'[t]+t*x[t]==4*t,{x[0]==2}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 4 - 2e^{-\frac{t^2}{2}}$$

4.3 problem 9.1 (iii)

Internal problem ID [11997]
Internal file name [`OUTPUT/10649_Saturday_September_02_2023_02_49_01_PM_25113563/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 9, First order linear equations and the integrating factor. Exercises page 86
Problem number: 9.1 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{z' - z\tan(y) = \sin(y)}$$

4.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$z' + p(y)z = q(y)$$

Where here

$$p(y) = -\tan(y)$$
$$q(y) = \sin(y)$$

Hence the ode is

$$z' - z\tan(y) = \sin(y)$$

The integrating factor μ is

$$\mu = e^{\int -\tan(y)dy}$$
$$= \cos(y)$$

The ode becomes

$$\frac{d}{dy}(\mu z) = (\mu)(\sin(y))$$
$$\frac{d}{dy}(\cos(y) z) = (\cos(y))(\sin(y))$$
$$d(\cos(y) z) = \left(\frac{\sin(2y)}{2}\right) dy$$

Integrating gives

$$\cos(y) z = \int \frac{\sin(2y)}{2} dy$$
$$\cos(y) z = -\frac{\cos(2y)}{4} + c_1$$

Dividing both sides by the integrating factor $\mu = \cos(y)$ results in

$$z = -\frac{\sec(y)\cos(2y)}{4} + c_1 \sec(y)$$

Summary

The solution(s) found are the following

$$z = -\frac{\sec(y)\cos(2y)}{4} + c_1 \sec(y) \tag{1}$$

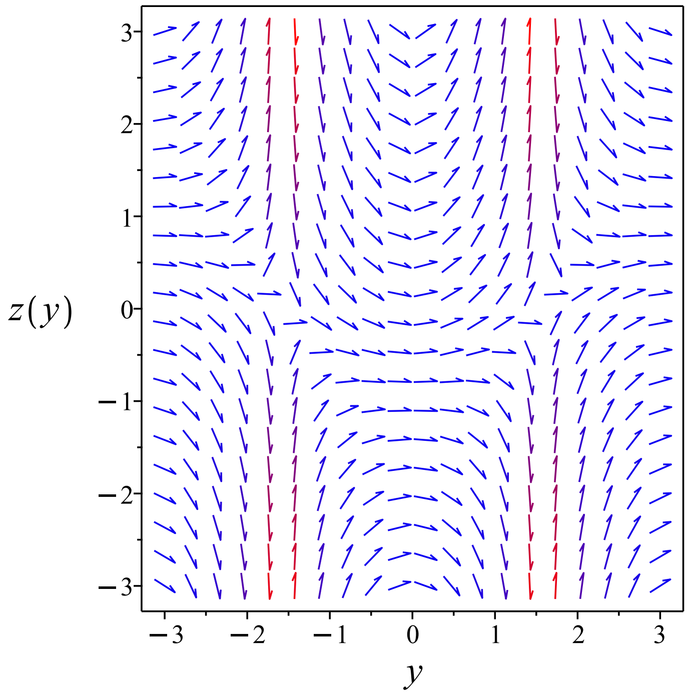


Figure 36: Slope field plot

Verification of solutions

$$z = -\frac{\sec(y)\cos(2y)}{4} + c_1 \sec(y)$$

Verified OK.

4.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$z' = z\tan(y) + \sin(y)$$
$$z' = \omega(y, z)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_z - \xi_y) - \omega^2 \xi_z - \omega_y \xi - \omega_z \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 47: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(y, z) &= 0 \\
\eta(y, z) &= \frac{1}{\cos(y)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, z) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dz}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial z}\right) S(y, z) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\cos(y)}} dy \end{aligned}$$

Which results in

$$S = \cos(y)\, z$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y,z) S_z}{R_y + \omega(y,z) R_z} \tag{2}$$

Where in the above R_y, R_z, S_y, S_z are all partial derivatives and $\omega(y,z)$ is the right hand side of the original ode given by

$$\omega(y,z) = z\tan(y) + \sin(y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_y &= 1 \\ R_z &= 0 \\ S_y &= -\sin(y)\, z \\ S_z &= \cos(y) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sin(2y)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, z in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\sin(2R)}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\cos(2R)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, z coordinates. This results in

$$z\cos(y) = -\frac{\cos(2y)}{4} + c_1$$

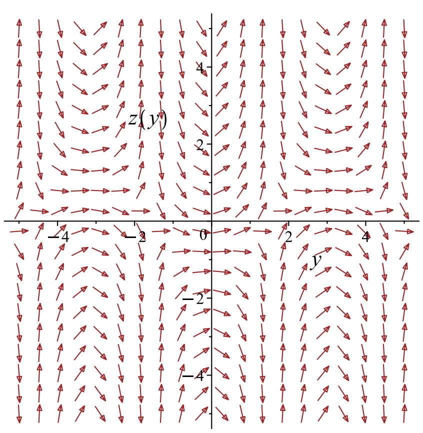
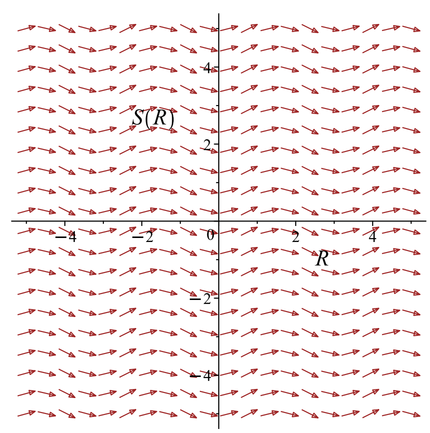
Which simplifies to

$$z\cos(y) = -\frac{\cos(2y)}{4} + c_1$$

Which gives

$$z = -\frac{\cos(2y) - 4c_1}{4\cos(y)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in y, z coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dz}{dy} = z\tan(y) + \sin(y)$ 	$R = y$ $S = \cos(y) z$	$\frac{dS}{dR} = \frac{\sin(2R)}{2}$ 

Summary

The solution(s) found are the following

$$z = -\frac{\cos(2y) - 4c_1}{4\cos(y)} \tag{1}$$

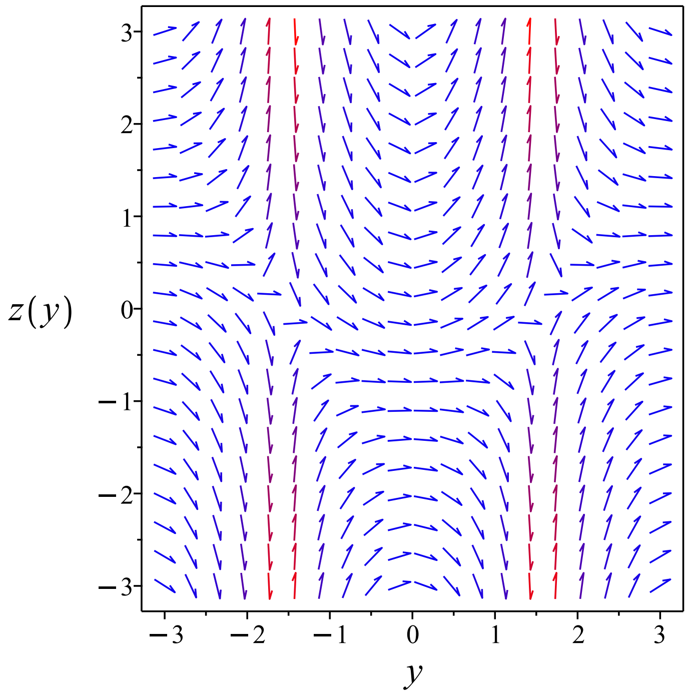


Figure 37: Slope field plot

Verification of solutions

$$z = -\frac{\cos(2y) - 4c_1}{4\cos(y)}$$

Verified OK.

4.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y, z) \, \mathrm{d}y + N(y, z) \, \mathrm{d}z = 0 \tag{1A}$$

Therefore

$$\mathrm{d}z = (z \tan(y) + \sin(y)) \, \mathrm{d}y$$
$$(-z \tan(y) - \sin(y)) \, \mathrm{d}y + \mathrm{d}z = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(y, z) = -z \tan(y) - \sin(y)$$
$$N(y, z) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial z} = \frac{\partial N}{\partial y}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial z} &= \frac{\partial}{\partial z}(-z \tan(y) - \sin(y)) \\ &= -\tan(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial y} &= \frac{\partial}{\partial y}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial z} \neq \frac{\partial N}{\partial y}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial z} - \frac{\partial N}{\partial y}\right) \\ &= 1((-\tan(y)) - (0)) \\ &= -\tan(y) \end{aligned}$$

Since A does not depend on z, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}y} \\ &= e^{\int -\tan(y) \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\cos(y))} \\ &= \cos(y) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \cos(y)(-z\tan(y) - \sin(y)) \\ &= \sin(y)(-\cos(y) - z) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \cos(y)(1) \\ &= \cos(y) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}z}{\mathrm{d}y} &= 0 \\ (\sin(y)(-\cos(y) - z)) + (\cos(y))\frac{\mathrm{d}z}{\mathrm{d}y} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(y, z)$

$$\frac{\partial \phi}{\partial y} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial z} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \overline{M} \, \mathrm{d}y$$
$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \sin(y)(-\cos(y) - z) \, \mathrm{d}y$$
$$\phi = \frac{\cos(y)(\cos(y) + 2z)}{2} + f(z) \tag{3}$$

Where $f(z)$ is used for the constant of integration since ϕ is a function of both y and z. Taking derivative of equation (3) w.r.t z gives

$$\frac{\partial \phi}{\partial z} = \cos(y) + f'(z) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial z} = \cos(y)$. Therefore equation (4) becomes

$$\cos(y) = \cos(y) + f'(z) \tag{5}$$

Solving equation (5) for $f'(z)$ gives

$$f'(z) = 0$$

Therefore

$$f(z) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(z)$ into equation (3) gives ϕ

$$\phi = \frac{\cos(y)(\cos(y) + 2z)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\cos(y)(\cos(y) + 2z)}{2}$$

The solution becomes

$$z = -\frac{\cos(y)^2 - 2c_1}{2\cos(y)}$$

Summary

The solution(s) found are the following

$$z = -\frac{\cos(y)^2 - 2c_1}{2\cos(y)} \tag{1}$$

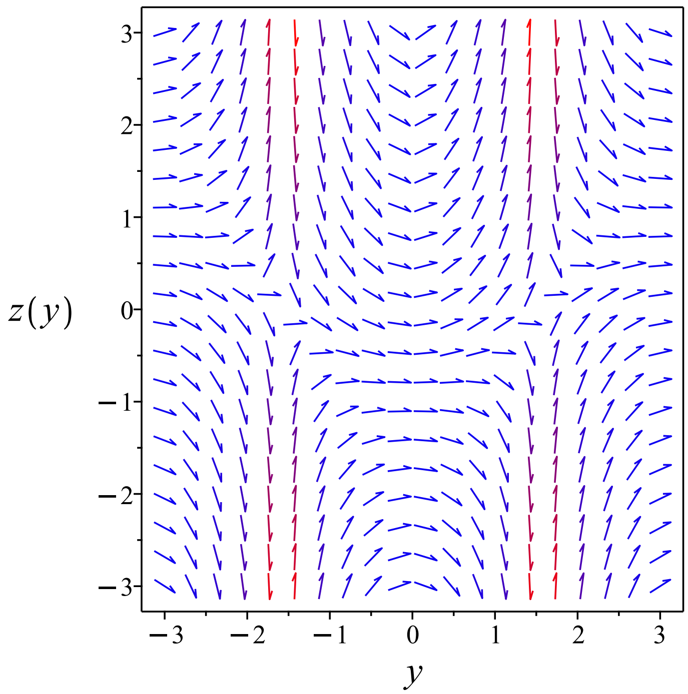


Figure 38: Slope field plot

Verification of solutions

$$z = -\frac{\cos(y)^2 - 2c_1}{2\cos(y)}$$

Verified OK.

4.3.4 Maple step by step solution

Let's solve

$z' - z\tan(y) = \sin(y)$

- Highest derivative means the order of the ODE is 1

  z'

- Isolate the derivative

$z' = z \tan(y) + \sin(y)$

- Group terms with z on the lhs of the ODE and the rest on the rhs of the ODE

  $z' - z \tan(y) = \sin(y)$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

  $\mu(y) (z' - z \tan(y)) = \mu(y) \sin(y)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y) z)$

  $\mu(y) (z' - z \tan(y)) = \mu'(y) z + \mu(y) z'$

- Isolate $\mu'(y)$

  $\mu'(y) = -\mu(y) \tan(y)$

- Solve to find the integrating factor

  $\mu(y) = \cos(y)$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}(\mu(y) z)\right) dy = \int \mu(y) \sin(y) dy + c_1$

- Evaluate the integral on the lhs

  $\mu(y) z = \int \mu(y) \sin(y) dy + c_1$

- Solve for z

  $z = \frac{\int \mu(y) \sin(y) dy + c_1}{\mu(y)}$

- Substitute $\mu(y) = \cos(y)$

  $z = \frac{\int \cos(y) \sin(y) dy + c_1}{\cos(y)}$

- Evaluate the integrals on the rhs

  $z = \frac{\frac{\sin(y)^2}{2} + c_1}{\cos(y)}$

- Simplify

  $z = \sec(y) \left(\frac{\sin(y)^2}{2} + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(z(y),y)=z(y)*tan(y)+sin(y),z(y), singsol=all)
```

$$z(y) = -\frac{\cos (y)}{2} + \sec (y) c_1 + \frac{\sec (y)}{4}$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 17

```
DSolve[z'[y]==z[y]*Tan[y]+Sin[y],z[y],y,IncludeSingularSolutions -> True]
```

$$z(y) \to -\frac{\cos(y)}{2} + c_1 \sec(y)$$

4.4 problem 9.1 (iv)

Internal problem ID [11998]
Internal file name [`OUTPUT/10650_Saturday_September_02_2023_02_49_02_PM_68971325/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 9, First order linear equations and the integrating factor. Exercises page 86
Problem number: 9.1 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \mathrm{e}^{-x} y = 1}$$

With initial conditions

$$[y(0) = \mathrm{e}]$$

4.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \mathrm{e}^{-x}$$
$$q(x) = 1$$

Hence the ode is

$$y' + e^{-x}y = 1$$

The domain of $p(x) = e^{-x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.4.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int e^{-x}dx} \\ &= e^{-e^{-x}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= \mu \\ \frac{d}{dx}\left(e^{-e^{-x}}y\right) &= e^{-e^{-x}} \\ d\left(e^{-e^{-x}}y\right) &= e^{-e^{-x}} dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-e^{-x}}y &= \int e^{-e^{-x}} dx \\ e^{-e^{-x}}y &= \text{expIntegral}_1\left(e^{-x}\right) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-e^{-x}}$ results in

$$y = e^{e^{-x}} \text{expIntegral}_1\left(e^{-x}\right) + c_1 e^{e^{-x}}$$

which simplifies to

$$y = e^{e^{-x}}\left(\text{expIntegral}_1\left(e^{-x}\right) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \mathrm{e}$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} = \mathrm{e}\ \mathrm{expIntegral}_1(1) + \mathrm{e}c_1$$

$$c_1 = 1 - \mathrm{expIntegral}_1(1)$$

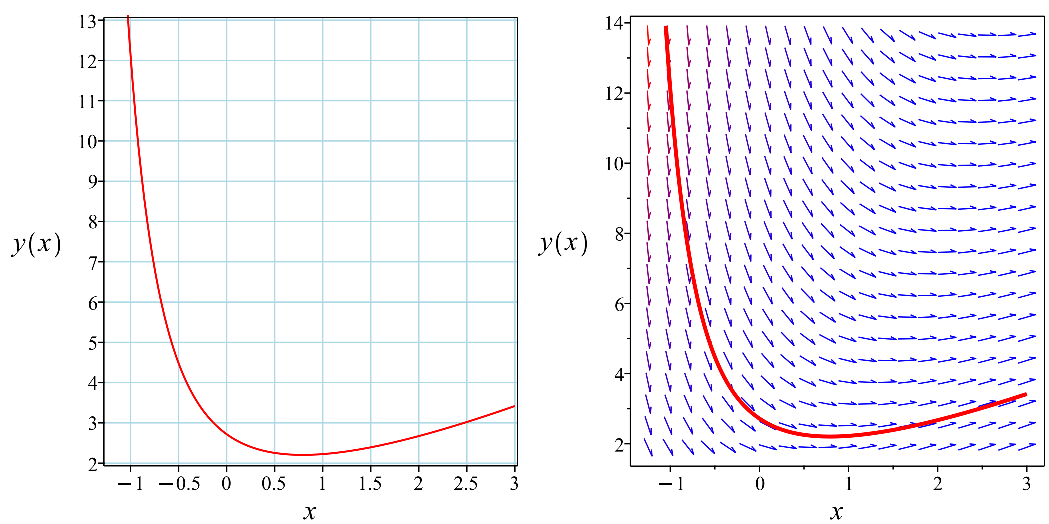
Substituting c_1 found above in the general solution gives

$$y = -\mathrm{e}^{\mathrm{e}^{-x}}\ \mathrm{expIntegral}_1(1) + \mathrm{e}^{\mathrm{e}^{-x}}\ \mathrm{expIntegral}_1\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{\mathrm{e}^{-x}}$$

Summary

The solution(s) found are the following

$$y = -\mathrm{e}^{\mathrm{e}^{-x}}\ \mathrm{expIntegral}_1(1) + \mathrm{e}^{\mathrm{e}^{-x}}\ \mathrm{expIntegral}_1\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{\mathrm{e}^{-x}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\mathrm{e}^{\mathrm{e}^{-x}}\ \mathrm{expIntegral}_1(1) + \mathrm{e}^{\mathrm{e}^{-x}}\ \mathrm{expIntegral}_1\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{\mathrm{e}^{-x}}$$

Verified OK.

4.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\mathrm{e}^{-x} y + 1 \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 50: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0\\
\eta(x,y) &= \mathrm{e}^{\mathrm{e}^{-x}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy\\
&= \int \frac{1}{\mathrm{e}^{\mathrm{e}^{-x}}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{-\mathrm{e}^{-x}} y
$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\mathrm{e}^{-x} y + 1
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= y\,\mathrm{e}^{-x-\mathrm{e}^{-x}}\\
S_y &= \mathrm{e}^{-\mathrm{e}^{-x}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-\mathrm{e}^{-x}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-\mathrm{e}^{-R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \text{expIntegral}_1\left(\mathrm{e}^{-R}\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\mathrm{e}^{-x}} y = \text{expIntegral}_1\left(\mathrm{e}^{-x}\right) + c_1$$

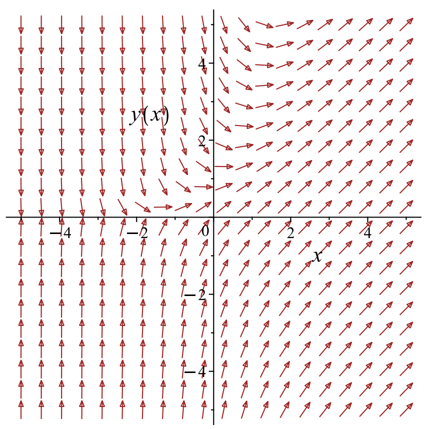
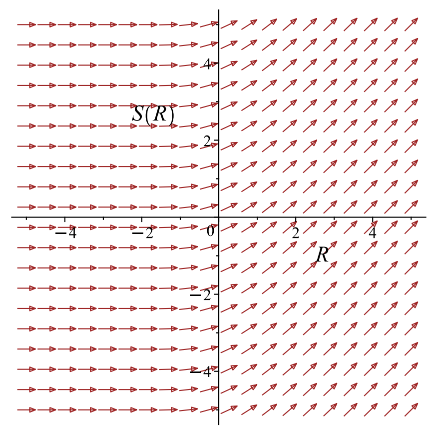
Which simplifies to

$$\mathrm{e}^{-\mathrm{e}^{-x}} y = \text{expIntegral}_1\left(\mathrm{e}^{-x}\right) + c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{e}^{-x}}\left(\text{expIntegral}_1\left(\mathrm{e}^{-x}\right) + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\mathrm{e}^{-x} y + 1$ 	$R = x$ $S = \mathrm{e}^{-\mathrm{e}^{-x}} y$	$\frac{dS}{dR} = \mathrm{e}^{-\mathrm{e}^{-R}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \mathrm{e}$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} = \mathrm{e}\ \mathrm{expIntegral}_1(1) + \mathrm{e} c_1$$

$$c_1 = 1 - \mathrm{expIntegral}_1(1)$$

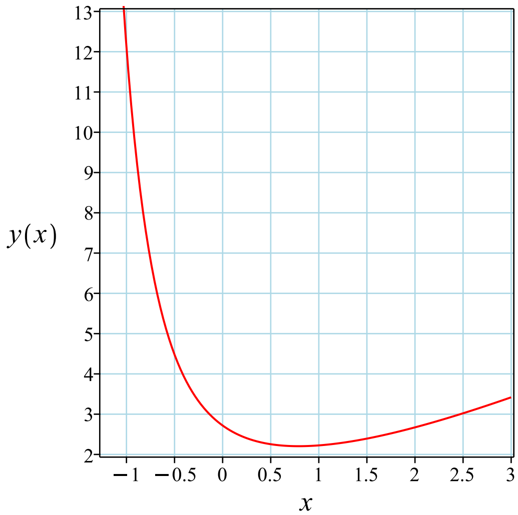
Substituting c_1 found above in the general solution gives

$$y = -\mathrm{e}^{\mathrm{e}^{-x}} \mathrm{expIntegral}_1(1) + \mathrm{e}^{\mathrm{e}^{-x}} \mathrm{expIntegral}_1\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{\mathrm{e}^{-x}}$$

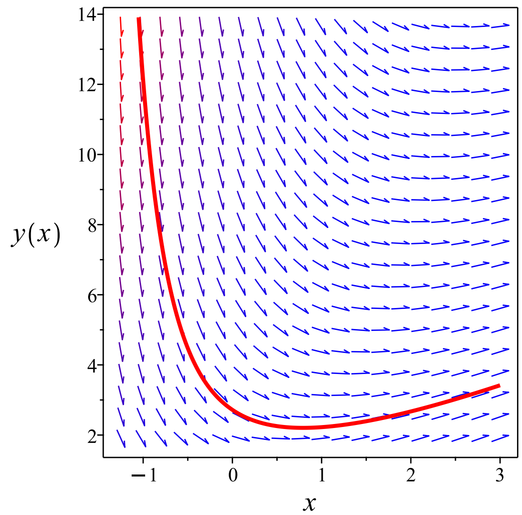
Summary

The solution(s) found are the following

$$y = -\mathrm{e}^{\mathrm{e}^{-x}} \mathrm{expIntegral}_1(1) + \mathrm{e}^{\mathrm{e}^{-x}} \mathrm{expIntegral}_1\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{\mathrm{e}^{-x}} \quad (1)$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -e^{e^{-x}} \operatorname{expIntegral}_1(1) + e^{e^{-x}} \operatorname{expIntegral}_1\left(e^{-x}\right) + e^{e^{-x}}$$

Verified OK.

4.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\mathrm{d}y &= \left(-\mathrm{e}^{-x}y + 1\right)\mathrm{d}x \\
\left(\mathrm{e}^{-x}y - 1\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= \mathrm{e}^{-x}y - 1 \\
N(x,y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{-x}y - 1\right) \\
&= \mathrm{e}^{-x}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\mathrm{e}^{-x}\right) - (0)\right) \\
&= \mathrm{e}^{-x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \mathrm{e}^{-x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\mathrm{e}^{-x}}\\ &= \mathrm{e}^{-\mathrm{e}^{-x}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-\mathrm{e}^{-x}}\left(\mathrm{e}^{-x}y-1\right)\\ &= \left(\mathrm{e}^{-x}y-1\right)\mathrm{e}^{-\mathrm{e}^{-x}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-\mathrm{e}^{-x}}(1)\\ &= \mathrm{e}^{-\mathrm{e}^{-x}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\left(\mathrm{e}^{-x}y-1\right)\mathrm{e}^{-\mathrm{e}^{-x}}\right)+\left(\mathrm{e}^{-\mathrm{e}^{-x}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\left(\mathrm{e}^{-x}y-1\right)\mathrm{e}^{-\mathrm{e}^{-x}}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\operatorname{expIntegral}_1\left(\mathrm{e}^{-x}\right)+\mathrm{e}^{-\mathrm{e}^{-x}}y+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\mathrm{e}^{-x}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\mathrm{e}^{-x}}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\mathrm{e}^{-x}} = \mathrm{e}^{-\mathrm{e}^{-x}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\operatorname{expIntegral}_1\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{-\mathrm{e}^{-x}} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\operatorname{expIntegral}_1\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{-\mathrm{e}^{-x}} y$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{e}^{-x}}\left(\operatorname{expIntegral}_1\left(\mathrm{e}^{-x}\right) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \mathrm{e}$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} = \mathrm{e} \operatorname{expIntegral}_1(1) + \mathrm{e} c_1$$

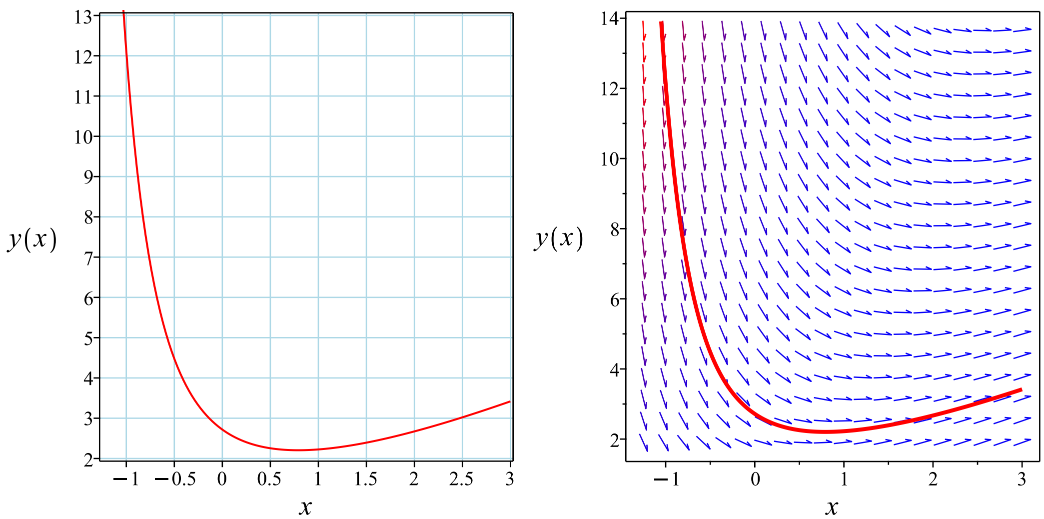
$$c_1 = 1 - \operatorname{expIntegral}_1(1)$$

Substituting c_1 found above in the general solution gives

$$y = -\mathrm{e}^{\mathrm{e}^{-x}} \operatorname{expIntegral}_1(1) + \mathrm{e}^{\mathrm{e}^{-x}} \operatorname{expIntegral}_1\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{\mathrm{e}^{-x}}$$

Summary
The solution(s) found are the following

$$y = -e^{e^{-x}} \operatorname{expIntegral}_1(1) + e^{e^{-x}} \operatorname{expIntegral}_1\left(e^{-x}\right) + e^{e^{-x}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -e^{e^{-x}} \operatorname{expIntegral}_1(1) + e^{e^{-x}} \operatorname{expIntegral}_1\left(e^{-x}\right) + e^{e^{-x}}$$

Verified OK.

4.4.5 Maple step by step solution

Let's solve

$[y' + e^{-x}y = 1, y(0) = e]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -e^{-x}y + 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + e^{-x}y = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y' + \mathrm{e}^{-x} y\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \mathrm{e}^{-x} y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\, \mathrm{e}^{-x}$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-\frac{1}{\mathrm{e}^x}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{-\frac{1}{\mathrm{e}^x}}$

  $y = \frac{\int \mathrm{e}^{-\frac{1}{\mathrm{e}^x}} dx + c_1}{\mathrm{e}^{-\frac{1}{\mathrm{e}^x}}}$

- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{Ei}_1\left(\frac{1}{\mathrm{e}^x}\right) + c_1}{\mathrm{e}^{-\frac{1}{\mathrm{e}^x}}}$

- Simplify

  $y = \mathrm{e}^{\mathrm{e}^{-x}} \left(\mathrm{Ei}_1(\mathrm{e}^{-x}) + c_1\right)$

- Use initial condition $y(0) = \mathrm{e}$

  $\mathrm{e} = \mathrm{e}(\mathrm{Ei}_1(1) + c_1)$

- Solve for c_1

  $c_1 = 1 - \mathrm{Ei}_1(1)$

- Substitute $c_1 = 1 - \mathrm{Ei}_1(1)$ into general solution and simplify

  $y = -(-\mathrm{Ei}_1(\mathrm{e}^{-x}) - 1 + \mathrm{Ei}_1(1))\, \mathrm{e}^{\mathrm{e}^{-x}}$

- Solution to the IVP

  $y = -(-\mathrm{Ei}_1(\mathrm{e}^{-x}) - 1 + \mathrm{Ei}_1(1))\, \mathrm{e}^{\mathrm{e}^{-x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 23

```
dsolve([diff(y(x),x)+exp(-x)*y(x)=1,y(0) = exp(1)],y(x), singsol=all)
```

$$y(x) = \left(\text{expIntegral}_1\left(\mathrm{e}^{-x}\right) + 1 - \text{expIntegral}_1\left(1\right)\right)\mathrm{e}^{\mathrm{e}^{-x}}$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 27

```
DSolve[{y'[x]+Exp[-x]*y[x]==1,{y[0]==Exp[1]}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{e^{-x}}\left(-\text{ExpIntegralEi}\left(-e^{-x}\right) + \text{ExpIntegralEi}(-1) + 1\right)$$

4.5 problem 9.1 (v)

Internal problem ID [11999]
Internal file name [`OUTPUT/10651_Saturday_September_02_2023_02_49_03_PM_78039918/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 9, First order linear equations and the integrating factor. Exercises page 86
Problem number: 9.1 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{x' + x \tanh(t) = 3}$$

4.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \tanh(t)$$
$$q(t) = 3$$

Hence the ode is

$$x' + x \tanh(t) = 3$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \tanh(t)dt} \\
&= \cosh(t)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)(3) \\
\frac{d}{dt}(\cosh(t)\, x) &= (\cosh(t))(3) \\
d(\cosh(t)\, x) &= (3\cosh(t))\, dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\cosh(t)\, x &= \int 3\cosh(t)\, dt \\
\cosh(t)\, x &= 3\sinh(t) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \cosh(t)$ results in

$$
x = 3\,\mathrm{sech}(t)\sinh(t) + c_1\,\mathrm{sech}(t)
$$

which simplifies to

$$
x = 3\tanh(t) + c_1\,\mathrm{sech}(t)
$$

<u>Summary</u>
The solution(s) found are the following

$$
x = 3\tanh(t) + c_1\,\mathrm{sech}(t) \tag{1}
$$

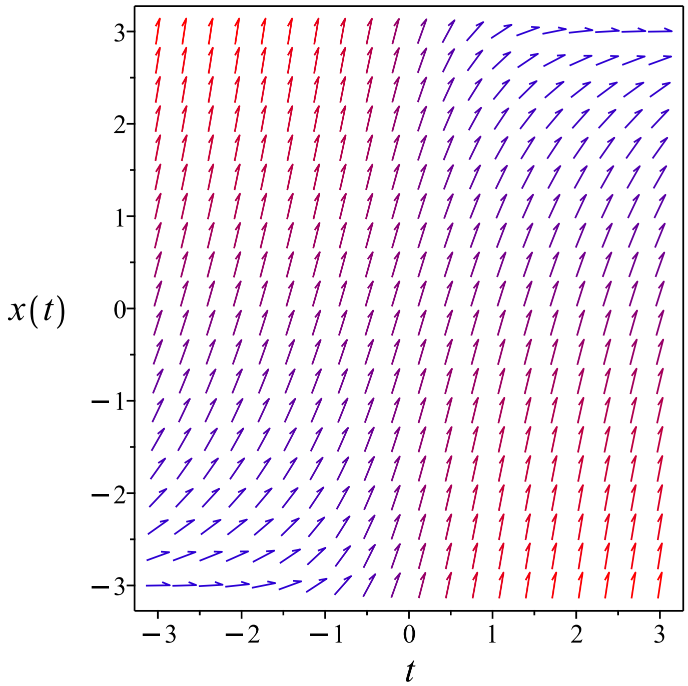


Figure 42: Slope field plot

Verification of solutions

$$x = 3\tanh(t) + c_1 \operatorname{sech}(t)$$

Verified OK.

4.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -x\tanh(t) + 3$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 53: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, x) &= 0 \\ \eta(t, x) &= \frac{1}{\cosh(t)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\cosh(t)}} dy \end{aligned}$$

Which results in

$$S = \cosh(t)\, x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -x \tanh(t) + 3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= x \sinh(t) \\ S_x &= \cosh(t) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3\cosh(t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3\cosh(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 3 \sinh(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$x \cosh(t) = 3 \sinh(t) + c_1$$

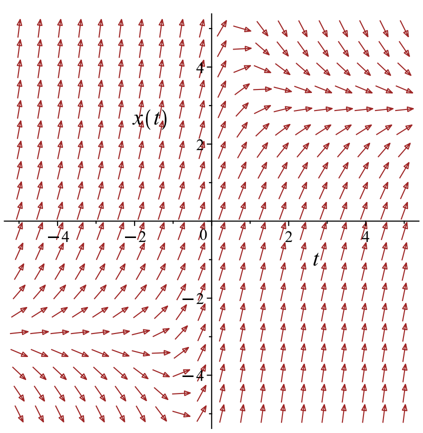
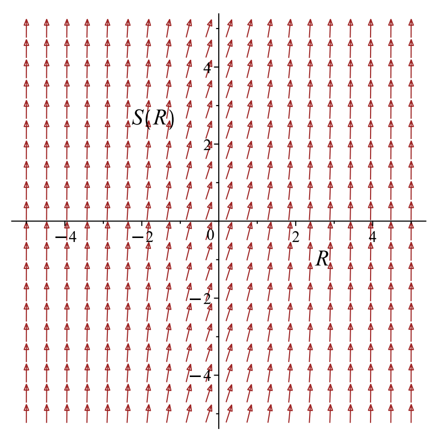
Which simplifies to

$$x \cosh(t) = 3 \sinh(t) + c_1$$

Which gives

$$x = \frac{3 \sinh(t) + c_1}{\cosh(t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -x \tanh(t) + 3$ 	$R = t$ $S = \cosh(t)\, x$	$\frac{dS}{dR} = 3 \cosh(R)$ 

Summary

The solution(s) found are the following

$$x = \frac{3 \sinh(t) + c_1}{\cosh(t)} \tag{1}$$

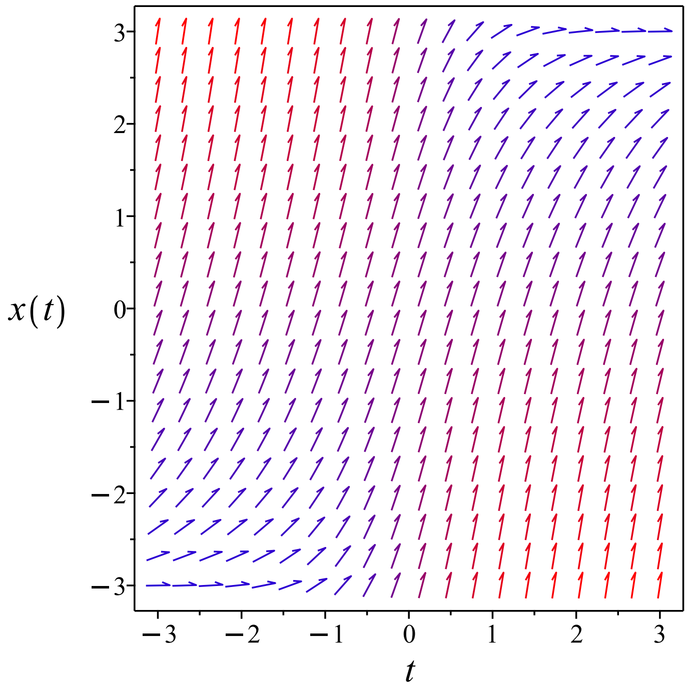


Figure 43: Slope field plot

Verification of solutions

$$x = \frac{3\sinh(t) + c_1}{\cosh(t)}$$

Verified OK.

4.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\, \mathrm{d}t + N(t, x)\, \mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = (-x \tanh(t) + 3)\, \mathrm{d}t$$
$$(x \tanh(t) - 3)\, \mathrm{d}t + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, x) = x \tanh(t) - 3$$
$$N(t, x) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}(x \tanh(t) - 3)$$
$$= \tanh(t)$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= 1((\tanh(t)) - (0)) \\
&= \tanh(t)
\end{aligned}
$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int \tanh(t) \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(\cosh(t))} \\
&= \cosh(t)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \cosh(t)\,(x\tanh(t) - 3) \\
&= x\sinh(t) - 3\cosh(t)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \cosh(t)\,(1) \\
&= \cosh(t)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
(x\sinh(t) - 3\cosh(t)) + (\cosh(t))\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int x \sinh(t) - 3\cosh(t) \, \mathrm{d}t$$

$$\phi = \cosh(t)\, x - 3\sinh(t) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \cosh(t) + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \cosh(t)$. Therefore equation (4) becomes

$$\cosh(t) = \cosh(t) + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \cosh(t)\, x - 3\sinh(t) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cosh(t)\, x - 3\sinh(t)$$

The solution becomes

$$x = \frac{3\sinh(t) + c_1}{\cosh(t)}$$

Summary

The solution(s) found are the following

$$x = \frac{3 \sinh(t) + c_1}{\cosh(t)} \tag{1}$$

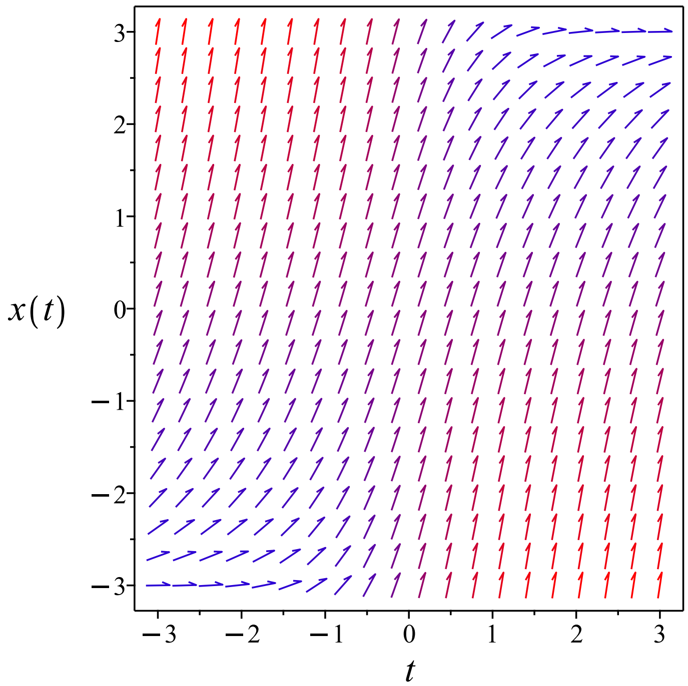


Figure 44: Slope field plot

Verification of solutions

$$x = \frac{3 \sinh(t) + c_1}{\cosh(t)}$$

Verified OK.

4.5.4 Maple step by step solution

Let's solve

$x' + x \tanh(t) = 3$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

$x' = -x \tanh(t) + 3$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

$x' + x \tanh(t) = 3$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)(x' + x \tanh(t)) = 3\mu(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) x)$

$\mu(t)(x' + x \tanh(t)) = \mu'(t) x + \mu(t) x'$

- Isolate $\mu'(t)$

$\mu'(t) = \mu(t) \tanh(t)$

- Solve to find the integrating factor

$\mu(t) = \cosh(t)$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t) x)\right) dt = \int 3\mu(t) dt + c_1$

- Evaluate the integral on the lhs

$\mu(t) x = \int 3\mu(t) dt + c_1$

- Solve for x

$x = \frac{\int 3\mu(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \cosh(t)$

$x = \frac{\int 3\cosh(t)dt + c_1}{\cosh(t)}$

- Evaluate the integrals on the rhs

$x = \frac{3\sinh(t) + c_1}{\cosh(t)}$

- Simplify

$x = 3\tanh(t) + c_1 \mathrm{sech}(t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(x(t),t)+x(t)*tanh(t)=3,x(t), singsol=all)
```

$$x(t) = 3\tanh(t) + \operatorname{sech}(t)\, c_1$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 15

```
DSolve[x'[t]+x[t]*Tanh[t]==3,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \operatorname{sech}(t)(3\sinh(t) + c_1)$$

4.6 problem 9.1 (vi)

Internal problem ID [12000]
Internal file name [`OUTPUT/10652_Saturday_September_02_2023_02_49_04_PM_11439773/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 9, First order linear equations and the integrating factor. Exercises page 86
Problem number: 9.1 (vi).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + 2y\cot(x) = 5}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 1\right]$$

4.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2\cot(x)$$
$$q(x) = 5$$

Hence the ode is

$$y' + 2y\cot(x) = 5$$

The domain of $p(x) = 2\cot(x)$ is

$$\{x < \pi\_Z118 \lor \pi\_Z118 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

4.6.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 2\cot(x)dx} \\ &= \sin(x)^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(5) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(x)^2 y\right) &= \left(\sin(x)^2\right)(5) \\ \mathrm{d}\left(\sin(x)^2 y\right) &= \left(5\sin(x)^2\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\sin(x)^2 y &= \int 5\sin(x)^2\,\mathrm{d}x \\ \sin(x)^2 y &= -\frac{5\cos(x)\sin(x)}{2} + \frac{5x}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sin(x)^2$ results in

$$y = \csc(x)^2\left(-\frac{5\cos(x)\sin(x)}{2} + \frac{5x}{2}\right) + c_1\csc(x)^2$$

which simplifies to

$$y = \frac{(2c_1 + 5x)\csc(x)^2}{2} - \frac{5\cot(x)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + \frac{5\pi}{4}$$

$$c_1 = -\frac{5\pi}{4} + 1$$

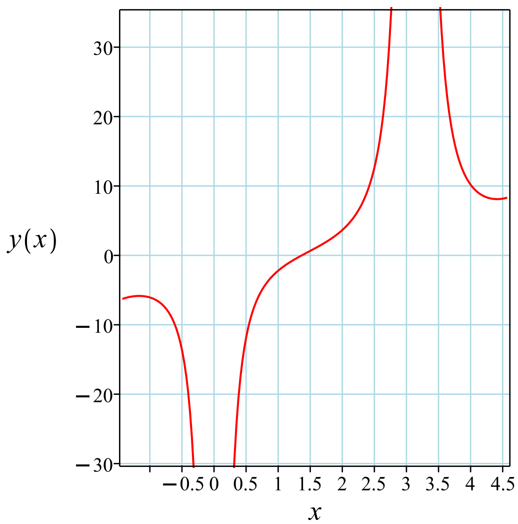
Substituting c_1 found above in the general solution gives

$$y = -\frac{5 \csc(x)^2 \pi}{4} + \csc(x)^2 + \frac{5 \csc(x)^2 x}{2} - \frac{5 \cot(x)}{2}$$

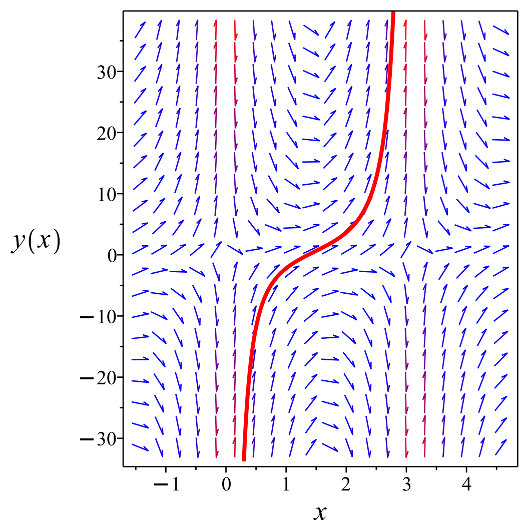
Summary

The solution(s) found are the following

$$y = -\frac{5 \csc(x)^2 \pi}{4} + \csc(x)^2 + \frac{5 \csc(x)^2 x}{2} - \frac{5 \cot(x)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{5 \csc(x)^2 \pi}{4} + \csc(x)^2 + \frac{5 \csc(x)^2 x}{2} - \frac{5 \cot(x)}{2}$$

Verified OK.

4.6.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -2y\cot(x) + 5 \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 56: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{1}{\sin(x)^2} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)^2}} dy \end{aligned}$$

Which results in

$$S = \sin(x)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2y\cot(x) + 5$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \sin(2x)\, y \\
S_y &= \sin(x)^2
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 5\sin(x)^2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 5\sin(R)^2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{5R}{2} + c_1 - \frac{5\sin(2R)}{4} \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\sin(x)^2\, y = \frac{5x}{2} + c_1 - \frac{5\sin(2x)}{4}
$$

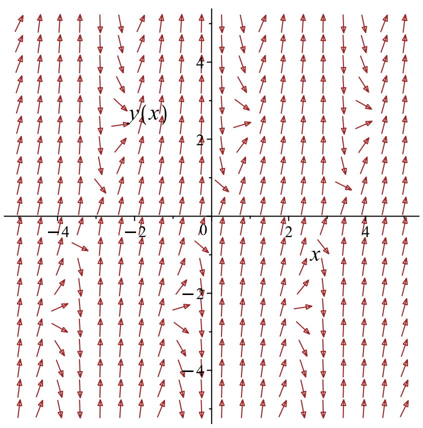
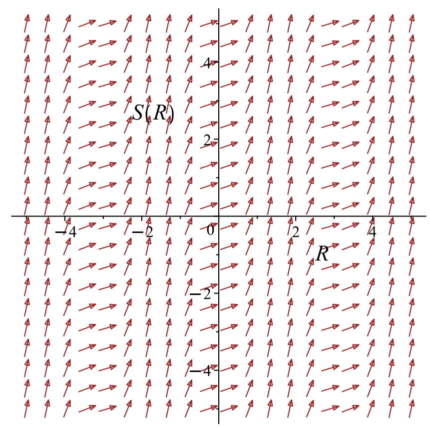
Which simplifies to

$$
\sin(x)^2\, y = \frac{5x}{2} + c_1 - \frac{5\sin(2x)}{4}
$$

Which gives

$$
y = -\frac{-4c_1 - 10x + 5\sin(2x)}{4\sin(x)^2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -2y\cot(x) + 5$

</td><td>$R = x$
$S = \sin(x)^2 y$</td><td>$\frac{dS}{dR} = 5\sin(R)^2$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + \frac{5\pi}{4}$$

$$c_1 = -\frac{5\pi}{4} + 1$$

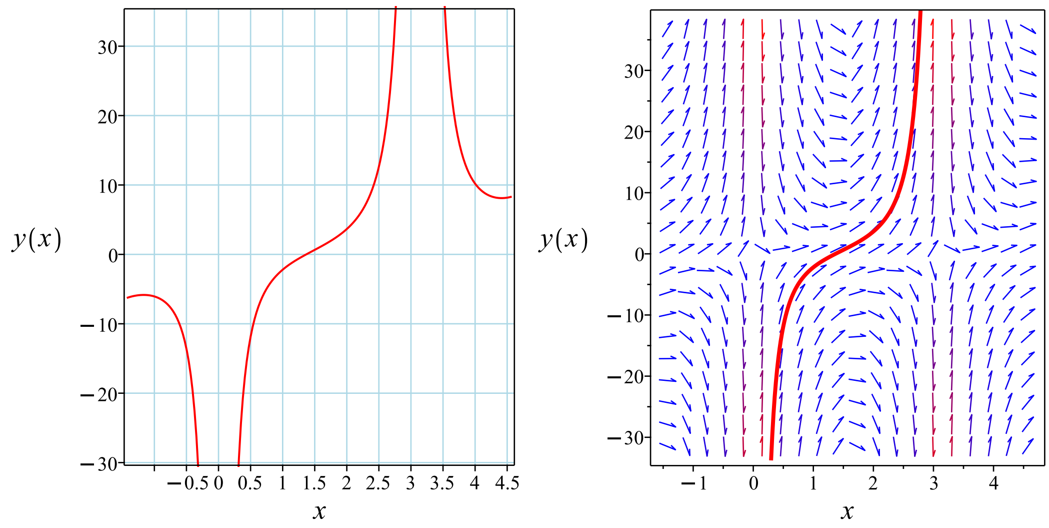
Substituting c_1 found above in the general solution gives

$$y = -\frac{5\csc(x)^2 \sin(x)\cos(x)}{2} - \frac{5\csc(x)^2 \pi}{4} + \frac{5\csc(x)^2 x}{2} + \csc(x)^2$$

Summary

The solution(s) found are the following

$$y = -\frac{5\csc(x)^2 \sin(x)\cos(x)}{2} - \frac{5\csc(x)^2 \pi}{4} + \frac{5\csc(x)^2 x}{2} + \csc(x)^2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{5\csc(x)^2 \sin(x)\cos(x)}{2} - \frac{5\csc(x)^2 \pi}{4} + \frac{5\csc(x)^2 x}{2} + \csc(x)^2$$

Verified OK.

4.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (-2y\cot(x) + 5)\,\mathrm{d}x\\ (2y\cot(x) - 5)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2y\cot(x) - 5\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y\cot(x) - 5)\\ &= 2\cot(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((2\cot(x)) - (0))\\ &= 2\cot(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 2\cot(x) \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2\ln(\sin(x))} \\ &= \sin(x)^2\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \sin(x)^2 (2y\cot(x) - 5) \\ &= (2y\cot(x) - 5)\sin(x)^2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \sin(x)^2 (1) \\ &= \sin(x)^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left((2y\cot(x) - 5)\sin(x)^2\right) + \left(\sin(x)^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int (2y\cot(x) - 5)\sin(x)^2 \, \mathrm{d}x\end{aligned}$$

$$\phi = -y\cos(x)^2 + \frac{5\cos(x)\sin(x)}{2} - \frac{5x}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\cos(x)^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)^2$. Therefore equation (4) becomes

$$\sin(x)^2 = -\cos(x)^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= \cos(x)^2 + \sin(x)^2 \\ &= 1 \end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned} \int f'(y)\, \mathrm{d}y &= \int (1)\, \mathrm{d}y \\ f(y) &= y + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -y\cos(x)^2 + \frac{5\cos(x)\sin(x)}{2} - \frac{5x}{2} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -y\cos(x)^2 + \frac{5\cos(x)\sin(x)}{2} - \frac{5x}{2} + y$$

The solution becomes

$$y = \frac{5\cos(x)\sin(x) - 2c_1 - 5x}{-2 + 2\cos(x)^2}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + \frac{5\pi}{4}$$

$$c_1 = -\frac{5\pi}{4} + 1$$

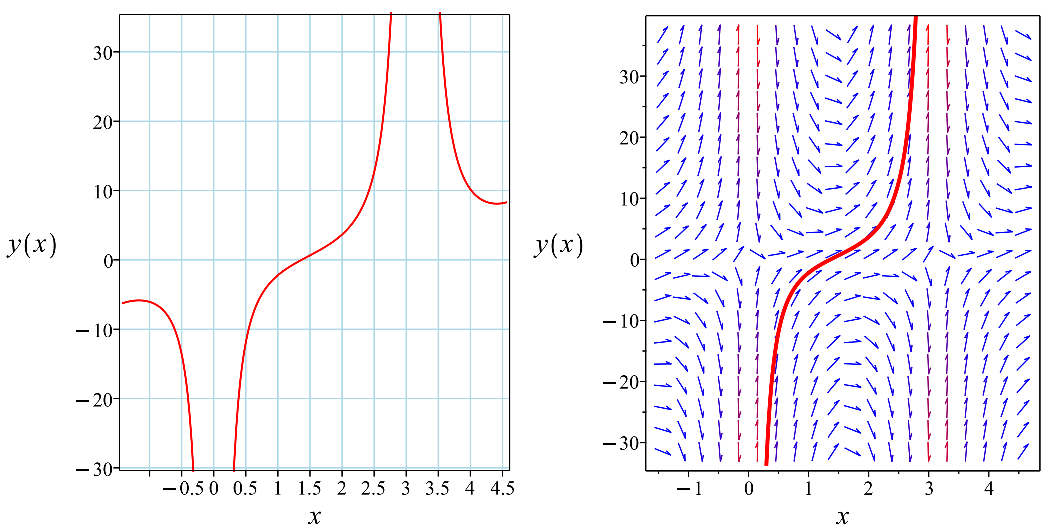
Substituting c_1 found above in the general solution gives

$$y = -\frac{5\csc(x)^2 \sin(x)\cos(x)}{2} - \frac{5\csc(x)^2 \pi}{4} + \frac{5\csc(x)^2 x}{2} + \csc(x)^2$$

Summary

The solution(s) found are the following

$$y = -\frac{5\csc(x)^2 \sin(x)\cos(x)}{2} - \frac{5\csc(x)^2 \pi}{4} + \frac{5\csc(x)^2 x}{2} + \csc(x)^2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{5\csc(x)^2 \sin(x)\cos(x)}{2} - \frac{5\csc(x)^2 \pi}{4} + \frac{5\csc(x)^2 x}{2} + \csc(x)^2$$

Verified OK.

4.6.5 Maple step by step solution

Let's solve

$\left[y' + 2y \cot(x) = 5, y\left(\frac{\pi}{2}\right) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2y \cot(x) + 5$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2y \cot(x) = 5$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' + 2y \cot(x)) = 5\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)(y' + 2y \cot(x)) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x) \cot(x)$

- Solve to find the integrating factor

  $\mu(x) = \sin(x)^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 5\mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 5\mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int 5\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sin(x)^2$

  $y = \frac{\int 5\sin(x)^2 dx + c_1}{\sin(x)^2}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{5\cos(x)\sin(x)}{2} + \frac{5x}{2} + c_1}{\sin(x)^2}$

- Simplify

$y = \frac{(2c_1+5x)\csc(x)^2}{2} - \frac{5\cot(x)}{2}$

- Use initial condition $y\left(\frac{\pi}{2}\right) = 1$

  $1 = c_1 + \frac{5\pi}{4}$

- Solve for c_1

  $c_1 = -\frac{5\pi}{4} + 1$

- Substitute $c_1 = -\frac{5\pi}{4} + 1$ into general solution and simplify

  $y = \frac{(-5\pi+10x+4)\csc(x)^2}{4} - \frac{5\cot(x)}{2}$

- Solution to the IVP

  $y = \frac{(-5\pi+10x+4)\csc(x)^2}{4} - \frac{5\cot(x)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 29

```
dsolve([diff(y(x),x)+2*y(x)*cot(x)=5,y(1/2*Pi) = 1],y(x), singsol=all)
```

$$y(x) = \frac{-10x + 5\sin(2x) - 4 + 5\pi}{-2 + 2\cos(2x)}$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 27

```
DSolve[{y'[x]+2*y[x]*Cot[x]==5,{y[Pi/2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}(10x - 5\sin(2x) - 5\pi + 4)\csc^2(x)$$

4.7 problem 9.1 (vii)

Internal problem ID [12001]
Internal file name [`OUTPUT/10653_Saturday_September_02_2023_02_49_05_PM_57665091/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 9, First order linear equations and the integrating factor. Exercises page 86
Problem number: 9.1 (vii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' + 5x = t}$$

4.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= t \end{aligned}$$

Hence the ode is

$$x' + 5x = t$$

The integrating factor μ is

$$\mu = e^{\int 5dt}$$
$$= e^{5t}$$

The ode becomes

$$\frac{d}{dt}(\mu x) = (\mu)(t)$$
$$\frac{d}{dt}(e^{5t}x) = (e^{5t})(t)$$
$$d(e^{5t}x) = (t\,e^{5t})\,dt$$

Integrating gives

$$e^{5t}x = \int t\,e^{5t}\,dt$$
$$e^{5t}x = \frac{(5t-1)\,e^{5t}}{25} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{5t}$ results in

$$x = \frac{e^{-5t}(5t-1)\,e^{5t}}{25} + c_1 e^{-5t}$$

which simplifies to

$$x = \frac{t}{5} - \frac{1}{25} + c_1 e^{-5t}$$

Summary
The solution(s) found are the following

$$x = \frac{t}{5} - \frac{1}{25} + c_1 e^{-5t} \tag{1}$$

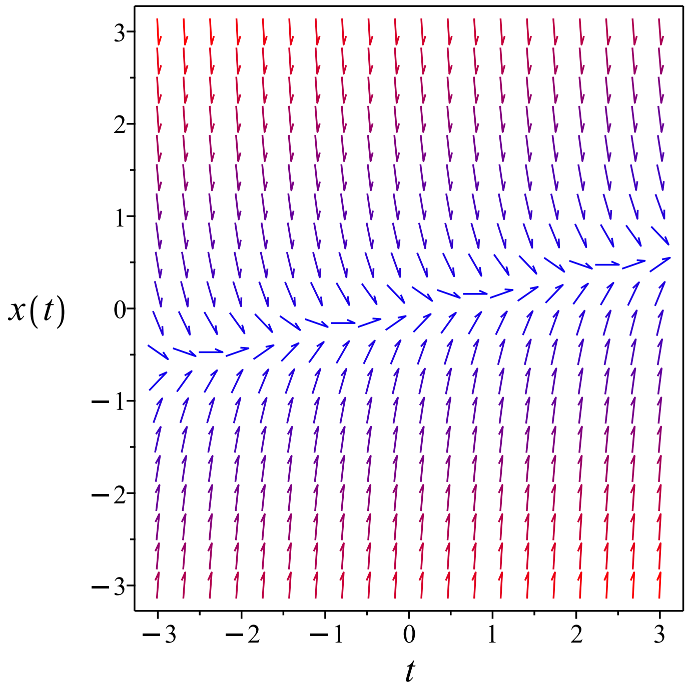


Figure 48: Slope field plot

Verification of solutions

$$x = \frac{t}{5} - \frac{1}{25} + c_1 e^{-5t}$$

Verified OK.

4.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} x' &= -5x + t \\ x' &= \omega(t, x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 59: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 0 \\
\eta(t, x) &= e^{-5t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-5t}} dy \end{aligned}$$

Which results in

$$S = e^{5t} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -5x + t$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 5\,e^{5t} x \\ S_x &= e^{5t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t\,e^{5t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\,e^{5R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{(5R-1)\,e^{5R}}{25} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$e^{5t}x = \frac{(5t-1)\,e^{5t}}{25} + c_1$$

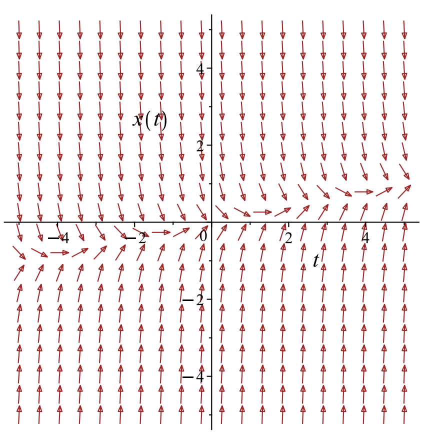
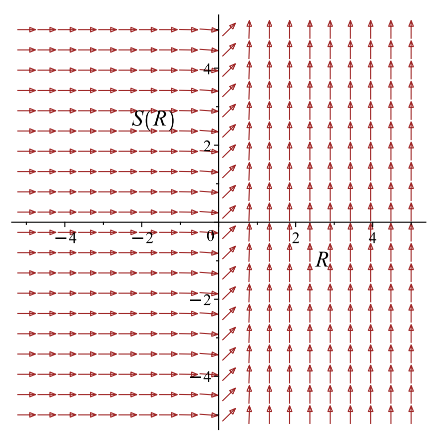
Which simplifies to

$$e^{5t}x = \frac{(5t-1)\,e^{5t}}{25} + c_1$$

Which gives

$$x = \frac{(5t\,e^{5t} - e^{5t} + 25c_1)\,e^{-5t}}{25}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -5x + t$ 	$R = t$ $S = e^{5t}x$	$\frac{dS}{dR} = R\,e^{5R}$ 

Summary

The solution(s) found are the following

$$x = \frac{(5t\,e^{5t} - e^{5t} + 25c_1)\,e^{-5t}}{25} \tag{1}$$

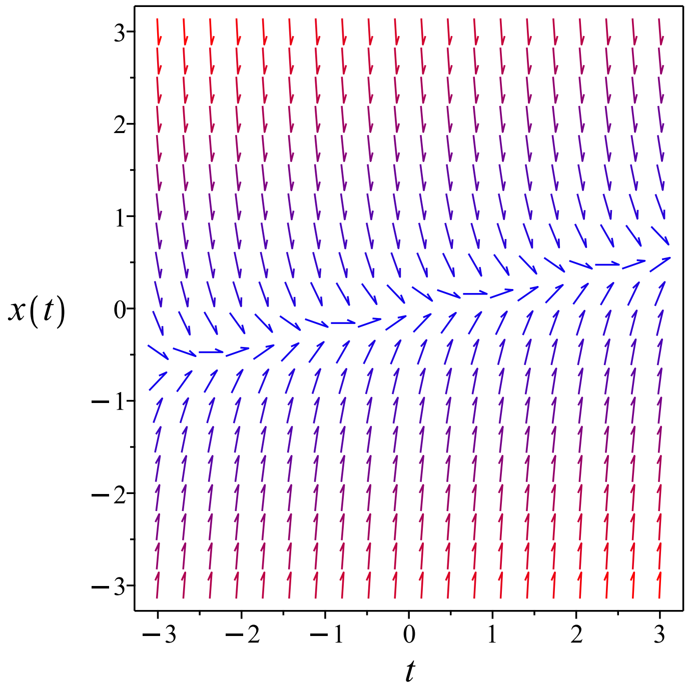


Figure 49: Slope field plot

<u>Verification of solutions</u>

$$x = \frac{(5t\,e^{5t} - e^{5t} + 25c_1)\,e^{-5t}}{25}$$

Verified OK.

4.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, x) \, \mathrm{d}t + N(t, x) \, \mathrm{d}x = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}x &= (-5x + t) \, \mathrm{d}t \\
(5x - t) \, \mathrm{d}t + \mathrm{d}x &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, x) &= 5x - t \\
N(t, x) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(5x - t) \\
&= 5
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\ &= 1((5) - (0)) \\ &= 5 \end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int 5 \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{5t} \\ &= \mathrm{e}^{5t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{5t}(5x - t) \\ &= -\mathrm{e}^{5t}(-5x + t) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{5t}(1) \\ &= \mathrm{e}^{5t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ \left(-\mathrm{e}^{5t}(-5x + t)\right) + \left(\mathrm{e}^{5t}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \overline{M} \,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int -\mathrm{e}^{5t}(-5x+t) \,\mathrm{d}t$$
$$\phi = -\frac{\mathrm{e}^{5t}\left(t-5x-\frac{1}{5}\right)}{5} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \mathrm{e}^{5t} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{5t}$. Therefore equation (4) becomes

$$\mathrm{e}^{5t} = \mathrm{e}^{5t} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{5t}\left(t-5x-\frac{1}{5}\right)}{5} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{5t}\left(t-5x-\frac{1}{5}\right)}{5}$$

The solution becomes

$$x = \frac{(5t\,\mathrm{e}^{5t} - \mathrm{e}^{5t} + 25c_1)\,\mathrm{e}^{-5t}}{25}$$

Summary

The solution(s) found are the following

$$x = \frac{\left(5t\,\mathrm{e}^{5t} - \mathrm{e}^{5t} + 25c_1\right)\mathrm{e}^{-5t}}{25} \tag{1}$$

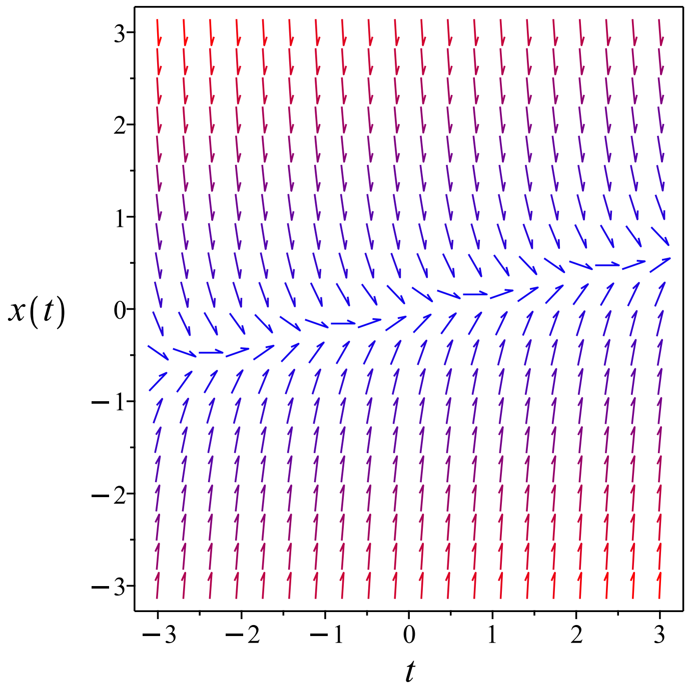


Figure 50: Slope field plot

Verification of solutions

$$x = \frac{\left(5t\,\mathrm{e}^{5t} - \mathrm{e}^{5t} + 25c_1\right)\mathrm{e}^{-5t}}{25}$$

Verified OK.

4.7.4 Maple step by step solution

Let's solve

$x' + 5x = t$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

$x' = -5x + t$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

$x' + 5x = t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(x' + 5x\right) = \mu(t)\, t$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$

$\mu(t)\left(x' + 5x\right) = \mu'(t)\, x + \mu(t)\, x'$

- Isolate $\mu'(t)$

$\mu'(t) = 5\mu(t)$

- Solve to find the integrating factor

$\mu(t) = \mathrm{e}^{5t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int \mu(t)\, t dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\, x = \int \mu(t)\, t dt + c_1$

- Solve for x

$x = \frac{\int \mu(t) t dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{5t}$

$x = \frac{\int t\,\mathrm{e}^{5t} dt + c_1}{\mathrm{e}^{5t}}$

- Evaluate the integrals on the rhs

$x = \frac{\frac{(5t-1)\mathrm{e}^{5t}}{25} + c_1}{\mathrm{e}^{5t}}$

- Simplify

$x = \frac{t}{5} - \frac{1}{25} + c_1 \mathrm{e}^{-5t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(x(t),t)+5*x(t)=t,x(t), singsol=all)
```

$$x(t) = \frac{t}{5} - \frac{1}{25} + \mathrm{e}^{-5t} c_1$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 22

```
DSolve[x'[t]+5*x[t]==t,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{t}{5} + c_1 e^{-5t} - \frac{1}{25}$$

4.8 problem 9.1 (viii)

Internal problem ID [12002]
Internal file name [`OUTPUT/10654_Saturday_September_02_2023_02_49_06_PM_74107094/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 9, First order linear equations and the integrating factor. Exercises page 86
Problem number: 9.1 (viii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x' + \left(a + \frac{1}{t}\right) x = b}$$

With initial conditions

$$[x(1) = x_0]$$

4.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -\frac{-ta-1}{t} \\ q(t) &= b \end{aligned}$$

Hence the ode is

$$x' - \frac{(-ta-1)\,x}{t} = b$$

The domain of $p(t) = -\frac{-ta-1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = b$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

4.8.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{-ta-1}{t}dt} \\ &= e^{ta+\ln(t)} \end{aligned}$$

Which simplifies to

$$\mu = t\,e^{ta}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu x) &= (\mu)\,(b) \\ \frac{d}{dt}\left(t\,e^{ta}x\right) &= \left(t\,e^{ta}\right)(b) \\ d\left(t\,e^{ta}x\right) &= \left(bt\,e^{ta}\right)\,dt \end{aligned}$$

Integrating gives

$$
\begin{aligned}
t\,\mathrm{e}^{ta} x &= \int b t\,\mathrm{e}^{ta}\,\mathrm{d}t \\
t\,\mathrm{e}^{ta} x &= \frac{(ta-1)\,b\,\mathrm{e}^{ta}}{a^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = t\,\mathrm{e}^{ta}$ results in

$$
x = \frac{\mathrm{e}^{-ta}(ta-1)\,b\,\mathrm{e}^{ta}}{t\,a^2} + \frac{c_1 \mathrm{e}^{-ta}}{t}
$$

which simplifies to

$$
x = \frac{c_1 \mathrm{e}^{-ta} a^2 + abt - b}{t\,a^2}
$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = x_0$ in the above solution gives an equation to solve for the constant of integration.

$$
x_0 = \frac{c_1 \mathrm{e}^{-a} a^2 + ab - b}{a^2}
$$

$$
c_1 = \frac{(x_0 a^2 - ab + b)\,\mathrm{e}^{a}}{a^2}
$$

Substituting c_1 found above in the general solution gives

$$
x = \frac{x_0 a^2 \mathrm{e}^{-a(t-1)} - ab\,\mathrm{e}^{-a(t-1)} + b\,\mathrm{e}^{-a(t-1)} + abt - b}{t\,a^2}
$$

Summary

The solution(s) found are the following

$$
x = \frac{x_0 a^2 \mathrm{e}^{-a(t-1)} - ab\,\mathrm{e}^{-a(t-1)} + b\,\mathrm{e}^{-a(t-1)} + abt - b}{t\,a^2} \tag{1}
$$

Verification of solutions

$$
x = \frac{x_0 a^2 \mathrm{e}^{-a(t-1)} - ab\,\mathrm{e}^{-a(t-1)} + b\,\mathrm{e}^{-a(t-1)} + abt - b}{t\,a^2}
$$

Verified OK.

4.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= -\frac{axt - bt + x}{t} \\
x' &= \omega(t, x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 62: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,x) = 0$$
$$\eta(t,x) = e^{-ta-\ln(t)} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right)S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-ta-\ln(t)}} dy \end{aligned}$$

Which results in

$$S = t\,e^{ta}x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{axt - bt + x}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= e^{ta}x(ta+1) \\ S_x &= t\,e^{ta} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = bt\,\mathrm{e}^{ta} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = bR\,\mathrm{e}^{Ra}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{(Ra-1)\,b\,\mathrm{e}^{Ra}}{a^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$t\,\mathrm{e}^{ta}x = \frac{(ta-1)\,b\,\mathrm{e}^{ta}}{a^2} + c_1$$

Which simplifies to

$$t\,\mathrm{e}^{ta}x = \frac{(ta-1)\,b\,\mathrm{e}^{ta}}{a^2} + c_1$$

Which gives

$$x = \frac{(bta\,\mathrm{e}^{ta} + c_1a^2 - b\,\mathrm{e}^{ta})\,\mathrm{e}^{-ta}}{t\,a^2}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = x_0$ in the above solution gives an equation to solve for the constant of integration.

$$x_0 = \frac{\mathrm{e}^{-a}\mathrm{e}^{a}ba + c_1\mathrm{e}^{-a}a^2 - \mathrm{e}^{-a}\mathrm{e}^{a}b}{a^2}$$

$$c_1 = \frac{(x_0a^2 - ab + b)\,\mathrm{e}^{a}}{a^2}$$

Substituting c_1 found above in the general solution gives

$$x = \frac{\mathrm{e}^{-ta}\mathrm{e}^{a}a^2x_0 + \mathrm{e}^{-ta}b\,\mathrm{e}^{ta}ta - \mathrm{e}^{-ta}\mathrm{e}^{a}ab + \mathrm{e}^{-ta}\mathrm{e}^{a}b - \mathrm{e}^{-ta}b\,\mathrm{e}^{ta}}{t\,a^2}$$

Summary
The solution(s) found are the following

$$x = \frac{e^{-ta} e^{a} a^{2} x_{0}+e^{-ta} b \,e^{ta} t a -e^{-ta} e^{a} a b +e^{-ta} e^{a} b -e^{-ta} b \,e^{ta}}{t \,a^{2}} \tag{1}$$

Verification of solutions

$$x = \frac{e^{-ta} e^{a} a^{2} x_{0}+e^{-ta} b \,e^{ta} t a -e^{-ta} e^{a} a b +e^{-ta} e^{a} b -e^{-ta} b \,e^{ta}}{t \,a^{2}}$$

Verified OK.

4.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = \left(-\left(a+\frac{1}{t}\right)x + b\right)\mathrm{d}t$$
$$\left(\left(a+\frac{1}{t}\right)x - b\right)\mathrm{d}t + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= \left(a+\frac{1}{t}\right)x - b\\ N(t,x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(\left(a+\frac{1}{t}\right)x - b\right)\\ &= a + \frac{1}{t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right)\\ &= 1\left(\left(a+\frac{1}{t}\right) - (0)\right)\\ &= a + \frac{1}{t}\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int a+\frac{1}{t}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{ta+\ln(t)}\\ &= t\,\mathrm{e}^{ta}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= t\,\mathrm{e}^{ta}\left(\left(a+\frac{1}{t}\right)x-b\right)\\ &= \mathrm{e}^{ta}((ax-b)\,t+x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= t\,\mathrm{e}^{ta}(1)\\ &= t\,\mathrm{e}^{ta}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\\ \left(\mathrm{e}^{ta}((ax-b)\,t+x)\right)+\left(t\,\mathrm{e}^{ta}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int\overline{M}\,\mathrm{d}t\\ \int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int\mathrm{e}^{ta}((ax-b)\,t+x)\,\mathrm{d}t\end{aligned}$$

$$\phi = \frac{(a^2tx-abt+b)\,\mathrm{e}^{ta}}{a^2}+f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = t\,\mathrm{e}^{ta} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = t\,\mathrm{e}^{ta}$. Therefore equation (4) becomes

$$t\,\mathrm{e}^{ta} = t\,\mathrm{e}^{ta} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{(a^2tx - abt + b)\,\mathrm{e}^{ta}}{a^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(a^2tx - abt + b)\,\mathrm{e}^{ta}}{a^2}$$

The solution becomes

$$x = \frac{(bta\,\mathrm{e}^{ta} + c_1a^2 - b\,\mathrm{e}^{ta})\,\mathrm{e}^{-ta}}{t\,a^2}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = x_0$ in the above solution gives an equation to solve for the constant of integration.

$$x_0 = \frac{\mathrm{e}^{-a}\mathrm{e}^{a}ba + c_1\mathrm{e}^{-a}a^2 - \mathrm{e}^{-a}\mathrm{e}^{a}b}{a^2}$$

$$c_1 = \frac{(x_0a^2 - ab + b)\,\mathrm{e}^{a}}{a^2}$$

Substituting c_1 found above in the general solution gives

$$x = \frac{e^{-ta} e^{a} a^{2} x_{0} + e^{-ta} b\, e^{ta} t a - e^{-ta} e^{a} a b + e^{-ta} e^{a} b - e^{-ta} b\, e^{ta}}{t\, a^{2}}$$

Summary

The solution(s) found are the following

$$x = \frac{e^{-ta} e^{a} a^{2} x_{0} + e^{-ta} b\, e^{ta} t a - e^{-ta} e^{a} a b + e^{-ta} e^{a} b - e^{-ta} b\, e^{ta}}{t\, a^{2}} \tag{1}$$

Verification of solutions

$$x = \frac{e^{-ta} e^{a} a^{2} x_{0} + e^{-ta} b\, e^{ta} t a - e^{-ta} e^{a} a b + e^{-ta} e^{a} b - e^{-ta} b\, e^{ta}}{t\, a^{2}}$$

Verified OK.

4.8.5 Maple step by step solution

Let's solve

$\left[x' + \left(a + \frac{1}{t}\right) x = b, x(1) = x_0\right]$

- Highest derivative means the order of the ODE is 1

  x'
- Isolate the derivative

  $x' = -\frac{(ta+1)x}{t} + b$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{(ta+1)x}{t} = b$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + \frac{(ta+1)x}{t}\right) = \mu(t)\, b$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$

  $\mu(t)\left(x' + \frac{(ta+1)x}{t}\right) = \mu'(t)\, x + \mu(t)\, x'$
- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)(ta+1)}{t}$
- Solve to find the integrating factor

  $\mu(t) = t\, e^{ta}$
- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int \mu(t)\, b dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\, x = \int \mu(t)\, b dt + c_1$$

- Solve for x

$$x = \frac{\int \mu(t) b dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = t \mathrm{e}^{ta}$

$$x = \frac{\int b t\, \mathrm{e}^{ta} dt + c_1}{t\, \mathrm{e}^{ta}}$$

- Evaluate the integrals on the rhs

$$x = \frac{\frac{(ta-1) b\, \mathrm{e}^{ta}}{a^2} + c_1}{t\, \mathrm{e}^{ta}}$$

- Simplify

$$x = \frac{c_1 \mathrm{e}^{-ta} a^2 + abt - b}{t\, a^2}$$

- Use initial condition $x(1) = x_0$

$$x_0 = \frac{c_1 \mathrm{e}^{-a} a^2 + ab - b}{a^2}$$

- Solve for c_1

$$c_1 = \frac{x_0 a^2 - ab + b}{\mathrm{e}^{-a} a^2}$$

- Substitute $c_1 = \frac{x_0 a^2 - ab + b}{\mathrm{e}^{-a} a^2}$ into general solution and simplify

$$x = \frac{\mathrm{e}^{-a(t-1)} (x_0 a^2 - ab + b) + b(ta-1)}{t\, a^2}$$

- Solution to the IVP

$$x = \frac{\mathrm{e}^{-a(t-1)} (x_0 a^2 - ab + b) + b(ta-1)}{t\, a^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 38

```
dsolve([diff(x(t),t)+(a+1/t)*x(t)=b,x(1) = x__0],x(t), singsol=all)
```

$$x(t) = \frac{(x_0 a^2 - ab + b)\,\mathrm{e}^{-a(t-1)} + b(at - 1)}{t\,a^2}$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 48

```
DSolve[{x'[t]+(a+1/t)*x[t]==b,{x[1]==x0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{e^{-at}(e^a a^2 \mathrm{x0} + be^{at}(at - 1) - (a - 1)e^a b)}{a^2 t}$$

4.9 problem 9.4

Internal problem ID [12003]
Internal file name [`OUTPUT/10655_Saturday_September_02_2023_02_56_32_PM_75656588/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 9, First order linear equations and the integrating factor. Exercises page 86
Problem number: 9.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{T' + k(T - \mu - a\cos(\omega(t - \phi))) = 0}$$

4.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$T' + p(t)T = q(t)$$

Where here

$$\begin{aligned} p(t) &= k \\ q(t) &= (a\cos(\omega(-t+\phi)) + \mu)\,k \end{aligned}$$

Hence the ode is

$$T' + Tk = (a\cos(\omega(-t+\phi)) + \mu)\,k$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int k dt}$$
$$= \mathrm{e}^{kt}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu T) = (\mu)\left((a\cos(\omega(-t+\phi)) + \mu) k\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{kt} T\right) = \left(\mathrm{e}^{kt}\right)\left((a\cos(\omega(-t+\phi)) + \mu) k\right)$$
$$\mathrm{d}\left(\mathrm{e}^{kt} T\right) = \left((a\cos(\omega(-t+\phi)) + \mu) k\,\mathrm{e}^{kt}\right)\,\mathrm{d}t$$

Integrating gives

$$\mathrm{e}^{kt} T = \int (a\cos(\omega(-t+\phi)) + \mu) k\,\mathrm{e}^{kt}\,\mathrm{d}t$$
$$\mathrm{e}^{kt} T = k\left(\frac{\mu\,\mathrm{e}^{kt}}{k} + a\left(\frac{k\,\mathrm{e}^{kt}\cos(\omega\phi - \omega t)}{k^2+\omega^2} - \frac{\omega\,\mathrm{e}^{kt}\sin(\omega\phi - \omega t)}{k^2+\omega^2}\right)\right) + c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{kt}$ results in

$$T = \mathrm{e}^{-kt} k\left(\frac{\mu\,\mathrm{e}^{kt}}{k} + a\left(\frac{k\,\mathrm{e}^{kt}\cos(\omega\phi - \omega t)}{k^2+\omega^2} - \frac{\omega\,\mathrm{e}^{kt}\sin(\omega\phi - \omega t)}{k^2+\omega^2}\right)\right) + c_1\mathrm{e}^{-kt}$$

which simplifies to

$$T = \frac{a\,k^2\cos(\omega(-t+\phi)) - ka\omega\sin(\omega(-t+\phi)) + (k^2+\omega^2)\left(c_1\mathrm{e}^{-kt} + \mu\right)}{k^2+\omega^2}$$

Summary

The solution(s) found are the following

$$T = \frac{a\,k^2\cos(\omega(-t+\phi)) - ka\omega\sin(\omega(-t+\phi)) + (k^2+\omega^2)\left(c_1\mathrm{e}^{-kt} + \mu\right)}{k^2+\omega^2} \tag{1}$$

Verification of solutions

$$T = \frac{a\,k^2\cos(\omega(-t+\phi)) - ka\omega\sin(\omega(-t+\phi)) + (k^2+\omega^2)\left(c_1\mathrm{e}^{-kt} + \mu\right)}{k^2+\omega^2}$$

Verified OK.

4.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
T' &= k(a\cos(\omega(t-\phi)) - T + \mu) \\
T' &= \omega(t,T)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_T - \xi_t) - \omega^2\xi_T - \omega_t\xi - \omega_T\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 65: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t,T) &= 0 \\ \eta(t,T) &= e^{-kt} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,T) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dT}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial T}\right) S(t,T) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-kt}} dy \end{aligned}$$

Which results in

$$S = e^{kt} T$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,T) S_T}{R_t + \omega(t,T) R_T} \tag{2}$$

Where in the above R_t, R_T, S_t, S_T are all partial derivatives and $\omega(t,T)$ is the right hand side of the original ode given by

$$\omega(t,T) = k(a\cos(\omega(t-\phi)) - T + \mu)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_T &= 0 \\ S_t &= k\, e^{kt} T \\ S_T &= e^{kt} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (a\cos(\omega(-t+\phi)) + \mu)\, k\, \mathrm{e}^{kt} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, T in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (a\cos(\omega(-R+\phi)) + \mu)\, k\, \mathrm{e}^{kR}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\mathrm{e}^{kR}\mu(k^2+\omega^2) + c_1(k^2+\omega^2) + \mathrm{e}^{kR}ak(\omega\sin(\omega(R-\phi)) + \cos(\omega(R-\phi))\, k)}{k^2+\omega^2} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, T coordinates. This results in

$$\mathrm{e}^{kt}T = \frac{\mathrm{e}^{kt}\mu(k^2+\omega^2) + c_1(k^2+\omega^2) + \mathrm{e}^{kt}ak(\omega\sin(\omega(t-\phi)) + k\cos(\omega(t-\phi)))}{k^2+\omega^2}$$

Which simplifies to

$$\mathrm{e}^{kt}T = \frac{\mathrm{e}^{kt}\mu(k^2+\omega^2) + c_1(k^2+\omega^2) + \mathrm{e}^{kt}ak(\omega\sin(\omega(t-\phi)) + k\cos(\omega(t-\phi)))}{k^2+\omega^2}$$

Which gives

$$T = -\frac{\mathrm{e}^{-kt}(a\omega\sin(\omega(-t+\phi))\, k\,\mathrm{e}^{kt} - \cos(\omega(-t+\phi))\,\mathrm{e}^{kt}a\,k^2 - \mathrm{e}^{kt}k^2\mu - \mathrm{e}^{kt}\mu\,\omega^2 - c_1k^2 - c_1\omega^2)}{k^2+\omega^2}$$

Summary

The solution(s) found are the following

$$T = -\frac{\mathrm{e}^{-kt}(a\omega\sin(\omega(-t+\phi))\, k\,\mathrm{e}^{kt} - \cos(\omega(-t+\phi))\,\mathrm{e}^{kt}a\,k^2 - \mathrm{e}^{kt}k^2\mu - \mathrm{e}^{kt}\mu\,\omega^2 - c_1k^2 - c_1\omega^2)}{k^2+\omega^2} \tag{1}$$

Verification of solutions

$$T = -\frac{\mathrm{e}^{-kt}(a\omega\sin(\omega(-t+\phi))\, k\,\mathrm{e}^{kt} - \cos(\omega(-t+\phi))\,\mathrm{e}^{kt}a\,k^2 - \mathrm{e}^{kt}k^2\mu - \mathrm{e}^{kt}\mu\,\omega^2 - c_1k^2 - c_1\omega^2)}{k^2+\omega^2}$$

Verified OK.

4.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,T)\,\mathrm{d}t + N(t,T)\,\mathrm{d}T = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}T &= (-k(T - \mu - a\cos(\omega(t-\phi))))\,\mathrm{d}t\\ (k(T - \mu - a\cos(\omega(t-\phi))))\,\mathrm{d}t + \mathrm{d}T &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,T) &= k(T - \mu - a\cos(\omega(t-\phi)))\\ N(t,T) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial T} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial T} &= \frac{\partial}{\partial T}(k(T - \mu - a\cos(\omega(t - \phi)))) \\ &= k\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial T} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial T} - \frac{\partial N}{\partial t}\right) \\ &= 1((k) - (0)) \\ &= k\end{aligned}$$

Since A does not depend on T, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int k \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{kt} \\ &= \mathrm{e}^{kt}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{kt}(k(T - \mu - a\cos(\omega(t - \phi)))) \\ &= (T - \mu - a\cos(\omega(-t + \phi)))\, k\,\mathrm{e}^{kt}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= e^{kt}(1) \\
&= e^{kt}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{dT}{dt} &= 0 \\
\left((T - \mu - a\cos(\omega(-t + \phi)))\, k\, e^{kt}\right) + \left(e^{kt}\right)\frac{dT}{dt} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, T)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial T} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\, dt &= \int \overline{M}\, dt \\
\int \frac{\partial \phi}{\partial t}\, dt &= \int (T - \mu - a\cos(\omega(-t + \phi)))\, k\, e^{kt}\, dt \\
\phi &= \frac{e^{kt}\left(-a\,k^2\cos(\omega(-t + \phi)) + ka\omega\sin(\omega(-t + \phi)) + (k^2 + \omega^2)(T - \mu)\right)}{k^2 + \omega^2} + f(T)
\end{aligned} \tag{3}
$$

Where $f(T)$ is used for the constant of integration since ϕ is a function of both t and T. Taking derivative of equation (3) w.r.t T gives

$$
\frac{\partial \phi}{\partial T} = e^{kt} + f'(T) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial T} = e^{kt}$. Therefore equation (4) becomes

$$
e^{kt} = e^{kt} + f'(T) \tag{5}
$$

Solving equation (5) for $f'(T)$ gives

$$
f'(T) = 0
$$

Therefore

$$f(T) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(T)$ into equation (3) gives ϕ

$$\phi = \frac{e^{kt}(-a\,k^2 \cos(\omega(-t+\phi)) + ka\omega \sin(\omega(-t+\phi)) + (k^2+\omega^2)(T-\mu))}{k^2+\omega^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{e^{kt}(-a\,k^2 \cos(\omega(-t+\phi)) + ka\omega \sin(\omega(-t+\phi)) + (k^2+\omega^2)(T-\mu))}{k^2+\omega^2}$$

The solution becomes

$$T = -\frac{e^{-kt}\left(a\omega \sin(\omega(-t+\phi))\,k\,e^{kt} - \cos(\omega(-t+\phi))\,e^{kt}a\,k^2 - e^{kt}k^2\mu - e^{kt}\mu\,\omega^2 - c_1k^2 - c_1\omega^2\right)}{k^2+\omega^2}$$

Summary

The solution(s) found are the following

$$T = -\frac{e^{-kt}\left(a\omega \sin(\omega(-t+\phi))\,k\,e^{kt} - \cos(\omega(-t+\phi))\,e^{kt}a\,k^2 - e^{kt}k^2\mu - e^{kt}\mu\,\omega^2 - c_1k^2 - c_1\omega^2\right)}{k^2+\omega^2} \tag{1}$$

Verification of solutions

$$T = -\frac{e^{-kt}\left(a\omega \sin(\omega(-t+\phi))\,k\,e^{kt} - \cos(\omega(-t+\phi))\,e^{kt}a\,k^2 - e^{kt}k^2\mu - e^{kt}\mu\,\omega^2 - c_1k^2 - c_1\omega^2\right)}{k^2+\omega^2}$$

Verified OK.

4.9.4 Maple step by step solution

Let's solve

$T' + k(T - \mu - a\cos(\omega(t-\phi))) = 0$

- Highest derivative means the order of the ODE is 1

  T'

- Isolate the derivative

  $T' = -Tk + (a\cos(\omega(-t+\phi)) + \mu)\, k$

- Group terms with T on the lhs of the ODE and the rest on the rhs of the ODE

  $T' + Tk = (a\cos(\omega(-t+\phi)) + \mu)\, k$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(T' + Tk) = \mu(t)\,(a\cos(\omega(-t+\phi)) + \mu)\, k$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, T)$

  $\mu(t)\,(T' + Tk) = \mu'(t)\, T + \mu(t)\, T'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)\, k$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{kt}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, T)\right) dt = \int \mu(t)\,(a\cos(\omega(-t+\phi)) + \mu)\, k dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, T = \int \mu(t)\,(a\cos(\omega(-t+\phi)) + \mu)\, k dt + c_1$

- Solve for T

  $T = \frac{\int \mu(t)(a\cos(\omega(-t+\phi))+\mu)k dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{kt}$

  $T = \frac{\int (a\cos(\omega(-t+\phi))+\mu)k\,\mathrm{e}^{kt} dt + c_1}{\mathrm{e}^{kt}}$

- Evaluate the integrals on the rhs

  $T = \frac{k\left(\frac{\mu\,\mathrm{e}^{kt}}{k} + a\left(\frac{k\,\mathrm{e}^{kt}\cos(\omega\phi-\omega t)}{k^2+\omega^2} - \frac{\omega\,\mathrm{e}^{kt}\sin(\omega\phi-\omega t)}{k^2+\omega^2}\right)\right) + c_1}{\mathrm{e}^{kt}}$

- Simplify

  $T = \frac{a\,k^2\cos(\omega(-t+\phi)) - ka\omega\sin(\omega(-t+\phi)) + (k^2+\omega^2)\,(c_1\mathrm{e}^{-kt}+\mu)}{k^2+\omega^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
dsolve(diff(T(t),t)=-k*(T(t)- (mu+a*cos( omega*(t-phi)))),T(t), singsol=all)
```

$$T(t) = \frac{\cos(\omega(-t+\phi))\, a\, k^2 - \sin(\omega(-t+\phi))\, ak\omega + (k^2+\omega^2)\left(\mathrm{e}^{-kt} c_1 + \mu\right)}{k^2+\omega^2}$$

✓ Solution by Mathematica

Time used: 0.511 (sec). Leaf size: 60

```
DSolve[T'[t]==-k*(T[t]- (mu+a*Cos[ omega*(t-phi)])),T[t],t,IncludeSingularSolutions -> True]
```

$$T(t) \to -\frac{ak\omega\sin(\omega(\phi-t))}{k^2+\omega^2} + \frac{ak^2\cos(\omega(\phi-t))}{k^2+\omega^2} + c_1 e^{-kt} + \mu$$

5 Chapter 10, Two tricks for nonlinear equations. Exercises page 97

5.1 problem 10.1 (i)

Internal problem ID [12004]
Internal file name [OUTPUT/10656_Saturday_September_02_2023_02_56_34_PM_18887529/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 10, Two tricks for nonlinear equations. Exercises page 97
Problem number: 10.1 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x),G(x)]`], [_Abel, `2
   nd type`, `class A`]]
```

$$\boxed{2yx + \left(x^2 + 2y\right) y' = \sec\left(x\right)^2}$$

5.1.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 + 2y)\,\mathrm{d}y &= \left(-2xy + \sec(x)^2\right)\mathrm{d}x \\ \left(2xy - \sec(x)^2\right)\mathrm{d}x + (x^2 + 2y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2xy - \sec(x)^2 \\ N(x, y) &= x^2 + 2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2xy - \sec(x)^2\right) \\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + 2y) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int 2xy-\sec(x)^2\,\mathrm{d}x$$
$$\phi=-\tan(x)+x^2y+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x^2+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x^2+2y$. Therefore equation (4) becomes

$$x^2+2y=x^2+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int (2y)\,\mathrm{d}y$$
$$f(y)=y^2+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=-\tan(x)+x^2y+y^2+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\tan(x) + x^2y + y^2$$

Summary

The solution(s) found are the following

$$-\tan(x) + x^2y + y^2 = c_1 \tag{1}$$

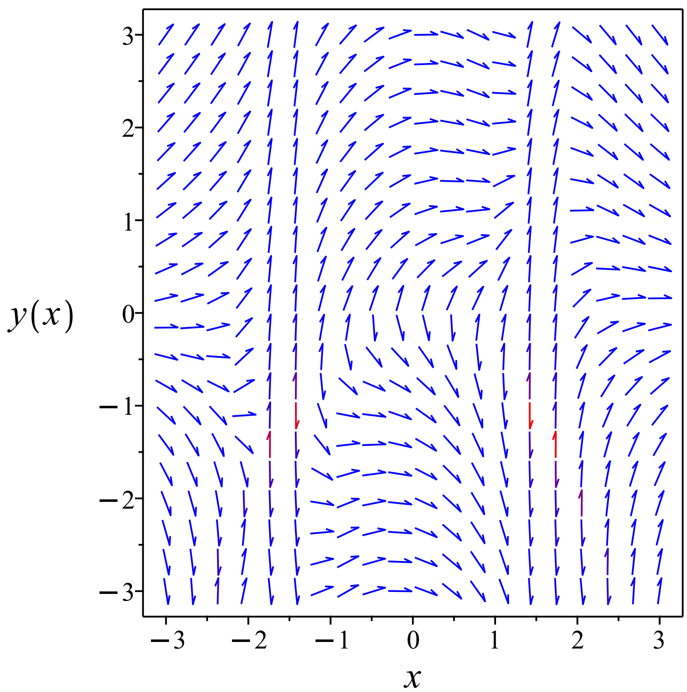


Figure 51: Slope field plot

Verification of solutions

$$-\tan(x) + x^2y + y^2 = c_1$$

Verified OK.

5.1.2 Maple step by step solution

Let's solve

$2yx + (x^2 + 2y)\, y' = \sec(x)^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ∘ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

- ∘ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- ∘ Evaluate derivatives

  $2x = 2x$

- ∘ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(2xy - \sec(x)^2\right) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = -\tan(x) + x^2 y + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x^2 + 2y = x^2 + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 2y$

- Solve for $f_1(y)$

  $f_1(y) = y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = -\tan(x) + x^2 y + y^2$$

- Substitute $F(x, y)$ into the solution of the ODE

$$-\tan(x) + x^2 y + y^2 = c_1$$

- Solve for y

$$\left\{y = -\frac{x^2}{2} - \frac{\sqrt{x^4 + 4\tan(x) + 4c_1}}{2}, y = -\frac{x^2}{2} + \frac{\sqrt{x^4 + 4\tan(x) + 4c_1}}{2}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
dsolve((2*x*y(x)- sec(x)^2)+(x^2+2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{2} - \frac{\sqrt{x^4 + 4\tan(x) - 4c_1}}{2}$$
$$y(x) = -\frac{x^2}{2} + \frac{\sqrt{x^4 + 4\tan(x) - 4c_1}}{2}$$

✓ Solution by Mathematica

Time used: 26.886 (sec). Leaf size: 90

```
DSolve[(2*x*y[x]- Sec[x]^2)+(x^2+2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-x^2 - \sqrt{\sec^2(x)}\sqrt{\cos^2(x)\left(x^4 + 4\tan(x) + 4c_1\right)}\right)$$
$$y(x) \to \frac{1}{2}\left(-x^2 + \sqrt{\sec^2(x)}\sqrt{\cos^2(x)\left(x^4 + 4\tan(x) + 4c_1\right)}\right)$$

5.2 problem 10.1 (ii)

Internal problem ID [12005]
Internal file name [`OUTPUT/10657_Saturday_September_02_2023_02_56_51_PM_2375291/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 10, Two tricks for nonlinear equations. Exercises page 97
Problem number: 10.1 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y\,e^{x}+yx\,e^{x}+\left(x\,e^{x}+2\right)y'=-1}$$

5.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{e^{x}(x+1)}{x\,e^{x}+2}$$
$$q(x)=-\frac{1}{x\,e^{x}+2}$$

Hence the ode is

$$y'+\frac{e^{x}(x+1)\,y}{x\,e^{x}+2}=-\frac{1}{x\,e^{x}+2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{e^x (x+1)}{x e^x + 2}dx}$$
$$= x e^x + 2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\frac{1}{x e^x + 2}\right)$$
$$\frac{d}{dx}((x e^x + 2) y) = (x e^x + 2)\left(-\frac{1}{x e^x + 2}\right)$$
$$d((x e^x + 2) y) = -1 \, dx$$

Integrating gives

$$(x e^x + 2) y = \int -1 \, dx$$
$$(x e^x + 2) y = -x + c_1$$

Dividing both sides by the integrating factor $\mu = x e^x + 2$ results in

$$y = -\frac{x}{x e^x + 2} + \frac{c_1}{x e^x + 2}$$

which simplifies to

$$y = \frac{-x + c_1}{x e^x + 2}$$

Summary
The solution(s) found are the following

$$y = \frac{-x + c_1}{x e^x + 2} \tag{1}$$

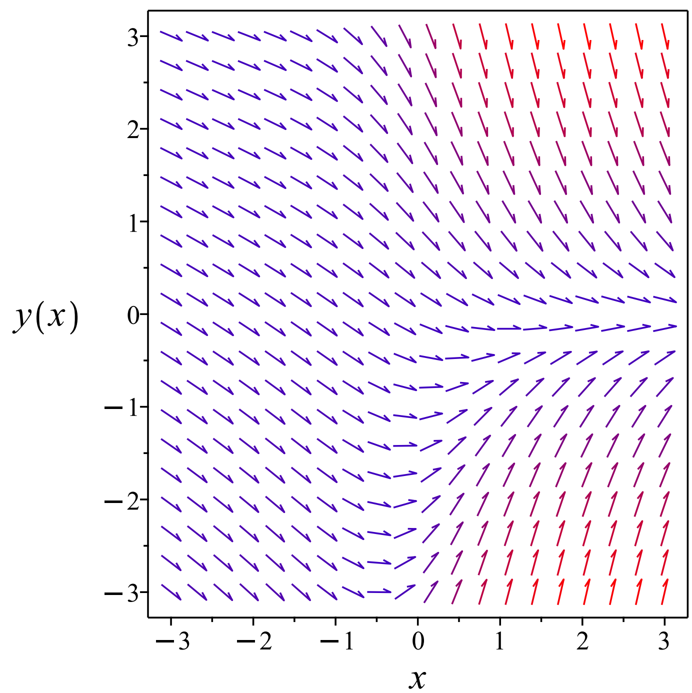


Figure 52: Slope field plot

Verification of solutions

$$y = \frac{-x + c_1}{x\,\mathrm{e}^{x} + 2}$$

Verified OK.

5.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{x\,\mathrm{e}^{x} y + \mathrm{e}^{x} y + 1}{x\,\mathrm{e}^{x} + 2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 69: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x\,\mathrm{e}^{x} + 2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x\,\mathrm{e}^{x}+2}} dy \end{aligned}$$

Which results in

$$S = (x\,\mathrm{e}^{x} + 2)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x\,\mathrm{e}^{x} y + \mathrm{e}^{x} y + 1}{x\,\mathrm{e}^{x} + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{x} y(x+1) \\ S_y &= x\,\mathrm{e}^{x} + 2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x\,e^x + 2)\,y = -x + c_1$$

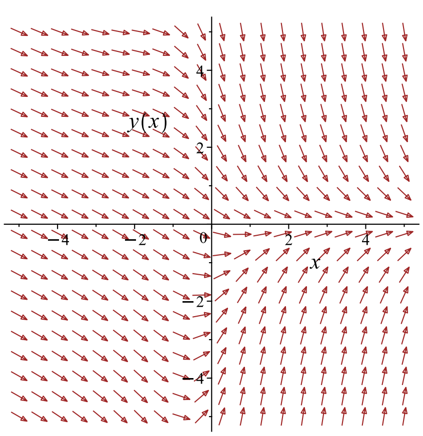
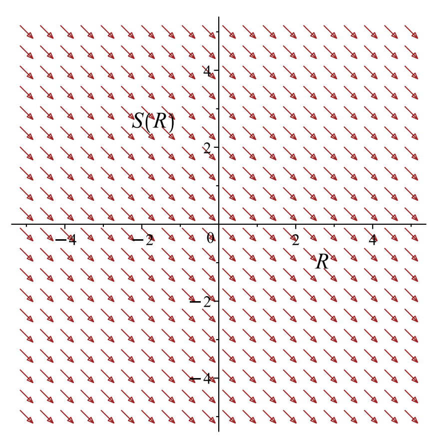
Which simplifies to

$$(x\,e^x + 2)\,y = -x + c_1$$

Which gives

$$y = \frac{-x + c_1}{x\,e^x + 2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x\,e^x y + e^x y + 1}{x\,e^x + 2}$ 	$R = x$ $S = (x\,e^x + 2)\,y$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = \frac{-x + c_1}{x\,e^x + 2} \tag{1}$$

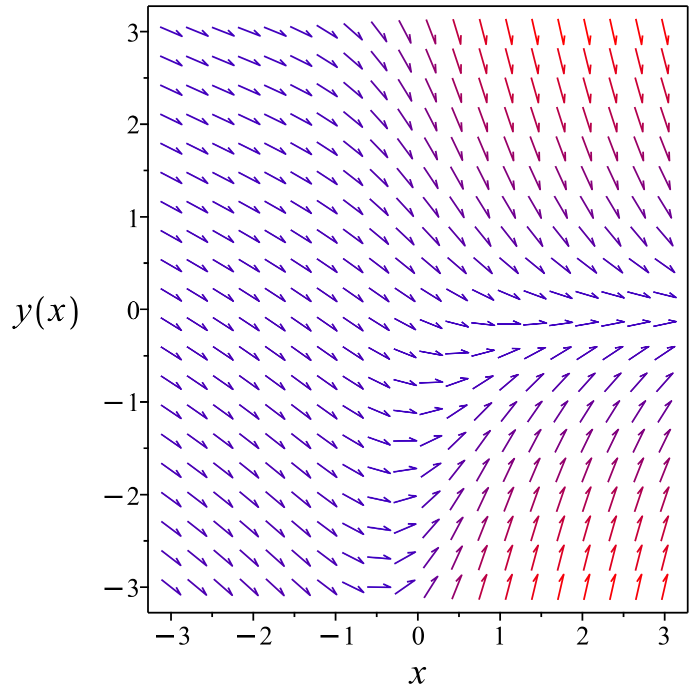


Figure 53: Slope field plot

Verification of solutions

$$y = \frac{-x + c_1}{x\,\mathrm{e}^x + 2}$$

Verified OK.

5.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x\,\mathrm{e}^x + 2)\,\mathrm{d}y = (-1 - \mathrm{e}^x y - x\,\mathrm{e}^x y)\,\mathrm{d}x$$
$$(x\,\mathrm{e}^x y + \mathrm{e}^x y + 1)\,\mathrm{d}x + (x\,\mathrm{e}^x + 2)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x\,\mathrm{e}^x y + \mathrm{e}^x y + 1$$
$$N(x, y) = x\,\mathrm{e}^x + 2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x\,\mathrm{e}^x y + \mathrm{e}^x y + 1) \\ &= \mathrm{e}^x(x + 1)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\,\mathrm{e}^x + 2) \\ &= \mathrm{e}^x(x + 1)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x \, \mathrm{e}^x y + \mathrm{e}^x y + 1 \, \mathrm{d}x$$
$$\phi = x(\mathrm{e}^x y + 1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x \, \mathrm{e}^x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x \, \mathrm{e}^x + 2$. Therefore equation (4) becomes

$$x \, \mathrm{e}^x + 2 = x \, \mathrm{e}^x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (2) \, \mathrm{d}y$$
$$f(y) = 2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(\mathrm{e}^x y + 1) + 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(\mathrm{e}^x y + 1) + 2y$$

The solution becomes

$$y = \frac{-x + c_1}{x\,\mathrm{e}^x + 2}$$

Summary

The solution(s) found are the following

$$y = \frac{-x + c_1}{x\,\mathrm{e}^x + 2} \tag{1}$$

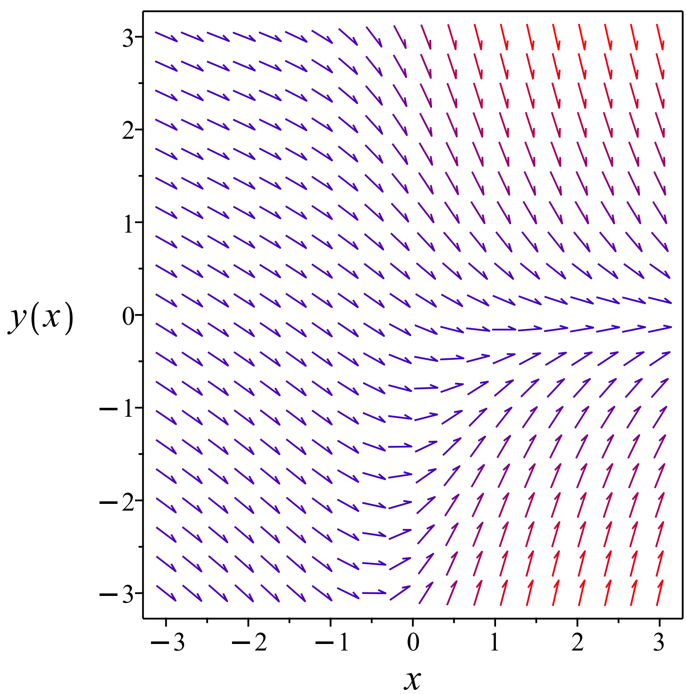


Figure 54: Slope field plot

Verification of solutions

$$y = \frac{-x + c_1}{x\,\mathrm{e}^x + 2}$$

Verified OK.

5.2.4 Maple step by step solution

Let's solve

$y\,\mathrm{e}^{x}+y x\,\mathrm{e}^{x}+\left(x\,\mathrm{e}^{x}+2\right) y'=-1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-\frac{\mathrm{e}^{x}(x+1) y}{x\,\mathrm{e}^{x}+2}-\frac{1}{x\,\mathrm{e}^{x}+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+\frac{\mathrm{e}^{x}(x+1) y}{x\,\mathrm{e}^{x}+2}=-\frac{1}{x\,\mathrm{e}^{x}+2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{\mathrm{e}^{x}(x+1) y}{x\,\mathrm{e}^{x}+2}\right)=-\frac{\mu(x)}{x\,\mathrm{e}^{x}+2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{\mathrm{e}^{x}(x+1) y}{x\,\mathrm{e}^{x}+2}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)\mathrm{e}^{x}(x+1)}{x\,\mathrm{e}^{x}+2}$

- Solve to find the integrating factor

  $\mu(x)=x\,\mathrm{e}^{x}+2$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int-\frac{\mu(x)}{x\,\mathrm{e}^{x}+2}dx+c_{1}$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int-\frac{\mu(x)}{x\,\mathrm{e}^{x}+2}dx+c_{1}$

- Solve for y

  $y=\frac{\int-\frac{\mu(x)}{x\,\mathrm{e}^{x}+2}dx+c_{1}}{\mu(x)}$

- Substitute $\mu(x)=x\mathrm{e}^{x}+2$

  $y=\frac{\int(-1)dx+c_{1}}{x\,\mathrm{e}^{x}+2}$

- Evaluate the integrals on the rhs

  $y=\frac{-x+c_{1}}{x\,\mathrm{e}^{x}+2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((1+exp(x)*y(x)+x*exp(x)*y(x))+(x*exp(x)+2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 - x}{e^x x + 2}$$

✓ Solution by Mathematica

Time used: 0.142 (sec). Leaf size: 21

```
DSolve[(1+Exp[x]*y[x]+x*Exp[x]*y[x])+(x*Exp[x]+2)*y'[x]==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{-x + c_1}{e^x x + 2}$$

5.3 problem 10.1 (iii)

Internal problem ID [12006]
Internal file name [`OUTPUT/10658_Saturday_September_02_2023_02_56_52_PM_22600988/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 10, Two tricks for nonlinear equations. Exercises page 97
Problem number: 10.1 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{(\cos(y)\,x+\cos(x))\,y'+\sin(y)-\sin(x)\,y=0}$$

5.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(y)\,x + \cos(x))\,\mathrm{d}y &= (-\sin(y) + \sin(x)\,y)\,\mathrm{d}x \\ (\sin(y) - \sin(x)\,y)\,\mathrm{d}x + (\cos(y)\,x + \cos(x))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \sin(y) - \sin(x)\,y \\ N(x, y) &= \cos(y)\,x + \cos(x) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\sin(y) - \sin(x)\,y) \\ &= \cos(y) - \sin(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(y)\,x + \cos(x)) \\ &= \cos(y) - \sin(x) \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \sin(y) - \sin(x) y \mathrm{d}x$$

$$\phi = \cos(x) y + \sin(y) x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(y) x + \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(y) x + \cos(x)$. Therefore equation (4) becomes

$$\cos(y) x + \cos(x) = \cos(y) x + \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x) y + \sin(y) x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x) y + \sin(y) x$$

Summary
The solution(s) found are the following

$$y \cos(x) + \sin(y) x = c_1 \tag{1}$$

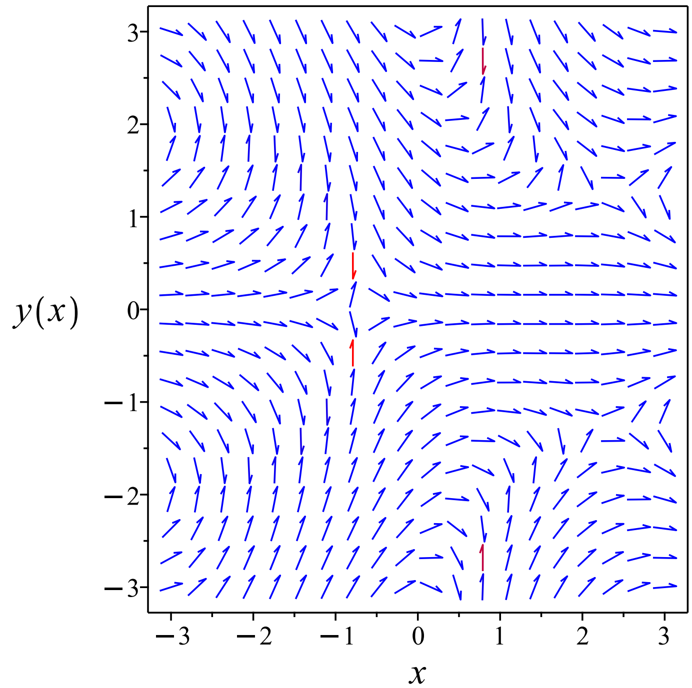


Figure 55: Slope field plot

Verification of solutions

$$y\cos(x) + \sin(y)\,x = c_1$$

Verified OK.

5.3.2 Maple step by step solution

Let's solve

$(\cos(y)\,x + \cos(x))\,y' + \sin(y) - \sin(x)\,y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right)y' = 0$

- Evaluate derivatives

  $\cos(y) - \sin(x) = \cos(y) - \sin(x)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (\sin(y) - \sin(x) y) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \cos(x) y + \sin(y) x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $\cos(y) x + \cos(x) = \cos(x) + \cos(y) x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \cos(x) y + \sin(y) x$

- Substitute $F(x, y)$ into the solution of the ODE

  $\cos(x) y + \sin(y) x = c_1$

- Solve for y

  $y = \mathit{RootOf}(-\cos(x)\_Z - \sin(\_Z) x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 15

```
dsolve((x*cos(y(x))+cos(x))*diff(y(x),x)+sin(y(x))-y(x)*sin(x)=0,y(x), singsol=all)
```

$$\cos(x)\, y(x) + x \sin(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.254 (sec). Leaf size: 17

```
DSolve[(x*Cos[y[x]]+Cos[x])*y'[x]+Sin[y[x]]-y[x]*Sin[x]==0,y[x],x,IncludeSingularSolutions -
```

$$\text{Solve}[x \sin(y(x)) + y(x) \cos(x) = c_1, y(x)]$$

5.4 problem 10.1 (iv)

Internal problem ID [12007]
Internal file name [`OUTPUT/10659_Saturday_September_02_2023_02_57_12_PM_55385050/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 10, Two tricks for nonlinear equations. Exercises page 97
Problem number: 10.1 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{e^{x} \sin (y)+y+\left(e^{x} \cos (y)+x+e^{y}\right) y'=0}$$

5.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(\mathrm{e}^x \cos(y) + x + \mathrm{e}^y)\, \mathrm{d}y &= (-\mathrm{e}^x \sin(y) - y)\, \mathrm{d}x \\
(\mathrm{e}^x \sin(y) + y)\, \mathrm{d}x + (\mathrm{e}^x \cos(y) + x + \mathrm{e}^y)\, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= \mathrm{e}^x \sin(y) + y \\
N(x, y) &= \mathrm{e}^x \cos(y) + x + \mathrm{e}^y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\mathrm{e}^x \sin(y) + y) \\
&= \mathrm{e}^x \cos(y) + 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^x \cos(y) + x + \mathrm{e}^y) \\
&= \mathrm{e}^x \cos(y) + 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \mathrm{e}^x \sin(y) + y \mathrm{d}x$$
$$\phi = xy + \mathrm{e}^x \sin(y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^x \cos(y) + x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^x \cos(y) + x + \mathrm{e}^y$. Therefore equation (4) becomes

$$\mathrm{e}^x \cos(y) + x + \mathrm{e}^y = \mathrm{e}^x \cos(y) + x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^y$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int (\mathrm{e}^y) \mathrm{d}y$$
$$f(y) = \mathrm{e}^y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = xy + \mathrm{e}^x \sin(y) + \mathrm{e}^y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy + \mathrm{e}^x \sin(y) + \mathrm{e}^y$$

Summary
The solution(s) found are the following

$$yx + e^x \sin(y) + e^y = c_1 \tag{1}$$

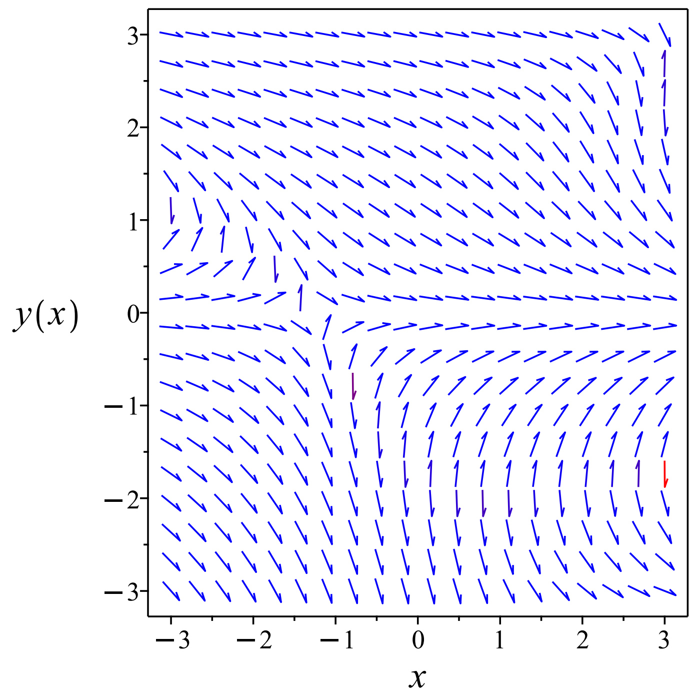


Figure 56: Slope field plot

Verification of solutions

$$yx + e^x \sin(y) + e^y = c_1$$

Verified OK.

5.4.2 Maple step by step solution

Let's solve

$e^x \sin(y) + y + (e^x \cos(y) + x + e^y) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x,y)=0$

- Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $e^{x}\cos(y)+1=e^{x}\cos(y)+1$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(e^{x}\sin(y)+y)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=xy+e^{x}\sin(y)+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $e^{x}\cos(y)+x+e^{y}=x+e^{x}\cos(y)+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=e^{y}$

- Solve for $f_1(y)$

  $f_1(y)=e^{y}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=xy+e^{x}\sin(y)+e^{y}$

- Substitute $F(x,y)$ into the solution of the ODE

  $xy+e^{x}\sin(y)+e^{y}=c_1$

- Solve for y

  $y=\mathit{RootOf}\left(-\sin(\_Z)e^{x}-\_Zx-e^{\_Z}+c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(exp(x)*sin(y(x))+y(x)+ (exp(x)*cos(y(x))+x+exp(y(x)))*diff(y(x),x)=0,y(x), singsol=al
```

$$y(x)\,x+\mathrm{e}^{x}\sin\left(y(x)\right)+\mathrm{e}^{y(x)}+c_1=0$$

✓ Solution by Mathematica

Time used: 0.637 (sec). Leaf size: 22

```
DSolve[Exp[x]*Sin[y[x]]+y[x]+ (Exp[x]*Cos[y[x]]+x+Exp[y[x]])*y'[x]==0,y[x],x,IncludeSingular
```

$$\text{Solve}\left[e^{y(x)}+xy(x)+e^{x}\sin(y(x))=c_1,y(x)\right]$$

5.5 problem 10.2

Internal problem ID [12008]
Internal file name [`OUTPUT/10660_Saturday_September_02_2023_02_57_34_PM_89564866/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 10, Two tricks for nonlinear equations. Exercises page 97
Problem number: 10.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`]]
```

$$\boxed{e^{-y} \sec(x) - e^{-y} y' = -2\cos(x)}$$

5.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(-\mathrm{e}^{-y}\right) \mathrm{d}y &= \left(-\mathrm{e}^{-y} \sec(x) - 2\cos(x)\right) \mathrm{d}x \\
\left(\mathrm{e}^{-y} \sec(x) + 2\cos(x)\right) \mathrm{d}x + \left(-\mathrm{e}^{-y}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= \mathrm{e}^{-y} \sec(x) + 2\cos(x) \\
N(x, y) &= -\mathrm{e}^{-y}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{-y} \sec(x) + 2\cos(x)\right) \\
&= -\mathrm{e}^{-y} \sec(x)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\mathrm{e}^{-y}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\mathrm{e}^{y}\left(\left(-\mathrm{e}^{-y} \sec(x)\right) - (0)\right) \\
&= \sec(x)
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \sec(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\sec(x)+\tan(x))}\\ &= \sec(x) + \tan(x)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \sec(x) + \tan(x)\left(\mathrm{e}^{-y}\sec(x) + 2\cos(x)\right)\\ &= \frac{-2\cos(x)^2 - \mathrm{e}^{-y}}{-1+\sin(x)}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \sec(x) + \tan(x)\left(-\mathrm{e}^{-y}\right)\\ &= -\mathrm{e}^{-y}(\sec(x) + \tan(x))\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-2\cos(x)^2 - \mathrm{e}^{-y}}{-1+\sin(x)}\right) + \left(-\mathrm{e}^{-y}(\sec(x)+\tan(x))\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \overline{M}\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \frac{-2\cos(x)^2 - \mathrm{e}^{-y}}{-1+\sin(x)}\, \mathrm{d}x$$

$$\phi = \frac{-4\cos\left(\frac{x}{2}\right)^3 + 4\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)^2 + (2x + 2\,\mathrm{e}^{-y})\cos\left(\frac{x}{2}\right) - 2\sin\left(\frac{x}{2}\right)x}{-\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)} + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2\,\mathrm{e}^{-y}\cos\left(\frac{x}{2}\right)}{-\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\mathrm{e}^{-y}(\sec(x) + \tan(x))$. Therefore equation (4) becomes

$$-\mathrm{e}^{-y}(\sec(x) + \tan(x)) = \frac{2\cos\left(\frac{x}{2}\right)\mathrm{e}^{-y}}{\sin\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{\mathrm{e}^{-y}\left(\sec(x)\sin\left(\frac{x}{2}\right) - \sec(x)\cos\left(\frac{x}{2}\right) + \tan(x)\sin\left(\frac{x}{2}\right) - \tan(x)\cos\left(\frac{x}{2}\right) + 2\cos\left(\frac{x}{2}\right)\right)}{\sin\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right)}$$

$$= \mathrm{e}^{-y}$$

Integrating the above w.r.t y results in

$$\int f'(y)\, \mathrm{d}y = \int (\mathrm{e}^{-y})\, \mathrm{d}y$$

$$f(y) = -\mathrm{e}^{-y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-4\cos\left(\frac{x}{2}\right)^3 + 4\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)^2 + (2x + 2\,\mathrm{e}^{-y})\cos\left(\frac{x}{2}\right) - 2\sin\left(\frac{x}{2}\right)x}{-\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)} - \mathrm{e}^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-4\cos\left(\frac{x}{2}\right)^3 + 4\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)^2 + \left(2x + 2\,\mathrm{e}^{-y}\right)\cos\left(\frac{x}{2}\right) - 2\sin\left(\frac{x}{2}\right)x}{-\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)} - \mathrm{e}^{-y}$$

The solution becomes

$$y = -\ln\left(-\frac{4\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)^2 - 4\cos\left(\frac{x}{2}\right)^3 + c_1\sin\left(\frac{x}{2}\right) - 2\sin\left(\frac{x}{2}\right)x - c_1\cos\left(\frac{x}{2}\right) + 2x\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)}\right)$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\ln\left(-\frac{4\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)^2 - 4\cos\left(\frac{x}{2}\right)^3 + c_1\sin\left(\frac{x}{2}\right) - 2\sin\left(\frac{x}{2}\right)x - c_1\cos\left(\frac{x}{2}\right) + 2x\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)}\right) \tag{1}$$

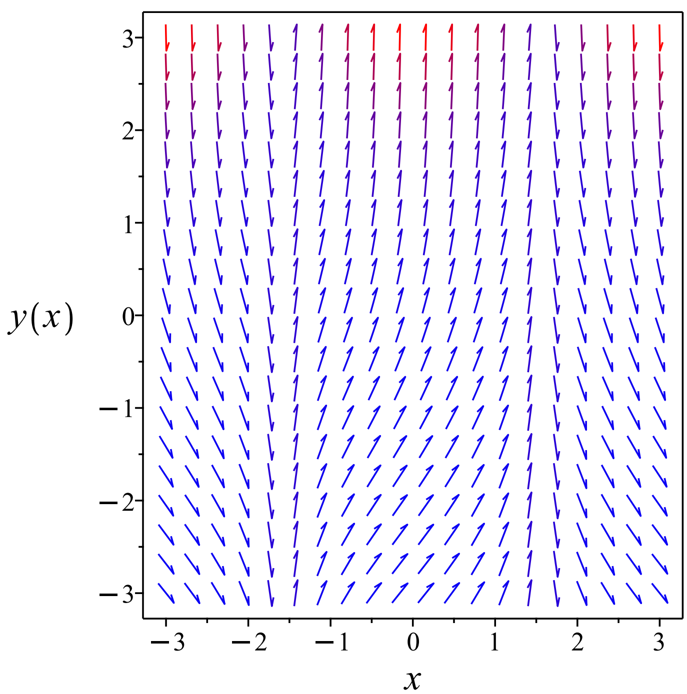


Figure 57: Slope field plot

Verification of solutions

$$y = -\ln\left(-\frac{4\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)^{2}-4\cos\left(\frac{x}{2}\right)^{3}+c_{1}\sin\left(\frac{x}{2}\right)-2\sin\left(\frac{x}{2}\right)x-c_{1}\cos\left(\frac{x}{2}\right)+2x\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 45

```
dsolve(exp(-y(x))*sec(x)+2*cos(x)-exp(-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \ln\left(-\frac{\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)^{2}}{\left(-4\cos\left(\frac{x}{2}\right)^{2}+c_{1}+2x\right)\left(2\cos\left(\frac{x}{2}\right)^{2}-1\right)}\right)$$

✓ Solution by Mathematica

Time used: 2.559 (sec). Leaf size: 33

```
DSolve[Exp[-y[x]]*Sec[x]+2*Cos[x]-Exp[-y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \log\left(\frac{e^{2\operatorname{arctanh}\left(\tan\left(\frac{x}{2}\right)\right)}}{2(-x+\cos(x)-2c_{1})}\right)$$

5.6 problem 10.3 (i)

Internal problem ID [12009]
Internal file name [`OUTPUT/10661_Saturday_September_02_2023_02_57_49_PM_60897075/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 10, Two tricks for nonlinear equations. Exercises page 97
Problem number: 10.3 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2yy' = -V'(x)}$$

5.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{D(V)(x)}{2y} \end{aligned}$$

Where $f(x) = -\frac{D(V)(x)}{2}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = -\frac{D(V)(x)}{2}\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int -\frac{D(V)(x)}{2}\,dx$$

$$\frac{y^2}{2} = -\frac{V(x)}{2} + c_1$$

Which results in

$$y = \sqrt{-V(x) + 2c_1}$$
$$y = -\sqrt{-V(x) + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-V(x) + 2c_1} \tag{1}$$
$$y = -\sqrt{-V(x) + 2c_1} \tag{2}$$

Verification of solutions

$$y = \sqrt{-V(x) + 2c_1}$$

Verified OK.

$$y = -\sqrt{-V(x) + 2c_1}$$

Verified OK.

5.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{V'(x)}{2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 74: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= -\frac{2}{V'(x)} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{2}{V'(x)}} dx \end{aligned}$$

Which results in

$$S = -\frac{V(x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{V'(x)}{2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{V'(x)}{2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{V(x)}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-\frac{V(x)}{2} = \frac{y^2}{2} + c_1$$

Summary

The solution(s) found are the following

$$-\frac{V(x)}{2} = \frac{y^2}{2} + c_1 \tag{1}$$

Verification of solutions

$$-\frac{V(x)}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

5.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(-2y) \, \mathrm{d}y &= (V'(x)) \, \mathrm{d}x \\
(-V'(x)) \, \mathrm{d}x + (-2y) \, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -V'(x) \\
N(x, y) &= -2y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-V'(x)) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2y) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int -V'(x) \, dx$$

$$\phi = -V(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -2y$. Therefore equation (4) becomes

$$-2y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (-2y) \, dy$$

$$f(y) = -y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -V(x) - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -V(x) - y^2$$

Summary
The solution(s) found are the following

$$-V(x) - y^2 = c_1 \tag{1}$$

Verification of solutions

$$-V(x) - y^2 = c_1$$

Verified OK.

5.6.4 Maple step by step solution

Let's solve
$2yy' = -V'(x)$

- Highest derivative means the order of the ODE is 1
  y'
- Integrate both sides with respect to x
  $\int 2yy' dx = \int -V'(x)\, dx + c_1$
- Evaluate integral
  $y^2 = -V(x) + c_1$
- Solve for y
  $\left\{y = \sqrt{-V(x) + c_1}, y = -\sqrt{-V(x) + c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve(diff(V(x),x)+2*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{-V(x) + c_1}$$
$$y(x) = -\sqrt{-V(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 37

```
DSolve[V'[x]+2*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-V(x) + 2c_1}$$
$$y(x) \to \sqrt{-V(x) + 2c_1}$$

5.7 problem 10.3 (ii)

Internal problem ID [12010]
Internal file name [`OUTPUT/10662_Saturday_September_02_2023_02_57_49_PM_66304589/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 10, Two tricks for nonlinear equations. Exercises page 97
Problem number: 10.3 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\left(\frac{1}{y}-a\right)y' = -\frac{2}{x}+b}$$

5.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(bx-2)}{(ay-1)\,x} \end{aligned}$$

Where $f(x) = -\frac{bx-2}{x}$ and $g(y) = \frac{y}{ay-1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{ay-1}}\,dy = -\frac{bx-2}{x}\,dx$$

$$
\int \frac{1}{\frac{y}{ay-1}}dy = \int -\frac{bx-2}{x}dx
$$
$$
ay - \ln(y) = -bx + 2\ln(x) + c_1
$$

Which results in

$$
y = -\frac{\mathrm{LambertW}\left(-\frac{a\,\mathrm{e}^{bx-c_1}}{x^2}\right)}{a}
$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. Which simplifies to

$$
y = -\frac{\mathrm{LambertW}\left(-\frac{a\,\mathrm{e}^{bx-c_1}}{x^2}\right)}{a}
$$

gives

$$
y = -\frac{\mathrm{LambertW}\left(-\frac{a\,\mathrm{e}^{bx}}{c_1x^2}\right)}{a}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\mathrm{LambertW}\left(-\frac{a\,\mathrm{e}^{bx}}{c_1x^2}\right)}{a} \tag{1}
$$

Verification of solutions

$$
y = -\frac{\mathrm{LambertW}\left(-\frac{a\,\mathrm{e}^{bx}}{c_1x^2}\right)}{a}
$$

Verified OK.

5.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
y' = -\frac{y(bx-2)}{(ay-1)\,x}
$$
$$
y' = \omega(x,y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 77: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{x}{bx - 2} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x}{bx-2}} dx \end{aligned}$$

Which results in

$$S = -bx + 2\ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(bx-2)}{(ay-1)\,x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{-bx+2}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{ay-1}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{Ra-1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = Ra - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-bx + 2\ln(x) = ay - \ln(y) + c_1$$

Which simplifies to

$$-bx + 2\ln(x) = ay - \ln(y) + c_1$$

Which gives

$$y = -\frac{\operatorname{LambertW}\left(-\frac{a\,e^{bx+c_1}}{x^2}\right)}{a}$$

Summary

The solution(s) found are the following

$$y = -\frac{\operatorname{LambertW}\left(-\frac{a\,e^{bx+c_1}}{x^2}\right)}{a} \tag{1}$$

Verification of solutions

$$y = -\frac{\operatorname{LambertW}\left(-\frac{a\,e^{bx+c_1}}{x^2}\right)}{a}$$

Verified OK.

5.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{ay-1}{y}\right)\mathrm{d}y&=\left(\frac{bx-2}{x}\right)\mathrm{d}x\\ \left(-\frac{bx-2}{x}\right)\mathrm{d}x+\left(-\frac{ay-1}{y}\right)\mathrm{d}y&=0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{bx-2}{x}\\ N(x,y)&=-\frac{ay-1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{bx-2}{x}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{ay-1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{bx-2}{x}\,\mathrm{d}x\end{aligned}$$

$$\phi = -bx + 2\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{ay-1}{y}$. Therefore equation (4) becomes

$$-\frac{ay-1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{ay-1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{-ay+1}{y}\right)\mathrm{d}y\\ f(y) &= -ay + \ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -bx + 2\ln(x) - ay + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -bx + 2\ln(x) - ay + \ln(y)$$

The solution becomes

$$y = -\frac{\text{LambertW}\left(-\frac{a\,e^{bx+c_1}}{x^2}\right)}{a}$$

Summary

The solution(s) found are the following

$$y = -\frac{\text{LambertW}\left(-\frac{a\,e^{bx+c_1}}{x^2}\right)}{a} \tag{1}$$

Verification of solutions

$$y = -\frac{\text{LambertW}\left(-\frac{a\,e^{bx+c_1}}{x^2}\right)}{a}$$

Verified OK.

5.7.4 Maple step by step solution

Let's solve

$\left(\frac{1}{y} - a\right) y' = -\frac{2}{x} + b$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(\frac{1}{y} - a\right) y' dx = \int \left(-\frac{2}{x} + b\right) dx + c_1$

- Evaluate integral

  $-ay + \ln(y) = bx - 2\ln(x) + c_1$

- Solve for y

$$y = \frac{e^{-LambertW\left(-\frac{a\,e^{bx+c_1}}{x^2}\right)+bx+c_1}}{x^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve((1/y(x)-a)*diff(y(x),x)+2/x-b=0,y(x), singsol=all)
```

$$y(x) = -\frac{\text{LambertW}\left(-\frac{a\,e^{bx}c_1}{x^2}\right)}{a}$$

✓ Solution by Mathematica

Time used: 6.296 (sec). Leaf size: 32

```
DSolve[(1/y[x]-a)*y'[x]+2/x-b==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{W\left(-\frac{ae^{bx-c_1}}{x^2}\right)}{a}$$
$$y(x) \to 0$$

5.8 problem 10.4 (i)

Internal problem ID [12011]
Internal file name [`OUTPUT/10663_Saturday_September_02_2023_02_57_50_PM_54190541/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 10, Two tricks for nonlinear equations. Exercises page 97
Problem number: 10.4 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{yx + y^2 - x^2y' = -x^2}$$

5.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x^2 + u(x)^2\,x^2 - x^2(u'(x)\,x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2+1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 + 1$. Integrating both sides gives

$$\frac{1}{u^2+1}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{u^2+1}\,du = \int \frac{1}{x}\,dx$$
$$\arctan(u) = \ln(x) + c_2$$

The solution is

$$\arctan(u(x)) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$
$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Summary
The solution(s) found are the following

$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

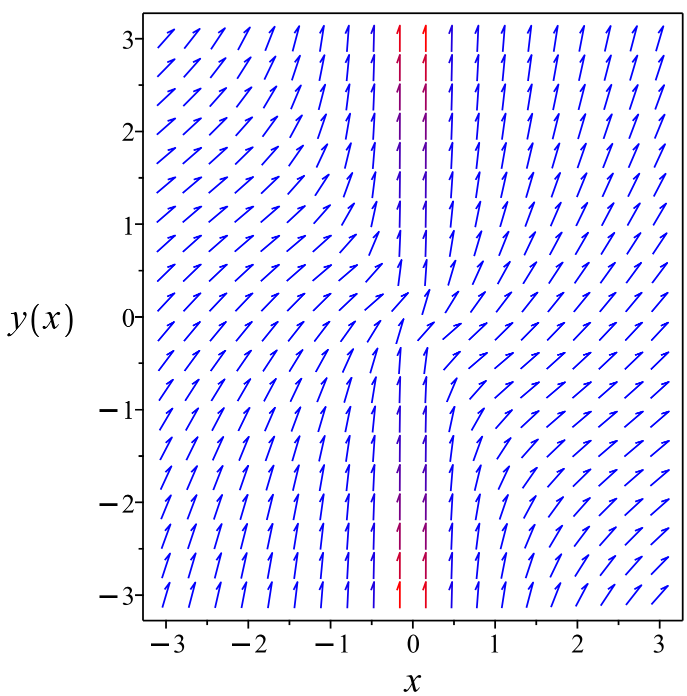


Figure 58: Slope field plot

Verification of solutions

$$\arctan\left(\frac{y}{x}\right)-\ln\left(x\right)-c_2=0$$

Verified OK.

5.8.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x^2+xy+y^2}{x^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\frac{(x^2+xy+y^2)(b_3-a_2)}{x^2}-\frac{(x^2+xy+y^2)^2 a_3}{x^4}-\left(\frac{2x+y}{x^2}-\frac{2(x^2+xy+y^2)}{x^3}\right)(xa_2+ya_3+a_1)-\frac{(x+2y)(xb_2+yb_3+b_1)}{x^2}=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4a_2+x^4a_3-x^4b_3+2x^3ya_3+2x^3yb_2-x^2y^2a_2+2x^2y^2a_3+x^2y^2b_3+y^4a_3+x^3b_1-x^2ya_1+2x^2yb_1-}{x^4}$$
$$=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
-x^{4} a_{2}-x^{4} a_{3}+x^{4} b_{3}-2 x^{3} y a_{3}-2 x^{3} y b_{2}+x^{2} y^{2} a_{2}-2 x^{2} y^{2} a_{3} \\
-x^{2} y^{2} b_{3}-y^{4} a_{3}-x^{3} b_{1}+x^{2} y a_{1}-2 x^{2} y b_{1}+2 x y^{2} a_{1}=0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x=v_{1}, y=v_{2}\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
-a_{2} v_{1}^{4}+a_{2} v_{1}^{2} v_{2}^{2}-a_{3} v_{1}^{4}-2 a_{3} v_{1}^{3} v_{2}-2 a_{3} v_{1}^{2} v_{2}^{2}-a_{3} v_{2}^{4}-2 b_{2} v_{1}^{3} v_{2} \\
+b_{3} v_{1}^{4}-b_{3} v_{1}^{2} v_{2}^{2}+a_{1} v_{1}^{2} v_{2}+2 a_{1} v_{1} v_{2}^{2}-b_{1} v_{1}^{3}-2 b_{1} v_{1}^{2} v_{2}=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_{1}, v_{2}\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
\left(-a_{2}-a_{3}+b_{3}\right) v_{1}^{4}+\left(-2 a_{3}-2 b_{2}\right) v_{1}^{3} v_{2}-b_{1} v_{1}^{3} \\
+\left(a_{2}-2 a_{3}-b_{3}\right) v_{1}^{2} v_{2}^{2}+\left(a_{1}-2 b_{1}\right) v_{1}^{2} v_{2}+2 a_{1} v_{1} v_{2}^{2}-a_{3} v_{2}^{4}=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2 a_{1} &=0 \\
-a_{3} &=0 \\
-b_{1} &=0 \\
a_{1}-2 b_{1} &=0 \\
-2 a_{3}-2 b_{2} &=0 \\
-a_{2}-a_{3}+b_{3} &=0 \\
a_{2}-2 a_{3}-b_{3} &=0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{x^2 + xy + y^2}{x^2}\right)(x) \\
&= \frac{-x^2 - y^2}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2 - y^2}{x}} dy
\end{aligned}
$$

Which results in

$$S = -\arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 + xy + y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x^2 + y^2} \\ S_y &= -\frac{x}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\arctan\left(\frac{y}{x}\right) = -\ln(x) + c_1$$

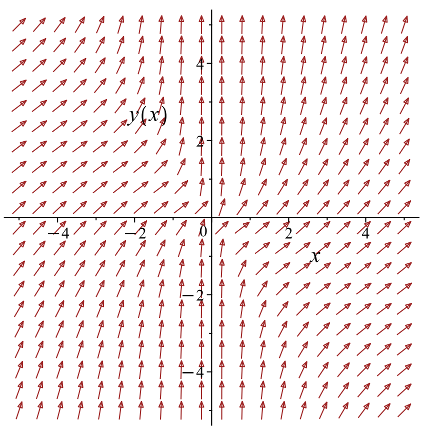
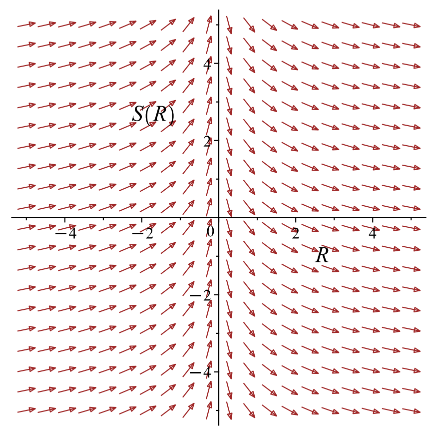
Which simplifies to

$$-\arctan\left(\frac{y}{x}\right) = -\ln\left(x\right) + c_1$$

Which gives

$$y = -\tan\left(-\ln\left(x\right) + c_1\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$

</td><td>$R = x$
$S = -\arctan\left(\frac{y}{x}\right)$</td><td>$\frac{dS}{dR} = -\frac{1}{R}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\tan\left(-\ln\left(x\right) + c_1\right) x \tag{1}$$

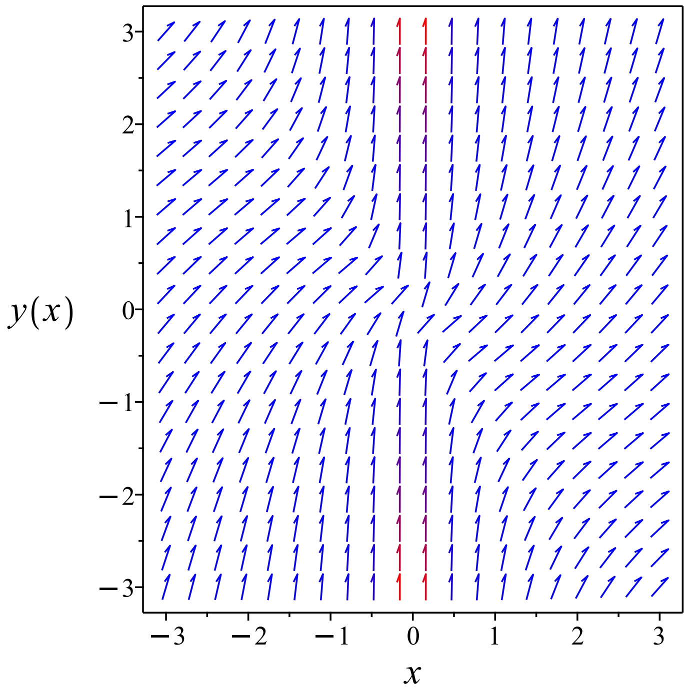


Figure 59: Slope field plot

Verification of solutions

$$y = -\tan(-\ln(x) + c_1) x$$

Verified OK.

5.8.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + xy + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 1 + \frac{y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 1$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= \frac{1}{x^3} \\ f_2^2 f_0 &= \frac{1}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{u'(x)}{x^3} + \frac{u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sin(\ln(x))\, c_1 + c_2 \cos(\ln(x))$$

The above shows that

$$u'(x) = \frac{\cos(\ln(x))\, c_1 - c_2 \sin(\ln(x))}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{x(\cos(\ln(x))\, c_1 - c_2 \sin(\ln(x)))}{\sin(\ln(x))\, c_1 + c_2 \cos(\ln(x))}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-\cos(\ln(x))\, c_3 + \sin(\ln(x)))\, x}{\sin(\ln(x))\, c_3 + \cos(\ln(x))}$$

Summary

The solution(s) found are the following

$$y = \frac{(-\cos(\ln(x))\, c_3 + \sin(\ln(x)))\, x}{\sin(\ln(x))\, c_3 + \cos(\ln(x))} \tag{1}$$

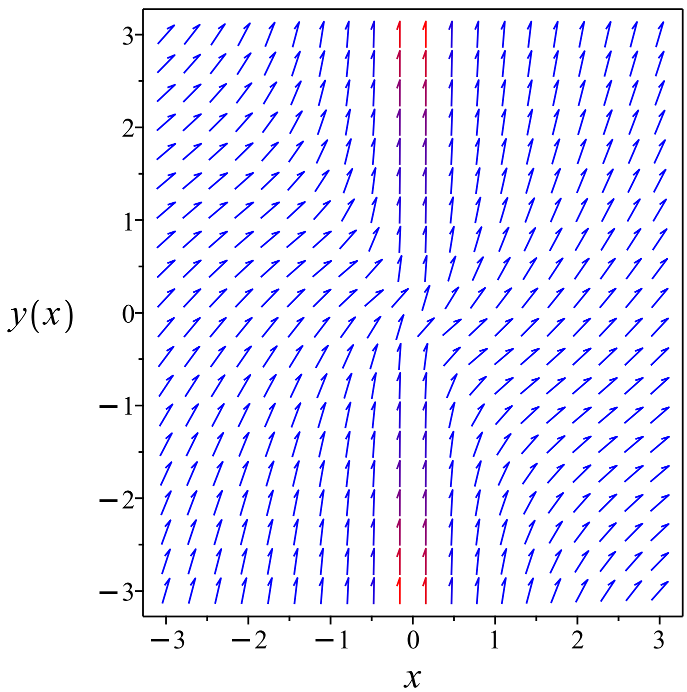


Figure 60: Slope field plot

Verification of solutions

$$y = \frac{(-\cos(\ln(x))\, c_3 + \sin(\ln(x)))\, x}{\sin(\ln(x))\, c_3 + \cos(\ln(x))}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(x*y(x)+y(x)^2+x^2-x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan(\ln(x) + c_1) x$$

✓ Solution by Mathematica

Time used: 0.314 (sec). Leaf size: 13

```
DSolve[x*y[x]+y[x]^2+x^2-x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan(\log(x) + c_1)$$

5.9 problem 10.4 (ii)

Internal problem ID [12012]
Internal file name [`OUTPUT/10664_Saturday_September_02_2023_02_57_51_PM_92539192/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 10, Two tricks for nonlinear equations. Exercises page 97
Problem number: 10.4 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{x' - \frac{x^2 + t\sqrt{x^2+t^2}}{xt} = 0}$$

5.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} x' &= \frac{x^2 + t\sqrt{t^2+x^2}}{xt} \\ x' &= \omega(t,x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ta_2 + xa_3 + a_1 \tag{1E}$$

$$\eta = tb_2 + xb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(x^2+t\sqrt{t^2+x^2}\right)(b_3-a_2)}{xt} - \frac{\left(x^2+t\sqrt{t^2+x^2}\right)^2 a_3}{x^2t^2} - \left(\frac{\sqrt{t^2+x^2}+\frac{t^2}{\sqrt{t^2+x^2}}}{xt} - \frac{x^2+t\sqrt{t^2+x^2}}{x\,t^2}\right)(ta_2+xa_3+a_1) - \left(\frac{2x+\frac{tx}{\sqrt{t^2+x^2}}}{xt} - \frac{x^2+t\sqrt{t^2+x^2}}{x^2t}\right)(tb_2+xb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\left(t^2+x^2\right)^{\frac{3}{2}} t^2 a_3 - t^5 b_2 + 2t^4 x a_2 - 2t^4 x b_3 + 3t^3x^2a_3 + t^2x^3a_2 - t^2x^3b_3 + 2t\,x^4a_3 + \sqrt{t^2+x^2}\,t\,x^2b_1 - \sqrt{t^2}}{\sqrt{t^2+x^2}\,x^2t^2} = 0$$

Setting the numerator to zero gives

$$-\left(t^2+x^2\right)^{\frac{3}{2}} t^2a_3 + t^5b_2 - 2t^4xa_2 + 2t^4xb_3 - 3t^3x^2a_3 - t^2x^3a_2 + t^2x^3b_3 - 2t\,x^4a_3 - \sqrt{t^2+x^2}\,t\,x^2b_1 + \sqrt{t^2+x^2}\,x^3a_1 + t^4b_1 - t^3xa_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-\left(t^2+x^2\right)^{\frac{3}{2}} t^2a_3 + \left(t^2+x^2\right)t^3b_2 - \left(t^2+x^2\right)t^2xa_2 + 2\left(t^2+x^2\right)t^2xb_3 - 2\left(t^2+x^2\right)t\,x^2a_3 - t^4xa_2 - t^3x^2a_3 - t^3x^2b_2 - t^2x^3b_3 + \left(t^2+x^2\right)t^2b_1 - \sqrt{t^2+x^2}\,t\,x^2b_1 + \sqrt{t^2+x^2}\,x^3a_1 - t^3xa_1 - t^2x^2b_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$t^5b_2 - 2t^4xa_2 + 2t^4xb_3 - t^4\sqrt{t^2+x^2}\,a_3 - 3t^3x^2a_3 - t^2x^3a_2 + t^2x^3b_3 - t^2x^2\sqrt{t^2+x^2}\,a_3 - 2t\,x^4a_3 + t^4b_1 - t^3xa_1 - \sqrt{t^2+x^2}\,t\,x^2b_1 + \sqrt{t^2+x^2}\,x^3a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\left\{t, x, \sqrt{t^2+x^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\left\{t = v_1, x = v_2, \sqrt{t^2+x^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$-2v_1^4v_2a_2 - v_1^2v_2^3a_2 - v_1^4v_3a_3 - 3v_1^3v_2^2a_3 - v_1^2v_2^2v_3a_3 - 2v_1v_2^4a_3 + v_1^5b_2 \\ + 2v_1^4v_2b_3 + v_1^2v_2^3b_3 - v_1^3v_2a_1 + v_3v_2^3a_1 + v_1^4b_1 - v_3v_1v_2^2b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$v_1^5b_2 + (-2a_2+2b_3)\,v_1^4v_2 - v_1^4v_3a_3 + v_1^4b_1 - 3v_1^3v_2^2a_3 - v_1^3v_2a_1 \\ + (b_3-a_2)\,v_1^2v_2^3 - v_1^2v_2^2v_3a_3 - 2v_1v_2^4a_3 - v_3v_1v_2^2b_1 + v_3v_2^3a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ -a_1 &= 0 \\ -3a_3 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2a_2+2b_3 &= 0 \\ b_3-a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= t \\
\eta &= x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(t,x)\,\xi \\
&= x - \left(\frac{x^2 + t\sqrt{t^2+x^2}}{xt}\right)(t) \\
&= -\frac{t\sqrt{t^2+x^2}}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{t\sqrt{t^2+x^2}}{x}} dy
\end{aligned}
$$

Which results in

$$S = -\frac{\sqrt{t^2 + x^2}}{t}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = \frac{x^2 + t\sqrt{t^2 + x^2}}{xt}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= \frac{x^2}{t^2 \sqrt{t^2 + x^2}} \\ S_x &= -\frac{x}{t\sqrt{t^2 + x^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

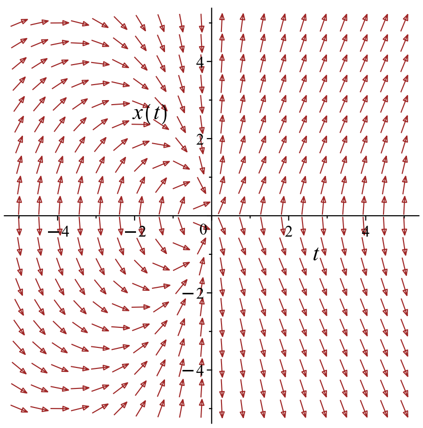
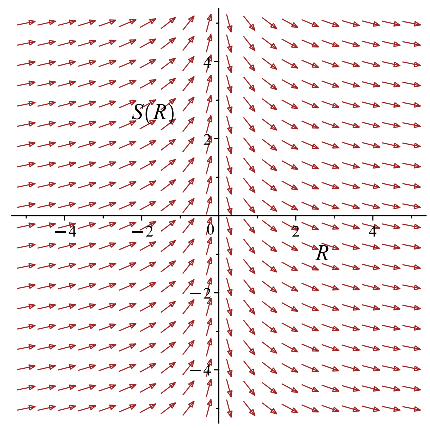
To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$-\frac{\sqrt{x^2 + t^2}}{t} = -\ln(t) + c_1$$

Which simplifies to

$$-\frac{\sqrt{x^2+t^2}}{t} = -\ln(t) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{x^2+t\sqrt{t^2+x^2}}{xt}$ 	$R = t$ $S = -\frac{\sqrt{t^2+x^2}}{t}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$-\frac{\sqrt{x^2+t^2}}{t} = -\ln(t) + c_1 \tag{1}$$

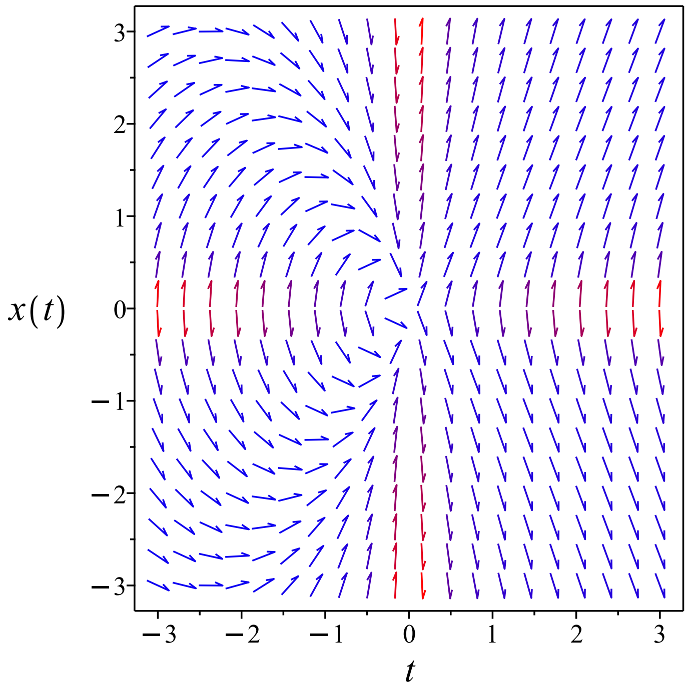


Figure 61: Slope field plot

Verification of solutions

$$-\frac{\sqrt{x^2+t^2}}{t} = -\ln(t) + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(x(t),t)=(x(t)^2+t*sqrt(t^2+x(t)^2))/(t*x(t)),x(t), singsol=all)
```

$$\frac{t \ln \left(t \right)-c_{1} t -\sqrt{t^{2}+x \left(t \right)^{2}}}{t} = 0$$

✓ Solution by Mathematica

Time used: 0.512 (sec). Leaf size: 54

```
DSolve[x'[t]==(x[t]^2+t*Sqrt[t^2+x[t]^2])/(t*x[t]),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -t\sqrt{\log^2(t) + 2c_1 \log(t) - 1 + c_1{}^2}$$
$$x(t) \to t\sqrt{\log^2(t) + 2c_1 \log(t) - 1 + c_1{}^2}$$

5.10 problem 10.5

Internal problem ID [12013]
Internal file name [`OUTPUT/10665_Saturday_September_02_2023_02_57_53_PM_72234192/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 10, Two tricks for nonlinear equations. Exercises page 97
Problem number: 10.5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' - kx + x^2 = 0}$$

5.10.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{kx - x^2} dx = \int dt$$
$$-\frac{\ln(-k + x)}{k} + \frac{\ln(x)}{k} = t + c_1$$

The above can be written as

$$\left(-\frac{1}{k}\right)(\ln(-k + x) - \ln(x)) = t + c_1$$
$$\ln(-k + x) - \ln(x) = (-k)(t + c_1)$$
$$= -k(t + c_1)$$

Raising both side to exponential gives

$$e^{\ln(-k+x)-\ln(x)} = -kc_1 e^{-kt}$$

Which simplifies to

$$\frac{-k+x}{x} = c_2 \mathrm{e}^{-kt}$$

Summary

The solution(s) found are the following

$$x = -\frac{k}{-1+c_2 \mathrm{e}^{-kt}} \tag{1}$$

Verification of solutions

$$x = -\frac{k}{-1+c_2 \mathrm{e}^{-kt}}$$

Verified OK.

5.10.2 Maple step by step solution

Let's solve

$x' - kx + x^2 = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{kx-x^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{kx-x^2} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{\ln(x-k)}{k} + \frac{\ln(x)}{k} = t + c_1$

- Solve for x

  $x = \frac{k\,\mathrm{e}^{c_1 k+kt}}{-1+\mathrm{e}^{c_1 k+kt}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(x(t),t)=k*x(t)-x(t)^2,x(t), singsol=all)
```

$$x(t) = \frac{k}{1 + \mathrm{e}^{-kt} c_1 k}$$

✓ Solution by Mathematica

Time used: 0.963 (sec). Leaf size: 37

```
DSolve[x'[t]==k*x[t]-x[t]^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{k e^{k(t+c_1)}}{-1 + e^{k(t+c_1)}}$$
$$x(t) \to 0$$
$$x(t) \to k$$

6 Chapter 12, Homogeneous second order linear equations. Exercises page 118

6.1 problem 12.1 (i)

Internal problem ID [12014]
Internal file name [`OUTPUT/10666_Saturday_September_02_2023_02_57_54_PM_18688578/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - 3x' + 2x = 0}$$

With initial conditions

$$[x(0) = 2, x'(0) = 6]$$

6.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - 3x' + 2x = 0$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.1.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 3\lambda \, e^{\lambda t} + 2 \, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)} \\
&= \frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$

$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{(2)t} + c_2 e^{(1)t}$$

Or

$$x = c_1 e^{2t} + c_2 e^{t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{2t} + c_2 e^{t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 2c_1 e^{2t} + c_2 e^{t}$$

substituting $x' = 6$ and $t = 0$ in the above gives

$$6 = 2c_1 + c_2 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

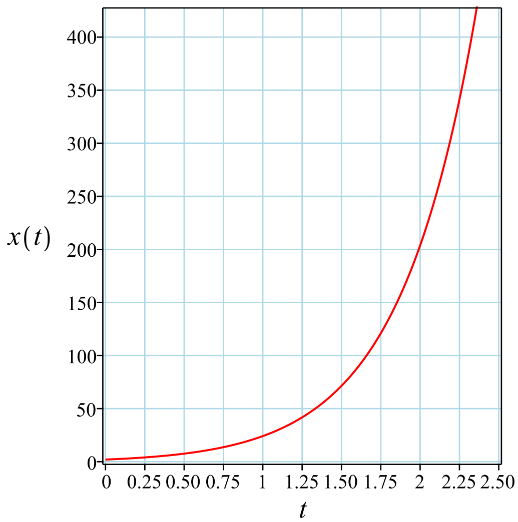
$$c_1 = 4$$
$$c_2 = -2$$

Substituting these values back in above solution results in

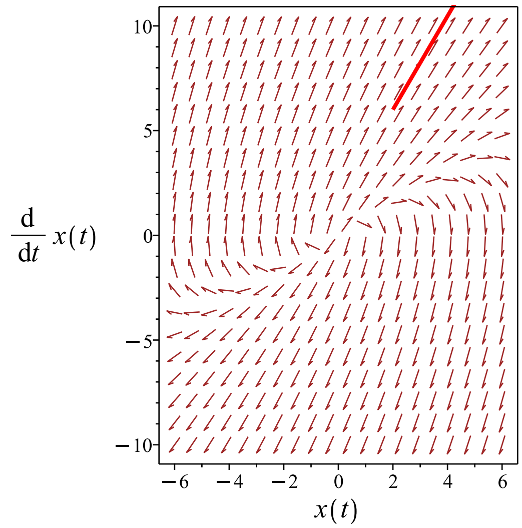
$$x = 4\,e^{2t} - 2\,e^{t}$$

Summary
The solution(s) found are the following

$$x = 4\,e^{2t} - 2\,e^{t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 4\,e^{2t} - 2\,e^{t}$$

Verified OK.

6.1.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 3x' + 2x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 81: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-\frac{t}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\, dt} \\
&= z_1 e^{\frac{3t}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-3}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{3t}}{(x_1)^2}\, dt \\
&= x_1 \left(\mathrm{e}^{t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 (e^t) + c_2 (e^t (e^t))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 e^t + c_2 e^{2t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 0$ in the above gives

$$
2 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = c_1 e^t + 2c_2 e^{2t}
$$

substituting $x' = 6$ and $t = 0$ in the above gives

$$
6 = 2c_2 + c_1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -2 \\
c_2 &= 4
\end{aligned}
$$

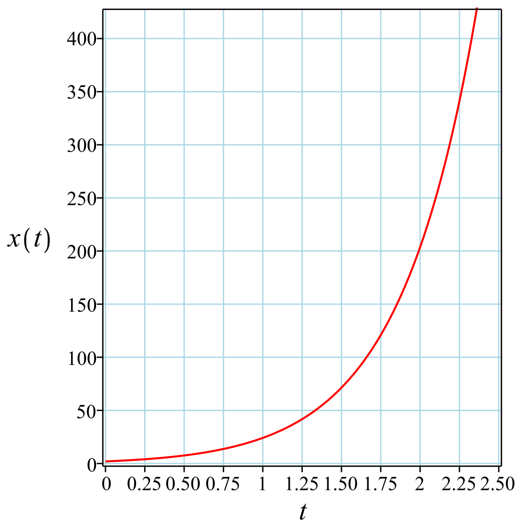
Substituting these values back in above solution results in

$$
x = 4 e^{2t} - 2 e^t
$$

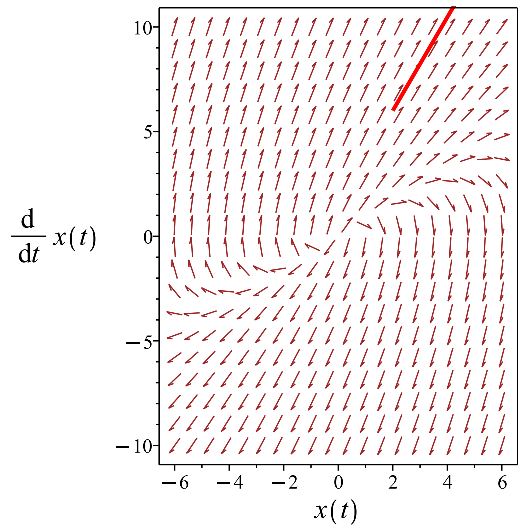
Summary

The solution(s) found are the following

$$
x = 4 e^{2t} - 2 e^t \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = 4\,e^{2t} - 2\,e^{t}$$

Verified OK.

6.1.4 Maple step by step solution

Let's solve

$$\left[x'' - 3x' + 2x = 0, x(0) = 2, x'\Big|_{\{t=0\}} = 6\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r - 1)\,(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the ODE

$x_1(t) = e^t$

- 2nd solution of the ODE

  $x_2(t) = e^{2t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 e^t + c_2 e^{2t}$

□ Check validity of solution $x = c_1 e^t + c_2 e^{2t}$

  - Use initial condition $x(0) = 2$

    $2 = c_1 + c_2$

  - Compute derivative of the solution

    $x' = c_1 e^t + 2c_2 e^{2t}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 6$

    $6 = 2c_2 + c_1$

  - Solve for c_1 and c_2

    $\{c_1 = -2, c_2 = 4\}$

  - Substitute constant values into general solution and simplify

    $x = 4\,e^{2t} - 2\,e^t$

- Solution to the IVP

  $x = 4\,e^{2t} - 2\,e^t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([diff(x(t),t$2)-3*diff(x(t),t)+2*x(t)=0,x(0) = 2, D(x)(0) = 6],x(t), singsol=all)
```

$$x(t) = -2\,e^{t} + 4\,e^{2t}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 17

```
DSolve[{x''[t]-3*x'[t]+2*x[t]==0,{x[0]==2,x'[0]==6}},x[t],t,IncludeSingularSolutions -> True
```

$$x(t) \to 2e^{t}(2e^{t} - 1)$$

6.2 problem 12.1 (ii)

Internal problem ID [12015]
Internal file name [`OUTPUT/10667_Saturday_September_02_2023_02_57_56_PM_3874535/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (ii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 4y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 3]$$

6.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 4y = 0$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.2.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\, \mathrm{e}^{\lambda x} + 4\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-4)^2 - (4)\,(1)\,(4)} \\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} x \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{2x} x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} + 2c_2 e^{2x} x + c_2 e^{2x}$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 3$$

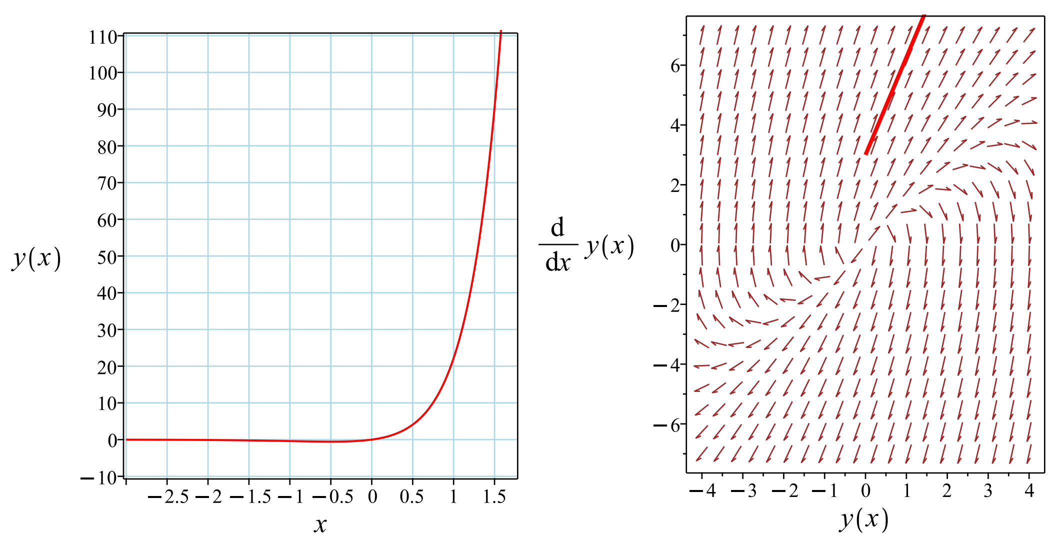
Substituting these values back in above solution results in

$$y = 3\,e^{2x} x$$

Summary

The solution(s) found are the following

$$y = 3\,e^{2x} x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{2x} x$$

Verified OK.

6.2.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^{2} + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -4\,dx} \\ &= \mathrm{e}^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{-2x} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-2x} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-2x} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-2x}}$$

Or

$$y = \mathrm{e}^{2x} c_1 x + c_2 \mathrm{e}^{2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{2x} c_1 x + c_2 \mathrm{e}^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{2x} + 2 e^{2x} c_1 x + 2 c_2 e^{2x}$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = 2c_2 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 0$$

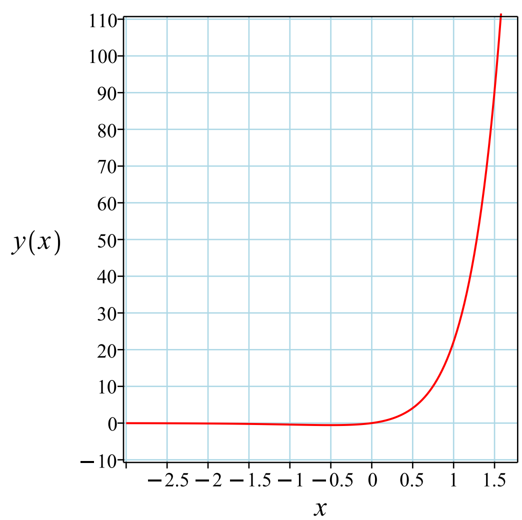
Substituting these values back in above solution results in

$$y = 3 e^{2x} x$$

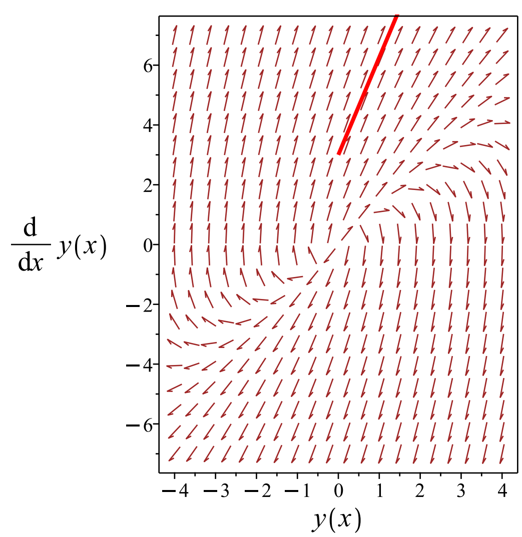
Summary

The solution(s) found are the following

$$y = 3 e^{2x} x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{2x}x$$

Verified OK.

6.2.4 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align*} y'' - 4y' + 4y &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 1 \\ B &= -4 \tag{3} \\ C &= 4 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{align*} s &= 0 \\ t &= 1 \end{align*}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 83: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} x \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} + 2c_2 e^{2x} x + c_2 e^{2x}$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 3$$

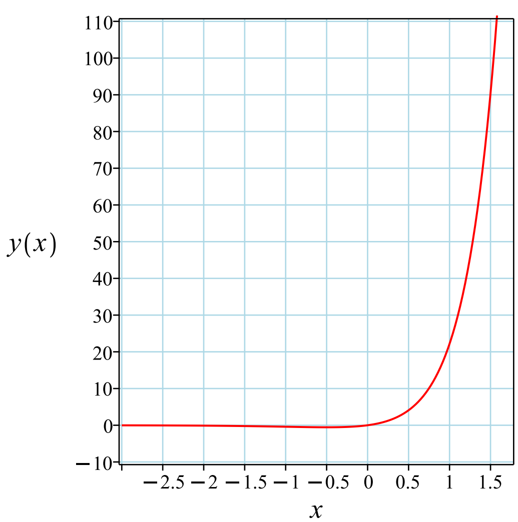
Substituting these values back in above solution results in

$$y = 3\,e^{2x} x$$

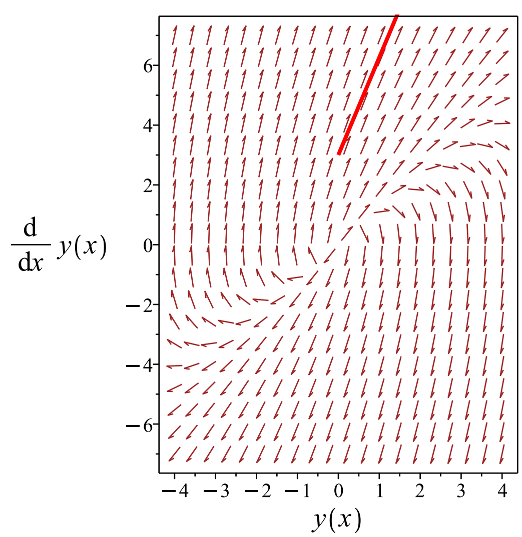
Summary

The solution(s) found are the following

$$y = 3\,e^{2x} x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\,e^{2x} x$$

Verified OK.

6.2.5 Maple step by step solution

Let's solve

$\left[y''-4y'+4y=0, y(0)=0, y'\Big|_{\{x=0\}}=3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2-4r+4=0$

- Factor the characteristic polynomial

  $(r-2)^2=0$

- Root of the characteristic polynomial

  $r=2$

- 1st solution of the ODE

  $y_1(x)=\mathrm{e}^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x)=\mathrm{e}^{2x}x$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{2x}+c_2\mathrm{e}^{2x}x$

□ Check validity of solution $y=c_1\mathrm{e}^{2x}+c_2\mathrm{e}^{2x}x$

  - Use initial condition $y(0)=0$

    $0=c_1$

  - Compute derivative of the solution

    $y'=2c_1\mathrm{e}^{2x}+2c_2\mathrm{e}^{2x}x+c_2\mathrm{e}^{2x}$

  - Use the initial condition $y'\Big|_{\{x=0\}}=3$

    $3=2c_1+c_2$

  - Solve for c_1 and c_2

    $\{c_1=0, c_2=3\}$

- Substitute constant values into general solution and simplify

  $$y = 3\,\mathrm{e}^{2x}x$$

- Solution to the IVP

  $$y = 3\,\mathrm{e}^{2x}x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=0,y(0) = 0, D(y)(0) = 3],y(x), singsol=all)
```

$$y(x) = 3x\,\mathrm{e}^{2x}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 13

```
DSolve[{y''[x]-4*y'[x]+4*y[x]==0,{y[0]==0,y'[0]==3}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to 3e^{2x}x$$

6.3 problem 12.1 (iii)

Internal problem ID [12016]
Internal file name [`OUTPUT/10668_Saturday_September_02_2023_02_57_58_PM_17663998/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (iii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{z'' - 4z' + 13z = 0}$$

With initial conditions

$$[z(0) = 7, z'(0) = 42]$$

6.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$z'' + p(t)z' + q(t)z = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 13 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$z'' - 4z' + 13z = 0$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 13$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.3.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Az''(t) + Bz'(t) + Cz(t) = 0$$

Where in the above $A = 1, B = -4, C = 13$. Let the solution be $z = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 4\lambda\, e^{\lambda t} + 13\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 4\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-4^2 - (4)(1)(13)}\\ &= 2 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + 3i\\ \lambda_2 &= 2 - 3i\end{aligned}$$

Which simplifies to

$$\lambda_1 = 2 + 3i$$
$$\lambda_2 = 2 - 3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$z = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$z = e^{2t}(c_1 \cos(3t) + c_2 \sin(3t))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$z = e^{2t}(c_1 \cos(3t) + c_2 \sin(3t)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 7$ and $t = 0$ in the above gives

$$7 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$z' = 2\,e^{2t}(c_1 \cos(3t) + c_2 \sin(3t)) + e^{2t}(-3c_1 \sin(3t) + 3c_2 \cos(3t))$$

substituting $z' = 42$ and $t = 0$ in the above gives

$$42 = 2c_1 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

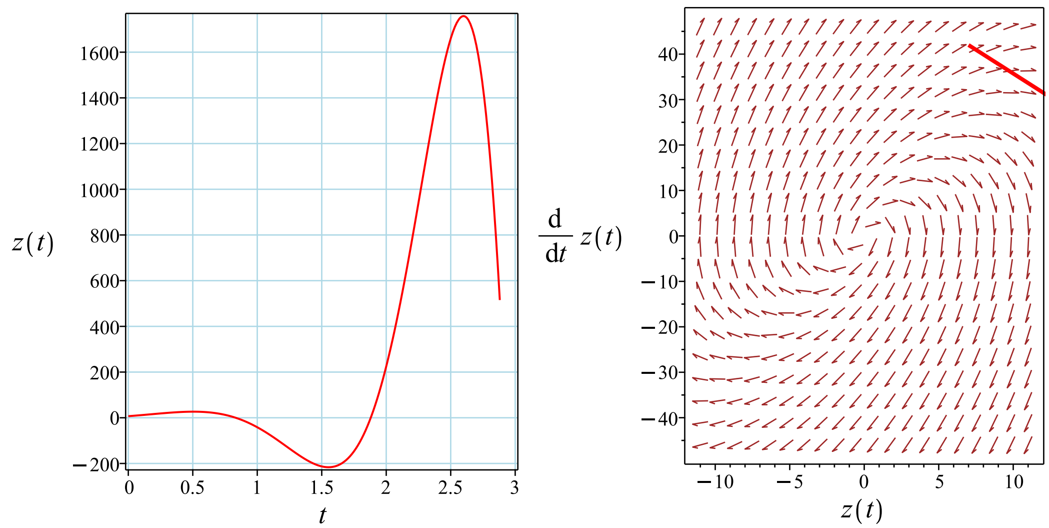
$$c_1 = 7$$
$$c_2 = \frac{28}{3}$$

Substituting these values back in above solution results in

$$z = \frac{7\,e^{2t}(3\cos(3t) + 4\sin(3t))}{3}$$

<u>Summary</u>
The solution(s) found are the following

$$z = \frac{7\,e^{2t}(3\cos(3t) + 4\sin(3t))}{3} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$z = \frac{7\,e^{2t}(3\cos(3t) + 4\sin(3t))}{3}$$

Verified OK.

6.3.3 Solving using Kovacic algorithm

Writing the ode as

$$z'' - 4z' + 13z = 0 \tag{1}$$
$$Az'' + Bz' + Cz = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -4 \\
C &= 13
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = ze^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-9}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -9 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = -9z \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then z is found using the inverse transformation

$$
z = ze^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 85: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in z is found from

$$\begin{aligned}z_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dt}\\ &= z_1 e^{2t}\\ &= z_1\left(\mathrm{e}^{2t}\right)\end{aligned}$$

Which simplifies to

$$z_1 = \mathrm{e}^{2t} \cos(3t)$$

The second solution z_2 to the original ode is found using reduction of order

$$z_2 = z_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{z_1^2}\, dt$$

Substituting gives

$$\begin{aligned} z_2 &= z_1 \int \frac{e^{\int -\frac{-4}{1}\, dt}}{(z_1)^2}\, dt \\ &= z_1 \int \frac{e^{4t}}{(z_1)^2}\, dt \\ &= z_1 \left(\frac{\tan(3t)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} z &= c_1 z_1 + c_2 z_2 \\ &= c_1 \left(\mathrm{e}^{2t} \cos(3t)\right) + c_2 \left(\mathrm{e}^{2t} \cos(3t) \left(\frac{\tan(3t)}{3}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$z = c_1 \mathrm{e}^{2t} \cos(3t) + \frac{c_2 \mathrm{e}^{2t} \sin(3t)}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 7$ and $t = 0$ in the above gives

$$7 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$z' = 2c_1 \mathrm{e}^{2t} \cos(3t) - 3c_1 \mathrm{e}^{2t} \sin(3t) + \frac{2c_2 \mathrm{e}^{2t} \sin(3t)}{3} + c_2 \mathrm{e}^{2t} \cos(3t)$$

substituting $z' = 42$ and $t = 0$ in the above gives

$$42 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 7$$
$$c_2 = 28$$

Substituting these values back in above solution results in

$$z = 7\,\mathrm{e}^{2t}\cos(3t) + \frac{28\,\mathrm{e}^{2t}\sin(3t)}{3}$$

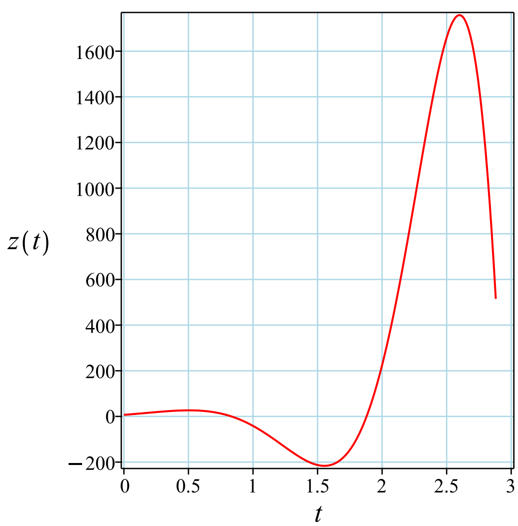
Which simplifies to

$$z = \frac{7\,\mathrm{e}^{2t}(3\cos(3t) + 4\sin(3t))}{3}$$

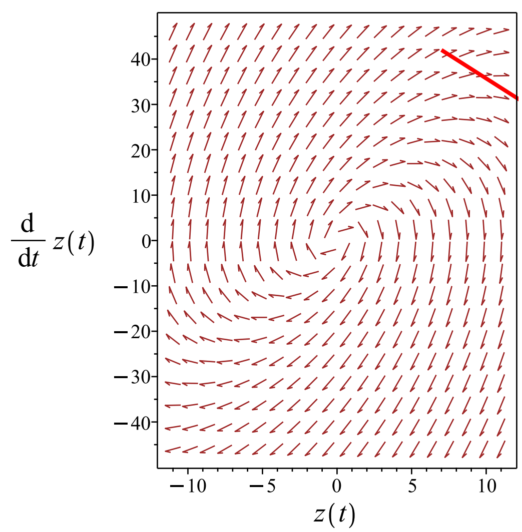
Summary

The solution(s) found are the following

$$z = \frac{7\,\mathrm{e}^{2t}(3\cos(3t) + 4\sin(3t))}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$z = \frac{7\,\mathrm{e}^{2t}(3\cos(3t) + 4\sin(3t))}{3}$$

Verified OK.

6.3.4 Maple step by step solution

Let's solve

$\left[z'' - 4z' + 13z = 0, z(0) = 7, z'\Big|_{\{t=0\}} = 42\right]$

- Highest derivative means the order of the ODE is 2

  z''

- Characteristic polynomial of ODE

  $r^2 - 4r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - 3\,\mathrm{I}, 2 + 3\,\mathrm{I})$

- 1st solution of the ODE

  $z_1(t) = \mathrm{e}^{2t} \cos(3t)$

- 2nd solution of the ODE

  $z_2(t) = \mathrm{e}^{2t} \sin(3t)$

- General solution of the ODE

  $z = c_1 z_1(t) + c_2 z_2(t)$

- Substitute in solutions

  $z = c_1 \mathrm{e}^{2t} \cos(3t) + c_2 \mathrm{e}^{2t} \sin(3t)$

☐ Check validity of solution $z = c_1 \mathrm{e}^{2t} \cos(3t) + c_2 \mathrm{e}^{2t} \sin(3t)$

- ○ Use initial condition $z(0) = 7$

  $7 = c_1$

- ○ Compute derivative of the solution

  $z' = 2c_1 \mathrm{e}^{2t} \cos(3t) - 3c_1 \mathrm{e}^{2t} \sin(3t) + 2c_2 \mathrm{e}^{2t} \sin(3t) + 3c_2 \mathrm{e}^{2t} \cos(3t)$

- ○ Use the initial condition $z'\Big|_{\{t=0\}} = 42$

  $42 = 2c_1 + 3c_2$

- ○ Solve for c_1 and c_2

  $\left\{c_1 = 7, c_2 = \frac{28}{3}\right\}$

- Substitute constant values into general solution and simplify

  $$z = \frac{7\,\mathrm{e}^{2t}(3\cos(3t)+4\sin(3t))}{3}$$

- Solution to the IVP

  $$z = \frac{7\,\mathrm{e}^{2t}(3\cos(3t)+4\sin(3t))}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(z(t),t$2)-4*diff(z(t),t)+13*z(t)=0,z(0) = 7, D(z)(0) = 42],z(t), singsol=all)
```

$$z(t) = \frac{7\,\mathrm{e}^{2t}(4\sin(3t)+3\cos(3t))}{3}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 27

```
DSolve[{z''[t]-4*z'[t]+13*z[t]==0,{z[0]==7,z'[0]==42}},z[t],t,IncludeSingularSolutions -> Tr
```

$$z(t) \to \frac{7}{3}e^{2t}(4\sin(3t)+3\cos(3t))$$

6.4 problem 12.1 (iv)

Internal problem ID [12017]
Internal file name [`OUTPUT/10669_Saturday_September_02_2023_02_58_00_PM_41522569/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (iv).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - 6y = 0}$$

With initial conditions

$$[y(0) = -1, y'(0) = 8]$$

6.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= -6 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y' - 6y = 0$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.4.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 1, C = -6$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 6 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-6)}\\ &= -\frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{5}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{5}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{(2)t} + c_2 e^{(-3)t}$$

Or

$$y = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-3t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-3t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $t = 0$ in the above gives

$$-1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 \mathrm{e}^{2t} - 3c_2 \mathrm{e}^{-3t}$$

substituting $y' = 8$ and $t = 0$ in the above gives

$$8 = 2c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

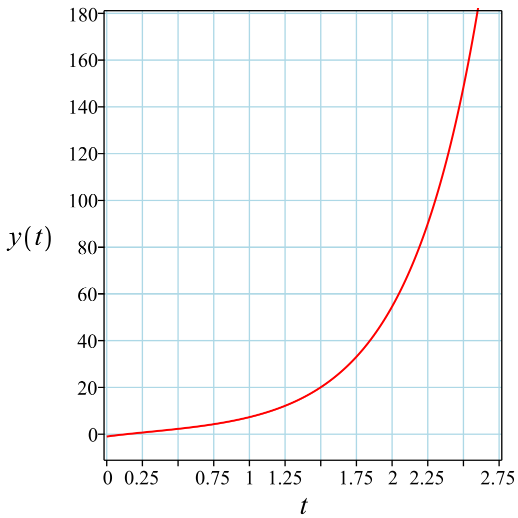
$$c_1 = 1$$
$$c_2 = -2$$

Substituting these values back in above solution results in

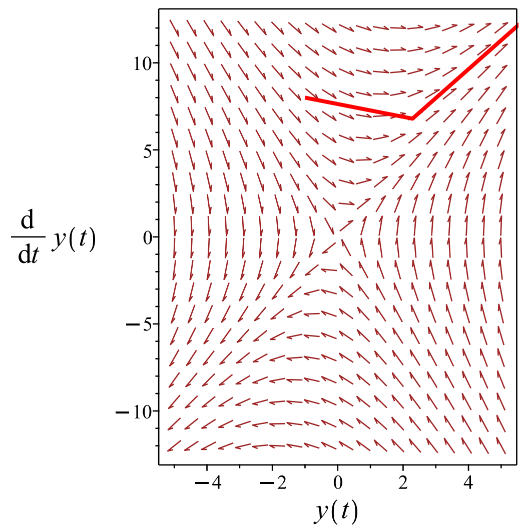
$$y = \mathrm{e}^{2t} - 2\,\mathrm{e}^{-3t}$$

Summary
The solution(s) found are the following

$$y = e^{2t} - 2\,e^{-3t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{2t} - 2\,e^{-3t}$$

Verified OK.

6.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 25$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \frac{25z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 87: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{25}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-\frac{5t}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-t}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{\mathrm{e}^{5t}}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t} \left(\frac{\mathrm{e}^{5t}}{5}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-3t} c_1 + \frac{c_2 \mathrm{e}^{2t}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $t = 0$ in the above gives

$$-1 = c_1 + \frac{c_2}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\,\mathrm{e}^{-3t} c_1 + \frac{2 c_2 \mathrm{e}^{2t}}{5}$$

substituting $y' = 8$ and $t = 0$ in the above gives

$$8 = -3c_1 + \frac{2c_2}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 \\ c_2 &= 5 \end{aligned}$$

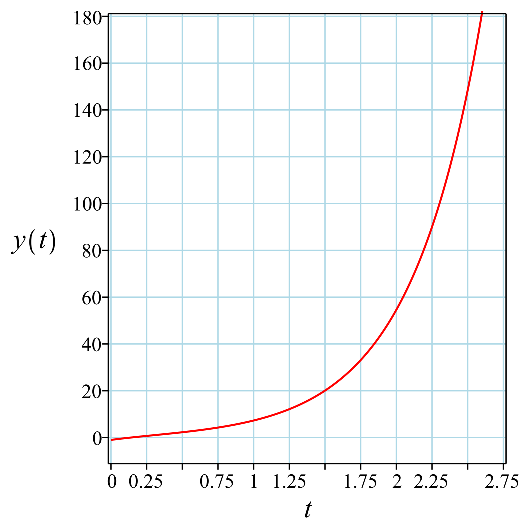
Substituting these values back in above solution results in

$$y = \mathrm{e}^{2t} - 2\,\mathrm{e}^{-3t}$$

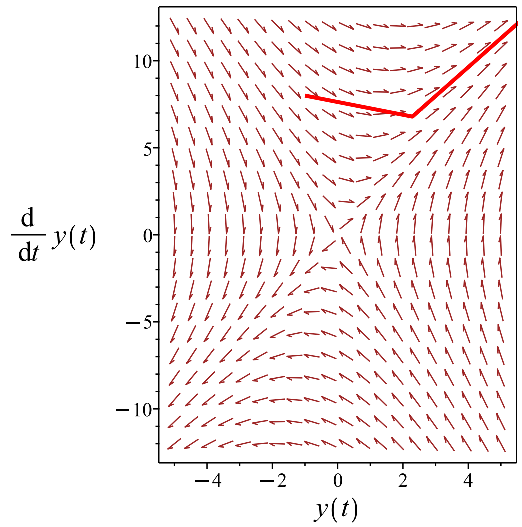
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2t} - 2\,\mathrm{e}^{-3t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{2t} - 2\,e^{-3t}$$

Verified OK.

6.4.4 Maple step by step solution

Let's solve

$$\left[y'' + y' - 6y = 0, y(0) = -1, y'\Big|_{\{t=0\}} = 8\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + r - 6 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 2)$

- 1st solution of the ODE

$y_1(t) = e^{-3t}$

- 2nd solution of the ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = e^{-3t} c_1 + c_2 e^{2t}$

□ Check validity of solution $y = e^{-3t} c_1 + c_2 e^{2t}$

  - Use initial condition $y(0) = -1$

    $-1 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -3\, e^{-3t} c_1 + 2c_2 e^{2t}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 8$

    $8 = -3c_1 + 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = -2, c_2 = 1\}$

  - Substitute constant values into general solution and simplify

    $y = (e^{5t} - 2)\, e^{-3t}$

- Solution to the IVP

  $y = (e^{5t} - 2)\, e^{-3t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$2)+diff(y(t),t)-6*y(t)=0,y(0) = -1, D(y)(0) = 8],y(t), singsol=all)
```

$$y(t) = \left(\mathrm{e}^{5t} - 2\right) \mathrm{e}^{-3t}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 18

```
DSolve[{y''[t]+y'[t]-6*y[t]==0,{y[0]==-1,y'[0]==8}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-3t}\left(e^{5t} - 2\right)$$

6.5 problem 12.1 (v)

Internal problem ID [12018]
Internal file name [`OUTPUT/10670_Sunday_September_03_2023_12_35_08_PM_75656588/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (v).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' = 0}$$

With initial conditions

$$[y(0) = 13, y'(0) = 0]$$

6.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 4y' = 0$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

6.5.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -4, C = 0$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 4\lambda\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 4\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 0$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(0)} \\ &= 2 \pm 2 \end{aligned}$$

Hence

$$\lambda_1 = 2 + 2$$
$$\lambda_2 = 2 - 2$$

Which simplifies to

$$\lambda_1 = 4$$
$$\lambda_2 = 0$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{(4)t} + c_2 e^{(0)t}$$

Or

$$y = c_1 \mathrm{e}^{4t} + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{4t} + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 13$ and $t = 0$ in the above gives

$$13 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 4c_1 \mathrm{e}^{4t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = 4c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 13$$

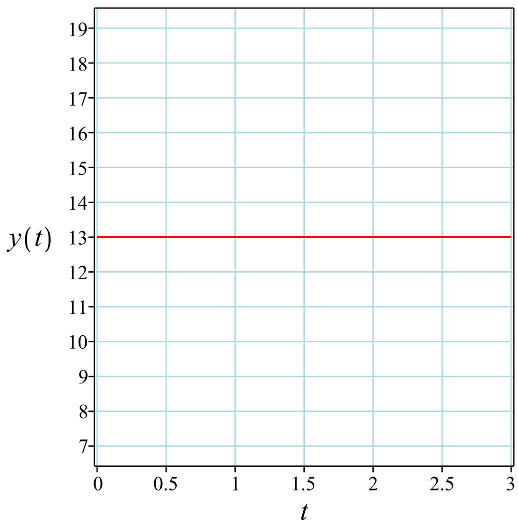
Substituting these values back in above solution results in

$$y = 13$$

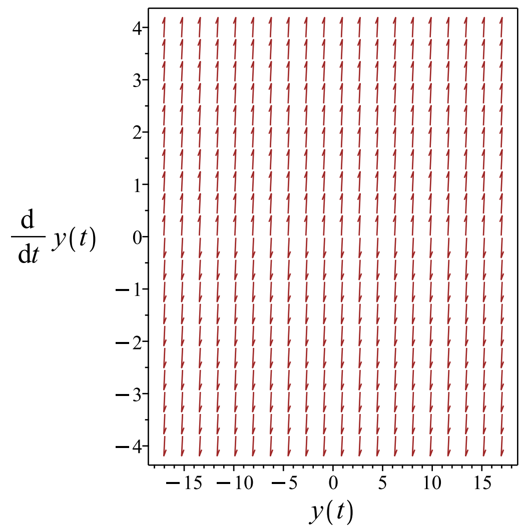
Summary

The solution(s) found are the following

$$y = 13 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 13$$

Verified OK.

6.5.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (y'' - 4y')\, dt = 0$$

$$-4y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{4y + c_1} dy = \int dt$$

$$\frac{\ln (4y + c_1)}{4} = t + c_2$$

Raising both side to exponential gives

$$(4y + c_1)^{\frac{1}{4}} = e^{t+c_2}$$

Which simplifies to

$$(4y + c_1)^{\frac{1}{4}} = c_3 e^t$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_3^4 e^{4t}}{4} - \frac{c_1}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 13$ and $t = 0$ in the above gives

$$13 = \frac{c_3^4}{4} - \frac{c_1}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_3^4 e^{4t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = c_3^4 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

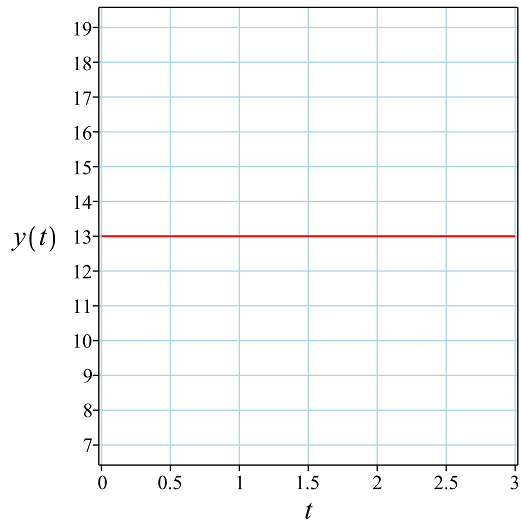
$$c_1 = -52$$
$$c_3 = 0$$

Substituting these values back in above solution results in

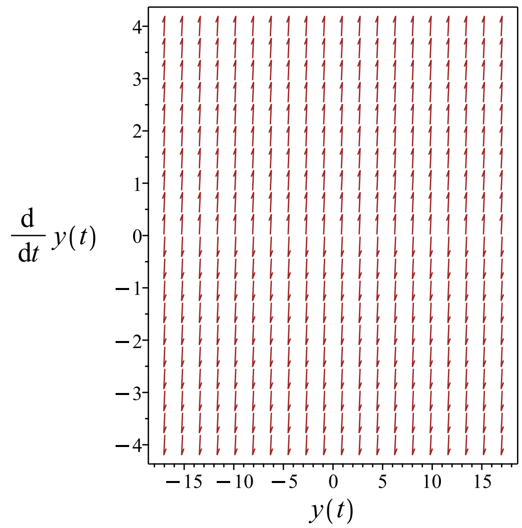
$$y = 13$$

Summary
The solution(s) found are the following

$$y = 13 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 13$$

Verified OK.

6.5.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$p'(t) - 4p(t) = 0$$

Which is now solve for $p(t)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{4p} dp = \int dt$$
$$\frac{\ln(p)}{4} = t + c_1$$

Raising both side to exponential gives

$$p^{\frac{1}{4}} = e^{t+c_1}$$

Which simplifies to

$$p^{\frac{1}{4}} = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $p = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2^4$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$p(t) = 0$$

Since $p = y'$ then the new first order ode to solve is

$$y' = 0$$

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, dt \\ &= c_3 \end{aligned}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $y = 13$ in the above solution gives an equation to solve for the constant of integration.

$$13 = c_3$$

$$c_3 = 13$$

Substituting c_3 found above in the general solution gives

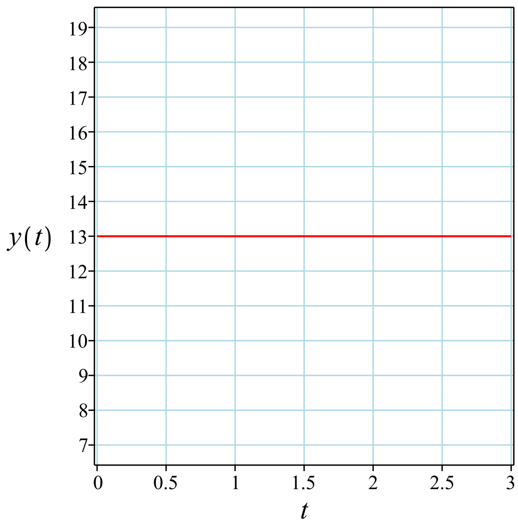
$$y = 13$$

Initial conditions are used to solve for the constants of integration.

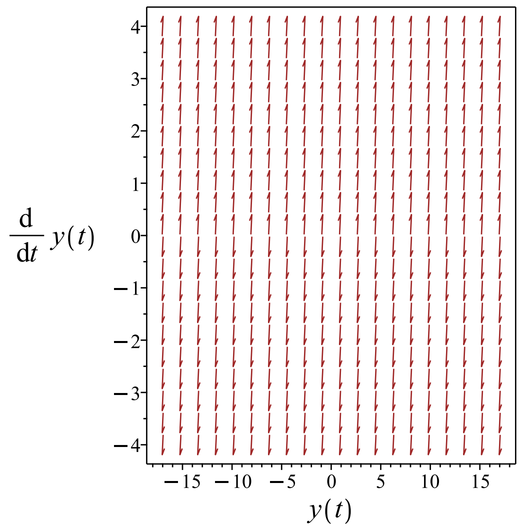
Summary

The solution(s) found are the following

$$y = 13 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 13$$

Verified OK.

6.5.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 4y' = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int (y'' - 4y')\, dt = 0$$

$$-4y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{4y + c_1} dy = \int dt$$

$$\frac{\ln (4y + c_1)}{4} = t + c_2$$

Raising both side to exponential gives

$$(4y + c_1)^{\frac{1}{4}} = \mathrm{e}^{t+c_2}$$

Which simplifies to

$$(4y + c_1)^{\frac{1}{4}} = c_3 e^t$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_3^4 e^{4t}}{4} - \frac{c_1}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 13$ and $t = 0$ in the above gives

$$13 = \frac{c_3^4}{4} - \frac{c_1}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_3^4 e^{4t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = c_3^4 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = -52$$
$$c_3 = 0$$

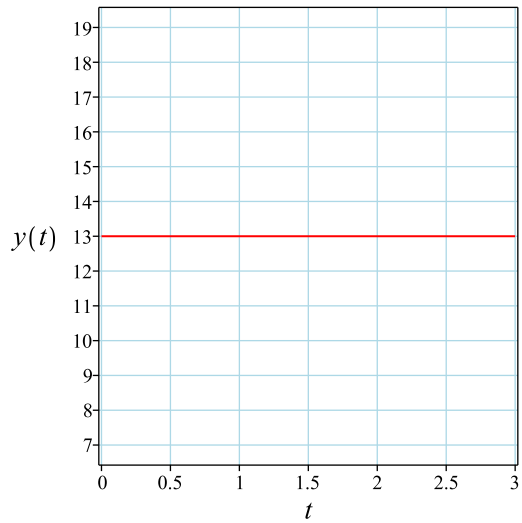
Substituting these values back in above solution results in

$$y = 13$$

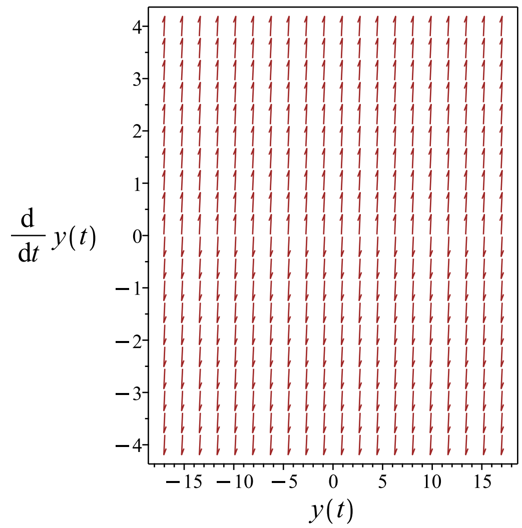
Summary

The solution(s) found are the following

$$y = 13 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 13$$

Verified OK.

6.5.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = 4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 89: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-2t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dt} \\ &= z_1 e^{2t} \\ &= z_1 \left(\mathrm{e}^{2t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{4t}}{(y_1)^2}\,dt \\ &= y_1 \left(\frac{\mathrm{e}^{4t}}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{e^{4t}}{4}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 + \frac{c_2 e^{4t}}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 13$ and $t = 0$ in the above gives

$$
13 = c_1 + \frac{c_2}{4} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_2 e^{4t}
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 13 \\
c_2 &= 0
\end{aligned}
$$

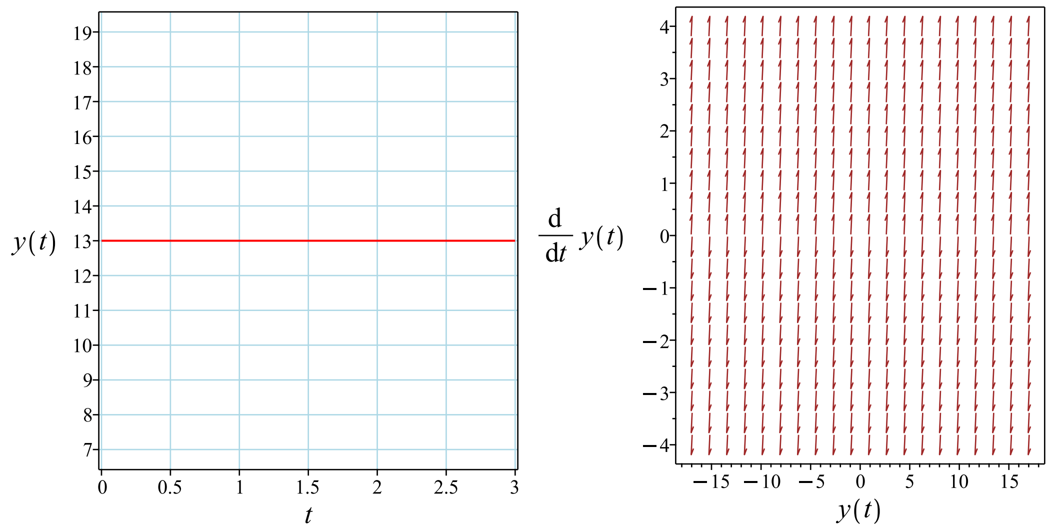
Substituting these values back in above solution results in

$$
y = 13
$$

Summary

The solution(s) found are the following

$$
y = 13 \tag{1}
$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 13$$

Verified OK.

6.5.7 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -4 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,y'+(q(t)-p'(t))\,y)'=s(x)$$

Integrating gives

$$p(t)\,y'+(q(t)-p'(t))\,y=\int s(t)\;dt$$

Substituting the above values for p,q,r,s gives

$$-4y+y'=c_1$$

We now have a first order ode to solve which is

$$-4y+y'=c_1$$

Integrating both sides gives

$$\begin{aligned}\int\frac{1}{4y+c_1}dy&=\int dt\\\frac{\ln(4y+c_1)}{4}&=t+c_2\end{aligned}$$

Raising both side to exponential gives

$$(4y+c_1)^{\frac{1}{4}}=\mathrm{e}^{t+c_2}$$

Which simplifies to

$$(4y+c_1)^{\frac{1}{4}}=c_3\mathrm{e}^{t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y=\frac{c_3^4\mathrm{e}^{4t}}{4}-\frac{c_1}{4}\tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=13$ and $t=0$ in the above gives

$$13=\frac{c_3^4}{4}-\frac{c_1}{4}\tag{1A}$$

Taking derivative of the solution gives

$$y' = c_3^4 e^{4t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = c_3^4 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

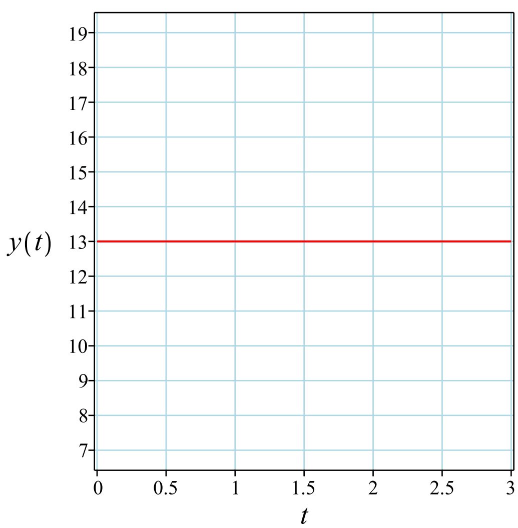
$$c_1 = -52$$
$$c_3 = 0$$

Substituting these values back in above solution results in

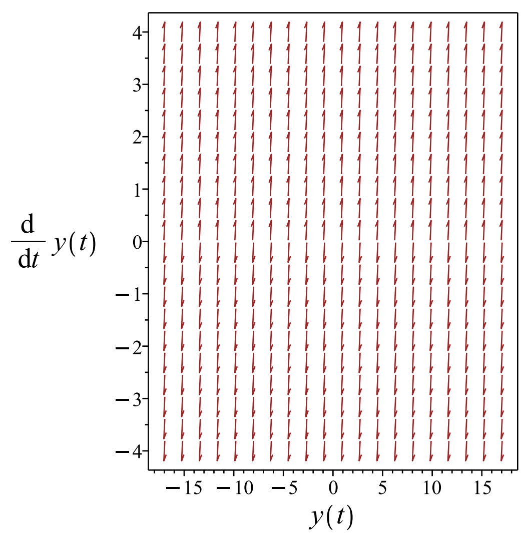
$$y = 13$$

Summary
The solution(s) found are the following

$$y = 13 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 13$$

Verified OK.

6.5.8 Maple step by step solution

Let's solve

$\left[y''-4y'=0, y(0)=13, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2-4r=0$

- Factor the characteristic polynomial

  $r(r-4)=0$

- Roots of the characteristic polynomial

  $r=(0,4)$

- 1st solution of the ODE

  $y_1(t)=1$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{4t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1+c_2\mathrm{e}^{4t}$

☐ Check validity of solution $y=c_1+c_2\mathrm{e}^{4t}$

  - Use initial condition $y(0)=13$

    $13=c_1+c_2$

  - Compute derivative of the solution

    $y'=4c_2\mathrm{e}^{4t}$

  - Use the initial condition $y'\Big|_{\{t=0\}}=0$

    $0=4c_2$

  - Solve for c_1 and c_2

    $\{c_1=13, c_2=0\}$

- Substitute constant values into general solution and simplify

  $$y = 13$$

- Solution to the IVP

  $$y = 13$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 5

```
dsolve([diff(y(t),t$2)-4*diff(y(t),t)=0,y(0) = 13, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = 13$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 6

```
DSolve[{y''[t]-4*y'[t]==0,{y[0]==13,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 13$$

6.6 problem 12.1 (vi)

Internal problem ID [12019]
Internal file name [`OUTPUT/10671_Sunday_September_03_2023_12_35_10_PM_68303959/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (vi).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{\theta'' + 4\theta = 0}$$

With initial conditions

$$[\theta(0) = 0, \theta'(0) = 10]$$

6.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$\theta'' + p(t)\theta' + q(t)\theta = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$\theta'' + 4\theta = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.6.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\theta''(t) + B\theta'(t) + C\theta(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $\theta = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\
&= \pm 2i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 2i \\
\lambda_2 &= -2i
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$
\theta = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))
$$

Which becomes

$$
\theta = e^{0}(c_1 \cos(2t) + c_2 \sin(2t))
$$

Or

$$
\theta = c_1 \cos(2t) + c_2 \sin(2t)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
\theta = c_1 \cos(2t) + c_2 \sin(2t) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $\theta = 0$ and $t = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
\theta' = -2c_1 \sin(2t) + 2c_2 \cos(2t)
$$

substituting $\theta' = 10$ and $t = 0$ in the above gives

$$
10 = 2c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

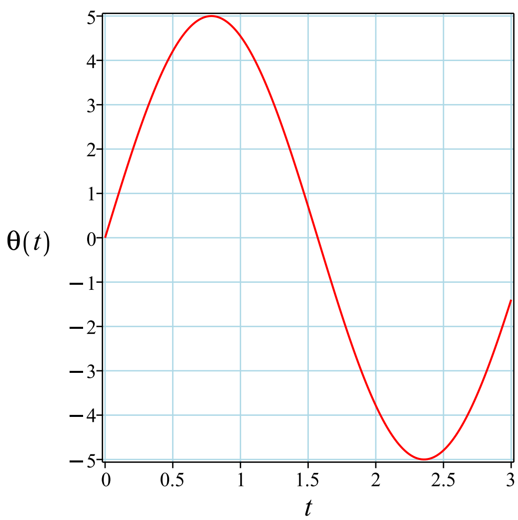
$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 5
\end{aligned}
$$

Substituting these values back in above solution results in

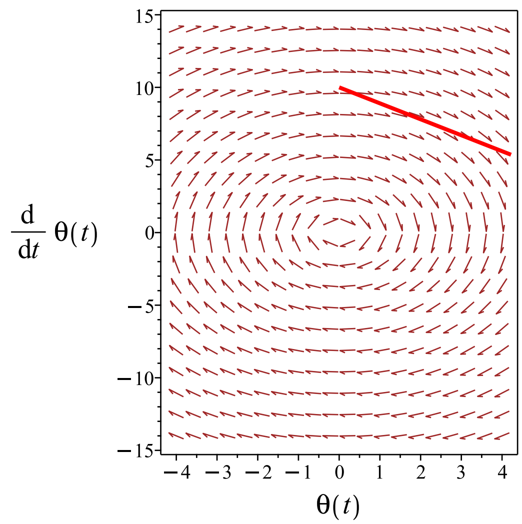
$$\theta = 5\sin(2t)$$

Summary
The solution(s) found are the following

$$\theta = 5\sin(2t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$\theta = 5\sin(2t)$$

Verified OK.

6.6.3 Solving as second order ode can be made integrable ode

Multiplying the ode by θ' gives

$$\theta'\theta'' + 4\theta'\theta = 0$$

Integrating the above w.r.t t gives

$$\int (\theta'\theta'' + 4\theta'\theta)\,dt = 0$$
$$\frac{\theta'^2}{2} + 2\theta^2 = c_2$$

Which is now solved for θ. Solving the given ode for θ' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\theta' = \sqrt{-4\theta^2 + 2c_1} \tag{1}$$

$$\theta' = -\sqrt{-4\theta^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{1}{\sqrt{-4\theta^2 + 2c_1}} d\theta = \int dt$$

$$\frac{\arctan\left(\frac{2\theta}{\sqrt{-4\theta^2+2c_1}}\right)}{2} = t + c_2$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-4\theta^2 + 2c_1}} d\theta = \int dt$$

$$-\frac{\arctan\left(\frac{2\theta}{\sqrt{-4\theta^2+2c_1}}\right)}{2} = t + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{2\theta}{\sqrt{-4\theta^2+2c_1}}\right)}{2} = t + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $\theta = 0$ and $t = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$\theta' = \frac{\left(2\tan(2t+2c_2)^2 + 2\right)\sqrt{2}\sqrt{\frac{c_1}{\tan(2t+2c_2)^2+1}}}{2} - \frac{\tan(2t+2c_2)^2\sqrt{2}\,c_1\left(2\tan(2t+2c_2)^2 + 2\right)}{2\sqrt{\frac{c_1}{\tan(2t+2c_2)^2+1}}\left(\tan(2t+2c_2)^2 + 1\right)^2}$$

substituting $\theta' = 10$ and $t = 0$ in the above gives

$$10 = \frac{\cos(2c_2)^2 \sqrt{2}\, c_1}{\sqrt{\cos(2c_2)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 50 \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$\frac{\arctan\left(\frac{\theta}{\sqrt{25-\theta^2}}\right)}{2} = t$$

Looking at the Second solution

$$-\frac{\arctan\left(\frac{2\theta}{\sqrt{-4\theta^2+2c_1}}\right)}{2} = t + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $\theta = 0$ and $t = 0$ in the above gives

$$0 = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$\theta' = -\frac{\left(2\tan(2t+2c_3)^2+2\right)\sqrt{2}\sqrt{\frac{c_1}{\tan(2t+2c_3)^2+1}}}{2} + \frac{\tan(2t+2c_3)^2 \sqrt{2}\, c_1 \left(2\tan(2t+2c_3)^2+2\right)}{2\sqrt{\frac{c_1}{\tan(2t+2c_3)^2+1}}\left(\tan(2t+2c_3)^2+1\right)^2}$$

substituting $\theta' = 10$ and $t = 0$ in the above gives

$$10 = -\frac{\cos(2c_3)^2 \sqrt{2}\, c_1}{\sqrt{\cos(2c_3)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{\theta}{\sqrt{25-\theta^2}}\right)}{2} = t \tag{1}$$

Verification of solutions

$$\frac{\arctan\left(\frac{\theta}{\sqrt{25-\theta^2}}\right)}{2} = t$$

Verified OK.

6.6.4 Solving using Kovacic algorithm

Writing the ode as

$$\theta'' + 4\theta = 0 \tag{1}$$
$$A\theta'' + B\theta' + C\theta = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = \theta e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then θ is found using the inverse transformation

$$\theta = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 91: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in θ is found from

$$\theta_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}\theta_1 &= z_1\\ &= \cos(2t)\end{aligned}$$

Which simplifies to

$$\theta_1 = \cos(2t)$$

The second solution θ_2 to the original ode is found using reduction of order

$$\theta_2 = \theta_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{\theta_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}\theta_2 &= \theta_1 \int \frac{1}{\theta_1^2}\,dt\\ &= \cos(2t) \int \frac{1}{\cos(2t)^2}\,dt\\ &= \cos(2t)\left(\frac{\tan(2t)}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}\theta &= c_1\theta_1 + c_2\theta_2\\ &= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right)\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$\theta = c_1\cos(2t) + \frac{c_2\sin(2t)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $\theta = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$\theta' = -2c_1 \sin(2t) + c_2 \cos(2t)$$

substituting $\theta' = 10$ and $t = 0$ in the above gives

$$10 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 10$$

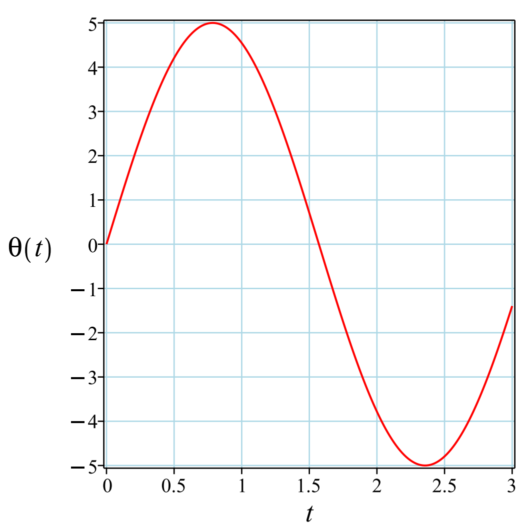
Substituting these values back in above solution results in

$$\theta = 5 \sin(2t)$$

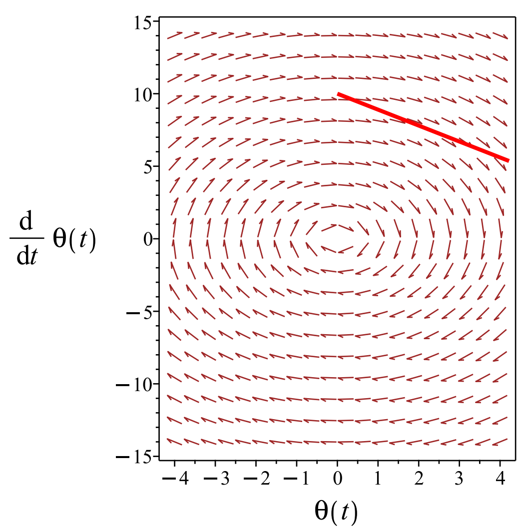
Summary

The solution(s) found are the following

$$\theta = 5 \sin(2t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$\theta = 5\sin(2t)$$

Verified OK.

6.6.5 Maple step by step solution

Let's solve

$\left[\theta'' + 4\theta = 0, \theta(0) = 0, \theta'\Big|_{\{t=0\}} = 10\right]$

- Highest derivative means the order of the ODE is 2

  θ''

- Characteristic polynomial of ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the ODE

  $\theta_1(t) = \cos(2t)$

- 2nd solution of the ODE

  $\theta_2(t) = \sin(2t)$

- General solution of the ODE

  $\theta = c_1\theta_1(t) + c_2\theta_2(t)$

- Substitute in solutions

  $\theta = c_1\cos(2t) + c_2\sin(2t)$

□ Check validity of solution $\theta = c_1\cos(2t) + c_2\sin(2t)$

  - Use initial condition $\theta(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $\theta' = -2c_1\sin(2t) + 2c_2\cos(2t)$

- Use the initial condition $\theta'\Big|_{\{t=0\}} = 10$

  $10 = 2c_2$
- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 5\}$
- Substitute constant values into general solution and simplify

  $\theta = 5\sin(2t)$

- Solution to the IVP

  $\theta = 5\sin(2t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 10

```
dsolve([diff(theta(t),t$2)+4*theta(t)=0,theta(0) = 0, D(theta)(0) = 10],theta(t), singsol=al
```

$$\theta(t) = 5\sin(2t)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 11

```
DSolve[{\[Theta]''[t]+4*\[Theta][t]==0,{\[Theta][0]==0,\[Theta]'[0]==10}},\[Theta][t],t,Incl
```

$$\theta(t) \to 5\sin(2t)$$

6.7 problem 12.1 (vii)

Internal problem ID [12020]
Internal file name [`OUTPUT/10672_Sunday_September_03_2023_12_35_12_PM_47658040/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (vii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 10y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 0]$$

6.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 10 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 10y = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.7.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 10$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + 10\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(10)} \\ &= -1 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 3i \\ \lambda_2 &= -1 - 3i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -1 + 3i$$
$$\lambda_2 = -1 - 3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-t}(c_1 \cos(3t) + c_2 \sin(3t)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $t = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\mathrm{e}^{-t}(c_1 \cos(3t) + c_2 \sin(3t)) + \mathrm{e}^{-t}(-3c_1 \sin(3t) + 3c_2 \cos(3t))$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 1$$

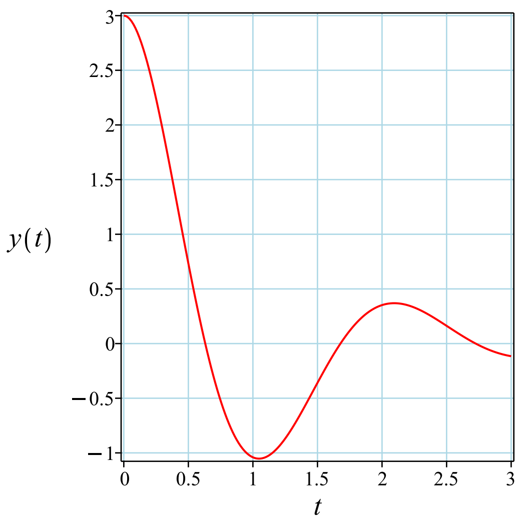
Substituting these values back in above solution results in

$$y = \mathrm{e}^{-t}(3\cos(3t) + \sin(3t))$$

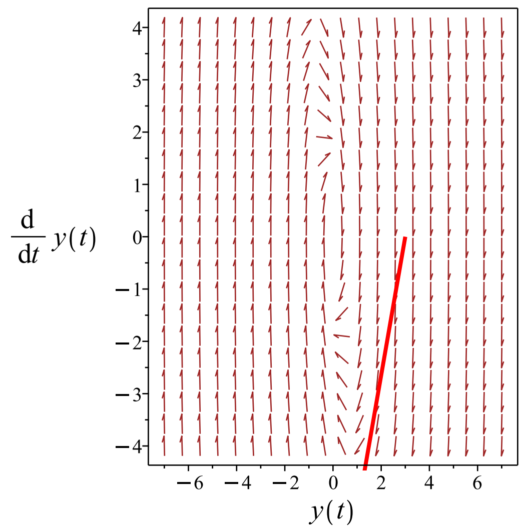
Summary

The solution(s) found are the following

$$y = e^{-t}(3\cos(3t) + \sin(3t)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-t}(3\cos(3t) + \sin(3t))$$

Verified OK.

6.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 10y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 93: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt}\\ &= z_1 e^{-t}\\ &= z_1\left(\mathrm{e}^{-t}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-t}\cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(y_1)^2}\,dt\\ &= y_1 \int \frac{e^{-2t}}{(y_1)^2}\,dt\\ &= y_1\left(\frac{\tan(3t)}{3}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-t} \cos (3t)\right) + c_2 \left(\mathrm{e}^{-t} \cos (3t) \left(\frac{\tan (3t)}{3}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-t} \cos (3t) + \frac{c_2 \mathrm{e}^{-t} \sin (3t)}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $t = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-t} \cos (3t) - 3c_1 \mathrm{e}^{-t} \sin (3t) - \frac{c_2 \mathrm{e}^{-t} \sin (3t)}{3} + c_2 \mathrm{e}^{-t} \cos (3t)
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = -c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= 3
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 3\,\mathrm{e}^{-t} \cos (3t) + \mathrm{e}^{-t} \sin (3t)
$$

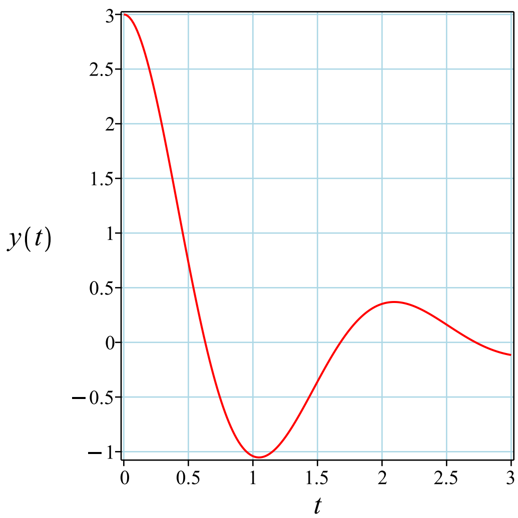
Which simplifies to

$$
y = \mathrm{e}^{-t}(3 \cos (3t) + \sin (3t))
$$

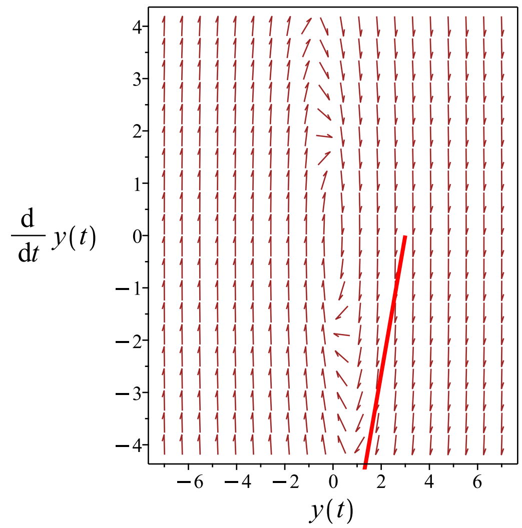
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-t}(3 \cos (3t) + \sin (3t)) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-t}(3\cos(3t) + \sin(3t))$$

Verified OK.

6.7.4 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + 10y = 0, y(0) = 3, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of ODE
  $r^2 + 2r + 10 = 0$
- Use quadratic formula to solve for r
  $r = \frac{(-2)\pm(\sqrt{-36})}{2}$
- Roots of the characteristic polynomial
  $r = (-1 - 3\,\mathrm{I}, -1 + 3\,\mathrm{I})$
- 1st solution of the ODE

$y_1(t) = \mathrm{e}^{-t}\cos(3t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-t}\sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1\mathrm{e}^{-t}\cos(3t) + c_2\mathrm{e}^{-t}\sin(3t)$

☐ Check validity of solution $y = c_1\mathrm{e}^{-t}\cos(3t) + c_2\mathrm{e}^{-t}\sin(3t)$

  ◦ Use initial condition $y(0) = 3$

    $3 = c_1$

  ◦ Compute derivative of the solution

    $y' = -c_1\mathrm{e}^{-t}\cos(3t) - 3c_1\mathrm{e}^{-t}\sin(3t) - c_2\mathrm{e}^{-t}\sin(3t) + 3c_2\mathrm{e}^{-t}\cos(3t)$

  ◦ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -c_1 + 3c_2$

  ◦ Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = 1\}$

  ◦ Substitute constant values into general solution and simplify

    $y = \mathrm{e}^{-t}(3\cos(3t) + \sin(3t))$

- Solution to the IVP

  $y = \mathrm{e}^{-t}(3\cos(3t) + \sin(3t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+10*y(t)=0,y(0) = 3, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = e^{-t}(\sin(3t) + 3\cos(3t))$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 22

```
DSolve[{y''[t]+2*y'[t]+10*y[t]==0,{y[0]==3,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to e^{-t}(\sin(3t) + 3\cos(3t))$$

6.8 problem 12.1 (viii)

Internal problem ID [12021]
Internal file name [`OUTPUT/10673_Sunday_September_03_2023_12_35_15_PM_81411984/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (viii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2z'' + 7z' - 4z = 0}$$

With initial conditions

$$[z(0) = 0, z'(0) = 9]$$

6.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$z'' + p(t)z' + q(t)z = F$$

Where here

$$\begin{aligned} p(t) &= \frac{7}{2} \\ q(t) &= -2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$z'' + \frac{7z'}{2} - 2z = 0$$

The domain of $p(t) = \frac{7}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.8.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Az''(t) + Bz'(t) + Cz(t) = 0$$

Where in the above $A = 2, B = 7, C = -4$. Let the solution be $z = e^{\lambda t}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda t} + 7\lambda\, e^{\lambda t} - 4\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$2\lambda^2 + 7\lambda - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 7, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-7}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{7^2 - (4)\,(2)\,(-4)} \\ &= -\frac{7}{4} \pm \frac{9}{4}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{7}{4} + \frac{9}{4}$$

$$\lambda_2 = -\frac{7}{4} - \frac{9}{4}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -4$$

Since roots are real and distinct, then the solution is

$$z = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$z = c_1 e^{\left(\frac{1}{2}\right)t} + c_2 e^{(-4)t}$$

Or

$$z = c_1 \mathrm{e}^{\frac{t}{2}} + c_2 \mathrm{e}^{-4t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$z = c_1 \mathrm{e}^{\frac{t}{2}} + c_2 \mathrm{e}^{-4t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$z' = \frac{c_1 \mathrm{e}^{\frac{t}{2}}}{2} - 4c_2 \mathrm{e}^{-4t}$$

substituting $z' = 9$ and $t = 0$ in the above gives

$$9 = \frac{c_1}{2} - 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$z = 2\,\mathrm{e}^{\frac{t}{2}} - 2\,\mathrm{e}^{-4t}$$

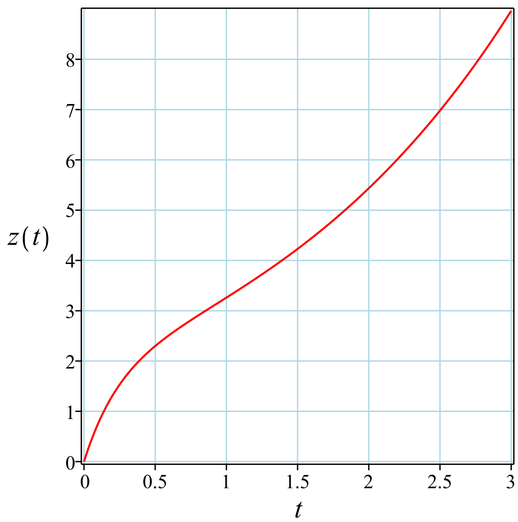
Which simplifies to

$$z = 2\left(e^{\frac{9t}{2}} - 1\right) e^{-4t}$$

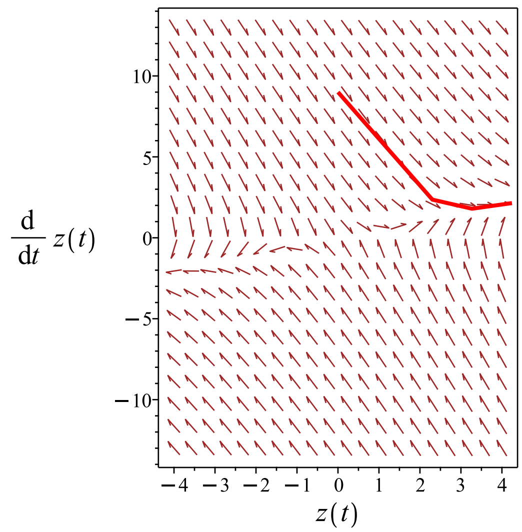
Summary

The solution(s) found are the following

$$z = 2\left(e^{\frac{9t}{2}} - 1\right) e^{-4t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$z = 2\left(e^{\frac{9t}{2}} - 1\right) e^{-4t}$$

Verified OK.

6.8.3 Solving using Kovacic algorithm

Writing the ode as

$$2z'' + 7z' - 4z = 0 \tag{1}$$
$$Az'' + Bz' + Cz = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 7 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ze^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{81}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 81 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{81z}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then z is found using the inverse transformation

$$z = ze^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 95: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{81}{16}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{9t}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in z is found from

$$\begin{aligned} z_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7}{2}\,dt} \\ &= z_1 e^{-\frac{7t}{4}} \\ &= z_1\left(\mathrm{e}^{-\frac{7t}{4}}\right) \end{aligned}$$

Which simplifies to

$$z_1 = \mathrm{e}^{-4t}$$

The second solution z_2 to the original ode is found using reduction of order

$$z_2 = z_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{z_1^2}\, dt$$

Substituting gives

$$\begin{aligned} z_2 &= z_1 \int \frac{e^{\int -\frac{7}{2}\, dt}}{(z_1)^2}\, dt \\ &= z_1 \int \frac{e^{-\frac{7t}{2}}}{(z_1)^2}\, dt \\ &= z_1 \left(\frac{2\,\mathrm{e}^{\frac{9t}{2}}}{9}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} z &= c_1 z_1 + c_2 z_2 \\ &= c_1 \left(\mathrm{e}^{-4t}\right) + c_2 \left(\mathrm{e}^{-4t} \left(\frac{2\,\mathrm{e}^{\frac{9t}{2}}}{9}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$z = c_1 \mathrm{e}^{-4t} + \frac{2c_2 \mathrm{e}^{\frac{t}{2}}}{9} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{2c_2}{9} \tag{1A}$$

Taking derivative of the solution gives

$$z' = -4c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{\frac{t}{2}}}{9}$$

substituting $z' = 9$ and $t = 0$ in the above gives

$$9 = -4c_1 + \frac{c_2}{9} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 9$$

Substituting these values back in above solution results in

$$z = 2\,e^{\frac{t}{2}} - 2\,e^{-4t}$$

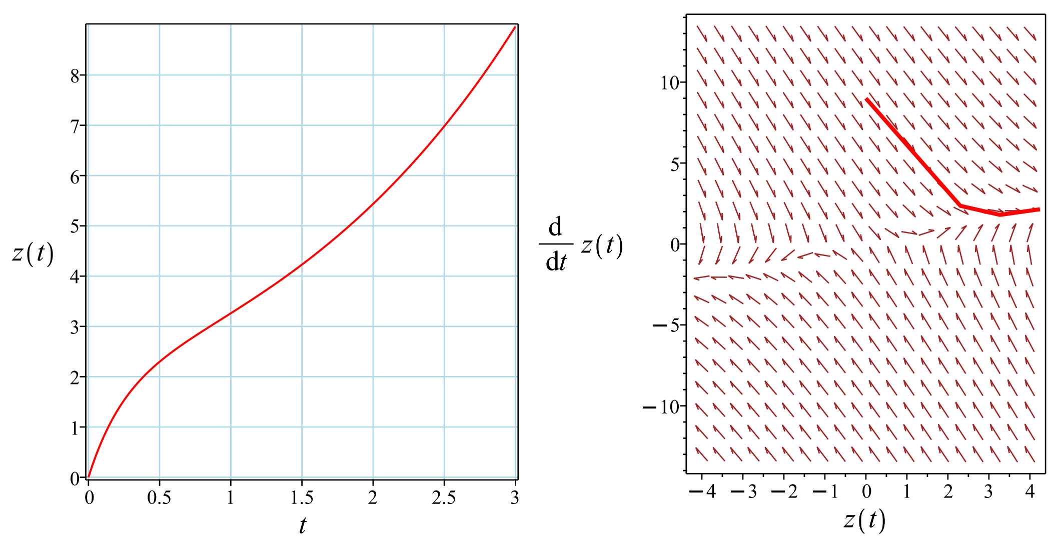
Which simplifies to

$$z = 2\left(e^{\frac{9t}{2}} - 1\right)e^{-4t}$$

Summary

The solution(s) found are the following

$$z = 2\left(e^{\frac{9t}{2}} - 1\right)e^{-4t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$z = 2\left(e^{\frac{9t}{2}} - 1\right)e^{-4t}$$

Verified OK.

6.8.4 Maple step by step solution

Let's solve

$\left[2z''+7z'-4z=0, z(0)=0, z'\Big|_{\{t=0\}}=9\right]$

- Highest derivative means the order of the ODE is 2

  z''

- Isolate 2nd derivative

  $z''=-\frac{7z'}{2}+2z$

- Group terms with z on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $z''+\frac{7z'}{2}-2z=0$

- Characteristic polynomial of ODE

  $r^2+\frac{7}{2}r-2=0$

- Factor the characteristic polynomial

  $\frac{(r+4)(2r-1)}{2}=0$

- Roots of the characteristic polynomial

  $r=\left(-4,\frac{1}{2}\right)$

- 1st solution of the ODE

  $z_1(t)=\mathrm{e}^{-4t}$

- 2nd solution of the ODE

  $z_2(t)=\mathrm{e}^{\frac{t}{2}}$

- General solution of the ODE

  $z=c_1z_1(t)+c_2z_2(t)$

- Substitute in solutions

  $z=c_1\mathrm{e}^{-4t}+c_2\mathrm{e}^{\frac{t}{2}}$

□ Check validity of solution $z=c_1\mathrm{e}^{-4t}+c_2\mathrm{e}^{\frac{t}{2}}$

  - Use initial condition $z(0)=0$

    $0=c_1+c_2$

  - Compute derivative of the solution

    $z'=-4c_1\mathrm{e}^{-4t}+\frac{c_2\mathrm{e}^{\frac{t}{2}}}{2}$

- Use the initial condition $z'\Big|_{\{t=0\}} = 9$

  $9 = -4c_1 + \frac{c_2}{2}$
- Solve for c_1 and c_2

  $\{c_1 = -2, c_2 = 2\}$
- Substitute constant values into general solution and simplify

  $z = 2\left(e^{\frac{9t}{2}} - 1\right) e^{-4t}$

- Solution to the IVP

  $z = 2\left(e^{\frac{9t}{2}} - 1\right) e^{-4t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([2*diff(z(t),t$2)+7*diff(z(t),t)-4*z(t)=0,z(0) = 0, D(z)(0) = 9],z(t), singsol=all)
```

$$z(t) = 2\left(e^{\frac{9t}{2}} - 1\right) e^{-4t}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 49

```
DSolve[{z''[t]+7*z'[t]-4*z[t]==0,{z[0]==3,z'[0]==9}},z[t],t,IncludeSingularSolutions -> True
```

$$z(t) \to \frac{3}{10} e^{-\frac{1}{2}\left(7+\sqrt{65}\right)t}\left(\left(5+\sqrt{65}\right)e^{\sqrt{65}t} + 5 - \sqrt{65}\right)$$

6.9 problem 12.1 (ix)

Internal problem ID [12022]
Internal file name [`OUTPUT/10674_Sunday_September_03_2023_12_35_35_PM_58129296/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (ix).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = -1]$$

6.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 2$$
$$q(t) = 1$$
$$F = 0$$

Hence the ode is

$$y'' + 2y' + y = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.9.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-t} + c_2 e^{-t} - c_2 t\, e^{-t}$$

substituting $y' = -1$ and $t = 0$ in the above gives

$$-1 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

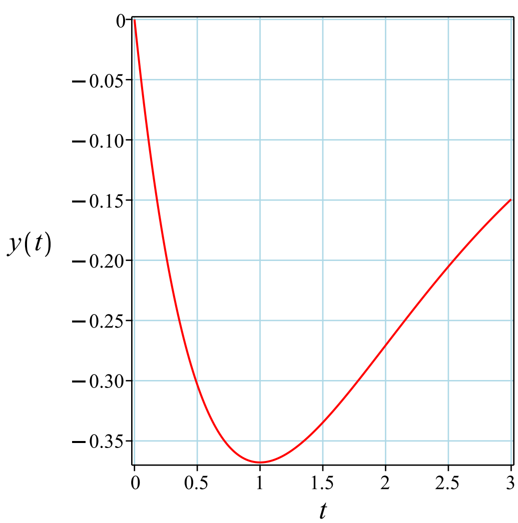
$$c_1 = 0$$
$$c_2 = -1$$

Substituting these values back in above solution results in

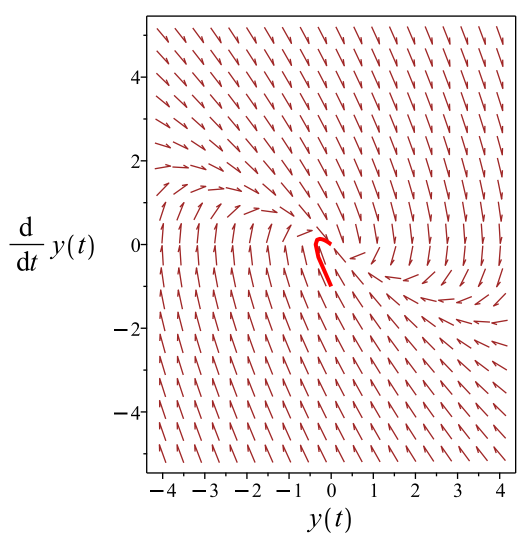
$$y = -t\, e^{-t}$$

Summary
The solution(s) found are the following

$$y = -t\, e^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -t\,\mathrm{e}^{-t}$$

Verified OK.

6.9.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\,y' + \frac{\left(p(t)^2 + p'(t)\right) y}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= \mathrm{e}^{t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{t} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{t} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{t} y\right) = c_1 t + c_2$$

Hence the solution is

$$y = \frac{c_1 t + c_2}{\mathrm{e}^{t}}$$

Or

$$y = t\,\mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{-t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = t\,\mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \mathrm{e}^{-t} - t\,\mathrm{e}^{-t} c_1 - c_2 \mathrm{e}^{-t}$$

substituting $y' = -1$ and $t = 0$ in the above gives

$$-1 = c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

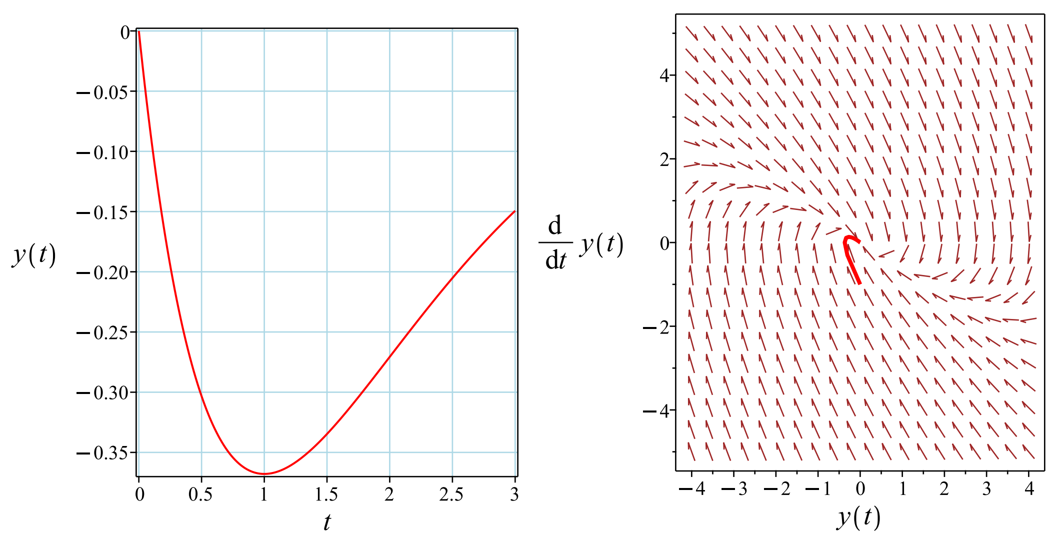
$$c_1 = -1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -t\,\mathrm{e}^{-t}$$

Summary
The solution(s) found are the following

$$y = -t\,\mathrm{e}^{-t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -t\,\mathrm{e}^{-t}$$

Verified OK.

6.9.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 97: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\
&= z_1 e^{-t} \\
&= z_1 \left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-2t}}{(y_1)^2}\,dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-t}\right) + c_2\left(\mathrm{e}^{-t}(t)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-t} + c_2 e^{-t} - c_2 t\, e^{-t}$$

substituting $y' = -1$ and $t = 0$ in the above gives

$$-1 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

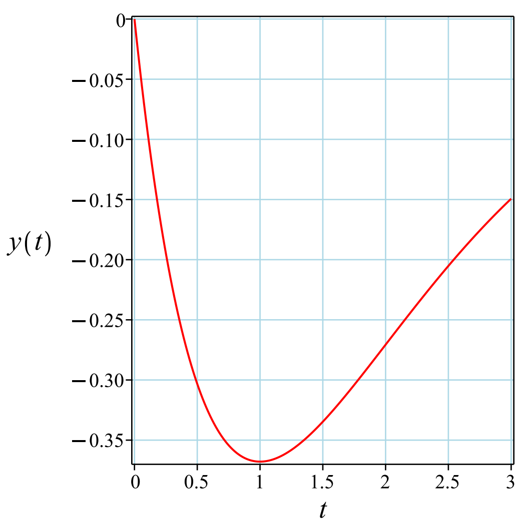
$$c_1 = 0$$
$$c_2 = -1$$

Substituting these values back in above solution results in

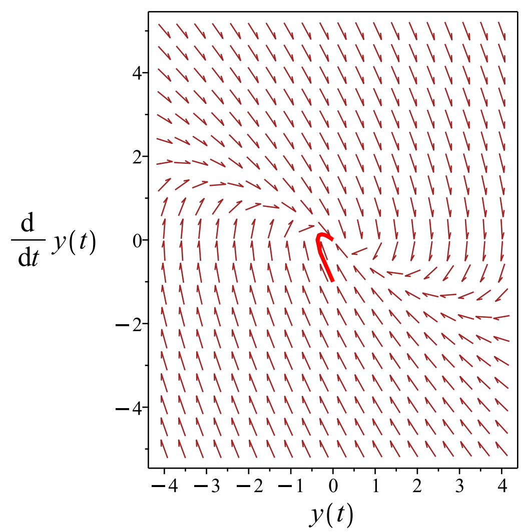
$$y = -t\, e^{-t}$$

Summary
The solution(s) found are the following

$$y = -t\, e^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -t\, e^{-t}$$

Verified OK.

6.9.5 Maple step by step solution

Let's solve

$\left[y'' + 2y' + y = 0, y(0) = 0, y'\Big|_{\{t=0\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the ODE

  $y_1(t) = e^{-t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{-t} + c_2 t\,e^{-t}$

□ Check validity of solution $y = c_1 e^{-t} + c_2 t e^{-t}$

  - Use initial condition $y(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $y' = -c_1 e^{-t} + c_2 e^{-t} - c_2 t\,e^{-t}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = -1$

    $-1 = -c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = -1\}$

- Substitute constant values into general solution and simplify

  $$y = -t\,\mathrm{e}^{-t}$$

- Solution to the IVP

  $$y = -t\,\mathrm{e}^{-t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t)=0,y(0) = 0, D(y)(0) = -1],y(t), singsol=all)
```

$$y(t) = -t\,\mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 13

```
DSolve[{y''[t]+2*y'[t]+y[t]==0,{y[0]==0,y'[0]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -e^{-t}t$$

6.10 problem 12.1 (x)

Internal problem ID [12023]
Internal file name [`OUTPUT/10675_Sunday_September_03_2023_12_35_36_PM_94584042/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (x).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + 6x' + 10x = 0}$$

With initial conditions

$$[x(0) = 3, x'(0) = 1]$$

6.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 10 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 6x' + 10x = 0$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.10.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 6, C = 10$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda\, e^{\lambda t} + 10\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{6^2 - (4)\,(1)\,(10)} \\ &= -3 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + i \\ \lambda_2 &= -3 - i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -3 + i$$
$$\lambda_2 = -3 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-3t}(c_1 \cos(t) + c_2 \sin(t))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \mathrm{e}^{-3t}(c_1 \cos(t) + c_2 \sin(t)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 3$ and $t = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3\,\mathrm{e}^{-3t}(c_1 \cos(t) + c_2 \sin(t)) + \mathrm{e}^{-3t}(-c_1 \sin(t) + c_2 \cos(t))$$

substituting $x' = 1$ and $t = 0$ in the above gives

$$1 = -3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

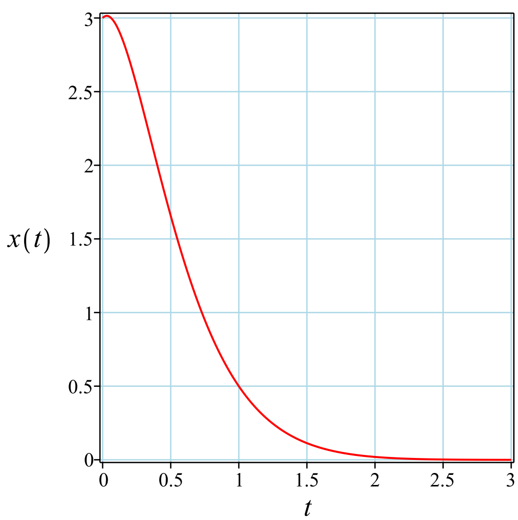
$$c_1 = 3$$
$$c_2 = 10$$

Substituting these values back in above solution results in

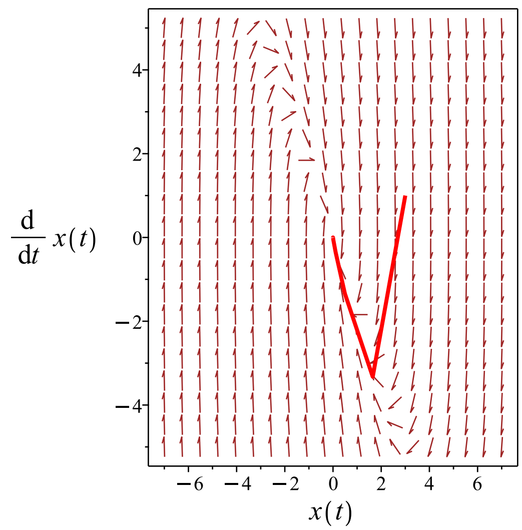
$$x = \mathrm{e}^{-3t}(3\cos(t) + 10\sin(t))$$

Summary
The solution(s) found are the following

$$x = e^{-3t}(3\cos(t) + 10\sin(t)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{-3t}(3\cos(t) + 10\sin(t))$$

Verified OK.

6.10.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 6x' + 10x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 99: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dt} \\ &= z_1 e^{-3t} \\ &= z_1\left(\mathrm{e}^{-3t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-3t}\cos(t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{6}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-6t}}{(x_1)^2}\,dt \\ &= x_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-3t} \cos(t)\right) + c_2 \left(\mathrm{e}^{-3t} \cos(t) (\tan(t))\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \mathrm{e}^{-3t} \cos(t) + c_2 \mathrm{e}^{-3t} \sin(t) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 3$ and $t = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -3c_1 \mathrm{e}^{-3t} \cos(t) - c_1 \mathrm{e}^{-3t} \sin(t) - 3c_2 \mathrm{e}^{-3t} \sin(t) + c_2 \mathrm{e}^{-3t} \cos(t)
$$

substituting $x' = 1$ and $t = 0$ in the above gives

$$
1 = -3c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= 10
\end{aligned}
$$

Substituting these values back in above solution results in

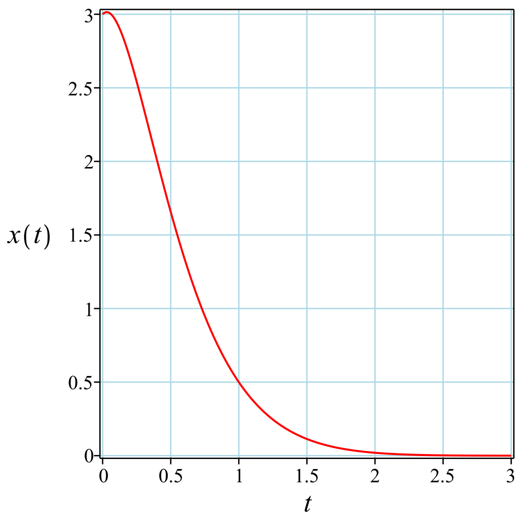
$$
x = 3\,\mathrm{e}^{-3t} \cos(t) + 10\,\mathrm{e}^{-3t} \sin(t)
$$

Which simplifies to

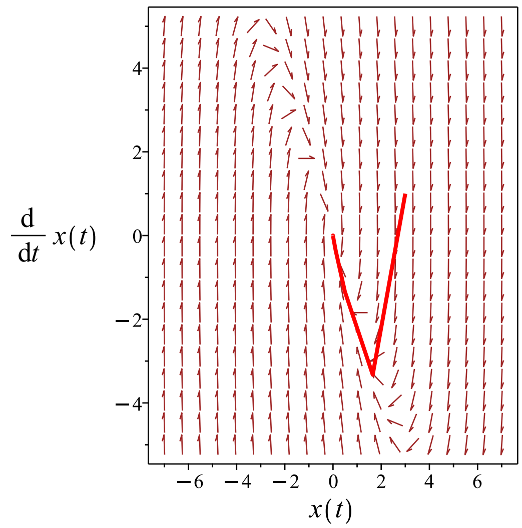
$$
x = \mathrm{e}^{-3t} (3 \cos(t) + 10 \sin(t))
$$

<u>Summary</u>
The solution(s) found are the following

$$
x = \mathrm{e}^{-3t} (3 \cos(t) + 10 \sin(t)) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{-3t}(3\cos(t) + 10\sin(t))$$

Verified OK.

6.10.4 Maple step by step solution

Let's solve

$$\left[x'' + 6x' + 10x = 0, x(0) = 3, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + 6r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - I, -3 + I)$

- 1st solution of the ODE

$x_1(t) = \mathrm{e}^{-3t} \cos(t)$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-3t} \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^{-3t} \cos(t) + c_2 \mathrm{e}^{-3t} \sin(t)$

□ Check validity of solution $x = c_1 \mathrm{e}^{-3t} \cos(t) + c_2 \mathrm{e}^{-3t} \sin(t)$

  - Use initial condition $x(0) = 3$

    $3 = c_1$

  - Compute derivative of the solution

    $x' = -3c_1 \mathrm{e}^{-3t} \cos(t) - c_1 \mathrm{e}^{-3t} \sin(t) - 3c_2 \mathrm{e}^{-3t} \sin(t) + c_2 \mathrm{e}^{-3t} \cos(t)$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 1$

    $1 = -3c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = 10\}$

  - Substitute constant values into general solution and simplify

    $x = \mathrm{e}^{-3t}(3\cos(t) + 10\sin(t))$

- Solution to the IVP

  $x = \mathrm{e}^{-3t}(3\cos(t) + 10\sin(t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([diff(x(t),t$2)+6*diff(x(t),t)+10*x(t)=0,x(0) = 3, D(x)(0) = 1],x(t), singsol=all)
```

$$x(t) = e^{-3t}(10\sin(t) + 3\cos(t))$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 20

```
DSolve[{x''[t]+6*x'[t]+10*x[t]==0,{x[0]==3,x'[0]==1}},x[t],t,IncludeSingularSolutions -> Tru
```

$$x(t) \to e^{-3t}(10\sin(t) + 3\cos(t))$$

6.11 problem 12.1 (xi)

Internal problem ID [12024]
Internal file name [OUTPUT/10676_Sunday_September_03_2023_12_35_39_PM_28976230/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (xi).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4x'' - 20x' + 21x = 0}$$

With initial conditions

$$[x(0) = -4, x'(0) = -12]$$

6.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -5 \\ q(t) &= \frac{21}{4} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - 5x' + \frac{21x}{4} = 0$$

The domain of $p(t) = -5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{21}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.11.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 4, B = -20, C = 21$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda t} - 20\lambda\, e^{\lambda t} + 21\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$4\lambda^2 - 20\lambda + 21 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = -20, C = 21$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{20}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{-20^2 - (4)(4)(21)} \\ &= \frac{5}{2} \pm 1\end{aligned}$$

Hence

$$\lambda_1 = \frac{5}{2} + 1$$

$$\lambda_2 = \frac{5}{2} - 1$$

Which simplifies to

$$\lambda_1 = \frac{7}{2}$$
$$\lambda_2 = \frac{3}{2}$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{\left(\frac{7}{2}\right)t} + c_2 e^{\left(\frac{3}{2}\right)t}$$

Or

$$x = c_1 \mathrm{e}^{\frac{7t}{2}} + c_2 \mathrm{e}^{\frac{3t}{2}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{\frac{7t}{2}} + c_2 \mathrm{e}^{\frac{3t}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -4$ and $t = 0$ in the above gives

$$-4 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{7c_1 \mathrm{e}^{\frac{7t}{2}}}{2} + \frac{3c_2 \mathrm{e}^{\frac{3t}{2}}}{2}$$

substituting $x' = -12$ and $t = 0$ in the above gives

$$-12 = \frac{7c_1}{2} + \frac{3c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -3$$
$$c_2 = -1$$

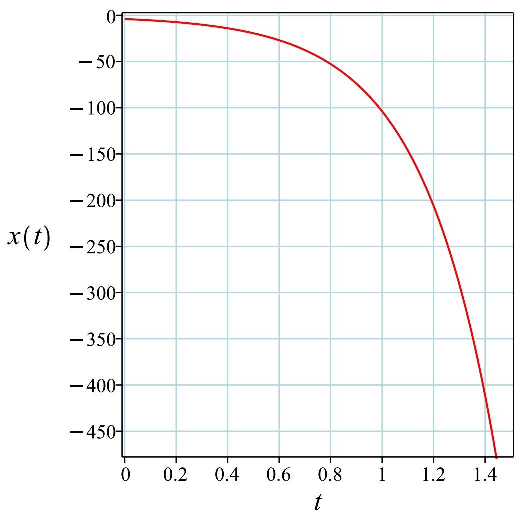
Substituting these values back in above solution results in

$$x = -3\,\mathrm{e}^{\frac{7t}{2}} - \mathrm{e}^{\frac{3t}{2}}$$

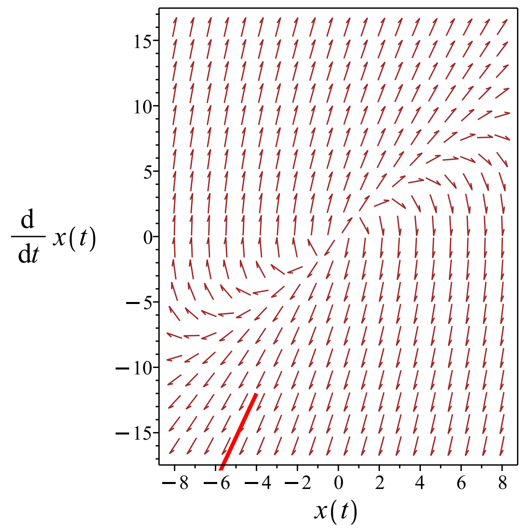
Summary

The solution(s) found are the following

$$x = -3\,\mathrm{e}^{\frac{7t}{2}} - \mathrm{e}^{\frac{3t}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -3\,\mathrm{e}^{\frac{7t}{2}} - \mathrm{e}^{\frac{3t}{2}}$$

Verified OK.

6.11.3 Solving using Kovacic algorithm

Writing the ode as

$$4x'' - 20x' + 21x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= -20 \\ C &= 21 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 101: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20}{4}\,dt} \\
&= z_1 e^{\frac{5t}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{5t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{\frac{3t}{2}}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-20}{4}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{5t}}{(x_1)^2}\,dt \\
&= x_1\left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned}x &= c_1 x_1 + c_2 x_2 \\ &= c_1\left(\mathrm{e}^{\frac{3t}{2}}\right) + c_2\left(\mathrm{e}^{\frac{3t}{2}}\left(\frac{\mathrm{e}^{2t}}{2}\right)\right)\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{\frac{3t}{2}} + \frac{c_2 \mathrm{e}^{\frac{7t}{2}}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -4$ and $t = 0$ in the above gives

$$-4 = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{3c_1 \mathrm{e}^{\frac{3t}{2}}}{2} + \frac{7c_2 \mathrm{e}^{\frac{7t}{2}}}{4}$$

substituting $x' = -12$ and $t = 0$ in the above gives

$$-12 = \frac{3c_1}{2} + \frac{7c_2}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= -1 \\ c_2 &= -6\end{aligned}$$

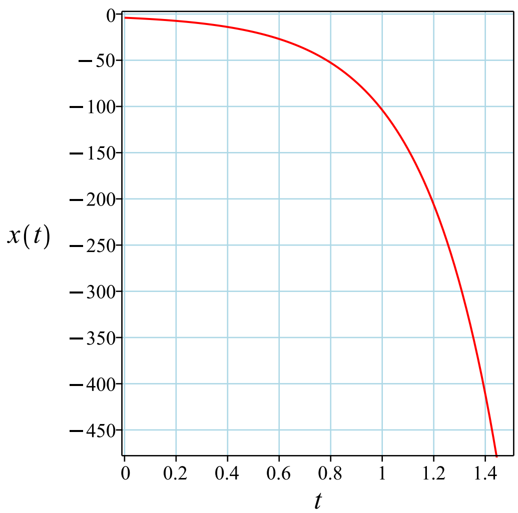
Substituting these values back in above solution results in

$$x = -3\,\mathrm{e}^{\frac{7t}{2}} - \mathrm{e}^{\frac{3t}{2}}$$

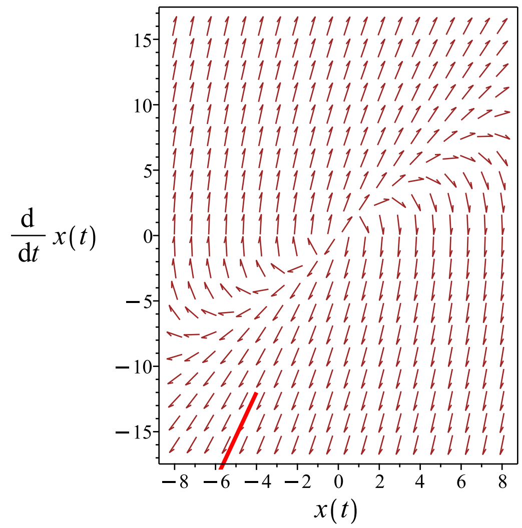
Summary

The solution(s) found are the following

$$x = -3\,\mathrm{e}^{\frac{7t}{2}} - \mathrm{e}^{\frac{3t}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -3\,\mathrm{e}^{\frac{7t}{2}} - \mathrm{e}^{\frac{3t}{2}}$$

Verified OK.

6.11.4 Maple step by step solution

Let's solve

$$\left[4x'' - 20x' + 21x = 0, x(0) = -4, x'\Big|_{\{t=0\}} = -12\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = 5x' - \frac{21x}{4}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' - 5x' + \frac{21x}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - 5r + \frac{21}{4} = 0$

- Factor the characteristic polynomial

$\frac{(2r-3)(2r-7)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{3}{2}, \frac{7}{2}\right)$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{\frac{3t}{2}}$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{\frac{7t}{2}}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^{\frac{3t}{2}} + c_2 \mathrm{e}^{\frac{7t}{2}}$

☐ Check validity of solution $x = c_1 \mathrm{e}^{\frac{3t}{2}} + c_2 \mathrm{e}^{\frac{7t}{2}}$

  - Use initial condition $x(0) = -4$

    $-4 = c_1 + c_2$

  - Compute derivative of the solution

    $x' = \frac{3c_1 \mathrm{e}^{\frac{3t}{2}}}{2} + \frac{7c_2 \mathrm{e}^{\frac{7t}{2}}}{2}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = -12$

    $-12 = \frac{3c_1}{2} + \frac{7c_2}{2}$

  - Solve for c_1 and c_2

    $\{c_1 = -1, c_2 = -3\}$

  - Substitute constant values into general solution and simplify

    $x = -3\,\mathrm{e}^{\frac{7t}{2}} - \mathrm{e}^{\frac{3t}{2}}$

- Solution to the IVP

  $x = -3\,\mathrm{e}^{\frac{7t}{2}} - \mathrm{e}^{\frac{3t}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([4*diff(x(t),t$2)-20*diff(x(t),t)+21*x(t)=0,x(0) = -4, D(x)(0) = -12],x(t), singsol=a
```

$$x(t) = -e^{\frac{3t}{2}} - 3\,e^{\frac{7t}{2}}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 23

```
DSolve[{4*x''[t]-20*x'[t]+21*x[t]==0,{x[0]==-4,x'[0]==-12}},x[t],t,IncludeSingularSolutions
```

$$x(t) \to -e^{3t/2}\left(3e^{2t} + 1\right)$$

6.12 problem 12.1 (xii)

Internal problem ID [12025]
Internal file name [`OUTPUT/10677_Sunday_September_03_2023_12_35_40_PM_28316033/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (xii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - 2y = 0}$$

With initial conditions

$$[y(0) = 4, y'(0) = -4]$$

6.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= -2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y' - 2y = 0$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.12.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \lambda\, e^{\lambda t} - 2\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-2)}\\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{3}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{(1)t} + c_2 e^{(-2)t}$$

Or

$$y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-2t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $t = 0$ in the above gives

$$4 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \mathrm{e}^t - 2c_2 \mathrm{e}^{-2t}$$

substituting $y' = -4$ and $t = 0$ in the above gives

$$-4 = c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{4}{3}$$
$$c_2 = \frac{8}{3}$$

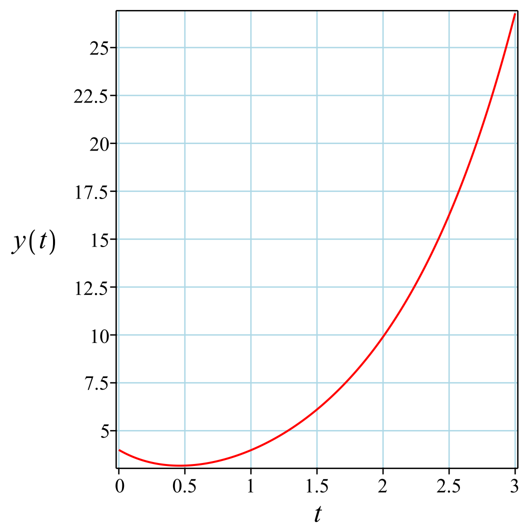
Substituting these values back in above solution results in

$$y = \frac{4\,\mathrm{e}^t}{3} + \frac{8\,\mathrm{e}^{-2t}}{3}$$

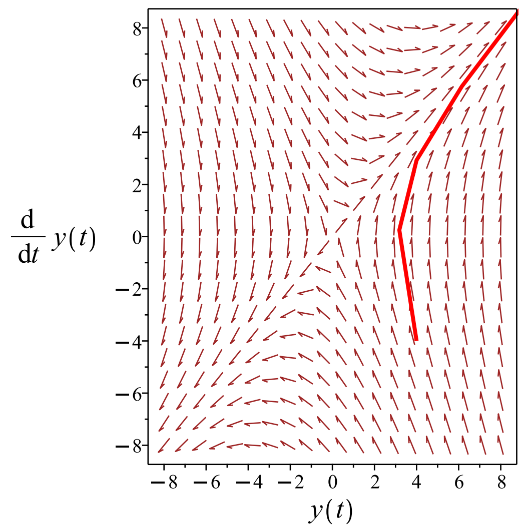
Summary

The solution(s) found are the following

$$y = \frac{4\,\mathrm{e}^{t}}{3} + \frac{8\,\mathrm{e}^{-2t}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4\,\mathrm{e}^{t}}{3} + \frac{8\,\mathrm{e}^{-2t}}{3}$$

Verified OK.

6.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{9z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 103: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{3t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\ &= z_1 e^{-\frac{t}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{-t}}{(y_1)^2} \, dt \\ &= y_1 \left(\frac{\mathrm{e}^{3t}}{3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2t} \right) + c_2 \left(\mathrm{e}^{-2t} \left(\frac{\mathrm{e}^{3t}}{3} \right) \right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-2t} + \frac{c_2 \mathrm{e}^{t}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $t = 0$ in the above gives

$$4 = c_1 + \frac{c_2}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \mathrm{e}^{-2t} + \frac{c_2 \mathrm{e}^{t}}{3}$$

substituting $y' = -4$ and $t = 0$ in the above gives

$$-4 = -2c_1 + \frac{c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

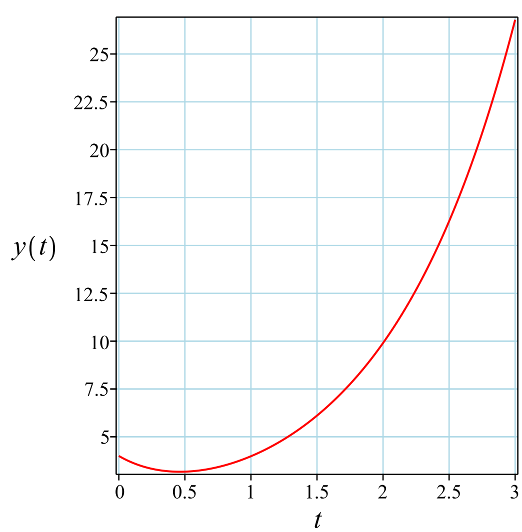
$$c_1 = \frac{8}{3}$$
$$c_2 = 4$$

Substituting these values back in above solution results in

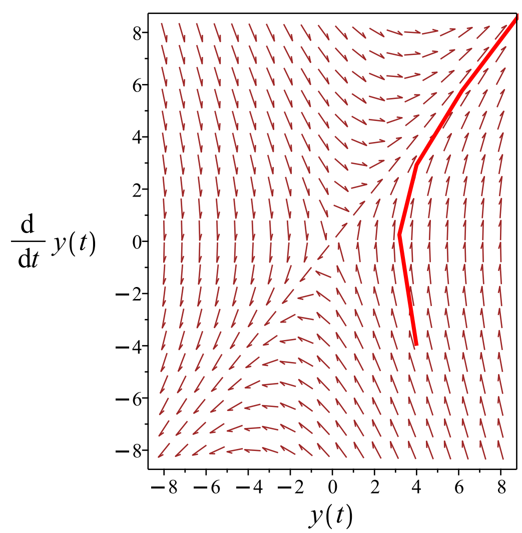
$$y = \frac{4\,\mathrm{e}^{t}}{3} + \frac{8\,\mathrm{e}^{-2t}}{3}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{4\,\mathrm{e}^{t}}{3} + \frac{8\,\mathrm{e}^{-2t}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{4\,\mathrm{e}^{t}}{3} + \frac{8\,\mathrm{e}^{-2t}}{3}$$

Verified OK.

6.12.4 Maple step by step solution

Let's solve

$$\left[y''+y'-2y=0, y(0)=4, y'\Big|_{\{t=0\}}=-4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2+r-2=0$

- Factor the characteristic polynomial

  $(r+2)(r-1)=0$

- Roots of the characteristic polynomial

  $r=(-2,1)$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-2t}$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{t}$

☐ Check validity of solution $y=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{t}$

- ○ Use initial condition $y(0)=4$

  $4=c_1+c_2$

- ○ Compute derivative of the solution

  $y'=-2c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{t}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}}=-4$

  $-4=-2c_1+c_2$

- ○ Solve for c_1 and c_2

  $\left\{c_1=\frac{8}{3}, c_2=\frac{4}{3}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{4(e^{3t}+2)e^{-2t}}{3}$

- Solution to the IVP

  $y = \frac{4(e^{3t}+2)e^{-2t}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$2)+diff(y(t),t)-2*y(t)=0,y(0) = 4, D(y)(0) = -4],y(t), singsol=all)
```

$$y(t) = \frac{4(e^{3t} + 2)\, e^{-2t}}{3}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 21

```
DSolve[{y''[t]+y'[t]-2*y[t]==0,{y[0]==4,y'[0]==-4}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{4}{3} e^{-2t} \left(e^{3t} + 2\right)$$

6.13 problem 12.1 (xiii)

Internal problem ID [12026]
Internal file name [`OUTPUT/10678_Sunday_September_03_2023_12_35_42_PM_80398571/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (xiii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 4y = 0}$$

With initial conditions

$$[y(0) = 10, y'(0) = 0]$$

6.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 4y = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.13.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 4\,e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\lambda_1 = +2$$
$$\lambda_2 = -2$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{(2)t} + c_2 e^{(-2)t}$$

Or

$$y = c_1 e^{2t} + c_2 e^{-2t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2t} + c_2 e^{-2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $t = 0$ in the above gives

$$10 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2t} - 2c_2 e^{-2t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = 2c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5$$
$$c_2 = 5$$

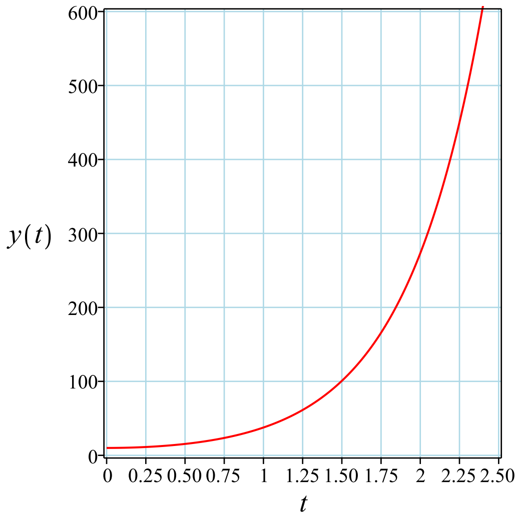
Substituting these values back in above solution results in

$$y = 5\,e^{2t} + 5\,e^{-2t}$$

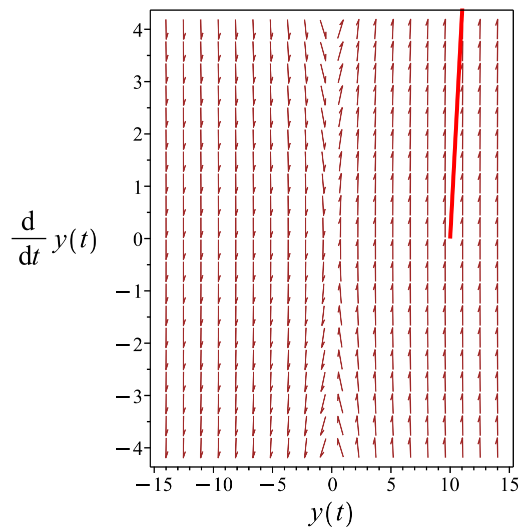
Summary

The solution(s) found are the following

$$y = 5\,\mathrm{e}^{2t} + 5\,\mathrm{e}^{-2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 5\,\mathrm{e}^{2t} + 5\,\mathrm{e}^{-2t}$$

Verified OK.

6.13.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - 4y'y = 0$$

Integrating the above w.r.t t gives

$$\int (y'y'' - 4y'y)\,dt = 0$$

$$\frac{y'^2}{2} - 2y^2 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{4y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{4y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{4y^2+2c_1}}dy = \int dt$$
$$\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4} = t+c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4}} = e^{t+c_2}$$

Which simplifies to

$$\sqrt{2y+\sqrt{4y^2+2c_1}} = c_3 e^t$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{4y^2+2c_1}}dy = \int dt$$
$$-\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4} = t+c_4$$

Raising both side to exponential gives

$$e^{-\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4}} = e^{t+c_4}$$

Which simplifies to

$$\frac{1}{\sqrt{2y+\sqrt{4y^2+2c_1}}} = c_5 e^t$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = \frac{\left(c_3^4 e^{4t}-2c_1\right)e^{-2t}}{4c_3^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $t = 0$ in the above gives

$$10 = \frac{c_3^4 - 2c_1}{4c_3^2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{(c_3^4 e^{4t} - 2c_1)\, e^{-2t}}{2c_3^2} + c_3^2 e^{2t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{c_3^4 + 2c_1}{2c_3^2} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Looking at the Second solution

$$y = -\frac{(2c_1 c_5^4 e^{4t} - 1)\, e^{-2t}}{4c_5^2} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $t = 0$ in the above gives

$$10 = \frac{-2c_1 c_5^4 + 1}{4c_5^2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_5^2 e^{2t} c_1 + \frac{(2c_1 c_5^4 e^{4t} - 1)\, e^{-2t}}{2c_5^2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{-2c_1 c_5^4 - 1}{2c_5^2} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_5\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

6.13.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-2t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{-2t} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dt}}{y_1^2} dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dt \\ &= e^{-2t} \int \frac{1}{e^{-4t}} dt \\ &= e^{-2t} \left(\frac{e^{4t}}{4} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-2t} \right) + c_2 \left(e^{-2t} \left(\frac{e^{4t}}{4} \right) \right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2t} + \frac{c_2 e^{2t}}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $t = 0$ in the above gives

$$10 = c_1 + \frac{c_2}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2t} + \frac{c_2 e^{2t}}{2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5$$
$$c_2 = 20$$

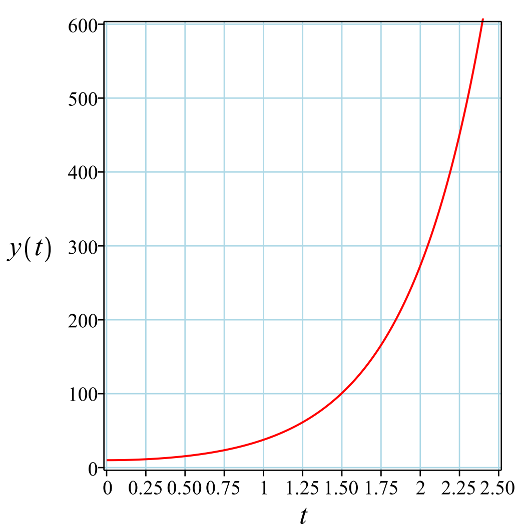
Substituting these values back in above solution results in

$$y = 5\,e^{2t} + 5\,e^{-2t}$$

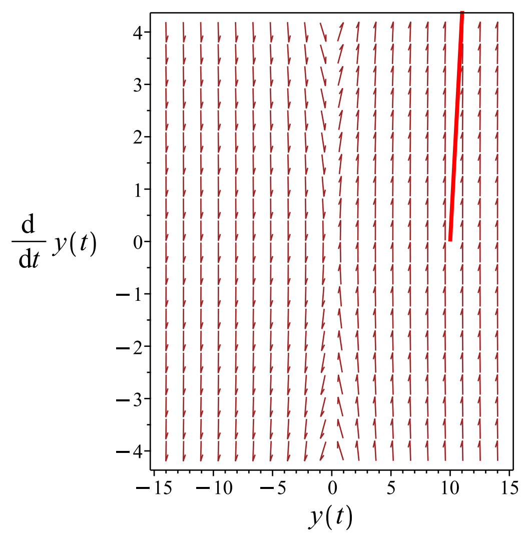
Summary

The solution(s) found are the following

$$y = 5\,e^{2t} + 5\,e^{-2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 5\,e^{2t} + 5\,e^{-2t}$$

Verified OK.

6.13.5 Maple step by step solution

Let's solve

$\left[y''-4y=0, y(0)=10, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2-4=0$

- Factor the characteristic polynomial

  $(r-2)(r+2)=0$

- Roots of the characteristic polynomial

  $r=(-2, 2)$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-2t}$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{2t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{2t}$

□ Check validity of solution $y=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{2t}$

  ◦ Use initial condition $y(0)=10$

    $10=c_1+c_2$

  ◦ Compute derivative of the solution

    $y'=-2c_1\mathrm{e}^{-2t}+2c_2\mathrm{e}^{2t}$

  ◦ Use the initial condition $y'\Big|_{\{t=0\}}=0$

    $0=-2c_1+2c_2$

  ◦ Solve for c_1 and c_2

    $\{c_1=5, c_2=5\}$

- Substitute constant values into general solution and simplify

  $$y = 5\,e^{2t} + 5\,e^{-2t}$$

- Solution to the IVP

  $$y = 5\,e^{2t} + 5\,e^{-2t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve([diff(y(t),t$2)-4*y(t)=0,y(0) = 10, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = 5\,e^{2t} + 5\,e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 19

```
DSolve[{y''[t]-4*y[t]==0,{y[0]==10,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 5e^{-2t}\left(e^{4t} + 1\right)$$

6.14 problem 12.1 (xiv)

Internal problem ID [12027]
Internal file name [`OUTPUT/10679_Sunday_September_03_2023_12_35_43_PM_25114971/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (xiv).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y' + 4y = 0}$$

With initial conditions

$$[y(0) = 27, y'(0) = -54]$$

6.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 4y = 0$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.14.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda\, e^{\lambda t} + 4\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(4)^2 - (4)\,(1)\,(4)}\\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 e^{-2t} + c_2 t\, e^{-2t} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-2t} + c_2 t\,\mathrm{e}^{-2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 27$ and $t = 0$ in the above gives

$$27 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-2t} - 2c_2 t\,\mathrm{e}^{-2t}$$

substituting $y' = -54$ and $t = 0$ in the above gives

$$-54 = -2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 27$$
$$c_2 = 0$$

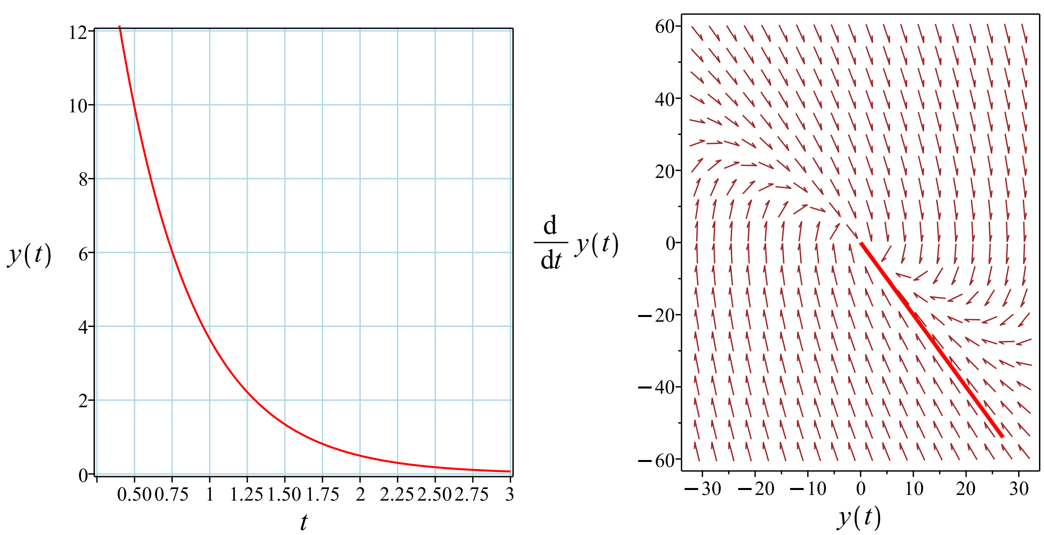
Substituting these values back in above solution results in

$$y = 27\,\mathrm{e}^{-2t}$$

Summary

The solution(s) found are the following

$$y = 27\,\mathrm{e}^{-2t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 27\,\mathrm{e}^{-2t}$$

Verified OK.

6.14.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\,y' + \frac{\left(p(t)^2 + p'(t)\right)y}{2} = f(t)$$

Where $p(t) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int 4\,dx} \\
&= \mathrm{e}^{2t}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\mathrm{e}^{2t}y\right)'' &= 0
\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{2t}y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{2t}y\right) = c_1t + c_2$$

Hence the solution is

$$y = \frac{c_1t + c_2}{\mathrm{e}^{2t}}$$

Or

$$y = c_1t\,\mathrm{e}^{-2t} + c_2\mathrm{e}^{-2t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1t\,\mathrm{e}^{-2t} + c_2\mathrm{e}^{-2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 27$ and $t = 0$ in the above gives

$$27 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \mathrm{e}^{-2t} - 2c_1 t\, \mathrm{e}^{-2t} - 2c_2 \mathrm{e}^{-2t}$$

substituting $y' = -54$ and $t = 0$ in the above gives

$$-54 = c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

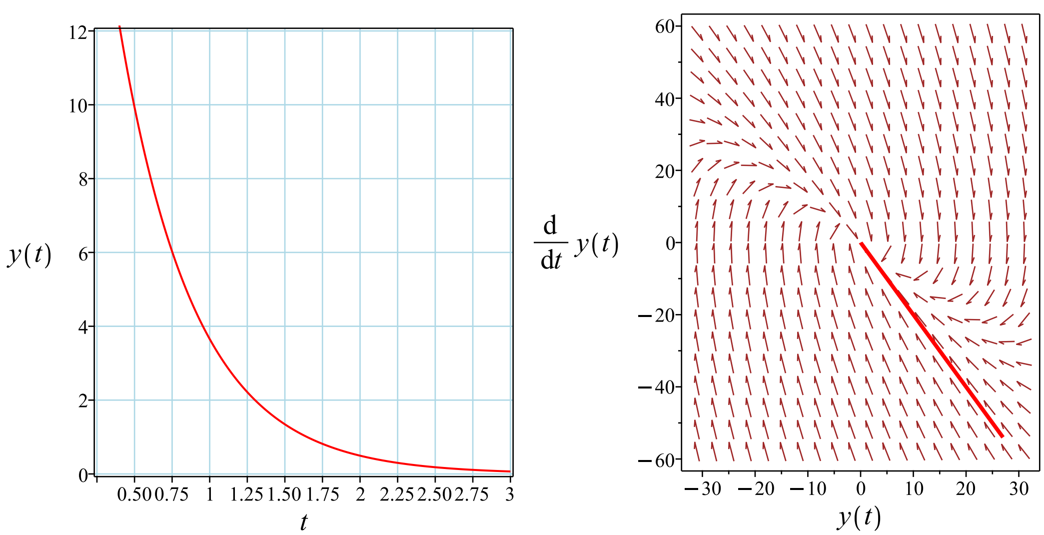
$$c_1 = 0$$
$$c_2 = 27$$

Substituting these values back in above solution results in

$$y = 27\, \mathrm{e}^{-2t}$$

Summary
The solution(s) found are the following

$$y = 27\, \mathrm{e}^{-2t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 27\,\mathrm{e}^{-2t}$$

Verified OK.

6.14.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\
&= z_1 e^{-2t} \\
&= z_1\left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2t}\right) + c_2\left(\mathrm{e}^{-2t}(t)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2t} + c_2 t\, \mathrm{e}^{-2t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 27$ and $t = 0$ in the above gives

$$
27 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t \, e^{-2t}$$

substituting $y' = -54$ and $t = 0$ in the above gives

$$-54 = -2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

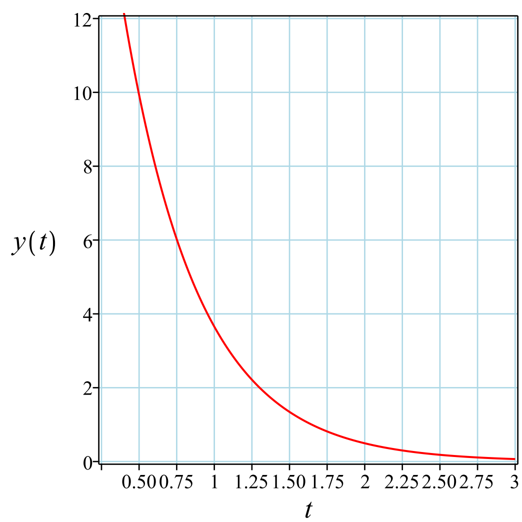
$$c_1 = 27$$
$$c_2 = 0$$

Substituting these values back in above solution results in

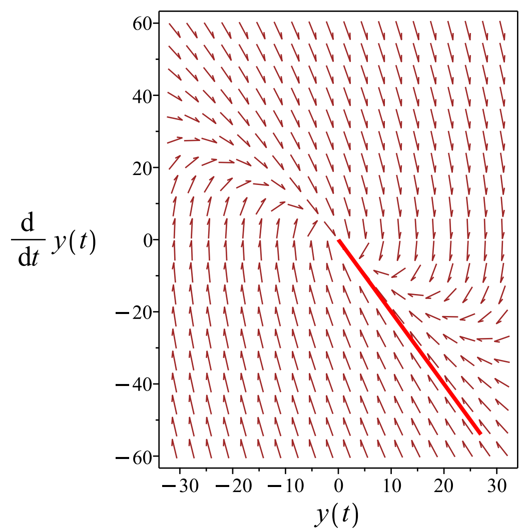
$$y = 27 \, e^{-2t}$$

Summary
The solution(s) found are the following

$$y = 27 \, e^{-2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 27 \, e^{-2t}$$

Verified OK.

6.14.5 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 4y = 0, y(0) = 27, y'\Big|_{\{t=0\}} = -54\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r+2)^2 = 0$

- Root of the characteristic polynomial

  $r = -2$

- 1st solution of the ODE

  $y_1(t) = e^{-2t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{-2t} + c_2 t\,e^{-2t}$

□ Check validity of solution $y = c_1 e^{-2t} + c_2 t e^{-2t}$

  ◦ Use initial condition $y(0) = 27$

    $27 = c_1$

  ◦ Compute derivative of the solution

    $y' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t\,e^{-2t}$

  ◦ Use the initial condition $y'\Big|_{\{t=0\}} = -54$

    $-54 = -2c_1 + c_2$

  ◦ Solve for c_1 and c_2

    $\{c_1 = 27, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = 27\,\mathrm{e}^{-2t}$

- Solution to the IVP

  $y = 27\,\mathrm{e}^{-2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+4*y(t)=0,y(0) = 27, D(y)(0) = -54],y(t), singsol=all)
```

$$y(t) = 27\,\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 12

```
DSolve[{y''[t]+4*y'[t]+4*y[t]==0,{y[0]==27,y'[0]==-54}},y[t],t,IncludeSingularSolutions -> T
```

$$y(t) \to 27e^{-2t}$$

6.15 problem 12.1 (xv)

Internal problem ID [12028]
Internal file name [`OUTPUT/10680_Sunday_September_03_2023_12_35_45_PM_96730303/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 12, Homogeneous second order linear equations. Exercises page 118
Problem number: 12.1 (xv).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + \omega^2 y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

6.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= \omega^2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \omega^2 y = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \omega^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.15.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = \omega^2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \omega^2 \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \omega^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \omega^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\omega^2)} \\ &= \pm\sqrt{-\omega^2}\end{aligned}$$

Hence

$$\lambda_1 = +\sqrt{-\omega^2}$$
$$\lambda_2 = -\sqrt{-\omega^2}$$

Which simplifies to

$$\lambda_1 = \sqrt{-\omega^2}$$
$$\lambda_2 = -\sqrt{-\omega^2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{\left(\sqrt{-\omega^2}\right)t} + c_2 e^{\left(-\sqrt{-\omega^2}\right)t}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t} - c_2 \sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

substituting $y' = 1$ and $t = 0$ in the above gives

$$1 = (c_1 - c_2)\sqrt{-\omega^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{\sqrt{-\omega^2}}{2\omega^2}$$
$$c_2 = \frac{\sqrt{-\omega^2}}{2\omega^2}$$

Substituting these values back in above solution results in

$$y = \frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} - \sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t}}{2\omega^2}$$

Which simplifies to

$$y = -\frac{\sqrt{-\omega^2}\left(-e^{-\sqrt{-\omega^2}\,t} + e^{\sqrt{-\omega^2}\,t}\right)}{2\omega^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{-\omega^2}\left(-e^{-\sqrt{-\omega^2}\,t} + e^{\sqrt{-\omega^2}\,t}\right)}{2\omega^2} \tag{1}$$

Verification of solutions

$$y = -\frac{\sqrt{-\omega^2}\left(-e^{-\sqrt{-\omega^2}\,t} + e^{\sqrt{-\omega^2}\,t}\right)}{2\omega^2}$$

Verified OK.

6.15.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + \omega^2 y'y = 0$$

Integrating the above w.r.t t gives

$$\int \left(y'y'' + \omega^2 y'y\right) dt = 0$$
$$\frac{y'^2}{2} + \frac{\omega^2 y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-\omega^2 y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-\omega^2 y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-\omega^2 y^2 + 2c_1}} dy = \int dt$$
$$\frac{\arctan\left(\frac{\sqrt{\omega^2}\,y}{\sqrt{-\omega^2 y^2 + 2c_1}}\right)}{\sqrt{\omega^2}} = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-\omega^2 y^2+2c_1}}dy = \int dt$$

$$-\frac{\arctan\left(\frac{\sqrt{\omega^2}\,y}{\sqrt{-\omega^2 y^2+2c_1}}\right)}{\sqrt{\omega^2}} = t + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{\sqrt{\omega^2}\,y}{\sqrt{-\omega^2 y^2+2c_1}}\right)}{\sqrt{\omega^2}} = t + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2\sqrt{2}\sqrt{\left(\tan\left(c_2\sqrt{\omega^2}+t\sqrt{\omega^2}\right)^2+1\right)c_1}\,\tan\left(c_2\sqrt{\omega^2}+t\sqrt{\omega^2}\right)^2\sqrt{\omega^2}}{\left(\tan\left(c_2\sqrt{\omega^2}+t\sqrt{\omega^2}\right)^2+1\right)\omega} + \frac{\sqrt{2}\tan\left(c_2\sqrt{\omega^2}+t\sqrt{\omega^2}\right)^2 \cdots}{\sqrt{\left(\tan\left(c_2\sqrt{\omega^2}+t\sqrt{\omega^2}\right)^2 \cdots}}$$

substituting $y' = 1$ and $t = 0$ in the above gives

$$1 = \left(-\cos(c_2\omega)^2 + 2\cos(c_2\,\mathrm{csgn}(\omega)\,\omega)^2\right)\sqrt{\sec(c_2\,\mathrm{csgn}(\omega)\,\omega)^2 c_1}\,\sqrt{2}\,\mathrm{csgn}(\omega) \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$\frac{\arctan\left(\frac{\sqrt{\omega^2}\,y}{\sqrt{-\omega^2 y^2+1}}\right)}{\sqrt{\omega^2}} = t$$

The above simplifies to

$$-t\sqrt{\omega^2} + \arctan\left(\frac{\sqrt{\omega^2}\, y}{\sqrt{-\omega^2 y^2 + 1}}\right) = 0$$

Which can be written as

$$\operatorname{csgn}(\omega)\left(-\omega t + \arctan\left(\frac{\omega y}{\sqrt{-\omega^2 y^2 + 1}}\right)\right) = 0$$

Looking at the Second solution

$$-\frac{\arctan\left(\frac{\sqrt{\omega^2}\, y}{\sqrt{-\omega^2 y^2 + 2c_1}}\right)}{\sqrt{\omega^2}} = t + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2\sqrt{2}\sqrt{\left(\tan\left(c_3\sqrt{\omega^2} + t\sqrt{\omega^2}\right)^2 + 1\right) c_1}\, \tan\left(c_3\sqrt{\omega^2} + t\sqrt{\omega^2}\right)^2 \sqrt{\omega^2}}{\left(\tan\left(c_3\sqrt{\omega^2} + t\sqrt{\omega^2}\right)^2 + 1\right)\omega} - \frac{\sqrt{2}\, \tan\left(c_3\sqrt{\omega^2} + t\sqrt{\omega^2}\right)^2 \sqrt{}}{\sqrt{\left(\tan\left(c_3\sqrt{\omega^2} + t\sqrt{\omega^2}\right)^2 + \right.}}$$

substituting $y' = 1$ and $t = 0$ in the above gives

$$1 = \left(\cos(c_3\omega)^2 - 2\cos(c_3 \operatorname{csgn}(\omega)\,\omega)^2\right)\sqrt{\sec(c_3 \operatorname{csgn}(\omega)\,\omega)^2 c_1}\,\sqrt{2}\, \operatorname{csgn}(\omega) \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_3 = 0$$

Substituting these values back in above solution results in

$$-\frac{\arctan\left(\frac{\sqrt{\omega^2}\, y}{\sqrt{-\omega^2 y^2 + 1}}\right)}{\sqrt{\omega^2}} = t$$

The above simplifies to

$$-t\sqrt{\omega^2}-\arctan\left(\frac{\sqrt{\omega^2}\, y}{\sqrt{-\omega^2 y^2+1}}\right)=0$$

Which can be written as

$$\operatorname{csgn}\left(\omega\right)\left(-\omega t-\arctan\left(\frac{\omega y}{\sqrt{-\omega^2 y^2+1}}\right)\right)=0$$

Summary

The solution(s) found are the following

$$\operatorname{csgn}\left(\omega\right)\left(-\omega t+\arctan\left(\frac{\omega y}{\sqrt{-\omega^2 y^2+1}}\right)\right)=0 \tag{1}$$

$$\operatorname{csgn}\left(\omega\right)\left(-\omega t-\arctan\left(\frac{\omega y}{\sqrt{-\omega^2 y^2+1}}\right)\right)=0 \tag{2}$$

Verification of solutions

$$\operatorname{csgn}\left(\omega\right)\left(-\omega t+\arctan\left(\frac{\omega y}{\sqrt{-\omega^2 y^2+1}}\right)\right)=0$$

Verified OK.

$$\operatorname{csgn}\left(\omega\right)\left(-\omega t-\arctan\left(\frac{\omega y}{\sqrt{-\omega^2 y^2+1}}\right)\right)=0$$

Verified OK.

6.15.4 Solving using Kovacic algorithm

Writing the ode as

$$y''+\omega^2 y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1\\ B &= 0\\ C &= \omega^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\omega^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\omega^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\omega^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\omega^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = e^{\sqrt{-\omega^2}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= e^{\sqrt{-\omega^2}\, t}
\end{aligned}$$

Which simplifies to

$$y_1 = e^{\sqrt{-\omega^2}\, t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\
&= e^{\sqrt{-\omega^2}\, t} \int \frac{1}{e^{2\sqrt{-\omega^2}\, t}}\, dt \\
&= e^{\sqrt{-\omega^2}\, t} \left(\frac{\sqrt{-\omega^2}\, e^{-2\sqrt{-\omega^2}\, t}}{2\omega^2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\sqrt{-\omega^2}\, t}\right) + c_2 \left(\mathrm{e}^{\sqrt{-\omega^2}\, t} \left(\frac{\sqrt{-\omega^2}\, \mathrm{e}^{-2\sqrt{-\omega^2}\, t}}{2\omega^2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = \frac{2c_1\omega^2 + \sqrt{-\omega^2}\, c_2}{2\omega^2} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 \sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t} + \frac{c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2}
$$

substituting $y' = 1$ and $t = 0$ in the above gives

$$
1 = \sqrt{-\omega^2}\, c_1 + \frac{c_2}{2} \tag{2A}
$$

Equations $\{1\text{A}, 2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{\sqrt{-\omega^2}}{2\omega^2} \\
c_2 &= 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} - \sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t}}{2\omega^2}
$$

Which simplifies to

$$
y = -\frac{\sqrt{-\omega^2}\left(-\mathrm{e}^{-\sqrt{-\omega^2}\, t} + \mathrm{e}^{\sqrt{-\omega^2}\, t}\right)}{2\omega^2}
$$

Summary
The solution(s) found are the following

$$y = -\frac{\sqrt{-\omega^2}\left(-e^{-\sqrt{-\omega^2}\, t} + e^{\sqrt{-\omega^2}\, t}\right)}{2\omega^2} \tag{1}$$

Verification of solutions

$$y = -\frac{\sqrt{-\omega^2}\left(-e^{-\sqrt{-\omega^2}\, t} + e^{\sqrt{-\omega^2}\, t}\right)}{2\omega^2}$$

Verified OK.

6.15.5 Maple step by step solution

Let's solve

$$\left[y'' + \omega^2 y = 0, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $\omega^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4\omega^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-\omega^2}, -\sqrt{-\omega^2}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{\sqrt{-\omega^2}\, t}$

- 2nd solution of the ODE

  $y_2(t) = e^{-\sqrt{-\omega^2}\, t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t}$

☐ Check validity of solution $y = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$
- Compute derivative of the solution

  $y' = c_1\sqrt{-\omega^2}\,\mathrm{e}^{\sqrt{-\omega^2}\,t} - c_2\sqrt{-\omega^2}\,\mathrm{e}^{-\sqrt{-\omega^2}\,t}$
- Use the initial condition $y'\big|_{\{t=0\}} = 1$

  $1 = \sqrt{-\omega^2}\,c_1 - \sqrt{-\omega^2}\,c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{\sqrt{-\omega^2}}{2\omega^2}, c_2 = \frac{\sqrt{-\omega^2}}{2\omega^2}\right\}$
- Substitute constant values into general solution and simplify

  $y = -\frac{\sqrt{-\omega^2}\left(-\mathrm{e}^{-\sqrt{-\omega^2}\,t} + \mathrm{e}^{\sqrt{-\omega^2}\,t}\right)}{2\omega^2}$

- Solution to the IVP

  $y = -\frac{\sqrt{-\omega^2}\left(-\mathrm{e}^{-\sqrt{-\omega^2}\,t} + \mathrm{e}^{\sqrt{-\omega^2}\,t}\right)}{2\omega^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([diff(y(t),t$2)+omega^2*y(t)=0,y(0) = 0, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{\sin(\omega t)}{\omega}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 13

```
DSolve[{y''[t]+w^2*y[t]==0,{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow \frac{\sin(tw)}{w}$$

7 Chapter 14, Inhomogeneous second order linear equations. Exercises page 140

7.1 problem 14.1 (i)

Internal problem ID [12029]
Internal file name [`OUTPUT/10681_Sunday_September_03_2023_12_35_48_PM_34773728/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.1 (i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x'' - 4x = t^2}$$

7.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = -4, f(t) = t^2$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - 4x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 4\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{(2)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-2t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}, e^{2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_3 t^2 - 4A_2 t - 4A_1 + 2A_3 = t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = 0, A_3 = -\frac{1}{4}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{t^2}{4} - \frac{1}{8}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{2t} + c_2 e^{-2t}\right) + \left(-\frac{t^2}{4} - \frac{1}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{2t} + c_2 e^{-2t} - \frac{t^2}{4} - \frac{1}{8} \tag{1}$$

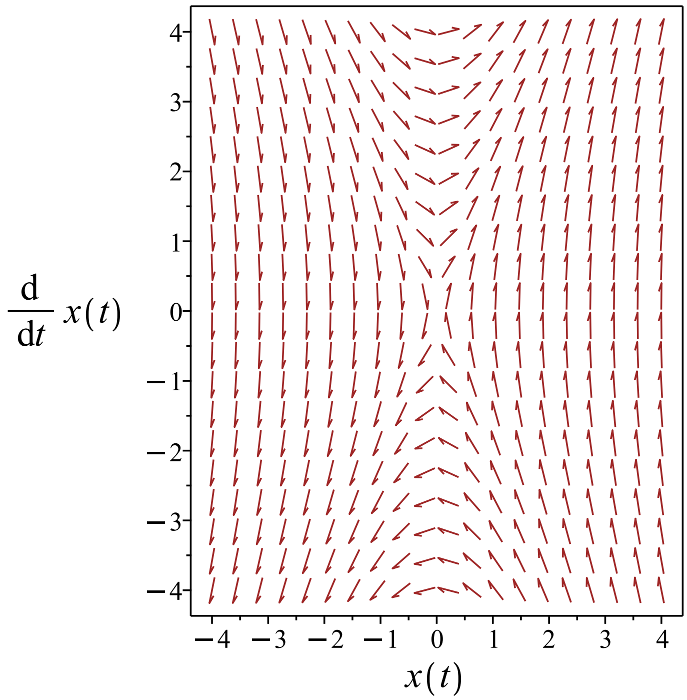


Figure 97: Slope field plot

Verification of solutions

$$x = c_1 e^{2t} + c_2 e^{-2t} - \frac{t^2}{4} - \frac{1}{8}$$

Verified OK.

7.1.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 4x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 111: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-2t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \mathrm{e}^{-2t}
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-2t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{x_1^2} \, dt
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2} \, dt \\
&= \mathrm{e}^{-2t} \int \frac{1}{\mathrm{e}^{-4t}} \, dt \\
&= \mathrm{e}^{-2t} \left(\frac{\mathrm{e}^{4t}}{4} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-2t}\right) + c_2 \left(\mathrm{e}^{-2t} \left(\frac{\mathrm{e}^{4t}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - 4x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-2t} + \frac{c_2 \mathrm{e}^{2t}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
t^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, t, t^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2t}}{4}, \mathrm{e}^{-2t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_3 t^2 + A_2 t + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_3t^2 - 4A_2t - 4A_1 + 2A_3 = t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = 0, A_3 = -\frac{1}{4}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{t^2}{4} - \frac{1}{8}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-2t} + \frac{c_2 e^{2t}}{4}\right) + \left(-\frac{t^2}{4} - \frac{1}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-2t} + \frac{c_2 e^{2t}}{4} - \frac{t^2}{4} - \frac{1}{8} \tag{1}$$

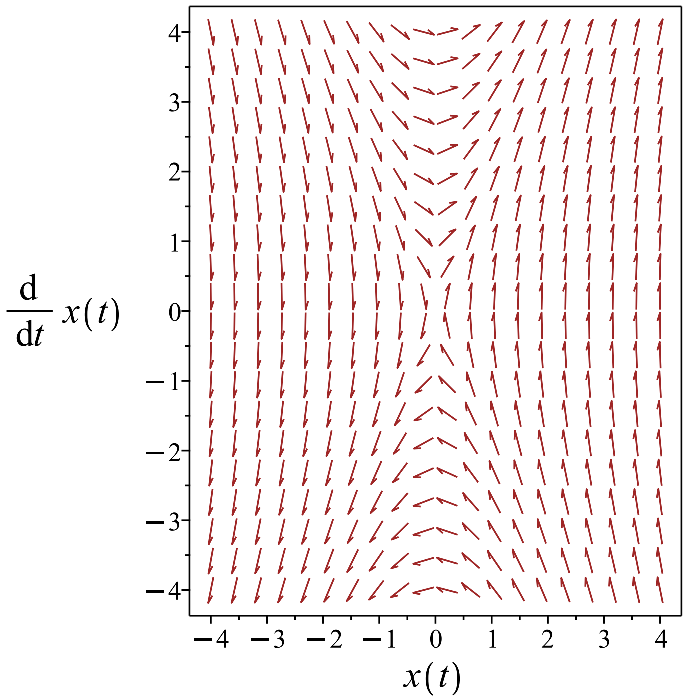


Figure 98: Slope field plot

Verification of solutions

$$x = c_1 e^{-2t} + \frac{c_2 e^{2t}}{4} - \frac{t^2}{4} - \frac{1}{8}$$

Verified OK.

7.1.3 Maple step by step solution

Let's solve

$x'' - 4x = t^2$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r - 2)(r + 2) = 0$

- Roots of the characteristic polynomial

$r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-2t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{2t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-2t} + c_2 e^{2t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = t^2\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-2t} & e^{2t} \\ -2\,e^{-2t} & 2\,e^{2t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = 4$

- ○ Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{e^{-2t}\left(\int e^{2t}t^2 dt\right)}{4} + \frac{e^{2t}\left(\int e^{-2t}t^2 dt\right)}{4}$

- ○ Compute integrals

  $x_p(t) = -\frac{t^2}{4} - \frac{1}{8}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{-2t} + c_2 e^{2t} - \frac{t^2}{4} - \frac{1}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(x(t),t$2)-4*x(t)=t^2,x(t), singsol=all)
```

$$x(t) = c_2 e^{2t} + e^{-2t} c_1 - \frac{t^2}{4} - \frac{1}{8}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 32

```
DSolve[x''[t]-4*x[t]==t^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{t^2}{4} + c_1 e^{2t} + c_2 e^{-2t} - \frac{1}{8}$$

7.2 problem 14.1 (ii)

Internal problem ID [12030]
Internal file name [`OUTPUT/10682_Sunday_September_03_2023_12_35_49_PM_63749259/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.1 (ii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x'' - 4x' = t^2}$$

7.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = -4, C = 0, f(t) = t^2$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''-4x'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t)+Bx'(t)+Cx(t)=0$$

Where in the above $A=1, B=-4, C=0$. Let the solution be $x=e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t}-4\lambda\,\mathrm{e}^{\lambda t}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2-4\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=-4, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{4}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{-4^2-(4)(1)(0)}\\&=2\pm 2\end{aligned}$$

Hence

$$\lambda_1=2+2$$
$$\lambda_2=2-2$$

Which simplifies to

$$\lambda_1=4$$
$$\lambda_2=0$$

Since roots are real and distinct, then the solution is

$$x=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}$$
$$x=c_1e^{(4)t}+c_2e^{(0)t}$$

Or

$$x = c_1 e^{4t} + c_2$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 e^{4t} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{4t}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t, t^2, t^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_3 t^3 + A_2 t^2 + A_1 t$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12t^2 A_3 - 8tA_2 + 6tA_3 - 4A_1 + 2A_2 = t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{32}, A_2 = -\frac{1}{16}, A_3 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{1}{12}t^3 - \frac{1}{16}t^2 - \frac{1}{32}t$$

Therefore the general solution is

$$\begin{aligned}x &= x_h + x_p \\ &= (c_1 e^{4t} + c_2) + \left(-\frac{1}{12}t^3 - \frac{1}{16}t^2 - \frac{1}{32}t\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{4t} + c_2 - \frac{t^3}{12} - \frac{t^2}{16} - \frac{t}{32} \tag{1}$$

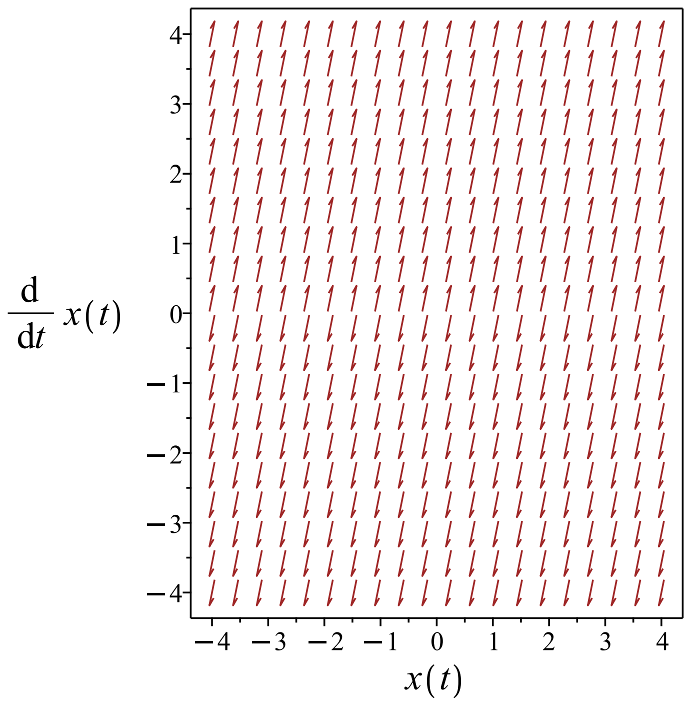


Figure 99: Slope field plot

Verification of solutions

$$x = c_1 e^{4t} + c_2 - \frac{t^3}{12} - \frac{t^2}{16} - \frac{t}{32}$$

Verified OK.

7.2.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' - 4x')\, dt = \int t^2 dt$$

$$-4x + x' = \frac{t^3}{3} + c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = -4$$
$$q(t) = \frac{t^3}{3} + c_1$$

Hence the ode is

$$-4x + x' = \frac{t^3}{3} + c_1$$

The integrating factor μ is

$$\mu = e^{\int (-4)dt}$$
$$= e^{-4t}$$

The ode becomes

$$\frac{d}{dt}(\mu x) = (\mu)\left(\frac{t^3}{3} + c_1\right)$$

$$\frac{d}{dt}\left(e^{-4t}x\right) = \left(e^{-4t}\right)\left(\frac{t^3}{3} + c_1\right)$$

$$d\left(e^{-4t}x\right) = \left(\frac{(t^3 + 3c_1)\, e^{-4t}}{3}\right) dt$$

Integrating gives

$$e^{-4t}x = \int \frac{(t^3 + 3c_1)\, e^{-4t}}{3}\, dt$$

$$e^{-4t}x = -\frac{(32t^3 + 24t^2 + 96c_1 + 12t + 3)\, e^{-4t}}{384} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-4t}$ results in

$$x = -\frac{e^{4t}(32t^3 + 24t^2 + 96c_1 + 12t + 3)\,e^{-4t}}{384} + c_2 e^{4t}$$

which simplifies to

$$x = -\frac{t^3}{12} - \frac{t^2}{16} - \frac{c_1}{4} - \frac{t}{32} - \frac{1}{128} + c_2 e^{4t}$$

Summary

The solution(s) found are the following

$$x = -\frac{t^3}{12} - \frac{t^2}{16} - \frac{c_1}{4} - \frac{t}{32} - \frac{1}{128} + c_2 e^{4t} \tag{1}$$

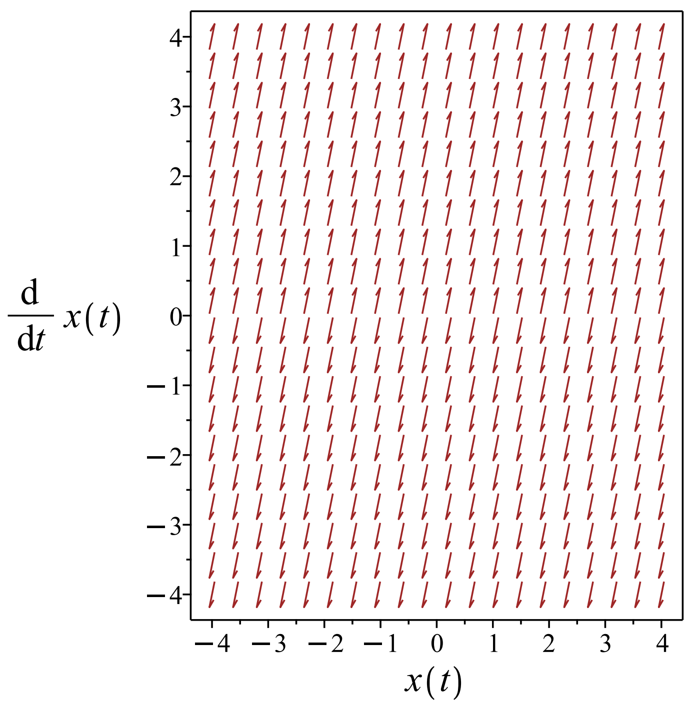


Figure 100: Slope field plot

Verification of solutions

$$x = -\frac{t^3}{12} - \frac{t^2}{16} - \frac{c_1}{4} - \frac{t}{32} - \frac{1}{128} + c_2 e^{4t}$$

Verified OK.

7.2.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) - 4p(t) - t^2 = 0$$

Which is now solve for $p(t)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(t) + p(t)p(t) = q(t)$$

Where here

$$\begin{aligned}p(t) &= -4\\ q(t) &= t^2\end{aligned}$$

Hence the ode is

$$p'(t) - 4p(t) = t^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-4)dt}\\ &= e^{-4t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu p) &= (\mu)\left(t^2\right)\\ \frac{d}{dt}\left(e^{-4t}p\right) &= \left(e^{-4t}\right)\left(t^2\right)\\ d\left(e^{-4t}p\right) &= \left(t^2 e^{-4t}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-4t}p &= \int t^2 e^{-4t}\, dt\\ e^{-4t}p &= -\frac{(8t^2 + 4t + 1)\, e^{-4t}}{32} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-4t}$ results in

$$p(t) = -\frac{e^{4t}(8t^2 + 4t + 1)\,e^{-4t}}{32} + c_1 e^{4t}$$

which simplifies to

$$p(t) = -\frac{t^2}{4} - \frac{t}{8} - \frac{1}{32} + c_1 e^{4t}$$

Since $p = x'$ then the new first order ode to solve is

$$x' = -\frac{t^2}{4} - \frac{t}{8} - \frac{1}{32} + c_1 e^{4t}$$

Integrating both sides gives

$$\begin{aligned} x &= \int -\frac{t^2}{4} - \frac{t}{8} - \frac{1}{32} + c_1 e^{4t}\, dt \\ &= -\frac{t^2}{16} - \frac{t}{32} - \frac{t^3}{12} + \frac{c_1 e^{4t}}{4} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$x = -\frac{t^2}{16} - \frac{t}{32} - \frac{t^3}{12} + \frac{c_1 e^{4t}}{4} + c_2 \tag{1}$$

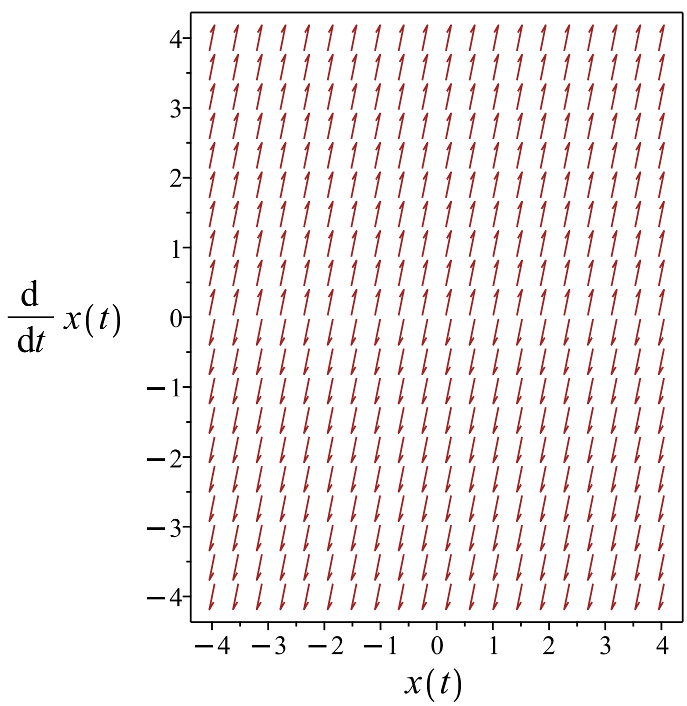


Figure 101: Slope field plot

Verification of solutions

$$x = -\frac{t^2}{16} - \frac{t}{32} - \frac{t^3}{12} + \frac{c_1 e^{4t}}{4} + c_2$$

Verified OK.

7.2.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x'' - 4x' = t^2$$

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' - 4x')\, dt = \int t^2 dt$$
$$-4x + x' = \frac{t^3}{3} + c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = -4$$
$$q(t) = \frac{t^3}{3} + c_1$$

Hence the ode is

$$-4x + x' = \frac{t^3}{3} + c_1$$

The integrating factor μ is

$$\mu = e^{\int (-4)dt}$$
$$= e^{-4t}$$

The ode becomes

$$\frac{d}{dt}(\mu x) = (\mu)\left(\frac{t^3}{3} + c_1\right)$$
$$\frac{d}{dt}\left(e^{-4t}x\right) = \left(e^{-4t}\right)\left(\frac{t^3}{3} + c_1\right)$$
$$d\left(e^{-4t}x\right) = \left(\frac{(t^3 + 3c_1)\, e^{-4t}}{3}\right) dt$$

Integrating gives

$$
\begin{aligned}
e^{-4t}x &= \int \frac{(t^3+3c_1)\,e^{-4t}}{3}\,dt \\
e^{-4t}x &= -\frac{(32t^3+24t^2+96c_1+12t+3)\,e^{-4t}}{384}+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-4t}$ results in

$$
x = -\frac{e^{4t}(32t^3+24t^2+96c_1+12t+3)\,e^{-4t}}{384}+c_2e^{4t}
$$

which simplifies to

$$
x = -\frac{t^3}{12}-\frac{t^2}{16}-\frac{c_1}{4}-\frac{t}{32}-\frac{1}{128}+c_2e^{4t}
$$

Summary

The solution(s) found are the following

$$
x = -\frac{t^3}{12}-\frac{t^2}{16}-\frac{c_1}{4}-\frac{t}{32}-\frac{1}{128}+c_2e^{4t} \tag{1}
$$

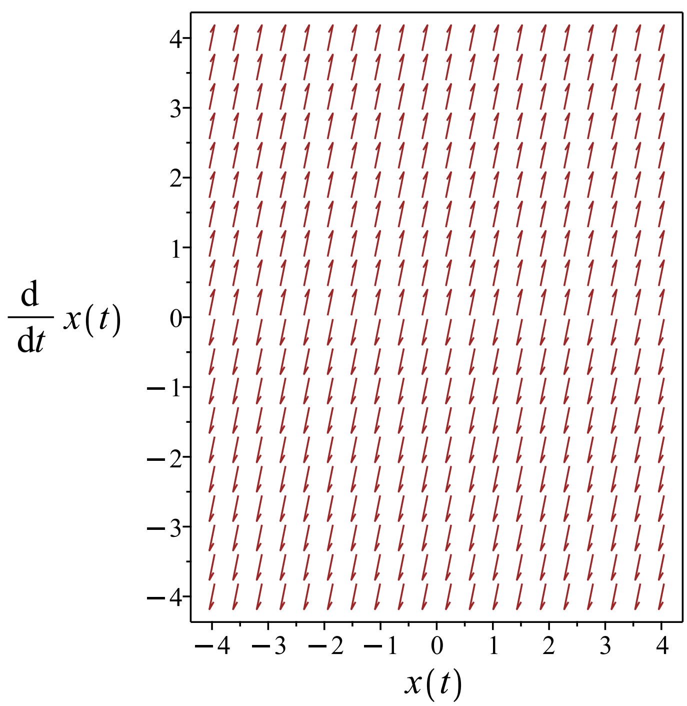


Figure 102: Slope field plot

Verification of solutions

$$x = -\frac{t^3}{12} - \frac{t^2}{16} - \frac{c_1}{4} - \frac{t}{32} - \frac{1}{128} + c_2 e^{4t}$$

Verified OK.

7.2.5 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 4x' = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 113: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-2t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dt} \\
&= z_1 e^{2t} \\
&= z_1\left(\mathrm{e}^{2t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = 1
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-4}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{4t}}{(x_1)^2}\,dt \\
&= x_1\left(\frac{\mathrm{e}^{4t}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(1) + c_2\left(1\left(\frac{\mathrm{e}^{4t}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$
x'' - 4x' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 + \frac{c_2 \mathrm{e}^{4t}}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, \frac{\mathrm{e}^{4t}}{4}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t, t^2, t^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_3 t^3 + A_2 t^2 + A_1 t$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12t^2 A_3 - 8t A_2 + 6t A_3 - 4A_1 + 2A_2 = t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{32}, A_2 = -\frac{1}{16}, A_3 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{1}{12}t^3 - \frac{1}{16}t^2 - \frac{1}{32}t$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 + \frac{c_2 e^{4t}}{4}\right) + \left(-\frac{1}{12}t^3 - \frac{1}{16}t^2 - \frac{1}{32}t\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
x = c_1 + \frac{c_2 e^{4t}}{4} - \frac{t^3}{12} - \frac{t^2}{16} - \frac{t}{32} \tag{1}
$$

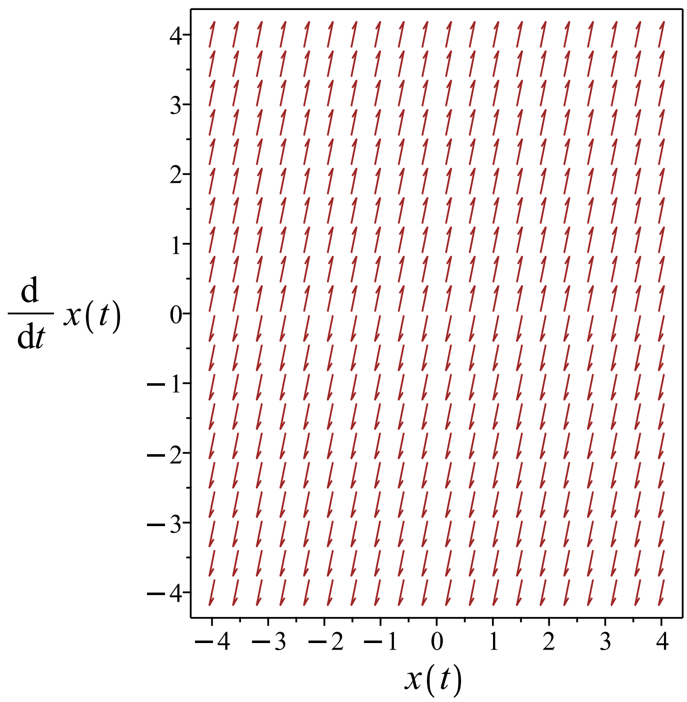


Figure 103: Slope field plot

<u>Verification of solutions</u>

$$
x = c_1 + \frac{c_2 e^{4t}}{4} - \frac{t^3}{12} - \frac{t^2}{16} - \frac{t}{32}
$$

Verified OK.

7.2.6 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,x'' + q(t)\,x' + r(t)\,x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -4 \\ r(x) &= 0 \\ s(x) &= t^2 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,x' + (q(t) - p'(t))\,x)' = s(x)$$

Integrating gives

$$p(t)\,x' + (q(t) - p'(t))\,x = \int s(t)\;dt$$

Substituting the above values for p, q, r, s gives

$$-4x + x' = \int t^2\,dt$$

We now have a first order ode to solve which is

$$-4x + x' = \frac{t^3}{3} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned}
p(t) &= -4 \\
q(t) &= \frac{t^3}{3} + c_1
\end{aligned}$$

Hence the ode is

$$-4x + x' = \frac{t^3}{3} + c_1$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int (-4)dt} \\
&= e^{-4t}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)\left(\frac{t^3}{3} + c_1\right) \\
\frac{d}{dt}\left(e^{-4t}x\right) &= \left(e^{-4t}\right)\left(\frac{t^3}{3} + c_1\right) \\
d\left(e^{-4t}x\right) &= \left(\frac{(t^3 + 3c_1)\,e^{-4t}}{3}\right) dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{-4t}x &= \int \frac{(t^3 + 3c_1)\,e^{-4t}}{3}\, dt \\
e^{-4t}x &= -\frac{(32t^3 + 24t^2 + 96c_1 + 12t + 3)\,e^{-4t}}{384} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-4t}$ results in

$$x = -\frac{e^{4t}(32t^3 + 24t^2 + 96c_1 + 12t + 3)\,e^{-4t}}{384} + c_2 e^{4t}$$

which simplifies to

$$x = -\frac{t^3}{12} - \frac{t^2}{16} - \frac{c_1}{4} - \frac{t}{32} - \frac{1}{128} + c_2 e^{4t}$$

Summary

The solution(s) found are the following

$$x = -\frac{t^3}{12} - \frac{t^2}{16} - \frac{c_1}{4} - \frac{t}{32} - \frac{1}{128} + c_2 e^{4t} \tag{1}$$

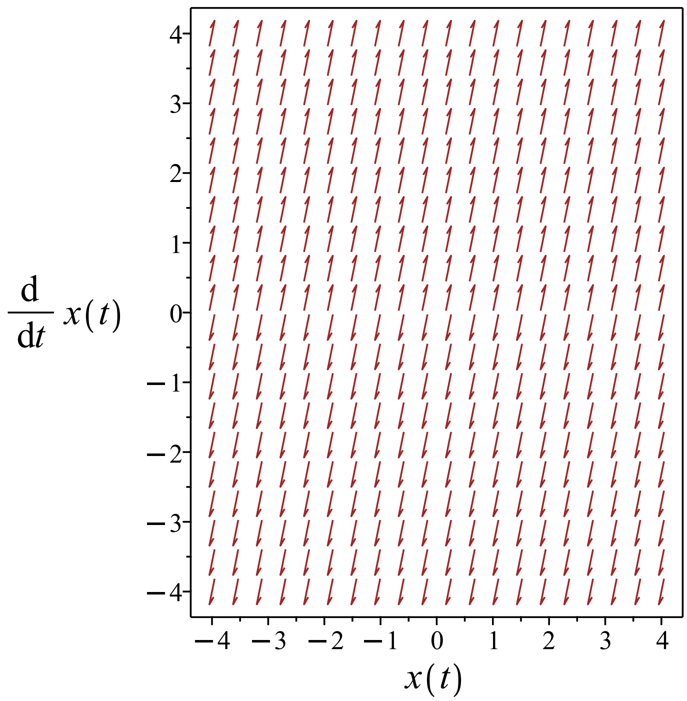


Figure 104: Slope field plot

Verification of solutions

$$x = -\frac{t^3}{12} - \frac{t^2}{16} - \frac{c_1}{4} - \frac{t}{32} - \frac{1}{128} + c_2 e^{4t}$$

Verified OK.

7.2.7 Maple step by step solution

Let's solve

$x'' - 4x' = t^2$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

$r^2 - 4r = 0$

- Factor the characteristic polynomial

  $r(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (0, 4)$

- 1st solution of the homogeneous ODE

  $x_1(t) = 1$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{4t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 + c_2 e^{4t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = t^2\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} 1 & e^{4t} \\ 0 & 4\,e^{4t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = 4\,e^{4t}$

- ○ Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{\left(\int t^2 dt\right)}{4} + \frac{e^{4t}\left(\int t^2 e^{-4t} dt\right)}{4}$

- ○ Compute integrals

  $x_p(t) = -\frac{1}{12}t^3 - \frac{1}{16}t^2 - \frac{1}{32}t - \frac{1}{128}$

- Substitute particular solution into general solution to ODE

  $x = c_1 + c_2 e^{4t} - \frac{t^3}{12} - \frac{t^2}{16} - \frac{t}{32} - \frac{1}{128}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^2+4*_b(_a), _b(_a)`   *** Sublevel 2
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(x(t),t$2)-4*diff(x(t),t)=t^2,x(t), singsol=all)
```

$$x(t) = -\frac{t^2}{16} - \frac{t^3}{12} + \frac{c_1 e^{4t}}{4} - \frac{t}{32} + c_2$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 36

```
DSolve[x''[t]-4*x'[t]==t^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{96}\left(-8t^3 - 6t^2 - 3t + 24c_1 e^{4t} + 96c_2\right)$$

7.3 problem 14.1 (iii)

Internal problem ID [12031]
Internal file name [`OUTPUT/10683_Sunday_September_03_2023_12_35_50_PM_49298837/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.1 (iii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' + x' - 2x = 3\,\mathrm{e}^{-t}}$$

7.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = -2, f(t) = 3\,\mathrm{e}^{-t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' - 2x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \lambda\, \mathrm{e}^{\lambda t} - 2\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-2)} \\
&= -\frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{3}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
x &= c_1 e^{(1)t} + c_2 e^{(-2)t}
\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-2t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\,\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{t}, \mathrm{e}^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \mathrm{e}^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \mathrm{e}^{-t} = 3\,\mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{3\,\mathrm{e}^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-2t}\right) + \left(-\frac{3\,\mathrm{e}^{-t}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-2t} - \frac{3\,\mathrm{e}^{-t}}{2} \tag{1}$$

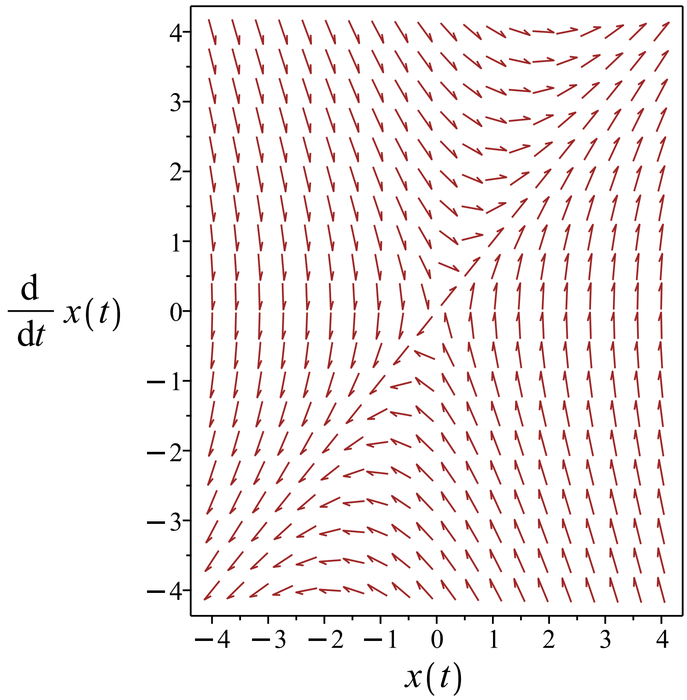


Figure 105: Slope field plot

Verification of solutions

$$x = c_1 e^t + c_2 e^{-2t} - \frac{3 e^{-t}}{2}$$

Verified OK.

7.3.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' - 2x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \frac{9z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 115: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{3t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dt} \\ &= z_1 e^{-\frac{t}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-2t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-t}}{(x_1)^2}\, dt \\ &= x_1 \left(\frac{\mathrm{e}^{3t}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-2t}\right) + c_2 \left(\mathrm{e}^{-2t} \left(\frac{\mathrm{e}^{3t}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + x' - 2x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-2t} + \frac{c_2 \mathrm{e}^{t}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
3\,\mathrm{e}^{-t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{t}}{3}, \mathrm{e}^{-2t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 \mathrm{e}^{-t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \mathrm{e}^{-t} = 3\,\mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{3\,\mathrm{e}^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-2t} + \frac{c_2 \mathrm{e}^{t}}{3}\right) + \left(-\frac{3\,\mathrm{e}^{-t}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{-2t} + \frac{c_2 \mathrm{e}^{t}}{3} - \frac{3\,\mathrm{e}^{-t}}{2} \tag{1}$$

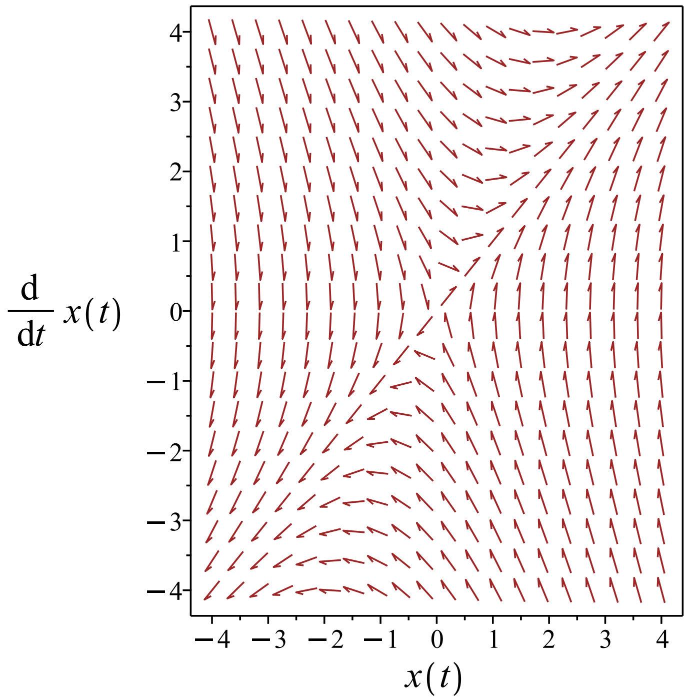


Figure 106: Slope field plot

Verification of solutions

$$x = c_1 e^{-2t} + \frac{c_2 e^t}{3} - \frac{3\,e^{-t}}{2}$$

Verified OK.

7.3.3 Maple step by step solution

Let's solve

$x'' + x' - 2x = 3\,e^{-t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 1) = 0$

- Roots of the characteristic polynomial

$r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 3\,\mathrm{e}^{-t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{t} \\ -2\,\mathrm{e}^{-2t} & \mathrm{e}^{t} \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 3\,\mathrm{e}^{-t}$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = \left(\mathrm{e}^{3t}\left(\int \mathrm{e}^{-2t} dt\right) - \left(\int \mathrm{e}^{t} dt\right)\right)\mathrm{e}^{-2t}$

  - Compute integrals

    $x_p(t) = -\frac{3\,\mathrm{e}^{-t}}{2}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{t} - \frac{3\,\mathrm{e}^{-t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(x(t),t$2)+diff(x(t),t)-2*x(t)=3*exp(-t),x(t), singsol=all)
```

$$x(t) = -\frac{(-2c_2 e^{3t} + 3\,e^{t} - 2c_1)\,e^{-2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 29

```
DSolve[x''[t]+x'[t]-2*x[t]==3*Exp[-t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{3e^{-t}}{2} + c_1 e^{-2t} + c_2 e^{t}$$

7.4 problem 14.1 (iv)

Internal problem ID [12032]
Internal file name [`OUTPUT/10684_Sunday_September_03_2023_12_35_52_PM_273934/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.1 (iv).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' + x' - 2x = e^{t}}$$

7.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = -2, f(t) = e^{t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' - 2x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \lambda\, \mathrm{e}^{\lambda t} - 2\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-2)}\\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{3}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{(1)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-2t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{t}, e^{-2t}\}$$

Since e^{t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,e^{t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t\,e^{t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{t} = e^{t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t\,e^{t}}{3}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{t} + c_2 e^{-2t}\right) + \left(\frac{t\,e^{t}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^t + c_2 e^{-2t} + \frac{t e^t}{3} \tag{1}$$

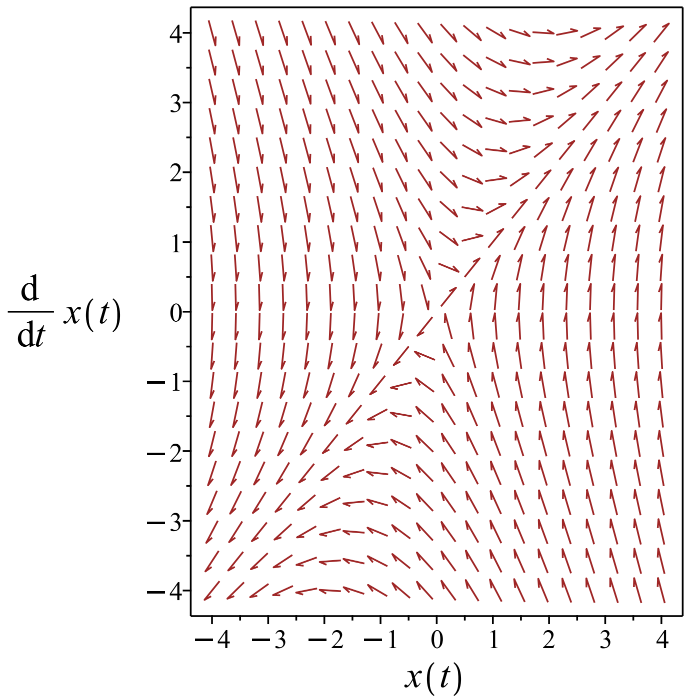


Figure 107: Slope field plot

Verification of solutions

$$x = c_1 e^t + c_2 e^{-2t} + \frac{t e^t}{3}$$

Verified OK.

7.4.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' - 2x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{9z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 117: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{3t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-2t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\, dt \\
&= x_1 \left(\frac{\mathrm{e}^{3t}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-2t}\right) + c_2 \left(\mathrm{e}^{-2t} \left(\frac{\mathrm{e}^{3t}}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' - 2x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-2t} + \frac{c_2 \mathrm{e}^{t}}{3}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \mathrm{e}^{-2t}$$
$$x_2 = \frac{\mathrm{e}^t}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$
$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2t} & \frac{\mathrm{e}^t}{3} \\ \frac{d}{dt}(\mathrm{e}^{-2t}) & \frac{d}{dt}\left(\frac{\mathrm{e}^t}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2t} & \frac{\mathrm{e}^t}{3} \\ -2\,\mathrm{e}^{-2t} & \frac{\mathrm{e}^t}{3} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-2t})\left(\frac{\mathrm{e}^t}{3}\right) - \left(\frac{\mathrm{e}^t}{3}\right)(-2\,\mathrm{e}^{-2t})$$

Which simplifies to

$$W = \mathrm{e}^t \mathrm{e}^{-2t}$$

Which simplifies to

$$W = \mathrm{e}^{-t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{2t}}{3}}{\mathrm{e}^{-t}}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^{3t}}{3} dt$$

Hence

$$u_1 = -\frac{\mathrm{e}^{3t}}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{t}\mathrm{e}^{-2t}}{\mathrm{e}^{-t}}\, dt$$

Which simplifies to

$$u_2 = \int 1 dt$$

Hence

$$u_2 = t$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} + \frac{t\,\mathrm{e}^{t}}{3}$$

Which simplifies to

$$x_p(t) = \frac{\mathrm{e}^{t}(-1+3t)}{9}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1\mathrm{e}^{-2t} + \frac{c_2\mathrm{e}^{t}}{3}\right) + \left(\frac{\mathrm{e}^{t}(-1+3t)}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-2t} + \frac{c_2 e^t}{3} + \frac{e^t(-1+3t)}{9} \tag{1}$$

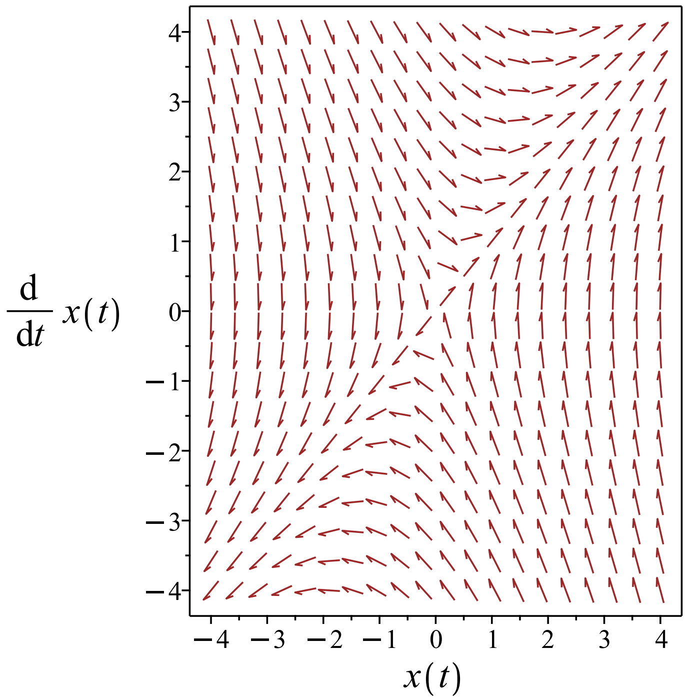


Figure 108: Slope field plot

Verification of solutions

$$x = c_1 e^{-2t} + \frac{c_2 e^t}{3} + \frac{e^t(-1+3t)}{9}$$

Verified OK.

7.4.3 Maple step by step solution

Let's solve

$x'' + x' - 2x = e^t$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

$r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \mathrm{e}^{t}\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{t} \\ -2\,\mathrm{e}^{-2t} & \mathrm{e}^{t} \end{bmatrix}$

  ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = 3\,\mathrm{e}^{-t}$

  ○ Substitute functions into equation for $x_p(t)$

  $x_p(t) = \frac{\left(\mathrm{e}^{3t}\left(\int 1 dt\right) - \left(\int \mathrm{e}^{3t} dt\right)\right)\mathrm{e}^{-2t}}{3}$

  ○ Compute integrals

  $x_p(t) = \frac{\mathrm{e}^{t}(-1+3t)}{9}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{t} + \frac{\mathrm{e}^{t}(-1+3t)}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(x(t),t$2)+diff(x(t),t)-2*x(t)=exp(t),x(t), singsol=all)
```

$$x(t) = \frac{e^{-2t}((t + 3c_2)\, e^{3t} + 3c_1)}{3}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 29

```
DSolve[x''[t]+x'[t]-2*x[t]==Exp[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{-2t} + e^t \left(\frac{t}{3} - \frac{1}{9} + c_2\right)$$

7.5 problem 14.1 (v)

Internal problem ID [12033]
Internal file name [`OUTPUT/10685_Sunday_September_03_2023_12_35_54_PM_54785160/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.1 (v).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' + 2x' + x = \mathrm{e}^{-t}}$$

7.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 2, C = 1, f(t) = \mathrm{e}^{-t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 2x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$x = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 e^{-t} + c_2 t\, e^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t\, e^{-t}, e^{-t}\}$$

Since e^{-t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,\mathrm{e}^{-t}\}]$$

Since $t\,\mathrm{e}^{-t}$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t^2\mathrm{e}^{-t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t^2 \mathrm{e}^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{-t} = \mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t^2\mathrm{e}^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1\mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t}\right) + \left(\frac{t^2\mathrm{e}^{-t}}{2}\right) \end{aligned}$$

Which simplifies to

$$x = \mathrm{e}^{-t}(c_2 t + c_1) + \frac{t^2\mathrm{e}^{-t}}{2}$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-t}(c_2 t + c_1) + \frac{t^2 \mathrm{e}^{-t}}{2} \tag{1}$$

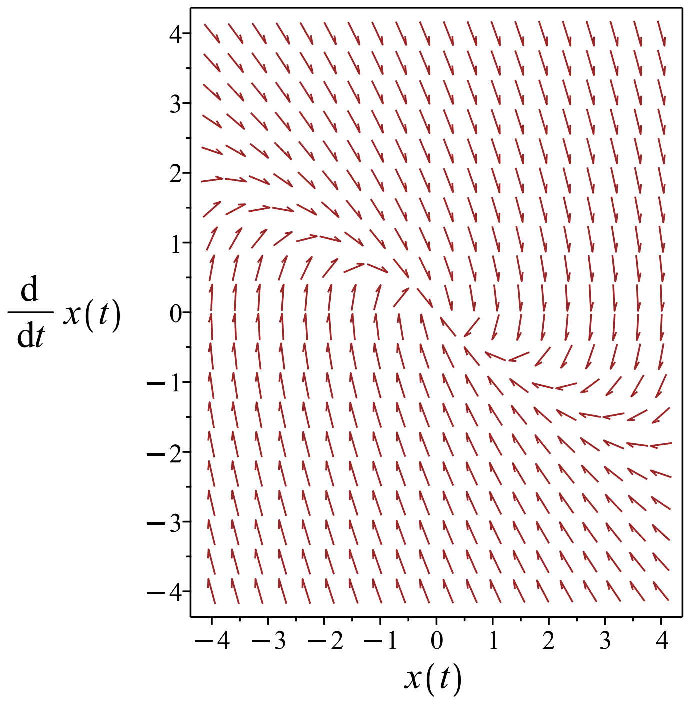


Figure 109: Slope field plot

Verification of solutions

$$x = \mathrm{e}^{-t}(c_2 t + c_1) + \frac{t^2 \mathrm{e}^{-t}}{2}$$

Verified OK.

7.5.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$x'' + p(t)\, x' + \frac{\left(p(t)^2 + p'(t)\right) x}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int 2\,dx} \\
&= \mathrm{e}^{t}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)x)'' &= \mathrm{e}^{t}\mathrm{e}^{-t} \\
\left(\mathrm{e}^{t}x\right)'' &= \mathrm{e}^{t}\mathrm{e}^{-t}
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{t}x\right)' = t + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{t}x\right) = \frac{t(t + 2c_1)}{2} + c_2
$$

Hence the solution is

$$
x = \frac{\frac{t(t+2c_1)}{2} + c_2}{\mathrm{e}^{t}}
$$

Or

$$
x = t\,\mathrm{e}^{-t}c_1 + \frac{t^2\mathrm{e}^{-t}}{2} + c_2\mathrm{e}^{-t}
$$

Summary

The solution(s) found are the following

$$
x = t\,\mathrm{e}^{-t}c_1 + \frac{t^2\mathrm{e}^{-t}}{2} + c_2\mathrm{e}^{-t} \tag{1}
$$

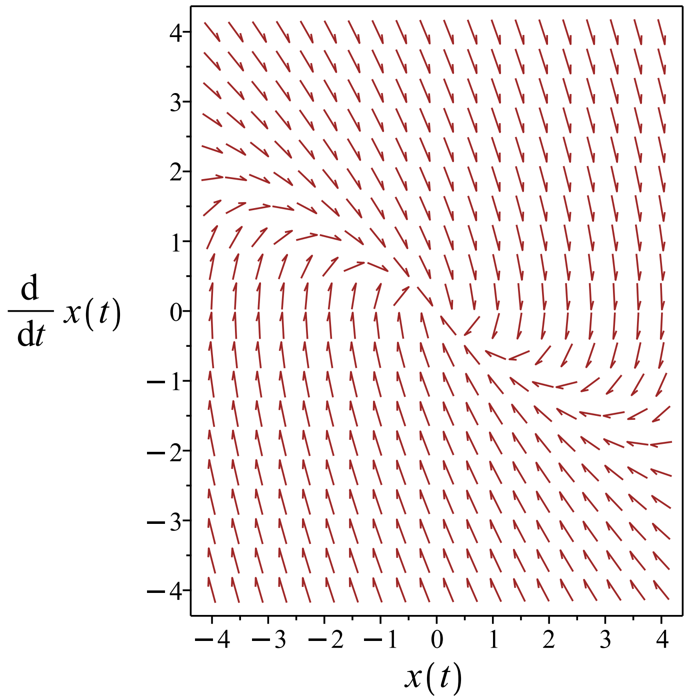


Figure 110: Slope field plot

Verification of solutions

$$x = t\,\mathrm{e}^{-t} c_1 + \frac{t^2 \mathrm{e}^{-t}}{2} + c_2 \mathrm{e}^{-t}$$

Verified OK.

7.5.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 2x' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$
$$B = 2 \tag{3}$$
$$C = 1$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 119: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\ &= z_1 e^{-t} \\ &= z_1 \left(\mathrm{e}^{-t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-2t}}{(x_1)^2}\,dt \\ &= x_1(t) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1\left(e^{-t}\right) + c_2\left(e^{-t}(t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 2x' + x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1e^{-t} + c_2t\,e^{-t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
e^{-t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{-t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{t\,e^{-t}, e^{-t}\}
$$

Since e^{-t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
[\{t\,e^{-t}\}]
$$

Since $t\,e^{-t}$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
[\{t^2e^{-t}\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t^2 \mathrm{e}^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{-t} = \mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t^2 \mathrm{e}^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t}\right) + \left(\frac{t^2 \mathrm{e}^{-t}}{2}\right) \end{aligned}$$

Which simplifies to

$$x = \mathrm{e}^{-t}(c_2 t + c_1) + \frac{t^2 \mathrm{e}^{-t}}{2}$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-t}(c_2 t + c_1) + \frac{t^2 \mathrm{e}^{-t}}{2} \tag{1}$$

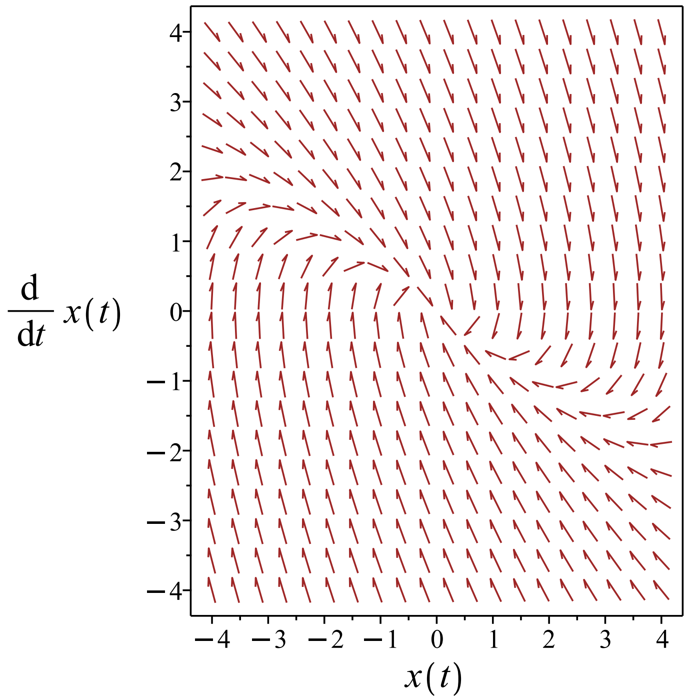


Figure 111: Slope field plot

Verification of solutions

$$x = e^{-t}(c_2 t + c_1) + \frac{t^2 e^{-t}}{2}$$

Verified OK.

7.5.4 Maple step by step solution

Let's solve

$x'' + 2x' + x = e^{-t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-t}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = t\,\mathrm{e}^{-t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1\mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \mathrm{e}^{-t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{-t} - t\,\mathrm{e}^{-t} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(x_1(t), x_2(t)) = \mathrm{e}^{-2t}$

  ◦ Substitute functions into equation for $x_p(t)$

  $x_p(t) = \mathrm{e}^{-t}\left(-\left(\int t dt\right) + \left(\int 1 dt\right) t\right)$

  ◦ Compute integrals

  $x_p(t) = \frac{t^2\mathrm{e}^{-t}}{2}$

- Substitute particular solution into general solution to ODE

  $x = c_2 t\,\mathrm{e}^{-t} + c_1\mathrm{e}^{-t} + \frac{t^2\mathrm{e}^{-t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(x(t),t$2)+2*diff(x(t),t)+x(t)=exp(-t),x(t), singsol=all)
```

$$x(t) = \mathrm{e}^{-t}\left(c_2 + c_1 t + \frac{1}{2}t^2\right)$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 27

```
DSolve[x''[t]+2*x'[t]+x[t]==Exp[-t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2}e^{-t}\left(t^2 + 2c_2 t + 2c_1\right)$$

7.6 problem 14.1 (vi)

Internal problem ID [12034]
Internal file name [`OUTPUT/10686_Sunday_September_03_2023_12_35_56_PM_9439109/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.1 (vi).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + \omega^2 x = \sin(\alpha t)}$$

7.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = \omega^2, f(t) = \sin(\alpha t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + \omega^2 x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = \omega^2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \omega^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \omega^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\omega^2)}\\ &= \pm\sqrt{-\omega^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-\omega^2}\\ \lambda_2 &= -\sqrt{-\omega^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-\omega^2}\\ \lambda_2 &= -\sqrt{-\omega^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{\left(\sqrt{-\omega^2}\right)t} + c_2 e^{\left(-\sqrt{-\omega^2}\right)t}\end{aligned}$$

Or

$$x = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(\alpha t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\alpha t), \sin(\alpha t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\sqrt{-\omega^2}\, t}, e^{-\sqrt{-\omega^2}\, t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(\alpha t) + A_2 \sin(\alpha t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\alpha^2 \cos(\alpha t) - A_2\alpha^2 \sin(\alpha t) + \omega^2(A_1 \cos(\alpha t) + A_2 \sin(\alpha t)) = \sin(\alpha t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{\alpha^2 - \omega^2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{\sin(\alpha t)}{\alpha^2 - \omega^2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t}\right) + \left(-\frac{\sin(\alpha t)}{\alpha^2 - \omega^2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t} - \frac{\sin(\alpha t)}{\alpha^2 - \omega^2} \tag{1}$$

Verification of solutions

$$x = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t} - \frac{\sin(\alpha t)}{\alpha^2 - \omega^2}$$

Verified OK.

7.6.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + \omega^2 x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \omega^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\omega^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\omega^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = (-\omega^2)\, z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 121: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r=-\omega^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z''=rz$ as one solution is

$$z_1(t) = \mathrm{e}^{\sqrt{-\omega^2}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B=0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{-\omega^2}\, t} \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{\sqrt{-\omega^2}\, t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B=0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= \mathrm{e}^{\sqrt{-\omega^2}\, t} \int \frac{1}{\mathrm{e}^{2\sqrt{-\omega^2}\, t}}\, dt \\ &= \mathrm{e}^{\sqrt{-\omega^2}\, t} \left(\frac{\sqrt{-\omega^2}\, \mathrm{e}^{-2\sqrt{-\omega^2}\, t}}{2\omega^2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(\mathrm{e}^{\sqrt{-\omega^2}\, t} \right) + c_2 \left(\mathrm{e}^{\sqrt{-\omega^2}\, t} \left(\frac{\sqrt{-\omega^2}\, \mathrm{e}^{-2\sqrt{-\omega^2}\, t}}{2\omega^2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + \omega^2 x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(\alpha t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\alpha t), \sin(\alpha t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2}, \mathrm{e}^{\sqrt{-\omega^2}\, t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(\alpha t) + A_2 \sin(\alpha t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\alpha^2 \cos(\alpha t) - A_2\alpha^2 \sin(\alpha t) + \omega^2(A_1 \cos(\alpha t) + A_2 \sin(\alpha t)) = \sin(\alpha t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{\alpha^2 - \omega^2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{\sin(\alpha t)}{\alpha^2 - \omega^2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2}\right) + \left(-\frac{\sin(\alpha t)}{\alpha^2 - \omega^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2} - \frac{\sin(\alpha t)}{\alpha^2 - \omega^2} \tag{1}$$

Verification of solutions

$$x = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2} - \frac{\sin(\alpha t)}{\alpha^2 - \omega^2}$$

Verified OK.

7.6.3 Maple step by step solution

Let's solve

$x'' + \omega^2 x = \sin(\alpha t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $\omega^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4\omega^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-\omega^2}, -\sqrt{-\omega^2}\right)$
- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{\sqrt{-\omega^2}\,t}$
- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-\sqrt{-\omega^2}\,t}$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{\sqrt{-\omega^2}\,t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\,t} + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \sin(\alpha t)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{\sqrt{-\omega^2}\,t} & \mathrm{e}^{-\sqrt{-\omega^2}\,t} \\ \sqrt{-\omega^2}\,\mathrm{e}^{\sqrt{-\omega^2}\,t} & -\sqrt{-\omega^2}\,\mathrm{e}^{-\sqrt{-\omega^2}\,t} \end{bmatrix}$
- ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = -2\sqrt{-\omega^2}$
- ○ Substitute functions into equation for $x_p(t)$

  $x_p(t) = \frac{\mathrm{e}^{\sqrt{-\omega^2}\,t}\left(\int \mathrm{e}^{-\sqrt{-\omega^2}\,t}\sin(\alpha t)dt\right) - \mathrm{e}^{-\sqrt{-\omega^2}\,t}\left(\int \sin(\alpha t)\mathrm{e}^{\sqrt{-\omega^2}\,t}dt\right)}{2\sqrt{-\omega^2}}$
- ○ Compute integrals

  $x_p(t) = -\frac{\sin(\alpha t)}{\alpha^2 - \omega^2}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{\sqrt{-\omega^2}\,t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\,t} - \frac{\sin(\alpha t)}{\alpha^2 - \omega^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(x(t),t$2)+omega^2*x(t)=sin(alpha*t),x(t), singsol=all)
```

$$x(t) = \sin(\omega t) c_2 + \cos(\omega t) c_1 + \frac{\sin(\alpha t)}{-\alpha^2 + \omega^2}$$

✓ Solution by Mathematica

Time used: 0.359 (sec). Leaf size: 56

```
DSolve[x''[t]+w^2*x[t]==Sin[a*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{-(c_1(a^2 - w^2)\cos(tw)) + c_2(w^2 - a^2)\sin(tw) + \sin(at)}{(w-a)(a+w)}$$

7.7 problem 14.1 (vii)

Internal problem ID [12035]
Internal file name [`OUTPUT/10687_Sunday_September_03_2023_12_35_58_PM_27378406/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.1 (vii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + \omega^2 x = \sin(\omega t)}$$

7.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = \omega^2, f(t) = \sin(\omega t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + \omega^2 x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = \omega^2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \omega^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \omega^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\omega^2)} \\ &= \pm\sqrt{-\omega^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-\omega^2} \\ \lambda_2 &= -\sqrt{-\omega^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-\omega^2} \\ \lambda_2 &= -\sqrt{-\omega^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ x &= c_1 e^{\left(\sqrt{-\omega^2}\right)t} + c_2 e^{\left(-\sqrt{-\omega^2}\right)t}\end{aligned}$$

Or

$$x = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \mathrm{e}^{\sqrt{-\omega^2}\, t}$$
$$x_2 = \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-\omega^2}\, t} & \mathrm{e}^{-\sqrt{-\omega^2}\, t} \\ \frac{d}{dt}\left(\mathrm{e}^{\sqrt{-\omega^2}\, t}\right) & \frac{d}{dt}\left(\mathrm{e}^{-\sqrt{-\omega^2}\, t}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-\omega^2}\, t} & \mathrm{e}^{-\sqrt{-\omega^2}\, t} \\ \sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t} & -\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{\sqrt{-\omega^2}\, t}\right)\left(-\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}\right) - \left(\mathrm{e}^{-\sqrt{-\omega^2}\, t}\right)\left(\sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t}\right)$$

Which simplifies to

$$W = -2\, \mathrm{e}^{\sqrt{-\omega^2}\, t} \sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

Which simplifies to

$$W = -2\sqrt{-\omega^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-\sqrt{-\omega^2}\, t} \sin(\omega t)}{-2\sqrt{-\omega^2}}\, dt$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-\sqrt{-\omega^2}\, t} \sin(\omega t)}{2\sqrt{-\omega^2}} dt$$

Hence

$$u_1 = -\frac{\frac{t\, e^{-\sqrt{-\omega^2}\, t}}{4} - \frac{t\, e^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4} - \frac{\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{4\omega^2} + \frac{\sqrt{-\omega^2}\, t\, e^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)}{2\omega} + \frac{\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4\omega^2}}{\omega \left(1 + \tan\left(\frac{\omega t}{2}\right)^2\right)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\sqrt{-\omega^2}\, t} \sin(\omega t)}{-2\sqrt{-\omega^2}}\, dt$$

Which simplifies to

$$u_2 = \int -\frac{e^{\sqrt{-\omega^2}\, t} \sin(\omega t)}{2\sqrt{-\omega^2}} dt$$

Hence

$$u_2 = \frac{-\frac{t\, e^{\sqrt{-\omega^2}\, t}}{4} + \frac{t\, e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4} - \frac{\sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t}}{4\omega^2} + \frac{\sqrt{-\omega^2}\, t\, e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)}{2\omega} + \frac{\sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4\omega^2}}{\omega \left(1 + \tan\left(\frac{\omega t}{2}\right)^2\right)}$$

Which simplifies to

$$u_1 = -\frac{e^{-\sqrt{-\omega^2}\, t} \left(\left(\sin(\omega t)\, \omega t - \cos(\omega t)\right) \sqrt{-\omega^2} + t\, \omega^2 \cos(\omega t)\right)}{4\omega^3}$$

$$u_2 = -\frac{e^{\sqrt{-\omega^2}\, t} \left(\left(-\sin(\omega t)\, \omega t + \cos(\omega t)\right) \sqrt{-\omega^2} + t\, \omega^2 \cos(\omega t)\right)}{4\omega^3}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
x_p(t) = &-\frac{e^{-\sqrt{-\omega^2}\,t}\left(\left(\sin\left(\omega t\right)\omega t-\cos\left(\omega t\right)\right)\sqrt{-\omega^2}+t\,\omega^2\cos\left(\omega t\right)\right)e^{\sqrt{-\omega^2}\,t}}{4\omega^3}\\
&-\frac{e^{\sqrt{-\omega^2}\,t}\left(\left(-\sin\left(\omega t\right)\omega t+\cos\left(\omega t\right)\right)\sqrt{-\omega^2}+t\,\omega^2\cos\left(\omega t\right)\right)e^{-\sqrt{-\omega^2}\,t}}{4\omega^3}
\end{aligned}
$$

Which simplifies to

$$
x_p(t) = -\frac{t\cos\left(\omega t\right)}{2\omega}
$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p\\
&= \left(c_1 e^{\sqrt{-\omega^2}\,t}+c_2 e^{-\sqrt{-\omega^2}\,t}\right)+\left(-\frac{t\cos\left(\omega t\right)}{2\omega}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 e^{\sqrt{-\omega^2}\,t}+c_2 e^{-\sqrt{-\omega^2}\,t}-\frac{t\cos\left(\omega t\right)}{2\omega} \tag{1}
$$

Verification of solutions

$$
x = c_1 e^{\sqrt{-\omega^2}\,t}+c_2 e^{-\sqrt{-\omega^2}\,t}-\frac{t\cos\left(\omega t\right)}{2\omega}
$$

Verified OK.

7.7.2 Solving using Kovacic algorithm

Writing the ode as

$$
x'' + \omega^2 x = 0 \tag{1}
$$

$$
Ax'' + Bx' + Cx = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= 0 \\
C &= \omega^2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\omega^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\omega^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = (-\omega^2)\, z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 123: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\omega^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{\sqrt{-\omega^2}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}x_1 &= z_1\\ &= \mathrm{e}^{\sqrt{-\omega^2}\, t}\end{aligned}$$

Which simplifies to

$$x_1 = e^{\sqrt{-\omega^2}\, t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= e^{\sqrt{-\omega^2}\, t} \int \frac{1}{e^{2\sqrt{-\omega^2}\, t}}\, dt \\ &= e^{\sqrt{-\omega^2}\, t} \left(\frac{\sqrt{-\omega^2}\, e^{-2\sqrt{-\omega^2}\, t}}{2\omega^2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(e^{\sqrt{-\omega^2}\, t} \right) + c_2 \left(e^{\sqrt{-\omega^2}\, t} \left(\frac{\sqrt{-\omega^2}\, e^{-2\sqrt{-\omega^2}\, t}}{2\omega^2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + \omega^2 x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 e^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \mathrm{e}^{\sqrt{-\omega^2}\, t}$$
$$x_2 = \frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-\omega^2}\, t} & \frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2} \\ \frac{d}{dt}\left(\mathrm{e}^{\sqrt{-\omega^2}\, t}\right) & \frac{d}{dt}\left(\frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-\omega^2}\, t} & \frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2} \\ \sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t} & \frac{\mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{\sqrt{-\omega^2}\, t}\right)\left(\frac{\mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2}\right) - \left(\frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2}\right)\left(\sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t}\right)$$

Which simplifies to

$$W = \mathrm{e}^{\sqrt{-\omega^2}\, t} \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} \sin(\omega t)}{2\omega^2}}{1} dt$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} \sin(\omega t)}{2\omega^2} dt$$

Hence

$$u_1 = -\frac{\frac{t\,\mathrm{e}^{-\sqrt{-\omega^2}\, t}}{4} - \frac{t\,\mathrm{e}^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4} - \frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{4\omega^2} + \frac{\sqrt{-\omega^2}\, t\,\mathrm{e}^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)}{2\omega} + \frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4\omega^2}}{\omega \left(1 + \tan\left(\frac{\omega t}{2}\right)^2\right)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{\sqrt{-\omega^2}\, t} \sin(\omega t)}{1} dt$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{\sqrt{-\omega^2}\, t} \sin(\omega t)\, dt$$

Hence

$$u_2 = \frac{t\,\mathrm{e}^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right) + \frac{\sqrt{-\omega^2}\, t\,\mathrm{e}^{\sqrt{-\omega^2}\, t}}{2\omega} - \frac{\mathrm{e}^{\sqrt{-\omega^2}\, t}}{2\omega} - \frac{\sqrt{-\omega^2}\, t\,\mathrm{e}^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{2\omega} + \frac{\mathrm{e}^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{2\omega}}{1 + \tan\left(\frac{\omega t}{2}\right)^2}$$

Which simplifies to

$$u_1 = -\frac{e^{-\sqrt{-\omega^2}\, t}\left(\left(\sin\left(\omega t\right)\omega t - \cos\left(\omega t\right)\right)\sqrt{-\omega^2} + t\,\omega^2\cos\left(\omega t\right)\right)}{4\omega^3}$$

$$u_2 = \frac{e^{\sqrt{-\omega^2}\, t}\left(\sin\left(\omega t\right)\omega t + \sqrt{-\omega^2}\, t\cos\left(\omega t\right) - \cos\left(\omega t\right)\right)}{2\omega}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{e^{-\sqrt{-\omega^2}\, t}\left(\left(\sin\left(\omega t\right)\omega t - \cos\left(\omega t\right)\right)\sqrt{-\omega^2} + t\,\omega^2\cos\left(\omega t\right)\right)e^{\sqrt{-\omega^2}\, t}}{4\omega^3} + \frac{e^{\sqrt{-\omega^2}\, t}\left(\sin\left(\omega t\right)\omega t + \sqrt{-\omega^2}\, t\cos\left(\omega t\right) - \cos\left(\omega t\right)\right)\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{4\omega^3}$$

Which simplifies to

$$x_p(t) = -\frac{t\cos\left(\omega t\right)}{2\omega}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\sqrt{-\omega^2}\, t} + \frac{c_2\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2}\right) + \left(-\frac{t\cos\left(\omega t\right)}{2\omega}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{\sqrt{-\omega^2}\, t} + \frac{c_2\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2} - \frac{t\cos\left(\omega t\right)}{2\omega} \tag{1}$$

Verification of solutions

$$x = c_1 e^{\sqrt{-\omega^2}\, t} + \frac{c_2\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2} - \frac{t\cos\left(\omega t\right)}{2\omega}$$

Verified OK.

7.7.3 Maple step by step solution

Let's solve

$x'' + \omega^2 x = \sin(\omega t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $\omega^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4\omega^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-\omega^2}, -\sqrt{-\omega^2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{\sqrt{-\omega^2}\, t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-\sqrt{-\omega^2}\, t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \sin(\omega t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{\sqrt{-\omega^2}\, t} & \mathrm{e}^{-\sqrt{-\omega^2}\, t} \\ \sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t} & -\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = -2\sqrt{-\omega^2}$

- ○ Substitute functions into equation for $x_p(t)$

$$x_p(t) = \frac{e^{\sqrt{-\omega^2}\, t}\left(\int e^{-\sqrt{-\omega^2}\, t} \sin(\omega t)dt\right) - e^{-\sqrt{-\omega^2}\, t}\left(\int e^{\sqrt{-\omega^2}\, t} \sin(\omega t)dt\right)}{2\sqrt{-\omega^2}}$$

- ◦ Compute integrals

  $x_p(t) = -\frac{t\cos(\omega t)}{2\omega}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t} - \frac{t\cos(\omega t)}{2\omega}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(x(t),t$2)+omega^2*x(t)=sin(omega*t),x(t), singsol=all)
```

$$x(t) = \frac{\sin(\omega t)(2c_2\omega^2 + 1) - \omega\cos(\omega t)(-2c_1\omega + t)}{2\omega^2}$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 29

```
DSolve[x''[t]+w^2*x[t]==Sin[w*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \left(-\frac{t}{2w} + c_1\right)\cos(tw) + c_2\sin(tw)$$

7.8 problem 14.1 (viii)

Internal problem ID [12036]
Internal file name [`OUTPUT/10688_Sunday_September_03_2023_12_36_04_PM_85575117/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.1 (viii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' + 2x' + 10x = \mathrm{e}^{-t}}$$

7.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 2, C = 10, f(t) = \mathrm{e}^{-t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 2x' + 10x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 2, C = 10$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 2\lambda\, \mathrm{e}^{\lambda t} + 10\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(10)} \\ &= -1 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 3i \\ \lambda_2 &= -1 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 3i \\ \lambda_2 &= -1 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t))$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-t}(c_1 \cos(3t) + c_2 \sin(3t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-t}\cos(3t), e^{-t}\sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 e^{-t} = e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{9}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{e^{-t}}{9}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-t}(c_1\cos(3t) + c_2\sin(3t))\right) + \left(\frac{e^{-t}}{9}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$x = e^{-t}(c_1\cos(3t) + c_2\sin(3t)) + \frac{e^{-t}}{9} \tag{1}$$

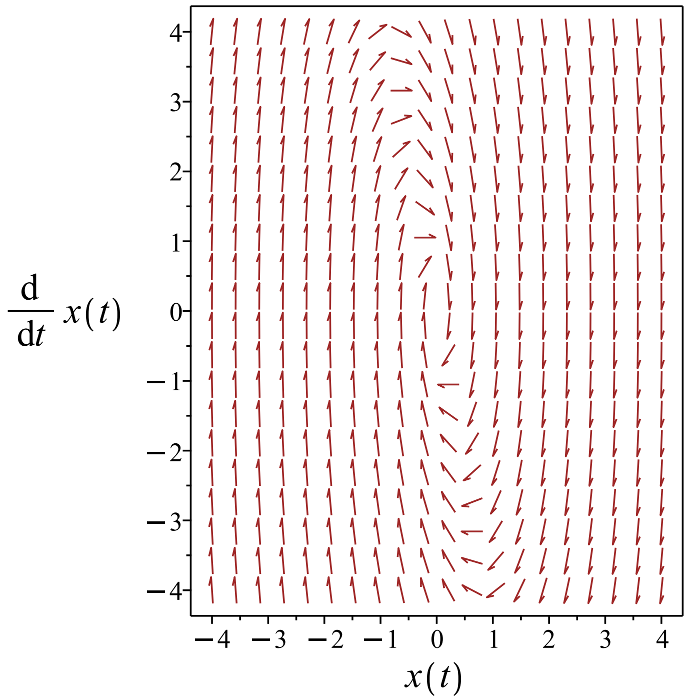


Figure 112: Slope field plot

Verification of solutions

$$x = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t)) + \frac{e^{-t}}{9}$$

Verified OK.

7.8.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 2x' + 10x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 125: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\ &= z_1 e^{-t} \\ &= z_1\left(\mathrm{e}^{-t}\right)\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}\cos(3t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}x_2 &= x_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-2t}}{(x_1)^2}\,dt \\ &= x_1\left(\frac{\tan(3t)}{3}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-t} \cos \left(3t\right)\right) + c_2 \left(\mathrm{e}^{-t} \cos \left(3t\right) \left(\frac{\tan \left(3t\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 2x' + 10x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-t} \cos \left(3t\right) + \frac{\mathrm{e}^{-t} \sin \left(3t\right) c_2}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{-t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-t} \cos \left(3t\right), \frac{\mathrm{e}^{-t} \sin \left(3t\right)}{3}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 \mathrm{e}^{-t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 e^{-t} = e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{9}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{e^{-t}}{9}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-t} \cos(3t) + \frac{e^{-t} \sin(3t) c_2}{3}\right) + \left(\frac{e^{-t}}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-t} \cos(3t) + \frac{e^{-t} \sin(3t) c_2}{3} + \frac{e^{-t}}{9} \tag{1}$$

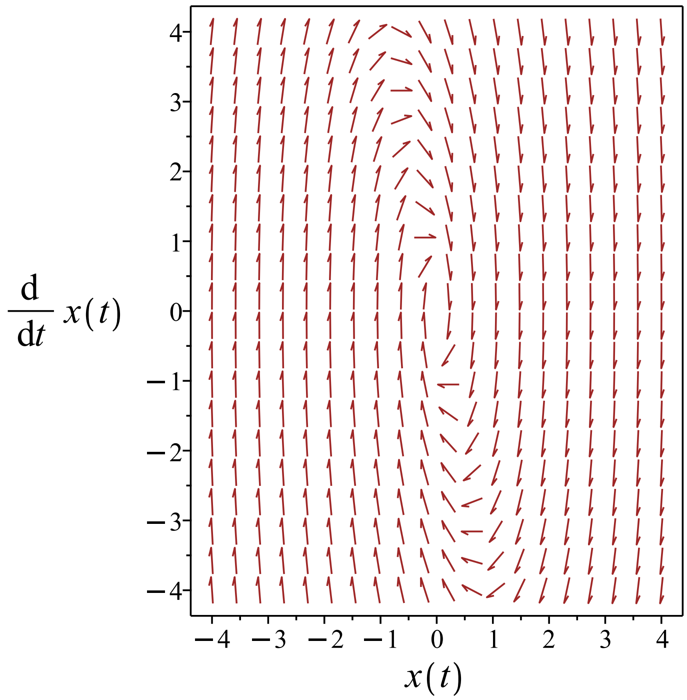


Figure 113: Slope field plot

Verification of solutions

$$x = c_1 e^{-t} \cos(3t) + \frac{e^{-t} \sin(3t) c_2}{3} + \frac{e^{-t}}{9}$$

Verified OK.

7.8.3 Maple step by step solution

Let's solve

$x'' + 2x' + 10x = e^{-t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-1 - 3\,\mathrm{I}, -1 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-t} \cos(3t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-t} \sin(3t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-t} \cos(3t) + \mathrm{e}^{-t} \sin(3t)\, c_2 + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \mathrm{e}^{-t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} \cos(3t) & \mathrm{e}^{-t} \sin(3t) \\ -\mathrm{e}^{-t} \cos(3t) - 3\,\mathrm{e}^{-t} \sin(3t) & -\mathrm{e}^{-t} \sin(3t) + 3\,\mathrm{e}^{-t} \cos(3t) \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(x_1(t), x_2(t)) = 3\,\mathrm{e}^{-2t}$

  ◦ Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{\mathrm{e}^{-t}(\cos(3t)(\int \sin(3t) dt) - \sin(3t)(\int \cos(3t) dt))}{3}$

  ◦ Compute integrals

  $x_p(t) = \frac{\mathrm{e}^{-t}}{9}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-t} \cos(3t) + \mathrm{e}^{-t} \sin(3t)\, c_2 + \frac{\mathrm{e}^{-t}}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(x(t),t$2)+2*diff(x(t),t)+10*x(t)=exp(-t),x(t), singsol=all)
```

$$x(t) = \frac{e^{-t}(9c_2 \sin(3t) + 9c_1 \cos(3t) + 1)}{9}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 32

```
DSolve[x''[t]+2*x'[t]+10*x[t]==Exp[-t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{9}e^{-t}(9c_2 \cos(3t) + 9c_1 \sin(3t) + 1)$$

7.9 problem 14.1 (ix)

Internal problem ID [12037]
Internal file name [`OUTPUT/10689_Sunday_September_03_2023_12_36_08_PM_22481411/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.1 (ix).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 2x' + 10x = \mathrm{e}^{-t}\cos(3t)}$$

7.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 2, C = 10, f(t) = \mathrm{e}^{-t}\cos(3t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 2x' + 10x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 2, C = 10$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 2\lambda \,\mathrm{e}^{\lambda t} + 10\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(10)}\\ &= -1 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 3i\\ \lambda_2 &= -1 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 3i\\ \lambda_2 &= -1 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t))$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-t}(c_1 \cos(3t) + c_2 \sin(3t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-t} \cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t} \cos(3t), e^{-t} \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-t} \cos(3t), e^{-t} \sin(3t)\}$$

Since $e^{-t} \cos(3t)$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,e^{-t} \cos(3t), t\,e^{-t} \sin(3t)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t\,e^{-t} \cos(3t) + A_2 t\,e^{-t} \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 e^{-t} \sin(3t) + 6A_2 e^{-t} \cos(3t) = e^{-t} \cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t\,e^{-t} \sin(3t)}{6}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-t}(c_1 \cos(3t) + c_2 \sin(3t))\right) + \left(\frac{t\,e^{-t} \sin(3t)}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t)) + \frac{t e^{-t} \sin(3t)}{6} \tag{1}$$

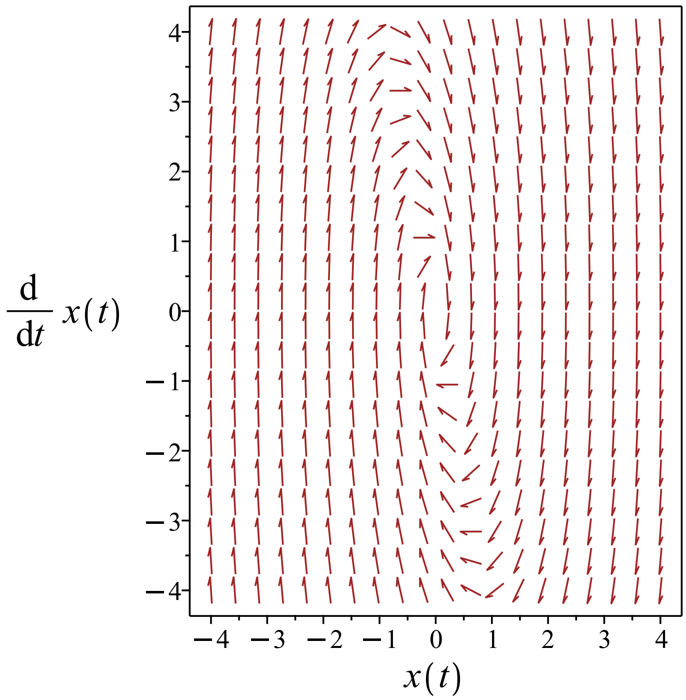


Figure 114: Slope field plot

Verification of solutions

$$x = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t)) + \frac{t e^{-t} \sin(3t)}{6}$$

Verified OK.

7.9.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 2x' + 10x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 2 \\
C &= 10
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = xe^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-9}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -9 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = -9z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 127: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(3t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dt} \\
&= z_1 e^{-t} \\
&= z_1\left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t} \cos(3t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{2}{1}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-2t}}{(x_1)^2}\, dt \\ &= x_1 \left(\frac{\tan(3t)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(\mathrm{e}^{-t} \cos(3t)\right) + c_2 \left(\mathrm{e}^{-t} \cos(3t) \left(\frac{\tan(3t)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 2x' + 10x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-t} \cos(3t) + \frac{\mathrm{e}^{-t} \sin(3t)\, c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-t} \cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\cos(3t), e^{-t}\sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-t}\cos(3t), \frac{e^{-t}\sin(3t)}{3}\right\}$$

Since $e^{-t}\cos(3t)$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,e^{-t}\cos(3t), t\,e^{-t}\sin(3t)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t\,e^{-t}\cos(3t) + A_2 t\,e^{-t}\sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 e^{-t}\sin(3t) + 6A_2 e^{-t}\cos(3t) = e^{-t}\cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{t\,e^{-t}\sin(3t)}{6}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-t}\cos(3t) + \frac{e^{-t}\sin(3t)\,c_2}{3}\right) + \left(\frac{t\,e^{-t}\sin(3t)}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-t} \cos(3t) + \frac{e^{-t} \sin(3t) c_2}{3} + \frac{t e^{-t} \sin(3t)}{6} \tag{1}$$

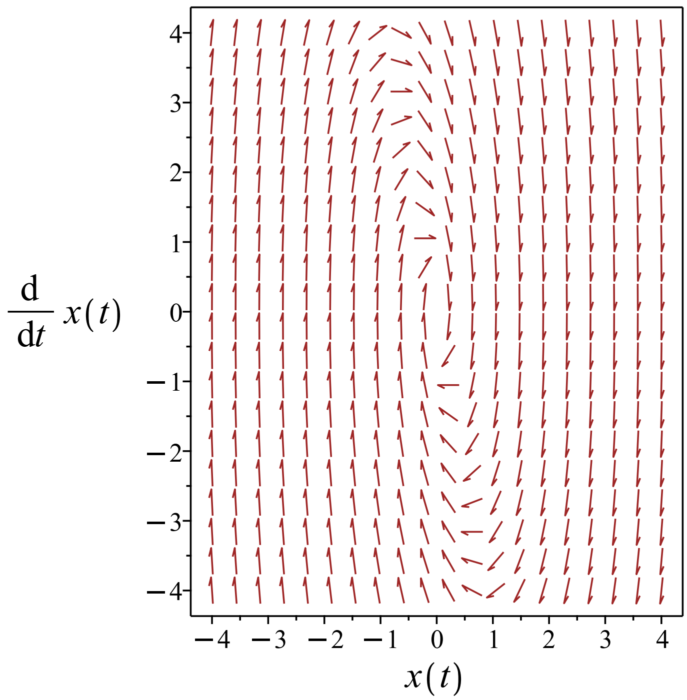


Figure 115: Slope field plot

Verification of solutions

$$x = c_1 e^{-t} \cos(3t) + \frac{e^{-t} \sin(3t) c_2}{3} + \frac{t e^{-t} \sin(3t)}{6}$$

Verified OK.

7.9.3 Maple step by step solution

Let's solve

$x'' + 2x' + 10x = e^{-t} \cos(3t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

$$r^2 + 2r + 10 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{(-2) \pm (\sqrt{-36})}{2}$$

- Roots of the characteristic polynomial

$$r = (-1 - 3\,\mathrm{I}, -1 + 3\,\mathrm{I})$$

- 1st solution of the homogeneous ODE

$$x_1(t) = \mathrm{e}^{-t} \cos(3t)$$

- 2nd solution of the homogeneous ODE

$$x_2(t) = \mathrm{e}^{-t} \sin(3t)$$

- General solution of the ODE

$$x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$$

- Substitute in solutions of the homogeneous ODE

$$x = c_1 \mathrm{e}^{-t} \cos(3t) + \mathrm{e}^{-t} \sin(3t)\, c_2 + x_p(t)$$

☐ Find a particular solution $x_p(t)$ of the ODE

- ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

$$\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \mathrm{e}^{-t} \cos(3t)\right]$$

- ◦ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} \cos(3t) & \mathrm{e}^{-t} \sin(3t) \\ -\mathrm{e}^{-t} \cos(3t) - 3\,\mathrm{e}^{-t} \sin(3t) & -\mathrm{e}^{-t} \sin(3t) + 3\,\mathrm{e}^{-t} \cos(3t) \end{bmatrix}$$

- ◦ Compute Wronskian

$$W(x_1(t), x_2(t)) = 3\,\mathrm{e}^{-2t}$$

- ◦ Substitute functions into equation for $x_p(t)$

$$x_p(t) = -\frac{\mathrm{e}^{-t}\left(\cos(3t)\left(\int \sin(6t) dt\right) - 2\sin(3t)\left(\int \cos(3t)^2 dt\right)\right)}{6}$$

- ◦ Compute integrals

$$x_p(t) = \frac{(6t \sin(3t) + \cos(3t))\mathrm{e}^{-t}}{36}$$

- Substitute particular solution into general solution to ODE

$$x = c_1 \mathrm{e}^{-t} \cos(3t) + \mathrm{e}^{-t} \sin(3t)\, c_2 + \frac{(6t \sin(3t) + \cos(3t))\mathrm{e}^{-t}}{36}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(x(t),t$2)+2*diff(x(t),t)+10*x(t)=exp(-t)*cos(3*t),x(t), singsol=all)
```

$$x(t) = \frac{\left(\left(6c_1 + \frac{1}{3}\right)\cos\left(3t\right) + \sin\left(3t\right)\left(t + 6c_2\right)\right)\mathrm{e}^{-t}}{6}$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 38

```
DSolve[x''[t]+2*x'[t]+10*x[t]==Exp[-t]*Cos[3*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{36}e^{-t}((1 + 36c_2)\cos(3t) + 6(t + 6c_1)\sin(3t))$$

7.10 problem 14.1 (x)

Internal problem ID [12038]
Internal file name [`OUTPUT/10690_Sunday_September_03_2023_12_36_16_PM_88302542/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.1 (x).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + 6x' + 10x = \cos(t)\,\mathrm{e}^{-2t}}$$

7.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 6, C = 10, f(t) = \cos(t)\,\mathrm{e}^{-2t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 6x' + 10x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A=1, B=6, C=10$. Let the solution be $x=e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 6\lambda\, \mathrm{e}^{\lambda t} + 10\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=6, C=10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(10)} \\ &= -3 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + i \\ \lambda_2 &= -3 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -3 + i \\ \lambda_2 &= -3 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-3t}(c_1 \cos(t) + c_2 \sin(t))$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-3t}(c_1 \cos(t) + c_2 \sin(t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)\,e^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t)\,e^{-2t}, \sin(t)\,e^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}\cos(t)\,, e^{-3t}\sin(t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(t)\,e^{-2t} + A_2 \sin(t)\,e^{-2t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(t)\,e^{-2t} - 2A_1 \sin(t)\,e^{-2t} + A_2 \sin(t)\,e^{-2t} + 2A_2 \cos(t)\,e^{-2t} = \cos(t)\,e^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = \frac{2}{5}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{\cos(t)\,e^{-2t}}{5} + \frac{2\sin(t)\,e^{-2t}}{5}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-3t}(c_1 \cos(t) + c_2 \sin(t))\right) + \left(\frac{\cos(t)\,e^{-2t}}{5} + \frac{2\sin(t)\,e^{-2t}}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = e^{-3t}(c_1 \cos(t) + c_2 \sin(t)) + \frac{\cos(t)\, e^{-2t}}{5} + \frac{2 \sin(t)\, e^{-2t}}{5} \tag{1}$$

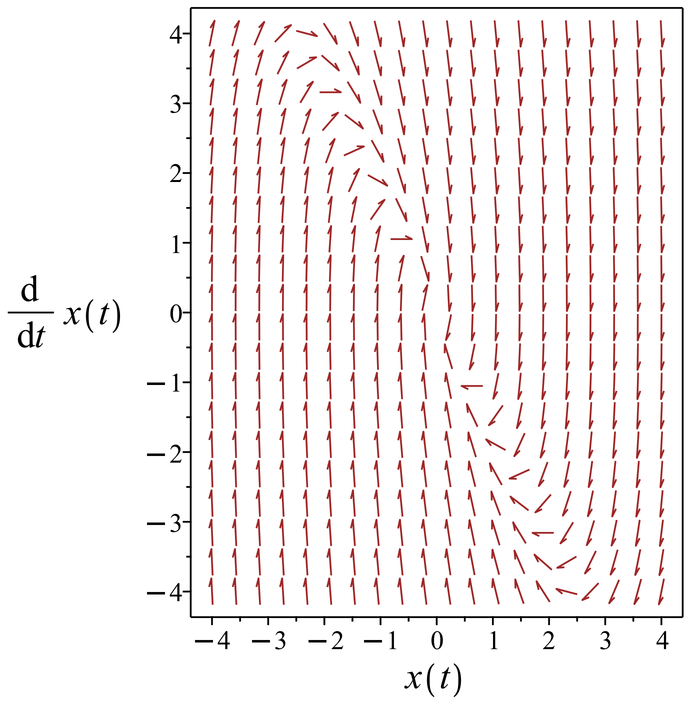


Figure 116: Slope field plot

Verification of solutions

$$x = e^{-3t}(c_1 \cos(t) + c_2 \sin(t)) + \frac{\cos(t)\, e^{-2t}}{5} + \frac{2 \sin(t)\, e^{-2t}}{5}$$

Verified OK.

7.10.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 6x' + 10x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 129: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\, dt} \\
&= z_1 e^{-3t} \\
&= z_1\left(\mathrm{e}^{-3t}\right)
\end{aligned}
$$

Which simplifies to

$$x_1 = \mathrm{e}^{-3t} \cos(t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{6}{1}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-6t}}{(x_1)^2}\, dt \\ &= x_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1\left(\mathrm{e}^{-3t} \cos(t)\right) + c_2\left(\mathrm{e}^{-3t} \cos(t)(\tan(t))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 6x' + 10x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-3t} \cos(t) + c_2 \mathrm{e}^{-3t} \sin(t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)\, \mathrm{e}^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t)\,\mathrm{e}^{-2t}, \sin(t)\,\mathrm{e}^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3t}\cos(t), \mathrm{e}^{-3t}\sin(t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\cos(t)\,\mathrm{e}^{-2t} + A_2\sin(t)\,\mathrm{e}^{-2t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\cos(t)\,\mathrm{e}^{-2t} - 2A_1\sin(t)\,\mathrm{e}^{-2t} + A_2\sin(t)\,\mathrm{e}^{-2t} + 2A_2\cos(t)\,\mathrm{e}^{-2t} = \cos(t)\,\mathrm{e}^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = \frac{2}{5}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{\cos(t)\,\mathrm{e}^{-2t}}{5} + \frac{2\sin(t)\,\mathrm{e}^{-2t}}{5}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1\mathrm{e}^{-3t}\cos(t) + c_2\mathrm{e}^{-3t}\sin(t)\right) + \left(\frac{\cos(t)\,\mathrm{e}^{-2t}}{5} + \frac{2\sin(t)\,\mathrm{e}^{-2t}}{5}\right) \end{aligned}$$

Which simplifies to

$$x = \mathrm{e}^{-3t}(c_1\cos(t) + c_2\sin(t)) + \frac{\cos(t)\,\mathrm{e}^{-2t}}{5} + \frac{2\sin(t)\,\mathrm{e}^{-2t}}{5}$$

Summary

The solution(s) found are the following

$$x = e^{-3t}(c_1 \cos(t) + c_2 \sin(t)) + \frac{\cos(t)\, e^{-2t}}{5} + \frac{2\sin(t)\, e^{-2t}}{5} \tag{1}$$

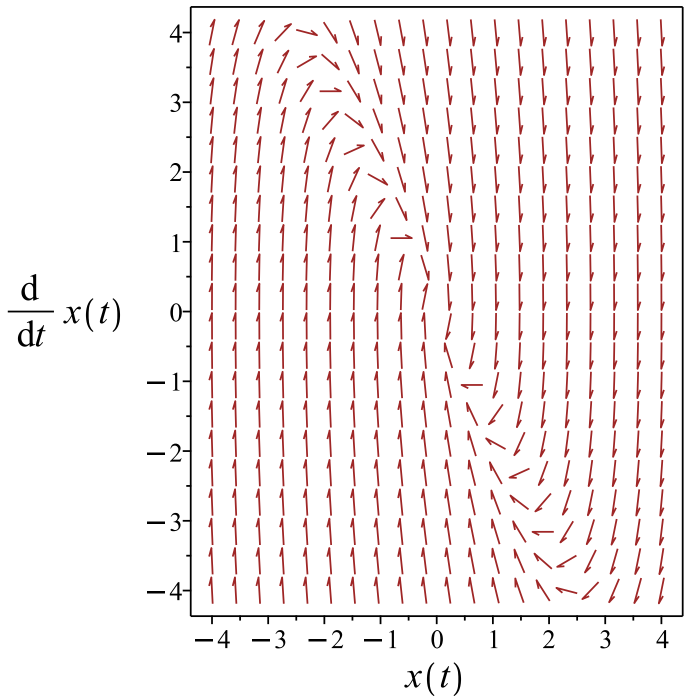


Figure 117: Slope field plot

Verification of solutions

$$x = e^{-3t}(c_1 \cos(t) + c_2 \sin(t)) + \frac{\cos(t)\, e^{-2t}}{5} + \frac{2\sin(t)\, e^{-2t}}{5}$$

Verified OK.

7.10.3 Maple step by step solution

Let's solve

$x'' + 6x' + 10x = \cos(t)\, e^{-2t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

$r^2 + 6r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - \mathrm{I}, -3 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-3t} \cos(t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-3t} \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-3t} \cos(t) + c_2 \mathrm{e}^{-3t} \sin(t) + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \cos(t)\,\mathrm{e}^{-2t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t}\cos(t) & \mathrm{e}^{-3t}\sin(t) \\ -3\,\mathrm{e}^{-3t}\cos(t) - \mathrm{e}^{-3t}\sin(t) & -3\,\mathrm{e}^{-3t}\sin(t) + \mathrm{e}^{-3t}\cos(t) \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = \mathrm{e}^{-6t}$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\frac{\mathrm{e}^{-3t}\left(\cos(t)\left(\int \mathrm{e}^{t}\sin(2t)dt\right) - 2\sin(t)\left(\int \cos(t)^2\mathrm{e}^{t}dt\right)\right)}{2}$

  - Compute integrals

    $x_p(t) = \frac{\mathrm{e}^{-2t}(\cos(t) + 2\sin(t))}{5}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-3t} \cos(t) + c_2 \mathrm{e}^{-3t} \sin(t) + \frac{\mathrm{e}^{-2t}(\cos(t) + 2\sin(t))}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(x(t),t$2)+6*diff(x(t),t)+10*x(t)=exp(-2*t)*cos(t),x(t), singsol=all)
```

$$x(t) = (\sin(t)\, c_2 + \cos(t)\, c_1)\, \mathrm{e}^{-3t} + \frac{\mathrm{e}^{-2t}(\cos(t) + 2\sin(t))}{5}$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 33

```
DSolve[x''[t]+6*x'[t]+10*x[t]==Exp[-3*t]*Cos[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2} e^{-3t}((1 + 2c_2)\cos(t) + (t + 2c_1)\sin(t))$$

7.11 problem 14.1 (xi)

Internal problem ID [12039]
Internal file name [`OUTPUT/10691_Sunday_September_03_2023_12_36_20_PM_16681166/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.1 (xi).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' + 4x' + 4x = e^{2t}}$$

7.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 4, C = 4, f(t) = e^{2t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 4x' + 4x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(4)^2 - (4)(1)(4)} \\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$x = c_1 e^{-2t} + c_2 t e^{-2t} \tag{1}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 e^{-2t} + c_2 t e^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t e^{-2t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 e^{2t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_1 e^{2t} = e^{2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{16}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{e^{2t}}{16}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-2t} + c_2 t\, e^{-2t}\right) + \left(\frac{e^{2t}}{16}\right) \end{aligned}$$

Which simplifies to

$$x = e^{-2t}(c_2 t + c_1) + \frac{e^{2t}}{16}$$

Summary

The solution(s) found are the following

$$x = e^{-2t}(c_2 t + c_1) + \frac{e^{2t}}{16} \tag{1}$$

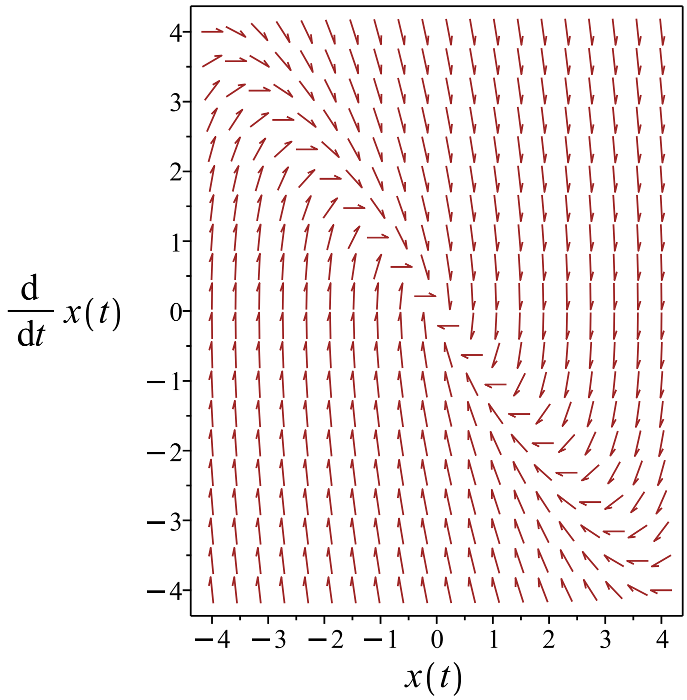


Figure 118: Slope field plot

Verification of solutions

$$x = e^{-2t}(c_2 t + c_1) + \frac{e^{2t}}{16}$$

Verified OK.

7.11.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$x'' + p(t)\, x' + \frac{\left(p(t)^2 + p'(t)\right) x}{2} = f(t)$$

Where $p(t) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 4\, dx} \\ &= e^{2t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)x)'' = e^{4t}$$
$$(e^{2t}x)'' = e^{4t}$$

Integrating once gives

$$(e^{2t}x)' = \frac{e^{4t}}{4} + c_1$$

Integrating again gives

$$(e^{2t}x) = c_1 t + \frac{e^{4t}}{16} + c_2$$

Hence the solution is

$$x = \frac{c_1 t + \frac{e^{4t}}{16} + c_2}{e^{2t}}$$

Or

$$x = \frac{e^{2t}}{16} + c_1 t\, e^{-2t} + c_2 e^{-2t}$$

Summary

The solution(s) found are the following

$$x = \frac{e^{2t}}{16} + c_1 t\, e^{-2t} + c_2 e^{-2t} \tag{1}$$

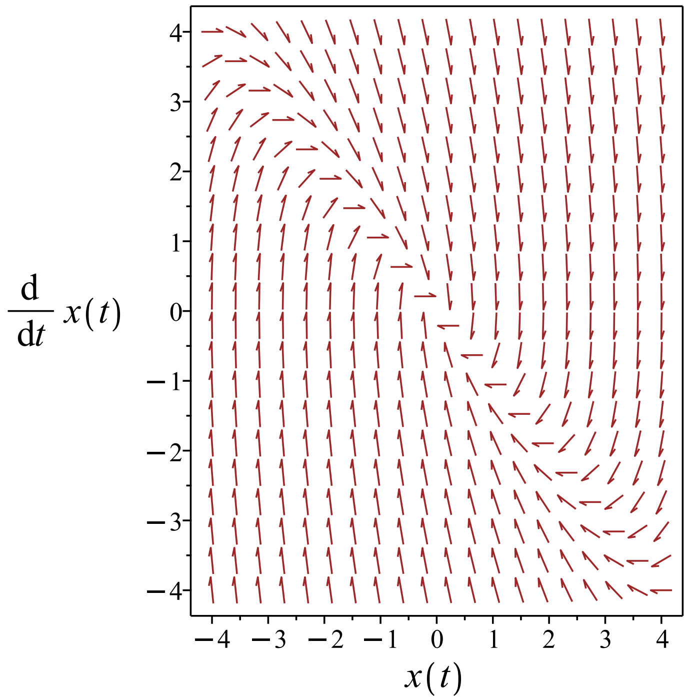


Figure 119: Slope field plot

Verification of solutions

$$x = \frac{e^{2t}}{16} + c_1 t\,e^{-2t} + c_2 e^{-2t}$$

Verified OK.

7.11.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 4x' + 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\ &= z_1 e^{-2t} \\ &= z_1 \left(\mathrm{e}^{-2t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-2t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-4t}}{(x_1)^2}\,dt \\ &= x_1(t) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-2t}\right) + c_2\left(\mathrm{e}^{-2t}(t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$
x'' + 4x' + 4x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-2t} + c_2 t\, \mathrm{e}^{-2t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{2t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{2t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{t\,\mathrm{e}^{-2t}, \mathrm{e}^{-2t}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 \mathrm{e}^{2t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
16A_1 \mathrm{e}^{2t} = \mathrm{e}^{2t}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{16}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{e^{2t}}{16}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-2t} + c_2 t\, e^{-2t}\right) + \left(\frac{e^{2t}}{16}\right) \end{aligned}$$

Which simplifies to

$$x = e^{-2t}(c_2 t + c_1) + \frac{e^{2t}}{16}$$

Summary
The solution(s) found are the following

$$x = e^{-2t}(c_2 t + c_1) + \frac{e^{2t}}{16} \tag{1}$$

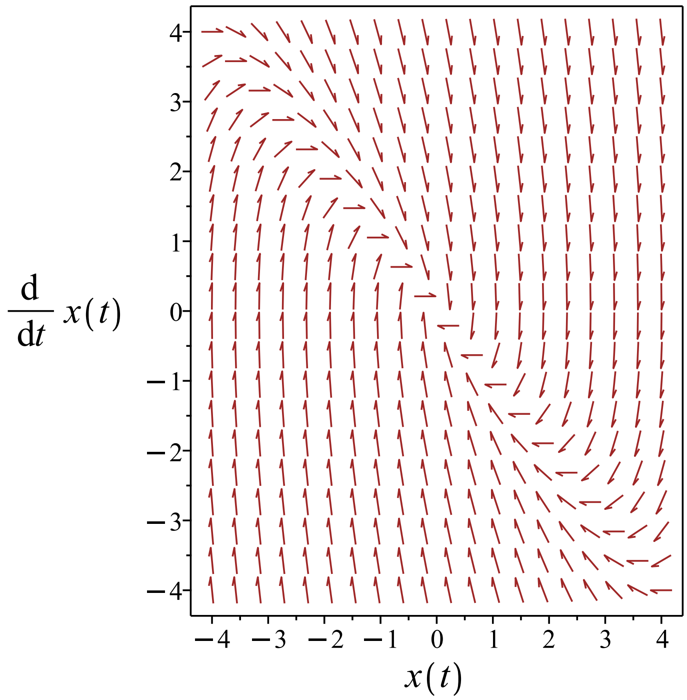


Figure 120: Slope field plot

Verification of solutions

$$x = e^{-2t}(c_2 t + c_1) + \frac{e^{2t}}{16}$$

Verified OK.

7.11.4 Maple step by step solution

Let's solve

$x'' + 4x' + 4x = e^{2t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r+2)^2 = 0$

- Root of the characteristic polynomial

$r = -2$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-2t}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = t\,\mathrm{e}^{-2t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-2t} + c_2 t\,\mathrm{e}^{-2t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \mathrm{e}^{2t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & t\,\mathrm{e}^{-2t} \\ -2\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t} - 2t\,\mathrm{e}^{-2t} \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = \mathrm{e}^{-4t}$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = \mathrm{e}^{-2t}\left(-\left(\int \mathrm{e}^{4t} t dt\right) + \left(\int \mathrm{e}^{4t} dt\right) t\right)$

  - Compute integrals

    $x_p(t) = \frac{\mathrm{e}^{2t}}{16}$

- Substitute particular solution into general solution to ODE

  $x = c_2 t\,\mathrm{e}^{-2t} + c_1 \mathrm{e}^{-2t} + \frac{\mathrm{e}^{2t}}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(x(t),t$2)+4*diff(x(t),t)+4*x(t)=exp(2*t),x(t), singsol=all)
```

$$x(t) = (c_1 t + c_2)\,\mathrm{e}^{-2t} + \frac{\mathrm{e}^{2t}}{16}$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 28

```
DSolve[x''[t]+4*x'[t]+4*x[t]==Exp[2*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{e^{2t}}{16} + e^{-2t}(c_2 t + c_1)$$

7.12 problem 14.2

Internal problem ID [12040]
Internal file name [`OUTPUT/10692_Sunday_September_03_2023_12_36_21_PM_42345115/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + x' - 2x = 12\,\mathrm{e}^{-t} - 6\,\mathrm{e}^{t}}$$

7.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 1, C = -2, f(t) = 12\,\mathrm{e}^{-t} - 6\,\mathrm{e}^{t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' - 2x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \lambda \,\mathrm{e}^{\lambda t} - 2\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-2)} \\
&= -\frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{3}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
x &= c_1 e^{(1)t} + c_2 e^{(-2)t}
\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-2t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\,\mathrm{e}^{-t}-6\,\mathrm{e}^{t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{t}\},\{\mathrm{e}^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{t},\mathrm{e}^{-2t}\}$$

Since e^{t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,\mathrm{e}^{t}\},\{\mathrm{e}^{-t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t\,\mathrm{e}^{t} + A_2\mathrm{e}^{-t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1\mathrm{e}^{t} - 2A_2\mathrm{e}^{-t} = 12\,\mathrm{e}^{-t} - 6\,\mathrm{e}^{t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = -6]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -2t\,\mathrm{e}^{t} - 6\,\mathrm{e}^{-t}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1\mathrm{e}^{t} + c_2\mathrm{e}^{-2t}\right) + \left(-2t\,\mathrm{e}^{t} - 6\,\mathrm{e}^{-t}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = c_1 e^t + c_2 e^{-2t} - 2t\, e^t - 6\, e^{-t} \tag{1}$$

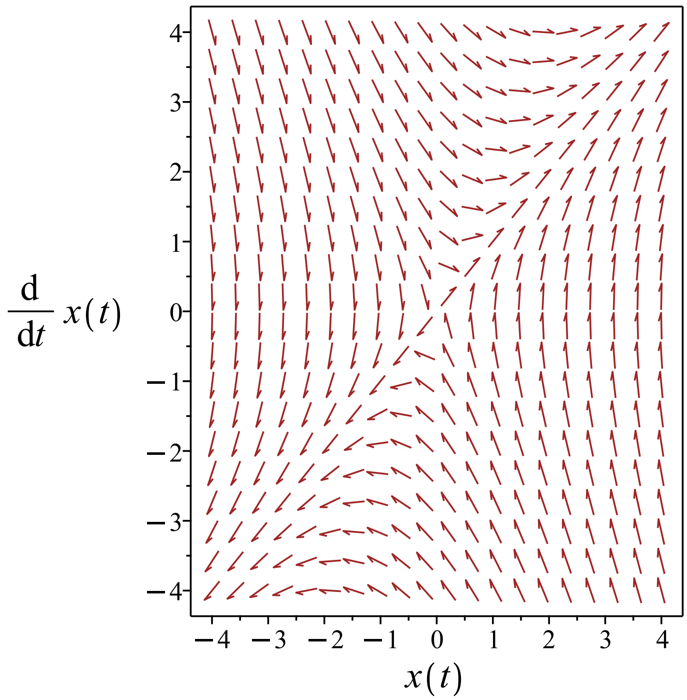


Figure 121: Slope field plot

Verification of solutions

$$x = c_1 e^t + c_2 e^{-2t} - 2t\, e^t - 6\, e^{-t}$$

Verified OK.

7.12.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' - 2x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{9z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 133: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{3t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt}\\ &= z_1 e^{-\frac{t}{2}}\\ &= z_1\left(\mathrm{e}^{-\frac{t}{2}}\right)\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-2t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\ &= x_1 \left(\frac{\mathrm{e}^{3t}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(\mathrm{e}^{-2t}\right) + c_2 \left(\mathrm{e}^{-2t} \left(\frac{\mathrm{e}^{3t}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + x' - 2x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \mathrm{e}^{-2t} + \frac{c_2 \mathrm{e}^{t}}{3}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \mathrm{e}^{-2t}$$
$$x_2 = \frac{\mathrm{e}^t}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$
$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2t} & \frac{\mathrm{e}^t}{3} \\ \frac{d}{dt}\left(\mathrm{e}^{-2t}\right) & \frac{d}{dt}\left(\frac{\mathrm{e}^t}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2t} & \frac{\mathrm{e}^t}{3} \\ -2\,\mathrm{e}^{-2t} & \frac{\mathrm{e}^t}{3} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2t}\right)\left(\frac{\mathrm{e}^t}{3}\right) - \left(\frac{\mathrm{e}^t}{3}\right)\left(-2\,\mathrm{e}^{-2t}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-2t}\mathrm{e}^t$$

Which simplifies to

$$W = e^{-t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^t(12\,e^{-t}-6\,e^t)}{3}}{e^{-t}}\,dt$$

Which simplifies to

$$u_1 = -\int \left(-2\,e^{3t} + 4\,e^t\right) dt$$

Hence

$$u_1 = -4\,e^t + \frac{2\,e^{3t}}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2t}(12\,e^{-t} - 6\,e^t)}{e^{-t}}\,dt$$

Which simplifies to

$$u_2 = \int \left(-6 + 12\,e^{-2t}\right) dt$$

Hence

$$u_2 = -6t - 6\,e^{-2t}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \left(-4\,e^t + \frac{2\,e^{3t}}{3}\right) e^{-2t} + \frac{(-6t - 6\,e^{-2t})\,e^t}{3}$$

Which simplifies to

$$x_p(t) = -6\,e^{-t} + \frac{2\,e^t}{3} - 2t\,e^t$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-2t} + \frac{c_2 e^t}{3}\right) + \left(-6\,e^{-t} + \frac{2\,e^t}{3} - 2t\,e^t\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-2t} + \frac{c_2 e^{t}}{3} - 6\,e^{-t} + \frac{2\,e^{t}}{3} - 2t\,e^{t} \tag{1}$$

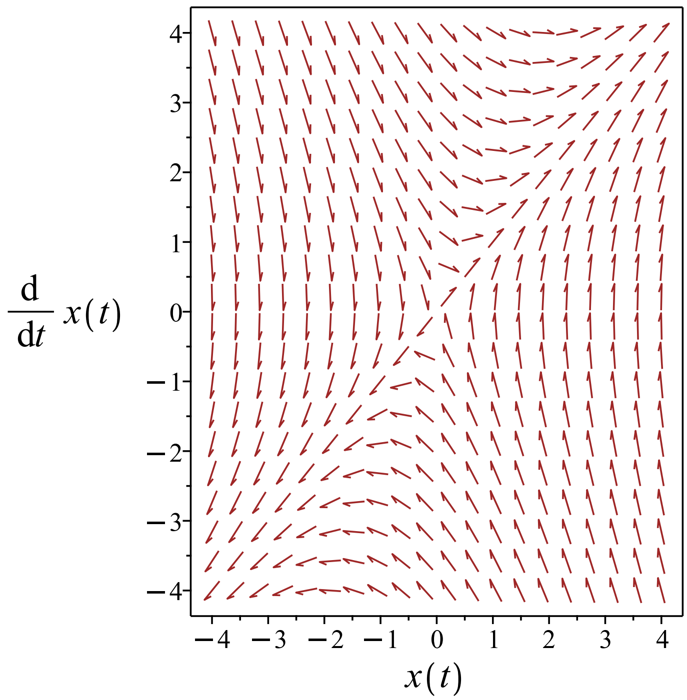


Figure 122: Slope field plot

Verification of solutions

$$x = c_1 e^{-2t} + \frac{c_2 e^{t}}{3} - 6\,e^{-t} + \frac{2\,e^{t}}{3} - 2t\,e^{t}$$

Verified OK.

7.12.3 Maple step by step solution

Let's solve

$x'' + x' - 2x = 12\,e^{-t} - 6\,e^{t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

$r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$
- Roots of the characteristic polynomial

  $r = (-2, 1)$
- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-2t}$
- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{t}$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 12\,\mathrm{e}^{-t} - 6\,\mathrm{e}^{t}\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{t} \\ -2\,\mathrm{e}^{-2t} & \mathrm{e}^{t} \end{bmatrix}$
- ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = 3\,\mathrm{e}^{-t}$
- ○ Substitute functions into equation for $x_p(t)$

  $x_p(t) = 2\left(\mathrm{e}^{3t}\left(\int (-1 + 2\,\mathrm{e}^{-2t})\, dt\right) - \left(\int (2\,\mathrm{e}^{t} - \mathrm{e}^{3t})\, dt\right)\right)\mathrm{e}^{-2t}$
- ○ Compute integrals

  $x_p(t) = -6\,\mathrm{e}^{-t} + \frac{2\,\mathrm{e}^{t}}{3} - 2t\,\mathrm{e}^{t}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{t} - 6\,\mathrm{e}^{-t} + \frac{2\,\mathrm{e}^{t}}{3} - 2t\,\mathrm{e}^{t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(x(t),t$2)+diff(x(t),t)-2*x(t)=12*exp(-t)-6*exp(t),x(t), singsol=all)
```

$$x(t) = -2\left(\left(t - \frac{c_2}{2} - \frac{1}{3}\right) e^{3t} - \frac{c_1}{2} + 3\,e^{t}\right) e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 34

```
DSolve[x''[t]+x'[t]-2*x[t]==12*Exp[-t]-6*Exp[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-2t}\left(-6e^{t} + e^{3t}\left(-2t + \frac{2}{3} + c_2\right) + c_1\right)$$

7.13 problem 14.3

Internal problem ID [12041]
Internal file name [`OUTPUT/10693_Sunday_September_03_2023_12_36_23_PM_24620877/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 14, Inhomogeneous second order linear equations. Exercises page 140
Problem number: 14.3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + 4x = 289t\,\mathrm{e}^{t} \sin(2t)}$$

7.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = 289t\,\mathrm{e}^{t} \sin(2t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 4x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0(c_1 \cos{(2t)} + c_2 \sin{(2t)})$$

Or

$$x = c_1 \cos{(2t)} + c_2 \sin{(2t)}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$289t\,\mathrm{e}^t \sin(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^t \cos(2t), \mathrm{e}^t \sin(2t), t\,\mathrm{e}^t \cos(2t), t\,\mathrm{e}^t \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2t), \sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\mathrm{e}^t \cos(2t) + A_2\mathrm{e}^t \sin(2t) + A_3 t\,\mathrm{e}^t \cos(2t) + A_4 t\,\mathrm{e}^t \sin(2t)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&A_1\mathrm{e}^t \cos(2t) - 4A_1\mathrm{e}^t \sin(2t) + A_2\mathrm{e}^t \sin(2t) + 4A_2\mathrm{e}^t \cos(2t) + 2A_3\mathrm{e}^t \cos(2t)\\
&- 4A_3\mathrm{e}^t \sin(2t) + A_3 t\,\mathrm{e}^t \cos(2t) - 4A_3 t\,\mathrm{e}^t \sin(2t) + 2A_4\mathrm{e}^t \sin(2t)\\
&+ 4A_4\mathrm{e}^t \cos(2t) + A_4 t\,\mathrm{e}^t \sin(2t) + 4A_4 t\,\mathrm{e}^t \cos(2t) = 289t\,\mathrm{e}^t \sin(2t)
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 76, A_2 = -2, A_3 = -68, A_4 = 17]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 76\,\mathrm{e}^t \cos(2t) - 2\,\mathrm{e}^t \sin(2t) - 68t\,\mathrm{e}^t \cos(2t) + 17t\,\mathrm{e}^t \sin(2t)$$

Therefore the general solution is

$$x = x_h + x_p$$
$$= (c_1 \cos(2t) + c_2 \sin(2t)) + \left(76\,\mathrm{e}^t \cos(2t) - 2\,\mathrm{e}^t \sin(2t) - 68t\,\mathrm{e}^t \cos(2t) + 17t\,\mathrm{e}^t \sin(2t)\right)$$

Summary

The solution(s) found are the following

$$x = c_1 \cos(2t) + c_2 \sin(2t) + 76\,\mathrm{e}^t \cos(2t) - 2\,\mathrm{e}^t \sin(2t) - 68t\,\mathrm{e}^t \cos(2t) + 17t\,\mathrm{e}^t \sin(2t) \tag{1}$$

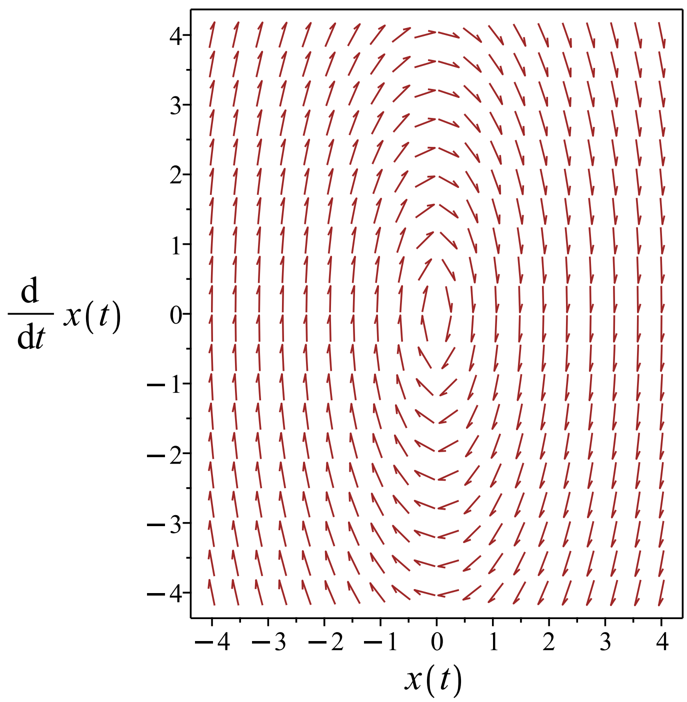


Figure 123: Slope field plot

Verification of solutions

$$x = c_1 \cos(2t) + c_2 \sin(2t) + 76\,\mathrm{e}^t \cos(2t) - 2\,\mathrm{e}^t \sin(2t) - 68t\,\mathrm{e}^t \cos(2t) + 17t\,\mathrm{e}^t \sin(2t)$$

Verified OK.

7.13.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 135: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(2t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \cos(2t)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \cos(2t)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\
&= \cos(2t) \int \frac{1}{\cos(2t)^2}\, dt \\
&= \cos(2t) \left(\frac{\tan(2t)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 4x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$289t\,e^t \sin(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^t \cos(2t), e^t \sin(2t), t\,e^t \cos(2t), t\,e^t \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2t)}{2}, \cos(2t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 e^t \cos(2t) + A_2 e^t \sin(2t) + A_3 t\,e^t \cos(2t) + A_4 t\,e^t \sin(2t)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&A_1 e^t \cos(2t) - 4A_1 e^t \sin(2t) + A_2 e^t \sin(2t) + 4A_2 e^t \cos(2t) + 2A_3 e^t \cos(2t)\\&- 4A_3 e^t \sin(2t) + A_3 t\,e^t \cos(2t) - 4A_3 t\,e^t \sin(2t) + 2A_4 e^t \sin(2t)\\&+ 4A_4 e^t \cos(2t) + A_4 t\,e^t \sin(2t) + 4A_4 t\,e^t \cos(2t) = 289t\,e^t \sin(2t)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 76, A_2 = -2, A_3 = -68, A_4 = 17]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 76\,e^t \cos(2t) - 2\,e^t \sin(2t) - 68t\,e^t \cos(2t) + 17t\,e^t \sin(2t)$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}\right) + \left(76\,\mathrm{e}^{t} \cos(2t) - 2\,\mathrm{e}^{t} \sin(2t) - 68t\,\mathrm{e}^{t} \cos(2t) + 17t\,\mathrm{e}^{t} \sin(2t)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
x = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} + 76\,\mathrm{e}^{t} \cos(2t) \\
- 2\,\mathrm{e}^{t} \sin(2t) - 68t\,\mathrm{e}^{t} \cos(2t) + 17t\,\mathrm{e}^{t} \sin(2t)
\end{aligned} \tag{1}
$$

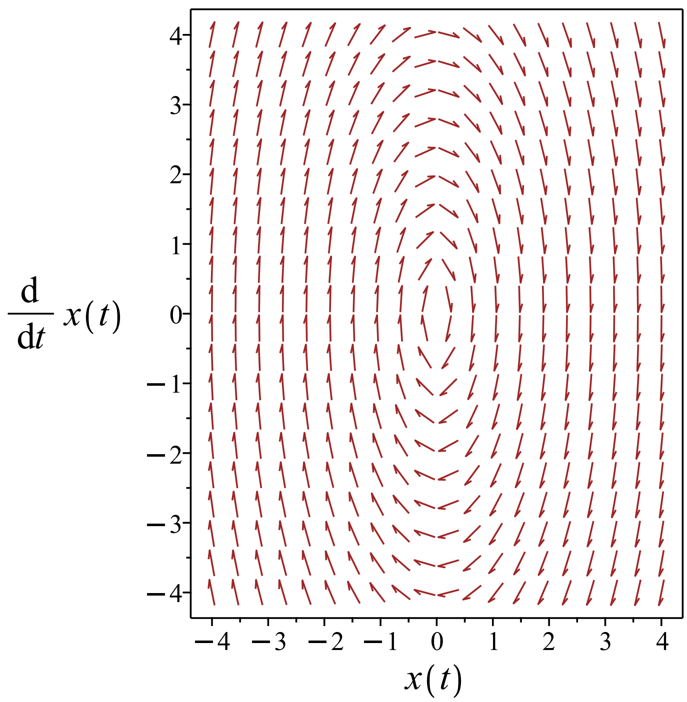


Figure 124: Slope field plot

Verification of solutions

$$
x = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} + 76\,\mathrm{e}^{t} \cos(2t) - 2\,\mathrm{e}^{t} \sin(2t) - 68t\,\mathrm{e}^{t} \cos(2t) + 17t\,\mathrm{e}^{t} \sin(2t)
$$

Verified OK.

7.13.3 Maple step by step solution

Let's solve

$x'' + 4x = 289t\,e^{t} \sin(2t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(2t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(2t) + c_2 \sin(2t) + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 289t\,e^{t} \sin(2t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

- ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = 2$

- ○ Substitute functions into equation for $x_p(t)$

$$x_p(t) = -\frac{289 \cos(2t) \left(\int \sin(2t)^2 t \,\mathrm{e}^t dt\right)}{2} + \frac{289 \sin(2t) \left(\int \sin(4t) t \,\mathrm{e}^t dt\right)}{4}$$

- ○ Compute integrals

  $$x_p(t) = \mathrm{e}^t (17 \sin(2t)\, t - 68 \cos(2t)\, t - 2 \sin(2t) + 76 \cos(2t))$$

- Substitute particular solution into general solution to ODE

  $$x = c_1 \cos(2t) + c_2 \sin(2t) + \mathrm{e}^t (17 \sin(2t)\, t - 68 \cos(2t)\, t - 2 \sin(2t) + 76 \cos(2t))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(x(t),t$2)+4*x(t)=289*t*exp(t)*sin(2*t),x(t), singsol=all)
```

$$x(t) = \left((-68t + 76)\,\mathrm{e}^t + c_1\right) \cos(2t) + 17 \sin(2t) \left(\mathrm{e}^t \left(t - \frac{2}{17}\right) + \frac{c_2}{17}\right)$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 40

```
DSolve[x''[t]+4*x[t]==289*t*Exp[t]*Sin[2*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \left(e^t(76 - 68t) + c_1\right) \cos(2t) + \left(e^t(17t - 2) + c_2\right) \sin(2t)$$

8 Chapter 15, Resonance. Exercises page 148

8.1 problem 15.1

Internal problem ID [12042]
Internal file name [`OUTPUT/10694_Sunday_September_03_2023_12_36_27_PM_89810358/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 15, Resonance. Exercises page 148
Problem number: 15.1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + \omega^2 x = \cos(\alpha t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

8.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= \omega^2 \\ F &= \cos(\alpha t) \end{aligned}$$

Hence the ode is

$$x'' + \omega^2 x = \cos(\alpha t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \omega^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(\alpha t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.1.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = \omega^2, f(t) = \cos(\alpha t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + \omega^2 x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = \omega^2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \omega^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \omega^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\omega^2)}\\ &= \pm\sqrt{-\omega^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-\omega^2}\\ \lambda_2 &= -\sqrt{-\omega^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-\omega^2}\\ \lambda_2 &= -\sqrt{-\omega^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{\left(\sqrt{-\omega^2}\right)t} + c_2 e^{\left(-\sqrt{-\omega^2}\right)t}\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(\alpha t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\alpha t), \sin(\alpha t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\sqrt{-\omega^2}\, t}, e^{-\sqrt{-\omega^2}\, t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(\alpha t) + A_2 \sin(\alpha t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\alpha^2 \cos(\alpha t) - A_2\alpha^2 \sin(\alpha t) + \omega^2(A_1 \cos(\alpha t) + A_2 \sin(\alpha t)) = \cos(\alpha t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{\alpha^2 - \omega^2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{\cos(\alpha t)}{\alpha^2 - \omega^2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t}\right) + \left(-\frac{\cos(\alpha t)}{\alpha^2 - \omega^2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t} - \frac{\cos(\alpha t)}{\alpha^2 - \omega^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \frac{(c_1 + c_2)\alpha^2 - 1 + (-c_1 - c_2)\omega^2}{\alpha^2 - \omega^2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1\sqrt{-\omega^2}\,\mathrm{e}^{\sqrt{-\omega^2}\,t} - c_2\sqrt{-\omega^2}\,\mathrm{e}^{-\sqrt{-\omega^2}\,t} + \frac{\alpha \sin(\alpha t)}{\alpha^2 - \omega^2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = (c_1 - c_2)\sqrt{-\omega^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{1}{2\alpha^2 - 2\omega^2} \\ c_2 &= \frac{1}{2\alpha^2 - 2\omega^2} \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{-2\cos(\alpha t) + \mathrm{e}^{\sqrt{-\omega^2}\,t} + \mathrm{e}^{-\sqrt{-\omega^2}\,t}}{2\alpha^2 - 2\omega^2}$$

Summary

The solution(s) found are the following

$$x = \frac{-2\cos(\alpha t) + \mathrm{e}^{\sqrt{-\omega^2}\,t} + \mathrm{e}^{-\sqrt{-\omega^2}\,t}}{2\alpha^2 - 2\omega^2} \tag{1}$$

Verification of solutions

$$x = \frac{-2\cos(\alpha t) + \mathrm{e}^{\sqrt{-\omega^2}\,t} + \mathrm{e}^{-\sqrt{-\omega^2}\,t}}{2\alpha^2 - 2\omega^2}$$

Verified OK.

8.1.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + \omega^2 x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \omega^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\omega^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\omega^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\omega^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 137: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\omega^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{\sqrt{-\omega^2}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{-\omega^2}\, t} \end{aligned}$$

Which simplifies to

$$x_1 = e^{\sqrt{-\omega^2}\, t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\
&= e^{\sqrt{-\omega^2}\, t} \int \frac{1}{e^{2\sqrt{-\omega^2}\, t}}\, dt \\
&= e^{\sqrt{-\omega^2}\, t} \left(\frac{\sqrt{-\omega^2}\, e^{-2\sqrt{-\omega^2}\, t}}{2\omega^2} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(e^{\sqrt{-\omega^2}\, t} \right) + c_2 \left(e^{\sqrt{-\omega^2}\, t} \left(\frac{\sqrt{-\omega^2}\, e^{-2\sqrt{-\omega^2}\, t}}{2\omega^2} \right) \right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + \omega^2 x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 e^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(\alpha t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\alpha t), \sin(\alpha t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{-\omega^2}\,\mathrm{e}^{-\sqrt{-\omega^2}\,t}}{2\omega^2}, \mathrm{e}^{\sqrt{-\omega^2}\,t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(\alpha t) + A_2 \sin(\alpha t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\alpha^2 \cos(\alpha t) - A_2\alpha^2 \sin(\alpha t) + \omega^2(A_1 \cos(\alpha t) + A_2 \sin(\alpha t)) = \cos(\alpha t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{\alpha^2 - \omega^2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{\cos(\alpha t)}{\alpha^2 - \omega^2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1\mathrm{e}^{\sqrt{-\omega^2}\,t} + \frac{c_2\sqrt{-\omega^2}\,\mathrm{e}^{-\sqrt{-\omega^2}\,t}}{2\omega^2}\right) + \left(-\frac{\cos(\alpha t)}{\alpha^2 - \omega^2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2} - \frac{\cos(\alpha t)}{\alpha^2 - \omega^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \frac{(\alpha^2 c_2 - c_2\omega^2)\sqrt{-\omega^2} + 2\omega^2(\alpha^2 c_1 - c_1\omega^2 - 1)}{2\alpha^2\omega^2 - 2\omega^4} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1\sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t} + \frac{c_2 e^{-\sqrt{-\omega^2}\, t}}{2} + \frac{\alpha \sin(\alpha t)}{\alpha^2 - \omega^2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = \sqrt{-\omega^2}\, c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2\alpha^2 - 2\omega^2}$$
$$c_2 = -\frac{\sqrt{-\omega^2}}{\alpha^2 - \omega^2}$$

Substituting these values back in above solution results in

$$x = \frac{-2\cos(\alpha t) + e^{\sqrt{-\omega^2}\, t} + e^{-\sqrt{-\omega^2}\, t}}{2\alpha^2 - 2\omega^2}$$

Summary

The solution(s) found are the following

$$x = \frac{-2\cos(\alpha t) + e^{\sqrt{-\omega^2}\, t} + e^{-\sqrt{-\omega^2}\, t}}{2\alpha^2 - 2\omega^2} \tag{1}$$

Verification of solutions

$$x = \frac{-2\cos(\alpha t) + e^{\sqrt{-\omega^2}\, t} + e^{-\sqrt{-\omega^2}\, t}}{2\alpha^2 - 2\omega^2}$$

Verified OK.

8.1.4 Maple step by step solution

Let's solve

$$\left[x''+\omega^2 x=\cos\left(\alpha t\right), x(0)=0, x'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $\omega^2+r^2=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm\left(\sqrt{-4\omega^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(\sqrt{-\omega^2}, -\sqrt{-\omega^2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t)=\mathrm{e}^{\sqrt{-\omega^2}\, t}$

- 2nd solution of the homogeneous ODE

  $x_2(t)=\mathrm{e}^{-\sqrt{-\omega^2}\, t}$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)+x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x=c_1\mathrm{e}^{\sqrt{-\omega^2}\, t}+c_2\mathrm{e}^{-\sqrt{-\omega^2}\, t}+x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t)=-x_1(t)\left(\int\frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int\frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t)=\cos\left(\alpha t\right)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t))=\left[\begin{array}{cc}\mathrm{e}^{\sqrt{-\omega^2}\, t} & \mathrm{e}^{-\sqrt{-\omega^2}\, t}\\ \sqrt{-\omega^2}\,\mathrm{e}^{\sqrt{-\omega^2}\, t} & -\sqrt{-\omega^2}\,\mathrm{e}^{-\sqrt{-\omega^2}\, t}\end{array}\right]$

- ○ Compute Wronskian

  $W(x_1(t), x_2(t))=-2\sqrt{-\omega^2}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = \frac{e^{\sqrt{-\omega^2}\,t}\left(\int e^{-\sqrt{-\omega^2}\,t}\cos(\alpha t)dt\right) - e^{-\sqrt{-\omega^2}\,t}\left(\int \cos(\alpha t)e^{\sqrt{-\omega^2}\,t}dt\right)}{2\sqrt{-\omega^2}}$

- Compute integrals

  $x_p(t) = -\frac{\cos(\alpha t)}{\alpha^2-\omega^2}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{\sqrt{-\omega^2}\,t} + c_2 e^{-\sqrt{-\omega^2}\,t} - \frac{\cos(\alpha t)}{\alpha^2-\omega^2}$

☐ Check validity of solution $x = c_1 e^{\sqrt{-\omega^2}\,t} + c_2 e^{-\sqrt{-\omega^2}\,t} - \frac{\cos(\alpha t)}{\alpha^2-\omega^2}$

- Use initial condition $x(0) = 0$

  $0 = c_1 + c_2 - \frac{1}{\alpha^2-\omega^2}$

- Compute derivative of the solution

  $x' = c_1\sqrt{-\omega^2}\,e^{\sqrt{-\omega^2}\,t} - c_2\sqrt{-\omega^2}\,e^{-\sqrt{-\omega^2}\,t} + \frac{\alpha\sin(\alpha t)}{\alpha^2-\omega^2}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = \sqrt{-\omega^2}\,c_1 - \sqrt{-\omega^2}\,c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{2(\alpha^2-\omega^2)}, c_2 = \frac{1}{2(\alpha^2-\omega^2)}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{-2\cos(\alpha t) + e^{\sqrt{-\omega^2}\,t} + e^{-\sqrt{-\omega^2}\,t}}{2\alpha^2-2\omega^2}$

- Solution to the IVP

  $x = \frac{-2\cos(\alpha t) + e^{\sqrt{-\omega^2}\,t} + e^{-\sqrt{-\omega^2}\,t}}{2\alpha^2-2\omega^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve([diff(x(t),t$2)+omega^2*x(t)=cos(alpha*t),x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{\cos (\omega t) - \cos (\alpha t)}{\alpha^2 - \omega^2}$$

✓ Solution by Mathematica

Time used: 0.25 (sec). Leaf size: 28

```
DSolve[{x''[t]+w^2*x[t]==Cos[a*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -> Tru
```

$$x(t) \to \frac{\cos(tw) - \cos(at)}{a^2 - w^2}$$

8.2 problem 15.3

Internal problem ID [12043]
Internal file name [`OUTPUT/10695_Sunday_September_03_2023_12_36_30_PM_82874852/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 15, Resonance. Exercises page 148
Problem number: 15.3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + \omega^2 x = \cos(\omega t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

8.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= \omega^2 \\ F &= \cos(\omega t) \end{aligned}$$

Hence the ode is

$$x'' + \omega^2 x = \cos(\omega t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \omega^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(\omega t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.2.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = \omega^2, f(t) = \cos(\omega t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + \omega^2 x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = \omega^2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \omega^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \omega^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\omega^2)} \\ &= \pm\sqrt{-\omega^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-\omega^2} \\ \lambda_2 &= -\sqrt{-\omega^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-\omega^2} \\ \lambda_2 &= -\sqrt{-\omega^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ x &= c_1 e^{\left(\sqrt{-\omega^2}\right)t} + c_2 e^{\left(-\sqrt{-\omega^2}\right)t}\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = e^{\sqrt{-\omega^2}\, t}$$

$$x_2 = e^{-\sqrt{-\omega^2}\, t}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\sqrt{-\omega^2}\, t} & e^{-\sqrt{-\omega^2}\, t} \\ \frac{d}{dt}\left(e^{\sqrt{-\omega^2}\, t}\right) & \frac{d}{dt}\left(e^{-\sqrt{-\omega^2}\, t}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\sqrt{-\omega^2}\, t} & e^{-\sqrt{-\omega^2}\, t} \\ \sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t} & -\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t} \end{vmatrix}$$

Therefore

$$W = \left(e^{\sqrt{-\omega^2}\, t}\right)\left(-\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}\right) - \left(e^{-\sqrt{-\omega^2}\, t}\right)\left(\sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t}\right)$$

Which simplifies to

$$W = -2\, e^{\sqrt{-\omega^2}\, t} \sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}$$

Which simplifies to

$$W = -2\sqrt{-\omega^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-\sqrt{-\omega^2}\,t}\cos(\omega t)}{-2\sqrt{-\omega^2}}\,dt$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-\sqrt{-\omega^2}\,t}\cos(\omega t)}{2\sqrt{-\omega^2}}dt$$

Hence

$$u_1 = -\frac{-\frac{t\,e^{-\sqrt{-\omega^2}\,t}\tan\left(\frac{\omega t}{2}\right)}{2} + \frac{\sqrt{-\omega^2}\,t\,e^{-\sqrt{-\omega^2}\,t}}{4\omega} - \frac{e^{-\sqrt{-\omega^2}\,t}}{4\omega} - \frac{\sqrt{-\omega^2}\,t\,e^{-\sqrt{-\omega^2}\,t}\tan\left(\frac{\omega t}{2}\right)^2}{4\omega} + \frac{e^{-\sqrt{-\omega^2}\,t}\tan\left(\frac{\omega t}{2}\right)^2}{4\omega}}{\omega\left(1+\tan\left(\frac{\omega t}{2}\right)^2\right)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\sqrt{-\omega^2}\,t}\cos(\omega t)}{-2\sqrt{-\omega^2}}\,dt$$

Which simplifies to

$$u_2 = \int -\frac{e^{\sqrt{-\omega^2}\,t}\cos(\omega t)}{2\sqrt{-\omega^2}}dt$$

Hence

$$u_2 = \frac{\frac{t\,e^{\sqrt{-\omega^2}\,t}\tan\left(\frac{\omega t}{2}\right)}{2} + \frac{\sqrt{-\omega^2}\,t\,e^{\sqrt{-\omega^2}\,t}}{4\omega} + \frac{e^{\sqrt{-\omega^2}\,t}}{4\omega} - \frac{\sqrt{-\omega^2}\,t\,e^{\sqrt{-\omega^2}\,t}\tan\left(\frac{\omega t}{2}\right)^2}{4\omega} - \frac{e^{\sqrt{-\omega^2}\,t}\tan\left(\frac{\omega t}{2}\right)^2}{4\omega}}{\omega\left(1+\tan\left(\frac{\omega t}{2}\right)^2\right)}$$

Which simplifies to

$$u_1 = \frac{e^{-\sqrt{-\omega^2}\,t}\left(-\sqrt{-\omega^2}\,t\cos(\omega t) + \sin(\omega t)\,\omega t + \cos(\omega t)\right)}{4\omega^2}$$

$$u_2 = \frac{e^{\sqrt{-\omega^2}\,t}\left(\sqrt{-\omega^2}\,t\cos(\omega t) + \sin(\omega t)\,\omega t + \cos(\omega t)\right)}{4\omega^2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{e^{-\sqrt{-\omega^2}\, t}\left(-\sqrt{-\omega^2}\, t\cos(\omega t)+\sin(\omega t)\,\omega t+\cos(\omega t)\right)e^{\sqrt{-\omega^2}\, t}}{4\omega^2} + \frac{e^{\sqrt{-\omega^2}\, t}\left(\sqrt{-\omega^2}\, t\cos(\omega t)+\sin(\omega t)\,\omega t+\cos(\omega t)\right)e^{-\sqrt{-\omega^2}\, t}}{4\omega^2}$$

Which simplifies to

$$x_p(t) = \frac{\sin(\omega t)\,\omega t+\cos(\omega t)}{2\omega^2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t}\right) + \left(\frac{\sin(\omega t)\,\omega t+\cos(\omega t)}{2\omega^2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t} + \frac{\sin(\omega t)\,\omega t+\cos(\omega t)}{2\omega^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \frac{1+(2c_1+2c_2)\,\omega^2}{2\omega^2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1\sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t} - c_2\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t} + \frac{t\cos(\omega t)}{2}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = (c_1 - c_2)\sqrt{-\omega^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{1}{4\omega^2} \\ c_2 &= -\frac{1}{4\omega^2} \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{2\sin(\omega t)\,\omega t + 2\cos(\omega t) - e^{\sqrt{-\omega^2}\,t} - e^{-\sqrt{-\omega^2}\,t}}{4\omega^2}$$

Summary

The solution(s) found are the following

$$x = \frac{2\sin(\omega t)\,\omega t + 2\cos(\omega t) - e^{\sqrt{-\omega^2}\,t} - e^{-\sqrt{-\omega^2}\,t}}{4\omega^2} \tag{1}$$

Verification of solutions

$$x = \frac{2\sin(\omega t)\,\omega t + 2\cos(\omega t) - e^{\sqrt{-\omega^2}\,t} - e^{-\sqrt{-\omega^2}\,t}}{4\omega^2}$$

Verified OK.

8.2.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + \omega^2 x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \omega^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\omega^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -\omega^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = (-\omega^2)\, z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 139: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\omega^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{\sqrt{-\omega^2}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{-\omega^2}\, t} \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{\sqrt{-\omega^2}\, t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= \mathrm{e}^{\sqrt{-\omega^2}\, t} \int \frac{1}{\mathrm{e}^{2\sqrt{-\omega^2}\, t}}\, dt \\ &= \mathrm{e}^{\sqrt{-\omega^2}\, t} \left(\frac{\sqrt{-\omega^2}\, \mathrm{e}^{-2\sqrt{-\omega^2}\, t}}{2\omega^2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1\left(\mathrm{e}^{\sqrt{-\omega^2}\, t}\right) + c_2\left(\mathrm{e}^{\sqrt{-\omega^2}\, t}\left(\frac{\sqrt{-\omega^2}\, \mathrm{e}^{-2\sqrt{-\omega^2}\, t}}{2\omega^2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + \omega^2 x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1\mathrm{e}^{\sqrt{-\omega^2}\, t} + \frac{c_2\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2}
$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
x_p(t) = u_1x_1 + u_2x_2 \tag{1}
$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
x_1 &= \mathrm{e}^{\sqrt{-\omega^2}\, t} \\
x_2 &= \frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}
$$

$$
u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\sqrt{-\omega^2}\,t} & \frac{\sqrt{-\omega^2}\,e^{-\sqrt{-\omega^2}\,t}}{2\omega^2} \\ \frac{d}{dt}\left(e^{\sqrt{-\omega^2}\,t}\right) & \frac{d}{dt}\left(\frac{\sqrt{-\omega^2}\,e^{-\sqrt{-\omega^2}\,t}}{2\omega^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\sqrt{-\omega^2}\,t} & \frac{\sqrt{-\omega^2}\,e^{-\sqrt{-\omega^2}\,t}}{2\omega^2} \\ \sqrt{-\omega^2}\,e^{\sqrt{-\omega^2}\,t} & \frac{e^{-\sqrt{-\omega^2}\,t}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{\sqrt{-\omega^2}\,t}\right)\left(\frac{e^{-\sqrt{-\omega^2}\,t}}{2}\right) - \left(\frac{\sqrt{-\omega^2}\,e^{-\sqrt{-\omega^2}\,t}}{2\omega^2}\right)\left(\sqrt{-\omega^2}\,e^{\sqrt{-\omega^2}\,t}\right)$$

Which simplifies to

$$W = e^{\sqrt{-\omega^2}\,t} e^{-\sqrt{-\omega^2}\,t}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{-\omega^2}\,e^{-\sqrt{-\omega^2}\,t}\cos(\omega t)}{2\omega^2}}{1}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-\omega^2}\,e^{-\sqrt{-\omega^2}\,t}\cos(\omega t)}{2\omega^2}dt$$

Hence

$$u_1 = -\frac{-\frac{t\,e^{-\sqrt{-\omega^2}\,t}\tan\left(\frac{\omega t}{2}\right)}{2} + \frac{\sqrt{-\omega^2}\,t\,e^{-\sqrt{-\omega^2}\,t}}{4\omega} - \frac{e^{-\sqrt{-\omega^2}\,t}}{4\omega} - \frac{\sqrt{-\omega^2}\,t\,e^{-\sqrt{-\omega^2}\,t}\tan\left(\frac{\omega t}{2}\right)^2}{4\omega} + \frac{e^{-\sqrt{-\omega^2}\,t}\tan\left(\frac{\omega t}{2}\right)^2}{4\omega}}{\omega\left(1+\tan\left(\frac{\omega t}{2}\right)^2\right)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\sqrt{-\omega^2}\, t} \cos(\omega t)}{1} dt$$

Which simplifies to

$$u_2 = \int e^{\sqrt{-\omega^2}\, t} \cos(\omega t)\, dt$$

Hence

$$u_2 = \frac{\frac{e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)}{\omega} + \frac{t\, e^{\sqrt{-\omega^2}\, t}}{2} - \frac{t\, e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{2} - \frac{\sqrt{-\omega^2}\, t\, e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)}{\omega}}{1 + \tan\left(\frac{\omega t}{2}\right)^2}$$

Which simplifies to

$$u_1 = \frac{e^{-\sqrt{-\omega^2}\, t} \left(-\sqrt{-\omega^2}\, t \cos(\omega t) + \sin(\omega t)\, \omega t + \cos(\omega t)\right)}{4\omega^2}$$

$$u_2 = \frac{e^{\sqrt{-\omega^2}\, t} \left(t \cos(\omega t)\, \omega - \sqrt{-\omega^2}\, t \sin(\omega t) + \sin(\omega t)\right)}{2\omega}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{e^{-\sqrt{-\omega^2}\, t} \left(-\sqrt{-\omega^2}\, t \cos(\omega t) + \sin(\omega t)\, \omega t + \cos(\omega t)\right) e^{\sqrt{-\omega^2}\, t}}{4\omega^2} + \frac{e^{\sqrt{-\omega^2}\, t} \left(t \cos(\omega t)\, \omega - \sqrt{-\omega^2}\, t \sin(\omega t) + \sin(\omega t)\right) \sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{4\omega^3}$$

Which simplifies to

$$x_p(t) = \frac{2 \sin(\omega t)\, \omega^2 t + \omega \cos(\omega t) + \sqrt{-\omega^2}\, \sin(\omega t)}{4\omega^3}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2}\right) + \left(\frac{2 \sin(\omega t)\, \omega^2 t + \omega \cos(\omega t) + \sqrt{-\omega^2}\, \sin(\omega t)}{4\omega^3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2} + \frac{2 \sin(\omega t)\, \omega^2 t + \omega \cos(\omega t) + \sqrt{-\omega^2}\, \sin(\omega t)}{4\omega^3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \frac{4c_1\omega^2 + 2\sqrt{-\omega^2}\, c_2 + 1}{4\omega^2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1 \sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t} + \frac{c_2 e^{-\sqrt{-\omega^2}\, t}}{2} + \frac{2\omega^3 \cos(\omega t)\, t + \sin(\omega t)\, \omega^2 + \sqrt{-\omega^2}\, \omega \cos(\omega t)}{4\omega^3}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = \frac{4\sqrt{-\omega^2}\, c_1\omega^2 + 2c_2\omega^2 + \sqrt{-\omega^2}}{4\omega^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{4\omega^2}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$x = \frac{2 \sin(\omega t)\, \omega^2 t + \omega \cos(\omega t) + \sqrt{-\omega^2}\, \sin(\omega t) - e^{\sqrt{-\omega^2}\, t}\omega}{4\omega^3}$$

Summary

The solution(s) found are the following

$$x = \frac{2 \sin(\omega t)\, \omega^2 t + \omega \cos(\omega t) + \sqrt{-\omega^2}\, \sin(\omega t) - e^{\sqrt{-\omega^2}\, t}\omega}{4\omega^3} \tag{1}$$

Verification of solutions

$$x = \frac{2 \sin(\omega t)\, \omega^2 t + \omega \cos(\omega t) + \sqrt{-\omega^2}\, \sin(\omega t) - e^{\sqrt{-\omega^2}\, t}\omega}{4\omega^3}$$

Verified OK.

8.2.4 Maple step by step solution

Let's solve

$$\left[x'' + \omega^2 x = \cos(\omega t), x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $\omega^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4\omega^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-\omega^2}, -\sqrt{-\omega^2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{\sqrt{-\omega^2}\, t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\sqrt{-\omega^2}\, t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \cos(\omega t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} e^{\sqrt{-\omega^2}\, t} & e^{-\sqrt{-\omega^2}\, t} \\ \sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t} & -\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = -2\sqrt{-\omega^2}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = \frac{e^{\sqrt{-\omega^2}\, t}\left(\int e^{-\sqrt{-\omega^2}\, t}\cos(\omega t)dt\right) - e^{-\sqrt{-\omega^2}\, t}\left(\int e^{\sqrt{-\omega^2}\, t}\cos(\omega t)dt\right)}{2\sqrt{-\omega^2}}$

- Compute integrals

  $x_p(t) = \frac{\sin(\omega t)\left(2\sqrt{-\omega^2}\, t-1\right)\omega+\sqrt{-\omega^2}\,\cos(\omega t)}{4\sqrt{-\omega^2}\,\omega^2}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t} + \frac{\sin(\omega t)\left(2\sqrt{-\omega^2}\, t-1\right)\omega+\sqrt{-\omega^2}\,\cos(\omega t)}{4\sqrt{-\omega^2}\,\omega^2}$

□ Check validity of solution $x = c_1 e^{\sqrt{-\omega^2}\, t} + c_2 e^{-\sqrt{-\omega^2}\, t} + \frac{\sin(\omega t)\left(2\sqrt{-\omega^2}\, t-1\right)\omega+\sqrt{-\omega^2}\,\cos(\omega t)}{4\sqrt{-\omega^2}\,\omega^2}$

- Use initial condition $x(0) = 0$

  $0 = c_1 + c_2 + \frac{1}{4\omega^2}$

- Compute derivative of the solution

  $x' = c_1\sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t} - c_2\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t} + \frac{\omega^2\cos(\omega t)\left(2\sqrt{-\omega^2}\, t-1\right)+\sin(\omega t)\sqrt{-\omega^2}\,\omega}{4\sqrt{-\omega^2}\,\omega^2}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = \sqrt{-\omega^2}\, c_1 - \sqrt{-\omega^2}\, c_2 - \frac{1}{4\sqrt{-\omega^2}}$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{4\omega^2}, c_2 = 0\right\}$

- Substitute constant values into general solution and simplify

  $x = -\frac{-\sin(\omega t)\left(2\sqrt{-\omega^2}\, t-1\right)\omega+\sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t}-\sqrt{-\omega^2}\,\cos(\omega t)}{4\sqrt{-\omega^2}\,\omega^2}$

- Solution to the IVP

  $x = -\frac{-\sin(\omega t)\left(2\sqrt{-\omega^2}\, t-1\right)\omega+\sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t}-\sqrt{-\omega^2}\,\cos(\omega t)}{4\sqrt{-\omega^2}\,\omega^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(x(t),t$2)+omega^2*x(t)=cos(omega*t),x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{\sin(\omega t)\, t}{2\omega}$$

✓ Solution by Mathematica

Time used: 0.099 (sec). Leaf size: 17

```
DSolve[{x''[t]+w^2*x[t]==Cos[w*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -> Tru
```

$$x(t) \to \frac{t \sin(tw)}{2w}$$

9 Chapter 16, Higher order linear equations with constant coefficients. Exercises page 153

9.1 problem 16.1 (i)

Internal problem ID [12044]
Internal file name [`OUTPUT/10696_Sunday_September_03_2023_12_36_40_PM_92005170/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 16, Higher order linear equations with constant coefficients. Exercises page 153
Problem number: 16.1 (i).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x''' - 6x'' + 11x' - 6x = e^{-t}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE And x_p is a particular solution to the nonhomogeneous ODE. x_h is the solution to

$$x''' - 6x'' + 11x' - 6x = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$x_h(t) = c_1 e^t + c_2 e^{2t} + e^{3t} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = e^t$$
$$x_2 = e^{2t}$$
$$x_3 = e^{3t}$$

Now the particular solution to the given ODE is found

$$x''' - 6x'' + 11x' - 6x = e^{-t}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^t, e^{2t}, e^{3t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-24A_1 e^{-t} = e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{24}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{e^{-t}}{24}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^t + c_2 e^{2t} + e^{3t} c_3\right) + \left(-\frac{e^{-t}}{24}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^t + c_2 e^{2t} + e^{3t} c_3 - \frac{e^{-t}}{24} \tag{1}$$

Verification of solutions

$$x = c_1 e^t + c_2 e^{2t} + e^{3t} c_3 - \frac{e^{-t}}{24}$$

Verified OK.

9.1.1 Maple step by step solution

Let's solve

$x''' - 6x'' + 11x' - 6x = e^{-t}$

- Highest derivative means the order of the ODE is 3

  x'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $x_1(t)$

    $x_1(t) = x$
  - Define new variable $x_2(t)$

    $x_2(t) = x'$
  - Define new variable $x_3(t)$

    $x_3(t) = x''$
  - Isolate for $x_3'(t)$ using original ODE

$x_3'(t) = \mathrm{e}^{-t} + 6x_3(t) - 11x_2(t) + 6x_1(t)$

Convert linear ODE into a system of first order ODEs

$[x_2(t) = x_1'(t), x_3(t) = x_2'(t), x_3'(t) = \mathrm{e}^{-t} + 6x_3(t) - 11x_2(t) + 6x_1(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_3 = \mathrm{e}^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

$$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + c_3\overrightarrow{x}_3 + \overrightarrow{x}_p(t)$$

☐ Fundamental matrix

  ◦ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} e^t & \frac{e^{2t}}{4} & \frac{e^{3t}}{9} \\ e^t & \frac{e^{2t}}{2} & \frac{e^{3t}}{3} \\ e^t & e^{2t} & e^{3t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^t & \frac{e^{2t}}{4} & \frac{e^{3t}}{9} \\ e^t & \frac{e^{2t}}{2} & \frac{e^{3t}}{3} \\ e^t & e^{2t} & e^{3t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{9} \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} 3e^t - 3e^{2t} + e^{3t} & -\frac{5e^t}{2} + 4e^{2t} - \frac{3e^{3t}}{2} & \frac{e^t}{2} - e^{2t} + \frac{e^{3t}}{2} \\ 3e^t - 6e^{2t} + 3e^{3t} & -\frac{5e^t}{2} + 8e^{2t} - \frac{9e^{3t}}{2} & \frac{e^t}{2} - 2e^{2t} + \frac{3e^{3t}}{2} \\ 3e^t - 12e^{2t} + 9e^{3t} & -\frac{5e^t}{2} + 16e^{2t} - \frac{27e^{3t}}{2} & \frac{e^t}{2} - 4e^{2t} + \frac{9e^{3t}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$
- Take the derivative of the particular solution

  $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{x}_p(t) = \left[\begin{array}{c} -\frac{e^{-t}}{24}+\frac{e^{t}}{4}-\frac{e^{2t}}{3}+\frac{e^{3t}}{8} \\ \frac{e^{-t}}{24}+\frac{e^{t}}{4}-\frac{2\,e^{2t}}{3}+\frac{3\,e^{3t}}{8} \\ -\frac{e^{-t}}{24}+\frac{e^{t}}{4}-\frac{4\,e^{2t}}{3}+\frac{9\,e^{3t}}{8} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + c_3\overrightarrow{x}_3 + \left[\begin{array}{c} -\frac{e^{-t}}{24}+\frac{e^{t}}{4}-\frac{e^{2t}}{3}+\frac{e^{3t}}{8} \\ \frac{e^{-t}}{24}+\frac{e^{t}}{4}-\frac{2\,e^{2t}}{3}+\frac{3\,e^{3t}}{8} \\ -\frac{e^{-t}}{24}+\frac{e^{t}}{4}-\frac{4\,e^{2t}}{3}+\frac{9\,e^{3t}}{8} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$x = \frac{(18c_2-24)e^{2t}}{72} + \frac{(8c_3+9)e^{3t}}{72} - \frac{e^{-t}}{24} + \frac{(72c_1+18)e^{t}}{72}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(x(t),t$3)-6*diff(x(t),t$2)+11*diff(x(t),t)-6*x(t)=exp(-t),x(t), singsol=all)
```

$$x(t) = -\frac{\mathrm{e}^{-t}}{24} + c_1\mathrm{e}^{t} + c_2\mathrm{e}^{2t} + c_3\mathrm{e}^{3t}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 37

```
DSolve[x'''[t]-6*x''[t]+11*x'[t]-6*x[t]==Exp[-t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{e^{-t}}{24} + c_1e^{t} + c_2e^{2t} + c_3e^{3t}$$

9.2 problem 16.1 (ii)

Internal problem ID [12045]
Internal file name [`OUTPUT/10697_Sunday_September_03_2023_12_36_41_PM_80785119/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 16, Higher order linear equations with constant coefficients. Exercises page 153
Problem number: 16.1 (ii).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 3y'' + 2y = \sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' + 2y = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1 - \sqrt{3}$$
$$\lambda_3 = 1 + \sqrt{3}$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + e^{(1+\sqrt{3})x} c_2 + e^{(1-\sqrt{3})x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$

$$y_2 = e^{(1+\sqrt{3})x}$$

$$y_3 = e^{(1-\sqrt{3})x}$$

Now the particular solution to the given ODE is found

$$y''' - 3y'' + 2y = \sin(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x, e^{(1-\sqrt{3})x}, e^{(1+\sqrt{3})x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \sin(x) - A_2 \cos(x) + 5A_1 \cos(x) + 5A_2 \sin(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{26}, A_2 = \frac{5}{26}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{26} + \frac{5\sin(x)}{26}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + e^{(1+\sqrt{3})x} c_2 + e^{(1-\sqrt{3})x} c_3\right) + \left(\frac{\cos(x)}{26} + \frac{5\sin(x)}{26}\right) \end{aligned}$$

Which simplifies to

$$y = e^x c_1 + e^{(1+\sqrt{3})x} c_2 + e^{-(\sqrt{3}-1)x} c_3 + \frac{\cos(x)}{26} + \frac{5\sin(x)}{26}$$

Summary

The solution(s) found are the following

$$y = e^x c_1 + e^{(1+\sqrt{3})x} c_2 + e^{-(\sqrt{3}-1)x} c_3 + \frac{\cos(x)}{26} + \frac{5\sin(x)}{26} \tag{1}$$

Verification of solutions

$$y = e^x c_1 + e^{(1+\sqrt{3})x} c_2 + e^{-(\sqrt{3}-1)x} c_3 + \frac{\cos(x)}{26} + \frac{5\sin(x)}{26}$$

Verified OK.

9.2.1 Maple step by step solution

Let's solve

$y''' - 3y'' + 2y = \sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

$y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = \sin(x) + 3y_3(x) - 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \sin(x) + 3y_3(x) - 2y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1-\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1-\sqrt{3}\right)^2} \\ \frac{1}{1-\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[1+\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1+\sqrt{3}\right)^2} \\ \frac{1}{1+\sqrt{3}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1-\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1-\sqrt{3}\right)^2} \\ \frac{1}{1-\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{\left(1-\sqrt{3}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(1-\sqrt{3}\right)^2} \\ \frac{1}{1-\sqrt{3}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1+\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1+\sqrt{3}\right)^2} \\ \frac{1}{1+\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{\left(1+\sqrt{3}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(1+\sqrt{3}\right)^2} \\ \frac{1}{1+\sqrt{3}} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \mathrm{e}^x & \frac{\mathrm{e}^{\left(1-\sqrt{3}\right)x}}{\left(1-\sqrt{3}\right)^2} & \frac{\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{\left(1+\sqrt{3}\right)^2} \\ \mathrm{e}^x & \frac{\mathrm{e}^{\left(1-\sqrt{3}\right)x}}{1-\sqrt{3}} & \frac{\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{1+\sqrt{3}} \\ \mathrm{e}^x & \mathrm{e}^{\left(1-\sqrt{3}\right)x} & \mathrm{e}^{\left(1+\sqrt{3}\right)x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \mathrm{e}^x & \frac{\mathrm{e}^{\left(1-\sqrt{3}\right)x}}{\left(1-\sqrt{3}\right)^2} & \frac{\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{\left(1+\sqrt{3}\right)^2} \\ \mathrm{e}^x & \frac{\mathrm{e}^{\left(1-\sqrt{3}\right)x}}{1-\sqrt{3}} & \frac{\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{1+\sqrt{3}} \\ \mathrm{e}^x & \mathrm{e}^{\left(1-\sqrt{3}\right)x} & \mathrm{e}^{\left(1+\sqrt{3}\right)x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{1}{\left(1-\sqrt{3}\right)^2} & \frac{1}{\left(1+\sqrt{3}\right)^2} \\ 1 & \frac{1}{1-\sqrt{3}} & \frac{1}{1+\sqrt{3}} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\left(1+\sqrt{3}\right)\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{6} + \frac{\left(1-\sqrt{3}\right)\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{6} + \frac{2\,\mathrm{e}^x}{3} & \frac{\left(-\sqrt{3}-2\right)\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{6} + \frac{\mathrm{e}^{\left(1+\sqrt{3}\right)x}\left(\sqrt{3}-2\right)}{6} + \frac{2\,\mathrm{e}^x}{3} & \\ \frac{2\,\mathrm{e}^x}{3} - \frac{\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{3} - \frac{\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{3} & \frac{\left(1+\sqrt{3}\right)\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{6} + \frac{\left(1-\sqrt{3}\right)\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{6} + \frac{2\,\mathrm{e}^x}{3} & (1 \\ \frac{\left(\sqrt{3}-1\right)\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{3} + \frac{\left(-1-\sqrt{3}\right)\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{3} + \frac{2\,\mathrm{e}^x}{3} & \frac{2\,\mathrm{e}^x}{3} - \frac{\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{3} - \frac{\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{3} & (2 \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{\left(5+2\sqrt{3}\right)\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{78} + \frac{\left(5-2\sqrt{3}\right)\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{78} + \frac{\cos(x)}{26} - \frac{\mathrm{e}^x}{6} + \frac{5\sin(x)}{26} \\ \frac{\left(-3\sqrt{3}-1\right)\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{78} + \frac{\left(3\sqrt{3}-1\right)\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{78} + \frac{5\cos(x)}{26} - \frac{\mathrm{e}^x}{6} - \frac{\sin(x)}{26} \\ \frac{\left(-2\sqrt{3}+8\right)\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{78} + \frac{\left(2\sqrt{3}+8\right)\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{78} - \frac{\cos(x)}{26} - \frac{\mathrm{e}^x}{6} - \frac{5\sin(x)}{26} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \begin{bmatrix} \frac{\left(5+2\sqrt{3}\right)\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{78} + \frac{\left(5-2\sqrt{3}\right)\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{78} + \frac{\cos(x)}{26} - \frac{\mathrm{e}^x}{6} + \frac{5\sin(x)}{26} \\ \frac{\left(-3\sqrt{3}-1\right)\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{78} + \frac{\left(3\sqrt{3}-1\right)\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{78} + \frac{5\cos(x)}{26} - \frac{\mathrm{e}^x}{6} - \frac{\sin(x)}{26} \\ \frac{\left(-2\sqrt{3}+8\right)\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{78} + \frac{\left(2\sqrt{3}+8\right)\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{78} - \frac{\cos(x)}{26} - \frac{\mathrm{e}^x}{6} - \frac{5\sin(x)}{26} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left((39c_2+2)\sqrt{3}+78c_2+5\right)\mathrm{e}^{-\left(\sqrt{3}-1\right)x}}{78} + \frac{\left((-39c_3-2)\sqrt{3}+78c_3+5\right)\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{78} + \frac{(-1+6c_1)\mathrm{e}^{x}}{6} + \frac{\cos(x)}{26} + \frac{5\sin(x)}{26}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+2*y(x)=sin(x),y(x), singsol=all)
```

$$y(x) = c_3\mathrm{e}^{-\left(\sqrt{3}-1\right)x} + c_1\mathrm{e}^{x} + c_2\mathrm{e}^{\left(1+\sqrt{3}\right)x} + \frac{5\sin(x)}{26} + \frac{\cos(x)}{26}$$

✓ Solution by Mathematica

Time used: 0.206 (sec). Leaf size: 49

```
DSolve[y'''[x]-3*y''[x]+2*y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{26}\left(5\sin(x) + \cos(x) + 26e^{x}\left(c_1e^{-\sqrt{3}x} + c_2e^{\sqrt{3}x} + c_3\right)\right)$$

9.3 problem 16.1 (iii)

Internal problem ID [12046]
Internal file name [`OUTPUT/10698_Sunday_September_03_2023_12_36_41_PM_90112867/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 16, Higher order linear equations with constant coefficients. Exercises page 153
Problem number: 16.1 (iii).
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'''' - 4x''' + 8x'' - 8x' + 4x = \sin(t)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE And x_p is a particular solution to the nonhomogeneous ODE. x_h is the solution to

$$x'''' - 4x''' + 8x'' - 8x' + 4x = 0$$

The characteristic equation is

$$\lambda^4 - 4\lambda^3 + 8\lambda^2 - 8\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1 - i$$
$$\lambda_2 = 1 + i$$
$$\lambda_3 = 1 - i$$
$$\lambda_4 = 1 + i$$

Therefore the homogeneous solution is

$$x_h(t) = e^{(1+i)t}c_1 + t\,e^{(1+i)t}c_2 + e^{(1-i)t}c_3 + t\,e^{(1-i)t}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = e^{(1+i)t}$$
$$x_2 = t\,e^{(1+i)t}$$
$$x_3 = e^{(1-i)t}$$
$$x_4 = t\,e^{(1-i)t}$$

Now the particular solution to the given ODE is found

$$x'''' - 4x''' + 8x'' - 8x' + 4x = \sin(t)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t\,e^{(1-i)t}, t\,e^{(1+i)t}, e^{(1-i)t}, e^{(1+i)t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(t) - 3A_2 \sin(t) + 4A_1 \sin(t) - 4A_2 \cos(t) = \sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{25}, A_2 = -\frac{3}{25}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{4\cos(t)}{25} - \frac{3\sin(t)}{25}$$

Therefore the general solution is

$$\begin{aligned}x &= x_h + x_p \\ &= \left(e^{(1+i)t}c_1 + t\,e^{(1+i)t}c_2 + e^{(1-i)t}c_3 + t\,e^{(1-i)t}c_4\right) + \left(\frac{4\cos(t)}{25} - \frac{3\sin(t)}{25}\right)\end{aligned}$$

Which simplifies to

$$x = (c_4 t + c_3)\,e^{(1-i)t} + e^{(1+i)t}(c_2 t + c_1) + \frac{4\cos(t)}{25} - \frac{3\sin(t)}{25}$$

Summary

The solution(s) found are the following

$$x = (c_4 t + c_3)\,e^{(1-i)t} + e^{(1+i)t}(c_2 t + c_1) + \frac{4\cos(t)}{25} - \frac{3\sin(t)}{25} \tag{1}$$

Verification of solutions

$$x = (c_4 t + c_3)\,e^{(1-i)t} + e^{(1+i)t}(c_2 t + c_1) + \frac{4\cos(t)}{25} - \frac{3\sin(t)}{25}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(x(t),t$4)-4*diff(x(t),t$3)+8*diff(x(t),t$2)-8*diff(x(t),t)+4*x(t)=sin(t),x(t), s
```

$$x(t) = ((c_3t + c_1)\cos(t) + \sin(t)(c_4t + c_2))\,\mathrm{e}^t + \frac{4\cos(t)}{25} - \frac{3\sin(t)}{25}$$

✓ Solution by Mathematica

Time used: 0.258 (sec). Leaf size: 42

```
DSolve[x''''[t]-4*x'''[t]+8*x''[t]-8*x'[t]+4*x[t]==Sin[t],x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to \left(\frac{4}{25} + e^t(c_4t + c_3)\right)\cos(t) + \left(-\frac{3}{25} + e^t(c_2t + c_1)\right)\sin(t)$$

9.4 problem 16.1 (iv)

Internal problem ID [12047]
Internal file name [`OUTPUT/10699_Sunday_September_03_2023_12_36_41_PM_2817466/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 16, Higher order linear equations with constant coefficients. Exercises page 153
Problem number: 16.1 (iv).
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _with_linear_symmetries]]
```

$$\boxed{x'''' - 5x'' + 4x = \mathrm{e}^{t}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE And x_p is a particular solution to the nonhomogeneous ODE. x_h is the solution to

$$x'''' - 5x'' + 4x = 0$$

The characteristic equation is

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = 1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$x_h(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-2t} + c_3 \mathrm{e}^{t} + \mathrm{e}^{2t} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = \mathrm{e}^{-t}$$
$$x_2 = \mathrm{e}^{-2t}$$
$$x_3 = \mathrm{e}^{t}$$
$$x_4 = \mathrm{e}^{2t}$$

Now the particular solution to the given ODE is found

$$x'''' - 5x'' + 4x = \mathrm{e}^{t}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{t}, \mathrm{e}^{-2t}, \mathrm{e}^{-t}, \mathrm{e}^{2t}\}$$

Since e^{t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,\mathrm{e}^{t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t\,\mathrm{e}^{t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \mathrm{e}^{t} = \mathrm{e}^{t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{6}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{t\,\mathrm{e}^t}{6}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{-2t} + c_3\mathrm{e}^t + \mathrm{e}^{2t}c_4\right) + \left(-\frac{t\,\mathrm{e}^t}{6}\right) \end{aligned}$$

<ins>Summary</ins>
The solution(s) found are the following

$$x = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{-2t} + c_3\mathrm{e}^t + \mathrm{e}^{2t}c_4 - \frac{t\,\mathrm{e}^t}{6} \tag{1}$$

<ins>Verification of solutions</ins>

$$x = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{-2t} + c_3\mathrm{e}^t + \mathrm{e}^{2t}c_4 - \frac{t\,\mathrm{e}^t}{6}$$

Verified OK.

9.4.1 Maple step by step solution

Let's solve

$x'''' - 5x'' + 4x = \mathrm{e}^t$

- Highest derivative means the order of the ODE is 4

  x''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $x_1(t)$

    $x_1(t) = x$

  - Define new variable $x_2(t)$

    $x_2(t) = x'$

- Define new variable $x_3(t)$

  $x_3(t) = x''$
- Define new variable $x_4(t)$

  $x_4(t) = x'''$
- Isolate for $x_4'(t)$ using original ODE

  $x_4'(t) = \mathrm{e}^t + 5x_3(t) - 4x_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[x_2(t) = x_1'(t), x_3(t) = x_2'(t), x_4(t) = x_3'(t), x_4'(t) = \mathrm{e}^t + 5x_3(t) - 4x_1(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathrm{e}^t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathrm{e}^t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = e^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_4 = e^{2t} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + c_4\vec{x}_4 + \vec{x}_p(t)$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{8} & -e^{-t} & e^{t} & \frac{e^{2t}}{8} \\ \frac{e^{-2t}}{4} & e^{-t} & e^{t} & \frac{e^{2t}}{4} \\ -\frac{e^{-2t}}{2} & -e^{-t} & e^{t} & \frac{e^{2t}}{2} \\ e^{-2t} & e^{-t} & e^{t} & e^{2t} \end{bmatrix}$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{8} & -e^{-t} & e^{t} & \frac{e^{2t}}{8} \\ \frac{e^{-2t}}{4} & e^{-t} & e^{t} & \frac{e^{2t}}{4} \\ -\frac{e^{-2t}}{2} & -e^{-t} & e^{t} & \frac{e^{2t}}{2} \\ e^{-2t} & e^{-t} & e^{t} & e^{2t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{8} & -1 & 1 & \frac{1}{8} \\ \frac{1}{4} & 1 & 1 & \frac{1}{4} \\ -\frac{1}{2} & -1 & 1 & \frac{1}{2} \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} -\frac{(e^{4t}-4e^{3t}-4e^{t}+1)e^{-2t}}{6} & -\frac{(e^{4t}-8e^{3t}+8e^{t}-1)e^{-2t}}{12} & -\frac{(-e^{4t}+e^{3t}+e^{t}-1)e^{-2t}}{6} & \frac{(e^{4t}-2e^{3t}+2e^{t}-1)e^{-2t}}{12} \\ -\frac{(e^{4t}-2e^{3t}+2e^{t}-1)e^{-2t}}{3} & -\frac{(e^{4t}-4e^{3t}-4e^{t}+1)e^{-2t}}{6} & \frac{(2e^{4t}-e^{3t}+e^{t}-2)e^{-2t}}{6} & -\frac{(-e^{4t}+e^{3t}+e^{t}-1)e^{-2t}}{6} \\ \frac{2(-e^{4t}+e^{3t}+e^{t}-1)e^{-2t}}{3} & -\frac{(e^{4t}-2e^{3t}+2e^{t}-1)e^{-2t}}{3} & -\frac{(-4e^{4t}+e^{3t}+e^{t}-4)e^{-2t}}{6} & \frac{(2e^{4t}-e^{3t}+e^{t}-2)e^{-2t}}{6} \\ -\frac{2(2e^{4t}-e^{3t}+e^{t}-2)e^{-2t}}{3} & \frac{2(-e^{4t}+e^{3t}+e^{t}-1)e^{-2t}}{3} & \frac{(8e^{4t}-e^{3t}+e^{t}-8)e^{-2t}}{6} & -\frac{(-4e^{4t}+e^{3t}+e^{t}-4)e^{-2t}}{6} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{(-3\,e^{4t}+6t\,e^{3t}+e^{3t}+3\,e^{t}-1)e^{-2t}}{36} \\ \frac{(6\,e^{4t}-6t\,e^{3t}-7\,e^{3t}+3\,e^{t}-2)e^{-2t}}{36} \\ \frac{(12\,e^{4t}-6t\,e^{3t}-13\,e^{3t}-3\,e^{t}+4)e^{-2t}}{36} \\ \frac{(24\,e^{4t}-6t\,e^{3t}-19\,e^{3t}+3\,e^{t}-8)e^{-2t}}{36} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + c_4\vec{x}_4 + \begin{bmatrix} -\frac{(-3\,e^{4t}+6t\,e^{3t}+e^{3t}+3\,e^{t}-1)e^{-2t}}{36} \\ \frac{(6\,e^{4t}-6t\,e^{3t}-7\,e^{3t}+3\,e^{t}-2)e^{-2t}}{36} \\ \frac{(12\,e^{4t}-6t\,e^{3t}-13\,e^{3t}-3\,e^{t}+4)e^{-2t}}{36} \\ \frac{(24\,e^{4t}-6t\,e^{3t}-19\,e^{3t}+3\,e^{t}-8)e^{-2t}}{36} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$x = -\frac{e^{-2t}\left(t-6c_3+\frac{1}{6}\right)e^{3t}}{6} - \frac{(72c_2e^{t}-9\,e^{4t}c_4+9c_1+6\,e^{t}-6\,e^{4t}-2)e^{-2t}}{72}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve(diff(x(t),t$4)-5*diff(x(t),t$2)+4*x(t)=exp(t),x(t), singsol=all)
```

$$x(t) = -\frac{e^{-2t}\left(\left(t-6c_1\right)e^{3t}-6c_3e^t-6c_4e^{4t}-6c_2\right)}{6}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 45

```
DSolve[x''''[t]-5*x''[t]+4*x[t]==Exp[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-2t}\left(c_2e^t + e^{3t}\left(-\frac{t}{6}-\frac{1}{36}+c_3\right)+c_4e^{4t}+c_1\right)$$

10 Chapter 17, Reduction of order. Exercises page 162

10.1 problem 17.1

Internal problem ID [12048]
Internal file name [`OUTPUT/10700_Sunday_September_03_2023_12_36_41_PM_98333301/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 17, Reduction of order. Exercises page 162
Problem number: 17.1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2y'' - (t^2 + 2t)\, y' + (t + 2)\, y = 0}$$

Given that one solution of the ode is

$$y_1 = t$$

Given one basis solution $y_1(t)$, then the second basis solution is given by

$$y_2(t) = y_1 \left(\int \frac{\mathrm{e}^{-(\int p dt)}}{y_1^2} dt \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(t)\, y' + q(t)\, y = f(t)$$

Looking at the ode to solve shows that

$$p(t) = \frac{-t^2 - 2t}{t^2}$$

Therefore

$$y_2(t) = t\left(\int \frac{\mathrm{e}^{-\left(\int \frac{-t^2-2t}{t^2}dt\right)}}{t^2}dt\right)$$

$$y_2(t) = t\int \frac{\mathrm{e}^{t+2\ln(t)}}{t^2}, dt$$

$$y_2(t) = t\left(\int \mathrm{e}^t dt\right)$$

$$y_2(t) = t\,\mathrm{e}^t$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(t) + c_2 y_2(t) \\ &= c_1 t + c_2 t\,\mathrm{e}^t \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 t + c_2 t\,\mathrm{e}^t \tag{1}$$

Verification of solutions

$$y = c_1 t + c_2 t\,\mathrm{e}^t$$

Verified OK.

10.1.1 Maple step by step solution

Let's solve

$t^2 y'' + (-t^2 - 2t)\,y' + (t+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(t+2)y}{t^2} + \frac{(t+2)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t+2)y'}{t} + \frac{(t+2)y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{t+2}{t}, P_3(t) = \frac{t+2}{t^2}\right]$
- $t \cdot P_2(t)$ is analytic at $t = 0$

  $$\left.(t \cdot P_2(t))\right|_{t=0} = -2$$
- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $$\left.(t^2 \cdot P_3(t))\right|_{t=0} = 2$$
- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $t^2y'' - t(t+2)\,y' + (t+2)\,y = 0$
- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k t^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $t^m \cdot y$ to series expansion for $m = 0..1$

  $$t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$$
- Shift index using $k- >k - m$

  $$t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$$
- Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

  $$t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$$
- Shift index using $k- >k + 1 - m$

  $$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, t^{k+r}$$
- Convert $t^2 \cdot y''$ to series expansion

  $$t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(-1+r)(-2+r)t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-2) - a_{k-1}(k+r-2)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)(a_k(k+r-1) - a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k+r-1)(a_{k+1}(k+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+r}$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+2}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^{1+k}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{1+k} = \frac{a_k}{1+k}, b_{1+k} = \frac{b_k}{k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 12

```
dsolve([t^2*diff(y(t),t$2)-(t^2+2*t)*diff(y(t),t)+(t+2)*y(t)=0,t],singsol=all)
```

$$y(t) = t(c_1 + c_2 e^t)$$

✓ Solution by Mathematica
Time used: 0.032 (sec). Leaf size: 16

```
DSolve[t^2*y''[t]-(t^2+2*t)*y'[t]+(t+2)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t(c_2 e^t + c_1)$$

10.2 problem 17.2

Internal problem ID [12049]
Internal file name [`OUTPUT/10701_Sunday_September_03_2023_12_36_42_PM_1254071/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 17, Reduction of order. Exercises page 162
Problem number: 17.2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-y'x+y=0}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{x}{x-1}$$

Therefore

$$y_2(x) = e^x \left(\int e^{-\left(\int -\frac{x}{x-1} dx\right)} e^{-2x} dx \right)$$

$$y_2(x) = e^x \int \frac{e^{x+\ln(x-1)}}{e^{2x}}, dx$$

$$y_2(x) = e^x \left(\int (x-1) e^{-x} dx \right)$$

$$y_2(x) = -e^x x e^{-x}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= e^x c_1 - c_2 e^x x e^{-x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x c_1 - c_2 e^x x e^{-x} \tag{1}$$

Verification of solutions

$$y = e^x c_1 - c_2 e^x x e^{-x}$$

Verified OK.

10.2.1 Maple step by step solution

Let's solve

$(x-1) y'' - y'x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

  $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

- $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $(x-1)\, y'' - y'x + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$

- Shift index using $k- >k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_{k+1}(k+1+r)-a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+1+r}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{k+1}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Revert the change of variables $u=x-1$

  $\left[y=\sum_{k=0}^{\infty}a_k(x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+3}$

- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Revert the change of variables $u=x-1$

  $\left[y=\sum_{k=0}^{\infty}a_k(x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k(x-1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(x-1)^{k+2}\right), a_{1+k}=\frac{a_k}{1+k}, b_{1+k}=\frac{b_k}{k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([(x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,exp(x)],singsol=all)
```

$$y(x) = c_2 e^x + c_1 x$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 x$$

10.3 problem 17.3

Internal problem ID [12050]
Internal file name [OUTPUT/10702_Sunday_September_03_2023_12_36_42_PM_41693741/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 17, Reduction of order. Exercises page 162
Problem number: 17.3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(\cos(t)\, t-\sin(t))\, x''-x' t \sin(t)-x \sin(t)=0}$$

Given that one solution of the ode is

$$x_1 = t$$

Given one basis solution $x_1(t)$, then the second basis solution is given by

$$x_2(t) = x_1\left(\int \frac{\mathrm{e}^{-\left(\int p dt\right)}}{x_1^2} dt\right)$$

Where $p(x)$ is the coefficient of x' when the ode is written in the normal form

$$x'' + p(t)\, x' + q(t)\, x = f(t)$$

Looking at the ode to solve shows that

$$p(t) = -\frac{\sin(t)\, t}{\cos(t)\, t-\sin(t)}$$

Therefore

$$x_2(t) = t\left(\int \frac{\mathrm{e}^{-\left(\int -\frac{\sin(t)t}{\cos(t)t-\sin(t)}dt\right)}}{t^2}dt\right)$$

$$x_2(t) = t\int \frac{\frac{1}{\cos(t)t-\sin(t)}}{t^2}, dt$$

$$x_2(t) = t\left(\int \frac{1}{\left(\cos\left(t\right)t-\sin\left(t\right)\right)t^2}dt\right)$$

$$x_2(t) = t\left(\int \frac{1}{\left(\cos\left(t\right)t-\sin\left(t\right)\right)t^2}dt\right)$$

Hence the solution is

$$\begin{aligned} x &= c_1x_1(t) + c_2x_2(t) \\ &= c_1t + c_2t\left(\int \frac{1}{\left(\cos\left(t\right)t-\sin\left(t\right)\right)t^2}dt\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$x = c_1t + c_2t\left(\int \frac{1}{\left(\cos\left(t\right)t-\sin\left(t\right)\right)t^2}dt\right) \tag{1}$$

<u>Verification of solutions</u>

$$x = c_1t + c_2t\left(\int \frac{1}{\left(\cos\left(t\right)t-\sin\left(t\right)\right)t^2}dt\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
```

✗ Solution by Maple

```
dsolve([(t*cos(t)-sin(t))*diff(x(t),t$2)-diff(x(t),t)*t*sin(t)-x(t)*sin(t)=0,t],singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(t*Cos[t]-Sin[t])*x''[t]-x'[t]*t*Sin[t]-x[t]*Sin[t]==0,x[t],t,IncludeSingularSolution
```

Not solved

10.4 problem 17.4

Internal problem ID [12051]
Internal file name [`OUTPUT/10703_Sunday_September_03_2023_12_36_42_PM_95706010/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 17, Reduction of order. Exercises page 162
Problem number: 17.4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction__of__order", "second__order__change__of__variable__on__y__method__2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{\left(-t^{2}+t\right) x''+\left(-t^{2}+2\right) x'+\left(-t+2\right) x=0}$$

Given that one solution of the ode is

$$x_1 = \mathrm{e}^{-t}$$

Given one basis solution $x_1(t)$, then the second basis solution is given by

$$x_2(t) = x_1\left(\int \frac{\mathrm{e}^{-\left(\int p dt\right)}}{x_1^2} dt\right)$$

Where $p(x)$ is the coefficient of x' when the ode is written in the normal form

$$x''+p(t)\,x'+q(t)\,x=f(t)$$

Looking at the ode to solve shows that

$$p(t) = \frac{-t^{2}+2}{-t^{2}+t}$$

Therefore

$$
\begin{aligned}
x_2(t) &= \mathrm{e}^{-t}\left(\int \mathrm{e}^{-\left(\int \frac{-t^2+2}{-t^2+t}dt\right)}\mathrm{e}^{2t}dt\right)\\
x_2(t) &= \mathrm{e}^{-t}\int \frac{\mathrm{e}^{-t+\ln(t-1)-2\ln(t)}}{\mathrm{e}^{-2t}}, dt\\
x_2(t) &= \mathrm{e}^{-t}\left(\int \frac{(t-1)\,\mathrm{e}^{t}}{t^2}dt\right)\\
x_2(t) &= \frac{\mathrm{e}^{-t}\mathrm{e}^{t}}{t}
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
x &= c_1x_1(t)+c_2x_2(t)\\
&= c_1\mathrm{e}^{-t}+\frac{c_2\mathrm{e}^{-t}\mathrm{e}^{t}}{t}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1\mathrm{e}^{-t}+\frac{c_2\mathrm{e}^{-t}\mathrm{e}^{t}}{t} \tag{1}
$$

Verification of solutions

$$
x = c_1\mathrm{e}^{-t}+\frac{c_2\mathrm{e}^{-t}\mathrm{e}^{t}}{t}
$$

Verified OK.

10.4.1 Maple step by step solution

Let's solve

$(-t^2+t)\,x''+(-t^2+2)\,x'+(-t+2)\,x=0$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x''=-\frac{(-2+t)x}{t(t-1)}-\frac{(t^2-2)x'}{t(t-1)}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$x'' + \frac{(t^2-2)x'}{t(t-1)} + \frac{(-2+t)x}{t(t-1)} = 0$

- ☐ Check to see if t_0 is a regular singular point
  - Define functions

    $\left[P_2(t) = \frac{t^2-2}{t(t-1)}, P_3(t) = \frac{-2+t}{t(t-1)}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = 2$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 0$
  - $t = 0$is a regular singular point

    Check to see if t_0 is a regular singular point

    $t_0 = 0$
- Multiply by denominators

  $x''t(t-1) + (t^2-2)\, x' + (-2+t)\, x = 0$
- Assume series solution for x

  $x = \sum_{k=0}^{\infty} a_k t^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $t^m \cdot x$ to series expansion for $m = 0..1$

    $t^m \cdot x = \sum_{k=0}^{\infty} a_k t^{k+r+m}$
  - Shift index using $k- >k-m$

    $t^m \cdot x = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$
  - Convert $t^m \cdot x'$ to series expansion for $m = 0..2$

    $t^m \cdot x' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $t^m \cdot x' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, t^{k+r}$
  - Convert $t^m \cdot x''$ to series expansion for $m = 1..2$

$$t^m \cdot x'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$t^m \cdot x'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(1+r)\, t^{-1+r} + (-a_1(1+r)(2+r) + a_0(1+r)(-2+r))\, t^r + \left(\sum_{k=1}^{\infty} (-a_{k+1}(k+r+1)(k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$-a_1(1+r)(2+r) + a_0(1+r)(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-a_{k+1}(k+r+1)(k+2+r) + a_k(k+r+1)(k+r-2) + a_{k-1}(k+r) = 0$$

- Shift index using $k->k+1$

$$-a_{k+2}(k+2+r)(k+3+r) + a_{k+1}(k+2+r)(k+r-1) + a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{k^2 a_{k+1} + 2kra_{k+1} + r^2 a_{k+1} + ka_k + ka_{k+1} + ra_k + ra_{k+1} + a_k - 2a_{k+1}}{(k+2+r)(k+3+r)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = \frac{k^2 a_{k+1} + ka_k - ka_{k+1} - 2a_{k+1}}{(k+1)(k+2)}$$

- Solution for $r = -1$

$$\left[x = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+2} = \frac{k^2 a_{k+1} + ka_k - ka_{k+1} - 2a_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{k^2 a_{k+1} + ka_k + ka_{k+1} + a_k - 2a_{k+1}}{(k+2)(k+3)}$$

- Solution for $r = 0$

$$\left[x = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{k^2 a_{k+1} + ka_k + ka_{k+1} + a_k - 2a_{k+1}}{(k+2)(k+3)}, -2a_1 - 2a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[x = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^k\right), a_{k+2} = \frac{k^2 a_{1+k} + k a_k - k a_{1+k} - 2a_{1+k}}{(1+k)(k+2)}, 0 = 0, b_{k+2} = \frac{k^2 b_{1+k} + k b_k + k b_{1+k} + b_k}{(k+2)(k+3)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([(t-t^2)*diff(x(t),t$2)+(2-t^2)*diff(x(t),t)+(2-t)*x(t)=0,exp(-t)],singsol=all)
```

$$x(t) = \frac{c_2 e^{-t} t + c_1}{t}$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 42

```
DSolve[(t-t^2)*x''[t]+(2-t^2)*x'[t]+(2-t)*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{e^{-t}\sqrt{1-t}(c_1 e^t - c_2 t)}{\sqrt{t-1}t}$$

10.5 problem 17.5

Internal problem ID [12052]
Internal file name [`OUTPUT/10704_Sunday_September_03_2023_12_36_43_PM_40557485/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 17, Reduction of order. Exercises page 162
Problem number: 17.5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Hermite]`

$$\boxed{y'' - y'x + y = 0}$$

Given that one solution of the ode is

$$y_1 = x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -x$$

Therefore

$$y_2(x) = x\left(\int \frac{e^{-\left(\int -x dx\right)}}{x^2} dx\right)$$

$$y_2(x) = x \int \frac{e^{\frac{x^2}{2}}}{x^2}, dx$$

$$y_2(x) = x\left(\int \frac{e^{\frac{x^2}{2}}}{x^2} dx\right)$$

$$y_2(x) = x\left(-\frac{e^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)$$

Hence the solution is

$$\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x + c_2 x\left(-\frac{e^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 x\left(-\frac{e^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2 x\left(-\frac{e^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)$$

Verified OK.

10.5.1 Maple step by step solution

Let's solve

$y'' - y'x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ∘ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ∘ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ∘ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k-1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - a_k(k-1) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k(k-1)}{k^2+3k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve([diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,x],singsol=all)
```

$$y(x) = c_2 e^{\frac{x^2}{2}} + \frac{\left(i c_2 \sqrt{2}\, \sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, x}{2}\right) + 2c_1\right) x}{2}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 61

```
DSolve[y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + c_2 e^{\frac{x^2}{2}} + \sqrt{2} c_1 x$$

10.6 problem 17.6

Internal problem ID [12053]
Internal file name [OUTPUT/10705_Sunday_September_03_2023_12_36_43_PM_33139359/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 17, Reduction of order. Exercises page 162
Problem number: 17.6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\tan(t)\, x'' - 3x' + (\tan(t) + 3\cot(t))\, x = 0}$$

Given that one solution of the ode is

$$x_1 = \sin(t)$$

Given one basis solution $x_1(t)$, then the second basis solution is given by

$$x_2(t) = x_1 \left(\int \frac{\mathrm{e}^{-\left(\int p dt\right)}}{x_1^2} dt \right)$$

Where $p(x)$ is the coefficient of x' when the ode is written in the normal form

$$x'' + p(t)\, x' + q(t)\, x = f(t)$$

Looking at the ode to solve shows that

$$p(t) = -\frac{3}{\tan(t)}$$

Therefore

$$
\begin{aligned}
x_2(t) &= \sin(t)\left(\int \frac{\mathrm{e}^{-\left(\int -\frac{3}{\tan(t)}dt\right)}}{\sin(t)^2}dt\right) \\
x_2(t) &= \sin(t)\int \frac{\sin(t)^3}{\sin(t)^2}, dt \\
x_2(t) &= \sin(t)\left(\int \sin(t)\,dt\right) \\
x_2(t) &= -\cos(t)\sin(t)
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
x &= c_1 x_1(t) + c_2 x_2(t) \\
&= c_1 \sin(t) - c_2 \cos(t)\sin(t)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 \sin(t) - c_2 \cos(t)\sin(t) \tag{1}
$$

Verification of solutions

$$
x = c_1 \sin(t) - c_2 \cos(t)\sin(t)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
   Change of variables used:
      [t = arcsin(t)]
   Linear ODE actually solved:
      (-2*t^2+3)*u(t)+(2*t^3-3*t)*diff(u(t),t)+(-t^4+t^2)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 13

```
dsolve([tan(t)*diff(x(t),t$2)-3*diff(x(t),t)+(tan(t)+3*cot(t))*x(t)=0,sin(t)],singsol=all)
```

$$x(t) = \sin(t)(c_1 + c_2\cos(t))$$

✓ Solution by Mathematica

Time used: 0.374 (sec). Leaf size: 24

```
DSolve[Tan[t]*x''[t]-3*x'[t]+(Tan[t]+3*Cot[t])*x[t]==0,x[t],t,IncludeSingularSolutions -> Tr
```

$$x(t) \to \sqrt{-\sin^2(t)}(c_2\cos(t) + c_1)$$

11 Chapter 18, The variation of constants formula. Exercises page 168

11.1 problem 18.1 (i)

Internal problem ID [12054]
Internal file name [`OUTPUT/10706_Sunday_September_03_2023_12_36_44_PM_10647444/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 18, The variation of constants formula. Exercises page 168
Problem number: 18.1 (i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' - 6y = e^{x}}$$

11.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -6, f(x) = e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda \, \mathrm{e}^{\lambda x} - 6 \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(-6)} \\ &= \frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{5}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(3)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^x}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 e^{-2x}\right) + \left(-\frac{e^x}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3x} + c_2 e^{-2x} - \frac{e^x}{6} \tag{1}$$

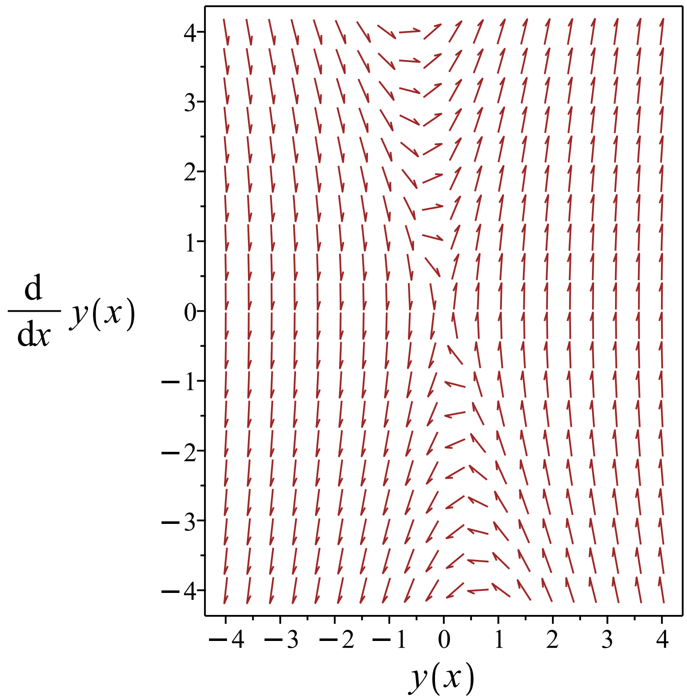


Figure 125: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{-2x} - \frac{e^x}{6}$$

Verified OK.

11.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 148: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{5x}}{5}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y' - 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{\mathrm{e}^{3x} c_2}{5}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{3x}}{5}, \mathrm{e}^{-2x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1\mathrm{e}^x = \mathrm{e}^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\mathrm{e}^x}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + \frac{\mathrm{e}^{3x}c_2}{5}\right) + \left(-\frac{\mathrm{e}^x}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-2x} + \frac{\mathrm{e}^{3x}c_2}{5} - \frac{\mathrm{e}^x}{6} \tag{1}$$

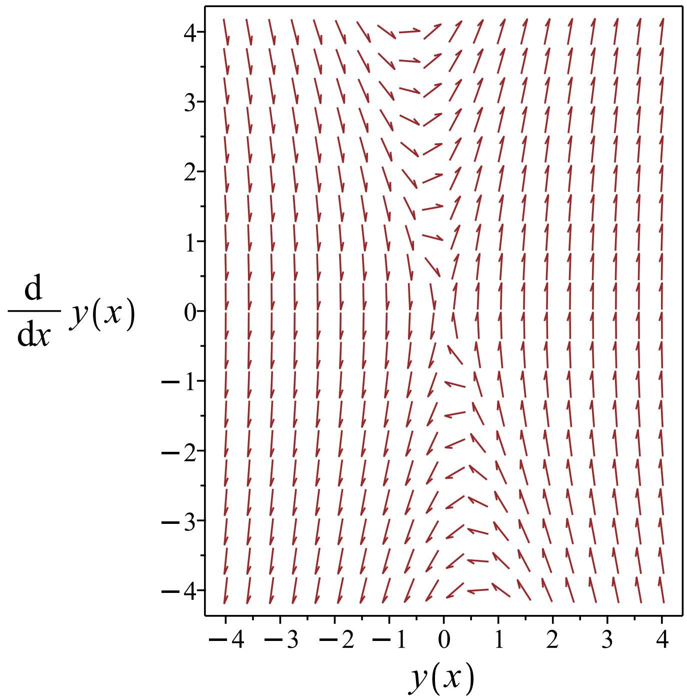


Figure 126: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + \frac{\mathrm{e}^{3x} c_2}{5} - \frac{\mathrm{e}^{x}}{6}$$

Verified OK.

11.1.3 Maple step by step solution

Let's solve

$y'' - y' - 6y = \mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-3) = 0$

- Roots of the characteristic polynomial

$r = (-2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{3x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{3x} \\ -2\,\mathrm{e}^{-2x} & 3\,\mathrm{e}^{3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 5\,\mathrm{e}^{x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{\left(\mathrm{e}^{5x}\left(\int \mathrm{e}^{-2x} dx\right) - \left(\int \mathrm{e}^{3x} dx\right)\right)\mathrm{e}^{-2x}}{5}$

  - Compute integrals

    $y_p(x) = -\frac{\mathrm{e}^{x}}{6}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{3x} c_2 - \frac{\mathrm{e}^{x}}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-6*y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = \frac{(6c_1 e^{5x} - e^{3x} + 6c_2)\, e^{-2x}}{6}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 29

```
DSolve[y''[x]-y'[x]-6*y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^x}{6} + c_1 e^{-2x} + c_2 e^{3x}$$

11.2 problem 18.1 (ii)

Internal problem ID [12055]
Internal file name [`OUTPUT/10707_Sunday_September_03_2023_12_36_47_PM_73449688/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 18, The variation of constants formula. Exercises page 168
Problem number: 18.1 (ii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' - x = \frac{1}{t}}$$

11.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = -1, f(t) = \frac{1}{t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ x &= c_1 e^{(1)t} + c_2 e^{(-1)t}\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-t}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= \mathrm{e}^t \\ x_2 &= \mathrm{e}^{-t} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^t & \mathrm{e}^{-t} \\ \frac{d}{dt}(\mathrm{e}^t) & \frac{d}{dt}(\mathrm{e}^{-t}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^t & \mathrm{e}^{-t} \\ \mathrm{e}^t & -\mathrm{e}^{-t} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^t)(-\mathrm{e}^{-t}) - (\mathrm{e}^{-t})(\mathrm{e}^t)$$

Which simplifies to

$$W = -2\,\mathrm{e}^t \mathrm{e}^{-t}$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-t}}{t}}{-2} dt$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-t}}{2t} dt$$

Hence

$$u_1 = -\frac{\text{expIntegral}_1(t)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{t}}{t}}{-2} dt$$

Which simplifies to

$$u_2 = \int -\frac{e^{t}}{2t} dt$$

Hence

$$u_2 = \frac{\text{expIntegral}_1(-t)}{2}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{\text{expIntegral}_1(t)\, e^{t}}{2} + \frac{\text{expIntegral}_1(-t)\, e^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{t} + c_2 e^{-t}\right) + \left(-\frac{\text{expIntegral}_1(t)\, e^{t}}{2} + \frac{\text{expIntegral}_1(-t)\, e^{-t}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{t} + c_2 e^{-t} - \frac{\text{expIntegral}_1(t)\, e^{t}}{2} + \frac{\text{expIntegral}_1(-t)\, e^{-t}}{2} \tag{1}$$

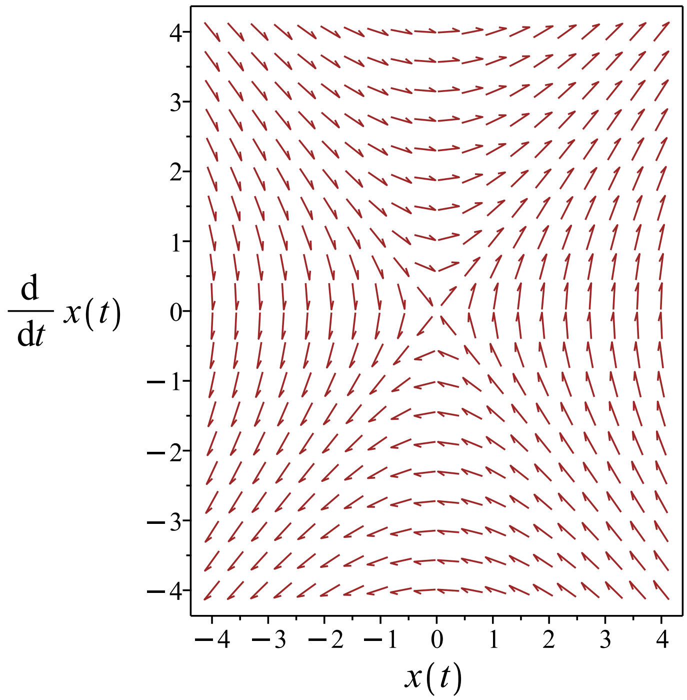


Figure 127: Slope field plot

Verification of solutions

$$x = c_1 e^t + c_2 e^{-t} - \frac{\text{expIntegral}_1(t)\, e^t}{2} + \frac{\text{expIntegral}_1(-t)\, e^{-t}}{2}$$

Verified OK.

11.2.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' - x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 150: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \mathrm{e}^{-t} \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= \mathrm{e}^{-t} \int \frac{1}{\mathrm{e}^{-2t}}\, dt \\ &= \mathrm{e}^{-t} \left(\frac{\mathrm{e}^{2t}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{t}}{2}
$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}
$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
x_1 &= \mathrm{e}^{-t} \\
x_2 &= \frac{\mathrm{e}^{t}}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}
$$

$$
u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-t} & \frac{\mathrm{e}^t}{2} \\ \frac{d}{dt}(\mathrm{e}^{-t}) & \frac{d}{dt}\left(\frac{\mathrm{e}^t}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-t} & \frac{\mathrm{e}^t}{2} \\ -\mathrm{e}^{-t} & \frac{\mathrm{e}^t}{2} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-t})\left(\frac{\mathrm{e}^t}{2}\right) - \left(\frac{\mathrm{e}^t}{2}\right)(-\mathrm{e}^{-t})$$

Which simplifies to

$$W = \mathrm{e}^t \mathrm{e}^{-t}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^t}{2t}}{1}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^t}{2t} dt$$

Hence

$$u_1 = \frac{\mathrm{expIntegral}_1(-t)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-t}}{t}}{1}\, dt$$

Which simplifies to

$$u_2 = \int \frac{e^{-t}}{t} dt$$

Hence

$$u_2 = -\operatorname{expIntegral}_1(t)$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = -\frac{\operatorname{expIntegral}_1(t)\, e^{t}}{2} + \frac{\operatorname{expIntegral}_1(-t)\, e^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-t} + \frac{c_2 e^{t}}{2}\right) + \left(-\frac{\operatorname{expIntegral}_1(t)\, e^{t}}{2} + \frac{\operatorname{expIntegral}_1(-t)\, e^{-t}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-t} + \frac{c_2 e^{t}}{2} - \frac{\operatorname{expIntegral}_1(t)\, e^{t}}{2} + \frac{\operatorname{expIntegral}_1(-t)\, e^{-t}}{2} \tag{1}$$

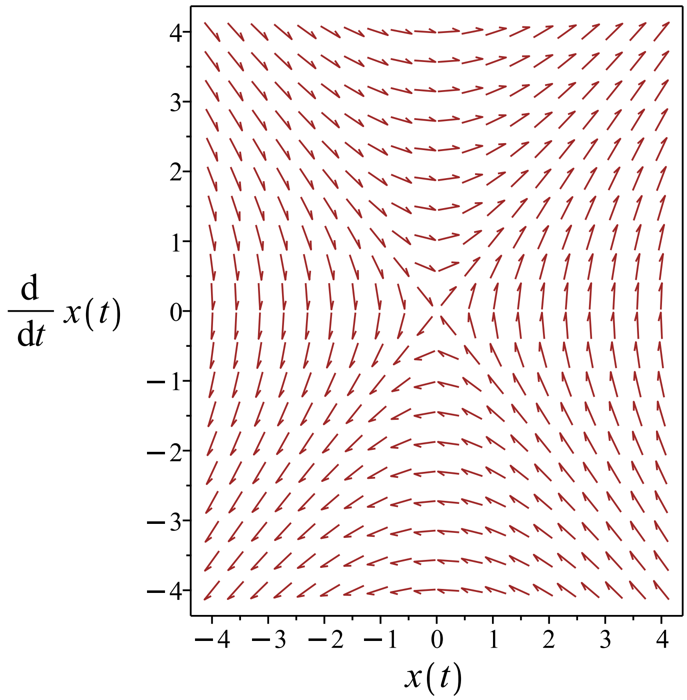


Figure 128: Slope field plot

Verification of solutions

$$x = c_1 e^{-t} + \frac{c_2 e^t}{2} - \frac{\text{expIntegral}_1(t)\, e^t}{2} + \frac{\text{expIntegral}_1(-t)\, e^{-t}}{2}$$

Verified OK.

11.2.3 Maple step by step solution

Let's solve

$x'' - x = \frac{1}{t}$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \frac{1}{t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{t} \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 2$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\frac{\mathrm{e}^{-t}\left(\int \frac{\mathrm{e}^{t}}{t}dt\right)}{2} + \frac{\mathrm{e}^{t}\left(\int \frac{\mathrm{e}^{-t}}{t}dt\right)}{2}$

  - Compute integrals

    $x_p(t) = -\frac{\mathrm{Ei}_1(t)\mathrm{e}^{t}}{2} + \frac{\mathrm{Ei}_1(-t)\mathrm{e}^{-t}}{2}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} - \frac{\mathrm{Ei}_1(t)\mathrm{e}^{t}}{2} + \frac{\mathrm{Ei}_1(-t)\mathrm{e}^{-t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(x(t),t$2)-x(t)=1/t,x(t), singsol=all)
```

$$x(t) = \frac{\text{expIntegral}_1\left(-t\right)\text{e}^{-t}}{2} + c_2\text{e}^{-t} + \text{e}^{t}\left(c_1 - \frac{\text{expIntegral}_1\left(t\right)}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 42

```
DSolve[x''[t]-x[t]==1/t,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2}e^{-t}\left(e^{2t}\,\text{ExpIntegralEi}(-t) - \text{ExpIntegralEi}(t) + 2\left(c_1e^{2t} + c_2\right)\right)$$

11.3 problem 18.1 (iii)

Internal problem ID [12056]
Internal file name [`OUTPUT/10708_Sunday_September_03_2023_12_36_48_PM_5467386/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 18, The variation of constants formula. Exercises page 168
Problem number: 18.1 (iii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y = \cot(2x)}$$

11.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \cot(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos{(2x)} + c_2 \sin{(2x)})$$

Or

$$y = c_1 \cos{(2x)} + c_2 \sin{(2x)}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$
$$y_2 = \sin(2x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}(\sin(2x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(2\cos(2x)) - (\sin(2x))(-2\sin(2x))$$

Which simplifies to

$$W = 2\cos(2x)^2 + 2\sin(2x)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(2x)\cot(2x)}{2}dx$$

Which simplifies to

$$u_1 = -\int \frac{\cos(2x)}{2}dx$$

Hence

$$u_1 = -\frac{\sin(2x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\cot(2x)}{2}dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(2x)\cot(2x)}{2}dx$$

Hence

$$u_2 = \frac{\cos(2x)}{4} + \frac{\ln(\csc(2x) - \cot(2x))}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\sin(2x)\cos(2x)}{4} + \left(\frac{\cos(2x)}{4} + \frac{\ln(\csc(2x) - \cot(2x))}{4}\right)\sin(2x)$$

Which simplifies to

$$y_p(x) = \frac{\sin(2x)\ln(\csc(2x) - \cot(2x))}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{\sin(2x)\ln(\csc(2x) - \cot(2x))}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(2x)\ln(\csc(2x) - \cot(2x))}{4} \tag{1}
$$

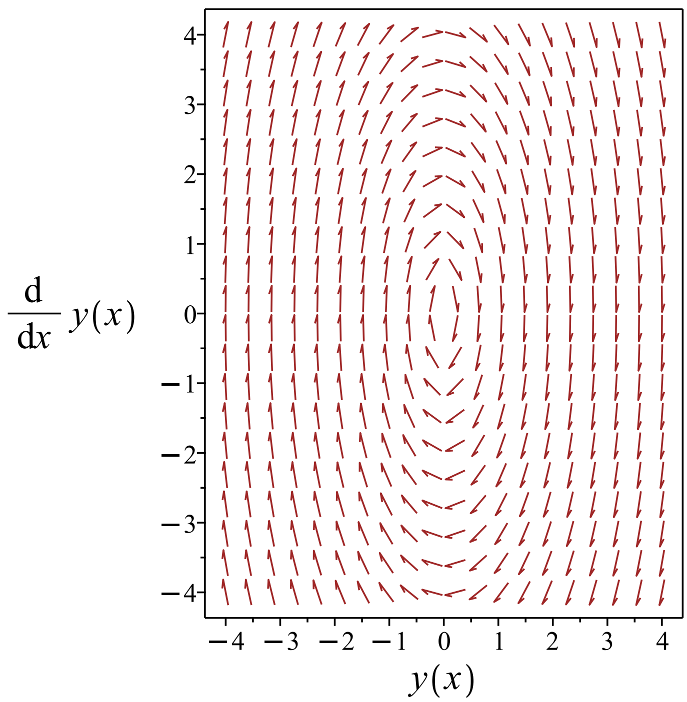


Figure 129: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(2x)\ln(\csc(2x) - \cot(2x))}{4}
$$

Verified OK.

11.3.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
y'' + 4y &= 0 && (1)\\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= 0 \qquad (3)\\
C &= 4
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4\\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 152: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(2x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\
&= \cos(2x) \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$

$$y_2 = \frac{\sin(2x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}\left(\frac{\sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ -2\sin(2x) & \cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(\cos(2x)) - \left(\frac{\sin(2x)}{2}\right)(-2\sin(2x))$$

Which simplifies to

$$W = \sin(2x)^2 + \cos(2x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(2x)\cot(2x)}{2}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{\cos(2x)}{2} dx$$

Hence

$$u_1 = -\frac{\sin(2x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\cot(2x)}{1} dx$$

Which simplifies to

$$u_2 = \int \cos(2x)\cot(2x)\, dx$$

Hence

$$u_2 = \frac{\cos(2x)}{2} + \frac{\ln(\csc(2x) - \cot(2x))}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\sin(2x)\cos(2x)}{4} + \frac{\left(\frac{\cos(2x)}{2} + \frac{\ln(\csc(2x)-\cot(2x))}{2}\right)\sin(2x)}{2}$$

Which simplifies to

$$y_p(x) = \frac{\sin(2x)\ln(\csc(2x) - \cot(2x))}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{\sin(2x)\ln(\csc(2x) - \cot(2x))}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\sin(2x)\ln(\csc(2x) - \cot(2x))}{4} \tag{1}$$

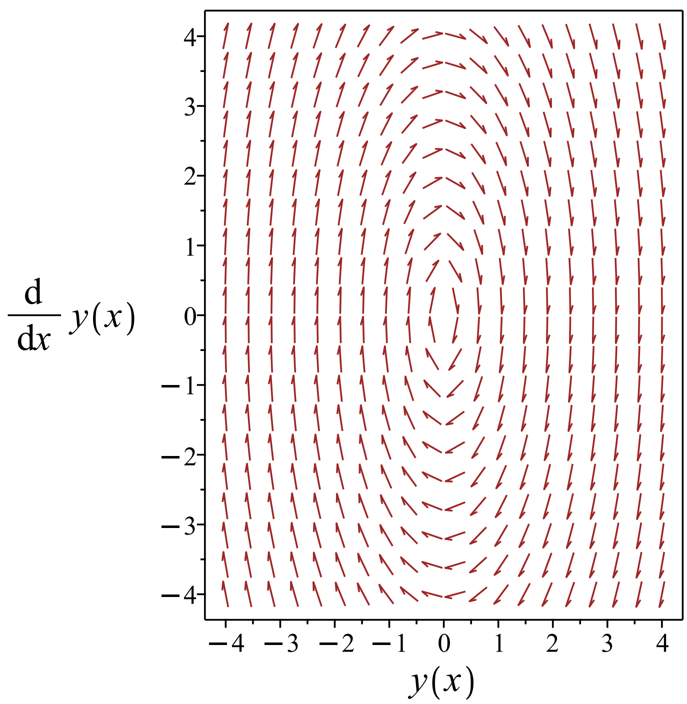


Figure 130: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\sin(2x)\ln(\csc(2x) - \cot(2x))}{4}$$

Verified OK.

11.3.3 Maple step by step solution

Let's solve

$y'' + 4y = \cot(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cot(2x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(2x)\left(\int \cos(2x)dx\right)}{2} + \frac{\sin(2x)\left(\int \cos(2x)\cot(2x)dx\right)}{2}$$

- ◦ Compute integrals

$$y_p(x) = \frac{\sin(2x)\ln(\csc(2x)-\cot(2x))}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(2x)\ln(\csc(2x)-\cot(2x))}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+4*y(x)=cot(2*x),y(x), singsol=all)
```

$$y(x) = c_2 \sin(2x) + \cos(2x)\, c_1 + \frac{\sin(2x)\ln(\csc(2x) - \cot(2x))}{4}$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 34

```
DSolve[y''[x]+4*y[x]==Cot[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(2x) + \frac{1}{4}\sin(2x)(\log(\sin(x)) - \log(\cos(x)) + 4c_2)$$

11.4 problem 18.1 (iv)

Internal problem ID [12057]
Internal file name [`OUTPUT/10709_Sunday_September_03_2023_12_36_53_PM_22442698/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 18, The variation of constants formula. Exercises page 168
Problem number: 18.1 (iv).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{t^2x'' - 2x = t^3}$$

11.4.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = t^2, B = 0, C = -2, f(t) = t^3$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. Solving for x_h from

$$t^2x'' - 2x = 0$$

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 0rt^{r-1} - 2t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 0\,t^r - 2t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$x = c_1x_1 + c_2x_2$$

Where $x_1 = t^{r_1}$ and $x_2 = t^{r_2}$. Hence

$$x = \frac{c_1}{t} + c_2t^2$$

Next, we find the particular solution to the ODE

$$t^2x'' - 2x = t^3$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1x_1 + u_2x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \frac{1}{t}$$
$$x_2 = t^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{t} & t^2 \\ \frac{d}{dt}\left(\frac{1}{t}\right) & \frac{d}{dt}(t^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t} & t^2 \\ -\frac{1}{t^2} & 2t \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t}\right)(2t) - \left(t^2\right)\left(-\frac{1}{t^2}\right)$$

Which simplifies to

$$W = 3$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{t^5}{3t^2}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{t^3}{3} dt$$

Hence

$$u_1 = -\frac{t^4}{12}$$

And Eq. (3) becomes

$$u_2 = \int \frac{t^2}{3t^2}\, dt$$

Which simplifies to

$$u_2 = \int \frac{1}{3} dt$$

Hence

$$u_2 = \frac{t}{3}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{t^3}{4}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \frac{t^3}{4} + \frac{c_1}{t} + c_2 t^2 \end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{t^3}{4} + \frac{c_1}{t} + c_2 t^2 \tag{1}$$

Verification of solutions

$$x = \frac{t^3}{4} + \frac{c_1}{t} + c_2 t^2$$

Verified OK.

11.4.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2x'' - 2x)\, dt = \int t^3 dt$$
$$t^2x' - 2xt = \frac{t^4}{4} + c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = -\frac{2}{t}$$
$$q(t) = \frac{t^4 + 4c_1}{4t^2}$$

Hence the ode is

$$x' - \frac{2x}{t} = \frac{t^4 + 4c_1}{4t^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{t}dt}$$
$$= \frac{1}{t^2}$$

The ode becomes

$$\frac{d}{dt}(\mu x) = (\mu)\left(\frac{t^4 + 4c_1}{4t^2}\right)$$
$$\frac{d}{dt}\left(\frac{x}{t^2}\right) = \left(\frac{1}{t^2}\right)\left(\frac{t^4 + 4c_1}{4t^2}\right)$$
$$d\left(\frac{x}{t^2}\right) = \left(\frac{t^4 + 4c_1}{4t^4}\right) dt$$

Integrating gives

$$\frac{x}{t^2} = \int \frac{t^4 + 4c_1}{4t^4} dt$$
$$\frac{x}{t^2} = \frac{t}{4} - \frac{c_1}{3t^3} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2$$

Summary

The solution(s) found are the following

$$x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2 \tag{1}$$

Verification of solutions

$$x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2$$

Verified OK.

11.4.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$t^2 x'' - 2x = t^3$$

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2 x'' - 2x)\, dt = \int t^3 dt$$
$$t^2 x' - 2xt = \frac{t^4}{4} + c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = -\frac{2}{t}$$
$$q(t) = \frac{t^4 + 4c_1}{4t^2}$$

Hence the ode is

$$x' - \frac{2x}{t} = \frac{t^4 + 4c_1}{4t^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{t}dt} = \frac{1}{t^2}$$

The ode becomes

$$\frac{d}{dt}(\mu x) = (\mu)\left(\frac{t^4+4c_1}{4t^2}\right)$$
$$\frac{d}{dt}\left(\frac{x}{t^2}\right) = \left(\frac{1}{t^2}\right)\left(\frac{t^4+4c_1}{4t^2}\right)$$
$$d\left(\frac{x}{t^2}\right) = \left(\frac{t^4+4c_1}{4t^4}\right) dt$$

Integrating gives

$$\frac{x}{t^2} = \int \frac{t^4+4c_1}{4t^4} dt$$
$$\frac{x}{t^2} = \frac{t}{4} - \frac{c_1}{3t^3} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2$$

Summary

The solution(s) found are the following

$$x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2 \tag{1}$$

Verification of solutions

$$x = t^2\left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2$$

Verified OK.

11.4.4 Solving using Kovacic algorithm

Writing the ode as

$$t^2x'' - 2x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{2}{t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 154: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2}{t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2}{t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -1$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{t} + (-)\,(0) \\ &= -\frac{1}{t} \\ &= -\frac{1}{t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{t}\right)(0) + \left(\left(\frac{1}{t^2}\right) + \left(-\frac{1}{t}\right)^2 - \left(\frac{2}{t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int -\frac{1}{t} dt} \\ &= \frac{1}{t} \end{aligned}$$

The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \frac{1}{t} \end{aligned}$$

Which simplifies to

$$x_1 = \frac{1}{t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= \frac{1}{t} \int \frac{1}{\frac{1}{t^2}}\, dt \\ &= \frac{1}{t}\left(\frac{t^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1\left(\frac{1}{t}\right) + c_2\left(\frac{1}{t}\left(\frac{t^3}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$t^2x'' - 2x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = \frac{c_1}{t} + \frac{c_2 t^2}{3}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = \frac{1}{t}$$

$$x_2 = \frac{t^2}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{t} & \frac{t^2}{3} \\ \frac{d}{dt}\left(\frac{1}{t}\right) & \frac{d}{dt}\left(\frac{t^2}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t} & \frac{t^2}{3} \\ -\frac{1}{t^2} & \frac{2t}{3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t}\right)\left(\frac{2t}{3}\right) - \left(\frac{t^2}{3}\right)\left(-\frac{1}{t^2}\right)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{t^5}{3}}{t^2}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{t^3}{3} dt$$

Hence

$$u_1 = -\frac{t^4}{12}$$

And Eq. (3) becomes

$$u_2 = \int \frac{t^2}{t^2}\, dt$$

Which simplifies to

$$u_2 = \int 1 dt$$

Hence

$$u_2 = t$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{t^3}{4}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(\frac{c_1}{t} + \frac{c_2 t^2}{3}\right) + \left(\frac{t^3}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = \frac{c_1}{t} + \frac{c_2 t^2}{3} + \frac{t^3}{4} \tag{1}$$

Verification of solutions

$$x = \frac{c_1}{t} + \frac{c_2 t^2}{3} + \frac{t^3}{4}$$

Verified OK.

11.4.5 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\, x'' + q(t)\, x' + r(t)\, x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= t^2 \\ q(x) &= 0 \\ r(x) &= -2 \\ s(x) &= t^3 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$2 - (0) + (-2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\, x' + (q(t) - p'(t))\, x)' = s(x)$$

Integrating gives

$$p(t)\, x' + (q(t) - p'(t))\, x = \int s(t)\, dt$$

Substituting the above values for p, q, r, s gives

$$t^2 x' - 2xt = \int t^3\, dt$$

We now have a first order ode to solve which is

$$t^2 x' - 2xt = \frac{t^4}{4} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -\frac{2}{t} \\ q(t) &= \frac{t^4 + 4c_1}{4t^2} \end{aligned}$$

Hence the ode is

$$x' - \frac{2x}{t} = \frac{t^4 + 4c_1}{4t^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{t} dt} \\ &= \frac{1}{t^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu) \left(\frac{t^4 + 4c_1}{4t^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{t^2}\right) &= \left(\frac{1}{t^2}\right) \left(\frac{t^4 + 4c_1}{4t^2}\right) \\ \mathrm{d}\left(\frac{x}{t^2}\right) &= \left(\frac{t^4 + 4c_1}{4t^4}\right) \mathrm{d}t \end{aligned}$$

Integrating gives

$$\frac{x}{t^2} = \int \frac{t^4 + 4c_1}{4t^4} \, dt$$
$$\frac{x}{t^2} = \frac{t}{4} - \frac{c_1}{3t^3} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$x = t^2 \left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2$$

Summary

The solution(s) found are the following

$$x = t^2 \left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2 \tag{1}$$

Verification of solutions

$$x = t^2 \left(\frac{t}{4} - \frac{c_1}{3t^3}\right) + c_2 t^2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(t^2*diff(x(t),t$2)-2*x(t)=t^3,x(t), singsol=all)
```

$$x(t) = c_2 t^2 + \frac{t^3}{4} + \frac{c_1}{t}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 25

```
DSolve[t^2*x''[t]-2*x[t]==t^3,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{t^3}{4} + c_2 t^2 + \frac{c_1}{t}$$

11.5 problem 18.1 (v)

Internal problem ID [12058]
Internal file name [`OUTPUT/10710_Sunday_September_03_2023_12_36_55_PM_84006246/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 18, The variation of constants formula. Exercises page 168
Problem number: 18.1 (v).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x'' - 4x' = \tan(t)}$$

11.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = -4, C = 0, f(t) = \tan(t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''-4x'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t)+Bx'(t)+Cx(t)=0$$

Where in the above $A=1, B=-4, C=0$. Let the solution be $x=e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t}-4\lambda\,\mathrm{e}^{\lambda t}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2-4\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=-4, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{4}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{-4^2-(4)\,(1)\,(0)}\\&=2\pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=2+2\\\lambda_2&=2-2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=4\\\lambda_2&=0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x&=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}\\x&=c_1e^{(4)t}+c_2e^{(0)t}\end{aligned}$$

Or

$$x = c_1 e^{4t} + c_2$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 e^{4t} + c_2$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = e^{4t}$$
$$x_2 = 1$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{4t} & 1 \\ \frac{d}{dt}(e^{4t}) & \frac{d}{dt}(1) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{4t} & 1 \\ 4\,e^{4t} & 0 \end{vmatrix}$$

Therefore

$$W = \left(e^{4t}\right)(0) - (1)\left(4\,e^{4t}\right)$$

Which simplifies to

$$W = -4\,e^{4t}$$

Which simplifies to

$$W = -4\,e^{4t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\tan(t)}{-4\,e^{4t}}\,dt$$

Which simplifies to

$$u_1 = -\int -\frac{\tan(t)\,e^{-4t}}{4}\,dt$$

Hence

$$u_1 = -\left(\int_0^t -\frac{\tan(\alpha)\,e^{-4\alpha}}{4}\,d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{4t}\tan(t)}{-4\,e^{4t}}\,dt$$

Which simplifies to

$$u_2 = \int -\frac{\tan(t)}{4}\,dt$$

Hence

$$u_2 = \frac{\ln(\cos(t))}{4}$$

Which simplifies to

$$u_1 = \frac{\left(\int_0^t \tan(\alpha)\,e^{-4\alpha}\,d\alpha\right)}{4}$$

$$u_2 = \frac{\ln(\cos(t))}{4}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{\left(\int_0^t \tan(\alpha)\, \mathrm{e}^{-4\alpha} d\alpha\right) \mathrm{e}^{4t}}{4} + \frac{\ln(\cos(t))}{4}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{4t} + c_2\right) + \left(\frac{\left(\int_0^t \tan(\alpha)\, \mathrm{e}^{-4\alpha} d\alpha\right) \mathrm{e}^{4t}}{4} + \frac{\ln(\cos(t))}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{4t} + c_2 + \frac{\left(\int_0^t \tan(\alpha)\, \mathrm{e}^{-4\alpha} d\alpha\right) \mathrm{e}^{4t}}{4} + \frac{\ln(\cos(t))}{4} \tag{1}$$

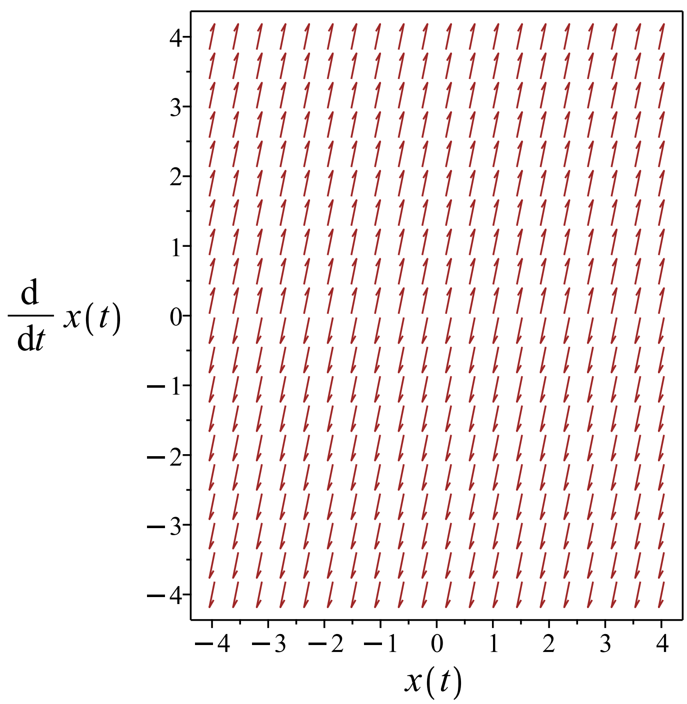


Figure 131: Slope field plot

Verification of solutions

$$
x = c_1 e^{4t} + c_2 + \frac{\left(\int_0^t \tan(\alpha)\, e^{-4\alpha} d\alpha\right) e^{4t}}{4} + \frac{\ln(\cos(t))}{4}
$$

Verified OK.

11.5.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$
\int (x'' - 4x')\, dt = \int \tan(t)\, dt
$$

$$
-4x + x' = -\ln(\cos(t)) + c_1
$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
x' + p(t)x = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= -4 \\
q(t) &= -\ln(\cos(t)) + c_1
\end{aligned}
$$

Hence the ode is

$$
-4x + x' = -\ln(\cos(t)) + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-4) dt} \\
&= e^{-4t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)\left(-\ln(\cos(t)) + c_1\right) \\
\frac{d}{dt}\left(e^{-4t} x\right) &= \left(e^{-4t}\right)\left(-\ln(\cos(t)) + c_1\right) \\
d\left(e^{-4t} x\right) &= \left(\left(-\ln(\cos(t)) + c_1\right) e^{-4t}\right)\, dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-4t} x &= \int \left(-\ln(\cos(t)) + c_1\right) e^{-4t}\, dt \\
e^{-4t} x &= \int \left(-\ln(\cos(t)) + c_1\right) e^{-4t} dt + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-4t}$ results in

$$x = e^{4t}\left(\int (-\ln(\cos(t)) + c_1) e^{-4t} dt\right) + c_2 e^{4t}$$

which simplifies to

$$x = e^{4t}\left(-\left(\int (\ln(\cos(t)) - c_1) e^{-4t} dt\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$x = e^{4t}\left(-\left(\int (\ln(\cos(t)) - c_1) e^{-4t} dt\right) + c_2\right) \tag{1}$$

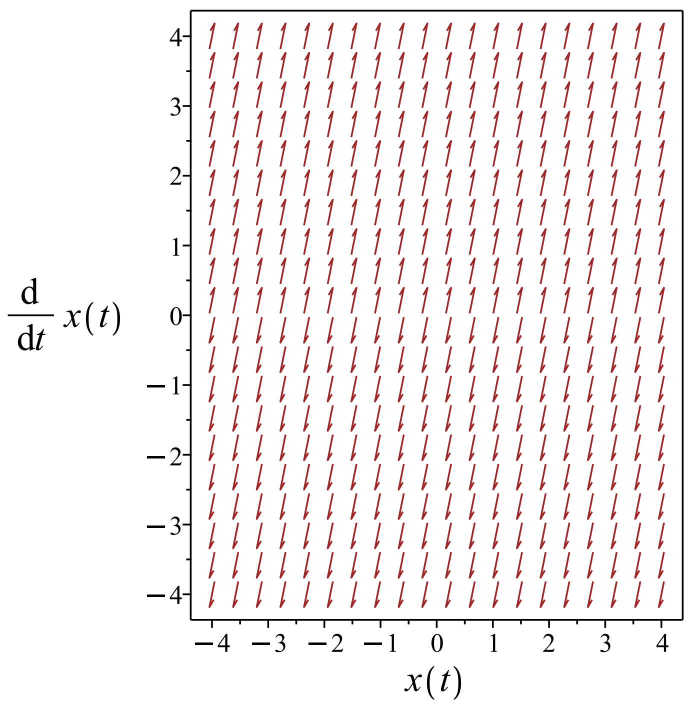


Figure 132: Slope field plot

Verification of solutions

$$x = e^{4t}\left(-\left(\int (\ln(\cos(t)) - c_1) e^{-4t} dt\right) + c_2\right)$$

Verified OK.

11.5.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) - 4p(t) - \tan(t) = 0$$

Which is now solve for $p(t)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(t) + p(t)p(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= \tan(t) \end{aligned}$$

Hence the ode is

$$p'(t) - 4p(t) = \tan(t)$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int (-4)dt} \\ &= \mathrm{e}^{-4t} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu p) &= (\mu)(\tan(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-4t}p\right) &= \left(\mathrm{e}^{-4t}\right)(\tan(t)) \\ \mathrm{d}\left(\mathrm{e}^{-4t}p\right) &= \left(\tan(t)\,\mathrm{e}^{-4t}\right)\,\mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{-4t}p &= \int \tan(t)\,\mathrm{e}^{-4t}\,\mathrm{d}t \\ \mathrm{e}^{-4t}p &= \frac{i\mathrm{e}^{-4t}}{4} - i\left(\int -\frac{2\,\mathrm{e}^{-4t}}{\mathrm{e}^{2it}+1}dt\right) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-4t}$ results in

$$p(t) = \mathrm{e}^{4t}\left(\frac{i\mathrm{e}^{-4t}}{4} - i\left(\int -\frac{2\,\mathrm{e}^{-4t}}{\mathrm{e}^{2it}+1}dt\right)\right) + c_1\mathrm{e}^{4t}$$

which simplifies to

$$p(t) = \frac{i}{4} + 2i\mathrm{e}^{4t}\left(\int \frac{\mathrm{e}^{-4t}}{\mathrm{e}^{2it}+1}dt\right) + c_1\mathrm{e}^{4t}$$

Since $p = x'$ then the new first order ode to solve is

$$x' = \frac{i}{4} + 2i\mathrm{e}^{4t}\left(\int \frac{\mathrm{e}^{-4t}}{\mathrm{e}^{2it}+1}dt\right) + c_1\mathrm{e}^{4t}$$

Integrating both sides gives

$$\begin{aligned} x &= \int \frac{i}{4} + 2i\mathrm{e}^{4t}\left(\int \frac{\mathrm{e}^{-4t}}{\mathrm{e}^{2it}+1}dt\right) + c_1\mathrm{e}^{4t}\,\mathrm{d}t \\ &= \int \left(\frac{i}{4} + 2i\mathrm{e}^{4t}\left(\int \frac{\mathrm{e}^{-4t}}{\mathrm{e}^{2it}+1}dt\right) + c_1\mathrm{e}^{4t}\right)dt + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$x = \int \left(\frac{i}{4} + 2i\mathrm{e}^{4t}\left(\int \frac{\mathrm{e}^{-4t}}{\mathrm{e}^{2it}+1}dt\right) + c_1\mathrm{e}^{4t}\right)dt + c_2 \tag{1}$$

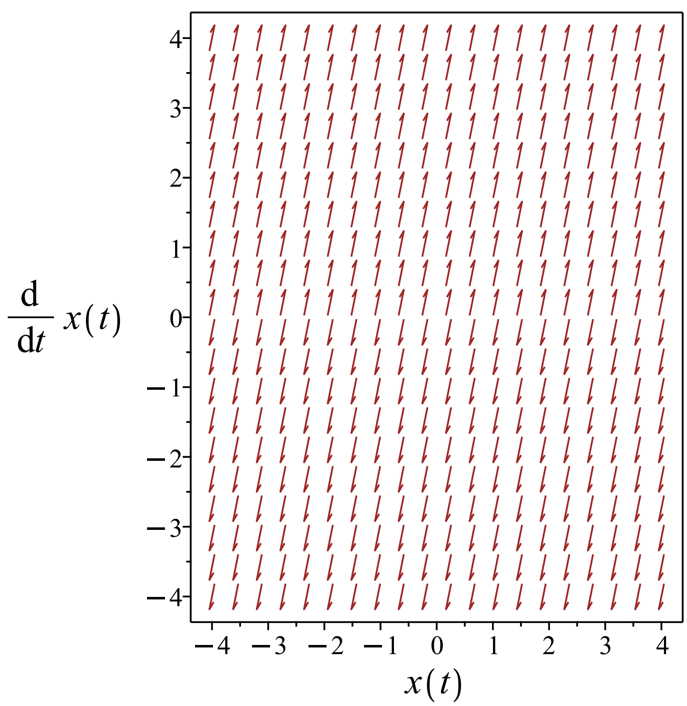


Figure 133: Slope field plot

Verification of solutions

$$x = \int \left(\frac{i}{4} + 2i\,e^{4t}\left(\int \frac{e^{-4t}}{e^{2it}+1}dt\right) + c_1 e^{4t}\right) dt + c_2$$

Verified OK.

11.5.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x'' - 4x' = \tan(t)$$

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' - 4x')\, dt = \int \tan(t)\, dt$$
$$-4x + x' = -\ln(\cos(t)) + c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = -4$$
$$q(t) = -\ln(\cos(t)) + c_1$$

Hence the ode is

$$-4x + x' = -\ln(\cos(t)) + c_1$$

The integrating factor μ is

$$\mu = e^{\int (-4)dt}$$
$$= e^{-4t}$$

The ode becomes

$$\frac{d}{dt}(\mu x) = (\mu)(-\ln(\cos(t)) + c_1)$$
$$\frac{d}{dt}\left(e^{-4t}x\right) = \left(e^{-4t}\right)(-\ln(\cos(t)) + c_1)$$
$$d\left(e^{-4t}x\right) = \left((-\ln(\cos(t)) + c_1)\, e^{-4t}\right)\, dt$$

Integrating gives

$$e^{-4t}x = \int \left(-\ln\left(\cos\left(t\right)\right) + c_1\right) e^{-4t}\, \mathrm{d}t$$
$$e^{-4t}x = \int \left(-\ln\left(\cos\left(t\right)\right) + c_1\right) e^{-4t} dt + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-4t}$ results in

$$x = e^{4t}\left(\int \left(-\ln\left(\cos\left(t\right)\right) + c_1\right) e^{-4t} dt\right) + c_2 e^{4t}$$

which simplifies to

$$x = e^{4t}\left(-\left(\int \left(\ln\left(\cos\left(t\right)\right) - c_1\right) e^{-4t} dt\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$x = e^{4t}\left(-\left(\int \left(\ln\left(\cos\left(t\right)\right) - c_1\right) e^{-4t} dt\right) + c_2\right) \tag{1}$$

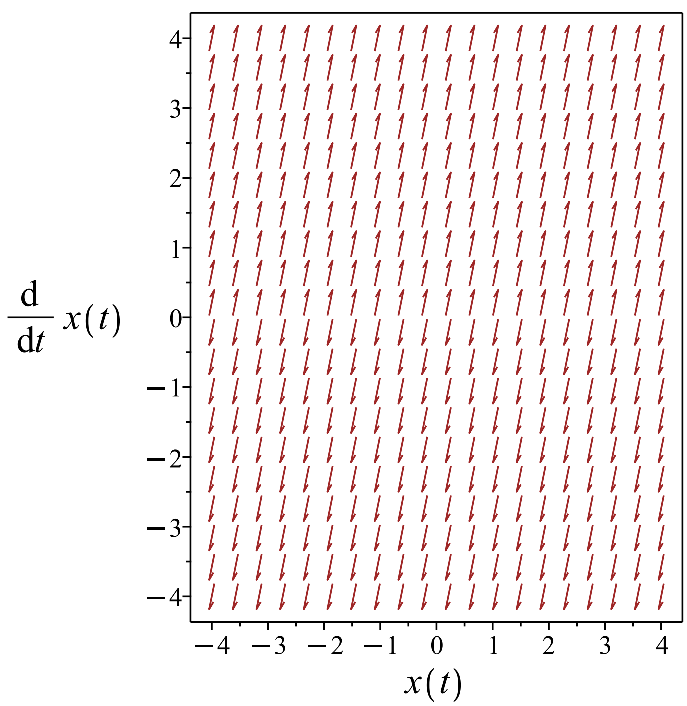


Figure 134: Slope field plot

Verification of solutions

$$x = e^{4t}\left(-\left(\int (\ln(\cos(t)) - c_1)\, e^{-4t} dt\right) + c_2\right)$$

Verified OK.

11.5.5 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 4x' = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 155: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-2t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dt} \\
&= z_1 e^{2t} \\
&= z_1 \left(\mathrm{e}^{2t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = 1
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-4}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{4t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{\mathrm{e}^{4t}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(1) + c_2\left(1\left(\frac{\mathrm{e}^{4t}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - 4x' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 + \frac{c_2 \mathrm{e}^{4t}}{4}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1 x_1 + u_2 x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \frac{\mathrm{e}^{4t}}{4} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{\mathrm{e}^{4t}}{4} \\ \frac{d}{dt}(1) & \frac{d}{dt}\left(\frac{\mathrm{e}^{4t}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{\mathrm{e}^{4t}}{4} \\ 0 & \mathrm{e}^{4t} \end{vmatrix}$$

Therefore

$$W = (1)\left(e^{4t}\right) - \left(\frac{e^{4t}}{4}\right)(0)$$

Which simplifies to

$$W = e^{4t}$$

Which simplifies to

$$W = e^{4t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{4t}\tan(t)}{4}}{e^{4t}}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{\tan(t)}{4} dt$$

Hence

$$u_1 = \frac{\ln(\cos(t))}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\tan(t)}{e^{4t}}\, dt$$

Which simplifies to

$$u_2 = \int \tan(t)\, e^{-4t} dt$$

Hence

$$u_2 = \int_0^t \tan(\alpha)\, e^{-4\alpha} d\alpha$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{\ln(\cos(t))}{4} + \frac{\left(\int_0^t \tan(\alpha)\, e^{-4\alpha} d\alpha\right) e^{4t}}{4}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(c_1 + \frac{c_2 \mathrm{e}^{4t}}{4}\right) + \left(\frac{\ln(\cos(t))}{4} + \frac{\left(\int_0^t \tan(\alpha)\,\mathrm{e}^{-4\alpha} d\alpha\right) \mathrm{e}^{4t}}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 + \frac{c_2 \mathrm{e}^{4t}}{4} + \frac{\ln(\cos(t))}{4} + \frac{\left(\int_0^t \tan(\alpha)\,\mathrm{e}^{-4\alpha} d\alpha\right) \mathrm{e}^{4t}}{4} \tag{1}
$$

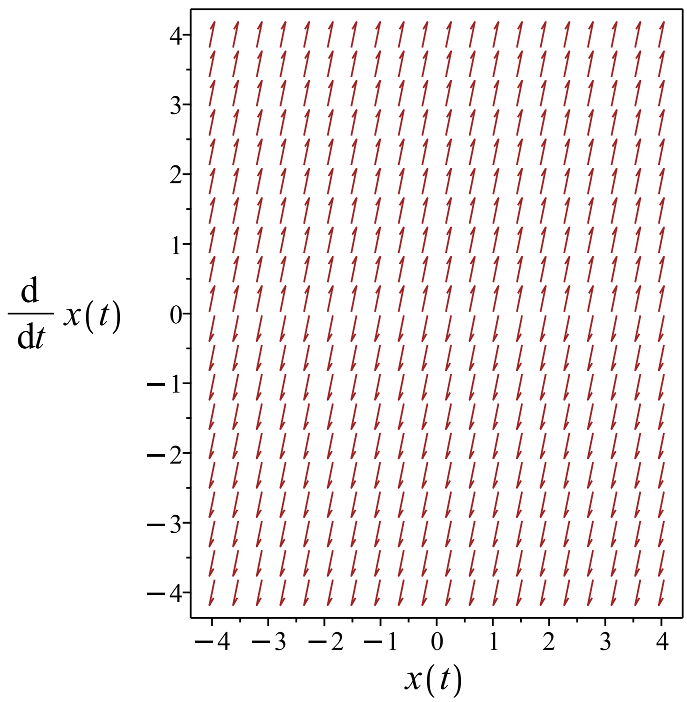


Figure 135: Slope field plot

Verification of solutions

$$
x = c_1 + \frac{c_2 \mathrm{e}^{4t}}{4} + \frac{\ln(\cos(t))}{4} + \frac{\left(\int_0^t \tan(\alpha)\,\mathrm{e}^{-4\alpha} d\alpha\right) \mathrm{e}^{4t}}{4}
$$

Verified OK.

11.5.6 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,x'' + q(t)\,x' + r(t)\,x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -4 \\ r(x) &= 0 \\ s(x) &= \tan(t) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,x' + (q(t) - p'(t))\,x)' = s(x)$$

Integrating gives

$$p(t)\,x' + (q(t) - p'(t))\,x = \int s(t)\,dt$$

Substituting the above values for p, q, r, s gives

$$-4x + x' = \int \tan(t)\,dt$$

We now have a first order ode to solve which is

$$-4x + x' = -\ln(\cos(t)) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= -\ln(\cos(t)) + c_1 \end{aligned}$$

Hence the ode is

$$-4x + x' = -\ln(\cos(t)) + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int (-4)dt} \\ &= \mathrm{e}^{-4t} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(-\ln(\cos(t)) + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-4t}x\right) &= \left(\mathrm{e}^{-4t}\right)\left(-\ln(\cos(t)) + c_1\right) \\ \mathrm{d}\left(\mathrm{e}^{-4t}x\right) &= \left(\left(-\ln(\cos(t)) + c_1\right)\mathrm{e}^{-4t}\right)\,\mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{-4t}x &= \int \left(-\ln(\cos(t)) + c_1\right)\mathrm{e}^{-4t}\,\mathrm{d}t \\ \mathrm{e}^{-4t}x &= \int \left(-\ln(\cos(t)) + c_1\right)\mathrm{e}^{-4t}dt + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-4t}$ results in

$$x = \mathrm{e}^{4t}\left(\int \left(-\ln(\cos(t)) + c_1\right)\mathrm{e}^{-4t}dt\right) + c_2\mathrm{e}^{4t}$$

which simplifies to

$$x = \mathrm{e}^{4t}\left(-\left(\int \left(\ln(\cos(t)) - c_1\right)\mathrm{e}^{-4t}dt\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{4t}\left(-\left(\int \left(\ln(\cos(t)) - c_1\right)\mathrm{e}^{-4t}dt\right) + c_2\right) \tag{1}$$

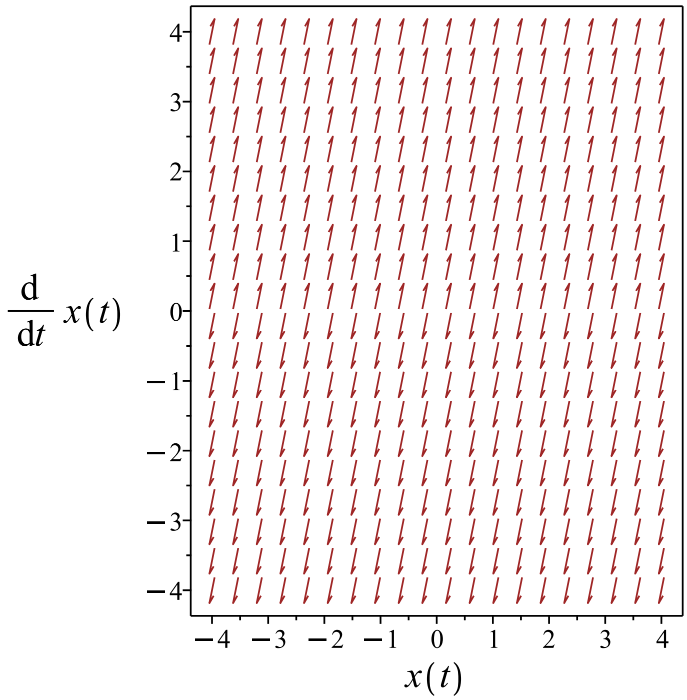


Figure 136: Slope field plot

Verification of solutions

$$x = e^{4t}\left(-\left(\int (\ln(\cos(t)) - c_1) e^{-4t} dt\right) + c_2\right)$$

Verified OK.

11.5.7 Maple step by step solution

Let's solve

$x'' - 4x' = \tan(t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r = 0$

- Factor the characteristic polynomial

  $r(r-4) = 0$

- Roots of the characteristic polynomial

$r = (0, 4)$

- 1st solution of the homogeneous ODE

  $x_1(t) = 1$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{4t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 + c_2 \mathrm{e}^{4t} + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \tan(t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} 1 & \mathrm{e}^{4t} \\ 0 & 4\,\mathrm{e}^{4t} \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 4\,\mathrm{e}^{4t}$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\frac{\left(\int \tan(t) dt\right)}{4} + \frac{\mathrm{e}^{4t} \left(\int \tan(t) \mathrm{e}^{-4t} dt\right)}{4}$

  - Compute integrals

    $x_p(t) = \frac{\mathrm{I}}{16} + \frac{\mathrm{I}\,\mathrm{e}^{4t} \left(\int \frac{\mathrm{e}^{-4t}}{\mathrm{e}^{2\,\mathrm{I}t} + 1} dt\right)}{2} + \frac{\ln(\cos(t))}{4}$

- Substitute particular solution into general solution to ODE

  $x = c_1 + c_2 \mathrm{e}^{4t} + \frac{\mathrm{I}}{16} + \frac{\mathrm{I}\,\mathrm{e}^{4t} \left(\int \frac{\mathrm{e}^{-4t}}{\mathrm{e}^{2\,\mathrm{I}t} + 1} dt\right)}{2} + \frac{\ln(\cos(t))}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 4*_b(_a)+tan(_a), _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(x(t),t$2)-4*diff(x(t),t)=tan(t),x(t), singsol=all)
```

$$x(t) = \int \left(\int \tan(t)\, \mathrm{e}^{-4t} dt + c_1\right) \mathrm{e}^{4t} dt + c_2$$

✓ Solution by Mathematica

Time used: 60.232 (sec). Leaf size: 82

```
DSolve[x''[t]-4*x'[t]==Tan[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \int_1^t \left(e^{4K[1]} c_1 + \frac{1}{20}\left(-5i \operatorname{Hypergeometric2F1}\left(2i, 1, 1+2i, -e^{2iK[1]}\right) - (2-4i)e^{2iK[1]} \operatorname{Hypergeometric2F1}\left(1, 1+2i, 2+2i, -e^{2iK[1]}\right)\right)\right) dK[1] + c_2$$

11.6 problem 18.1 (vi)

Internal problem ID [12059]
Internal file name [`OUTPUT/10711_Sunday_September_03_2023_12_37_01_PM_62615434/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 18, The variation of constants formula. Exercises page 168
Problem number: 18.1 (vi).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{(\tan(x)^2 - 1)\, y'' - 4y' \tan(x)^3 + 2y \sec(x)^4 = (\tan(x)^2 - 1)\,(1 - 2\sin(x)^2)}$$

Given that one solution of the ode is

$$y_1 = \sec(x)^2$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = -2+\sec(x)^2, B = -4\tan(x)^3, C = 2\sec(x)^4, f(x) = -\sec(x)^2\,(2\cos(x)^2 - 1)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y''(-2 + \sec(x)^2) - 4y' \tan(x)^3 + 2y \sec(x)^4 = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{4 \tan(x)^3}{-2 + \sec(x)^2}$$

Therefore

$$y_2(x) = \sec(x)^2 \left(\int \frac{e^{-\left(\int -\frac{4\tan(x)^3}{-2+\sec(x)^2}dx\right)}}{\sec(x)^4}dx\right)$$

$$y_2(x) = \sec(x)^2 \int \frac{e^{\ln\left(2\cos(x)^2-1\right)-4\ln(\cos(x))}}{\sec(x)^4}, dx$$

$$y_2(x) = \sec(x)^2 \left(\int \cos(2x)\, dx\right)$$

$$y_2(x) = \frac{\sec(x)^2 \sin(2x)}{2}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \sec(x)^2 c_1 + \frac{c_2 \sec(x)^2 \sin(2x)}{2} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = \sec(x)^2 c_1 + \frac{c_2 \sec(x)^2 \sin(2x)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sec(x)^2$$

$$y_2 = \frac{\sec(x)^2 \sin(2x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sec(x)^2 & \frac{\sec(x)^2 \sin(2x)}{2} \\ \frac{d}{dx}\left(\sec(x)^2\right) & \frac{d}{dx}\left(\frac{\sec(x)^2 \sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sec(x)^2 & \frac{\sec(x)^2 \sin(2x)}{2} \\ 2\tan(x)\sec(x)^2 & \sec(x)^2 \sin(2x)\tan(x) + \cos(2x)\sec(x)^2 \end{vmatrix}$$

Therefore

$$W = \left(\sec(x)^2\right)\left(\sec(x)^2 \sin(2x)\tan(x) + \cos(2x)\sec(x)^2\right) - \left(\frac{\sec(x)^2 \sin(2x)}{2}\right)\left(2\tan(x)\sec(x)^2\right)$$

Which simplifies to

$$W = \sec(x)^4 \cos(2x)$$

Which simplifies to

$$W = \sec(x)^4 \cos(2x)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\sec(x)^4 \sin(2x)\left(2\cos(x)^2-1\right)^2}{2}}{\left(-2+\sec\left(x\right)^2\right)\sec\left(x\right)^4\cos\left(2x\right)}\,dx$$

Which simplifies to

$$u_1 = -\int \cos\left(x\right)^3 \sin\left(x\right) dx$$

Hence

$$u_1 = \frac{\cos\left(x\right)^4}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\sec\left(x\right)^4 \left(2\cos\left(x\right)^2-1\right)^2}{\left(-2+\sec\left(x\right)^2\right)\sec\left(x\right)^4\cos\left(2x\right)}\,dx$$

Which simplifies to

$$u_2 = \int \sec\left(2x\right)\cos\left(x\right)^2 \left(2\cos\left(x\right)^2-1\right) dx$$

Hence

$$u_2 = \frac{\tan\left(x\right)}{2\tan\left(x\right)^2+2} + \frac{x}{2}$$

Which simplifies to

$$u_1 = \frac{\cos\left(x\right)^4}{4}$$

$$u_2 = \frac{\sin\left(2x\right)}{4} + \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sec\left(x\right)^2\cos\left(x\right)^4}{4} + \frac{\left(\frac{\sin(2x)}{4}+\frac{x}{2}\right)\sec\left(x\right)^2\sin\left(2x\right)}{2}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)^2}{4} + \frac{x\tan(x)}{2} + \frac{1}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\sec(x)^2 c_1 + \frac{c_2 \sec(x)^2 \sin(2x)}{2}\right) + \left(-\frac{\cos(x)^2}{4} + \frac{x\tan(x)}{2} + \frac{1}{2}\right) \end{aligned}$$

Which simplifies to

$$y = \tan(x) c_2 + \sec(x)^2 c_1 - \frac{\cos(x)^2}{4} + \frac{x\tan(x)}{2} + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \tan(x) c_2 + \sec(x)^2 c_1 - \frac{\cos(x)^2}{4} + \frac{x\tan(x)}{2} + \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = \tan(x) c_2 + \sec(x)^2 c_1 - \frac{\cos(x)^2}{4} + \frac{x\tan(x)}{2} + \frac{1}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
         Reducible group (found another exponential solution)
      <- Kovacics algorithm successful
      Change of variables used:
         [x = arcsin(t)]
      Linear ODE actually solved:
         2*u(t)+(6*t^5-7*t^3+t)*diff(u(t),t)+(2*t^6-5*t^4+4*t^2-1)*diff(diff(u(t),t),t) = 0
   <- change of variables successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.25 (sec). Leaf size: 29

```
dsolve([(tan(x)^2-1)*diff(y(x),x$2)-4*tan(x)^3*diff(y(x),x)+2*y(x)*sec(x)^4=(tan(x)^2-1)*(1-
```

$$y(x) = \frac{(4c_1 + 2x)\tan(x)}{4} + \sec(x)^2 c_2 - \frac{\cos(x)^2}{4} + \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 0.764 (sec). Leaf size: 66

```
DSolve[(Tan[x]^2-1)*y''[x]-4*Tan[x]^3*y'[x]+2*y[x]*Sec[x]^4==(Tan[x]^2-1)*(1-2*Sin[x]^2),y[x
```

$$
\begin{aligned}
y(x) \rightarrow & \sqrt{\sin^2(x)} \sec(x) \arctan\left(\frac{\cos(x)}{1-\sqrt{\sin^2(x)}}\right) \\
& -\frac{1}{4}\cos^2(x)+c_1\sec^2(x)+c_2\sqrt{\sin^2(x)}\sec(x)+\frac{1}{2}
\end{aligned}
$$

12 Chapter 19, CauchyEuler equations. Exercises page 174

12.1 problem 19.1 (i)

Internal problem ID [12060]
Internal file name [`OUTPUT/10712_Monday_September_11_2023_12_49_27_AM_75656588/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 19, CauchyEuler equations. Exercises page 174
Problem number: 19.1 (i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 4y'x + 6y = 0}$$

With initial conditions

$$[y(1) = 0, y'(1) = 1]$$

12.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{4y'}{x} + \frac{6y}{x^2} = 0$$

The domain of $p(x) = -\frac{4}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{6}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

12.1.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^3 + c_1x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2x^3 + c_1x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_2x^2 + 2c_1x$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = 3c_2 + 2c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = x^3 - x^2$$

Which simplifies to

$$y = x^2(x - 1)$$

Summary

The solution(s) found are the following

$$y = x^2(x-1) \tag{1}$$

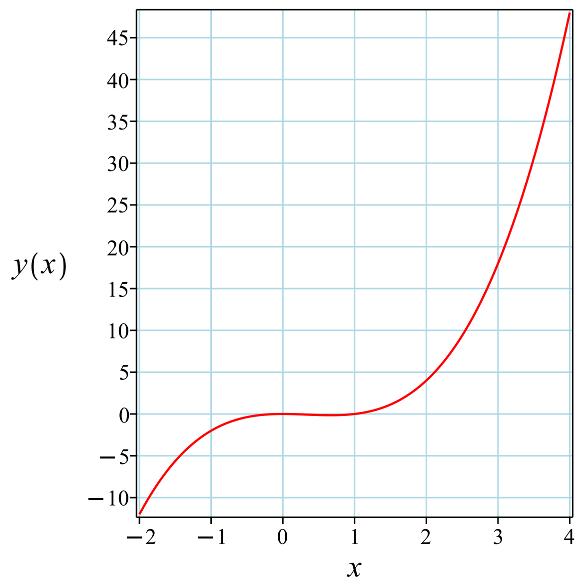


Figure 137: Solution plot

Verification of solutions

$$y = x^2(x-1)$$

Verified OK.

12.1.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{4}{x}\,dx} \\ &= \frac{1}{x^2} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$
$$\left(\frac{y}{x^2}\right)'' = 0$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1x^3 + c_2x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1x^3 + c_2x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1x^2 + 2c_2x$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = 3c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = x^3 - x^2$$

Which simplifies to

$$y = x^2(x - 1)$$

Summary

The solution(s) found are the following

$$y = x^2(x - 1) \tag{1}$$

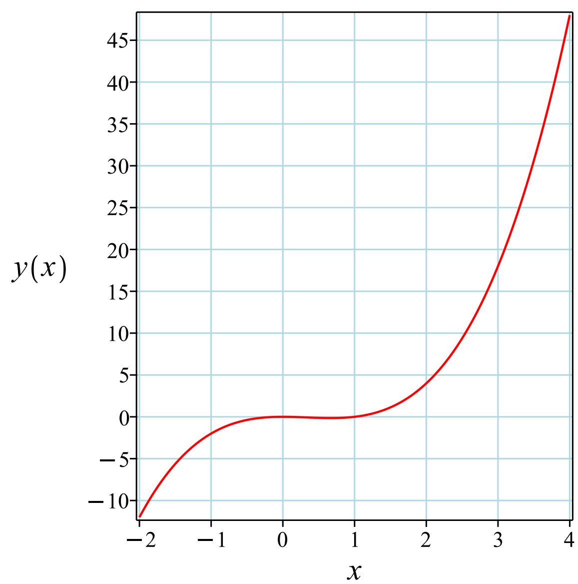


Figure 138: Solution plot

Verification of solutions

$$y = x^2(x - 1)$$

Verified OK.

12.1.4 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(x)dx\right)}dx \\ &= \int e^{-\left(\int -\frac{4}{x}dx\right)}dx \\ &= \int e^{4\ln(x)}\,dx \\ &= \int x^4 dx \\ &= \frac{x^5}{5} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{6}{x^2}}{x^8} \\ &= \frac{6}{x^{10}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{6}{x^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{5}$$
$$r_2 = \frac{3}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{2}{5}} + c_2 \tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = \frac{5^{\frac{2}{5}} \left(5^{\frac{1}{5}} c_1 + c_2\right)}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2 c_1 5^{\frac{3}{5}} x^4}{5 (x^5)^{\frac{3}{5}}} + \frac{3 c_2 5^{\frac{2}{5}} x^4}{5 (x^5)^{\frac{2}{5}}}$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = \frac{\left(2\, 5^{\frac{1}{5}} c_1 + 3 c_2\right) 5^{\frac{2}{5}}}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -5^{\frac{2}{5}}$$
$$c_2 = 5^{\frac{3}{5}}$$

Substituting these values back in above solution results in

$$y = (x^5)^{\frac{3}{5}} - (x^5)^{\frac{2}{5}}$$

Summary
The solution(s) found are the following

$$y = (x^5)^{\frac{3}{5}} - (x^5)^{\frac{2}{5}} \tag{1}$$

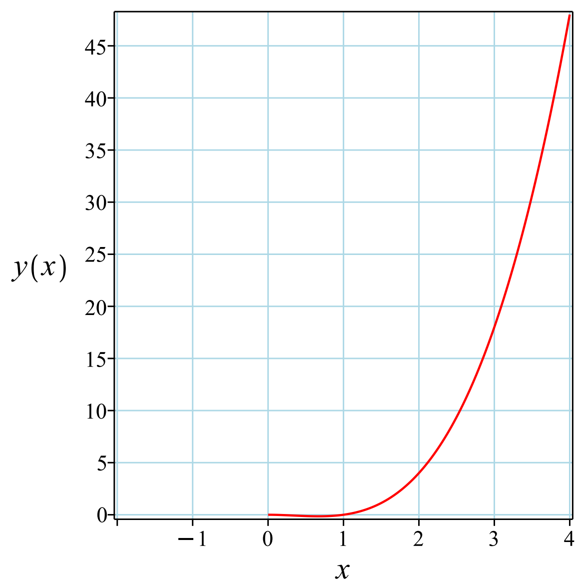


Figure 139: Solution plot

Verification of solutions

$$y = (x^5)^{\frac{3}{5}} - (x^5)^{\frac{2}{5}}$$

Verified OK.

12.1.5 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\, x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{5c\sqrt{6}}{6} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau} y(\tau)\right)}{6} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\, c\tau}{12}} \left(c_1 \cosh\left(\frac{\sqrt{6}\, c\tau}{12}\right) + i c_2 \sinh\left(\frac{\sqrt{6}\, c\tau}{12}\right) \right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int \sqrt{6}\, \sqrt{\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{6}\, \sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{5x^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right)}{2} + x^{\frac{5}{2}} \left(\frac{c_1 \sinh\left(\frac{\ln(x)}{2}\right)}{2x} + \frac{i c_2 \cosh\left(\frac{\ln(x)}{2}\right)}{2x} \right)
$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = \frac{5c_1}{2} + \frac{ic_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= -2i \end{aligned}$$

Substituting these values back in above solution results in

$$y = 2x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right)$$

Summary

The solution(s) found are the following

$$y = 2x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right) \tag{1}$$

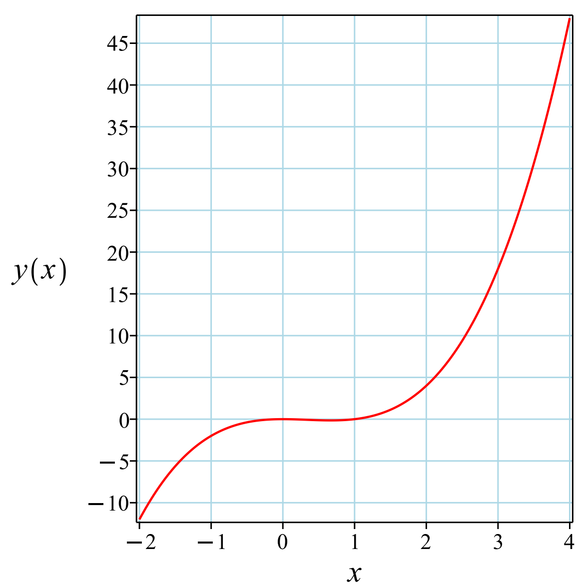


Figure 140: Solution plot

Verification of solutions

$$y = 2x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right)$$

Verified OK.

12.1.6 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{4}{x}}{2}} \\ &= x^2 \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$x^4 v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 x + c_2)\, x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1 x + c_2)\, x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 x^2 + 2(c_1 x + c_2)\, x$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = 3c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = x^2(x - 1)$$

Summary

The solution(s) found are the following

$$y = x^2(x-1) \tag{1}$$

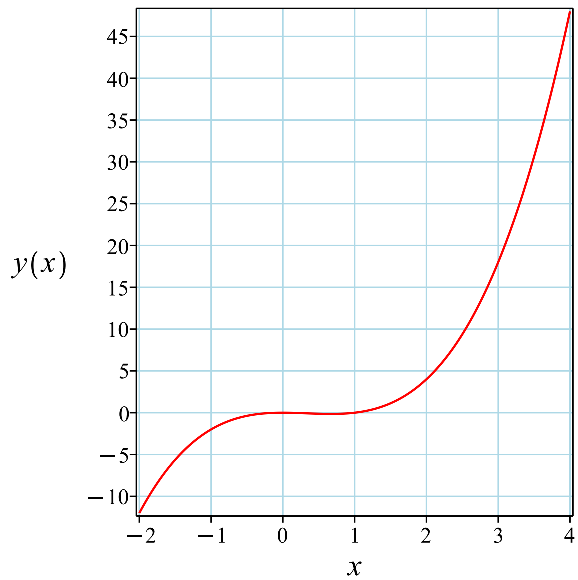


Figure 141: Solution plot

Verification of solutions

$$y = x^2(x-1)$$

Verified OK.

12.1.7 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{6}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x}\,dx\\
\int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx\\
\ln(u) &= -2\ln(x) + c_1\\
u &= \mathrm{e}^{-2\ln(x)+c_1}\\
&= \frac{c_1}{x^2}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx + c_2\\
&= -\frac{c_1}{x} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n\\
&= \left(-\frac{c_1}{x} + c_2\right)x^3\\
&= (c_2 x - c_1)\,x^2
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(-\frac{c_1}{x} + c_2\right)x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = -c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 x + 3\left(-\frac{c_1}{x} + c_2\right)x^2$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = -2c_1 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = x^2(x - 1)$$

Summary

The solution(s) found are the following

$$y = x^2(x - 1) \tag{1}$$

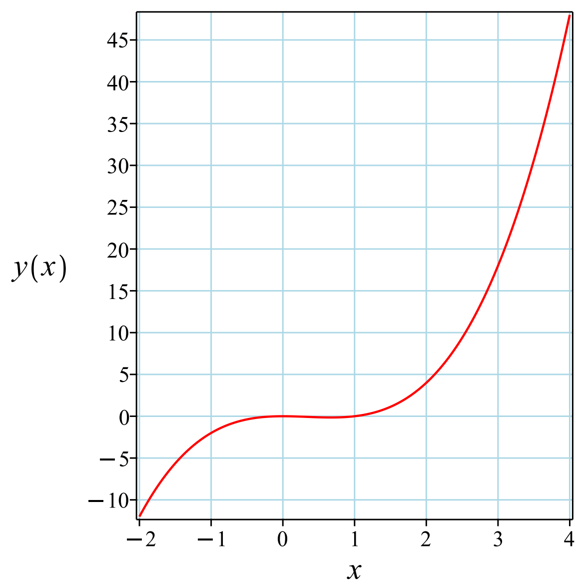


Figure 142: Solution plot

Verification of solutions

$$y = x^2(x - 1)$$

Verified OK.

12.1.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 157: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{2\ln(x)} \\
&= z_1 (x^2)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x^2) + c_2 (x^2(x))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_2 x^3 + c_1 x^2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$
0 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 3c_2 x^2 + 2c_1 x
$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = 3c_2 + 2c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = x^3 - x^2$$

Which simplifies to

$$y = x^2(x - 1)$$

<u>Summary</u>
The solution(s) found are the following

$$y = x^2(x - 1) \tag{1}$$

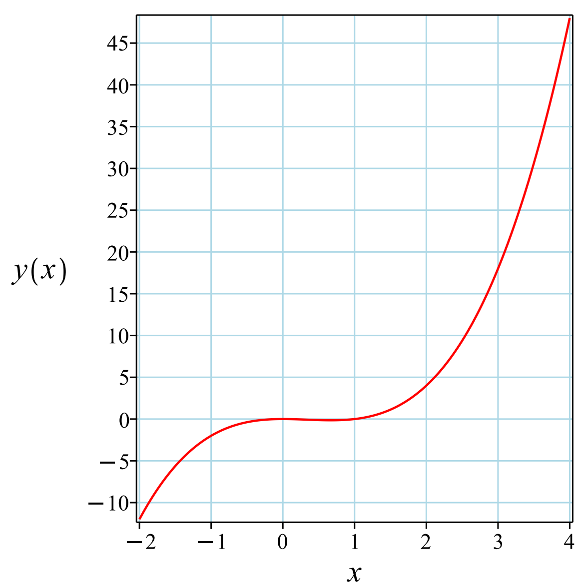


Figure 143: Solution plot

<u>Verification of solutions</u>

$$y = x^2(x - 1)$$

Verified OK.

12.1.9 Maple step by step solution

Let's solve

$$\left[y''x^2 - 4y'x + 6y = 0, y(1) = 0, y'\Big|_{\{x=1\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{x} - \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 4y'x + 6y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 4\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 6y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{2t} + c_2 e^{3t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^3 + c_1 x^2$

- Simplify

  $y = x^2(c_2 x + c_1)$

☐ Check validity of solution $y = x^2(c_2 x + c_1)$

- ○ Use initial condition $y(1) = 0$

  $0 = c_1 + c_2$

- ○ Compute derivative of the solution

  $y' = 2x(c_2 x + c_1) + c_2 x^2$

- ○ Use the initial condition $y'\Big|_{\{x=1\}} = 1$

  $1 = 3c_2 + 2c_1$

- ○ Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = 1\}$

- ○ Substitute constant values into general solution and simplify

  $y = x^2(x - 1)$

- Solution to the IVP

$$y = x^2(x-1)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve([x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(1) = 0, D(y)(1) = 1],y(x), singsol=al
```

$$y(x) = x^2(-1+x)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 12

```
DSolve[{x^2*y''[x]-4*x*y'[x]+6*y[x]==0,{y[1]==0,y'[1]==1}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \rightarrow (x-1)x^2$$

12.2 problem 19.1 (ii)

Internal problem ID [12061]
Internal file name [`OUTPUT/10713_Monday_September_11_2023_12_49_30_AM_35355913/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 19, CauchyEuler equations. Exercises page 174
Problem number: 19.1 (ii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{4x^2y'' + y = 0}$$

With initial conditions

$$[y(1) = 1, y'(1) = 0]$$

12.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{1}{4x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{y}{4x^2} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{1}{4x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

12.2.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$4x^2(r(r-1))x^{r-2} + 0rx^{r-1} + x^r = 0$$

Simplifying gives

$$4r(r-1)\,x^r + 0\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1\sqrt{x} + c_2\sqrt{x}\,\ln(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\sqrt{x} + c_2\sqrt{x}\,\ln(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{2\sqrt{x}} + \frac{c_2\ln(x)}{2\sqrt{x}} + \frac{c_2}{\sqrt{x}}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = \frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -\frac{1}{2}$$

Substituting these values back in above solution results in

$$y = -\frac{\sqrt{x}\,\ln(x)}{2} + \sqrt{x}$$

Which simplifies to

$$y = \left(-\frac{\ln(x)}{2} + 1\right)\sqrt{x}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{\ln(x)}{2} + 1\right)\sqrt{x} \tag{1}$$

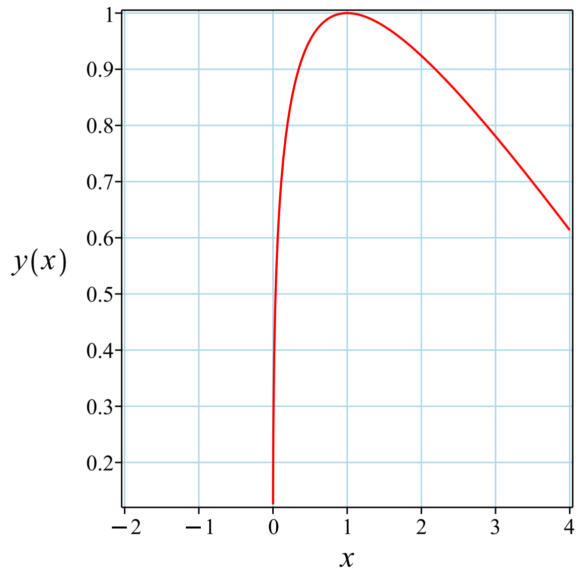


Figure 144: Solution plot

Verification of solutions

$$y = \left(-\frac{\ln(x)}{2} + 1\right)\sqrt{x}$$

Verified OK.

12.2.3 Solving using Kovacic algorithm

Writing the ode as

$$4x^2y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 159: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx}\\ &= \mathrm{e}^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \sqrt{x}\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B=0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \sqrt{x} \int \frac{1}{x}\,dx \\ &= \sqrt{x}(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(\ln(x))\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\sqrt{x} + c_2\sqrt{x}\,\ln(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=1$ and $x=1$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{2\sqrt{x}} + \frac{c_2\ln(x)}{2\sqrt{x}} + \frac{c_2}{\sqrt{x}}$$

substituting $y'=0$ and $x=1$ in the above gives

$$0 = \frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -\frac{1}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{\sqrt{x}\,\ln(x)}{2} + \sqrt{x}$$

Which simplifies to

$$y = \left(-\frac{\ln(x)}{2} + 1\right)\sqrt{x}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{\ln(x)}{2} + 1\right)\sqrt{x} \tag{1}$$

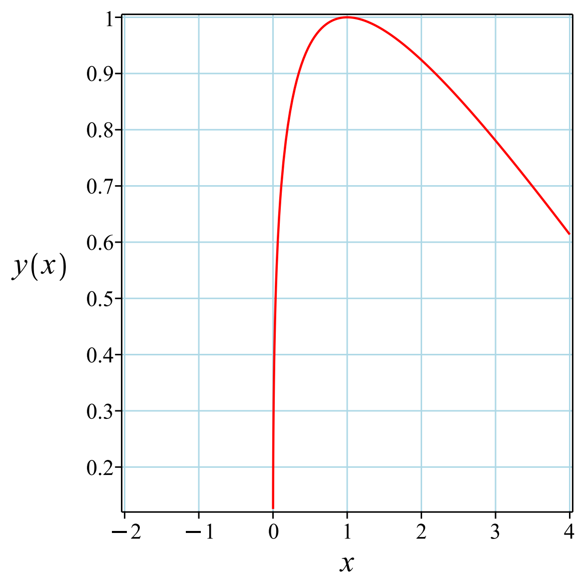


Figure 145: Solution plot

Verification of solutions

$$y = \left(-\frac{\ln(x)}{2} + 1\right)\sqrt{x}$$

Verified OK.

12.2.4 Maple step by step solution

Let's solve

$$\left[4y''x^2+y=0, y(1)=1, y'\Big|_{\{x=1\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y}{4x^2}=0$

- Multiply by denominators of the ODE

  $4y''x^2+y=0$

- Make a change of variables

  $t=\ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y'=\left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y'=\frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y''=\left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2+t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y''=\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $4\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x^2+y(t)=0$

- Simplify

  $4\frac{d^2}{dt^2}y(t)-4\frac{d}{dt}y(t)+y(t)=0$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t) = \frac{d}{dt}y(t) - \frac{y(t)}{4}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) + \frac{y(t)}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = \frac{1}{2}$

- 1st solution of the ODE

  $y_1(t) = e^{\frac{t}{2}}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{\frac{t}{2}}$

- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$

- Substitute in solutions

  $y(t) = c_1e^{\frac{t}{2}} + c_2t\,e^{\frac{t}{2}}$

- Change variables back using $t = \ln(x)$

  $y = c_1\sqrt{x} + c_2\sqrt{x}\,\ln(x)$

- Simplify

  $y = (c_1 + \ln(x)\,c_2)\sqrt{x}$

☐ Check validity of solution $y = (c_1 + \ln(x)\,c_2)\sqrt{x}$

  - Use initial condition $y(1) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $y' = \frac{c_2}{\sqrt{x}} + \frac{c_1+\ln(x)c_2}{2\sqrt{x}}$

  - Use the initial condition $y'\Big|_{\{x=1\}} = 0$

    $0 = \frac{c_1}{2} + c_2$

  - Solve for c_1 and c_2

$\{c_1 = 1, c_2 = -\frac{1}{2}\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{(\ln(x)-2)\sqrt{x}}{2}$

- Solution to the IVP

  $y = -\frac{(\ln(x)-2)\sqrt{x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve([4*x^2*diff(y(x),x$2)+y(x)=0,y(1) = 1, D(y)(1) = 0],y(x), singsol=all)
```

$$y(x) = \sqrt{x}\left(1 - \frac{\ln(x)}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 47

```
DSolve[{x^2*y''[x]+y[x]==0,{y[1]==1,y'[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{3}\sqrt{x}\left(\sqrt{3}\sin\left(\frac{1}{2}\sqrt{3}\log(x)\right) - 3\cos\left(\frac{1}{2}\sqrt{3}\log(x)\right)\right)$$

12.3 problem 19.1 (iii)

Internal problem ID [12062]
Internal file name [`OUTPUT/10714_Monday_September_11_2023_12_49_31_AM_58476720/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 19, CauchyEuler equations. Exercises page 174
Problem number: 19.1 (iii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2x'' - 5tx' + 10x = 0}$$

With initial conditions

$$[x(1) = 2, x'(1) = 1]$$

12.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -\frac{5}{t} \\ q(t) &= \frac{10}{t^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - \frac{5x'}{t} + \frac{10x}{t^2} = 0$$

The domain of $p(t) = -\frac{5}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = \frac{10}{t^2}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

12.3.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - 5trt^{r-1} + 10t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - 5r\,t^r + 10t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - 5r + 10 = 0$$

Or

$$r^2 - 6r + 10 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}r_1 &= 3 - i \\ r_2 &= 3 + i\end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned}r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta\end{aligned}$$

Where in this case $\alpha = 3$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned}x &= c_1 t^{r_1} + c_2 t^{r_2} \\ &= c_1 t^{\alpha+i\beta} + c_2 t^{\alpha-i\beta} \\ &= t^{\alpha}\left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\ &= t^{\alpha}\left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\ &= t^{\alpha}\left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right)\end{aligned}$$

Using the values for $\alpha = 3, \beta = -1$, the above becomes

$$x = t^3\left(c_1 e^{-i\ln(t)} + c_2 e^{i\ln(t)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$x = t^3(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = t^3(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 1$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 3t^2(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))) + t^3\left(-\frac{c_1 \sin(\ln(t))}{t} + \frac{c_2 \cos(\ln(t))}{t}\right)$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -5$$

Substituting these values back in above solution results in

$$x = 2t^3 \cos(\ln(t)) - 5t^3 \sin(\ln(t))$$

Which simplifies to

$$x = (-5\sin(\ln(t)) + 2\cos(\ln(t)))\, t^3$$

Summary

The solution(s) found are the following

$$x = (-5\sin(\ln(t)) + 2\cos(\ln(t)))\, t^3 \tag{1}$$

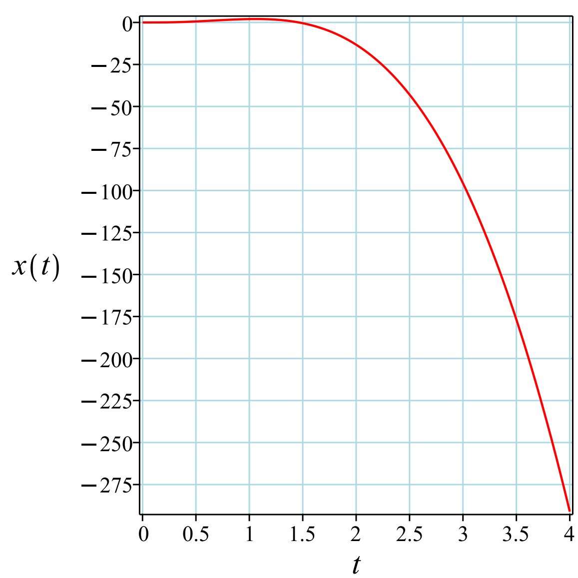


Figure 146: Solution plot

Verification of solutions

$$x = (-5\sin(\ln(t)) + 2\cos(\ln(t)))\, t^3$$

Verified OK.

12.3.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2x'' - 5tx' + 10x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{5}{t} \\ q(t) &= \frac{10}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)} dt \\ &= \int \mathrm{e}^{-\left(\int -\frac{5}{t}dt\right)} dt \\ &= \int e^{5\ln(t)}\, dt \\ &= \int t^5 dt \\ &= \frac{t^6}{6} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{10}{t^2}}{t^{10}} \\ &= \frac{10}{t^{12}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}x(\tau) + q_1 x(\tau) &= 0 \\ \frac{d^2}{d\tau^2}x(\tau) + \frac{10x(\tau)}{t^{12}} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{10}{t^{12}} = \frac{5}{18\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}x(\tau) + \frac{5x(\tau)}{18\tau^2} = 0$$

The above ode is now solved for $x(\tau)$. The ode can be written as

$$18\left(\frac{d^2}{d\tau^2}x(\tau)\right)\tau^2 + 5x(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $x(\tau) = \tau^r$, then $x' = r\tau^{r-1}$ and $x'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$18\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0$$

Simplifying gives

$$18r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$18r(r-1) + 0 + 5 = 0$$

Or

$$18r^2 - 18r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2} - \frac{i}{6} \\
r_2 &= \frac{1}{2} + \frac{i}{6}
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{6}$. Hence the solution becomes

$$
\begin{aligned}
x(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\
&= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\
&= \tau^{\alpha}\left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\
&= \tau^{\alpha}\left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})}\right) \\
&= \tau^{\alpha}\left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right)
\end{aligned}
$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{1}{6}$, the above becomes

$$
x(\tau) = \tau^{\frac{1}{2}}\left(c_1 e^{-\frac{i \ln(\tau)}{6}} + c_2 e^{\frac{i \ln(\tau)}{6}}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
x(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\frac{\ln(\tau)}{6}\right) + c_2 \sin\left(\frac{\ln(\tau)}{6}\right)\right)
$$

The above solution is now transformed back to x using (6) which results in

$$
x = \frac{\sqrt{6}\,\sqrt{t^6}\left(c_1 \cos\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(t^6)}{6}\right) + c_2 \sin\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(t^6)}{6}\right)\right)}{6}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = \frac{\sqrt{6}\,\sqrt{t^6}\left(c_1 \cos\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(t^6)}{6}\right) + c_2 \sin\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(t^6)}{6}\right)\right)}{6} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 1$ in the above gives

$$2 = \frac{\left(c_1 \cos\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right) - c_2 \sin\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right)\right)\sqrt{6}}{6} \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{\sqrt{6}\left(c_1 \cos\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(t^6)}{6}\right) + c_2 \sin\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(t^6)}{6}\right)\right) t^5}{2\sqrt{t^6}} + \frac{\sqrt{6}\sqrt{t^6}\left(-\frac{c_1 \sin\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6}\right.}{t}\right.}{}$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = \frac{\left(\left(c_1 + \frac{c_2}{3}\right)\cos\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right) + \frac{\sin\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right)(c_1 - 3c_2)}{3}\right)\sqrt{6}}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \sqrt{6}\left(-5\sin\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right) + 2\cos\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right)\right)$$
$$c_2 = \left(-5\cos\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right) - 2\sin\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right)\right)\sqrt{6}$$

Substituting these values back in above solution results in

$$x = 2\sqrt{t^6}\cos\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(t^6)}{6}\right)\cos\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right) - 5\sqrt{t^6}\cos\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(t^6)}{6}\right.$$

Which simplifies to

$$x = 2\left(\cos\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(t^6)}{6}\right)\left(\cos\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right) - \frac{5\sin\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right)}{2}\right) - \frac{5\left(\cos\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right) + \frac{2\sin\left(\frac{\ln(2)}{6} + \frac{\ln(3)}{6}\right)}{5}\right)\sin\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(t^6)}{6}\right)}{2}\right)\sqrt{t^6}$$

Summary

The solution(s) found are the following

$$x = 2\left(\cos\left(-\frac{\ln(2)}{6}-\frac{\ln(3)}{6}+\frac{\ln(t^6)}{6}\right)\left(\cos\left(\frac{\ln(2)}{6}+\frac{\ln(3)}{6}\right)-\frac{5\sin\left(\frac{\ln(2)}{6}+\frac{\ln(3)}{6}\right)}{2}\right) - \frac{5\left(\cos\left(\frac{\ln(2)}{6}+\frac{\ln(3)}{6}\right)+\frac{2\sin\left(\frac{\ln(2)}{6}+\frac{\ln(3)}{6}\right)}{5}\right)\sin\left(-\frac{\ln(2)}{6}-\frac{\ln(3)}{6}+\frac{\ln(t^6)}{6}\right)}{2}\right)\sqrt{t^6} \tag{1}$$

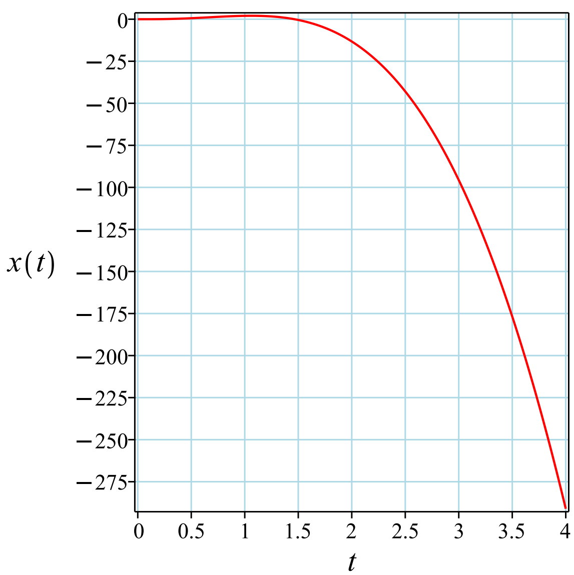


Figure 147: Solution plot

Verification of solutions

$$x = 2\left(\cos\left(-\frac{\ln(2)}{6}-\frac{\ln(3)}{6}+\frac{\ln(t^6)}{6}\right)\left(\cos\left(\frac{\ln(2)}{6}+\frac{\ln(3)}{6}\right)-\frac{5\sin\left(\frac{\ln(2)}{6}+\frac{\ln(3)}{6}\right)}{2}\right)\right.$$
$$\left. - \frac{5\left(\cos\left(\frac{\ln(2)}{6}+\frac{\ln(3)}{6}\right)+\frac{2\sin\left(\frac{\ln(2)}{6}+\frac{\ln(3)}{6}\right)}{5}\right)\sin\left(-\frac{\ln(2)}{6}-\frac{\ln(3)}{6}+\frac{\ln(t^6)}{6}\right)}{2}\right)\sqrt{t^6}$$

Verified OK.

12.3.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2x'' - 5tx' + 10x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{5}{t} \\ q(t) &= \frac{10}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}}{c} \\
\tau'' &= -\frac{\sqrt{10}}{c\sqrt{\frac{1}{t^2}}\,t^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{\sqrt{10}}{c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{5}{t}\frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\
&= -\frac{3c\sqrt{10}}{5}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
x(\tau)'' + p_1 x(\tau)' + q_1 x(\tau) &= 0 \\
\frac{d^2}{d\tau^2}x(\tau) - \frac{3c\sqrt{10}\left(\frac{d}{d\tau}x(\tau)\right)}{5} + c^2 x(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $x(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$x(\tau) = e^{\frac{3\sqrt{10}\,c\tau}{10}}\left(c_1 \cos\left(\frac{\sqrt{10}\,c\tau}{10}\right) + c_2 \sin\left(\frac{\sqrt{10}\,c\tau}{10}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int \sqrt{10}\,\sqrt{\frac{1}{t^2}}dt}{c} \\
&= \frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$x = t^3(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = t^3(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 1$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 3t^2(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))) + t^3\left(-\frac{c_1 \sin(\ln(t))}{t} + \frac{c_2 \cos(\ln(t))}{t}\right)$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2 \\ c_2 &= -5 \end{aligned}$$

Substituting these values back in above solution results in

$$x = 2t^3 \cos(\ln(t)) - 5t^3 \sin(\ln(t))$$

Which simplifies to

$$x = (-5 \sin(\ln(t)) + 2 \cos(\ln(t)))\, t^3$$

Summary

The solution(s) found are the following

$$x = (-5 \sin(\ln(t)) + 2 \cos(\ln(t)))\, t^3 \tag{1}$$

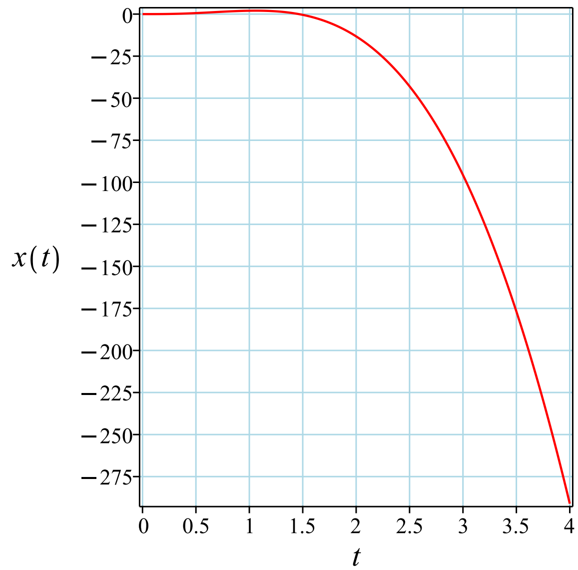


Figure 148: Solution plot

Verification of solutions

$$x = (-5 \sin(\ln(t)) + 2 \cos(\ln(t))) t^3$$

Verified OK.

12.3.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2 x'' - 5 t x' + 10 x = 0 \tag{1}$$

Becomes

$$x'' + p(t) x' + q(t) x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{5}{t} \\ q(t) &= \frac{10}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $x = v(t) t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2}+\frac{np}{t}+q=0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2}-\frac{5n}{t^2}+\frac{10}{t^2}=0 \tag{5}$$

Solving (5) for n gives

$$n=3+i \tag{6}$$

Substituting this value in (3) gives

$$v''(t)+\left(\frac{6+2i}{t}-\frac{5}{t}\right)v'(t)=0$$

$$v''(t)+\frac{(1+2i)\,v'(t)}{t}=0 \tag{7}$$

Using the substitution

$$u(t)=v'(t)$$

Then (7) becomes

$$u'(t)+\frac{(1+2i)\,u(t)}{t}=0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(t,u)\\ &= f(t)g(u)\\ &= \frac{(-1-2i)\,u}{t}\end{aligned}$$

Where $f(t)=\frac{-1-2i}{t}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{-1-2i}{t}\,dt\\ \int\frac{1}{u}\,du &= \int\frac{-1-2i}{t}\,dt\\ \ln(u) &= (-1-2i)\ln(t)+c_1\\ u &= \mathrm{e}^{(-1-2i)\ln(t)+c_1}\\ &= c_1\mathrm{e}^{(-1-2i)\ln(t)}\end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_1 t^{-2i}}{t}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\,dt + c_2 \\ &= \frac{i c_1 t^{-2i}}{2} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} x &= v(t)\,t^n \\ &= \left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^{3+i} \\ &= c_2 t^{3+i} + \frac{i c_1 t^{3-i}}{2} \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^{3+i} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 1$ in the above gives

$$2 = \frac{i c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{c_1 t^{-2i} t^{3+i}}{t} + \frac{(3+i)\left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^{3+i}}{t}$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = \left(\frac{1}{2} + \frac{3i}{2}\right) c_1 + (3+i)\,c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -5-2i \\
c_2 &= 1+\frac{5i}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = -\frac{5i t^{3+i} t^{-2i}}{2} + \frac{5i t^{3+i}}{2} + t^{3+i} t^{-2i} + t^{3+i}
$$

Summary

The solution(s) found are the following

$$
x = \left(1-\frac{5i}{2}\right) t^{3-i} + \left(1+\frac{5i}{2}\right) t^{3+i} \tag{1}
$$

Verification of solutions

$$
x = \left(1-\frac{5i}{2}\right) t^{3-i} + \left(1+\frac{5i}{2}\right) t^{3+i}
$$

Verified OK.

12.3.6 Solving using Kovacic algorithm

Writing the ode as

$$
t^2 x'' - 5t x' + 10x = 0 \tag{1}
$$
$$
A x'' + B x' + C x = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= t^2 \\
B &= -5t \\
C &= 10
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = x e^{\int \frac{B}{2A} \, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{5}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 161: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{4t^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+i$	$\frac{1}{2}-i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+i$	$\frac{1}{2}-i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - i$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - i}{t} + (-)(0) \\
&= \frac{\frac{1}{2} - i}{t} \\
&= \frac{\frac{1}{2} - i}{t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - i}{t}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{t^2}\right) + \left(\frac{\frac{1}{2} - i}{t}\right)^2 - \left(-\frac{5}{4t^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \frac{\frac{1}{2} - i}{t} dt} \\ &= t^{\frac{1}{2} - i} \end{aligned}$$

The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-5t}{t^2} \, dt} \\ &= z_1 e^{\frac{5 \ln(t)}{2}} \\ &= z_1 \left(t^{\frac{5}{2}}\right) \end{aligned}$$

Which simplifies to

$$x_1 = t^{3-i}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{x_1^2} \, dt$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-5t}{t^2}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{5\ln(t)}}{(x_1)^2}\, dt \\
&= x_1 \left(-\frac{i t^{2i}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(t^{3-i}\right) + c_2 \left(t^{3-i} \left(-\frac{i t^{2i}}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 t^{3-i} - \frac{i c_2 t^{3+i}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 1$ in the above gives

$$
2 = c_1 - \frac{i c_2}{2} \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = \frac{(3-i)\, c_1 t^{3-i}}{t} + \frac{\left(\frac{1}{2} - \frac{3i}{2}\right) c_2 t^{3+i}}{t}
$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$
1 = (3-i)\, c_1 + \left(\frac{1}{2} - \frac{3i}{2}\right) c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 - \frac{5i}{2} \\
c_2 &= -5 + 2i
\end{aligned}
$$

Substituting these values back in above solution results in

$$x=-\frac{5i t^{3-i}}{2}+\frac{5i t^{3+i}}{2}+t^{3-i}+t^{3+i}$$

Summary

The solution(s) found are the following

$$x=\left(1-\frac{5i}{2}\right)t^{3-i}+\left(1+\frac{5i}{2}\right)t^{3+i} \tag{1}$$

Verification of solutions

$$x=\left(1-\frac{5i}{2}\right)t^{3-i}+\left(1+\frac{5i}{2}\right)t^{3+i}$$

Verified OK.

12.3.7 Maple step by step solution

Let's solve

$\left[t^2x''-5tx'+10x=0, x(1)=2, x'\Big|_{\{t=1\}}=1\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x''=\frac{5x'}{t}-\frac{10x}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x''-\frac{5x'}{t}+\frac{10x}{t^2}=0$

- Multiply by denominators of the ODE

  $t^2x''-5tx'+10x=0$

- Make a change of variables

  $s=\ln(t)$

☐ Substitute the change of variables back into the ODE

- ◦ Calculate the 1st derivative of x with respect to t , using the chain rule

  $x'=\left(\frac{d}{ds}x(s)\right)s'(t)$

- ◦ Compute derivative

$x' = \frac{\frac{d}{ds}x(s)}{t}$

- Calculate the 2nd derivative of x with respect to t , using the chain rule

  $x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}x(s)\right)$

- Compute derivative

  $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}\right) - 5\frac{d}{ds}x(s) + 10x(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}x(s) - 6\frac{d}{ds}x(s) + 10x(s) = 0$

- Characteristic polynomial of ODE

  $r^2 - 6r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{6\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (3 - \mathrm{I}, 3 + \mathrm{I})$

- 1st solution of the ODE

  $x_1(s) = \mathrm{e}^{3s}\cos(s)$

- 2nd solution of the ODE

  $x_2(s) = \mathrm{e}^{3s}\sin(s)$

- General solution of the ODE

  $x(s) = c_1x_1(s) + c_2x_2(s)$

- Substitute in solutions

  $x(s) = c_1\mathrm{e}^{3s}\cos(s) + c_2\mathrm{e}^{3s}\sin(s)$

- Change variables back using $s = \ln(t)$

  $x = c_1t^3\cos(\ln(t)) + \sin(\ln(t))\,c_2t^3$

- Simplify

  $x = t^3(c_1\cos(\ln(t)) + c_2\sin(\ln(t)))$

□ Check validity of solution $x = t^3(c_1\cos(\ln(t)) + c_2\sin(\ln(t)))$

- Use initial condition $x(1) = 2$

  $2 = c_1$

- Compute derivative of the solution

  $x' = 3t^2(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))) + t^3\left(-\frac{c_1 \sin(\ln(t))}{t} + \frac{c_2 \cos(\ln(t))}{t}\right)$

- Use the initial condition $x'\Big|_{\{t=1\}} = 1$

  $1 = 3c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = -5\}$

- Substitute constant values into general solution and simplify

  $x = (-5 \sin(\ln(t)) + 2 \cos(\ln(t)))\, t^3$

- Solution to the IVP

  $x = (-5 \sin(\ln(t)) + 2 \cos(\ln(t)))\, t^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 19

```
dsolve([t^2*diff(x(t),t$2)-5*t*diff(x(t),t)+10*x(t)=0,x(1) = 2, D(x)(1) = 1],x(t), singsol=a
```

$$x(t) = t^3(-5 \sin(\ln(t)) + 2 \cos(\ln(t)))$$

✓ Solution by Mathematica

Time used: 0.197 (sec). Leaf size: 256

```
DSolve[{t^2*x''[t]-5*t*x[t]+10*x[t]==0,{x[1]==2,x'[1]==1}},x[t],t,IncludeSingularSolutions -
```

$$x(t) \to \frac{2\sqrt{t}\left(\left(\text{BesselI}\left(-1-i\sqrt{39},2\sqrt{5}\right)+\text{BesselI}\left(1-i\sqrt{39},2\sqrt{5}\right)\right)\text{BesselI}\left(i\sqrt{39},2\sqrt{5}\sqrt{t}\right)-\left(\text{BesselI}\left(-1+\right.\right.\right.}{\text{BesselI}\left(i\sqrt{39},2\sqrt{5}\right)\left(\text{BesselI}\left(-1-i\sqrt{39},2\sqrt{5}\right)+\text{BesselI}\left(1-i\sqrt{39},2\sqrt{5}\right)\right)-\text{BesselI}\left(-i\sqrt{39},2\right.}$$

12.4 problem 19.1 (iv)

Internal problem ID [12063]
Internal file name [`OUTPUT/10715_Monday_September_11_2023_12_49_34_AM_38090367/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 19, CauchyEuler equations. Exercises page 174
Problem number: 19.1 (iv).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{t^2x'' + tx' - x = 0}$$

With initial conditions

$$[x(1) = 1, x'(1) = 1]$$

12.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= -\frac{1}{t^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + \frac{x'}{t} - \frac{x}{t^2} = 0$$

The domain of $p(t) = \frac{1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = -\frac{1}{t^2}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

12.4.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + trt^{r-1} - t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + r\,t^r - t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= -1 \\
r_2 &= 1
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$x = c_1 x_1 + c_2 x_2$$

Where $x_1 = t^{r_1}$ and $x_2 = t^{r_2}$. Hence

$$x = \frac{c_1}{t} + c_2 t$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{c_1}{t} + c_2 t \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 1$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{c_1}{t^2} + c_2$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$x = t$$

Summary
The solution(s) found are the following

$$x = t \tag{1}$$

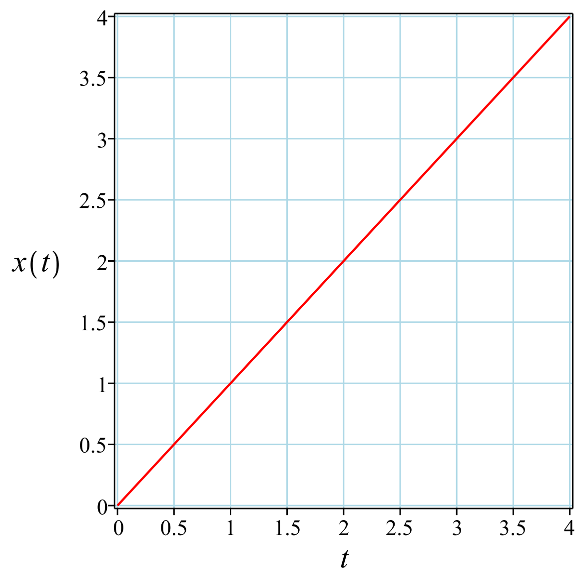


Figure 149: Solution plot

Verification of solutions

$$x = t$$

Verified OK.

12.4.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2x'' + tx' - x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$p(t) = \frac{1}{t}$$
$$q(t) = -\frac{1}{t^2}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^{2}}{d\tau^{2}}x(\tau) + p_{1}\left(\frac{d}{d\tau}x(\tau)\right) + q_{1}x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_{1}(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^{2}} \tag{4}$$

$$q_{1}(\tau) = \frac{q(t)}{\tau'(t)^{2}} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(t)dt\right)}dt \\
&= \int e^{-\left(\int \frac{1}{t}dt\right)}dt \\
&= \int e^{-\ln(t)}\,dt \\
&= \int \frac{1}{t}dt \\
&= \ln(t)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_{1}(\tau) &= \frac{q(t)}{\tau'(t)^{2}} \\
&= \frac{-\frac{1}{t^{2}}}{\frac{1}{t^{2}}} \\
&= -1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^{2}}{d\tau^{2}}x(\tau) + q_{1}x(\tau) &= 0 \\
\frac{d^{2}}{d\tau^{2}}x(\tau) - x(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $x(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(\tau) + Bx'(\tau) + Cx(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $x(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} - \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}\\ x(\tau) &= c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}\end{aligned}$$

Or

$$x(\tau) = c_1 \mathrm{e}^{\tau} + c_2 \mathrm{e}^{-\tau}$$

The above solution is now transformed back to x using (6) which results in

$$x = \frac{c_1 t^2 + c_2}{t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{c_1 t^2 + c_2}{t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 1$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 2c_1 - \frac{c_1 t^2 + c_2}{t^2}$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$x = t$$

Summary

The solution(s) found are the following

$$x = t \tag{1}$$

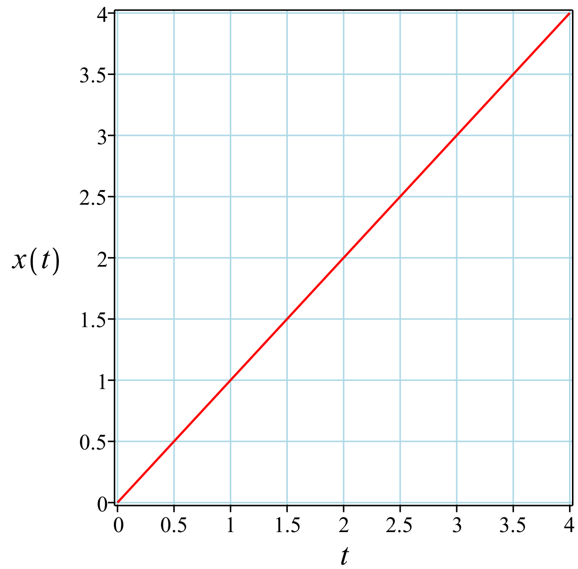


Figure 150: Solution plot

Verification of solutions

$$x = t$$

Verified OK.

12.4.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2x'' + tx' - x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$p(t) = \frac{1}{t}$$
$$q(t) = -\frac{1}{t^2}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{t^2}}}{c} \qquad (6) \\ \tau'' &= \frac{1}{c\sqrt{-\frac{1}{t^2}}\, t^3}\end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\ &= \frac{\frac{1}{c\sqrt{-\frac{1}{t^2}}\, t^3} + \frac{1}{t}\frac{\sqrt{-\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{t^2}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}x(\tau)'' + p_1 x(\tau)' + q_1 x(\tau) &= 0 \\ \frac{d^2}{d\tau^2}x(\tau) + c^2 x(\tau) &= 0 \qquad (7)\end{aligned}$$

The above ode is now solved for $x(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$x(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\ &= \frac{\int \sqrt{-\frac{1}{t^2}}dt}{c} \\ &= \frac{\sqrt{-\frac{1}{t^2}}\, t \ln(t)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$x = \frac{(ic_2 + c_1)\, t^2 - ic_2 + c_1}{2t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{(ic_2 + c_1)\, t^2 - ic_2 + c_1}{2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 1$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = ic_2 + c_1 - \frac{(ic_2 + c_1)\, t^2 - ic_2 + c_1}{2t^2}$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = ic_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -i$$

Substituting these values back in above solution results in

$$x = t$$

Summary
The solution(s) found are the following

$$x = t \tag{1}$$

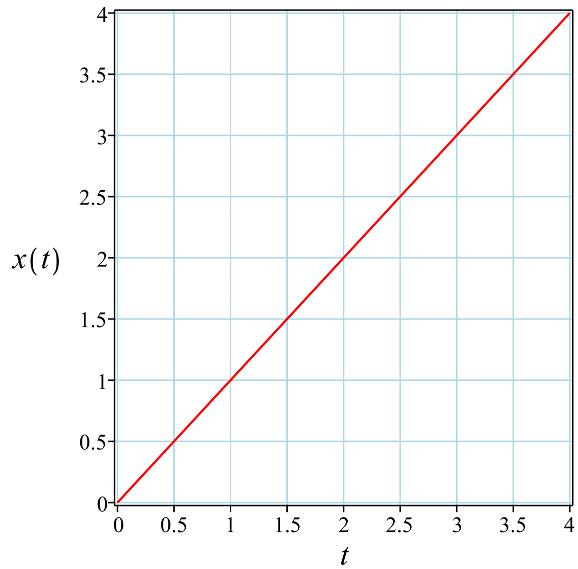


Figure 151: Solution plot

Verification of solutions

$$x = t$$

Verified OK.

12.4.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2 x'' + t x' - x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$p(t) = \frac{1}{t}$$
$$q(t) = -\frac{1}{t^2}$$

Applying change of variables on the depndent variable $x = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{n}{t^2} - \frac{1}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{3v'(t)}{t} &= 0 \\ v''(t) + \frac{3v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{3u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{3u}{t} \end{aligned}$$

Where $f(t) = -\frac{3}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{t}\, dt \\ \int \frac{1}{u}\, du &= \int -\frac{3}{t}\, dt \\ \ln(u) &= -3\ln(t) + c_1 \\ u &= \mathrm{e}^{-3\ln(t)+c_1} \\ &= \frac{c_1}{t^3} \end{aligned}$$

Now that $u(t)$ is known, then

$$
\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\, dt + c_2 \\
&= -\frac{c_1}{2t^2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
x &= v(t)\, t^n \\
&= \left(-\frac{c_1}{2t^2} + c_2\right) t \\
&= \left(-\frac{c_1}{2t^2} + c_2\right) t
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = \left(-\frac{c_1}{2t^2} + c_2\right) t \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 1$ in the above gives

$$
1 = -\frac{c_1}{2} + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = \frac{c_1}{2t^2} + c_2
$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$
1 = \frac{c_1}{2} + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = t
$$

Summary
The solution(s) found are the following

$$x = t \tag{1}$$

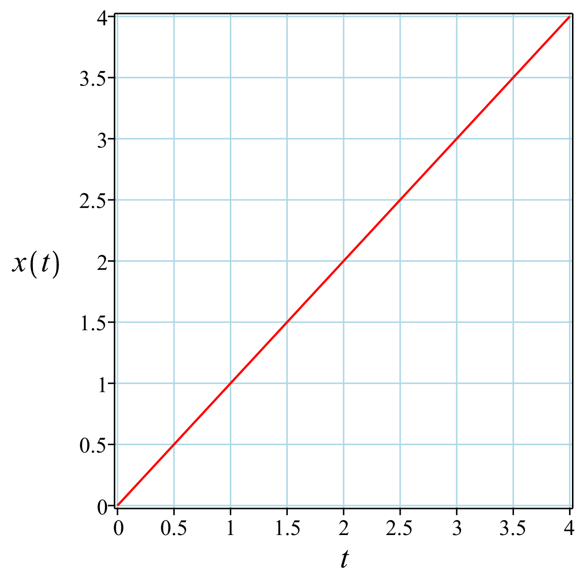


Figure 152: Solution plot

Verification of solutions

$$x = t$$

Verified OK.

12.4.6 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2x'' + tx' - x) \, dt = 0$$
$$t^2x' - xt = c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= -\frac{1}{t} \\
q(t) &= \frac{c_1}{t^2}
\end{aligned}
$$

Hence the ode is

$$
x' - \frac{x}{t} = \frac{c_1}{t^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{1}{t} dt} \\
&= \frac{1}{t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)\left(\frac{c_1}{t^2}\right) \\
\frac{d}{dt}\left(\frac{x}{t}\right) &= \left(\frac{1}{t}\right)\left(\frac{c_1}{t^2}\right) \\
d\left(\frac{x}{t}\right) &= \left(\frac{c_1}{t^3}\right) dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{x}{t} &= \int \frac{c_1}{t^3} dt \\
\frac{x}{t} &= -\frac{c_1}{2t^2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t}$ results in

$$
x = -\frac{c_1}{2t} + c_2 t
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = -\frac{c_1}{2t} + c_2 t \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 1$ in the above gives

$$
1 = -\frac{c_1}{2} + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$x' = \frac{c_1}{2t^2} + c_2$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = \frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$x = t$$

Summary

The solution(s) found are the following

$$x = t \tag{1}$$

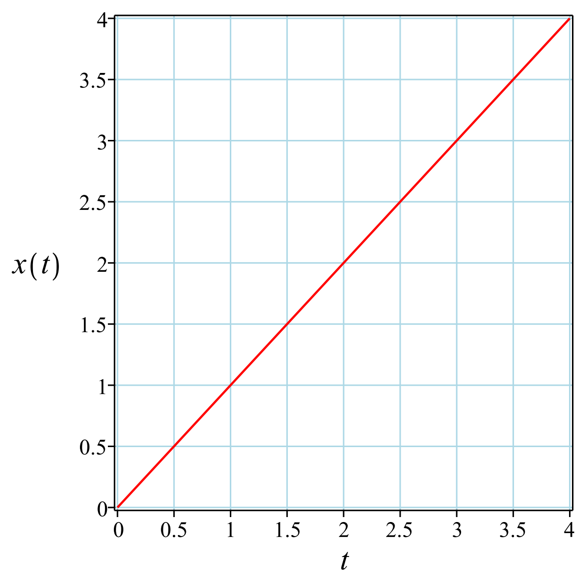


Figure 153: Solution plot

Verification of solutions

$$x = t$$

Verified OK.

12.4.7 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ax'' + Bx' + Cx = F(t)$$

This method reduces the order ode the ODE by one by applying the transformation

$$x = Bv$$

This results in

$$\begin{aligned} x' &= B'v + v'B \\ x'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $x = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= t^2 \\ B &= t \\ C &= -1 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(t^2\right)(0) + (t)(1) + (-1)(t) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$t^3 v'' + (3t^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$t^2(u'(t)\, t + 3u(t)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= -\frac{3u}{t} \end{aligned}$$

Where $f(t) = -\frac{3}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{t}\, dt \\ \int \frac{1}{u}\, du &= \int -\frac{3}{t}\, dt \\ \ln(u) &= -3\ln(t) + c_1 \\ u &= \mathrm{e}^{-3\ln(t)+c_1} \\ &= \frac{c_1}{t^3} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{t^3} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(t) &= \int \frac{c_1}{t^3}\, \mathrm{d}t \\ &= -\frac{c_1}{2t^2} + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x(t) &= Bv \\ &= (t)\left(-\frac{c_1}{2t^2} + c_2\right) \\ &= \left(-\frac{c_1}{2t^2} + c_2\right) t \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \left(-\frac{c_1}{2t^2} + c_2\right) t \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 1$ in the above gives

$$1 = -\frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{c_1}{2t^2} + c_2$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = \frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$x = t$$

Summary
The solution(s) found are the following

$$x = t \tag{1}$$

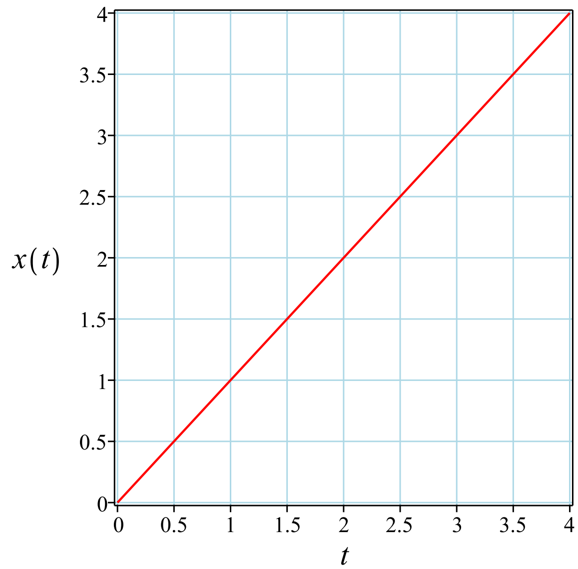


Figure 154: Solution plot

Verification of solutions

$$x = t$$

Verified OK.

12.4.8 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$t^2x'' + tx' - x = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2x'' + tx' - x)\, dt = 0$$
$$t^2x' - xt = c_1$$

Which is now solved for x.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= -\frac{1}{t} \\
q(t) &= \frac{c_1}{t^2}
\end{aligned}
$$

Hence the ode is

$$
x' - \frac{x}{t} = \frac{c_1}{t^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{t}dt} \\
&= \frac{1}{t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{c_1}{t^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{t}\right) &= \left(\frac{1}{t}\right)\left(\frac{c_1}{t^2}\right) \\
\mathrm{d}\left(\frac{x}{t}\right) &= \left(\frac{c_1}{t^3}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{x}{t} &= \int \frac{c_1}{t^3}\,\mathrm{d}t \\
\frac{x}{t} &= -\frac{c_1}{2t^2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t}$ results in

$$
x = -\frac{c_1}{2t} + c_2 t
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = -\frac{c_1}{2t} + c_2 t \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 1$ in the above gives

$$
1 = -\frac{c_1}{2} + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$x' = \frac{c_1}{2t^2} + c_2$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = \frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$x = t$$

Summary

The solution(s) found are the following

$$x = t \tag{1}$$

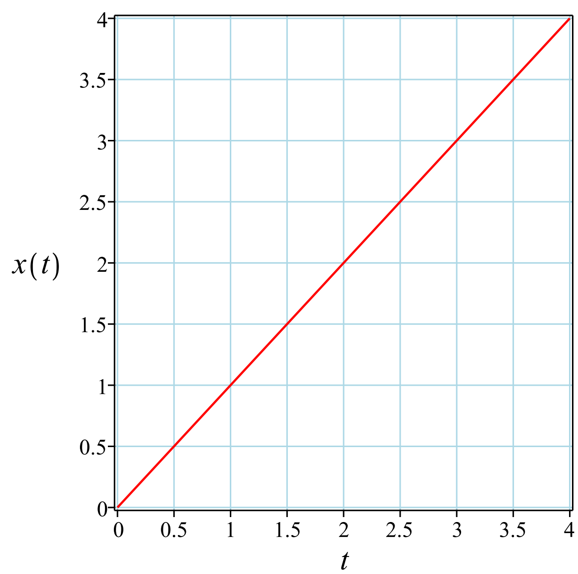


Figure 155: Solution plot

Verification of solutions

$$x = t$$

Verified OK.

12.4.9 Solving using Kovacic algorithm

Writing the ode as

$$t^2x'' + tx' - x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 163: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4t^2}
$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + (-)(0) \\ &= -\frac{1}{2t} \\ &= -\frac{1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(-\frac{1}{2t}\right)^2 - \left(\frac{3}{4t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int -\frac{1}{2t} dt} \\ &= \frac{1}{\sqrt{t}} \end{aligned}$$

The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{t^2}\,dt} \\
&= z_1 e^{-\frac{\ln(t)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \frac{1}{t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{t}{t^2}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-\ln(t)}}{(x_1)^2}\,dt \\
&= x_1\left(\frac{t^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\frac{1}{t}\right) + c_2\left(\frac{1}{t}\left(\frac{t^2}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = \frac{c_1}{t} + \frac{c_2 t}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 1$ in the above gives

$$1 = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{c_1}{t^2} + \frac{c_2}{2}$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = -c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$x = t$$

Summary

The solution(s) found are the following

$$x = t \tag{1}$$

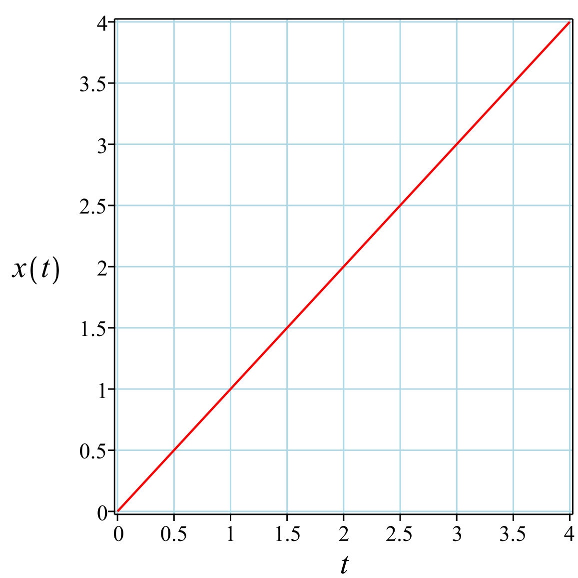


Figure 156: Solution plot

Verification of solutions

$$x = t$$

Verified OK.

12.4.10 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,x'' + q(t)\,x' + r(t)\,x = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= t^2 \\ q(x) &= t \\ r(x) &= -1 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 1 \end{aligned}$$

Therefore (1) becomes

$$2 - (1) + (-1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,x' + (q(t) - p'(t))\,x)' = s(x)$$

Integrating gives

$$p(t)\,x' + (q(t) - p'(t))\,x = \int s(t)\;dt$$

Substituting the above values for p, q, r, s gives

$$t^2 x' - xt = c_1$$

We now have a first order ode to solve which is

$$t^2x' - xt = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -\frac{1}{t} \\ q(t) &= \frac{c_1}{t^2} \end{aligned}$$

Hence the ode is

$$x' - \frac{x}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{t}dt} \\ &= \frac{1}{t} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{c_1}{t^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{t}\right) &= \left(\frac{1}{t}\right)\left(\frac{c_1}{t^2}\right) \\ \mathrm{d}\left(\frac{x}{t}\right) &= \left(\frac{c_1}{t^3}\right)\mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{x}{t} &= \int \frac{c_1}{t^3}\,\mathrm{d}t \\ \frac{x}{t} &= -\frac{c_1}{2t^2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t}$ results in

$$x = -\frac{c_1}{2t} + c_2 t$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = -\frac{c_1}{2t} + c_2 t \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 1$ in the above gives

$$1 = -\frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{c_1}{2t^2} + c_2$$

substituting $x' = 1$ and $t = 1$ in the above gives

$$1 = \frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$x = t$$

Summary

The solution(s) found are the following

$$x = t \tag{1}$$

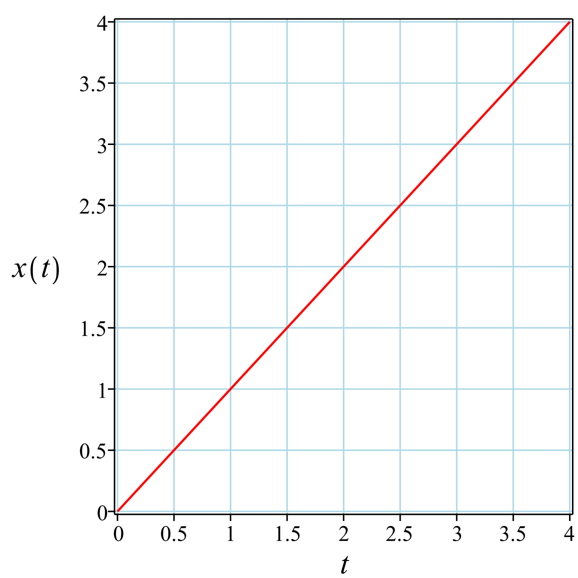


Figure 157: Solution plot

Verification of solutions

$$x = t$$

Verified OK.

12.4.11 Maple step by step solution

Let's solve

$$\left[t^2 x'' + t x' - x = 0, x(1) = 1, x'\Big|_{\{t=1\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -\frac{x'}{t} + \frac{x}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{x'}{t} - \frac{x}{t^2} = 0$

- Multiply by denominators of the ODE

  $t^2 x'' + t x' - x = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of x with respect to t , using the chain rule

    $x' = \left(\frac{d}{ds} x(s)\right) s'(t)$

  - Compute derivative

    $x' = \frac{\frac{d}{ds} x(s)}{t}$

  - Calculate the 2nd derivative of x with respect to t , using the chain rule

    $x'' = \left(\frac{d^2}{ds^2} x(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds} x(s)\right)$

  - Compute derivative

    $x'' = \frac{\frac{d^2}{ds^2} x(s)}{t^2} - \frac{\frac{d}{ds} x(s)}{t^2}$

  Substitute the change of variables back into the ODE

$t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}\right) + \frac{d}{ds}x(s) - x(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}x(s) - x(s) = 0$

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the ODE

  $x_1(s) = \mathrm{e}^{-s}$

- 2nd solution of the ODE

  $x_2(s) = \mathrm{e}^{s}$

- General solution of the ODE

  $x(s) = c_1 x_1(s) + c_2 x_2(s)$

- Substitute in solutions

  $x(s) = c_1 \mathrm{e}^{-s} + c_2 \mathrm{e}^{s}$

- Change variables back using $s = \ln(t)$

  $x = \frac{c_1}{t} + c_2 t$

- Simplify

  $x = \frac{c_1}{t} + c_2 t$

☐ Check validity of solution $x = \frac{c_1}{t} + c_2 t$

  - Use initial condition $x(1) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $x' = -\frac{c_1}{t^2} + c_2$

  - Use the initial condition $x'\Big|_{\{t=1\}} = 1$

    $1 = -c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 1\}$
- Substitute constant values into general solution and simplify

  $x = t$

- Solution to the IVP

  $x = t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([t^2*diff(x(t),t$2)+t*diff(x(t),t)-x(t)=0,x(1) = 1, D(x)(1) = 1],x(t), singsol=all)
```

$$x(t) = t$$

✓ Solution by Mathematica

Time used: 0.119 (sec). Leaf size: 172

```
DSolve[{t^2*x''[t]+t*x[t]-x[t]==0,{x[1]==1,x'[1]==1}},x[t],t,IncludeSingularSolutions -> Tru
```

$$x(t) \to \frac{\sqrt{t}\left(\left(\text{BesselJ}\left(\sqrt{5},2\right) - \text{BesselJ}\left(-1+\sqrt{5},2\right) + \text{BesselJ}\left(1+\sqrt{5},2\right)\right) \text{BesselJ}\left(-\sqrt{5},2\sqrt{t}\right) - \left(\text{BesselJ}\left(-\right.\right.\right.}{\text{BesselJ}\left(\sqrt{5},2\right)\left(\text{BesselJ}\left(-1-\sqrt{5},2\right) - \text{BesselJ}\left(1-\sqrt{5},2\right)\right) + \text{BesselJ}\left(-\sqrt{}\right.}$$

12.5 problem 19.1 (v)

Internal problem ID [12064]
Internal file name [`OUTPUT/10716_Monday_September_11_2023_12_49_36_AM_24676024/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 19, CauchyEuler equations. Exercises page 174
Problem number: 19.1 (v).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2z'' + 3xz' + 4z = 0}$$

With initial conditions

$$[z(1) = 0, z'(1) = 5]$$

12.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$z'' + p(x)z' + q(x)z = F$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{4}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$z'' + \frac{3z'}{x} + \frac{4z}{x^2} = 0$$

The domain of $p(x) = \frac{3}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{4}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

12.5.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $z = x^r$, then $z' = rx^{r-1}$ and $z'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 3r + 4 = 0$$

Or

$$r^2 + 2r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= -i\sqrt{3} - 1 \\
r_2 &= i\sqrt{3} - 1
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = -1$ and $\beta = -\sqrt{3}$. Hence the solution becomes

$$
\begin{aligned}
z &= c_1 x^{r_1} + c_2 x^{r_2} \\
&= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\
&= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\
&= x^{\alpha}\left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\
&= x^{\alpha}\left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right)
\end{aligned}
$$

Using the values for $\alpha = -1, \beta = -\sqrt{3}$, the above becomes

$$
z = x^{-1}\left(c_1 e^{-i\sqrt{3}\ln(x)} + c_2 e^{i\sqrt{3}\ln(x)}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
z = \frac{1}{x}\left(c_1 \cos\left(\sqrt{3}\ln(x)\right) + c_2 \sin\left(\sqrt{3}\ln(x)\right)\right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
z = \frac{c_1 \cos\left(\sqrt{3}\ln(x)\right) + c_2 \sin\left(\sqrt{3}\ln(x)\right)}{x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 0$ and $x = 1$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$z' = -\frac{c_1 \cos\left(\sqrt{3}\, \ln\left(x\right)\right) + c_2 \sin\left(\sqrt{3}\, \ln\left(x\right)\right)}{x^2} + \frac{-\frac{c_1 \sqrt{3}\, \sin\left(\sqrt{3}\, \ln(x)\right)}{x} + \frac{c_2 \sqrt{3}\, \cos\left(\sqrt{3}\, \ln(x)\right)}{x}}{x}$$

substituting $z' = 5$ and $x = 1$ in the above gives

$$5 = c_2 \sqrt{3} - c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = \frac{5\sqrt{3}}{3}$$

Substituting these values back in above solution results in

$$z = \frac{5\sqrt{3}\, \sin\left(\sqrt{3}\, \ln\left(x\right)\right)}{3x}$$

Summary

The solution(s) found are the following

$$z = \frac{5\sqrt{3}\, \sin\left(\sqrt{3}\, \ln\left(x\right)\right)}{3x} \tag{1}$$

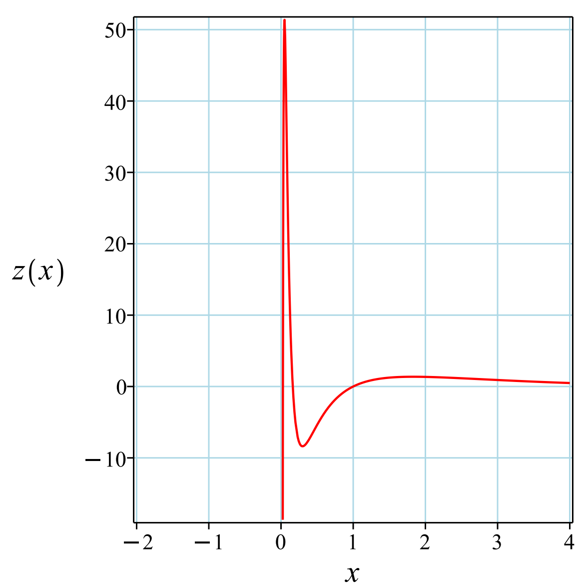


Figure 158: Solution plot

Verification of solutions

$$z = \frac{5\sqrt{3}\,\sin\left(\sqrt{3}\,\ln\left(x\right)\right)}{3x}$$

Verified OK.

12.5.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2z'' + 3xz' + 4z = 0 \tag{1}$$

Becomes

$$z'' + p(x)\,z' + q(x)\,z = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}z(\tau) + p_1\left(\frac{d}{d\tau}z(\tau)\right) + q_1z(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{3}{x}dx\right)}dx \\ &= \int e^{-3\ln(x)}\,dx \\ &= \int \frac{1}{x^3}dx \\ &= -\frac{1}{2x^2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{4}{x^2}}{\frac{1}{x^6}} \\ &= 4x^4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}z(\tau) + q_1 z(\tau) = 0$$
$$\frac{d^2}{d\tau^2}z(\tau) + 4x^4 z(\tau) = 0$$

But in terms of τ

$$4x^4 = \frac{1}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}z(\tau) + \frac{z(\tau)}{\tau^2} = 0$$

The above ode is now solved for $z(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}z(\tau)\right)\tau^2 + z(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $z(\tau) = \tau^r$, then $z' = r\tau^{r-1}$ and $z'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 + 1 = 0$$

Or

$$r^2 - r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$r_2 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{\sqrt{3}}{2}$. Hence the solution becomes

$$\begin{aligned} z(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\ &= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\ &= \tau^{\alpha} \left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta} \right) \\ &= \tau^{\alpha} \left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})} \right) \\ &= \tau^{\alpha} \left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)} \right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{\sqrt{3}}{2}$, the above becomes

$$z(\tau) = \tau^{\frac{1}{2}} \left(c_1 e^{-\frac{i\sqrt{3}\,\ln(\tau)}{2}} + c_2 e^{\frac{i\sqrt{3}\,\ln(\tau)}{2}} \right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$z(\tau) = \sqrt{\tau} \left(c_1 \cos\left(\frac{\sqrt{3}\,\ln(\tau)}{2} \right) + c_2 \sin\left(\frac{\sqrt{3}\,\ln(\tau)}{2} \right) \right)$$

The above solution is now transformed back to z using (6) which results in

$$z = \frac{\left(c_1 \cos\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2} \right) + c_2 \sin\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2} \right) \right) \sqrt{2}\sqrt{-\frac{1}{x^2}}}{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$z = \frac{\left(c_1 \cos\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2} \right) + c_2 \sin\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2} \right) \right) \sqrt{2}\sqrt{-\frac{1}{x^2}}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 0$ and $x = 1$ in the above gives

$$0 = \frac{\sqrt{2}\left(i c_1 \cosh\left(\frac{\sqrt{3}\,(\pi+i\ln(2))}{2}\right) - c_2 \sinh\left(\frac{\sqrt{3}\,(\pi+i\ln(2))}{2}\right)\right)}{2} \tag{1A}$$

Taking derivative of the solution gives

$$z' = \frac{\left(\frac{c_1\sqrt{3}\sin\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2}\right)}{x} - \frac{c_2\sqrt{3}\cos\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2}\right)}{x}\right)\sqrt{2}\sqrt{-\frac{1}{x^2}}}{2} + \frac{\left(c_1\cos\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right.\right.}{2}\right.\right.}{}$$

substituting $z' = 5$ and $x = 1$ in the above gives

$$5 = -\frac{\sqrt{2}\left(i\left(c_2\sqrt{3}+c_1\right)\cosh\left(\frac{\sqrt{3}\,(\pi+i\ln(2))}{2}\right) + \sinh\left(\frac{\sqrt{3}\,(\pi+i\ln(2))}{2}\right)\left(\sqrt{3}\,c_1 - c_2\right)\right)}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{5\sinh\left(\frac{\sqrt{3}\,(\pi+i\ln(2))}{2}\right)\sqrt{6}}{3}$$
$$c_2 = \frac{5i\cosh\left(\frac{\sqrt{3}\,(\pi+i\ln(2))}{2}\right)\sqrt{6}}{3}$$

Substituting these values back in above solution results in

$$z = \frac{5i\sin\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2}\right)\cosh\left(\frac{\sqrt{3}\,(\pi+i\ln(2))}{2}\right)\sqrt{2}\sqrt{6}\sqrt{-\frac{1}{x^2}}}{6} + \frac{5\sqrt{2}\sqrt{-\frac{1}{x^2}}\cos\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2}\right.}{6}$$

Summary

The solution(s) found are the following

$$z = \frac{5\sqrt{3}\left(i\sin\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2}\right)\cosh\left(\frac{\sqrt{3}\,(\pi+i\ln(2))}{2}\right) + \cos\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2}\right)\sinh\left(\frac{\sqrt{3}\,(\pi+i\ln(2))}{2}\right)\right)\sqrt{}}{3} \tag{1}$$

Verification of solutions

$$z = \frac{5\sqrt{3}\left(i\sin\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2}\right)\cosh\left(\frac{\sqrt{3}\,(\pi+i\ln(2))}{2}\right) + \cos\left(\frac{\sqrt{3}\left(-\ln(2)+\ln\left(-\frac{1}{x^2}\right)\right)}{2}\right)\sinh\left(\frac{\sqrt{3}\,(\pi+i\ln(2))}{2}\right)\right)\sqrt{}}{3}$$

Verified OK.

12.5.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2z'' + 3xz' + 4z = 0 \tag{1}$$

Becomes

$$z'' + p(x)\, z' + q(x)\, z = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}z(\tau) + p_1\left(\frac{d}{d\tau}z(\tau)\right) + q_1 z(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3} + \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= c \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
z(\tau)'' + p_1 z(\tau)' + q_1 z(\tau) &= 0 \\
\frac{d^2}{d\tau^2} z(\tau) + c\left(\frac{d}{d\tau} z(\tau)\right) + c^2 z(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $z(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
z(\tau) = \mathrm{e}^{-\frac{c\tau}{2}} \left(c_1 \cos\left(\frac{c\sqrt{3}\,\tau}{2}\right) + c_2 \sin\left(\frac{c\sqrt{3}\,\tau}{2}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
z = \frac{c_1 \cos\left(\sqrt{3}\, \ln(x)\right) + c_2 \sin\left(\sqrt{3}\, \ln(x)\right)}{x}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
z = \frac{c_1 \cos\left(\sqrt{3}\, \ln(x)\right) + c_2 \sin\left(\sqrt{3}\, \ln(x)\right)}{x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 0$ and $x = 1$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
z' = -\frac{c_1 \cos\left(\sqrt{3}\, \ln(x)\right) + c_2 \sin\left(\sqrt{3}\, \ln(x)\right)}{x^2} + \frac{-\frac{c_1\sqrt{3}\, \sin\left(\sqrt{3}\, \ln(x)\right)}{x} + \frac{c_2\sqrt{3}\, \cos\left(\sqrt{3}\, \ln(x)\right)}{x}}{x}
$$

substituting $z' = 5$ and $x = 1$ in the above gives

$$5 = c_2\sqrt{3} - c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = \frac{5\sqrt{3}}{3}$$

Substituting these values back in above solution results in

$$z = \frac{5\sqrt{3}\,\sin\left(\sqrt{3}\,\ln\left(x\right)\right)}{3x}$$

Summary
The solution(s) found are the following

$$z = \frac{5\sqrt{3}\,\sin\left(\sqrt{3}\,\ln\left(x\right)\right)}{3x} \tag{1}$$

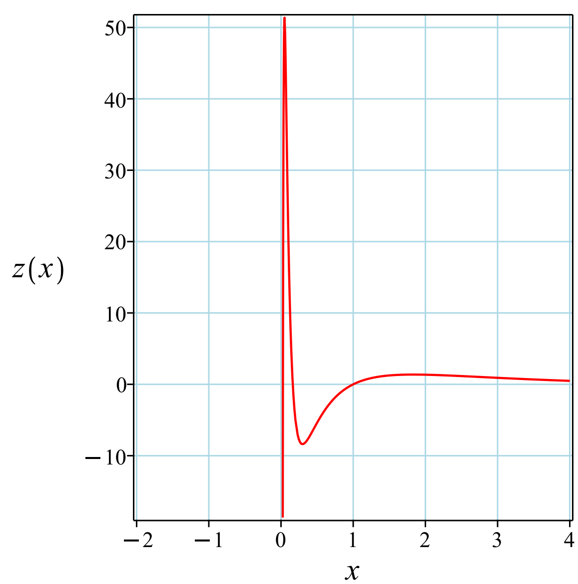


Figure 159: Solution plot

Verification of solutions

$$z = \frac{5\sqrt{3}\,\sin\left(\sqrt{3}\,\ln\left(x\right)\right)}{3x}$$

Verified OK.

12.5.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2z'' + 3xz' + 4z = 0 \tag{1}$$

Becomes

$$z'' + p(x)\, z' + q(x)\, z = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $z = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not z.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = i\sqrt{3} - 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2i\sqrt{3}-2}{x} + \frac{3}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(2i\sqrt{3}+1)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{\left(2i\sqrt{3}+1\right) u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\left(-2i\sqrt{3}-1\right) u}{x} \end{aligned}$$

Where $f(x) = \frac{-2i\sqrt{3}-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-2i\sqrt{3}-1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{-2i\sqrt{3}-1}{x}\, dx \\ \ln(u) &= \left(-2i\sqrt{3}-1\right) \ln(x) + c_1 \\ u &= \mathrm{e}^{\left(-2i\sqrt{3}-1\right) \ln(x) + c_1} \\ &= c_1 \mathrm{e}^{\left(-2i\sqrt{3}-1\right) \ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-2i\sqrt{3}}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= \frac{i\sqrt{3}\, c_1 x^{-2i\sqrt{3}}}{6} + c_2 \end{aligned}$$

Hence

$$\begin{aligned}
z &= v(x)\, x^n \\
&= \left(\frac{i\sqrt{3}\, c_1 x^{-2i\sqrt{3}}}{6} + c_2\right) x^{i\sqrt{3}-1} \\
&= \frac{x^{i\sqrt{3}} c_2 + \frac{i x^{-i\sqrt{3}} \sqrt{3}\, c_1}{6}}{x}
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$z = \left(\frac{i\sqrt{3}\, c_1 x^{-2i\sqrt{3}}}{6} + c_2\right) x^{i\sqrt{3}-1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 0$ and $x = 1$ in the above gives

$$0 = \frac{i\sqrt{3}\, c_1}{6} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$z' = \frac{c_1 x^{-2i\sqrt{3}} x^{i\sqrt{3}-1}}{x} + \frac{\left(\frac{i\sqrt{3}\, c_1 x^{-2i\sqrt{3}}}{6} + c_2\right) x^{i\sqrt{3}-1} \left(i\sqrt{3} - 1\right)}{x}$$

substituting $z' = 5$ and $x = 1$ in the above gives

$$5 = \frac{i(-c_1 + 6c_2)\sqrt{3}}{6} + \frac{c_1}{2} - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}
c_1 &= 5 \\
c_2 &= -\frac{5i\sqrt{3}}{6}
\end{aligned}$$

Substituting these values back in above solution results in

$$z = \frac{5i\sqrt{3}\, x^{i\sqrt{3}-1} x^{-2i\sqrt{3}}}{6} - \frac{5i\sqrt{3}\, x^{i\sqrt{3}-1}}{6}$$

Summary

The solution(s) found are the following

$$z = -\frac{5i\sqrt{3}\left(-x^{-i\sqrt{3}} + x^{i\sqrt{3}}\right)}{6x} \tag{1}$$

Verification of solutions

$$z = -\frac{5i\sqrt{3}\left(-x^{-i\sqrt{3}} + x^{i\sqrt{3}}\right)}{6x}$$

Verified OK.

12.5.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2z'' + 3xz' + 4z = 0 \tag{1}$$
$$Az'' + Bz' + Cz = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ze^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-13}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -13 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{13}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then z is found using the inverse transformation

$$
z = ze^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 165: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{13}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{13}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i\sqrt{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i\sqrt{3}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{13}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{13}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i\sqrt{3} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i\sqrt{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{13}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i\sqrt{3}$	$\frac{1}{2} - i\sqrt{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+i\sqrt{3}$	$\frac{1}{2}-i\sqrt{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}-i\sqrt{3}$ then

$$\begin{aligned}d&=\alpha_\infty^--\left(\alpha_{c_1}^-\right)\\&=\frac{1}{2}-i\sqrt{3}-\left(\frac{1}{2}-i\sqrt{3}\right)\\&=0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\&=\frac{\frac{1}{2}-i\sqrt{3}}{x}+(-)(0)\\&=\frac{\frac{1}{2}-i\sqrt{3}}{x}\\&=\frac{-2i\sqrt{3}+1}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{\frac{1}{2}-i\sqrt{3}}{x}\right)(0)+\left(\left(-\frac{\frac{1}{2}-i\sqrt{3}}{x^{2}}\right)+\left(\frac{\frac{1}{2}-i\sqrt{3}}{x}\right)^{2}-\left(-\frac{13}{4x^{2}}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2}-i\sqrt{3}}{x} dx} \\
&= x^{\frac{1}{2}-i\sqrt{3}}
\end{aligned}$$

The first solution to the original ode in z is found from

$$\begin{aligned}
z_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1\left(\frac{1}{x^{\frac{3}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$z_1 = x^{-i\sqrt{3}-1}$$

The second solution z_2 to the original ode is found using reduction of order

$$z_2 = z_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{z_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
z_2 &= z_1 \int \frac{e^{\int -\frac{3x}{x^2}\, dx}}{(z_1)^2}\, dx \\
&= z_1 \int \frac{e^{-3\ln(x)}}{(z_1)^2}\, dx \\
&= z_1\left(-\frac{i x^{2i\sqrt{3}}\sqrt{3}}{6}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
z &= c_1 z_1 + c_2 z_2 \\
&= c_1 \left(x^{-i\sqrt{3}-1}\right) + c_2 \left(x^{-i\sqrt{3}-1} \left(-\frac{i x^{2i\sqrt{3}} \sqrt{3}}{6}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
z = c_1 x^{-i\sqrt{3}-1} - \frac{i c_2 \sqrt{3}\, x^{i\sqrt{3}-1}}{6} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 0$ and $x = 1$ in the above gives

$$
0 = -\frac{i c_2 \sqrt{3}}{6} + c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
z' = \frac{c_1 x^{-i\sqrt{3}-1} \left(-i\sqrt{3} - 1\right)}{x} - \frac{i c_2 \sqrt{3}\, x^{i\sqrt{3}-1} \left(i\sqrt{3} - 1\right)}{6x}
$$

substituting $z' = 5$ and $x = 1$ in the above gives

$$
5 = \frac{i(-6c_1 + c_2)\sqrt{3}}{6} - c_1 + \frac{c_2}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{5i\sqrt{3}}{6} \\
c_2 &= 5
\end{aligned}
$$

Substituting these values back in above solution results in

$$
z = \frac{5i\sqrt{3}\, x^{-i\sqrt{3}-1}}{6} - \frac{5i\sqrt{3}\, x^{i\sqrt{3}-1}}{6}
$$

Summary

The solution(s) found are the following

$$
z = -\frac{5i\sqrt{3} \left(-x^{-i\sqrt{3}} + x^{i\sqrt{3}}\right)}{6x} \tag{1}
$$

Verification of solutions

$$z = -\frac{5i\sqrt{3}\left(-x^{-i\sqrt{3}}+x^{i\sqrt{3}}\right)}{6x}$$

Verified OK.

12.5.7 Maple step by step solution

Let's solve

$\left[x^2z''+3xz'+4z=0, z(1)=0, z'\Big|_{\{x=1\}}=5\right]$

- Highest derivative means the order of the ODE is 2

  z''

- Isolate 2nd derivative

  $z''=-\frac{3z'}{x}-\frac{4z}{x^2}$

- Group terms with z on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $z''+\frac{3z'}{x}+\frac{4z}{x^2}=0$

- Multiply by denominators of the ODE

  $x^2z''+3xz'+4z=0$

- Make a change of variables

  $t=\ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of z with respect to x , using the chain rule

    $z'=\left(\frac{d}{dt}z(t)\right)t'(x)$

  - Compute derivative

    $z'=\frac{\frac{d}{dt}z(t)}{x}$

  - Calculate the 2nd derivative of z with respect to x , using the chain rule

    $z''=\left(\frac{d^2}{dt^2}z(t)\right)t'(x)^2+t''(x)\left(\frac{d}{dt}z(t)\right)$

  - Compute derivative

    $z''=\frac{\frac{d^2}{dt^2}z(t)}{x^2}-\frac{\frac{d}{dt}z(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$x^2\left(\frac{\frac{d^2}{dt^2}z(t)}{x^2}-\frac{\frac{d}{dt}z(t)}{x^2}\right)+3\frac{d}{dt}z(t)+4z(t)=0$$

- Simplify

$$\frac{d^2}{dt^2}z(t)+2\frac{d}{dt}z(t)+4z(t)=0$$

- Characteristic polynomial of ODE

$$r^2+2r+4=0$$

- Use quadratic formula to solve for r

$$r=\frac{(-2)\pm(\sqrt{-12})}{2}$$

- Roots of the characteristic polynomial

$$r=\left(-\mathrm{I}\sqrt{3}-1,\mathrm{I}\sqrt{3}-1\right)$$

- 1st solution of the ODE

$$z_1(t)=\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right)$$

- 2nd solution of the ODE

$$z_2(t)=\mathrm{e}^{-t}\sin\left(\sqrt{3}\,t\right)$$

- General solution of the ODE

$$z(t)=c_1z_1(t)+c_2z_2(t)$$

- Substitute in solutions

$$z(t)=c_1\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t\right)+c_2\mathrm{e}^{-t}\sin\left(\sqrt{3}\,t\right)$$

- Change variables back using $t=\ln(x)$

$$z=\frac{c_1\cos\left(\sqrt{3}\ln(x)\right)}{x}+\frac{c_2\sin\left(\sqrt{3}\ln(x)\right)}{x}$$

- Simplify

$$z=\frac{c_1\cos\left(\sqrt{3}\ln(x)\right)}{x}+\frac{c_2\sin\left(\sqrt{3}\ln(x)\right)}{x}$$

□ Check validity of solution $z=\frac{c_1\cos\left(\sqrt{3}\ln(x)\right)}{x}+\frac{c_2\sin\left(\sqrt{3}\ln(x)\right)}{x}$

  - Use initial condition $z(1)=0$

    $$0=c_1$$

  - Compute derivative of the solution

    $$z'=-\frac{c_1\cos\left(\sqrt{3}\ln(x)\right)}{x^2}-\frac{c_1\sqrt{3}\sin\left(\sqrt{3}\ln(x)\right)}{x^2}+\frac{c_2\sqrt{3}\cos\left(\sqrt{3}\ln(x)\right)}{x^2}-\frac{c_2\sin\left(\sqrt{3}\ln(x)\right)}{x^2}$$

  - Use the initial condition $z'\Big|_{\{x=1\}}=5$

$$5 = c_2\sqrt{3} - c_1$$

- Solve for c_1 and c_2

$$\left\{c_1 = 0, c_2 = \frac{5\sqrt{3}}{3}\right\}$$

- Substitute constant values into general solution and simplify

$$z = \frac{5\sqrt{3}\sin\left(\sqrt{3}\ln(x)\right)}{3x}$$

- Solution to the IVP

$$z = \frac{5\sqrt{3}\sin\left(\sqrt{3}\ln(x)\right)}{3x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve([x^2*diff(z(x),x$2)+3*x*diff(z(x),x)+4*z(x)=0,z(1) = 0, D(z)(1) = 5],z(x), singsol=al
```

$$z(x) = \frac{5\sqrt{3}\sin\left(\sqrt{3}\ln(x)\right)}{3x}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 220

```
DSolve[{x^2*z''[x]+3*x*z[x]+4*z[x]==0,{z[1]==0,z'[1]==5}},z[x],x,IncludeSingularSolutions ->
```

$$z(x) \rightarrow \frac{10\sqrt{x}\left(\text{BesselJ}\left(i\sqrt{15},2\sqrt{3}\right)\text{BesselJ}\left(-i\sqrt{15},2\sqrt{3}\sqrt{x}\right)-\text{BesselJ}\left(-i\sqrt{15}\right.\right.}{\sqrt{3}\left(\text{BesselJ}\left(i\sqrt{15},2\sqrt{3}\right)\left(\text{BesselJ}\left(-1-i\sqrt{15},2\sqrt{3}\right)-\text{BesselJ}\left(1-i\sqrt{15},2\sqrt{3}\right)\right)+\text{BesselJ}\left(-i\sqrt{15},\right.\right.}$$

12.6 problem 19.1 (vi)

Internal problem ID [12065]
Internal file name [`OUTPUT/10717_Monday_September_11_2023_12_49_48_AM_75081633/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 19, CauchyEuler equations. Exercises page 174
Problem number: 19.1 (vi).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' - y'x - 3y = 0}$$

With initial conditions

$$[y(1) = 1, y'(1) = -1]$$

12.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -\frac{3}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{y'}{x} - \frac{3y}{x^2} = 0$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -\frac{3}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

12.6.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} - 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r - 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r - 3 = 0$$

Or

$$r^2 - 2r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2 x^3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x} + c_2 x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{x^2} + 3c_2 x^2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = -c_1 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = \frac{1}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{x} \tag{1}$$

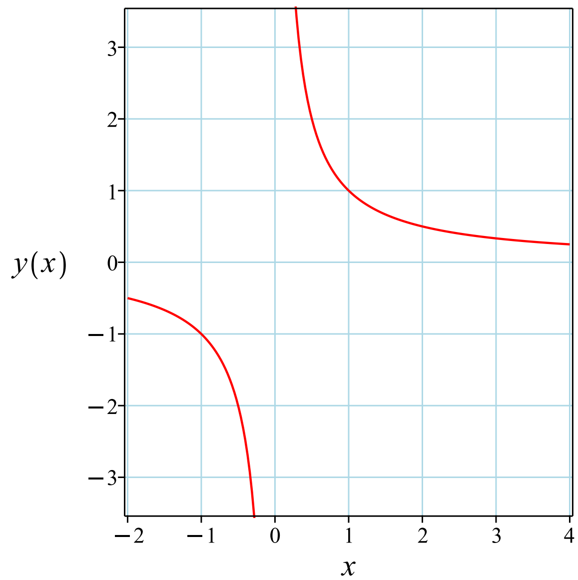


Figure 160: Solution plot

Verification of solutions

$$y = \frac{1}{x}$$

Verified OK.

12.6.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - y'x - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = -\frac{3}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\ &= \int e^{\ln(x)}\,dx \\ &= \int x dx \\ &= \frac{x^2}{2}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-\frac{3}{x^2}}{x^2} \\ &= -\frac{3}{x^4}\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{3y(\tau)}{x^4} &= 0\end{aligned}$$

But in terms of τ

$$-\frac{3}{x^4} = -\frac{3}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{3y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 3\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r - 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 - 3 = 0$$

Or

$$4r^2 - 4r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{2}$$
$$r_2 = \frac{3}{2}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\sqrt{\tau}} + c_2\tau^{\frac{3}{2}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,(c_2x^4 + 4c_1)}{4x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{\sqrt{2}\,(c_2 x^4 + 4c_1)}{4x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = \sqrt{2}\left(c_1 + \frac{c_2}{4}\right) \tag{1A}$$

Taking derivative of the solution gives

$$y' = \sqrt{2}\,c_2 x^2 - \frac{\sqrt{2}\,(c_2 x^4 + 4c_1)}{4x^2}$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \sqrt{2}\left(-c_1 + \frac{3c_2}{4}\right) \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{\sqrt{2}}{2} \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{1}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{x} \tag{1}$$

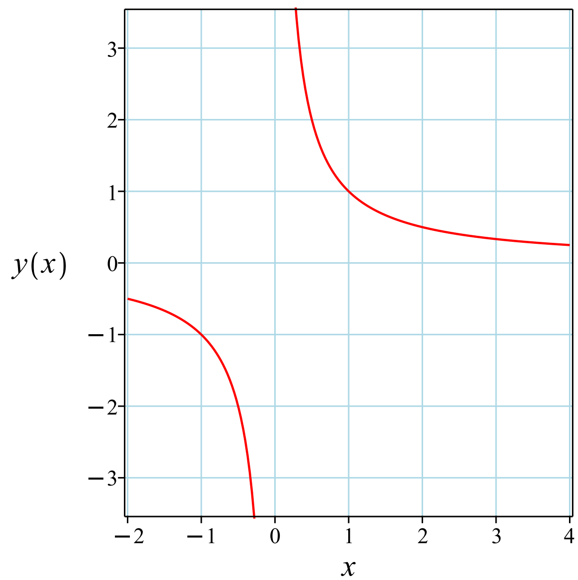


Figure 161: Solution plot

Verification of solutions

$$y = \frac{1}{x}$$

Verified OK.

12.6.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - y'x - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = -\frac{3}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} - \frac{3}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{5v'(x)}{x} &= 0 \\ v''(x) + \frac{5v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{5u}{x} \end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{5}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{5}{x}\, dx \\ \ln(u) &= -5\ln(x) + c_1 \\ u &= e^{-5\ln(x)+c_1} \\ &= \frac{c_1}{x^5} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{4x^4} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{4x^4} + c_2\right) x^3 \\
&= \frac{4c_2 x^4 - c_1}{4x}
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(-\frac{c_1}{4x^4} + c_2\right) x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = -\frac{c_1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{x^2} + 3\left(-\frac{c_1}{4x^4} + c_2\right) x^2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{c_1}{4} + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}
c_1 &= -4 \\
c_2 &= 0
\end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{1}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{x} \tag{1}$$

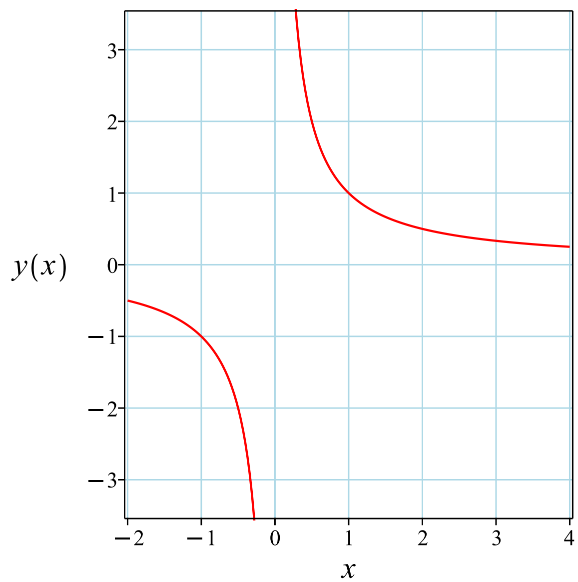


Figure 162: Solution plot

Verification of solutions

$$y = \frac{1}{x}$$

Verified OK.

12.6.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2 y'' - y'x - 3y)\, dx = 0$$
$$x^2 y' - 3yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{3y}{x} = \frac{c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{3}{x}dx} \\
&= \frac{1}{x^3}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(\frac{c_1}{x^2}\right) \\
\mathrm{d}\left(\frac{y}{x^3}\right) &= \left(\frac{c_1}{x^5}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x^3} &= \int \frac{c_1}{x^5}\,\mathrm{d}x \\
\frac{y}{x^3} &= -\frac{c_1}{4x^4} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$
y = -\frac{c_1}{4x} + c_2 x^3
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = -\frac{c_1}{4x} + c_2 x^3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$
1 = -\frac{c_1}{4} + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{4x^2} + 3c_2x^2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{c_1}{4} + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -4$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = \frac{1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x} \tag{1}$$

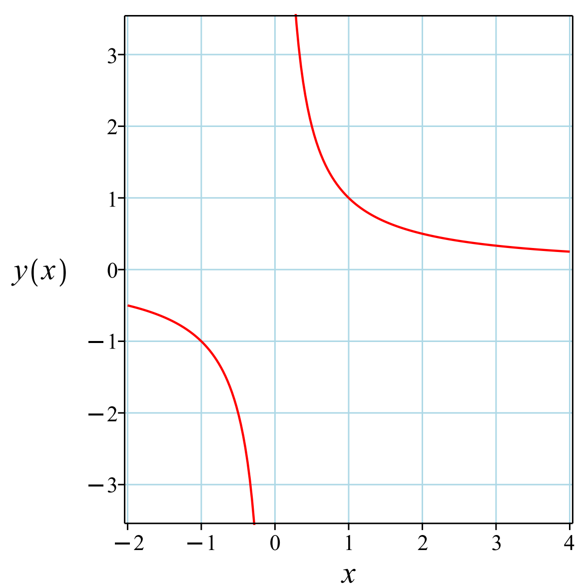


Figure 163: Solution plot

Verification of solutions

$$y = \frac{1}{x}$$

Verified OK.

12.6.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' - y'x - 3y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2y'' - y'x - 3y\right) dx = 0$$
$$x^2y' - 3yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' - \frac{3y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{x}dx}$$
$$= \frac{1}{x^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^3}\right) = \left(\frac{1}{x^3}\right)\left(\frac{c_1}{x^2}\right)$$
$$d\left(\frac{y}{x^3}\right) = \left(\frac{c_1}{x^5}\right) dx$$

Integrating gives

$$\frac{y}{x^3} = \int \frac{c_1}{x^5}\,\mathrm{d}x$$
$$\frac{y}{x^3} = -\frac{c_1}{4x^4} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = -\frac{c_1}{4x} + c_2 x^3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{c_1}{4x} + c_2 x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = -\frac{c_1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{4x^2} + 3c_2 x^2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{c_1}{4} + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -4$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = \frac{1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x} \tag{1}$$

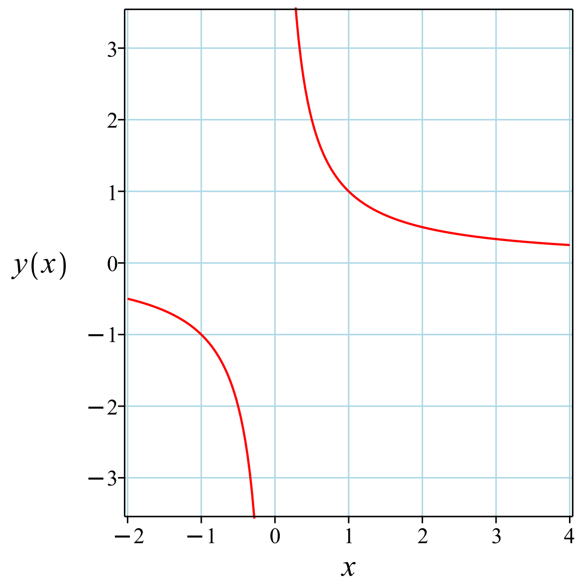


Figure 164: Solution plot

Verification of solutions

$$y = \frac{1}{x}$$

Verified OK.

12.6.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - y'x - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 167: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + (-)(0) \\ &= -\frac{3}{2x} \\ &= -\frac{3}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{3}{2x} dx} \\ &= \frac{1}{x^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^4}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^4}{4}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1}{x} + \frac{c_2 x^3}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$
1 = c_1 + \frac{c_2}{4} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{c_1}{x^2} + \frac{3c_2 x^2}{4}
$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$
-1 = -c_1 + \frac{3c_2}{4} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = \frac{1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x} \tag{1}$$

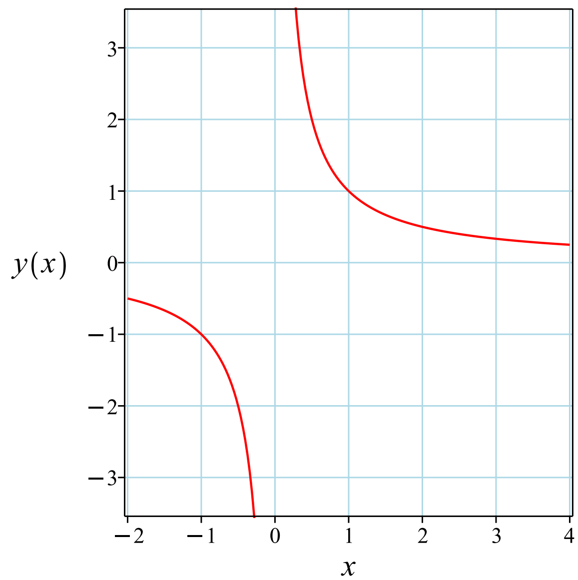


Figure 165: Solution plot

Verification of solutions

$$y = \frac{1}{x}$$

Verified OK.

12.6.8 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x^2 \\
q(x) &= -x \\
r(x) &= -3 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= -1
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (-1) + (-3) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
x^2 y' - 3yx = c_1
$$

We now have a first order ode to solve which is

$$
x^2 y' - 3yx = c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$y' - \frac{3y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\ \frac{d}{dx}\left(\frac{y}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(\frac{c_1}{x^2}\right) \\ d\left(\frac{y}{x^3}\right) &= \left(\frac{c_1}{x^5}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^3} &= \int \frac{c_1}{x^5}\, dx \\ \frac{y}{x^3} &= -\frac{c_1}{4x^4} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = -\frac{c_1}{4x} + c_2 x^3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{c_1}{4x} + c_2 x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = -\frac{c_1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{4x^2} + 3c_2 x^2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{c_1}{4} + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -4 \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x} \tag{1}$$

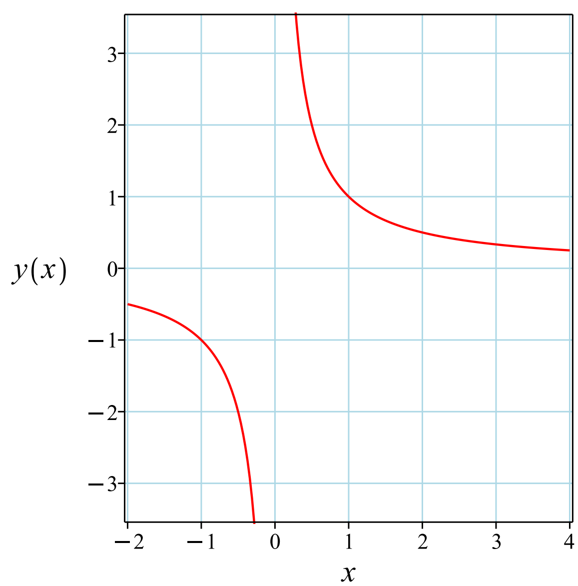


Figure 166: Solution plot

Verification of solutions

$$y = \frac{1}{x}$$

Verified OK.

12.6.9 Maple step by step solution

Let's solve

$\left[y''x^2 - y'x - 3y = 0, y(1) = 1, y'\Big|_{\{x=1\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} + \frac{3y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} - \frac{3y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - y'x - 3y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - \frac{d}{dt}y(t) - 3y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) - 3y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{-t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{-t} + c_2 e^{3t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + c_2 x^3$

- Simplify

  $y = \frac{c_1}{x} + c_2 x^3$

☐ Check validity of solution $y = \frac{c_1}{x} + c_2 x^3$

  - Use initial condition $y(1) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -\frac{c_1}{x^2} + 3c_2 x^2$

  - Use the initial condition $y'\Big|_{\{x=1\}} = -1$

    $-1 = -c_1 + 3c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

    $y = \frac{1}{x}$

- Solution to the IVP

$$y = \frac{1}{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 7

```
dsolve([x^2*diff(y(x),x$2)-x*diff(y(x),x)-3*y(x)=0,y(1) = 1, D(y)(1) = -1],y(x), singsol=all
```

$$y(x) = \frac{1}{x}$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 169

```
DSolve[{x^2*y''[x]-x*y[x]-3*y[x]==0,{y[1]==1,y'[1]==-1}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{\sqrt{x}\left(\left(3\,\text{BesselI}\left(-\sqrt{13},2\right)+\text{BesselI}\left(-1-\sqrt{13},2\right)+\text{BesselI}\left(1-\sqrt{13},2\right)\right)\text{BesselI}\left(\sqrt{13},2\sqrt{x}\right)-(3\,\text{Be}}{\text{BesselI}\left(\sqrt{13},2\right)\left(\text{BesselI}\left(-1-\sqrt{13},2\right)+\text{BesselI}\left(1-\sqrt{13},2\right)\right)-\text{BesselI}}$$

12.7 problem 19.1 (vii)

Internal problem ID [12066]
Internal file name [`OUTPUT/10718_Monday_September_11_2023_12_49_50_AM_91865300/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 19, CauchyEuler equations. Exercises page 174
Problem number: 19.1 (vii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{4t^2x'' + 8tx' + 5x = 0}$$

With initial conditions

$$[x(1) = 2, x'(1) = 0]$$

12.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= \frac{2}{t} \\ q(t) &= \frac{5}{4t^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + \frac{2x'}{t} + \frac{5x}{4t^2} = 0$$

The domain of $p(t) = \frac{2}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = \frac{5}{4t^2}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

12.7.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$4t^2(r(r-1))t^{r-2} + 8trt^{r-1} + 5t^r = 0$$

Simplifying gives

$$4r(r-1)\,t^r + 8r\,t^r + 5t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$4r(r-1) + 8r + 5 = 0$$

Or

$$4r^2 + 4r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= -\frac{1}{2} - i \\
r_2 &= -\frac{1}{2} + i
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = -\frac{1}{2}$ and $\beta = -1$. Hence the solution becomes

$$
\begin{aligned}
x &= c_1 t^{r_1} + c_2 t^{r_2} \\
&= c_1 t^{\alpha+i\beta} + c_2 t^{\alpha-i\beta} \\
&= t^{\alpha}\left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\
&= t^{\alpha}\left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\
&= t^{\alpha}\left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right)
\end{aligned}
$$

Using the values for $\alpha = -\frac{1}{2}, \beta = -1$, the above becomes

$$
x = t^{-\frac{1}{2}}\left(c_1 e^{-i\ln(t)} + c_2 e^{i\ln(t)}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
x = \frac{1}{\sqrt{t}}(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)))
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = \frac{c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))}{\sqrt{t}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 1$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$x' = -\frac{c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))}{2t^{\frac{3}{2}}} + \frac{-\frac{c_1 \sin(\ln(t))}{t} + \frac{c_2 \cos(\ln(t))}{t}}{\sqrt{t}}$$

substituting $x' = 0$ and $t = 1$ in the above gives

$$0 = -\frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$x = \frac{2\cos(\ln(t)) + \sin(\ln(t))}{\sqrt{t}}$$

Summary
The solution(s) found are the following

$$x = \frac{2\cos(\ln(t)) + \sin(\ln(t))}{\sqrt{t}} \tag{1}$$

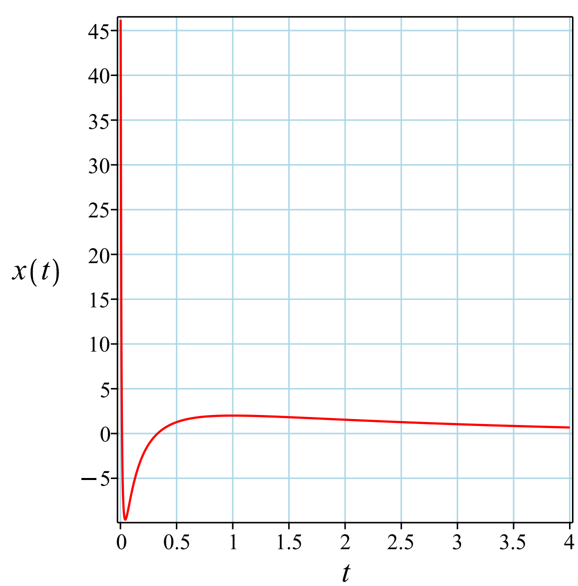


Figure 167: Solution plot

Verification of solutions

$$x = \frac{2\cos(\ln(t)) + \sin(\ln(t))}{\sqrt{t}}$$

Verified OK.

12.7.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$4t^2x'' + 8tx' + 5x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{2}{t} \\ q(t) &= \frac{5}{4t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int \frac{2}{t}dt\right)}dt \\ &= \int e^{-2\ln(t)}\,dt \\ &= \int \frac{1}{t^2}dt \\ &= -\frac{1}{t} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{5}{4t^2}}{\frac{1}{t^4}} \\ &= \frac{5t^2}{4} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}x(\tau) + q_1 x(\tau) = 0$$
$$\frac{d^2}{d\tau^2}x(\tau) + \frac{5t^2 x(\tau)}{4} = 0$$

But in terms of τ

$$\frac{5t^2}{4} = \frac{5}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}x(\tau) + \frac{5x(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $x(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}x(\tau)\right)\tau^2 + 5x(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $x(\tau) = \tau^r$, then $x' = r\tau^{r-1}$ and $x'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 5 = 0$$

Or

$$4r^2 - 4r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - i$$
$$r_2 = \frac{1}{2} + i$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned} x(\tau) &= c_1\tau^{r_1} + c_2\tau^{r_2} \\ &= c_1\tau^{\alpha+i\beta} + c_2\tau^{\alpha-i\beta} \\ &= \tau^{\alpha}\left(c_1\tau^{i\beta} + c_2\tau^{-i\beta}\right) \\ &= \tau^{\alpha}\left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})}\right) \\ &= \tau^{\alpha}\left(c_1 e^{i(\beta\ln\tau)} + c_2 e^{-i(\beta\ln\tau)}\right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -1$, the above becomes

$$x(\tau) = \tau^{\frac{1}{2}}\left(c_1 e^{-i\ln(\tau)} + c_2 e^{i\ln(\tau)}\right)$$

Using Euler relation, the expression $c_1e^{iA} + c_2e^{-iA}$ is transformed to $c_1\cos A + c_1\sin A$ where the constants are free to change. Applying this to the above result gives

$$x(\tau) = \sqrt{\tau}(c_1\cos(\ln(\tau)) + c_2\sin(\ln(\tau)))$$

The above solution is now transformed back to x using (6) which results in

$$x = \sqrt{-\frac{1}{t}}\left(c_1\cos\left(\ln\left(-\frac{1}{t}\right)\right) + c_2\sin\left(\ln\left(-\frac{1}{t}\right)\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \sqrt{-\frac{1}{t}}\left(c_1\cos\left(\ln\left(-\frac{1}{t}\right)\right) + c_2\sin\left(\ln\left(-\frac{1}{t}\right)\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 1$ in the above gives

$$2 = ic_1 \cosh(\pi) - c_2 \sinh(\pi) \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{c_1 \cos\left(\ln\left(-\frac{1}{t}\right)\right) + c_2 \sin\left(\ln\left(-\frac{1}{t}\right)\right)}{2\sqrt{-\frac{1}{t}}\, t^2} + \sqrt{-\frac{1}{t}}\left(\frac{c_1 \sin\left(\ln\left(-\frac{1}{t}\right)\right)}{t} - \frac{c_2 \cos\left(\ln\left(-\frac{1}{t}\right)\right)}{t}\right)$$

substituting $x' = 0$ and $t = 1$ in the above gives

$$0 = \frac{i(-c_1 - 2c_2)\cosh(\pi)}{2} - \left(c_1 - \frac{c_2}{2}\right)\sinh(\pi) \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2i\cosh(\pi) + \sinh(\pi)$$
$$c_2 = i\cosh(\pi) + 2\sinh(\pi)$$

Substituting these values back in above solution results in

$$x = -2i\cos\left(\ln\left(-\frac{1}{t}\right)\right)\cosh(\pi)\sqrt{-\frac{1}{t}} + i\sqrt{-\frac{1}{t}}\sin\left(\ln\left(-\frac{1}{t}\right)\right)\cosh(\pi) + \cos\left(\ln\left(-\frac{1}{t}\right)\right)\sinh(\pi)\sqrt{}$$

Which simplifies to

$$x = \left(i\left(-2\cos\left(\ln\left(-\frac{1}{t}\right)\right) + \sin\left(\ln\left(-\frac{1}{t}\right)\right)\right)\cosh(\pi) + \left(\cos\left(\ln\left(-\frac{1}{t}\right)\right) + 2\sin\left(\ln\left(-\frac{1}{t}\right)\right)\right)\sinh(\pi)\right)\sqrt{-\frac{1}{t}}$$

Summary

The solution(s) found are the following

$$x = \left(i\left(-2\cos\left(\ln\left(-\frac{1}{t}\right)\right) + \sin\left(\ln\left(-\frac{1}{t}\right)\right)\right)\cosh(\pi) + \left(\cos\left(\ln\left(-\frac{1}{t}\right)\right) + 2\sin\left(\ln\left(-\frac{1}{t}\right)\right)\right)\sinh(\pi)\right)\sqrt{-\frac{1}{t}} \tag{1}$$

Verification of solutions

$$x = \left(i\left(-2\cos\left(\ln\left(-\frac{1}{t}\right)\right) + \sin\left(\ln\left(-\frac{1}{t}\right)\right)\right)\cosh(\pi) + \left(\cos\left(\ln\left(-\frac{1}{t}\right)\right) + 2\sin\left(\ln\left(-\frac{1}{t}\right)\right)\right)\sinh(\pi)\right)\sqrt{-\frac{1}{t}}$$

Verified OK.

12.7.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$4t^2x'' + 8tx' + 5x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{2}{t} \\ q(t) &= \frac{5}{4t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1 x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}}{2c} \qquad (6) \\ \tau'' &= -\frac{\sqrt{5}}{2c\sqrt{\frac{1}{t^2}}\,t^3} \end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\ &= \frac{-\frac{\sqrt{5}}{2c\sqrt{\frac{1}{t^2}}\,t^3} + \frac{2}{t}\frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}}{2c}}{\left(\frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}}{2c}\right)^2} \\ &= \frac{2c\sqrt{5}}{5} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
x(\tau)'' + p_1 x(\tau)' + q_1 x(\tau) &= 0 \\
\frac{d^2}{d\tau^2}x(\tau) + \frac{2c\sqrt{5}\left(\frac{d}{d\tau}x(\tau)\right)}{5} + c^2 x(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $x(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
x(\tau) = \mathrm{e}^{-\frac{\sqrt{5}\,c\tau}{5}}\left(c_1\cos\left(\frac{2\sqrt{5}\,c\tau}{5}\right) + c_2\sin\left(\frac{2\sqrt{5}\,c\tau}{5}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int \frac{\sqrt{5}\sqrt{\frac{1}{t^2}}}{2}dt}{c} \\
&= \frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}\,t\ln(t)}{2c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
x = \frac{c_1\cos(\ln(t)) + c_2\sin(\ln(t))}{\sqrt{t}}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = \frac{c_1\cos(\ln(t)) + c_2\sin(\ln(t))}{\sqrt{t}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 1$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -\frac{c_1\cos(\ln(t)) + c_2\sin(\ln(t))}{2t^{\frac{3}{2}}} + \frac{-\frac{c_1\sin(\ln(t))}{t} + \frac{c_2\cos(\ln(t))}{t}}{\sqrt{t}}
$$

substituting $x' = 0$ and $t = 1$ in the above gives

$$0 = -\frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$x = \frac{2\cos(\ln(t)) + \sin(\ln(t))}{\sqrt{t}}$$

Summary
The solution(s) found are the following

$$x = \frac{2\cos(\ln(t)) + \sin(\ln(t))}{\sqrt{t}} \tag{1}$$

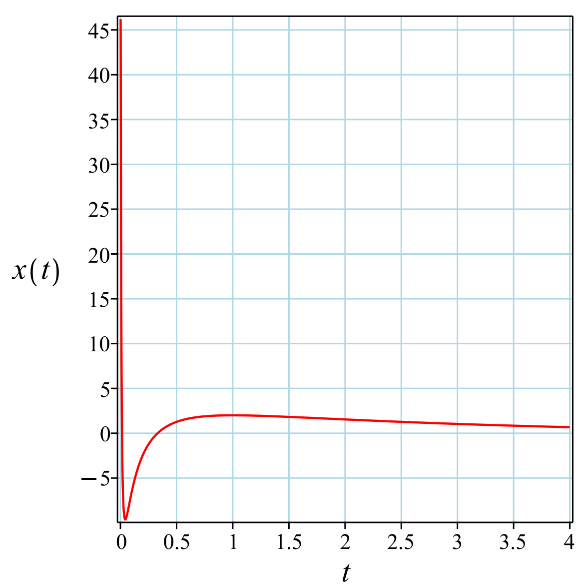


Figure 168: Solution plot

Verification of solutions

$$x = \frac{2\cos(\ln(t)) + \sin(\ln(t))}{\sqrt{t}}$$

Verified OK.

12.7.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$4t^2x'' + 8tx' + 5x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{2}{t} \\ q(t) &= \frac{5}{4t^2} \end{aligned}$$

Applying change of variables on the depndent variable $x = v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{2n}{t^2} + \frac{5}{4t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -\frac{1}{2} + i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \left(\frac{-1+2i}{t} + \frac{2}{t}\right) v'(t) &= 0 \\ v''(t) + \frac{(1+2i)\,v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t)+\frac{(1+2i)\,u(t)}{t}=0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(t,u)\\ &= f(t)g(u)\\ &= \frac{(-1-2i)\,u}{t}\end{aligned}$$

Where $f(t)=\frac{-1-2i}{t}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{-1-2i}{t}\,dt\\ \int \frac{1}{u}\,du &= \int \frac{-1-2i}{t}\,dt\\ \ln(u) &= (-1-2i)\ln(t)+c_1\\ u &= \mathrm{e}^{(-1-2i)\ln(t)+c_1}\\ &= c_1\mathrm{e}^{(-1-2i)\ln(t)}\end{aligned}$$

Which simplifies to

$$u(t)=\frac{c_1t^{-2i}}{t}$$

Now that $u(t)$ is known, then

$$\begin{aligned}v'(t) &= u(t)\\ v(t) &= \int u(t)\,dt+c_2\\ &= \frac{ic_1t^{-2i}}{2}+c_2\end{aligned}$$

Hence

$$\begin{aligned}x &= v(t)\,t^n\\ &= \left(\frac{ic_1t^{-2i}}{2}+c_2\right)t^{-\frac{1}{2}+i}\\ &= \frac{t^{-\frac{1}{2}-i}(2c_2t^{2i}+ic_1)}{2}\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \left(\frac{i c_1 t^{-2i}}{2}+c_2\right) t^{-\frac{1}{2}+i} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 1$ in the above gives

$$2 = \frac{i c_1}{2}+c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{c_1 t^{-2i} t^{-\frac{1}{2}+i}}{t}+\frac{\left(-\frac{1}{2}+i\right) \left(\frac{i c_1 t^{-2i}}{2}+c_2\right) t^{-\frac{1}{2}+i}}{t}$$

substituting $x' = 0$ and $t = 1$ in the above gives

$$0 = \left(\frac{1}{2}-\frac{i}{4}\right) c_1 +\left(-\frac{1}{2}+i\right) c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1-2i$$
$$c_2 = 1-\frac{i}{2}$$

Substituting these values back in above solution results in

$$x = \frac{i t^{-\frac{1}{2}+i} t^{-2i}}{2}-\frac{i t^{-\frac{1}{2}+i}}{2}+t^{-\frac{1}{2}+i} t^{-2i}+t^{-\frac{1}{2}+i}$$

Summary

The solution(s) found are the following

$$x = \left(1+\frac{i}{2}\right) t^{-\frac{1}{2}-i}+\left(1-\frac{i}{2}\right) t^{-\frac{1}{2}+i} \tag{1}$$

Verification of solutions

$$x = \left(1+\frac{i}{2}\right) t^{-\frac{1}{2}-i}+\left(1-\frac{i}{2}\right) t^{-\frac{1}{2}+i}$$

Verified OK.

12.7.6 Solving using Kovacic algorithm

Writing the ode as

$$4t^2x'' + 8tx' + 5x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4t^2 \\ B &= 8t \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{5}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 169: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{5}{4t^2}
$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + i \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{5}{4t^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + i \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{5}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - i$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2} - i}{t} + (-)\,(0) \\
&= \frac{\frac{1}{2} - i}{t} \\
&= \frac{\frac{1}{2} - i}{t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - i}{t}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{t^2}\right) + \left(\frac{\frac{1}{2} - i}{t}\right)^2 - \left(-\frac{5}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \frac{\frac{1}{2} - i}{t} dt} \\
&= t^{\frac{1}{2} - i}
\end{aligned}
$$

The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8t}{4t^2}\, dt} \\
&= z_1 e^{-\ln(t)} \\
&= z_1 \left(\frac{1}{t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = t^{-\frac{1}{2}-i}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{8t}{4t^2}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{-2\ln(t)}}{(x_1)^2}\, dt \\
&= x_1 \left(-\frac{i t^{2i}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(t^{-\frac{1}{2}-i}\right) + c_2 \left(t^{-\frac{1}{2}-i} \left(-\frac{i t^{2i}}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 t^{-\frac{1}{2}-i} - \frac{i c_2 t^{-\frac{1}{2}+i}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 1$ in the above gives

$$2 = c_1 - \frac{i c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{\left(-\frac{1}{2} - i\right) c_1 t^{-\frac{1}{2}-i}}{t} + \frac{\left(\frac{1}{2} + \frac{i}{4}\right) c_2 t^{-\frac{1}{2}+i}}{t}$$

substituting $x' = 0$ and $t = 1$ in the above gives

$$0 = \left(-\frac{1}{2} - i\right) c_1 + \left(\frac{1}{2} + \frac{i}{4}\right) c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 + \frac{i}{2} \\ c_2 &= 1 + 2i \end{aligned}$$

Substituting these values back in above solution results in

$$x = \frac{i t^{-\frac{1}{2}-i}}{2} - \frac{i t^{-\frac{1}{2}+i}}{2} + t^{-\frac{1}{2}-i} + t^{-\frac{1}{2}+i}$$

Summary
The solution(s) found are the following

$$x = \left(1 + \frac{i}{2}\right) t^{-\frac{1}{2}-i} + \left(1 - \frac{i}{2}\right) t^{-\frac{1}{2}+i} \tag{1}$$

Verification of solutions

$$x = \left(1 + \frac{i}{2}\right) t^{-\frac{1}{2}-i} + \left(1 - \frac{i}{2}\right) t^{-\frac{1}{2}+i}$$

Verified OK.

12.7.7 Maple step by step solution

Let's solve

$$\left[4t^2x''+8tx'+5x=0, x(1)=2, x'\Big|_{\{t=1\}}=0\right]$$

- Highest derivative means the order of the ODE is 2
  x''
- Isolate 2nd derivative
  $x''=-\frac{2x'}{t}-\frac{5x}{4t^2}$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $x''+\frac{2x'}{t}+\frac{5x}{4t^2}=0$
- Multiply by denominators of the ODE
  $4t^2x''+8tx'+5x=0$
- Make a change of variables
  $s=\ln(t)$

☐ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of x with respect to t , using the chain rule
    $x'=\left(\frac{d}{ds}x(s)\right)s'(t)$
  - Compute derivative
    $x'=\frac{\frac{d}{ds}x(s)}{t}$
  - Calculate the 2nd derivative of x with respect to t , using the chain rule
    $x''=\left(\frac{d^2}{ds^2}x(s)\right)s'(t)^2+s''(t)\left(\frac{d}{ds}x(s)\right)$
  - Compute derivative
    $x''=\frac{\frac{d^2}{ds^2}x(s)}{t^2}-\frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE
  $4t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2}-\frac{\frac{d}{ds}x(s)}{t^2}\right)+8\frac{d}{ds}x(s)+5x(s)=0$

- Simplify
  $4\frac{d^2}{ds^2}x(s)+4\frac{d}{ds}x(s)+5x(s)=0$
- Isolate 2nd derivative
  $\frac{d^2}{ds^2}x(s)=-\frac{d}{ds}x(s)-\frac{5x(s)}{4}$
- Group terms with $x(s)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{ds^2}x(s) + \frac{d}{ds}x(s) + \frac{5x(s)}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 + r + \frac{5}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \mathrm{I}, -\frac{1}{2} + \mathrm{I}\right)$

- 1st solution of the ODE

  $x_1(s) = \mathrm{e}^{-\frac{s}{2}} \cos(s)$

- 2nd solution of the ODE

  $x_2(s) = \mathrm{e}^{-\frac{s}{2}} \sin(s)$

- General solution of the ODE

  $x(s) = c_1 x_1(s) + c_2 x_2(s)$

- Substitute in solutions

  $x(s) = c_1 \mathrm{e}^{-\frac{s}{2}} \cos(s) + c_2 \mathrm{e}^{-\frac{s}{2}} \sin(s)$

- Change variables back using $s = \ln(t)$

  $x = \frac{c_1 \cos(\ln(t))}{\sqrt{t}} + \frac{c_2 \sin(\ln(t))}{\sqrt{t}}$

- Simplify

  $x = \frac{c_1 \cos(\ln(t))}{\sqrt{t}} + \frac{c_2 \sin(\ln(t))}{\sqrt{t}}$

☐ Check validity of solution $x = \frac{c_1 \cos(\ln(t))}{\sqrt{t}} + \frac{c_2 \sin(\ln(t))}{\sqrt{t}}$

  - Use initial condition $x(1) = 2$

    $2 = c_1$

  - Compute derivative of the solution

    $x' = -\frac{c_1 \cos(\ln(t))}{2t^{\frac{3}{2}}} - \frac{c_1 \sin(\ln(t))}{t^{\frac{3}{2}}} - \frac{c_2 \sin(\ln(t))}{2t^{\frac{3}{2}}} + \frac{c_2 \cos(\ln(t))}{t^{\frac{3}{2}}}$

  - Use the initial condition $x'\Big|_{\{t=1\}} = 0$

    $0 = -\frac{c_1}{2} + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 2, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $x = \frac{2\cos(\ln(t))+\sin(\ln(t))}{\sqrt{t}}$

- Solution to the IVP

  $x = \frac{2\cos(\ln(t))+\sin(\ln(t))}{\sqrt{t}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve([4*t^2*diff(x(t),t$2)+8*t*diff(x(t),t)+5*x(t)=0,x(1) = 2, D(x)(1) = 0],x(t), singsol=
```

$$x(t) = \frac{\sin(\ln(t)) + 2\cos(\ln(t))}{\sqrt{t}}$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 232

```
DSolve[{4*t^2*x''[t]+8*t*x[t]+5*x[t]==0,{x[1]==2,x'[1]==0}},x[t],t,IncludeSingularSolutions
```

$x(t)$

$$\rightarrow \frac{\sqrt{t}\left(\left(2\,\mathrm{BesselJ}\left(-1+2i,2\sqrt{2}\right)+\sqrt{2}\,\mathrm{BesselJ}\left(2i,2\sqrt{2}\right)-2\,\mathrm{BesselJ}\left(1+2i,2\sqrt{2}\right)\right)\mathrm{BesselJ}\left(-2i,2\sqrt{2}\sqrt{t}\right.\right.}{\mathrm{BesselJ}\left(-1+2i,2\sqrt{2}\right)\mathrm{BesselJ}\left(-2i,2\sqrt{2}\right)-\mathrm{BesselJ}\left(-1-2i,2\sqrt{2}\right)\mathrm{BesselJ}\left(2i,2\sqrt{2}\right.}$$

12.8 problem 19.1 (viii)

Internal problem ID [12067]
Internal file name [`OUTPUT/10719_Monday_September_11_2023_12_49_53_AM_25239577/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 19, CauchyEuler equations. Exercises page 174
Problem number: 19.1 (viii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - 5y'x + 5y = 0}$$

With initial conditions

$$[y(1) = -2, y'(1) = 1]$$

12.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{5}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{5y'}{x} + \frac{5y}{x^2} = 0$$

The domain of $p(x) = -\frac{5}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{5}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

12.8.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 5xrx^{r-1} + 5x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 5r\,x^r + 5x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 5r + 5 = 0$$

Or

$$r^2 - 6r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= 1 \\
r_2 &= 5
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
y = c_1 y_1 + c_2 y_2
$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$
y = c_2 x^5 + c_1 x
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_2 x^5 + c_1 x \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 1$ in the above gives

$$
-2 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 5c_2 x^4 + c_1
$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$
1 = 5c_2 + c_1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{11}{4} \\
c_2 &= \frac{3}{4}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{3}{4} x^5 - \frac{11}{4} x
$$

Summary

The solution(s) found are the following

$$y = \frac{3}{4}x^5 - \frac{11}{4}x \tag{1}$$

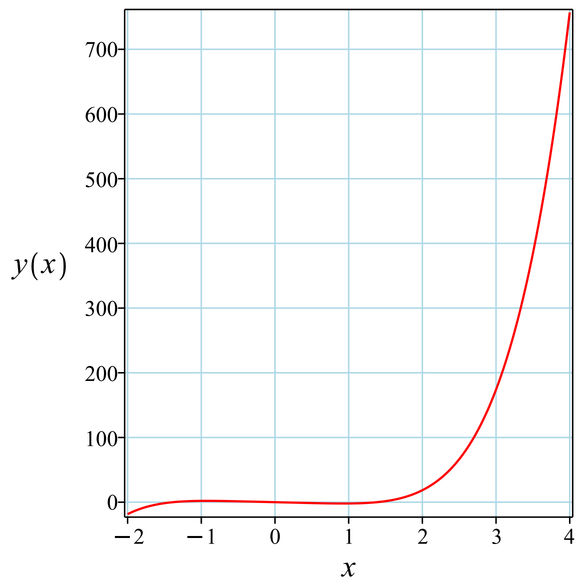


Figure 169: Solution plot

Verification of solutions

$$y = \frac{3}{4}x^5 - \frac{11}{4}x$$

Verified OK.

12.8.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 5y'x + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{5}{x}$$
$$q(x) = \frac{5}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{5}{x}dx\right)} dx \\
&= \int e^{5\ln(x)}\, dx \\
&= \int x^5 dx \\
&= \frac{x^6}{6}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{5}{x^2}}{x^{10}} \\
&= \frac{5}{x^{12}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{x^{12}} = 0$$

But in terms of τ

$$\frac{5}{x^{12}} = \frac{5}{36\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{36\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$36\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 5y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$36\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0$$

Simplifying gives

$$36r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$36r(r-1) + 0 + 5 = 0$$

Or

$$36r^2 - 36r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{6}$$
$$r_2 = \frac{5}{6}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{6}} + c_2\tau^{\frac{5}{6}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 6^{\frac{5}{6}}(x^6)^{\frac{1}{6}}}{6} + \frac{c_2 6^{\frac{1}{6}}(x^6)^{\frac{5}{6}}}{6}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 6^{\frac{5}{6}} (x^6)^{\frac{1}{6}}}{6} + \frac{c_2 6^{\frac{1}{6}} (x^6)^{\frac{5}{6}}}{6} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 1$ in the above gives

$$-2 = \frac{6^{\frac{1}{6}} \left(c_1 6^{\frac{2}{3}} + c_2\right)}{6} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 6^{\frac{5}{6}} x^5}{6 (x^6)^{\frac{5}{6}}} + \frac{5 c_2 6^{\frac{1}{6}} x^5}{6 (x^6)^{\frac{1}{6}}}$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = \frac{6^{\frac{1}{6}} \left(c_1 6^{\frac{2}{3}} + 5 c_2\right)}{6} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{11\, 6^{\frac{1}{6}}}{4}$$
$$c_2 = \frac{3\, 6^{\frac{5}{6}}}{4}$$

Substituting these values back in above solution results in

$$y = \frac{3(x^6)^{\frac{5}{6}}}{4} - \frac{11(x^6)^{\frac{1}{6}}}{4}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{3(x^6)^{\frac{5}{6}}}{4} - \frac{11(x^6)^{\frac{1}{6}}}{4} \tag{1}$$

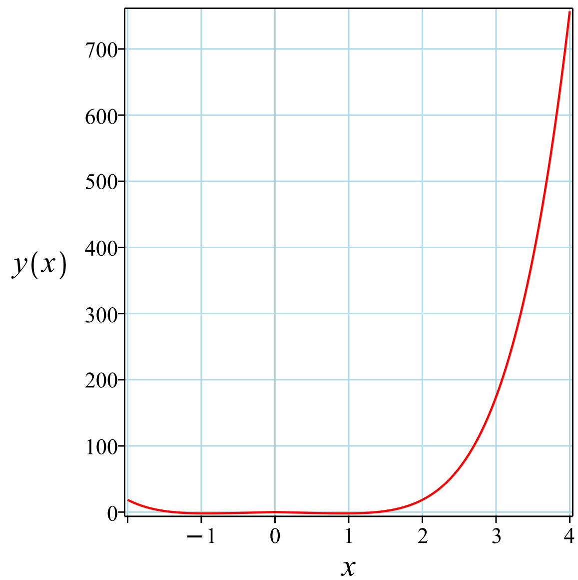


Figure 170: Solution plot

Verification of solutions

$$y = \frac{3(x^6)^{\frac{5}{6}}}{4} - \frac{11(x^6)^{\frac{1}{6}}}{4}$$

Verified OK.

12.8.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - 5y'x + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{5}{x}$$
$$q(x) = \frac{5}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{5}}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{5}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{5}{x}\frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{6c\sqrt{5}}{5} \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{6c\sqrt{5}\left(\frac{d}{d\tau}y(\tau)\right)}{5} + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{\frac{3\sqrt{5}\,c\tau}{5}}\left(c_1\cosh\left(\frac{2\sqrt{5}\,c\tau}{5}\right) + ic_2\sinh\left(\frac{2\sqrt{5}\,c\tau}{5}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{5}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^3(c_1\cosh(2\ln(x)) + ic_2\sinh(2\ln(x)))
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x^3(c_1\cosh(2\ln(x)) + ic_2\sinh(2\ln(x))) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 1$ in the above gives

$$
-2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 3x^2(c_1\cosh(2\ln(x)) + ic_2\sinh(2\ln(x))) + x^3\left(\frac{2c_1\sinh(2\ln(x))}{x} + \frac{2ic_2\cosh(2\ln(x))}{x}\right)
$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$
1 = 2ic_2 + 3c_1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -2 \\
c_2 &= -\frac{7i}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{7x^3\sinh(2\ln(x))}{2} - 2\cosh(2\ln(x))\,x^3
$$

Which simplifies to

$$y = \left(\frac{7 \sinh\left(2 \ln\left(x\right)\right)}{2} - 2 \cosh\left(2 \ln\left(x\right)\right)\right) x^3$$

Summary

The solution(s) found are the following

$$y = \left(\frac{7 \sinh\left(2 \ln\left(x\right)\right)}{2} - 2 \cosh\left(2 \ln\left(x\right)\right)\right) x^3 \tag{1}$$

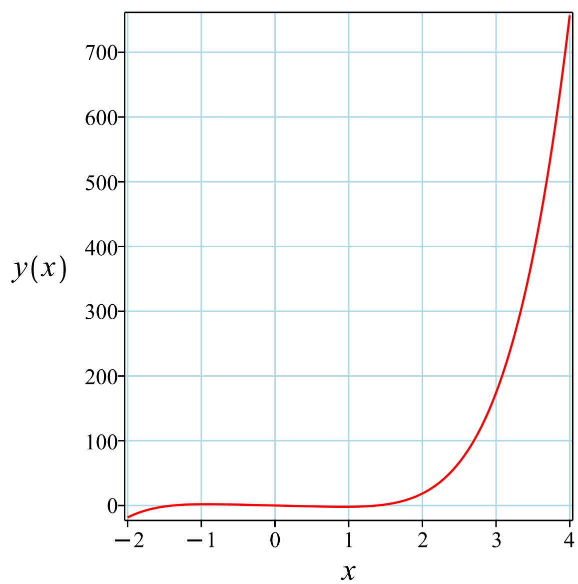


Figure 171: Solution plot

Verification of solutions

$$y = \left(\frac{7 \sinh\left(2 \ln\left(x\right)\right)}{2} - 2 \cosh\left(2 \ln\left(x\right)\right)\right) x^3$$

Verified OK.

12.8.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 5y'x + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{5}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{5n}{x^2} + \frac{5}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 5 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{5v'(x)}{x} &= 0 \\ v''(x) + \frac{5v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{5u}{x} \end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{5}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{5}{x}\,dx \\ \ln(u) &= -5\ln(x) + c_1 \\ u &= \mathrm{e}^{-5\ln(x)+c_1} \\ &= \frac{c_1}{x^5} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{4x^4} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(-\frac{c_1}{4x^4} + c_2\right)x^5 \\ &= c_2x^5 - \frac{1}{4}c_1x \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(-\frac{c_1}{4x^4} + c_2\right)x^5 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 1$ in the above gives

$$-2 = -\frac{c_1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 + 5\left(-\frac{c_1}{4x^4} + c_2\right) x^4$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = -\frac{c_1}{4} + 5c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 11$$
$$c_2 = \frac{3}{4}$$

Substituting these values back in above solution results in

$$y = \frac{x(3x^4 - 11)}{4}$$

Which simplifies to

$$y = \frac{3}{4}x^5 - \frac{11}{4}x$$

Summary

The solution(s) found are the following

$$y = \frac{3}{4}x^5 - \frac{11}{4}x \tag{1}$$

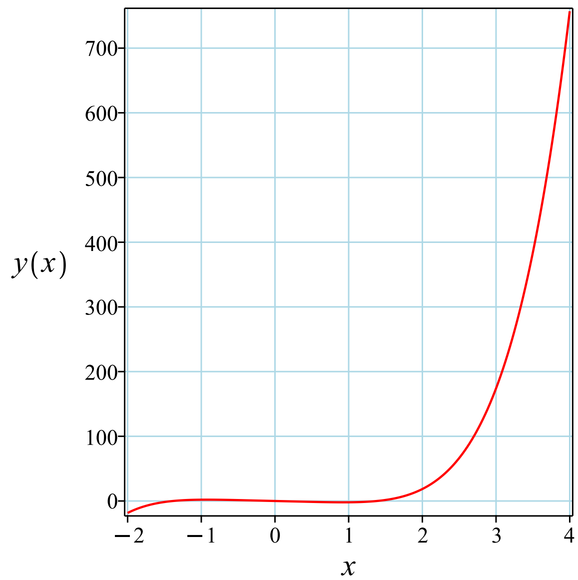


Figure 172: Solution plot

Verification of solutions

$$y = \frac{3}{4}x^5 - \frac{11}{4}x$$

Verified OK.

12.8.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv = 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v = 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 \\
B &= -5x \\
C &= 5 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (-5x)(-5) + (5)(-5x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-5x^3 v'' + \left(15x^2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$-5x^2 (u'(x)\, x - 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= \frac{3u}{x}
\end{aligned}$$

Where $f(x) = \frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= \frac{3}{x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{3}{x}\, dx \\
\ln(u) &= 3\ln(x) + c_1 \\
u &= e^{3\ln(x) + c_1} \\
&= c_1 x^3
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1 x^3 \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int c_1 x^3 \, \mathrm{d}x \\ &= \frac{c_1 x^4}{4} + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (-5x)\left(\frac{c_1 x^4}{4} + c_2\right) \\ &= -\frac{5x(c_1 x^4 + 4c_2)}{4} \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{5x(c_1 x^4 + 4c_2)}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 1$ in the above gives

$$-2 = -\frac{5c_1}{4} - 5c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{25c_1 x^4}{4} - 5c_2$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = -\frac{25c_1}{4} - 5c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{3}{5} \\ c_2 &= \frac{11}{20} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{x(3x^4 - 11)}{4}$$

Which simplifies to

$$y = \frac{3}{4}x^5 - \frac{11}{4}x$$

Summary

The solution(s) found are the following

$$y = \frac{3}{4}x^5 - \frac{11}{4}x \tag{1}$$

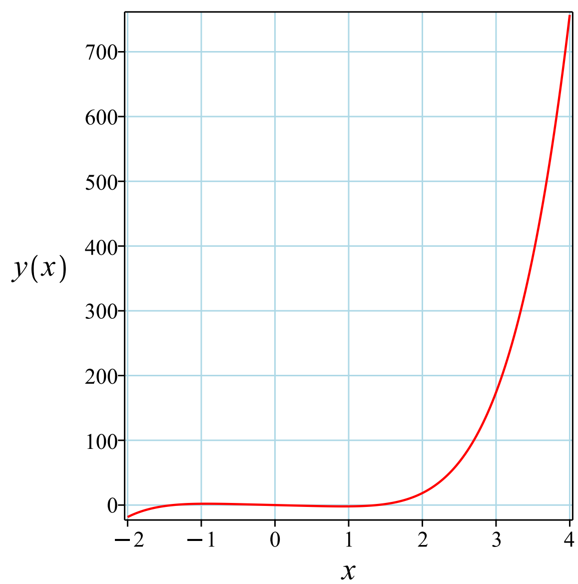


Figure 173: Solution plot

Verification of solutions

$$y = \frac{3}{4}x^5 - \frac{11}{4}x$$

Verified OK.

12.8.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 5y'x + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -5x \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 171: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{15}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{15}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{3}{2x} + (-)\,(0) \\ &= -\frac{3}{2x} \\ &= -\frac{3}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{2x} dx} \\ &= \frac{1}{x^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x}{x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}} \\
&= z_1 \left(x^{\frac{5}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^4}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{x^4}{4}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 x + \frac{1}{4} c_2 x^5 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 1$ in the above gives

$$-2 = c_1 + \frac{c_2}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 + \frac{5c_2x^4}{4}$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = c_1 + \frac{5c_2}{4} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{11}{4}$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = \frac{3}{4}x^5 - \frac{11}{4}x$$

Summary

The solution(s) found are the following

$$y = \frac{3}{4}x^5 - \frac{11}{4}x \tag{1}$$

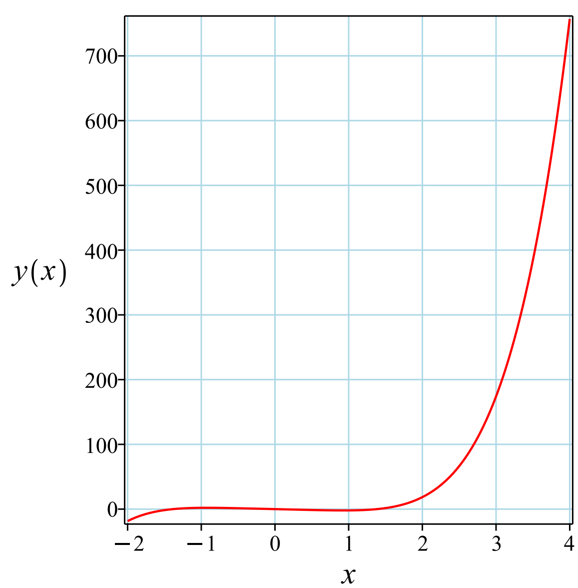


Figure 174: Solution plot

Verification of solutions

$$y = \frac{3}{4}x^5 - \frac{11}{4}x$$

Verified OK.

12.8.8 Maple step by step solution

Let's solve

$\left[y''x^2 - 5y'x + 5y = 0, y(1) = -2, y'\Big|_{\{x=1\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{x} - \frac{5y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{x} + \frac{5y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 5y'x + 5y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 5\frac{d}{dt}y(t) + 5y(t) = 0$$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 6\frac{d}{dt}y(t) + 5y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 6r + 5 = 0$
- Factor the characteristic polynomial

  $(r-1)(r-5) = 0$
- Roots of the characteristic polynomial

  $r = (1, 5)$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^t$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{5t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1\mathrm{e}^t + c_2\mathrm{e}^{5t}$
- Change variables back using $t = \ln(x)$

  $y = c_2 x^5 + c_1 x$
- Simplify

  $y = x(c_2 x^4 + c_1)$

☐ Check validity of solution $y = x(c_2 x^4 + c_1)$

  - Use initial condition $y(1) = -2$

    $-2 = c_1 + c_2$
  - Compute derivative of the solution

    $y' = 5c_2 x^4 + c_1$
  - Use the initial condition $y'\Big|_{\{x=1\}} = 1$

    $1 = 5c_2 + c_1$

- Solve for c_1 and c_2

  $\{c_1 = -\frac{11}{4}, c_2 = \frac{3}{4}\}$

- Substitute constant values into general solution and simplify

  $y = \frac{3}{4}x^5 - \frac{11}{4}x$

- Solution to the IVP

  $y = \frac{3}{4}x^5 - \frac{11}{4}x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([x^2*diff(y(x),x$2)-5*x*diff(y(x),x)+5*y(x)=0,y(1) = -2, D(y)(1) = 1],y(x), singsol=a
```

$$y(x) = \frac{3}{4}x^5 - \frac{11}{4}x$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 17

```
DSolve[{x^2*y''[x]-5*x*y'[x]+5*y[x]==0,{y[1]==-2,y'[1]==1}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{4}x(3x^4 - 11)$$

12.9 problem 19.1 (ix)

Internal problem ID [12068]
Internal file name [`OUTPUT/10720_Monday_September_11_2023_12_49_54_AM_21117854/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 19, CauchyEuler equations. Exercises page 174
Problem number: 19.1 (ix).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{3x^2z'' + 5xz' - z = 0}$$

With initial conditions

$$[z(1) = 2, z'(1) = -1]$$

12.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$z'' + p(x)z' + q(x)z = F$$

Where here

$$\begin{aligned} p(x) &= \frac{5}{3x} \\ q(x) &= -\frac{1}{3x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$z'' + \frac{5z'}{3x} - \frac{z}{3x^2} = 0$$

The domain of $p(x) = \frac{5}{3x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -\frac{1}{3x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

12.9.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $z = x^r$, then $z' = rx^{r-1}$ and $z'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$3x^2(r(r-1))x^{r-2} + 5xrx^{r-1} - x^r = 0$$

Simplifying gives

$$3r(r-1)\,x^r + 5r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$3r(r-1) + 5r - 1 = 0$$

Or

$$3r^2 + 2r - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = \frac{1}{3}$$

Since the roots are real and distinct, then the general solution is

$$z = c_1 z_1 + c_2 z_2$$

Where $z_1 = x^{r_1}$ and $z_2 = x^{r_2}$. Hence

$$z = \frac{c_1}{x} + c_2 x^{\frac{1}{3}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$z = \frac{c_1}{x} + c_2 x^{\frac{1}{3}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 2$ and $x = 1$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$z' = -\frac{c_1}{x^2} + \frac{c_2}{3x^{\frac{2}{3}}}$$

substituting $z' = -1$ and $x = 1$ in the above gives

$$-1 = -c_1 + \frac{c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{5}{4}$$
$$c_2 = \frac{3}{4}$$

Substituting these values back in above solution results in

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x}$$

Summary
The solution(s) found are the following

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x} \tag{1}$$

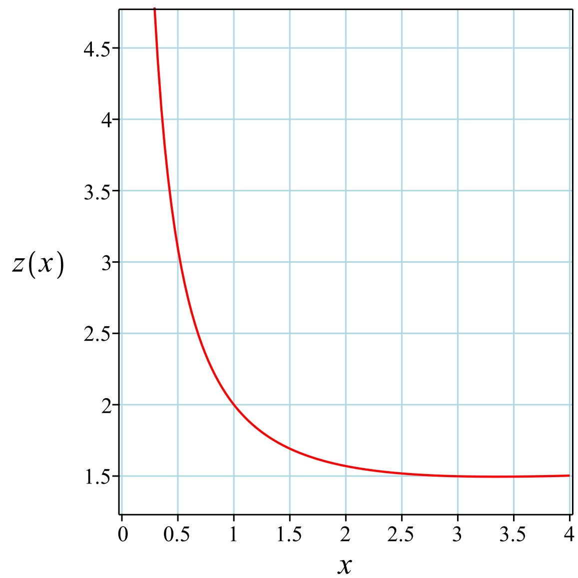


Figure 175: Solution plot

Verification of solutions

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x}$$

Verified OK.

12.9.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$3x^2z'' + 5xz' - z = 0 \tag{1}$$

Becomes

$$z'' + p(x)\, z' + q(x)\, z = 0 \tag{2}$$

Where

$$p(x) = \frac{5}{3x}$$
$$q(x) = -\frac{1}{3x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}z(\tau) + p_1\left(\frac{d}{d\tau}z(\tau)\right) + q_1 z(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int \frac{5}{3x}dx\right)} dx \\
&= \int e^{-\frac{5\ln(x)}{3}} dx \\
&= \int \frac{1}{x^{\frac{5}{3}}} dx \\
&= -\frac{3}{2x^{\frac{2}{3}}}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{3x^2}}{\frac{1}{x^{\frac{10}{3}}}} \\
&= -\frac{x^{\frac{4}{3}}}{3}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}z(\tau) + q_1 z(\tau) = 0$$

$$\frac{d^2}{d\tau^2}z(\tau) - \frac{x^{\frac{4}{3}}z(\tau)}{3} = 0$$

But in terms of τ

$$-\frac{x^{\frac{4}{3}}}{3} = -\frac{3}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}z(\tau) - \frac{3z(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $z(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}z(\tau)\right)\tau^2 - 3z(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $z(\tau) = \tau^r$, then $z' = r\tau^{r-1}$ and $z'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 3\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r - 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 - 3 = 0$$

Or

$$4r^2 - 4r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{2}$$
$$r_2 = \frac{3}{2}$$

Since the roots are real and distinct, then the general solution is

$$z(\tau) = c_1z_1 + c_2z_2$$

Where $z_1 = \tau^{r_1}$ and $z_2 = \tau^{r_2}$. Hence

$$z(\tau) = \frac{c_1}{\sqrt{\tau}} + c_2\tau^{\frac{3}{2}}$$

The above solution is now transformed back to z using (6) which results in

$$z = \frac{\left(4c_1 x^{\frac{4}{3}} + 9c_2\right)\sqrt{6}}{12 x^{\frac{4}{3}} \sqrt{-\frac{1}{x^{\frac{2}{3}}}}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$z = \frac{\left(4c_1 x^{\frac{4}{3}} + 9c_2\right)\sqrt{6}}{12 x^{\frac{4}{3}} \sqrt{-\frac{1}{x^{\frac{2}{3}}}}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 2$ and $x = 1$ in the above gives

$$2 = -\frac{i(4c_1 + 9c_2)\sqrt{6}}{12} \tag{1A}$$

Taking derivative of the solution gives

$$z' = \frac{4c_1\sqrt{6}}{9x\sqrt{-\frac{1}{x^{\frac{2}{3}}}}} - \frac{\left(4c_1 x^{\frac{4}{3}} + 9c_2\right)\sqrt{6}}{9x^{\frac{7}{3}}\sqrt{-\frac{1}{x^{\frac{2}{3}}}}} - \frac{\left(4c_1 x^{\frac{4}{3}} + 9c_2\right)\sqrt{6}}{36x^3\left(-\frac{1}{x^{\frac{2}{3}}}\right)^{\frac{3}{2}}}$$

substituting $z' = -1$ and $x = 1$ in the above gives

$$-1 = -\frac{i\sqrt{6}\,(4c_1 - 27c_2)}{36} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3i\sqrt{6}}{8}$$
$$c_2 = \frac{5i\sqrt{6}}{18}$$

Substituting these values back in above solution results in

$$z = \frac{3i x^{\frac{4}{3}} + 5i}{4x^{\frac{4}{3}}\sqrt{-\frac{1}{x^{\frac{2}{3}}}}}$$

Summary

The solution(s) found are the following

$$z = \frac{i\left(3x^{\frac{4}{3}}+5\right)}{4x^{\frac{4}{3}}\sqrt{-\frac{1}{x^{\frac{2}{3}}}}} \tag{1}$$

Verification of solutions

$$z = \frac{i\left(3x^{\frac{4}{3}}+5\right)}{4x^{\frac{4}{3}}\sqrt{-\frac{1}{x^{\frac{2}{3}}}}}$$

Verified OK.

12.9.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$3x^2z'' + 5xz' - z = 0 \tag{1}$$

Becomes

$$z'' + p(x)\, z' + q(x)\, z = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{3x} \\ q(x) &= -\frac{1}{3x^2} \end{aligned}$$

Applying change of variables on the depndent variable $z = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not z.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{5n}{3x^2} - \frac{1}{3x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{1}{3} \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{7v'(x)}{3x} = 0$$
$$v''(x) + \frac{7v'(x)}{3x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{7u(x)}{3x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{7u}{3x} \end{aligned}$$

Where $f(x) = -\frac{7}{3x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{7}{3x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{7}{3x}\,dx \\ \ln(u) &= -\frac{7\ln(x)}{3} + c_1 \\ u &= e^{-\frac{7\ln(x)}{3} + c_1} \\ &= \frac{c_1}{x^{\frac{7}{3}}} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{3c_1}{4x^{\frac{4}{3}}} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
z &= v(x)\,x^n \\
&= \left(-\frac{3c_1}{4x^{\frac{4}{3}}}+c_2\right)x^{\frac{1}{3}} \\
&= \frac{4c_2x^{\frac{4}{3}}-3c_1}{4x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
z = \left(-\frac{3c_1}{4x^{\frac{4}{3}}}+c_2\right)x^{\frac{1}{3}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 2$ and $x = 1$ in the above gives

$$
2 = -\frac{3c_1}{4}+c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
z' = \frac{c_1}{x^2}+\frac{-\frac{3c_1}{4x^{\frac{4}{3}}}+c_2}{3x^{\frac{2}{3}}}
$$

substituting $z' = -1$ and $x = 1$ in the above gives

$$
-1 = \frac{3c_1}{4}+\frac{c_2}{3} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{5}{3} \\
c_2 &= \frac{3}{4}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
z = \frac{3x^{\frac{4}{3}}+5}{4x}
$$

Summary

The solution(s) found are the following

$$
z = \frac{3x^{\frac{4}{3}}+5}{4x} \tag{1}
$$

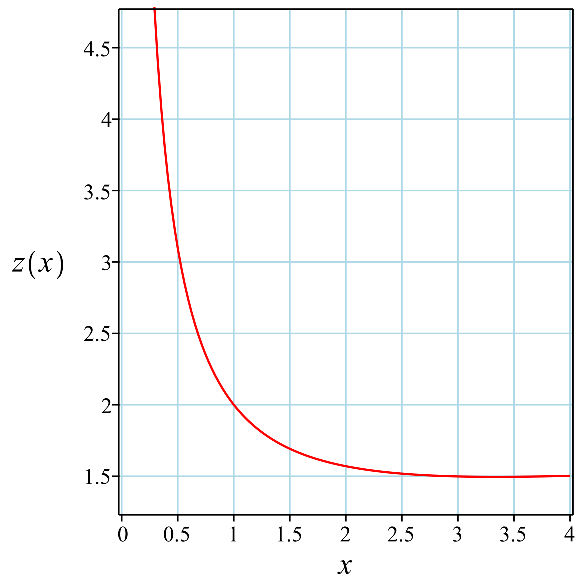


Figure 176: Solution plot

Verification of solutions

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x}$$

Verified OK.

12.9.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(3x^2 z'' + 5xz' - z\right) dx = 0$$
$$-xz + 3x^2 z' = c_1$$

Which is now solved for z.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$z' + p(x)z = q(x)$$

Where here

$$p(x) = -\frac{1}{3x}$$
$$q(x) = \frac{c_1}{3x^2}$$

Hence the ode is

$$z' - \frac{z}{3x} = \frac{c_1}{3x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{3x}dx} \\ &= \frac{1}{x^{\frac{1}{3}}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu z) &= (\mu)\left(\frac{c_1}{3x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{z}{x^{\frac{1}{3}}}\right) &= \left(\frac{1}{x^{\frac{1}{3}}}\right)\left(\frac{c_1}{3x^2}\right) \\ \mathrm{d}\left(\frac{z}{x^{\frac{1}{3}}}\right) &= \left(\frac{c_1}{3x^{\frac{7}{3}}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{z}{x^{\frac{1}{3}}} &= \int \frac{c_1}{3x^{\frac{7}{3}}}\,\mathrm{d}x \\ \frac{z}{x^{\frac{1}{3}}} &= -\frac{c_1}{4x^{\frac{4}{3}}} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^{\frac{1}{3}}}$ results in

$$z = -\frac{c_1}{4x} + c_2 x^{\frac{1}{3}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$z = -\frac{c_1}{4x} + c_2 x^{\frac{1}{3}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 2$ and $x = 1$ in the above gives

$$2 = -\frac{c_1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$z' = \frac{c_1}{4x^2} + \frac{c_2}{3x^{\frac{2}{3}}}$$

substituting $z' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{c_1}{4} + \frac{c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -5$$
$$c_2 = \frac{3}{4}$$

Substituting these values back in above solution results in

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x}$$

Summary

The solution(s) found are the following

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x} \tag{1}$$

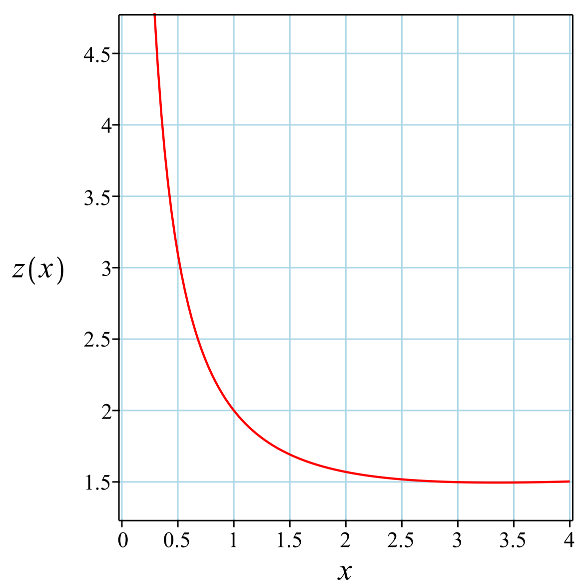


Figure 177: Solution plot

Verification of solutions

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x}$$

Verified OK.

12.9.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$3x^2z'' + 5xz' - z = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int \left(3x^2z'' + 5xz' - z\right) dx &= 0\\ -xz + 3x^2z' &= c_1\end{aligned}$$

Which is now solved for z.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$z' + p(x)z = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{3x}\\ q(x) &= \frac{c_1}{3x^2}\end{aligned}$$

Hence the ode is

$$z' - \frac{z}{3x} = \frac{c_1}{3x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{3x}dx}\\ &= \frac{1}{x^{\frac{1}{3}}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu z) &= (\mu)\left(\frac{c_1}{3x^2}\right)\\ \frac{d}{dx}\left(\frac{z}{x^{\frac{1}{3}}}\right) &= \left(\frac{1}{x^{\frac{1}{3}}}\right)\left(\frac{c_1}{3x^2}\right)\\ d\left(\frac{z}{x^{\frac{1}{3}}}\right) &= \left(\frac{c_1}{3x^{\frac{7}{3}}}\right) dx\end{aligned}$$

Integrating gives

$$
\begin{aligned}
\frac{z}{x^{\frac{1}{3}}} &= \int \frac{c_1}{3x^{\frac{7}{3}}}\,\mathrm{d}x \\
\frac{z}{x^{\frac{1}{3}}} &= -\frac{c_1}{4x^{\frac{4}{3}}} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^{\frac{1}{3}}}$ results in

$$
z = -\frac{c_1}{4x} + c_2 x^{\frac{1}{3}}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
z = -\frac{c_1}{4x} + c_2 x^{\frac{1}{3}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 2$ and $x = 1$ in the above gives

$$
2 = -\frac{c_1}{4} + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
z' = \frac{c_1}{4x^2} + \frac{c_2}{3x^{\frac{2}{3}}}
$$

substituting $z' = -1$ and $x = 1$ in the above gives

$$
-1 = \frac{c_1}{4} + \frac{c_2}{3} \tag{2A}
$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -5 \\
c_2 &= \frac{3}{4}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
z = \frac{3x^{\frac{4}{3}} + 5}{4x}
$$

<u>Summary</u>
The solution(s) found are the following

$$
z = \frac{3x^{\frac{4}{3}} + 5}{4x} \tag{1}
$$

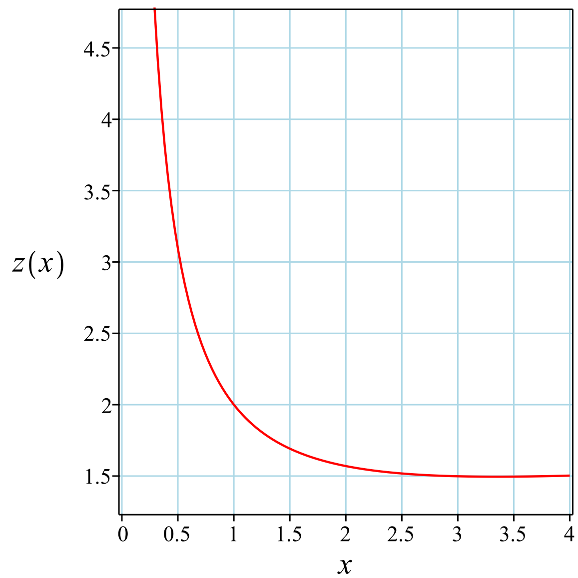


Figure 178: Solution plot

Verification of solutions

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x}$$

Verified OK.

12.9.7 Solving using Kovacic algorithm

Writing the ode as

$$3x^2z'' + 5xz' - z = 0 \tag{1}$$
$$Az'' + Bz' + Cz = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= 5x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ze^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then z is found using the inverse transformation

$$z = ze^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 173: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7}{36x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{6x} + (-)\,(0) \\ &= -\frac{1}{6x} \\ &= -\frac{1}{6x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6x}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x}\right)^2 - \left(\frac{7}{36x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{6x} dx} \\ &= \frac{1}{x^{\frac{1}{6}}} \end{aligned}$$

The first solution to the original ode in z is found from

$$\begin{aligned} z_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x}{3x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln(x)}{6}} \\ &= z_1 \left(\frac{1}{x^{\frac{5}{6}}}\right) \end{aligned}$$

Which simplifies to

$$z_1 = \frac{1}{x}$$

The second solution z_2 to the original ode is found using reduction of order

$$z_2 = z_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{z_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
z_2 &= z_1 \int \frac{e^{\int -\frac{5x}{3x^2}\, dx}}{(z_1)^2}\, dx \\
&= z_1 \int \frac{e^{-\frac{5\ln(x)}{3}}}{(z_1)^2}\, dx \\
&= z_1 \left(\frac{3x^{\frac{4}{3}}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
z &= c_1 z_1 + c_2 z_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{3x^{\frac{4}{3}}}{4}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
z = \frac{c_1}{x} + \frac{3c_2 x^{\frac{1}{3}}}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 2$ and $x = 1$ in the above gives

$$
2 = \frac{3c_2}{4} + c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
z' = -\frac{c_1}{x^2} + \frac{c_2}{4x^{\frac{2}{3}}}
$$

substituting $z' = -1$ and $x = 1$ in the above gives

$$
-1 = -c_1 + \frac{c_2}{4} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{5}{4} \\
c_2 &= 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x}$$

Summary
The solution(s) found are the following

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x} \tag{1}$$

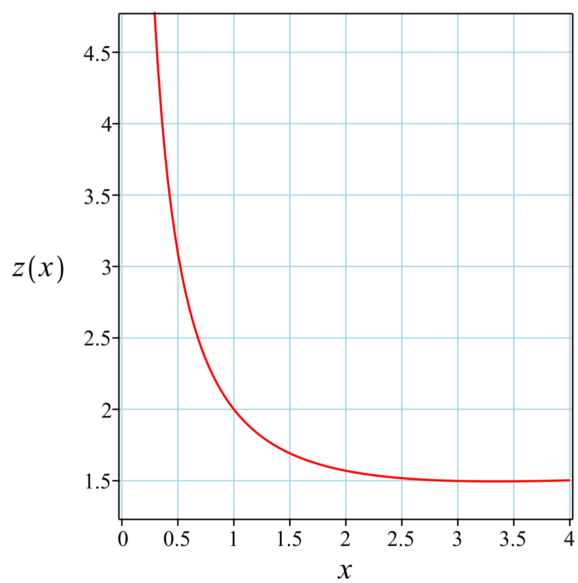


Figure 179: Solution plot

Verification of solutions

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x}$$

Verified OK.

12.9.8 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, z'' + q(x)\, z' + r(x)\, z = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 3x^2 \\
q(x) &= 5x \\
r(x) &= -1 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 6 \\
q'(x) &= 5
\end{aligned}
$$

Therefore (1) becomes

$$
6 - (5) + (-1) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, z' + (q(x) - p'(x))\, z)' = s(x)
$$

Integrating gives

$$
p(x)\, z' + (q(x) - p'(x))\, z = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
-xz + 3x^2 z' = c_1
$$

We now have a first order ode to solve which is

$$
-xz + 3x^2 z' = c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
z' + p(x)z = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{3x} \\
q(x) &= \frac{c_1}{3x^2}
\end{aligned}
$$

Hence the ode is

$$z' - \frac{z}{3x} = \frac{c_1}{3x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{3x}dx} \\ &= \frac{1}{x^{\frac{1}{3}}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu z) &= (\mu)\left(\frac{c_1}{3x^2}\right) \\ \frac{d}{dx}\left(\frac{z}{x^{\frac{1}{3}}}\right) &= \left(\frac{1}{x^{\frac{1}{3}}}\right)\left(\frac{c_1}{3x^2}\right) \\ d\left(\frac{z}{x^{\frac{1}{3}}}\right) &= \left(\frac{c_1}{3x^{\frac{7}{3}}}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{z}{x^{\frac{1}{3}}} &= \int \frac{c_1}{3x^{\frac{7}{3}}} dx \\ \frac{z}{x^{\frac{1}{3}}} &= -\frac{c_1}{4x^{\frac{4}{3}}} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^{\frac{1}{3}}}$ results in

$$z = -\frac{c_1}{4x} + c_2 x^{\frac{1}{3}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$z = -\frac{c_1}{4x} + c_2 x^{\frac{1}{3}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $z = 2$ and $x = 1$ in the above gives

$$2 = -\frac{c_1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$z' = \frac{c_1}{4x^2} + \frac{c_2}{3x^{\frac{2}{3}}}$$

substituting $z' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{c_1}{4} + \frac{c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -5$$
$$c_2 = \frac{3}{4}$$

Substituting these values back in above solution results in

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x}$$

Summary

The solution(s) found are the following

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x} \tag{1}$$

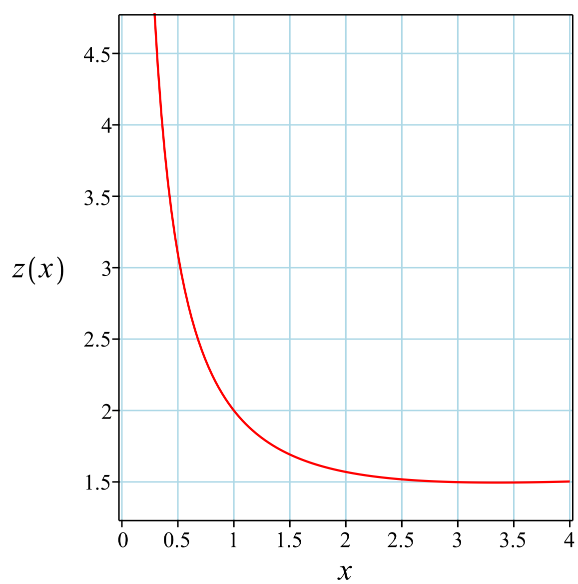


Figure 180: Solution plot

Verification of solutions

$$z = \frac{3x^{\frac{4}{3}} + 5}{4x}$$

Verified OK.

12.9.9 Maple step by step solution

Let's solve

$\left[3x^2z'' + 5xz' - z = 0, z(1) = 2, z'\Big|_{\{x=1\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  z''

- Isolate 2nd derivative

  $z'' = -\frac{5z'}{3x} + \frac{z}{3x^2}$

- Group terms with z on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $z'' + \frac{5z'}{3x} - \frac{z}{3x^2} = 0$

- Multiply by denominators of the ODE

  $3x^2z'' + 5xz' - z = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of z with respect to x , using the chain rule

    $z' = \left(\frac{d}{dt}z(t)\right)t'(x)$

  ◦ Compute derivative

    $z' = \frac{\frac{d}{dt}z(t)}{x}$

  ◦ Calculate the 2nd derivative of z with respect to x , using the chain rule

    $z'' = \left(\frac{d^2}{dt^2}z(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}z(t)\right)$

  ◦ Compute derivative

    $z'' = \frac{\frac{d^2}{dt^2}z(t)}{x^2} - \frac{\frac{d}{dt}z(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $3x^2\left(\frac{\frac{d^2}{dt^2}z(t)}{x^2} - \frac{\frac{d}{dt}z(t)}{x^2}\right) + 5\frac{d}{dt}z(t) - z(t) = 0$

- Simplify

  $3\frac{d^2}{dt^2}z(t) + 2\frac{d}{dt}z(t) - z(t) = 0$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}z(t) = -\frac{2\frac{d}{dt}z(t)}{3} + \frac{z(t)}{3}$

- Group terms with $z(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dt^2}z(t) + \frac{2\frac{d}{dt}z(t)}{3} - \frac{z(t)}{3} = 0$

- Characteristic polynomial of ODE

$r^2 + \frac{2}{3}r - \frac{1}{3} = 0$

- Factor the characteristic polynomial

$\frac{(r+1)(3r-1)}{3} = 0$

- Roots of the characteristic polynomial

$r = \left(-1, \frac{1}{3}\right)$

- 1st solution of the ODE

$z_1(t) = e^{-t}$

- 2nd solution of the ODE

$z_2(t) = e^{\frac{t}{3}}$

- General solution of the ODE

$z(t) = c_1 z_1(t) + c_2 z_2(t)$

- Substitute in solutions

$z(t) = c_1 e^{-t} + c_2 e^{\frac{t}{3}}$

- Change variables back using $t = \ln(x)$

$z = \frac{c_1}{x} + c_2 x^{\frac{1}{3}}$

- Simplify

$z = \frac{c_1}{x} + c_2 x^{\frac{1}{3}}$

☐ Check validity of solution $z = \frac{c_1}{x} + c_2 x^{\frac{1}{3}}$

  - Use initial condition $z(1) = 2$

    $2 = c_1 + c_2$

  - Compute derivative of the solution

    $z' = -\frac{c_1}{x^2} + \frac{c_2}{3x^{\frac{2}{3}}}$

  - Use the initial condition $z'\Big|_{\{x=1\}} = -1$

    $-1 = -c_1 + \frac{c_2}{3}$

- Solve for c_1 and c_2

  $\{c_1 = \frac{5}{4}, c_2 = \frac{3}{4}\}$

- Substitute constant values into general solution and simplify

  $z = \frac{3x^{\frac{4}{3}}+5}{4x}$

- Solution to the IVP

  $z = \frac{3x^{\frac{4}{3}}+5}{4x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([3*x^2*diff(z(x),x$2)+5*x*diff(z(x),x)-z(x)=0,z(1) = 2, D(z)(1) = -1],z(x), singsol=a
```

$$z(x) = \frac{3x^{\frac{4}{3}} + 5}{4x}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 21

```
DSolve[{3*x^2*z''[x]+5*x*z'[x]-z[x]==0,{z[1]==2,z'[1]==-1}},z[x],x,IncludeSingularSolutions
```

$$z(x) \to \frac{3x^{4/3} + 5}{4x}$$

12.10 problem 19.1 (x)

Internal problem ID [12069]
Internal file name [`OUTPUT/10721_Monday_September_11_2023_12_49_57_AM_97028781/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 19, CauchyEuler equations. Exercises page 174
Problem number: 19.1 (x).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2x'' + 3tx' + 13x = 0}$$

With initial conditions

$$[x(1) = -1, x'(1) = 2]$$

12.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= \frac{13}{t^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + \frac{3x'}{t} + \frac{13x}{t^2} = 0$$

The domain of $p(t) = \frac{3}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = \frac{13}{t^2}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

12.10.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $x = t^r$, then $x' = rt^{r-1}$ and $x'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 3trt^{r-1} + 13t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 3r\,t^r + 13t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 3r + 13 = 0$$

Or

$$r^2 + 2r + 13 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -2i\sqrt{3} - 1$$
$$r_2 = 2i\sqrt{3} - 1$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = -1$ and $\beta = -2\sqrt{3}$. Hence the solution becomes

$$\begin{aligned} x &= c_1 t^{r_1} + c_2 t^{r_2} \\ &= c_1 t^{\alpha+i\beta} + c_2 t^{\alpha-i\beta} \\ &= t^{\alpha}\left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\ &= t^{\alpha}\left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\ &= t^{\alpha}\left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right) \end{aligned}$$

Using the values for $\alpha = -1, \beta = -2\sqrt{3}$, the above becomes

$$x = t^{-1}\left(c_1 e^{-2i\sqrt{3}\,\ln(t)} + c_2 e^{2i\sqrt{3}\,\ln(t)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$x = \frac{1}{t}\left(c_1 \cos\left(2\sqrt{3}\,\ln(t)\right) + c_2 \sin\left(2\sqrt{3}\,\ln(t)\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{c_1 \cos\left(2\sqrt{3}\,\ln(t)\right) + c_2 \sin\left(2\sqrt{3}\,\ln(t)\right)}{t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 1$ in the above gives

$$-1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{c_1 \cos\left(2\sqrt{3}\,\ln(t)\right) + c_2 \sin\left(2\sqrt{3}\,\ln(t)\right)}{t^2} + \frac{-\frac{2c_1\sqrt{3}\,\sin\left(2\sqrt{3}\,\ln(t)\right)}{t} + \frac{2c_2\sqrt{3}\,\cos\left(2\sqrt{3}\,\ln(t)\right)}{t}}{t}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = 2c_2\sqrt{3} - c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = \frac{\sqrt{3}}{6}$$

Substituting these values back in above solution results in

$$x = \frac{\sin\left(2\sqrt{3}\,\ln(t)\right)\sqrt{3} - 6\cos\left(2\sqrt{3}\,\ln(t)\right)}{6t}$$

Summary
The solution(s) found are the following

$$x = \frac{\sin\left(2\sqrt{3}\,\ln(t)\right)\sqrt{3} - 6\cos\left(2\sqrt{3}\,\ln(t)\right)}{6t} \tag{1}$$

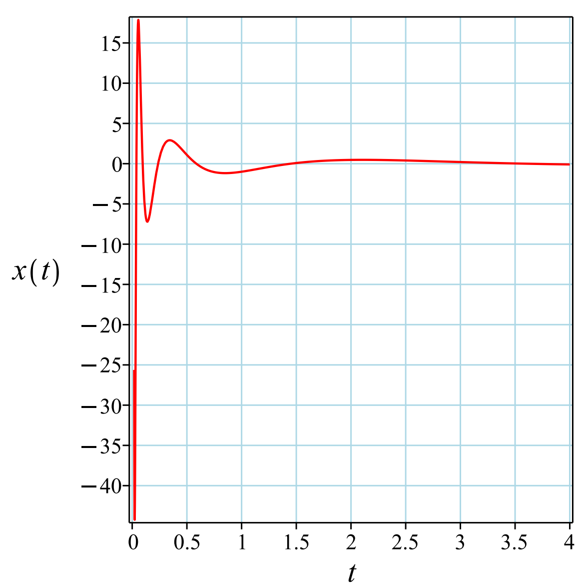


Figure 181: Solution plot

Verification of solutions

$$x = \frac{\sin\left(2\sqrt{3}\,\ln\left(t\right)\right)\sqrt{3}-6\cos\left(2\sqrt{3}\,\ln\left(t\right)\right)}{6t}$$

Verified OK.

12.10.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2x''+3tx'+13x=0 \tag{1}$$

Becomes

$$x''+p(t)\,x'+q(t)\,x=0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= \frac{13}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}x(\tau)+p_1\left(\frac{d}{d\tau}x(\tau)\right)+q_1x(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t)+p(t)\,\tau'(t)}{\tau'\left(t\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'\left(t\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t)+p(t)\,\tau'(t)=0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int \frac{3}{t}dt\right)}dt \\ &= \int e^{-3\ln(t)}\,dt \\ &= \int \frac{1}{t^3}dt \\ &= -\frac{1}{2t^2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{13}{t^2}}{\frac{1}{t^6}} \\ &= 13t^4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}x(\tau) + q_1 x(\tau) = 0$$
$$\frac{d^2}{d\tau^2}x(\tau) + 13t^4 x(\tau) = 0$$

But in terms of τ

$$13t^4 = \frac{13}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}x(\tau) + \frac{13x(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $x(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}x(\tau)\right)\tau^2 + 13x(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $x(\tau) = \tau^r$, then $x' = r\tau^{r-1}$ and $x'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 13\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + 13\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 13 = 0$$

Or

$$4r^2 - 4r + 13 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - i\sqrt{3}$$
$$r_2 = \frac{1}{2} + i\sqrt{3}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\sqrt{3}$. Hence the solution becomes

$$\begin{aligned}
x(\tau) &= c_1\tau^{r_1} + c_2\tau^{r_2} \\
&= c_1\tau^{\alpha+i\beta} + c_2\tau^{\alpha-i\beta} \\
&= \tau^{\alpha}\left(c_1\tau^{i\beta} + c_2\tau^{-i\beta}\right) \\
&= \tau^{\alpha}\left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})}\right) \\
&= \tau^{\alpha}\left(c_1 e^{i(\beta\ln\tau)} + c_2 e^{-i(\beta\ln\tau)}\right)
\end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\sqrt{3}$, the above becomes

$$x(\tau) = \tau^{\frac{1}{2}}\left(c_1 e^{-i\sqrt{3}\,\ln(\tau)} + c_2 e^{i\sqrt{3}\,\ln(\tau)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$x(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\sqrt{3}\,\ln(\tau)\right) + c_2 \sin\left(\sqrt{3}\,\ln(\tau)\right)\right)$$

The above solution is now transformed back to x using (6) which results in

$$x = \frac{\left(c_1 \cos\left(\sqrt{3}\left(\ln(2) - \ln\left(-\frac{1}{t^2}\right)\right)\right) - c_2 \sin\left(\sqrt{3}\left(\ln(2) - \ln\left(-\frac{1}{t^2}\right)\right)\right)\right)\sqrt{2}\,\sqrt{-\frac{1}{t^2}}}{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \frac{\left(c_1 \cos\left(\sqrt{3}\left(\ln(2) - \ln\left(-\frac{1}{t^2}\right)\right)\right) - c_2 \sin\left(\sqrt{3}\left(\ln(2) - \ln\left(-\frac{1}{t^2}\right)\right)\right)\right)\sqrt{2}\,\sqrt{-\frac{1}{t^2}}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 1$ in the above gives

$$-1 = \frac{\sqrt{2}\left(i c_1 \cosh\left(\sqrt{3}\,(\pi + i\ln(2))\right) - c_2 \sinh\left(\sqrt{3}\,(\pi + i\ln(2))\right)\right)}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{\left(-\frac{2c_1\sqrt{3}\,\sin\left(\sqrt{3}\left(\ln(2)-\ln\left(-\frac{1}{t^2}\right)\right)\right)}{t} - \frac{2c_2\sqrt{3}\,\cos\left(\sqrt{3}\left(\ln(2)-\ln\left(-\frac{1}{t^2}\right)\right)\right)}{t}\right)\sqrt{2}\,\sqrt{-\frac{1}{t^2}}}{2} + \frac{\left(c_1\cos\left(\sqrt{3}\left(\ln(2) - \ln\left(-\right.\right.\right.\right.}{}$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = -\frac{\sqrt{2}\left(i\left(2c_2\sqrt{3} + c_1\right)\cosh\left(\sqrt{3}\,(\pi + i\ln(2))\right) + 2\sinh\left(\sqrt{3}\,(\pi + i\ln(2))\right)\left(\sqrt{3}\,c_1 - \frac{c_2}{2}\right)\right)}{2} \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{\sqrt{2}\left(6i\cosh\left(\sqrt{3}\,(\pi + i\ln(2))\right) + \sqrt{3}\,\sinh\left(\sqrt{3}\,(\pi + i\ln(2))\right)\right)}{6}$$
$$c_2 = \frac{\sqrt{2}\left(i\sqrt{3}\,\cosh\left(\sqrt{3}\,(\pi + i\ln(2))\right) - 6\sinh\left(\sqrt{3}\,(\pi + i\ln(2))\right)\right)}{6}$$

Substituting these values back in above solution results in

$$x = -\frac{i\sin\left(\sqrt{3}\left(\ln(2) - \ln\left(-\frac{1}{t^2}\right)\right)\right)\sqrt{3}\,\sqrt{-\frac{1}{t^2}}\,\cosh\left(\sqrt{3}\,(\pi + i\ln(2))\right)}{6} + i\cos\left(\sqrt{3}\left(\ln(2) - \ln\left(-\frac{1}{t^2}\right)\right)\right.$$

Summary

The solution(s) found are the following

$$x = -\frac{\sqrt{-\frac{1}{t^2}}\left(\left(-6i\cosh\left(\sqrt{3}\,(\pi + i\ln(2))\right) - \sqrt{3}\,\sinh\left(\sqrt{3}\,(\pi + i\ln(2))\right)\right)\cos\left(\sqrt{3}\left(\ln(2) - \ln\left(-\frac{1}{t^2}\right)\right)\right) + (i\right.}{6} \tag{1}$$

Verification of solutions

$$x = -\frac{\sqrt{-\frac{1}{t^2}}\left(\left(-6i\cosh\left(\sqrt{3}\,(\pi + i\ln(2))\right) - \sqrt{3}\,\sinh\left(\sqrt{3}\,(\pi + i\ln(2))\right)\right)\cos\left(\sqrt{3}\left(\ln(2) - \ln\left(-\frac{1}{t^2}\right)\right)\right) + (i\right.}{6}$$

Verified OK.

12.10.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2x'' + 3tx' + 13x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\,x' + q(t)\,x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= \frac{13}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}x(\tau) + p_1\left(\frac{d}{d\tau}x(\tau)\right) + q_1x(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'\,(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'\,(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{13}\,\sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{\sqrt{13}}{c\sqrt{\frac{1}{t^2}}\,t^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'\,(t)^2} \\ &= \frac{-\frac{\sqrt{13}}{c\sqrt{\frac{1}{t^2}}\,t^3} + \frac{3}{t}\frac{\sqrt{13}\,\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{13}\,\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\ &= \frac{2c\sqrt{13}}{13} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
x(\tau)'' + p_1 x(\tau)' + q_1 x(\tau) &= 0 \\
\frac{d^2}{d\tau^2}x(\tau) + \frac{2c\sqrt{13}\left(\frac{d}{d\tau}x(\tau)\right)}{13} + c^2 x(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $x(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
x(\tau) = \mathrm{e}^{-\frac{\sqrt{13}\,c\tau}{13}}\left(c_1 \cos\left(\frac{2c\sqrt{39}\,\tau}{13}\right) + c_2 \sin\left(\frac{2c\sqrt{39}\,\tau}{13}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int \sqrt{13}\,\sqrt{\frac{1}{t^2}}dt}{c} \\
&= \frac{\sqrt{13}\,\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
x = \frac{c_1 \cos\left(2\sqrt{3}\,\ln(t)\right) + c_2 \sin\left(2\sqrt{3}\,\ln(t)\right)}{t}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = \frac{c_1 \cos\left(2\sqrt{3}\,\ln(t)\right) + c_2 \sin\left(2\sqrt{3}\,\ln(t)\right)}{t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 1$ in the above gives

$$
-1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -\frac{c_1 \cos\left(2\sqrt{3}\,\ln(t)\right) + c_2 \sin\left(2\sqrt{3}\,\ln(t)\right)}{t^2} + \frac{-\frac{2c_1\sqrt{3}\,\sin\left(2\sqrt{3}\,\ln(t)\right)}{t} + \frac{2c_2\sqrt{3}\,\cos\left(2\sqrt{3}\,\ln(t)\right)}{t}}{t}
$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$2 = 2c_2\sqrt{3} - c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = \frac{\sqrt{3}}{6}$$

Substituting these values back in above solution results in

$$x = \frac{\sin\left(2\sqrt{3}\,\ln\left(t\right)\right)\sqrt{3} - 6\cos\left(2\sqrt{3}\,\ln\left(t\right)\right)}{6t}$$

Summary

The solution(s) found are the following

$$x = \frac{\sin\left(2\sqrt{3}\,\ln\left(t\right)\right)\sqrt{3} - 6\cos\left(2\sqrt{3}\,\ln\left(t\right)\right)}{6t} \tag{1}$$

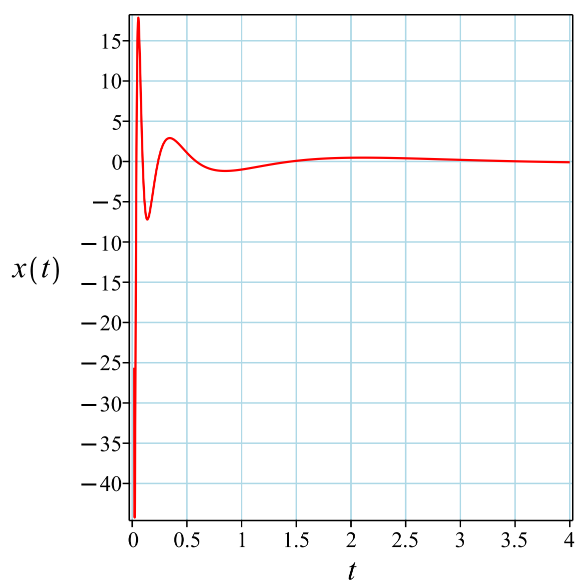


Figure 182: Solution plot

Verification of solutions

$$x = \frac{\sin\left(2\sqrt{3}\,\ln\left(t\right)\right)\sqrt{3} - 6\cos\left(2\sqrt{3}\,\ln\left(t\right)\right)}{6t}$$

Verified OK.

12.10.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2x'' + 3tx' + 13x = 0 \tag{1}$$

Becomes

$$x'' + p(t)\, x' + q(t)\, x = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= \frac{13}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $x = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not x.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{3n}{t^2} + \frac{13}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2i\sqrt{3} - 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \left(\frac{4i\sqrt{3}-2}{t} + \frac{3}{t}\right) v'(t) &= 0 \\ v''(t) + \frac{\left(4i\sqrt{3}+1\right) v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{\left(4i\sqrt{3}+1\right) u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= \frac{\left(-4i\sqrt{3}-1\right) u}{t} \end{aligned}$$

Where $f(t) = \frac{-4i\sqrt{3}-1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{-4i\sqrt{3}-1}{t}\,dt \\ \int \frac{1}{u}\,du &= \int \frac{-4i\sqrt{3}-1}{t}\,dt \\ \ln(u) &= \left(-4i\sqrt{3}-1\right)\ln(t) + c_1 \\ u &= \mathrm{e}^{\left(-4i\sqrt{3}-1\right)\ln(t)+c_1} \\ &= c_1 \mathrm{e}^{\left(-4i\sqrt{3}-1\right)\ln(t)} \end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_1 t^{-4i\sqrt{3}}}{t}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\,dt + c_2 \\ &= \frac{i\sqrt{3}\,c_1 t^{-4i\sqrt{3}}}{12} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
x &= v(t)\, t^n \\
&= \left(\frac{i\sqrt{3}\, c_1 t^{-4i\sqrt{3}}}{12} + c_2\right) t^{2i\sqrt{3}-1} \\
&= \frac{t^{2i\sqrt{3}} c_2 + \frac{i t^{-2i\sqrt{3}} \sqrt{3}\, c_1}{12}}{t}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = \left(\frac{i\sqrt{3}\, c_1 t^{-4i\sqrt{3}}}{12} + c_2\right) t^{2i\sqrt{3}-1} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 1$ in the above gives

$$
-1 = \frac{i\sqrt{3}\, c_1}{12} + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = \frac{c_1 t^{-4i\sqrt{3}} t^{2i\sqrt{3}-1}}{t} + \frac{\left(\frac{i\sqrt{3}\, c_1 t^{-4i\sqrt{3}}}{12} + c_2\right) t^{2i\sqrt{3}-1} (2i\sqrt{3} - 1)}{t}
$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$
2 = \frac{i(-c_1 + 24 c_2)\sqrt{3}}{12} + \frac{c_1}{2} - c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2i\sqrt{3} + 1 \\
c_2 &= -\frac{i\sqrt{3}}{12} - \frac{1}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = \frac{i t^{2i\sqrt{3}-1} \sqrt{3}\, t^{-4i\sqrt{3}}}{12} - \frac{i t^{2i\sqrt{3}-1} \sqrt{3}}{12} - \frac{t^{2i\sqrt{3}-1} t^{-4i\sqrt{3}}}{2} - \frac{t^{2i\sqrt{3}-1}}{2}
$$

Summary
The solution(s) found are the following

$$x = \frac{i\sqrt{3}\, t^{-2i\sqrt{3}} - i t^{2i\sqrt{3}}\sqrt{3} - 6t^{-2i\sqrt{3}} - 6t^{2i\sqrt{3}}}{12t} \tag{1}$$

Verification of solutions

$$x = \frac{i\sqrt{3}\, t^{-2i\sqrt{3}} - i t^{2i\sqrt{3}}\sqrt{3} - 6t^{-2i\sqrt{3}} - 6t^{2i\sqrt{3}}}{12t}$$

Verified OK.

12.10.6 Solving using Kovacic algorithm

Writing the ode as

$$t^2x'' + 3tx' + 13x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 3t \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-49}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -49 \\
t &= 4t^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(-\frac{49}{4t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 175: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{49}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{49}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i\sqrt{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i\sqrt{3}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{49}{4t^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{49}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i\sqrt{3} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i\sqrt{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{49}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + 2i\sqrt{3}$	$\frac{1}{2} - 2i\sqrt{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}+2i\sqrt{3}$	$\frac{1}{2}-2i\sqrt{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}-2i\sqrt{3}$ then

$$\begin{aligned}
d&=\alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\
&=\frac{1}{2}-2i\sqrt{3}-\left(\frac{1}{2}-2i\sqrt{3}\right)\\
&=0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{t-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&=\frac{\frac{1}{2}-2i\sqrt{3}}{t}+(-)\,(0)\\
&=\frac{\frac{1}{2}-2i\sqrt{3}}{t}\\
&=\frac{-4i\sqrt{3}+1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(t)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{\frac{1}{2}-2i\sqrt{3}}{t}\right)(0)+\left(\left(-\frac{\frac{1}{2}-2i\sqrt{3}}{t^2}\right)+\left(\frac{\frac{1}{2}-2i\sqrt{3}}{t}\right)^2-\left(-\frac{49}{4t^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(t)&=pe^{\int \omega\,dt}\\&=\mathrm{e}^{\int \frac{\frac{1}{2}-2i\sqrt{3}}{t}dt}\\&=t^{\frac{1}{2}-2i\sqrt{3}}\end{aligned}$$

The first solution to the original ode in x is found from

$$\begin{aligned}x_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\&=z_1e^{-\int \frac{1}{2}\frac{3t}{t^2}\,dt}\\&=z_1e^{-\frac{3\ln(t)}{2}}\\&=z_1\left(\frac{1}{t^{\frac{3}{2}}}\right)\end{aligned}$$

Which simplifies to

$$x_1=t^{-2i\sqrt{3}-1}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2=x_1\int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}x_2&=x_1\int \frac{e^{\int -\frac{3t}{t^2}\,dt}}{(x_1)^2}\,dt\\&=x_1\int \frac{e^{-3\ln(t)}}{(x_1)^2}\,dt\\&=x_1\left(-\frac{it^{4i\sqrt{3}}\sqrt{3}}{12}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1\left(t^{-2i\sqrt{3}-1}\right) + c_2\left(t^{-2i\sqrt{3}-1}\left(-\frac{i t^{4i\sqrt{3}}\sqrt{3}}{12}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 t^{-2i\sqrt{3}-1} - \frac{i c_2 t^{2i\sqrt{3}-1}\sqrt{3}}{12} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -1$ and $t = 1$ in the above gives

$$
-1 = -\frac{i\sqrt{3}\, c_2}{12} + c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = \frac{c_1 t^{-2i\sqrt{3}-1}\left(-2i\sqrt{3}-1\right)}{t} - \frac{i c_2 t^{2i\sqrt{3}-1}\left(2i\sqrt{3}-1\right)\sqrt{3}}{12t}
$$

substituting $x' = 2$ and $t = 1$ in the above gives

$$
2 = \frac{i(-24c_1 + c_2)\sqrt{3}}{12} - c_1 + \frac{c_2}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{12} \\
c_2 &= -2i\sqrt{3} + 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = \frac{i\sqrt{3}\, t^{-2i\sqrt{3}-1}}{12} - \frac{i t^{2i\sqrt{3}-1}\sqrt{3}}{12} - \frac{t^{-2i\sqrt{3}-1}}{2} - \frac{t^{2i\sqrt{3}-1}}{2}
$$

Summary

The solution(s) found are the following

$$
x = \frac{i\sqrt{3}\, t^{-2i\sqrt{3}} - i t^{2i\sqrt{3}}\sqrt{3} - 6t^{-2i\sqrt{3}} - 6t^{2i\sqrt{3}}}{12t} \tag{1}
$$

Verification of solutions

$$x = \frac{i\sqrt{3}\, t^{-2i\sqrt{3}} - i t^{2i\sqrt{3}}\sqrt{3} - 6t^{-2i\sqrt{3}} - 6t^{2i\sqrt{3}}}{12t}$$

Verified OK.

12.10.7 Maple step by step solution

Let's solve

$\left[t^2x'' + 3tx' + 13x = 0, x(1) = -1, x'\Big|_{\{t=1\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -\frac{3x'}{t} - \frac{13x}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{3x'}{t} + \frac{13x}{t^2} = 0$

- Multiply by denominators of the ODE

  $t^2x'' + 3tx' + 13x = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of x with respect to t , using the chain rule

    $x' = \left(\frac{d}{ds}x(s)\right) s'(t)$

  - Compute derivative

    $x' = \frac{\frac{d}{ds}x(s)}{t}$

  - Calculate the 2nd derivative of x with respect to t , using the chain rule

    $x'' = \left(\frac{d^2}{ds^2}x(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}x(s)\right)$

  - Compute derivative

    $x'' = \frac{\frac{d^2}{ds^2}x(s)}{t^2} - \frac{\frac{d}{ds}x(s)}{t^2}$

  Substitute the change of variables back into the ODE

$$t^2\left(\frac{\frac{d^2}{ds^2}x(s)}{t^2}-\frac{\frac{d}{ds}x(s)}{t^2}\right)+3\frac{d}{ds}x(s)+13x(s)=0$$

- Simplify

  $\frac{d^2}{ds^2}x(s)+2\frac{d}{ds}x(s)+13x(s)=0$

- Characteristic polynomial of ODE

  $r^2+2r+13=0$

- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm(\sqrt{-48})}{2}$

- Roots of the characteristic polynomial

  $r=\left(-2\,\mathrm{I}\sqrt{3}-1, 2\,\mathrm{I}\sqrt{3}-1\right)$

- 1st solution of the ODE

  $x_1(s)=\mathrm{e}^{-s}\cos\left(2\sqrt{3}\,s\right)$

- 2nd solution of the ODE

  $x_2(s)=\mathrm{e}^{-s}\sin\left(2\sqrt{3}\,s\right)$

- General solution of the ODE

  $x(s)=c_1x_1(s)+c_2x_2(s)$

- Substitute in solutions

  $x(s)=c_1\mathrm{e}^{-s}\cos\left(2\sqrt{3}\,s\right)+c_2\mathrm{e}^{-s}\sin\left(2\sqrt{3}\,s\right)$

- Change variables back using $s=\ln(t)$

  $x=\frac{c_1\cos\left(2\sqrt{3}\,\ln(t)\right)}{t}+\frac{c_2\sin\left(2\sqrt{3}\,\ln(t)\right)}{t}$

- Simplify

  $x=\frac{c_1\cos\left(2\sqrt{3}\,\ln(t)\right)}{t}+\frac{c_2\sin\left(2\sqrt{3}\,\ln(t)\right)}{t}$

□ Check validity of solution $x=\frac{c_1\cos\left(2\sqrt{3}\,\ln(t)\right)}{t}+\frac{c_2\sin\left(2\sqrt{3}\,\ln(t)\right)}{t}$

  - Use initial condition $x(1)=-1$

    $-1=c_1$

  - Compute derivative of the solution

    $x'=-\frac{c_1\cos\left(2\sqrt{3}\,\ln(t)\right)}{t^2}-\frac{2c_1\sqrt{3}\,\sin\left(2\sqrt{3}\,\ln(t)\right)}{t^2}-\frac{c_2\sin\left(2\sqrt{3}\,\ln(t)\right)}{t^2}+\frac{2c_2\sqrt{3}\,\cos\left(2\sqrt{3}\,\ln(t)\right)}{t^2}$

  - Use the initial condition $x'\Big|_{\{t=1\}}=2$

$$2 = 2c_2\sqrt{3} - c_1$$

- Solve for c_1 and c_2

$$\left\{c_1 = -1, c_2 = \frac{\sqrt{3}}{6}\right\}$$

- Substitute constant values into general solution and simplify

$$x = \frac{\sin\left(2\sqrt{3}\,\ln(t)\right)\sqrt{3} - 6\cos\left(2\sqrt{3}\,\ln(t)\right)}{6t}$$

- Solution to the IVP

$$x = \frac{\sin\left(2\sqrt{3}\,\ln(t)\right)\sqrt{3} - 6\cos\left(2\sqrt{3}\,\ln(t)\right)}{6t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
dsolve([t^2*diff(x(t),t$2)+3*t*diff(x(t),t)+13*x(t)=0,x(1) = -1, D(x)(1) = 2],x(t), singsol=
```

$$x(t) = \frac{\sqrt{3}\,\sin\left(2\sqrt{3}\,\ln(t)\right) - 6\cos\left(2\sqrt{3}\,\ln(t)\right)}{6t}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 41

```
DSolve[{t^2*x''[t]+3*t*x'[t]+13*x[t]==0,{x[1]==-1,x'[1]==2}},x[t],t,IncludeSingularSolutions
```

$$x(t) \to \frac{\sqrt{3}\sin\left(2\sqrt{3}\log(t)\right) - 6\cos\left(2\sqrt{3}\log(t)\right)}{6t}$$

12.11 problem 19.2

Internal problem ID [12070]
Internal file name [`OUTPUT/10722_Monday_September_11_2023_12_50_02_AM_59016790/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 19, CauchyEuler equations. Exercises page 174
Problem number: 19.2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{ay'' + (b-a)\,y' + cy = 0}$$

12.11.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(z) + By'(z) + Cy(z) = 0$$

Where in the above $A = a, B = b - a, C = c$. Let the solution be $y = e^{\lambda z}$. Substituting this into the ODE gives

$$a\,\lambda^2 \mathrm{e}^{\lambda z} + (b-a)\,\lambda\,\mathrm{e}^{\lambda z} + c\,\mathrm{e}^{\lambda z} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda z}$ gives

$$a\,\lambda^2 + (b-a)\,\lambda + c = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = a, B = b - a, C = c$ into the above gives

$$\lambda_{1,2} = \frac{a-b}{(2)(a)} \pm \frac{1}{(2)(a)}\sqrt{b-a^2-(4)(a)(c)}$$
$$= -\frac{b-a}{2a} \pm \frac{\sqrt{(b-a)^2-4ac}}{2a}$$

Hence

$$\lambda_1 = -\frac{b-a}{2a} + \frac{\sqrt{(b-a)^2-4ac}}{2a}$$
$$\lambda_2 = -\frac{b-a}{2a} - \frac{\sqrt{(b-a)^2-4ac}}{2a}$$

Which simplifies to

$$\lambda_1 = \frac{a-b+\sqrt{a^2+(-2b-4c)a+b^2}}{2a}$$
$$\lambda_2 = \frac{a-b-\sqrt{a^2+(-2b-4c)a+b^2}}{2a}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z}$$
$$y = c_1 e^{\left(\frac{a-b+\sqrt{a^2+(-2b-4c)a+b^2}}{2a}\right)z} + c_2 e^{\left(\frac{a-b-\sqrt{a^2+(-2b-4c)a+b^2}}{2a}\right)z}$$

Or

$$y = c_1 \mathrm{e}^{\frac{\left(a-b+\sqrt{a^2+(-2b-4c)a+b^2}\right)z}{2a}} + c_2 \mathrm{e}^{\frac{\left(a-b-\sqrt{a^2+(-2b-4c)a+b^2}\right)z}{2a}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{\left(a-b+\sqrt{a^2+(-2b-4c)a+b^2}\right)z}{2a}} + c_2 \mathrm{e}^{\frac{\left(a-b-\sqrt{a^2+(-2b-4c)a+b^2}\right)z}{2a}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{\left(a-b+\sqrt{a^2+(-2b-4c)a+b^2}\right)z}{2a}} + c_2 \mathrm{e}^{\frac{\left(a-b-\sqrt{a^2+(-2b-4c)a+b^2}\right)z}{2a}}$$

Verified OK.

12.11.2 Solving using Kovacic algorithm

Writing the ode as

$$ay'' + (b - a)\,y' + cy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= a \\ B &= b - a \\ C &= c \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2 - 2ab - 4ac + b^2}{4a^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2 - 2ab - 4ac + b^2 \\ t &= 4a^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{a^2 - 2ab - 4ac + b^2}{4a^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\,e^{-\int \frac{B}{2A}\,dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 177: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{a^2-2ab-4ac+b^2}{4a^2}$ is not a function of z, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(z) = \mathrm{e}^{\frac{\sqrt{\frac{a^2-2ab-4ac+b^2}{a^2}}\, z}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\
&= z_1 e^{-\int \frac{1}{2}\frac{b-a}{a}\,dz} \\
&= z_1 e^{-\frac{(b-a)z}{2a}} \\
&= z_1 \left(\mathrm{e}^{\frac{(a-b)z}{2a}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{z\left(\sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}\,a+a-b\right)}{2a}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{b-a}{a}\,dz}}{(y_1)^2}\,dz \\
&= y_1 \int \frac{e^{\frac{(a-b)z}{a}}}{(y_1)^2}\,dz \\
&= y_1 \left(-\frac{a^2 \mathrm{e}^{-\sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}\,z}\sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}}{a^2+(-2b-4c)\,a+b^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{z\left(\sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}\,a+a-b\right)}{2a}}\right) \\
&\quad + c_2 \left(\mathrm{e}^{\frac{z\left(\sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}\,a+a-b\right)}{2a}} \left(-\frac{a^2 \mathrm{e}^{-\sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}\,z}\sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}}{a^2+(-2b-4c)\,a+b^2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{z\left(\sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}\, a+a-b\right)}{2a}} - \frac{c_2 a^2 \mathrm{e}^{-\frac{z\left(\sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}\, a-a+b\right)}{2a}} \sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}}{a^2+(-2b-4c)\, a+b^2} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{z\left(\sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}\, a+a-b\right)}{2a}} - \frac{c_2 a^2 \mathrm{e}^{-\frac{z\left(\sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}\, a-a+b\right)}{2a}} \sqrt{\frac{a^2+(-2b-4c)a+b^2}{a^2}}}{a^2+(-2b-4c)\, a+b^2}$$

Verified OK.

12.11.3 Maple step by step solution

Let's solve

$a y'' + (b-a)\, y' + c y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{c y}{a} + \frac{(a-b) y'}{a}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(a-b) y'}{a} + \frac{c y}{a} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{(a-b) r}{a} + \frac{c}{a} = 0$

- Factor the characteristic polynomial

  $\frac{r^2 a - r a + r b + c}{a} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{a-b+\sqrt{a^2-2ab-4ac+b^2}}{2a}, -\frac{-a+b+\sqrt{a^2-2ab-4ac+b^2}}{2a}\right)$

- 1st solution of the ODE

  $y_1(z) = \mathrm{e}^{\frac{\left(a-b+\sqrt{a^2-2ab-4ac+b^2}\right) z}{2a}}$

- 2nd solution of the ODE

$$y_2(z) = \mathrm{e}^{-\frac{\left(-a+b+\sqrt{a^2-2ab-4ac+b^2}\right)z}{2a}}$$

- General solution of the ODE

$$y = c_1 y_1(z) + c_2 y_2(z)$$

- Substitute in solutions

$$y = c_1 \mathrm{e}^{\frac{\left(a-b+\sqrt{a^2-2ab-4ac+b^2}\right)z}{2a}} + c_2 \mathrm{e}^{-\frac{\left(-a+b+\sqrt{a^2-2ab-4ac+b^2}\right)z}{2a}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 69

```
dsolve(a*diff(y(z),z$2)+(b-a)*diff(y(z),z)+c*y(z)=0,y(z), singsol=all)
```

$$y(z) = c_1 \mathrm{e}^{\frac{\left(a-b+\sqrt{a^2+(-2b-4c)a+b^2}\right)z}{2a}} + \mathrm{e}^{-\frac{\left(-a+b+\sqrt{a^2+(-2b-4c)a+b^2}\right)z}{2a}} c_2$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 72

```
DSolve[a*y''[z]+(b-a)*y'[z]+c*y[z]==0,y[z],z,IncludeSingularSolutions -> True]
```

$$y(z) \to \left(c_2 e^{\frac{z\sqrt{a^2-2a(b+2c)+b^2}}{a}} + c_1\right) \exp\left(-\frac{z\left(\sqrt{a^2-2a(b+2c)+b^2}-a+b\right)}{2a}\right)$$

13 Chapter 20, Series solutions of second order linear equations. Exercises page 195

13.1 problem 20.1

Internal problem ID [12071]
Internal file name [OUTPUT/10723_Monday_September_11_2023_12_50_05_AM_96468558/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 20, Series solutions of second order linear equations. Exercises page 195
Problem number: 20.1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Gegenbauer]
```

$$\boxed{\left(-x^2+1\right) y''-2y'x+n(n+1)\, y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2} y''(x_0)+\frac{(x-x_0)^3}{3!} y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2} f|_{x_0,y_0,y_0'}+\frac{x^3}{3!} f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (255)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (256)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{yn^2 + yn - 2y'x}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(n^2x^2 + n\,x^2 - n^2 + 6x^2 - n + 2)\, y' - 4ynx(n+1)}{(x^2-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(x+1)\,(x-1)\,(-8((n^2+n+3)\,x^2 - n^2 - n + 3)\,xy' + yn((n^2+n+18)\,x^2 - n^2 - n + 6)\,(n+1))}{(x^2-1)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x+1)\,(((n^4 + 2n^3 + 59n^2 + 58n + 120)\,x^4 + (-2n^4 - 4n^3 - 46n^2 - 44n + 240)\,x^2 + n^4 + 2n^3 - 1}{(x^2-1)^5} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(-18x\left(\left(n^4 + 2n^3 + \frac{77}{3}n^2 + \frac{74}{3}n + 40\right)x^4 - 2\left(n^2 + n - \frac{20}{3}\right)(n^2 + n + 10)\,x^2 + n^4 + 2n^3 - 17n^2 - 1\right.\right.}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)\, n(n+1) \\
F_1 &= -y'(0)\, n^2 - y'(0)\, n + 2y'(0) \\
F_2 &= y(0)\, n^4 + 2y(0)\, n^3 - 5y(0)\, n^2 - 6y(0)\, n \\
F_3 &= y'(0)\, n^4 + 2y'(0)\, n^3 - 13y'(0)\, n^2 - 14y'(0)\, n + 24y'(0) \\
F_4 &= -y(0)\, n^6 - 3y(0)\, n^5 + 23y(0)\, n^4 + 51y(0)\, n^3 - 94y(0)\, n^2 - 120y(0)\, n
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1-\frac{1}{2}n^{2} x^{2}-\frac{1}{2}n\,x^{2}+\frac{1}{24}n^{4} x^{4}+\frac{1}{12}x^{4} n^{3}-\frac{5}{24}n^{2} x^{4}-\frac{1}{4}n\,x^{4}-\frac{1}{720}x^{6} n^{6}-\frac{1}{240}n^{5} x^{6} + \frac{23}{720}n^{4} x^{6}+\frac{17}{240}n^{3} x^{6}-\frac{47}{360}n^{2} x^{6}-\frac{1}{6}x^{6} n\right) y(0) + \left(x-\frac{1}{6}n^{2} x^{3}-\frac{1}{6}n\,x^{3}+\frac{1}{3}x^{3}+\frac{1}{120}x^{5} n^{4}+\frac{1}{60}x^{5} n^{3}-\frac{13}{120}x^{5} n^{2}-\frac{7}{60}x^{5} n+\frac{1}{5}x^{5}\right) y^{\prime}(0) + O\left(x^{6}\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-x^{2}+1\right) y^{\prime \prime}-2 y^{\prime} x+\left(n^{2}+n \right) y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n =0}^{\infty} a_{n} x^{n}$$

Then

$$y^{\prime} = \sum_{n =1}^{\infty} n a_{n} x^{n -1}$$

$$y^{\prime \prime} = \sum_{n =2}^{\infty} n \left(n -1\right) a_{n} x^{n -2}$$

Substituting the above back into the ode gives

$$\left(-x^{2}+1\right) \left(\sum_{n =2}^{\infty} n \left(n -1\right) a_{n} x^{n -2}\right)-2\left(\sum_{n =1}^{\infty} n a_{n} x^{n -1}\right) x +\left(n^{2}+n \right) \left(\sum_{n =0}^{\infty} a_{n} x^{n}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n =2}^{\infty}\left(-x^{n} a_{n} n \left(n -1\right)\right)+\left(\sum_{n =2}^{\infty} n \left(n -1\right) a_{n} x^{n -2}\right) + \sum_{n =1}^{\infty}\left(-2 n a_{n} x^{n}\right)+\left(\sum_{n =0}^{\infty} \left(n^{2}+n \right) a_{n} x^{n}\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-2n a_n x^n\right) + \left(\sum_{n=0}^{\infty} \left(n^2+n\right) a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 n(n+1) = 0$$

$$a_2 = -\frac{1}{2}a_0 n^2 - \frac{1}{2}a_0 n$$

$n = 1$ gives

$$6a_3 - 2a_1 + a_1 n(n+1) = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{1}{6}a_1 n^2 - \frac{1}{6}a_1 n + \frac{1}{3}a_1$$

For $2 \leq n$, the recurrence equation is

$$-n a_n(n-1) + (n+2)\, a_{n+2}(n+1) - 2n a_n + a_n n(n+1) = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - n^2 + n - n)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-6a_2 + 12a_4 + a_2 n(n+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5}{24}a_0 n^2 - \frac{1}{4}a_0 n + \frac{1}{24}a_0 n^4 + \frac{1}{12}a_0 n^3$$

For $n = 3$ the recurrence equation gives

$$-12a_3 + 20a_5 + a_3 n(n+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{13}{120}a_1 n^2 - \frac{7}{60}a_1 n + \frac{1}{5}a_1 + \frac{1}{120}a_1 n^4 + \frac{1}{60}a_1 n^3$$

For $n = 4$ the recurrence equation gives

$$-20a_4 + 30a_6 + a_4 n(n+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{47}{360}a_0 n^2 - \frac{1}{6}a_0 n + \frac{23}{720}a_0 n^4 + \frac{17}{240}a_0 n^3 - \frac{1}{720}a_0 n^6 - \frac{1}{240}a_0 n^5$$

For $n = 5$ the recurrence equation gives

$$-30a_5 + 42a_7 + a_5 n(n+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{5}{63}a_1 n^2 - \frac{37}{420}a_1 n + \frac{1}{7}a_1 + \frac{41}{5040}a_1 n^4 + \frac{29}{1680}a_1 n^3 - \frac{1}{5040}a_1 n^6 - \frac{1}{1680}a_1 n^5$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y = {} & a_0 + a_1 x + \left(-\frac{1}{2} a_0 n^2 - \frac{1}{2} a_0 n\right) x^2 + \left(-\frac{1}{6} a_1 n^2 - \frac{1}{6} a_1 n + \frac{1}{3} a_1\right) x^3 \\
& + \left(-\frac{5}{24} a_0 n^2 - \frac{1}{4} a_0 n + \frac{1}{24} a_0 n^4 + \frac{1}{12} a_0 n^3\right) x^4 \\
& + \left(-\frac{13}{120} a_1 n^2 - \frac{7}{60} a_1 n + \frac{1}{5} a_1 + \frac{1}{120} a_1 n^4 + \frac{1}{60} a_1 n^3\right) x^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = {} & \left(1 + \left(-\frac{1}{2} n^2 - \frac{1}{2} n\right) x^2 + \left(-\frac{5}{24} n^2 - \frac{1}{4} n + \frac{1}{24} n^4 + \frac{1}{12} n^3\right) x^4\right) a_0 \\
& + \left(x + \left(-\frac{1}{6} n^2 - \frac{1}{6} n + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120} n^2 - \frac{7}{60} n + \frac{1}{5} + \frac{1}{120} n^4 + \frac{1}{60} n^3\right) x^5\right) a_1 + O(x^6)
\end{aligned} \tag{3}
$$

At $x = 0$ the solution above becomes

$$
\begin{aligned}
y = {} & \left(1 + \left(-\frac{1}{2} n^2 - \frac{1}{2} n\right) x^2 + \left(-\frac{5}{24} n^2 - \frac{1}{4} n + \frac{1}{24} n^4 + \frac{1}{12} n^3\right) x^4\right) c_1 \\
& + \left(x + \left(-\frac{1}{6} n^2 - \frac{1}{6} n + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120} n^2 - \frac{7}{60} n + \frac{1}{5} + \frac{1}{120} n^4 + \frac{1}{60} n^3\right) x^5\right) c_2 + O(x^6)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1 - \frac{1}{2} n^2 x^2 - \frac{1}{2} n\, x^2 + \frac{1}{24} n^4 x^4 + \frac{1}{12} x^4 n^3 - \frac{5}{24} n^2 x^4 - \frac{1}{4} n\, x^4 - \frac{1}{720} x^6 n^6 - \frac{1}{240} n^5 x^6 \right. \\
& \left. + \frac{23}{720} n^4 x^6 + \frac{17}{240} n^3 x^6 - \frac{47}{360} n^2 x^6 - \frac{1}{6} x^6 n\right) y(0) + \left(x - \frac{1}{6} n^2 x^3 - \frac{1}{6} n\, x^3 + \frac{1}{3} x^3 \right. \\
& \left. + \frac{1}{120} x^5 n^4 + \frac{1}{60} x^5 n^3 - \frac{13}{120} x^5 n^2 - \frac{7}{60} x^5 n + \frac{1}{5} x^5\right) y'(0) + O(x^6)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
y = {} & \left(1 + \left(-\frac{1}{2} n^2 - \frac{1}{2} n\right) x^2 + \left(-\frac{5}{24} n^2 - \frac{1}{4} n + \frac{1}{24} n^4 + \frac{1}{12} n^3\right) x^4\right) c_1 \\
& + \left(x + \left(-\frac{1}{6} n^2 - \frac{1}{6} n + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120} n^2 - \frac{7}{60} n + \frac{1}{5} + \frac{1}{120} n^4 + \frac{1}{60} n^3\right) x^5\right) c_2 \\
& + O(x^6)
\end{aligned} \tag{2}
$$

Verification of solutions

$$y = \left(1-\frac{1}{2}n^2x^2-\frac{1}{2}n\,x^2+\frac{1}{24}n^4x^4+\frac{1}{12}x^4n^3-\frac{5}{24}n^2x^4-\frac{1}{4}n\,x^4-\frac{1}{720}x^6n^6-\frac{1}{240}n^5x^6 +\frac{23}{720}n^4x^6+\frac{17}{240}n^3x^6-\frac{47}{360}n^2x^6-\frac{1}{6}x^6n\right)y(0) +\left(x-\frac{1}{6}n^2x^3-\frac{1}{6}n\,x^3+\frac{1}{3}x^3+\frac{1}{120}x^5n^4+\frac{1}{60}x^5n^3-\frac{13}{120}x^5n^2-\frac{7}{60}x^5n+\frac{1}{5}x^5\right)y'(0) +O(x^6)$$

Verified OK.

$$y = \left(1+\left(-\frac{1}{2}n^2-\frac{1}{2}n\right)x^2+\left(-\frac{5}{24}n^2-\frac{1}{4}n+\frac{1}{24}n^4+\frac{1}{12}n^3\right)x^4\right)c_1 +\left(x+\left(-\frac{1}{6}n^2-\frac{1}{6}n+\frac{1}{3}\right)x^3+\left(-\frac{13}{120}n^2-\frac{7}{60}n+\frac{1}{5}+\frac{1}{120}n^4+\frac{1}{60}n^3\right)x^5\right)c_2+O(x^6)$$

Verified OK.

13.1.1 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-2y'x+(n^2+n)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{n(n+1)y}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^2-1}-\frac{n(n+1)y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{n(n+1)}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((x+1)\cdot P_2(x))\right|_{x=-1}=1$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''(x^2-1) + 2y'x - n(n+1)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) + (-n^2-n)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2 + a_k(r+1+n+k)\,(r-n+k)\right)u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(1+n+k)(-n+k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 101

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+n*(n+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{n(n+1)\,x^2}{2}+\frac{n(n^3+2n^2-5n-6)\,x^4}{24}\right)y(0) + \left(x-\frac{(n^2+n-2)\,x^3}{6}+\frac{(n^4+2n^3-13n^2-14n+24)\,x^5}{120}\right)D(y)\,(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 120

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-2*x*y'[x]+n*(n+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{1}{120}(n^2+n)^2x^5+\frac{7}{60}(-n^2-n)\,x^5+\frac{1}{6}(-n^2-n)\,x^3+\frac{x^5}{5}+\frac{x^3}{3}+x\right) + c_1\left(\frac{1}{24}(n^2+n)^2x^4+\frac{1}{4}(-n^2-n)\,x^4+\frac{1}{2}(-n^2-n)\,x^2+1\right)$$

13.2 problem 20.2 (i)

Internal problem ID [12072]
Internal file name [`OUTPUT/10724_Monday_September_11_2023_12_50_06_AM_90167203/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 20, Series solutions of second order linear equations. Exercises page 195
Problem number: 20.2 (i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Hermite]`

$$\boxed{y'' - y'x + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (258) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (259) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= y'x - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= (y'x - y)\, x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^2+1\right)(y'x - y) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x\left(x^2+3\right)(y'x - y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= (y'x - y)\left(x^4 + 6x^2 + 3\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= 0 \\
F_2 &= -y(0) \\
F_3 &= 0 \\
F_4 &= -3y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{240}x^6\right) y(0) + y'(0)\, x + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n (n-1)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 0$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{24} a_0 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) a_0 + a_1 x + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) c_1 + c_2 x + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{240}x^6\right) y(0) + y'(0)\, x + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) c_1 + c_2 x + O(x^6) \tag{2}$$

<u>Verification of solutions</u>

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{240}x^6\right) y(0) + y'(0)\, x + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) c_1 + c_2 x + O(x^6)$$

Verified OK.

13.2.1 Maple step by step solution

Let's solve

$y'' = y'x - y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y'x + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k-1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - a_k(k-1) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k(k-1)}{k^2+3k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) y(0) + x D(y)\left(0\right) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 27

```
AsymptoticDSolveValue[y''[x]-x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^4}{24} - \frac{x^2}{2} + 1\right) + c_2 x$$

13.3 problem 20.2 (ii)

Internal problem ID [12073]
Internal file name [`OUTPUT/10725_Monday_September_11_2023_12_50_06_AM_88369984/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 20, Series solutions of second order linear equations. Exercises page 195
Problem number: 20.2 (ii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(x^2+1)\,y''+y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (261) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (262) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n \big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-x^2 y' + 2yx - y'}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{4y'x^3 - 5x^2 y + 4y'x + 3y}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(-17x^4 - 10x^2 + 7)\, y' + 16xy(x^2-2)}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(84x^5 - 24x^3 - 108x)\, y' + (-63x^4 + 282x^2 - 39)\, y}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= 3y(0) \\
F_3 &= 7y'(0) \\
F_4 &= -39y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 1)\, y'' + y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\,a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - n + 1)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$3a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$13a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{13a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$21a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{7a_1}{240}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{6} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{7}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{2}x^{2}+\frac{1}{8}x^{4}-\frac{13}{240}x^{6}\right)y(0)+\left(x-\frac{1}{6}x^{3}+\frac{7}{120}x^{5}\right)y'(0)+O(x^{6})$$

Verified OK.

$$y = \left(1-\frac{1}{2}x^{2}+\frac{1}{8}x^{4}\right)c_{1}+\left(x-\frac{1}{6}x^{3}+\frac{7}{120}x^{5}\right)c_{2}+O(x^{6})$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[(1+x^2)*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{7x^5}{120} - \frac{x^3}{6} + x\right) + c_1\left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right)$$

13.4 problem 20.2 (iii)

Internal problem ID [12074]
Internal file name [`OUTPUT/10726_Monday_September_11_2023_12_50_06_AM_57487830/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 20, Series solutions of second order linear equations. Exercises page 195
Problem number: 20.2 (iii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{2xy'' + y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + y' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = -\frac{1}{x}$$

Table 181: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + r\,x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_n(n+r) - 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{2a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{2r^2+3r+1}$	$\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{2}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{2r^2+3r+1}$	$\frac{2}{3}$
a_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{15}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{4}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{2r^2+3r+1}$	$\frac{2}{3}$
a_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{15}$
a_3	$\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{4}{315}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{2}{2835}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{2r^2+3r+1}$	$\frac{2}{3}$
a_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{15}$
a_3	$\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{4}{315}$
a_4	$\frac{16}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{2}{2835}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r^2 +}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{4}{155925}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{2r^2+3r+1}$	$\frac{2}{3}$
a_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{15}$
a_3	$\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{4}{315}$
a_4	$\frac{16}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{2}{2835}$
a_5	$\frac{32}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$\frac{4}{155925}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{2x}{3} + \frac{2x^2}{15} + \frac{4x^3}{315} + \frac{2x^4}{2835} + \frac{4x^5}{155925} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + (n+r)\, b_n - 2b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{2b_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{2b_{n-1}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{2r^2+3r+1}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{2r^2+3r+1}$	2
b_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{4}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{2r^2+3r+1}$	2
b_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{3}$
b_3	$\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{4}{45}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{2}{315}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{2r^2+3r+1}$	2
b_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{3}$
b_3	$\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{4}{45}$
b_4	$\frac{16}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{2}{315}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r^2 + \cdots}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{4}{14175}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{2r^2+3r+1}$	2
b_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{3}$
b_3	$\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{4}{45}$
b_4	$\frac{16}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{2}{315}$
b_5	$\frac{32}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$\frac{4}{14175}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + 2x + \frac{2x^2}{3} + \frac{4x^3}{45} + \frac{2x^4}{315} + \frac{4x^5}{14175} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x} \left(1 + \frac{2x}{3} + \frac{2x^2}{15} + \frac{4x^3}{315} + \frac{2x^4}{2835} + \frac{4x^5}{155925} + O(x^6)\right) \\ &\quad + c_2 \left(1 + 2x + \frac{2x^2}{3} + \frac{4x^3}{45} + \frac{2x^4}{315} + \frac{4x^5}{14175} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1+\frac{2x}{3}+\frac{2x^2}{15}+\frac{4x^3}{315}+\frac{2x^4}{2835}+\frac{4x^5}{155925}+O\left(x^6\right)\right) \\
&\quad + c_2\left(1+2x+\frac{2x^2}{3}+\frac{4x^3}{45}+\frac{2x^4}{315}+\frac{4x^5}{14175}+O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1\sqrt{x}\left(1+\frac{2x}{3}+\frac{2x^2}{15}+\frac{4x^3}{315}+\frac{2x^4}{2835}+\frac{4x^5}{155925}+O\left(x^6\right)\right) \\
+ c_2\left(1+2x+\frac{2x^2}{3}+\frac{4x^3}{45}+\frac{2x^4}{315}+\frac{4x^5}{14175}+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1\sqrt{x}\left(1+\frac{2x}{3}+\frac{2x^2}{15}+\frac{4x^3}{315}+\frac{2x^4}{2835}+\frac{4x^5}{155925}+O\left(x^6\right)\right) \\
+ c_2\left(1+2x+\frac{2x^2}{3}+\frac{4x^3}{45}+\frac{2x^4}{315}+\frac{4x^5}{14175}+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

13.4.1 Maple step by step solution

Let's solve

$2y''x+y'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y'}{2x}+\frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{2x}-\frac{y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{1}{2x}, P_3(x)=-\frac{1}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $$a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) - 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1}-2a_{k}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_{k}}{(2k+1+2r)(k+1+r)}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{2a_{k}}{(2k+1)(k+1)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k}, a_{k+1}=\frac{2a_{k}}{(2k+1)(k+1)}\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{2a_{k}}{(2k+2)\left(k+\frac{3}{2}\right)}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{2}}, a_{k+1}=\frac{2a_{k}}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_{k} x^{k}\right)+\left(\sum_{k=0}^{\infty} b_{k} x^{k+\frac{1}{2}}\right), a_{1+k}=\frac{2a_{k}}{(2k+1)(1+k)}, b_{1+k}=\frac{2b_{k}}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 44

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)+diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{2}{3}x+\frac{2}{15}x^2+\frac{4}{315}x^3+\frac{2}{2835}x^4+\frac{4}{155925}x^5+\mathrm{O}\left(x^6\right)\right) + c_2\left(1+2x+\frac{2}{3}x^2+\frac{4}{45}x^3+\frac{2}{315}x^4+\frac{4}{14175}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 83

```
AsymptoticDSolveValue[2*x*y''[x]+y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{4x^5}{155925}+\frac{2x^4}{2835}+\frac{4x^3}{315}+\frac{2x^2}{15}+\frac{2x}{3}+1\right) + c_2\left(\frac{4x^5}{14175}+\frac{2x^4}{315}+\frac{4x^3}{45}+\frac{2x^2}{3}+2x+1\right)$$

13.5 problem 20.2 (iv) (k=-2)

Internal problem ID [12075]
Internal file name [OUTPUT/10727_Monday_September_11_2023_12_50_07_AM_16624974/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 20, Series solutions of second order linear equations. Exercises page 195
Problem number: 20.2 (iv) (k=-2).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y'x - 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (265)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (266)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2y'x + 4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4x^2 y' + 8yx + 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 4(2x^3 + 7x)\, y' + 16y(x^2 + 2) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= (16x^4 + 96x^2 + 60)\, y' + (32x^3 + 144x)\, y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= (32x^5 + 288x^3 + 456x)\, y' + 64y\left(x^4 + \frac{15}{2}x^2 + 6\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 4y(0) \\
F_1 &= 6y'(0) \\
F_2 &= 32y(0) \\
F_3 &= 60y'(0) \\
F_4 &= 384y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + 2x^2 + \frac{4}{3}x^4 + \frac{8}{15}x^6\right) y(0) + \left(x + x^3 + \frac{1}{2}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 4a_0 = 0$$

$$a_2 = 2a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2n a_n - 4 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2a_n}{n+1} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 6a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = a_1$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 8a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{4a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 10a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{2}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{8a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 14a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{6}$$

And so on. Therefore the solution is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 2a_0 x^2 + a_1 x^3 + \frac{4}{3} a_0 x^4 + \frac{1}{2} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 2x^2 + \frac{4}{3}x^4\right) a_0 + \left(x + x^3 + \frac{1}{2}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + 2x^2 + \frac{4}{3}x^4\right) c_1 + \left(x + x^3 + \frac{1}{2}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + 2x^2 + \frac{4}{3}x^4 + \frac{8}{15}x^6\right) y(0) + \left(x + x^3 + \frac{1}{2}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + 2x^2 + \frac{4}{3}x^4\right) c_1 + \left(x + x^3 + \frac{1}{2}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + 2x^2 + \frac{4}{3}x^4 + \frac{8}{15}x^6\right) y(0) + \left(x + x^3 + \frac{1}{2}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + 2x^2 + \frac{4}{3}x^4\right) c_1 + \left(x + x^3 + \frac{1}{2}x^5\right) c_2 + O(x^6)$$

Verified OK.

13.5.1 Maple step by step solution

Let's solve

$y'' = 2y'x + 4y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2y'x - 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$

  - Shift index using $k- > k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) - 2a_k(k+2)) \, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)(ka_{k+2} - 2a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
Order:=6;
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + 2x^2 + \frac{4}{3}x^4\right) y(0) + \left(x + x^3 + \frac{1}{2}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[y''[x]-2*x*y'[x]-4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{2} + x^3 + x\right) + c_1\left(\frac{4x^4}{3} + 2x^2 + 1\right)$$

13.6 problem 20.2 (iv) (k=2)

Internal problem ID [12076]
Internal file name [`OUTPUT/10728_Monday_September_11_2023_12_50_08_AM_72086316/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 20, Series solutions of second order linear equations. Exercises page 195
Problem number: 20.2 (iv) (k=2).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y'x + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (268)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (269)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= 2y'x-4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= 4x^2y'-8yx-2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \left(8x^3-4x\right)y'-16x^2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= 16x^4y'-32yx^3-16yx-4y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= 32\left(\left(x^2-\frac{1}{2}\right)y'-2yx\right)x\left(x^2+\frac{3}{2}\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -4y(0) \\
F_1 &= -2y'(0) \\
F_2 &= 0 \\
F_3 &= -4y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y=\left(-2x^2+1\right)y(0)+\left(x-\frac{1}{3}x^3-\frac{1}{30}x^5\right)y'(0)+O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 4\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{3}$$

$n=0$ gives

$$2a_2 + 4a_0 = 0$$

$$a_2 = -2a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2n a_n + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2a_n(n-2)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{30}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{210}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 2a_0 x^2 - \frac{1}{3} a_1 x^3 - \frac{1}{30} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(-2x^2 + 1\right) a_0 + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(-2x^2 + 1\right) c_1 + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(-2x^2 + 1\right) y(0) + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(-2x^2 + 1\right) c_1 + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(-2x^2 + 1\right) y(0) + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) y'(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(-2x^2 + 1\right) c_1 + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) c_2 + O\left(x^6\right)$$

Verified OK.

13.6.1 Maple step by step solution

Let's solve

$y'' = 2y'x - 4y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2y'x + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(k-2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - 2a_k(k-2) = 0$

- Recursion relation; series terminates at $k = 2$

  $a_{k+2} = \frac{2a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

  $a_2 = -2a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y = A_2x^2 + A_1x - 2a_0$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
Order:=6;
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (-2x^2+1)\,y(0) + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 33

```
AsymptoticDSolveValue[y''[x]-2*x*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1(1-2x^2) + c_2\left(-\frac{x^5}{30} - \frac{x^3}{3} + x\right)$$

13.7 problem 20.3

Internal problem ID [12077]
Internal file name [`OUTPUT/10729_Monday_September_11_2023_12_50_08_AM_75123953/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 20, Series solutions of second order linear equations. Exercises page 195
Problem number: 20.3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(1-x)\,y''-3y'x-y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^2+x\right)y''-3y'x-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{x-1}$$
$$q(x)=\frac{1}{x\,(x-1)}$$

Table 185: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x-1}$	
singularity	type
$x = 1$	"regular"

$q(x) = \frac{1}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-1) - 3y'x - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x(x-1) - 3\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-a_{n-1}(n+r-1)(n+r-2)+a_n(n+r)(n+r-1)-3a_{n-1}(n+r-1)-a_{n-1}=0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r)\, a_{n-1}}{n+r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(n+1)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1+r}{r}$$

Which for the root $r = 1$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2+r}{r}$$

Which for the root $r = 1$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{3+r}{r}$$

Which for the root $r = 1$ becomes

$$a_3 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4+r}{r}$$

Which for the root $r = 1$ becomes

$$a_4 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4
a_4	$\frac{4+r}{r}$	5

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{5+r}{r}$$

Which for the root $r = 1$ becomes

$$a_5 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4
a_4	$\frac{4+r}{r}$	5
a_5	$\frac{5+r}{r}$	6

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{1+r}{r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1+r}{r} &= \lim_{r \to 0} \frac{1+r}{r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $-y''x(x-1) - 3y'x - y = 0$ gives

$$\begin{aligned}
&-\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x(x-1) \\
&- 3\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) x \\
&- Cy_1(x)\ln(x) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

Which can be written as

$$
\begin{aligned}
&\left(\left(-y_1''(x)\, x(x-1) - 3y_1'(x)\, x - y_1(x)\right) \ln(x) - \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x(x-1) \right. \\
&\left. - 3y_1(x)\right) C - \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x(x-1) \\
&- 3\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) x - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
-y_1''(x)\, x(x-1) - 3y_1'(x)\, x - y_1(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(-\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x(x-1) - 3y_1(x)\right) C \\
&- \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x(x-1) \\
&- 3\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) x - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\frac{\left(-2x(x-1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) + (-2x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} \\
&+ \frac{(-x^3+x^2)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) - 3\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x^2 - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} \\
&= 0
\end{aligned}
\tag{9}
$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(-2x(x-1)\left(\sum_{n=0}^{\infty} x^n a_n(n+1)\right) + (-2x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} + \frac{(-x^3+x^2)\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) - 3\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x^2 - \left(\sum_{n=0}^{\infty} b_n x^n\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{aligned}\sum_{n=0}^{\infty} \left(-2C\, x^{n+1} a_n(n+1)\right) + \left(\sum_{n=0}^{\infty} 2C\, x^n a_n(n+1)\right) \\ + \sum_{n=0}^{\infty} \left(-2C\, x^{n+1} a_n\right) + \sum_{n=0}^{\infty} \left(-C a_n x^n\right) + \sum_{n=0}^{\infty} \left(-x^n b_n n(n-1)\right) \\ + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n(n-1)\right) + \sum_{n=0}^{\infty} \left(-3x^n b_n n\right) + \sum_{n=0}^{\infty} \left(-b_n x^n\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=0}^{\infty} \left(-2C\, x^{n+1} a_n(n+1)\right) &= \sum_{n=2}^{\infty} \left(-2C a_{-2+n}(n-1)\, x^{n-1}\right) \\ \sum_{n=0}^{\infty} 2C\, x^n a_n(n+1) &= \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1} \\ \sum_{n=0}^{\infty} \left(-2C\, x^{n+1} a_n\right) &= \sum_{n=2}^{\infty} \left(-2C a_{-2+n} x^{n-1}\right) \\ \sum_{n=0}^{\infty} \left(-C a_n x^n\right) &= \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) \\ \sum_{n=0}^{\infty} \left(-x^n b_n n(n-1)\right) &= \sum_{n=1}^{\infty} \left(-(n-1)\, b_{n-1}(-2+n)\, x^{n-1}\right)\end{aligned}$$

$$\sum_{n=0}^{\infty}\left(-3x^{n} b_{n} n\right) = \sum_{n=1}^{\infty}\left(-3(n-1) b_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-b_{n} x^{n}\right) = \sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(-2C a_{-2+n}(n-1)\, x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right) \\
&+ \sum_{n=2}^{\infty}\left(-2C a_{-2+n} x^{n-1}\right) + \sum_{n=1}^{\infty}\left(-C a_{n-1} x^{n-1}\right) \\
&+ \sum_{n=1}^{\infty}\left(-(n-1)\, b_{n-1}(-2+n)\, x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_{n}(n-1)\right) \\
&+ \sum_{n=1}^{\infty}\left(-3(n-1)\, b_{n-1} x^{n-1}\right) + \sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C - 1 = 0$$

Which is solved for C. Solving for C gives

$$C = 1$$

For $n = 2$, Eq (2B) gives

$$\left(-4a_0 + 3a_1\right) C - 4b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 + 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -1$$

For $n = 3$, Eq (2B) gives

$$\left(-6a_1 + 5a_2\right) C - 9b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12 + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -2$$

For $n = 4$, Eq (2B) gives

$$(-8a_2 + 7a_3) C - 16b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$36 + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -3$$

For $n = 5$, Eq (2B) gives

$$(-10a_3 + 9a_4) C - 25b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$80 + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -4$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = {} & 1\left(x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O\left(x^6\right)\right)\right) \ln(x) \\ & + 1 - x^2 - 2x^3 - 3x^4 - 4x^5 + O\left(x^6\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O\left(x^6\right)\right) \\ &\quad + c_2\left(1\left(x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O\left(x^6\right)\right)\right) \ln(x) + 1 - x^2 - 2x^3 \right. \\ &\quad \left. - 3x^4 - 4x^5 + O\left(x^6\right)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O\left(x^6\right)\right) \\
&\quad + c_2\left(x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O\left(x^6\right)\right)\ln\left(x\right)+1-x^2-2x^3-3x^4-4x^5 \right. \\
&\qquad \left. +O\left(x^6\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O\left(x^6\right)\right) \\
&\quad + c_2\left(x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O\left(x^6\right)\right)\ln\left(x\right)+1-x^2-2x^3-3x^4 \right. \\
&\qquad \left. -4x^5+O\left(x^6\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O\left(x^6\right)\right) \\
&\quad + c_2\left(x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O\left(x^6\right)\right)\ln\left(x\right)+1-x^2-2x^3-3x^4-4x^5 \right. \\
&\qquad \left. +O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

13.7.1 Maple step by step solution

Let's solve

$-y''x(x-1)-3y'x-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x(x-1)}-\frac{3y'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{x-1}+\frac{y}{x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3}{x-1}, P_3(x)=\frac{1}{x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$y''x(x-1) + 3y'x + y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+r+1)\,(k+r) + a_k (k+r+1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r+1)\left(-a_{k+1}(k+r) + a_k(k+r+1)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+1)}{k+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+1)}{k}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(k+1)}{k}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k(k+2)}{k+1}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k(k+2)}{k+1}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{1+k}\right), a_{1+k} = \frac{a_k(1+k)}{k}, b_{1+k} = \frac{b_k(k+2)}{1+k}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 60

```
Order:=6;
dsolve(x*(1-x)*diff(y(x),x$2)-3*x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & \ln(x)\left(x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \mathrm{O}\left(x^6\right)\right) c_2 \\
& + c_1 x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + \left(1 + 3x + 5x^2 + 7x^3 + 9x^4 + 11x^5 + \mathrm{O}\left(x^6\right)\right) c_2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 63

```
AsymptoticDSolveValue[x*(1-x)*y''[x]-3*x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(x^4+x^3+x^2+\left(4x^3+3x^2+2x+1\right)x\log(x)+x+1\right) + c_2\left(5x^5+4x^4+3x^3+2x^2+x\right)$$

13.8 problem 20.4

Internal problem ID [12078]
Internal file name [`OUTPUT/10730_Monday_September_11_2023_12_50_10_AM_73939356/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 20, Series solutions of second order linear equations. Exercises page 195
Problem number: 20.4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x - x^2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The ODE is

$$x^2y'' + y'x - x^2y = 0$$

Or

$$x(xy'' - yx + y') = 0$$

For $x \neq 0$ the above simplifies to

$$xy'' - yx + y' = 0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y'x - x^2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -1$$

Table 187: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + y' x - x^2 y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2 + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x - x^2 \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-x^{2+n+r} a_n\right)=0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{2+n+r} a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right)+\sum_{n=2}^{\infty}\left(-a_{n-2} x^{n+r}\right)=0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1)+x^{n+r} a_n(n+r)=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)+x^r a_0 r=0$$

Or

$$\left(x^r r(-1+r)+x^r r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r^2=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=0$$
$$r_2=0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+2)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$	$-\frac{2}{(r+2)^3}$	$-\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$	$\frac{-4r-12}{(r+2)^3(r+4)^3}$	$-\frac{3}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) - \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) + c_2\left(\left(1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) - \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) + c_2\left(\left(1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) - \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\left(1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \\
& + c_2\left(\left(1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) - \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\left(1+\frac{x^2}{4}+\frac{x^4}{64}+O\left(x^6\right)\right)+c_2\left(\left(1+\frac{x^2}{4}+\frac{x^4}{64}+O\left(x^6\right)\right)\ln\left(x\right)-\frac{x^2}{4}-\frac{3x^4}{128}+O\left(x^6\right)\right)$$

Verified OK.

13.8.1 Maple step by step solution

Let's solve

$y''x^2 + y'x - x^2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x - yx + y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + a_1 (1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1} (k+r+1)^2 - a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term must be 0

  $a_1 (1+r)^2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1} (k+1)^2 - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2} (k+2)^2 - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+2)^2}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{(k+2)^2}$
- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{(k+2)^2}, a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 41

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_1 + c_2 \ln(x)) \left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-\frac{1}{4}x^2 - \frac{3}{128}x^4 + \mathrm{O}\left(x^6\right)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]-x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{64}+\frac{x^2}{4}+1\right)+c_2\left(-\frac{3x^4}{128}-\frac{x^2}{4}+\left(\frac{x^4}{64}+\frac{x^2}{4}+1\right)\log(x)\right)$$

13.9 problem 20.5

Internal problem ID [12079]
Internal file name [`OUTPUT/10731_Monday_September_11_2023_12_50_10_AM_71107924/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 20, Series solutions of second order linear equations. Exercises page 195
Problem number: 20.5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Bessel]`

$$\boxed{x^2y'' + y'x + y(x^2 - 1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y'x + y(x^2 - 1) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 - 1}{x^2}$$

Table 189: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + y'x + y(x^2 - 1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x^2 + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) (x^2 - 1) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n=\sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1)+x^{n+r} a_n(n+r)-a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)+x^r a_0 r-a_0 x^r=0$$

Or

$$\left(x^r r(-1+r)+x^r r-x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-1\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2-1=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_n(n + r) + a_{n-2} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 4r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+3)^2 (1+r)(r+5)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{(r+3)^2(1+r)(r+5)}$	$\frac{1}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r+3}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{(r+3)^2(1+r)(r+5)}$	$\frac{1}{192}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= -\frac{1}{r^2 + 4r + 3} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{1}{r^2 + 4r + 3} &= \lim_{r\to -1} -\frac{1}{r^2 + 4r + 3} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln (x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln (x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n + r_2)}{x}\right) \\ &= C y_1'(x) \ln (x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n + r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln (x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n + r_2)}{x^2}\right) \\ &= C y_1''(x) \ln (x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n + r_2)(-1 + n + r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x^2y'' + y'x + y(x^2 - 1) = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2 + \left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)x + \left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right)(x^2-1) = 0$$

Which can be written as

$$\left(\left(y_1''(x)\,x^2 + y_1'(x)\,x + y_1(x)\,(x^2-1)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x^2 + y_1(x)\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2 + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)(x^2-1) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x^2 + y_1'(x)\,x + y_1(x)\,(x^2-1) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x^2 + y_1(x)\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2 + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)(x^2-1) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&2x\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) C + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}$$

Since $r_1 = 1$ and $r_2 = -1$ then the above becomes

$$\begin{aligned}
&2x\left(\sum_{n=0}^{\infty} x^n a_n (n+1)\right) C + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-1)(n-2)\right) x^2 \\
&+ \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-1)\right) x - \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1)\right) + \left(\sum_{n=0}^{\infty} x^{n+1} b_n\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n^2 - 3n + 2)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1) = \sum_{n=2}^{\infty} 2C a_{n-2} (n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} x^{n+1} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=2}^{\infty} 2Ca_{n-2}(n-1)\, x^{n-1}\right)+\left(\sum_{n=2}^{\infty} b_{n-2}x^{n-1}\right)$$
$$+\left(\sum_{n=0}^{\infty} x^{n-1}b_n\left(n^2-3n+2\right)\right)+\left(\sum_{n=0}^{\infty} x^{n-1}b_n(n-1)\right)+\sum_{n=0}^{\infty}\left(-b_n x^{n-1}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1=0$$

Solving the above for b_1 gives

$$b_1=0$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C+1=0$$

Which is solved for C. Solving for C gives

$$C=-\frac{1}{2}$$

For $n=3$, Eq (2B) gives

$$4Ca_1+b_1+3b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3=0$$

Solving the above for b_3 gives

$$b_3=0$$

For $n=4$, Eq (2B) gives

$$6Ca_2+b_2+8b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4+\frac{3}{8}=0$$

Solving the above for b_4 gives

$$b_4 = -\frac{3}{64}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + b_3 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\right)\ln(x) + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{1}{2}\left(x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\right)\ln(x) + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\ln(x)}{2} + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x \left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) + c_2 \left(-\frac{x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\ln(x)}{2} + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) + c_2 \left(-\frac{x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\ln(x)}{2} + \frac{1 - \frac{3x^4}{64} + O(x^6)}{x}\right)$$

Verified OK.

13.9.1 Maple step by step solution

Let's solve

$y''x^2 + y'x + y(x^2 - 1) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-1)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2 + y'x + y(x^2 - 1) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + r)\,(-1 + r)\, x^r + a_1(2 + r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k + r + 1)\,(k + r - 1) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + r)\,(-1 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term must be 0

  $a_1(2 + r)\, r = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-1) + a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+3+r)(k+r+1) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+3+r)(k+r+1)}$

- Recursion relation for $r=-1$

  $a_{k+2} = -\frac{a_k}{(k+2)k}$

- Solution for $r=-1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+2)k}, a_1 = 0\right]$

- Recursion relation for $r=1$

  $a_{k+2} = -\frac{a_k}{(k+4)(k+2)}$

- Solution for $r=1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+4)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{1+k}\right), a_{k+2} = -\frac{a_k}{(k+2)k}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+4)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1x^2\left(1-\frac{1}{8}x^2+\frac{1}{192}x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(\ln\left(x\right)\left(x^2-\frac{1}{8}x^4+\mathrm{O}\left(x^6\right)\right)+\left(-2+\frac{3}{32}x^4+\mathrm{O}\left(x^6\right)\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 58

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{192}-\frac{x^3}{8}+x\right)+c_1\left(\frac{1}{16}x\left(x^2-8\right)\log(x)-\frac{5x^4-16x^2-64}{64x}\right)$$

13.10 problem 20.7

Internal problem ID [12080]
Internal file name [OUTPUT/10732_Monday_September_11_2023_12_50_12_AM_11152359/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 20, Series solutions of second order linear equations. Exercises page 195
Problem number: 20.7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[_Bessel]`

$$\boxed{x^2y'' + y'x + (-n^2 + x^2)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y'x + (-n^2 + x^2)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{n^2 - x^2}{x^2}$$

Table 191: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{n^2 - x^2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + y'x + (-n^2 + x^2) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x^2 + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x + (-n^2 + x^2) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} n^2 a_n\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} n^2 a_n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - x^{n+r} n^2 a_n = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - x^r n^2 a_0 = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r n^2\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(-n^2 + r^2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$-n^2 + r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= n \\
r_2 &= -n
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(-n^2 + r^2\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Assuming the roots differ by non-integer Since $r_1 - r_2 = 2n$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+n} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-n}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_n(n+r) - a_n n^2 + a_{n-2} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-2}}{n^2 - n^2 - 2nr - r^2} \tag{4}
$$

Which for the root $r = n$ becomes

$$
a_n = -\frac{a_{n-2}}{n(2n+n)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = n$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{n^2 - r^2 - 4r - 4}$$

Which for the root $r = n$ becomes

$$a_2 = \frac{1}{-4n - 4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{n^2-r^2-4r-4}$	$\frac{1}{-4n-4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{n^2-r^2-4r-4}$	$\frac{1}{-4n-4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(n^2 - r^2 - 4r - 4)(n^2 - r^2 - 8r - 16)}$$

Which for the root $r = n$ becomes

$$a_4 = \frac{1}{32\,(n+1)\,(2+n)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{n^2-r^2-4r-4}$	$\frac{1}{-4n-4}$
a_3	0	0
a_4	$\frac{1}{(n^2-r^2-4r-4)(n^2-r^2-8r-16)}$	$\frac{1}{32(n+1)(2+n)}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{n^2-r^2-4r-4}$	$\frac{1}{-4n-4}$
a_3	0	0
a_4	$\frac{1}{(n^2-r^2-4r-4)(n^2-r^2-8r-16)}$	$\frac{1}{32(n+1)(2+n)}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^n \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^n \left(1 + \frac{x^2}{-4n-4} + \frac{x^4}{32\,(n+1)\,(2+n)} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) - n^2 b_n + b_{n-2} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{n^2 - n^2 - 2nr - r^2} \tag{4}$$

Which for the root $r = -n$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-n)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -n$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{n^2 - r^2 - 4r - 4}$$

Which for the root $r = -n$ becomes

$$b_2 = \frac{1}{4n - 4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{n^2-r^2-4r-4}$	$\frac{1}{4n-4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{n^2-r^2-4r-4}$	$\frac{1}{4n-4}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(n^2 - r^2 - 4r - 4)(n^2 - r^2 - 8r - 16)}$$

Which for the root $r = -n$ becomes

$$b_4 = \frac{1}{32(n-1)(n-2)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{n^2-r^2-4r-4}$	$\frac{1}{4n-4}$
b_3	0	0
b_4	$\frac{1}{(n^2-r^2-4r-4)(n^2-r^2-8r-16)}$	$\frac{1}{32(n-1)(n-2)}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{n^2-r^2-4r-4}$	$\frac{1}{4n-4}$
b_3	0	0
b_4	$\frac{1}{(n^2-r^2-4r-4)(n^2-r^2-8r-16)}$	$\frac{1}{32(n-1)(n-2)}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^n \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{-n} \left(1 + \frac{x^2}{4n-4} + \frac{x^4}{32(n-1)(n-2)} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^n \left(1 + \frac{x^2}{-4n-4} + \frac{x^4}{32(n+1)(2+n)} + O(x^6)\right) \\
&\quad + c_2 x^{-n} \left(1 + \frac{x^2}{4n-4} + \frac{x^4}{32(n-1)(n-2)} + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^n \left(1 + \frac{x^2}{-4n-4} + \frac{x^4}{32(n+1)(2+n)} + O(x^6)\right) \\
&\quad + c_2 x^{-n} \left(1 + \frac{x^2}{4n-4} + \frac{x^4}{32(n-1)(n-2)} + O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x^n \left(1 + \frac{x^2}{-4n-4} + \frac{x^4}{32(n+1)(2+n)} + O(x^6)\right) \\
&\quad + c_2 x^{-n} \left(1 + \frac{x^2}{4n-4} + \frac{x^4}{32(n-1)(n-2)} + O(x^6)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 x^n \left(1 + \frac{x^2}{-4n-4} + \frac{x^4}{32(n+1)(2+n)} + O(x^6)\right) \\
&\quad + c_2 x^{-n} \left(1 + \frac{x^2}{4n-4} + \frac{x^4}{32(n-1)(n-2)} + O(x^6)\right)
\end{aligned}$$

Verified OK.

13.10.1 Maple step by step solution

Let's solve

$y''x^2 + y'x + (-n^2 + x^2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(n^2 - x^2)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(n^2 - x^2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{n^2 - x^2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -n^2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2 + y'x + (-n^2 + x^2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(n+r)\,(-n+r)\, x^r + a_1(1+n+r)\,(1-n+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+n+r)\,(k-n+r) + a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(n+r)\,(-n+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{n, -n\}$

- Each term must be 0

  $a_1(1+n+r)\,(1-n+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+n+r)\,(k-n+r) + a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+2+n+r)\,(k+2-n+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+n+r)(k+2-n+r)}$

- Recursion relation for $r = n$

  $a_{k+2} = -\frac{a_k}{(k+2+2n)(k+2)}$

- Solution for $r = n$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+n}, a_{k+2} = -\frac{a_k}{(k+2+2n)(k+2)}, a_1 = 0\right]$

- Recursion relation for $r = -n$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+2-2n)}$
- Solution for $r = -n$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-n}, a_{k+2} = -\frac{a_k}{(k+2)(k+2-2n)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+n}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-n}\right), a_{k+2} = -\frac{a_k}{(k+2+2n)(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+2-2n)}, b_1 =\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 77

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-n^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^{-n}\left(1 + \frac{1}{4n-4}x^2 + \frac{1}{32}\frac{1}{(n-2)(n-1)}x^4 + \mathrm{O}\left(x^6\right)\right)c_1 + c_2 x^n\left(1 - \frac{1}{4n+4}x^2 + \frac{1}{32}\frac{1}{(n+2)(n+1)}x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 160

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-n^2)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\ & c_2\left(\frac{x^4}{(-n^2-n+(1-n)(2-n)+2)\,(-n^2-n+(3-n)(4-n)+4)}\right.\\
&\left.-\frac{x^2}{-n^2-n+(1-n)(2-n)+2}+1\right)x^{-n}\\
&+c_1\left(\frac{x^4}{(-n^2+n+(n+1)(n+2)+2)\,(-n^2+n+(n+3)(n+4)+4)}\right.\\
&\left.-\frac{x^2}{-n^2+n+(n+1)(n+2)+2}+1\right)x^{n}
\end{aligned}
$$

14 Chapter 26, Explicit solutions of coupled linear systems. Exercises page 257

14.1 problem 26.1 (i)

Internal problem ID [12081]
Internal file name [`OUTPUT/10733_Monday_September_11_2023_12_50_12_AM_56986172/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 26, Explicit solutions of coupled linear systems. Exercises page 257
Problem number: 26.1 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 4x(t) - y(t) \\
y'(t) &= 2x(t) + y(t) + t^2
\end{aligned}
$$

With initial conditions

$$[x(0) = 0, y(0) = 1]$$

14.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ t^2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -e^{2t} + 2e^{3t} & -e^{3t} + e^{2t} \\ 2e^{3t} - 2e^{2t} & 2e^{2t} - e^{3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} -e^{2t} + 2e^{3t} & -e^{3t} + e^{2t} \\ 2e^{3t} - 2e^{2t} & 2e^{2t} - e^{3t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -e^{3t} + e^{2t} \\ 2e^{2t} - e^{3t} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} -e^{-3t}(e^{t} - 2) & e^{-3t}(e^{t} - 1) \\ -2e^{-3t}(e^{t} - 1) & e^{-3t}(2e^{t} - 1) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} -e^{2t} + 2e^{3t} & -e^{3t} + e^{2t} \\ 2e^{3t} - 2e^{2t} & 2e^{2t} - e^{3t} \end{bmatrix} \int \begin{bmatrix} -e^{-3t}(e^{t} - 2) & e^{-3t}(e^{t} - 1) \\ -2e^{-3t}(e^{t} - 1) & e^{-3t}(2e^{t} - 1) \end{bmatrix} \begin{bmatrix} 0 \\ t^2 \end{bmatrix} dt \\ &= \begin{bmatrix} -e^{2t} + 2e^{3t} & -e^{3t} + e^{2t} \\ 2e^{3t} - 2e^{2t} & 2e^{2t} - e^{3t} \end{bmatrix} \begin{bmatrix} \frac{(9t^2+6t+2)e^{-3t}}{27} - \frac{(t^2+t+\frac{1}{2})e^{-2t}}{2} \\ \frac{(9t^2+6t+2)e^{-3t}}{27} - \left(t^2 + t + \frac{1}{2}\right)e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{6}t^2 - \frac{5}{18}t - \frac{19}{108} \\ -\frac{2}{3}t^2 - \frac{7}{9}t - \frac{23}{54} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} -e^{3t} + e^{2t} - \frac{t^2}{6} - \frac{5t}{18} - \frac{19}{108} \\ 2e^{2t} - e^{3t} - \frac{2t^2}{3} - \frac{7t}{9} - \frac{23}{54} \end{bmatrix} \end{aligned}$$

14.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ t^2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 5\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \implies \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{3t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{2t}}{2} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{e^{2t}}{2} & e^{3t} \\ e^{2t} & e^{3t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -2\,e^{-2t} & 2\,e^{-2t} \\ 2\,e^{-3t} & -e^{-3t} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^{2t}}{2} & e^{3t} \\ e^{2t} & e^{3t} \end{bmatrix} \int \begin{bmatrix} -2\,e^{-2t} & 2\,e^{-2t} \\ 2\,e^{-3t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ t^2 \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{e^{2t}}{2} & e^{3t} \\ e^{2t} & e^{3t} \end{bmatrix} \int \begin{bmatrix} 2\,e^{-2t}t^2 \\ -e^{-3t}t^2 \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{e^{2t}}{2} & e^{3t} \\ e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} -\frac{(2t^2+2t+1)e^{-2t}}{2} \\ \frac{(9t^2+6t+2)e^{-3t}}{27} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{6}t^2 - \frac{5}{18}t - \frac{19}{108} \\ -\frac{2}{3}t^2 - \frac{7}{9}t - \frac{23}{54} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{2t}}{2} \\ c_1 e^{2t} \end{bmatrix} + \begin{bmatrix} c_2 e^{3t} \\ c_2 e^{3t} \end{bmatrix} + \begin{bmatrix} -\frac{1}{6}t^2 - \frac{5}{18}t - \frac{19}{108} \\ -\frac{2}{3}t^2 - \frac{7}{9}t - \frac{23}{54} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{2t}}{2} + c_2 e^{3t} - \frac{t^2}{6} - \frac{5t}{18} - \frac{19}{108} \\ c_1 e^{2t} + c_2 e^{3t} - \frac{2t^2}{3} - \frac{7t}{9} - \frac{23}{54} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 0 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

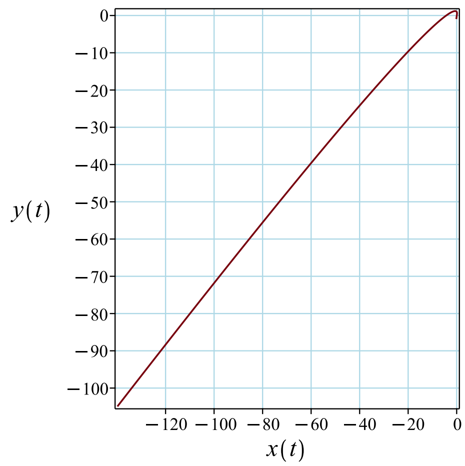
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{c_1}{2} + c_2 - \frac{19}{108} \\ c_1 + c_2 - \frac{23}{54} \end{bmatrix}$$

Solving for the constants of integrations gives

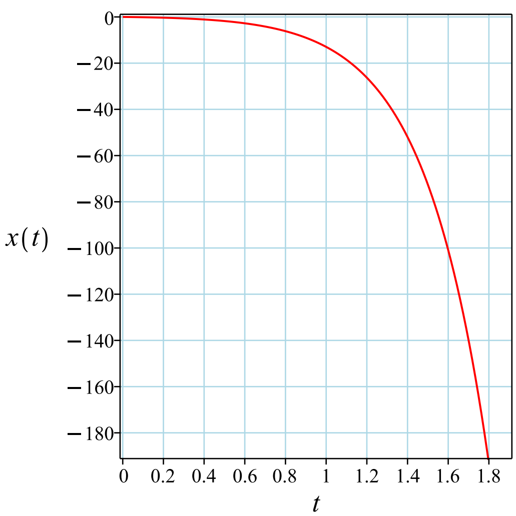
$$\begin{bmatrix} c_1 = \frac{5}{2} \\ c_2 = -\frac{29}{27} \end{bmatrix}$$

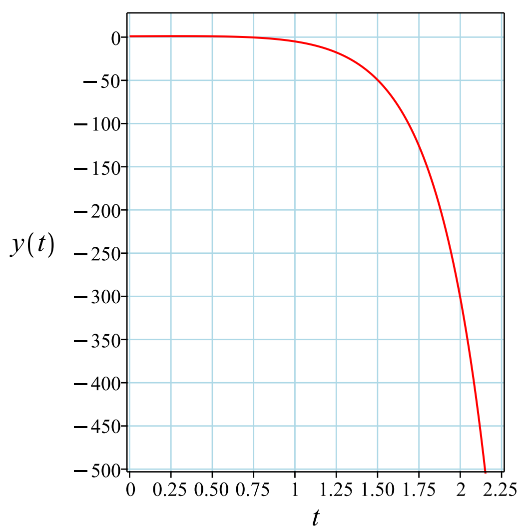
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{5\,e^{2t}}{4} - \frac{29\,e^{3t}}{27} - \frac{t^2}{6} - \frac{5t}{18} - \frac{19}{108} \\ \frac{5\,e^{2t}}{2} - \frac{29\,e^{3t}}{27} - \frac{2t^2}{3} - \frac{7t}{9} - \frac{23}{54} \end{bmatrix}$$



The following are plots of each solution.





✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 52

```
dsolve([diff(x(t),t) = 4*x(t)-y(t), diff(y(t),t) = 2*x(t)+y(t)+t^2, x(0) = 0, y(0) = 1], sin
```

$$x(t) = -\frac{29\,\mathrm{e}^{3t}}{27} + \frac{5\,\mathrm{e}^{2t}}{4} - \frac{t^2}{6} - \frac{5t}{18} - \frac{19}{108}$$
$$y(t) = -\frac{29\,\mathrm{e}^{3t}}{27} + \frac{5\,\mathrm{e}^{2t}}{2} - \frac{7t}{9} - \frac{23}{54} - \frac{2t^2}{3}$$

✓ Solution by Mathematica

Time used: 0.128 (sec). Leaf size: 64

```
DSolve[{x'[t]==4*x[t]-y[t],y'[t]==2*x[t]+y[t]+t^2},{x[0]==0,y[0]==1},{x[t],y[t]},t,IncludeSi
```

$$x(t) \to \frac{1}{108}\left(-18t^2 - 30t + 135e^{2t} - 116e^{3t} - 19\right)$$
$$y(t) \to \frac{1}{54}\left(-36t^2 - 42t + 135e^{2t} - 58e^{3t} - 23\right)$$

14.2 problem 26.1 (ii)

Internal problem ID [12082]
Internal file name [`OUTPUT/10734_Monday_September_11_2023_12_50_13_AM_85594900/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 26, Explicit solutions of coupled linear systems. Exercises page 257
Problem number: 26.1 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) - 4y(t) + 2\cos(t)^2 - 1 \\
y'(t) &= x(t) + y(t)
\end{aligned}
$$

With initial conditions

$$[x(0) = 1, y(0) = 1]$$

14.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 2\cos(t)^2 - 1 \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^t \cos(2t) & -2\,e^t \sin(2t) \\ \frac{e^t \sin(2t)}{2} & e^t \cos(2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^t \cos(2t) & -2\,e^t \sin(2t) \\ \frac{e^t \sin(2t)}{2} & e^t \cos(2t) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^t \cos(2t) - 2\,e^t \sin(2t) \\ \frac{e^t \sin(2t)}{2} + e^t \cos(2t) \end{bmatrix} \\
&= \begin{bmatrix} e^t(\cos(2t) - 2\sin(2t)) \\ \frac{e^t(\sin(2t)+2\cos(2t))}{2} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} e^{-t} \cos(2t) & 2\,e^{-t} \sin(2t) \\ -\frac{e^{-t} \sin(2t)}{2} & e^{-t} \cos(2t) \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^t \cos(2t) & -2\,e^t \sin(2t) \\ \frac{e^t \sin(2t)}{2} & e^t \cos(2t) \end{bmatrix} \int \begin{bmatrix} e^{-t} \cos(2t) & 2\,e^{-t} \sin(2t) \\ -\frac{e^{-t} \sin(2t)}{2} & e^{-t} \cos(2t) \end{bmatrix} \begin{bmatrix} 2\cos(t)^2 - 1 \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} e^t \cos(2t) & -2\,e^t \sin(2t) \\ \frac{e^t \sin(2t)}{2} & e^t \cos(2t) \end{bmatrix} \begin{bmatrix} -\frac{(17+\cos(4t)-4\sin(4t))e^{-t}}{34} \\ \frac{e^{-t}(4\cos(4t)+\sin(4t))}{68} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{9\cos(2t)}{17} + \frac{2\sin(2t)}{17} \\ -\frac{4\sin(2t)}{17} + \frac{\cos(2t)}{17} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \frac{(17\,\mathrm{e}^{t}-9)\cos(2t)}{17} + \frac{(-34\,\mathrm{e}^{t}+2)\sin(2t)}{17} \\ \frac{(34\,\mathrm{e}^{t}+2)\cos(2t)}{34} + \frac{\sin(2t)(17\,\mathrm{e}^{t}-8)}{34} \end{bmatrix}
\end{aligned}
$$

14.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 2\cos(t)^2 - 1 \\ 0 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & -4 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 2\lambda + 5 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 + 2i$	1	complex eigenvalue
$1 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} - (1 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2i & -4 & 0 \\ 1 & 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{iR_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2it\}$

Hence the solution is

$$\begin{bmatrix} -2\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -2it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2\,\mathrm{I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} - (1 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2i & -4 & 0 \\ 1 & -2i & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{iR_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2it\}$

Hence the solution is

$$\begin{bmatrix} 2\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 2it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2\,\mathrm{I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$1 + 2i$	1	1	No	$\begin{bmatrix} 2i \\ 1 \end{bmatrix}$
$1 - 2i$	1	1	No	$\begin{bmatrix} -2i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2i\mathrm{e}^{(1+2i)t} \\ \mathrm{e}^{(1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} -2i\mathrm{e}^{(1-2i)t} \\ \mathrm{e}^{(1-2i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 2i e^{(1+2i)t} & -2i e^{(1-2i)t} \\ e^{(1+2i)t} & e^{(1-2i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{i e^{(-1-2i)t}}{4} & \frac{e^{(-1-2i)t}}{2} \\ \frac{i e^{(-1+2i)t}}{4} & \frac{e^{(-1+2i)t}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 2i e^{(1+2i)t} & -2i e^{(1-2i)t} \\ e^{(1+2i)t} & e^{(1-2i)t} \end{bmatrix} \int \begin{bmatrix} -\frac{i e^{(-1-2i)t}}{4} & \frac{e^{(-1-2i)t}}{2} \\ \frac{i e^{(-1+2i)t}}{4} & \frac{e^{(-1+2i)t}}{2} \end{bmatrix} \begin{bmatrix} 2\cos(t)^2 - 1 \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} 2i e^{(1+2i)t} & -2i e^{(1-2i)t} \\ e^{(1+2i)t} & e^{(1-2i)t} \end{bmatrix} \int \begin{bmatrix} -\frac{i\cos(2t) e^{(-1-2i)t}}{4} \\ \frac{i\cos(2t) e^{(-1+2i)t}}{4} \end{bmatrix} dt \\
&= \begin{bmatrix} 2i e^{(1+2i)t} & -2i e^{(1-2i)t} \\ e^{(1+2i)t} & e^{(1-2i)t} \end{bmatrix} \begin{bmatrix} \frac{((2+9i)\cos(2t)+(-8-2i)\sin(2t)) e^{(-1-2i)t}}{68} \\ -\frac{9\left(\left(-\frac{2}{9}+i\right)\cos(2t)+\left(\frac{8}{9}-\frac{2i}{9}\right)\sin(2t)\right) e^{(-1+2i)t}}{68} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{9\cos(2t)}{17} + \frac{2\sin(2t)}{17} \\ -\frac{4\sin(2t)}{17} + \frac{\cos(2t)}{17} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2i c_1 e^{(1+2i)t} \\ c_1 e^{(1+2i)t} \end{bmatrix} + \begin{bmatrix} -2i c_2 e^{(1-2i)t} \\ c_2 e^{(1-2i)t} \end{bmatrix} + \begin{bmatrix} -\frac{9\cos(2t)}{17} + \frac{2\sin(2t)}{17} \\ -\frac{4\sin(2t)}{17} + \frac{\cos(2t)}{17} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2ic_1 e^{(1+2i)t} - 2ic_2 e^{(1-2i)t} - \frac{9\cos(2t)}{17} + \frac{2\sin(2t)}{17} \\ c_1 e^{(1+2i)t} + c_2 e^{(1-2i)t} - \frac{4\sin(2t)}{17} + \frac{\cos(2t)}{17} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

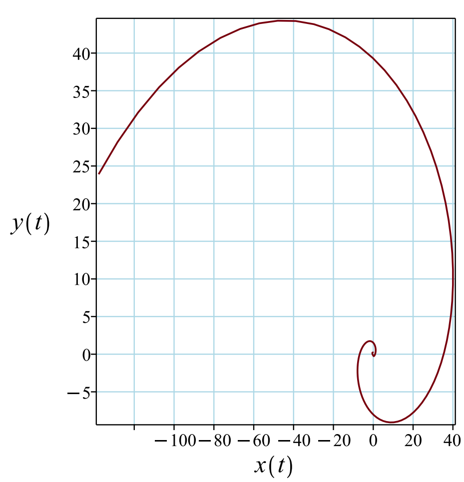
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2ic_1 - 2ic_2 - \frac{9}{17} \\ c_1 + c_2 + \frac{1}{17} \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{8}{17} - \frac{13i}{34} \\ c_2 = \frac{8}{17} + \frac{13i}{34} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{13}{17} + \frac{16i}{17}\right) e^{(1+2i)t} + \left(\frac{13}{17} - \frac{16i}{17}\right) e^{(1-2i)t} - \frac{9\cos(2t)}{17} + \frac{2\sin(2t)}{17} \\ \left(\frac{8}{17} - \frac{13i}{34}\right) e^{(1+2i)t} + \left(\frac{8}{17} + \frac{13i}{34}\right) e^{(1-2i)t} - \frac{4\sin(2t)}{17} + \frac{\cos(2t)}{17} \end{bmatrix}$$



The following are plots of each solution.

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 66

```
dsolve([diff(x(t),t) = x(t)-4*y(t)+cos(2*t), diff(y(t),t) = x(t)+y(t), x(0) = 1, y(0) = 1],
```

$$x(t) = \frac{26\,e^{t}\cos(2t)}{17} - \frac{32\,e^{t}\sin(2t)}{17} + \frac{2\sin(2t)}{17} - \frac{9\cos(2t)}{17}$$
$$y(t) = \frac{13\,e^{t}\sin(2t)}{17} + \frac{16\,e^{t}\cos(2t)}{17} + \frac{\cos(2t)}{17} - \frac{4\sin(2t)}{17}$$

✓ Solution by Mathematica

Time used: 0.122 (sec). Leaf size: 67

```
DSolve[{x'[t]==x[t]-4*y[t]+Cos[2*t],y'[t]==x[t]+y[t]},{x[0]==1,y[0]==1},{x[t],y[t]},t,Includ
```

$$x(t) \to \frac{1}{17}\left(\left(26e^{t}-9\right)\cos(2t) - 2\left(16e^{t}-1\right)\sin(2t)\right)$$
$$y(t) \to \frac{1}{17}\left(\left(13e^{t}-4\right)\sin(2t) + \left(16e^{t}+1\right)\cos(2t)\right)$$

14.3 problem 26.1 (iii)

Internal problem ID [12083]
Internal file name [`OUTPUT/10735_Monday_September_11_2023_12_50_14_AM_41076842/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 26, Explicit solutions of coupled linear systems. Exercises page 257
Problem number: 26.1 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + 2y(t) \\
y'(t) &= 6x(t) + 3y(t) + \mathrm{e}^{t}
\end{aligned}
$$

With initial conditions

$$
[x(0) = 0, y(0) = 1]
$$

14.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{e}^{t} \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{4\,e^{-t}}{7} + \frac{3\,e^{6t}}{7} & \frac{2\,e^{6t}}{7} - \frac{2\,e^{-t}}{7} \\ \frac{6\,e^{6t}}{7} - \frac{6\,e^{-t}}{7} & \frac{3\,e^{-t}}{7} + \frac{4\,e^{6t}}{7} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{4\,e^{-t}}{7} + \frac{3\,e^{6t}}{7} & \frac{2\,e^{6t}}{7} - \frac{2\,e^{-t}}{7} \\ \frac{6\,e^{6t}}{7} - \frac{6\,e^{-t}}{7} & \frac{3\,e^{-t}}{7} + \frac{4\,e^{6t}}{7} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2\,e^{6t}}{7} - \frac{2\,e^{-t}}{7} \\ \frac{3\,e^{-t}}{7} + \frac{4\,e^{6t}}{7} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{(4\,e^{7t}+3)\,e^{-6t}}{7} & -\frac{2(e^{7t}-1)\,e^{-6t}}{7} \\ -\frac{6(e^{7t}-1)\,e^{-6t}}{7} & \frac{(3\,e^{7t}+4)\,e^{-6t}}{7} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{4\,e^{-t}}{7} + \frac{3\,e^{6t}}{7} & \frac{2\,e^{6t}}{7} - \frac{2\,e^{-t}}{7} \\ \frac{6\,e^{6t}}{7} - \frac{6\,e^{-t}}{7} & \frac{3\,e^{-t}}{7} + \frac{4\,e^{6t}}{7} \end{bmatrix} \int \begin{bmatrix} \frac{(4\,e^{7t}+3)\,e^{-6t}}{7} & -\frac{2(e^{7t}-1)\,e^{-6t}}{7} \\ -\frac{6(e^{7t}-1)\,e^{-6t}}{7} & \frac{(3\,e^{7t}+4)\,e^{-6t}}{7} \end{bmatrix} \begin{bmatrix} 0 \\ e^{t} \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{4\,e^{-t}}{7} + \frac{3\,e^{6t}}{7} & \frac{2\,e^{6t}}{7} - \frac{2\,e^{-t}}{7} \\ \frac{6\,e^{6t}}{7} - \frac{6\,e^{-t}}{7} & \frac{3\,e^{-t}}{7} + \frac{4\,e^{6t}}{7} \end{bmatrix} \begin{bmatrix} -\frac{(5\,e^{7t}+2)\,e^{-5t}}{35} \\ \frac{(15\,e^{7t}-8)\,e^{-5t}}{70} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{e^{t}}{5} \\ \frac{e^{t}}{10} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \frac{2\,e^{6t}}{7} - \frac{2\,e^{-t}}{7} - \frac{e^{t}}{5} \\ \frac{3\,e^{-t}}{7} + \frac{4\,e^{6t}}{7} + \frac{e^{t}}{10} \end{bmatrix} \end{aligned}$$

14.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{e}^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 2 \\ 6 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 5\lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 6 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 6 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{6t} \\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{6t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{2\,\mathrm{e}^{-t}}{3} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{6t}}{2} \\ \mathrm{e}^{6t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\frac{2\,\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{6t}}{2} \\ \mathrm{e}^{-t} & \mathrm{e}^{6t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{6\,\mathrm{e}^{t}}{7} & \frac{3\,\mathrm{e}^{t}}{7} \\ \frac{6\,\mathrm{e}^{-6t}}{7} & \frac{4\,\mathrm{e}^{-6t}}{7} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{2\,e^{-t}}{3} & \frac{e^{6t}}{2} \\ e^{-t} & e^{6t} \end{bmatrix} \int \begin{bmatrix} -\frac{6\,e^{t}}{7} & \frac{3\,e^{t}}{7} \\ \frac{6\,e^{-6t}}{7} & \frac{4\,e^{-6t}}{7} \end{bmatrix} \begin{bmatrix} 0 \\ e^{t} \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{2\,e^{-t}}{3} & \frac{e^{6t}}{2} \\ e^{-t} & e^{6t} \end{bmatrix} \int \begin{bmatrix} \frac{3\,e^{2t}}{7} \\ \frac{4\,e^{-5t}}{7} \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{2\,e^{-t}}{3} & \frac{e^{6t}}{2} \\ e^{-t} & e^{6t} \end{bmatrix} \begin{bmatrix} \frac{3\,e^{2t}}{14} \\ -\frac{4\,e^{-5t}}{35} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^{t}}{5} \\ \frac{e^{t}}{10} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_1 e^{-t}}{3} \\ c_1 e^{-t} \end{bmatrix} + \begin{bmatrix} \frac{c_2 e^{6t}}{2} \\ c_2 e^{6t} \end{bmatrix} + \begin{bmatrix} -\frac{e^{t}}{5} \\ \frac{e^{t}}{10} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_1 e^{-t}}{3} + \frac{c_2 e^{6t}}{2} - \frac{e^{t}}{5} \\ c_1 e^{-t} + c_2 e^{6t} + \frac{e^{t}}{10} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 0 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

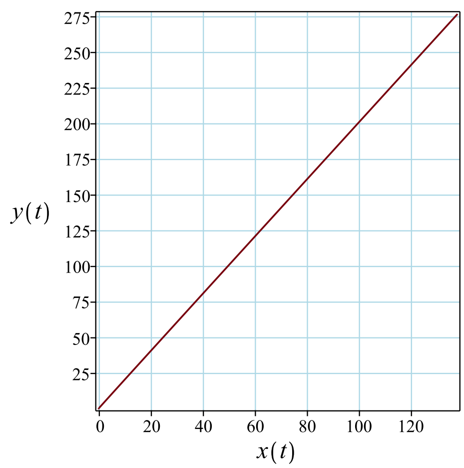
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2c_1}{3} + \frac{c_2}{2} - \frac{1}{5} \\ c_1 + c_2 + \frac{1}{10} \end{bmatrix}$$

Solving for the constants of integrations gives

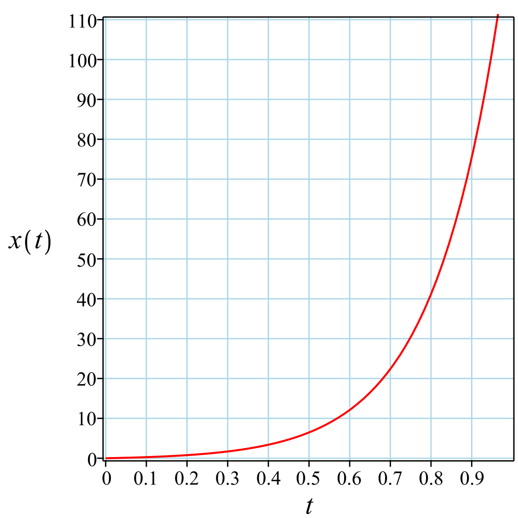
$$\begin{bmatrix} c_1 = \frac{3}{14} \\ c_2 = \frac{24}{35} \end{bmatrix}$$

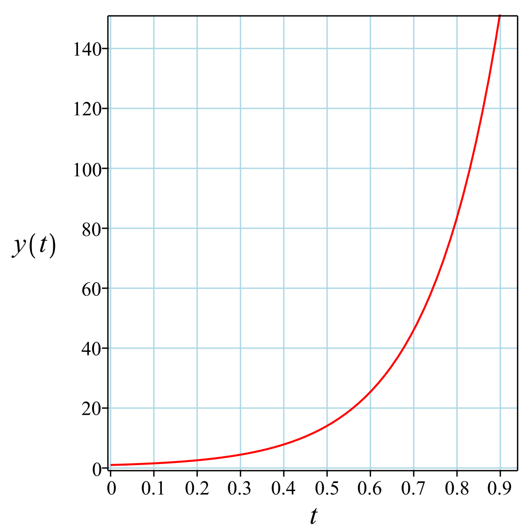
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{e^{-t}}{7} + \frac{12\,e^{6t}}{35} - \frac{e^{t}}{5} \\ \frac{3\,e^{-t}}{14} + \frac{24\,e^{6t}}{35} + \frac{e^{t}}{10} \end{bmatrix}$$



The following are plots of each solution.





✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 42

```
dsolve([diff(x(t),t) = 2*x(t)+2*y(t), diff(y(t),t) = 6*x(t)+3*y(t)+exp(t), x(0) = 0, y(0) =
```

$$x(t) = \frac{12\,\mathrm{e}^{6t}}{35} - \frac{\mathrm{e}^{-t}}{7} - \frac{\mathrm{e}^{t}}{5}$$
$$y(t) = \frac{24\,\mathrm{e}^{6t}}{35} + \frac{3\,\mathrm{e}^{-t}}{14} + \frac{\mathrm{e}^{t}}{10}$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 58

```
DSolve[{x'[t]==2*x[t]+2*y[t],y'[t]==6*x[t]+3*y[t]+Exp[t]},{x[0]==0,y[0]==1},{x[t],y[t]},t,In
```

$$x(t) \to \frac{1}{35}e^{-t}\left(-7e^{2t} + 12e^{7t} - 5\right)$$
$$y(t) \to \frac{1}{70}e^{-t}\left(7e^{2t} + 48e^{7t} + 15\right)$$

14.4 problem 26.1 (iv)

Internal problem ID [12084]
Internal file name [`OUTPUT/10736_Monday_September_11_2023_12_50_15_AM_74835345/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 26, Explicit solutions of coupled linear systems. Exercises page 257
Problem number: 26.1 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 5x(t) - 4y(t) + e^{3t} \\
y'(t) &= x(t) + y(t)
\end{aligned}
$$

With initial conditions

$$[x(0) = 1, y(0) = -1]$$

14.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t}(2t+1) & -4t\,e^{3t} \\ t\,e^{3t} & e^{3t}(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} e^{3t}(2t+1) & -4t\,e^{3t} \\ t\,e^{3t} & e^{3t}(1-2t) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} e^{3t}(2t+1) + 4t\,e^{3t} \\ t\,e^{3t} - e^{3t}(1-2t) \end{bmatrix} \\ &= \begin{bmatrix} e^{3t}(1+6t) \\ e^{3t}(-1+3t) \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} e^{-3t}(1-2t) & 4t\,e^{-3t} \\ -t\,e^{-3t} & e^{-3t}(2t+1) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} e^{3t}(2t+1) & -4t\,e^{3t} \\ t\,e^{3t} & e^{3t}(1-2t) \end{bmatrix} \int \begin{bmatrix} e^{-3t}(1-2t) & 4t\,e^{-3t} \\ -t\,e^{-3t} & e^{-3t}(2t+1) \end{bmatrix} \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix} dt \\ &= \begin{bmatrix} e^{3t}(2t+1) & -4t\,e^{3t} \\ t\,e^{3t} & e^{3t}(1-2t) \end{bmatrix} \begin{bmatrix} -t(t-1) \\ -\frac{t^2}{2} \end{bmatrix} \\ &= \begin{bmatrix} e^{3t}(1+t)\,t \\ \frac{t^2 e^{3t}}{2} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} e^{3t}(t^2 + 7t + 1) \\ e^{3t}\left(-1 + 3t + \frac{1}{2}t^2\right) \end{bmatrix}$$

14.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & -4 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

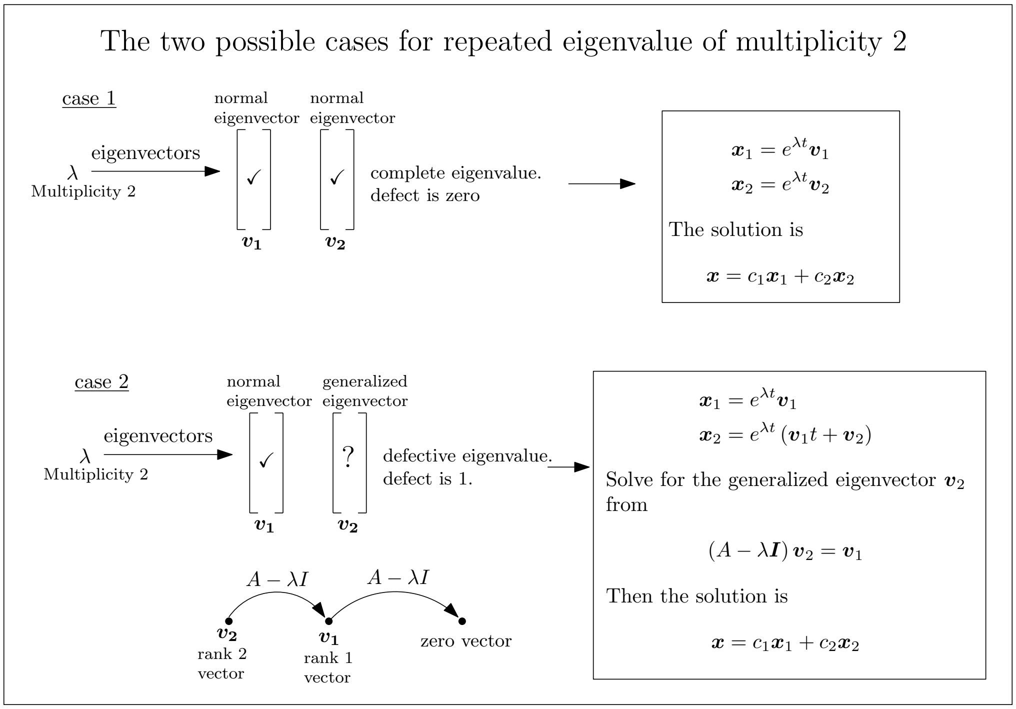


Figure 183: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^{3t}\\ &= \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) \mathrm{e}^{3t}\\ &= \begin{bmatrix} \mathrm{e}^{3t}(2t+3) \\ \mathrm{e}^{3t}(1+t) \end{bmatrix}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{3t}(2t+3) \\ \mathrm{e}^{3t}(1+t) \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 2\,\mathrm{e}^{3t} & \mathrm{e}^{3t}(2t+3) \\ \mathrm{e}^{3t} & \mathrm{e}^{3t}(1+t) \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -e^{-3t}(1+t) & e^{-3t}(2t+3) \\ e^{-3t} & -2\,e^{-3t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 2\,e^{3t} & e^{3t}(2t+3) \\ e^{3t} & e^{3t}(1+t) \end{bmatrix} \int \begin{bmatrix} -e^{-3t}(1+t) & e^{-3t}(2t+3) \\ e^{-3t} & -2\,e^{-3t} \end{bmatrix} \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} 2\,e^{3t} & e^{3t}(2t+3) \\ e^{3t} & e^{3t}(1+t) \end{bmatrix} \int \begin{bmatrix} -1-t \\ 1 \end{bmatrix} dt \\
&= \begin{bmatrix} 2\,e^{3t} & e^{3t}(2t+3) \\ e^{3t} & e^{3t}(1+t) \end{bmatrix} \begin{bmatrix} -\frac{t(t+2)}{2} \\ t \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(1+t)\,t \\ \frac{t^2 e^{3t}}{2} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} 2c_1 e^{3t} \\ c_1 e^{3t} \end{bmatrix} + \begin{bmatrix} c_2 e^{3t}(2t+3) \\ c_2 e^{3t}(1+t) \end{bmatrix} + \begin{bmatrix} e^{3t}(1+t)\,t \\ \frac{t^2 e^{3t}}{2} \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{3t}(t^2 + 2c_2 t + t + 2c_1 + 3c_2) \\ e^{3t}\left(c_1 + c_2 t + c_2 + \frac{1}{2}t^2\right) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = -1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

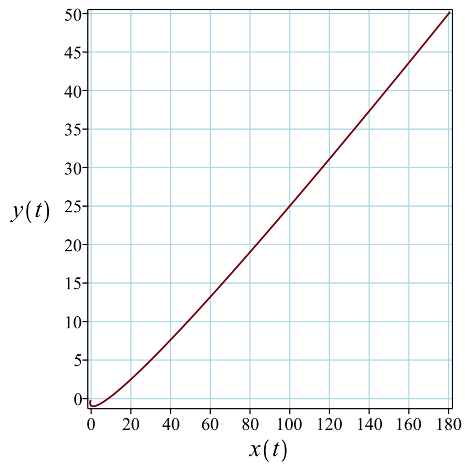
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2c_1 + 3c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

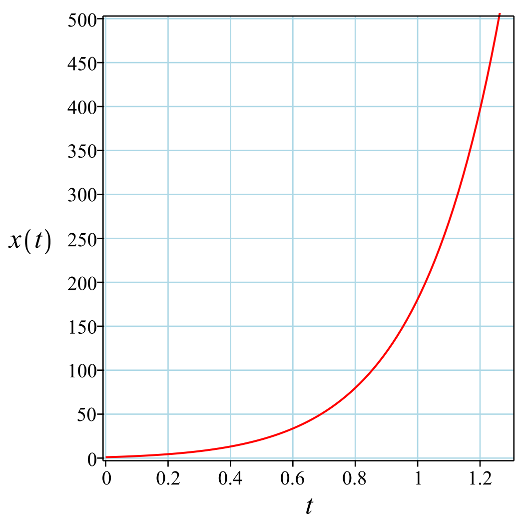
$$\begin{bmatrix} c_1 = -4 \\ c_2 = 3 \end{bmatrix}$$

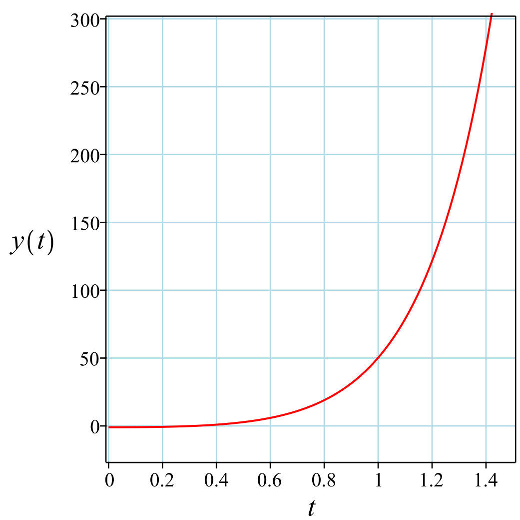
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{3t}(t^2 + 7t + 1) \\ e^{3t}\left(-1 + 3t + \frac{1}{2}t^2\right) \end{bmatrix}$$



The following are plots of each solution.





✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
dsolve([diff(x(t),t) = 5*x(t)-4*y(t)+exp(3*t), diff(y(t),t) = x(t)+y(t), x(0) = 1, y(0) = -1
```

$$x(t) = e^{3t}(t^2 + 7t + 1)$$
$$y(t) = \frac{e^{3t}(t^2 + 6t - 2)}{2}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 39

```
DSolve[{x'[t]==5*x[t]-4*y[t]+Exp[3*t],y'[t]==x[t]+y[t]},{x[0]==1,y[0]==-1},{x[t],y[t]},t,Inc
```

$$x(t) \to e^{3t}(t^2 + 7t + 1)$$
$$y(t) \to \frac{1}{2}e^{3t}(t^2 + 6t - 2)$$

14.5 problem 26.1 (v)

Internal problem ID [12085]
Internal file name [`OUTPUT/10737_Monday_September_11_2023_12_50_16_AM_73901759/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 26, Explicit solutions of coupled linear systems. Exercises page 257
Problem number: 26.1 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2x(t) + 5y(t)$$
$$y'(t) = -2x(t) + 4\cos(t)^3 - 3\cos(t)$$

With initial conditions

$$[x(0) = 2, y(0) = -1]$$

14.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 4\cos(t)^3 - 3\cos(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^{t} \cos\left(3t\right)+\frac{\mathrm{e}^{t} \sin\left(3t\right)}{3} & \frac{5\,\mathrm{e}^{t} \sin\left(3t\right)}{3} \\ -\frac{2\,\mathrm{e}^{t} \sin\left(3t\right)}{3} & \mathrm{e}^{t} \cos\left(3t\right)-\frac{\mathrm{e}^{t} \sin\left(3t\right)}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{t} \left(3\cos\left(3t\right)+\sin\left(3t\right)\right)}{3} & \frac{5\,\mathrm{e}^{t} \sin\left(3t\right)}{3} \\ -\frac{2\,\mathrm{e}^{t} \sin\left(3t\right)}{3} & \frac{\mathrm{e}^{t} \left(3\cos\left(3t\right)-\sin\left(3t\right)\right)}{3} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{\mathrm{e}^{t} \left(3\cos\left(3t\right)+\sin\left(3t\right)\right)}{3} & \frac{5\,\mathrm{e}^{t} \sin\left(3t\right)}{3} \\ -\frac{2\,\mathrm{e}^{t} \sin\left(3t\right)}{3} & \frac{\mathrm{e}^{t} \left(3\cos\left(3t\right)-\sin\left(3t\right)\right)}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{t} \left(3\cos\left(3t\right)+\sin\left(3t\right)\right)}{3}-\frac{5\,\mathrm{e}^{t} \sin\left(3t\right)}{3} \\ -\frac{4\,\mathrm{e}^{t} \sin\left(3t\right)}{3}-\frac{\mathrm{e}^{t} \left(3\cos\left(3t\right)-\sin\left(3t\right)\right)}{3} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t} \left(2\cos\left(3t\right)-\sin\left(3t\right)\right) \\ -\mathrm{e}^{t} \left(\sin\left(3t\right)+\cos\left(3t\right)\right) \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{\left(3\cos\left(3t\right)-\sin\left(3t\right)\right)\mathrm{e}^{-t}}{3} & -\frac{5\,\mathrm{e}^{-t} \sin\left(3t\right)}{3} \\ \frac{2\,\mathrm{e}^{-t} \sin\left(3t\right)}{3} & \frac{\mathrm{e}^{-t} \left(3\cos\left(3t\right)+\sin\left(3t\right)\right)}{3} \end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{\mathrm{e}^{t} \left(3\cos\left(3t\right)+\sin\left(3t\right)\right)}{3} & \frac{5\,\mathrm{e}^{t} \sin\left(3t\right)}{3} \\ -\frac{2\,\mathrm{e}^{t} \sin\left(3t\right)}{3} & \frac{\mathrm{e}^{t} \left(3\cos\left(3t\right)-\sin\left(3t\right)\right)}{3} \end{bmatrix} \int \begin{bmatrix} \frac{\left(3\cos\left(3t\right)-\sin\left(3t\right)\right)\mathrm{e}^{-t}}{3} & -\frac{5\,\mathrm{e}^{-t} \sin\left(3t\right)}{3} \\ \frac{2\,\mathrm{e}^{-t} \sin\left(3t\right)}{3} & \frac{\mathrm{e}^{-t} \left(3\cos\left(3t\right)+\sin\left(3t\right)\right)}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 4\cos\left(t\right)^{3}- \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{t} \left(3\cos\left(3t\right)+\sin\left(3t\right)\right)}{3} & \frac{5\,\mathrm{e}^{t} \sin\left(3t\right)}{3} \\ -\frac{2\,\mathrm{e}^{t} \sin\left(3t\right)}{3} & \frac{\mathrm{e}^{t} \left(3\cos\left(3t\right)-\sin\left(3t\right)\right)}{3} \end{bmatrix} \begin{bmatrix} \frac{5\,\mathrm{e}^{-t} \left(\sin\left(6t\right)+6\cos\left(6t\right)\right)}{222} \\ -\frac{\left(111-17\sin\left(6t\right)+9\cos\left(6t\right)\right)\mathrm{e}^{-t}}{222} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{30\sin\left(3t\right)}{37}+\frac{5\cos\left(3t\right)}{37} \\ \frac{9\sin\left(3t\right)}{37}-\frac{20\cos\left(3t\right)}{37} \end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \left[\begin{array}{c} \frac{(74\,\mathrm{e}^{t}+5)\cos(3t)}{37} + \frac{(-37\,\mathrm{e}^{t}-30)\sin(3t)}{37} \\ \frac{(-37\,\mathrm{e}^{t}-20)\cos(3t)}{37} + \frac{(-37\,\mathrm{e}^{t}+9)\sin(3t)}{37} \end{array}\right]
\end{aligned}
$$

14.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\left[\begin{array}{c} x'(t) \\ y'(t) \end{array}\right] = \left[\begin{array}{cc} 2 & 5 \\ -2 & 0 \end{array}\right] \left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ 4\cos(t)^{3} - 3\cos(t) \end{array}\right]
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\left[\begin{array}{cc} 2 & 5 \\ -2 & 0 \end{array}\right] - \lambda \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right) = 0
$$

Therefore

$$
\det\left(\left[\begin{array}{cc} 2-\lambda & 5 \\ -2 & -\lambda \end{array}\right]\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 2\lambda + 10 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 3i$$
$$\lambda_2 = 1 - 3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 + 3i$	1	complex eigenvalue
$1 - 3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 5 \\ -2 & 0 \end{bmatrix} - (1-3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+3i & 5 \\ -2 & -1+3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+3i & 5 & 0 \\ -2 & -1+3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{5} - \frac{3i}{5}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+3i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+3i & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} + \frac{3i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{3\,\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}+\frac{3i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{3\,\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2}+\frac{3i}{2} \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{3\,\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}+\frac{3i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{3\,\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -1+3i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1+3i$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 5 \\ -2 & 0 \end{bmatrix} - (1+3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1-3i & 5 \\ -2 & -1-3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-3i & 5 & 0 \\ -2 & -1-3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{5}+\frac{3i}{5}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} 1-3i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-3i & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{3i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{3i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} - \frac{3i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{3i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1-3i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$1+3i$	1	1	No	$\left[\begin{array}{c} -\frac{1}{2}-\frac{3i}{2} \\ 1 \end{array}\right]$
$1-3i$	1	1	No	$\left[\begin{array}{c} -\frac{1}{2}+\frac{3i}{2} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} \left(-\frac{1}{2}-\frac{3i}{2}\right) e^{(1+3i)t} \\ e^{(1+3i)t} \end{array}\right] + c_2 \left[\begin{array}{c} \left(-\frac{1}{2}+\frac{3i}{2}\right) e^{(1-3i)t} \\ e^{(1-3i)t} \end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{cc} \left(-\frac{1}{2}-\frac{3i}{2}\right) e^{(1+3i)t} & \left(-\frac{1}{2}+\frac{3i}{2}\right) e^{(1-3i)t} \\ e^{(1+3i)t} & e^{(1-3i)t} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \left[\begin{array}{cc} \frac{i e^{(-1-3i)t}}{3} & \left(\frac{1}{2}+\frac{i}{6}\right) e^{(-1-3i)t} \\ -\frac{i e^{(-1+3i)t}}{3} & \left(\frac{1}{2}-\frac{i}{6}\right) e^{(-1+3i)t} \end{array}\right]$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \left(-\frac{1}{2}-\frac{3i}{2}\right) e^{(1+3i)t} & \left(-\frac{1}{2}+\frac{3i}{2}\right) e^{(1-3i)t} \\ e^{(1+3i)t} & e^{(1-3i)t} \end{bmatrix} \int \begin{bmatrix} \frac{i e^{(-1-3i)t}}{3} & \left(\frac{1}{2}+\frac{i}{6}\right) e^{(-1-3i)t} \\ -\frac{i e^{(-1+3i)t}}{3} & \left(\frac{1}{2}-\frac{i}{6}\right) e^{(-1+3i)t} \end{bmatrix} \begin{bmatrix} 0 \\ 4\cos(t)^3 - 3\cos \end{bmatrix}$$

$$= \begin{bmatrix} \left(-\frac{1}{2}-\frac{3i}{2}\right) e^{(1+3i)t} & \left(-\frac{1}{2}+\frac{3i}{2}\right) e^{(1-3i)t} \\ e^{(1+3i)t} & e^{(1-3i)t} \end{bmatrix} \int \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{6}\right) e^{(-1-3i)t}\cos(3t) \\ \left(\frac{1}{2}-\frac{i}{6}\right) e^{(-1+3i)t}\cos(3t) \end{bmatrix} dt$$

$$= \begin{bmatrix} \left(-\frac{1}{2}-\frac{3i}{2}\right) e^{(1+3i)t} & \left(-\frac{1}{2}+\frac{3i}{2}\right) e^{(1-3i)t} \\ e^{(1+3i)t} & e^{(1-3i)t} \end{bmatrix} \begin{bmatrix} \left(\frac{3}{74}+\frac{i}{74}\right)\left(\left(-\frac{19}{3}+i\right)\cos(3t)+(1-6i)\sin(3t)\right) e^{(-} \\ \left(-\frac{3}{74}+\frac{i}{74}\right) e^{(-1+3i)t}\left(\left(\frac{19}{3}+i\right)\cos(3t)+(-1-6i)\sin \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{30\sin(3t)}{37}+\frac{5\cos(3t)}{37} \\ \frac{9\sin(3t)}{37}-\frac{20\cos(3t)}{37} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{3i}{2}\right) c_1 e^{(1+3i)t} \\ c_1 e^{(1+3i)t} \end{bmatrix} + \begin{bmatrix} \left(-\frac{1}{2}+\frac{3i}{2}\right) c_2 e^{(1-3i)t} \\ c_2 e^{(1-3i)t} \end{bmatrix} + \begin{bmatrix} -\frac{30\sin(3t)}{37}+\frac{5\cos(3t)}{37} \\ \frac{9\sin(3t)}{37}-\frac{20\cos(3t)}{37} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{3i}{2}\right) c_1 e^{(1+3i)t} + \left(-\frac{1}{2}+\frac{3i}{2}\right) c_2 e^{(1-3i)t} - \frac{30\sin(3t)}{37}+\frac{5\cos(3t)}{37} \\ c_1 e^{(1+3i)t} + c_2 e^{(1-3i)t} + \frac{9\sin(3t)}{37}-\frac{20\cos(3t)}{37} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0)=2 \\ y(0)=-1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t=0$ gives

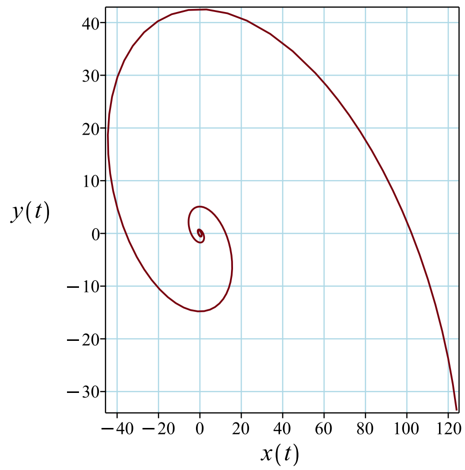
$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{3i}{2}\right) c_1 + \frac{5}{37} + \left(-\frac{1}{2}+\frac{3i}{2}\right) c_2 \\ c_1 + c_2 - \frac{20}{37} \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -\frac{17}{74}+\frac{121i}{222} \\ c_2 = -\frac{17}{74}-\frac{121i}{222} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{69}{74}+\frac{8i}{111}\right) e^{(1+3i)t} + \left(\frac{69}{74}-\frac{8i}{111}\right) e^{(1-3i)t} - \frac{30\sin(3t)}{37} + \frac{5\cos(3t)}{37} \\ \left(-\frac{17}{74}+\frac{121i}{222}\right) e^{(1+3i)t} + \left(-\frac{17}{74}-\frac{121i}{222}\right) e^{(1-3i)t} + \frac{9\sin(3t)}{37} - \frac{20\cos(3t)}{37} \end{bmatrix}$$



The following are plots of each solution.

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 66

```
dsolve([diff(x(t),t) = 2*x(t)+5*y(t), diff(y(t),t) = -2*x(t)+cos(3*t), x(0) = 2, y(0) = -1],
```

$$x(t) = -\frac{16\,e^{t}\sin(3t)}{111} + \frac{69\,e^{t}\cos(3t)}{37} + \frac{5\cos(3t)}{37} - \frac{30\sin(3t)}{37}$$
$$y(t) = -\frac{121\,e^{t}\sin(3t)}{111} - \frac{17\,e^{t}\cos(3t)}{37} + \frac{9\sin(3t)}{37} - \frac{20\cos(3t)}{37}$$

✓ Solution by Mathematica

Time used: 0.363 (sec). Leaf size: 70

```
DSolve[{x'[t]==2*x[t]+5*y[t],y'[t]==-2*x[t]+Cos[3*t]},{x[0]==2,y[0]==-1},{x[t],y[t]},t,Inclu
```

$$x(t) \to \frac{1}{111}\left(3\left(69e^{t}+5\right)\cos(3t) - 2\left(8e^{t}+45\right)\sin(3t)\right)$$
$$y(t) \to \frac{1}{111}\left(-\left(121e^{t}-27\right)\sin(3t) - 3\left(17e^{t}+20\right)\cos(3t)\right)$$

14.6 problem 26.1 (vi)

Internal problem ID [12086]
Internal file name [`OUTPUT/10738_Monday_September_11_2023_12_50_17_AM_80096652/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 26, Explicit solutions of coupled linear systems. Exercises page 257
Problem number: 26.1 (vi).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = x(t) + y(t) + \mathrm{e}^{-t}$$
$$y'(t) = 4x(t) - 2y(t) + \mathrm{e}^{2t}$$

With initial conditions

$$[x(0) = 1, y(0) = -1]$$

14.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(4\,e^{5t}+1)e^{-3t}}{5} & \frac{(e^{5t}-1)e^{-3t}}{5} \\ \frac{4(e^{5t}-1)e^{-3t}}{5} & \frac{(e^{5t}+4)e^{-3t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{(4\,e^{5t}+1)e^{-3t}}{5} & \frac{(e^{5t}-1)e^{-3t}}{5} \\ \frac{4(e^{5t}-1)e^{-3t}}{5} & \frac{(e^{5t}+4)e^{-3t}}{5} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(4\,e^{5t}+1)e^{-3t}}{5} - \frac{(e^{5t}-1)e^{-3t}}{5} \\ \frac{4(e^{5t}-1)e^{-3t}}{5} - \frac{(e^{5t}+4)e^{-3t}}{5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{(3\,e^{5t}+2)e^{-3t}}{5} \\ \frac{(3\,e^{5t}-8)e^{-3t}}{5} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{(e^{5t}+4)e^{-2t}}{5} & -\frac{(e^{5t}-1)e^{-2t}}{5} \\ -\frac{4(e^{5t}-1)e^{-2t}}{5} & \frac{(4\,e^{5t}+1)e^{-2t}}{5} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{(4\,e^{5t}+1)e^{-3t}}{5} & \frac{(e^{5t}-1)e^{-3t}}{5} \\ \frac{4(e^{5t}-1)e^{-3t}}{5} & \frac{(e^{5t}+4)e^{-3t}}{5} \end{bmatrix} \int \begin{bmatrix} \frac{(e^{5t}+4)e^{-2t}}{5} & -\frac{(e^{5t}-1)e^{-2t}}{5} \\ -\frac{4(e^{5t}-1)e^{-2t}}{5} & \frac{(4\,e^{5t}+1)e^{-2t}}{5} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{2t} \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{(4\,e^{5t}+1)e^{-3t}}{5} & \frac{(e^{5t}-1)e^{-3t}}{5} \\ \frac{4(e^{5t}-1)e^{-3t}}{5} & \frac{(e^{5t}+4)e^{-3t}}{5} \end{bmatrix} \begin{bmatrix} \frac{(-6\,e^{8t}+15\,e^{5t}+30t\,e^{3t}-40)e^{-3t}}{150} \\ -\frac{(-12\,e^{8t}+30\,e^{5t}-15t\,e^{3t}+20)e^{-3t}}{75} \end{bmatrix} \\ &= \begin{bmatrix} \frac{(5t-1)e^{2t}}{25} - \frac{e^{-t}}{6} \\ \frac{(4+5t)e^{2t}}{25} - \frac{2\,e^{-t}}{3} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} -\frac{(-30t\,e^{5t}-84\,e^{5t}+25\,e^{2t}-60)e^{-3t}}{150} \\ -\frac{(-15t\,e^{5t}-57\,e^{5t}+50\,e^{2t}+120)e^{-3t}}{75} \end{bmatrix}$$

14.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} e^{-t} \\ e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 4R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{2t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-3t}$$
$$= \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} e^{-3t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{e^{-3t}}{4} \\ e^{-3t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{2t} & -\frac{e^{-3t}}{4} \\ e^{2t} & e^{-3t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{4\,e^{-2t}}{5} & \frac{e^{-2t}}{5} \\ -\frac{4\,e^{3t}}{5} & \frac{4\,e^{3t}}{5} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{2t} & -\frac{e^{-3t}}{4} \\ e^{2t} & e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{4\,e^{-2t}}{5} & \frac{e^{-2t}}{5} \\ -\frac{4\,e^{3t}}{5} & \frac{4\,e^{3t}}{5} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{2t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{2t} & -\frac{e^{-3t}}{4} \\ e^{2t} & e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{4\,e^{-3t}}{5} + \frac{1}{5} \\ -\frac{4\,e^{2t}}{5} + \frac{4\,e^{5t}}{5} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{2t} & -\frac{e^{-3t}}{4} \\ e^{2t} & e^{-3t} \end{bmatrix} \begin{bmatrix} \frac{t}{5} - \frac{4\,e^{-3t}}{15} \\ \frac{4\,e^{5t}}{25} - \frac{2\,e^{2t}}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(5t-1)e^{2t}}{25} - \frac{e^{-t}}{6} \\ \frac{(4+5t)e^{2t}}{25} - \frac{2\,e^{-t}}{3} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ c_1 e^{2t} \end{bmatrix} + \begin{bmatrix} -\frac{c_2 e^{-3t}}{4} \\ c_2 e^{-3t} \end{bmatrix} + \begin{bmatrix} \frac{(5t-1)e^{2t}}{25} - \frac{e^{-t}}{6} \\ \frac{(4+5t)e^{2t}}{25} - \frac{2\,e^{-t}}{3} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{-3t}\left(\left(t+5c_1-\frac{1}{5}\right)e^{5t}-\frac{5c_2}{4}-\frac{5\,e^{2t}}{6}\right)}{5} \\ \frac{e^{-3t}\left(\left(t+5c_1+\frac{4}{5}\right)e^{5t}+5c_2-\frac{10\,e^{2t}}{3}\right)}{5} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = -1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

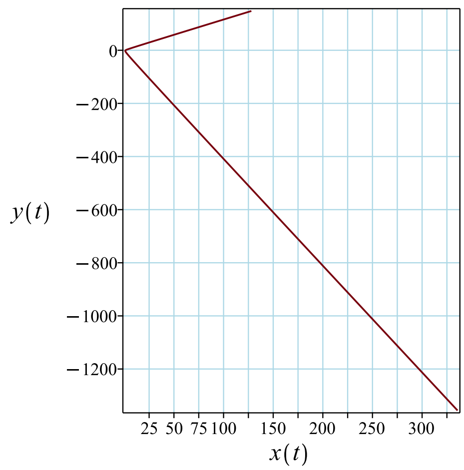
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{31}{150} + c_1 - \frac{c_2}{4} \\ -\frac{38}{75} + c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

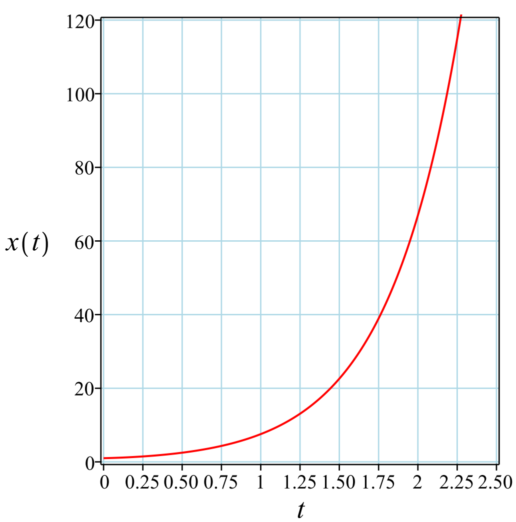
$$\begin{bmatrix} c_1 = \frac{13}{15} \\ c_2 = -\frac{34}{25} \end{bmatrix}$$

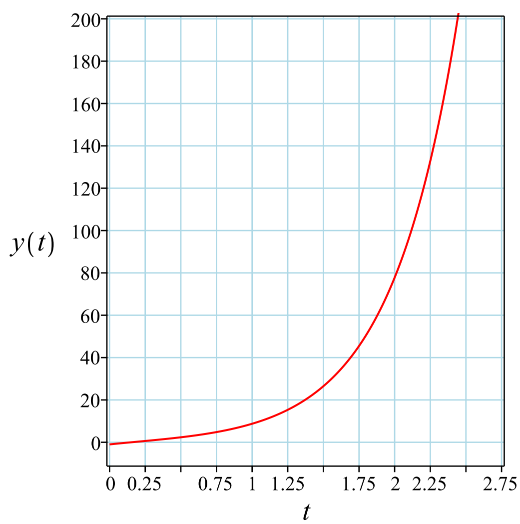
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{-3t}\left(\left(t+\frac{62}{15}\right)e^{5t}+\frac{17}{10}-\frac{5\,e^{2t}}{6}\right)}{5} \\ \frac{e^{-3t}\left(\left(t+\frac{77}{15}\right)e^{5t}-\frac{34}{5}-\frac{10\,e^{2t}}{3}\right)}{5} \end{bmatrix}$$



The following are plots of each solution.





✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 60

```
dsolve([diff(x(t),t) = x(t)+y(t)+exp(-t), diff(y(t),t) = 4*x(t)-2*y(t)+exp(2*t), x(0) = 1, y
```

$$x(t) = \frac{62\,e^{2t}}{75} + \frac{17\,e^{-3t}}{50} + \frac{e^{2t}t}{5} - \frac{e^{-t}}{6}$$
$$y(t) = \frac{77\,e^{2t}}{75} - \frac{34\,e^{-3t}}{25} + \frac{e^{2t}t}{5} - \frac{2\,e^{-t}}{3}$$

✓ Solution by Mathematica

Time used: 0.69 (sec). Leaf size: 67

```
DSolve[{x'[t]==x[t]+y[t]+Exp[-t],y'[t]==4*x[t]-2*y[t]+Exp[2*t]},{x[0]==1,y[0]==-1},{x[t],y[t
```

$$x(t) \to \frac{1}{150}e^{-3t}\left(2e^{5t}(15t+62) - 25e^{2t} + 51\right)$$
$$y(t) \to \frac{1}{75}e^{-3t}\left(e^{5t}(15t+77) - 50e^{2t} - 102\right)$$

14.7 problem 26.1 (vii)

Internal problem ID [12087]
Internal file name [`OUTPUT/10739_Monday_September_11_2023_12_50_18_AM_68308895/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 26, Explicit solutions of coupled linear systems. Exercises page 257
Problem number: 26.1 (vii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 8x(t) + 14y(t)$$
$$y'(t) = 7x(t) + y(t)$$

With initial conditions

$$[x(0) = 1, y(0) = 1]$$

14.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(2\,\mathrm{e}^{21t}+1)\,\mathrm{e}^{-6t}}{3} & \frac{2(\mathrm{e}^{21t}-1)\,\mathrm{e}^{-6t}}{3} \\ \frac{(\mathrm{e}^{21t}-1)\,\mathrm{e}^{-6t}}{3} & \frac{(\mathrm{e}^{21t}+2)\,\mathrm{e}^{-6t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{(2\,\mathrm{e}^{21t}+1)\mathrm{e}^{-6t}}{3} & \frac{2(\mathrm{e}^{21t}-1)\mathrm{e}^{-6t}}{3} \\ \frac{(\mathrm{e}^{21t}-1)\mathrm{e}^{-6t}}{3} & \frac{(\mathrm{e}^{21t}+2)\mathrm{e}^{-6t}}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2\,\mathrm{e}^{21t}+1)\mathrm{e}^{-6t}}{3} + \frac{2(\mathrm{e}^{21t}-1)\mathrm{e}^{-6t}}{3} \\ \frac{(\mathrm{e}^{21t}-1)\mathrm{e}^{-6t}}{3} + \frac{(\mathrm{e}^{21t}+2)\mathrm{e}^{-6t}}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{(4\,\mathrm{e}^{21t}-1)\mathrm{e}^{-6t}}{3} \\ \frac{(2\,\mathrm{e}^{21t}+1)\mathrm{e}^{-6t}}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 8-\lambda & 14 \\ 7 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 9\lambda - 90 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -6$$
$$\lambda_2 = 15$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-6	1	real eigenvalue
15	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} - (-6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 14 & 14 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 14 & 14 & 0 \\ 7 & 7 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{cc|c} 14 & 14 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 14 & 14 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 15$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} - (15) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 14 \\ 7 & -14 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -7 & 14 & 0 \\ 7 & -14 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -7 & 14 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -7 & 14 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-6	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
15	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -6 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-6t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$$

Since eigenvalue 15 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{15t}$$
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{15t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-6t} \\ \mathrm{e}^{-6t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{15t} \\ \mathrm{e}^{15t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -(-2c_2 \mathrm{e}^{21t} + c_1)\,\mathrm{e}^{-6t} \\ (c_2 \mathrm{e}^{21t} + c_1)\,\mathrm{e}^{-6t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_2 - c_1 \\ c_2 + c_1 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{1}{3} \\ c_2 = \frac{2}{3} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\left(-\frac{4\,e^{21t}}{3}+\frac{1}{3}\right)e^{-6t} \\ \left(\frac{2\,e^{21t}}{3}+\frac{1}{3}\right)e^{-6t} \end{bmatrix}$$

The following is the phase plot of the system.

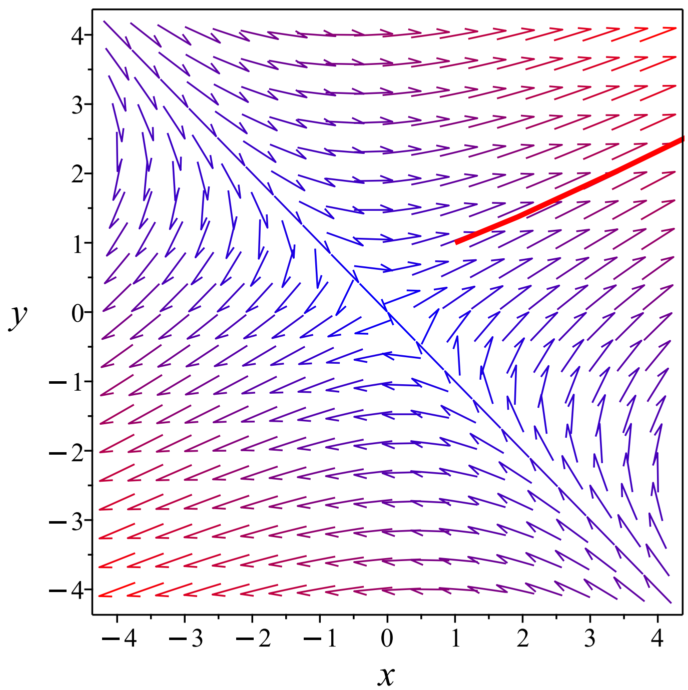
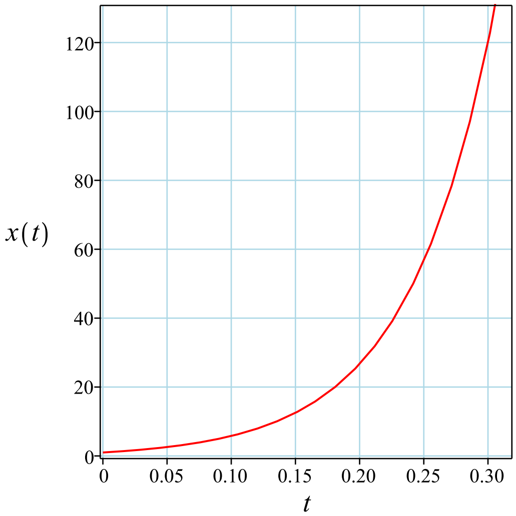
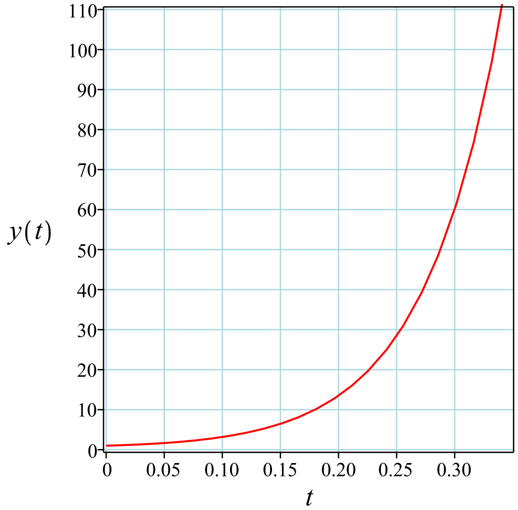


Figure 184: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve([diff(x(t),t) = 8*x(t)+14*y(t), diff(y(t),t) = 7*x(t)+y(t), x(0) = 1, y(0) = 1], sing
```

$$x(t) = -\frac{e^{-6t}}{3} + \frac{4\,e^{15t}}{3}$$
$$y(t) = \frac{e^{-6t}}{3} + \frac{2\,e^{15t}}{3}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 44

```
DSolve[{x'[t]==8*x[t]+14*y[t],y'[t]==7*x[t]+y[t]},{x[0]==1,y[0]==1},{x[t],y[t]},t,IncludeSin
```

$$x(t) \to \frac{1}{3}e^{-6t}\left(4e^{21t} - 1\right)$$
$$y(t) \to \frac{1}{3}e^{-6t}\left(2e^{21t} + 1\right)$$

15 Chapter 27, Eigenvalues and eigenvectors. Exercises page 267

15.1 problem 27.1 (ii)

Internal problem ID [12088]
Internal file name [OUTPUT/10740_Monday_September_11_2023_12_50_18_AM_9897443/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 27, Eigenvalues and eigenvectors. Exercises page 267
Problem number: 27.1 (ii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 2 & 2 \\ 0 & -4 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 2 & 2 \\ 0 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 2-\lambda & 2 \\ 0 & -4-\lambda \end{bmatrix} = 0$$
$$(-2+\lambda)(4+\lambda) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = -4$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
-4	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 2 & 2 \\ 0 & -4 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 2 & 2 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 2 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & -6 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + 3R_1 \implies \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$
\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = -4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 2 & 2 \\ 0 & -4 \end{bmatrix} - (-4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 2 & 2 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 6 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	1	2	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
-4	1	2	No	$\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 2 & 2 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}^{-1}$$

15.2 problem 27.1 (iii)

Internal problem ID [12089]
Internal file name [`OUTPUT/10741_Monday_September_11_2023_12_50_19_AM_8761059/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 27, Eigenvalues and eigenvectors. Exercises page 267
Problem number: 27.1 (iii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 7 & -2 \\ 26 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 7 & -2 \\ 26 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 7-\lambda & -2 \\ 26 & -1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 6\lambda + 45 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 3 + 6i$$
$$\lambda_2 = 3 - 6i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$3 - 6i$	1	complex eigenvalue
$3 + 6i$	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 3 - 6i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 7 & -2 \\ 26 & -1 \end{bmatrix} - (3-6i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 7 & -2 \\ 26 & -1 \end{bmatrix} - \begin{bmatrix} 3-6i & 0 \\ 0 & 3-6i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 4+6i & -2 \\ 26 & -4+6i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 4+6i & -2 & 0 \\ 26 & -4+6i & 0 \end{array}\right]
$$

$$
R_2 = R_2 + (-2+3i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 4+6i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 4+6i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{2}{13} - \frac{3i}{13}\right)t\right\}$

Hence the solution is

$$
\begin{bmatrix} \left(\frac{2}{13} - \frac{3\,\mathrm{I}}{13}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{13} - \frac{3i}{13}\right)t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{2}{13} - \frac{3\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{13} - \frac{3i}{13} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \left(\frac{2}{13} - \frac{3\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{13} - \frac{3i}{13} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \left(\frac{2}{13} - \frac{3\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 2 - 3i \\ 13 \end{bmatrix}$$

Considering $\lambda = 3 + 6i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 7 & -2 \\ 26 & -1 \end{bmatrix} - (3+6i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 7 & -2 \\ 26 & -1 \end{bmatrix} - \begin{bmatrix} 3+6i & 0 \\ 0 & 3+6i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 4-6i & -2 \\ 26 & -4-6i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 4-6i & -2 & 0 \\ 26 & -4-6i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-2-3i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 4-6i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4-6i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{2}{13} + \frac{3i}{13}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{2}{13} + \frac{3\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{13} + \frac{3i}{13}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{2}{13} + \frac{3\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{13} + \frac{3i}{13} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \left(\frac{2}{13} + \frac{3\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{13} + \frac{3i}{13} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \left(\frac{2}{13} + \frac{3\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 2+3i \\ 13 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$3-6i$	1	2	No	$\begin{bmatrix} 2-3i \\ 13 \end{bmatrix}$
$3+6i$	1	2	No	$\begin{bmatrix} 2+3i \\ 13 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 3-6i & 0 \\ 0 & 3+6i \end{bmatrix}$$
$$P = \begin{bmatrix} 2-3i & 2+3i \\ 13 & 13 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 7 & -2 \\ 26 & -1 \end{bmatrix} = \begin{bmatrix} 2-3i & 2+3i \\ 13 & 13 \end{bmatrix} \begin{bmatrix} 3-6i & 0 \\ 0 & 3+6i \end{bmatrix} \begin{bmatrix} 2-3i & 2+3i \\ 13 & 13 \end{bmatrix}^{-1}$$

15.3 problem 27.1 (iv)

Internal problem ID [12090]
Internal file name [`OUTPUT/10742_Monday_September_11_2023_12_50_19_AM_71352917/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 27, Eigenvalues and eigenvectors. Exercises page 267
Problem number: 27.1 (iv).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 9-\lambda & 2 \\ 2 & 6-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 15\lambda + 50 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 10$$
$$\lambda_2 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
5	1	real eigenvalue
10	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$
\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Considering $\lambda = 10$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix} - (10) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
5	1	2	No	$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$
10	1	2	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$$

15.4 problem 27.1 (v)

Internal problem ID [12091]
Internal file name [OUTPUT/10743_Monday_September_11_2023_12_50_19_AM_24070263/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 27, Eigenvalues and eigenvectors. Exercises page 267
Problem number: 27.1 (v).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 7 & 1 \\ -4 & 11 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 7 & 1 \\ -4 & 11 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 7-\lambda & 1 \\ -4 & 11-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 18\lambda + 81 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 9$$
$$\lambda_2 = 9$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
9	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 9$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 7 & 1 \\ -4 & 11 \end{bmatrix} - (9)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 7 & 1 \\ -4 & 11 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ -4 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
9	2	2	No	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 7 & 1 \\ -4 & 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}^{-1}$$

15.5 problem 27.1 (vi)

Internal problem ID [12092]
Internal file name [OUTPUT/10744_Monday_September_11_2023_12_50_19_AM_98924648/index.tex]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 27, Eigenvalues and eigenvectors. Exercises page 267
Problem number: 27.1 (vi).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det \begin{bmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 4\lambda + 13 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2 + 3i$$
$$\lambda_2 = 2 - 3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2 - 3i$	1	complex eigenvalue
$2 + 3i$	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2 - 3i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} - (2-3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 2-3i & 0 \\ 0 & 2-3i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 3i & -3 \\ 3 & 3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 3i & -3 & 0 \\ 3 & 3i & 0 \end{array}\right]
$$

$$
R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 3i & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 3i & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$
\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering $\lambda = 2 + 3i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} - (2+3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 2+3i & 0 \\ 0 & 2+3i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -3i & -3 & 0 \\ 3 & -3i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -3i & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3i & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$2-3i$	1	2	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$2+3i$	1	2	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2-3i & 0 \\ 0 & 2+3i \end{bmatrix}$$

$$P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2-3i & 0 \\ 0 & 2+3i \end{bmatrix} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}^{-1}$$

15.6 problem 27.1 (vii)

Internal problem ID [12093]
Internal file name [`OUTPUT/10745_Monday_September_11_2023_12_50_20_AM_917783/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 27, Eigenvalues and eigenvectors. Exercises page 267
Problem number: 27.1 (vii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 6 & 0 \\ 0 & -13 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 6 & 0 \\ 0 & -13 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 6-\lambda & 0 \\ 0 & -13-\lambda \end{bmatrix} = 0$$
$$(-6+\lambda)(13+\lambda) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = -13$$
$$\lambda_2 = 6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
6	1	real eigenvalue
-13	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 6$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 6 & 0 \\ 0 & -13 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 6 & 0 \\ 0 & -13 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 \\ 0 & -19 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -19 & 0 \end{array}\right]$$

Since the current pivot $A(1, 2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 0 & -19 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -19 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = -13$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 6 & 0 \\ 0 & -13 \end{bmatrix} - (-13) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 6 & 0 \\ 0 & -13 \end{bmatrix} - \begin{bmatrix} -13 & 0 \\ 0 & -13 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 19 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 19 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 19 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
6	1	2	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
-13	1	2	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 6 & 0 \\ 0 & -13 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 6 & 0 \\ 0 & -13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -13 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

15.7 problem 27.1 (viii)

Internal problem ID [12094]
Internal file name [`OUTPUT/10746_Monday_September_11_2023_12_50_20_AM_17291308/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 27, Eigenvalues and eigenvectors. Exercises page 267
Problem number: 27.1 (viii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det \begin{bmatrix} 4-\lambda & -2 \\ 1 & 2-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 6\lambda + 10 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 3 + i$$
$$\lambda_2 = 3 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$3 + i$	1	complex eigenvalue
$3 - i$	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 3 + i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix} - (3+i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3+i & 0 \\ 0 & 3+i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} - \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = t\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Considering $\lambda = 3 - i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix} - (3-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3-i & 0 \\ 0 & 3-i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+i & -2 & 0 \\ 1 & -1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$3+i$	1	2	No	$\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$
$3-i$	1	2	No	$\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 3+i & 0 \\ 0 & 3-i \end{bmatrix}$$

$$P = \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3+i & 0 \\ 0 & 3-i \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix}^{-1}$$

15.8 problem 27.1 (ix)

Internal problem ID [12095]
Internal file name [`OUTPUT/10747_Monday_September_11_2023_12_50_20_AM_3171660/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 27, Eigenvalues and eigenvectors. Exercises page 267
Problem number: 27.1 (ix).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 4\lambda + 4 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$
\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	2	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{-1}$$

15.9 problem 27.1 (x)

Internal problem ID [12096]
Internal file name [`OUTPUT/10748_Monday_September_11_2023_12_50_21_AM_81019018/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 27, Eigenvalues and eigenvectors. Exercises page 267
Problem number: 27.1 (x).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} -7 & 6 \\ 12 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} -7 & 6 \\ 12 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} -7-\lambda & 6 \\ 12 & -1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 + 8\lambda - 65 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 5$$
$$\lambda_2 = -13$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
5	1	real eigenvalue
-13	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} -7 & 6 \\ 12 & -1 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -7 & 6 \\ 12 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -12 & 6 \\ 12 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -12 & 6 & 0 \\ 12 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -12 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -12 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Considering $\lambda = -13$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} -7 & 6 \\ 12 & -1 \end{bmatrix} - (-13) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} -7 & 6 \\ 12 & -1 \end{bmatrix} - \begin{bmatrix} -13 & 0 \\ 0 & -13 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 6 & 6 \\ 12 & 12 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & 6 & 0 \\ 12 & 12 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 6 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
5	1	2	No	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
-13	1	2	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -13 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} -7 & 6 \\ 12 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -13 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1}$$

16 Chapter 28, Distinct real eigenvalues. Exercises page 282

16.1 problem 28.2 (i)

Internal problem ID [12097]
Internal file name [`OUTPUT/10749_Monday_September_11_2023_12_50_21_AM_19334543/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 28, Distinct real eigenvalues. Exercises page 282
Problem number: 28.2 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 8x(t) + 14y(t) \\
y'(t) &= 7x(t) + y(t)
\end{aligned}
$$

16.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(2\,\mathrm{e}^{21t}+1)\,\mathrm{e}^{-6t}}{3} & \frac{2(\mathrm{e}^{21t}-1)\,\mathrm{e}^{-6t}}{3} \\ \frac{(\mathrm{e}^{21t}-1)\,\mathrm{e}^{-6t}}{3} & \frac{(\mathrm{e}^{21t}+2)\,\mathrm{e}^{-6t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(2\,\mathrm{e}^{21t}+1)\,\mathrm{e}^{-6t}}{3} & \frac{2(\mathrm{e}^{21t}-1)\,\mathrm{e}^{-6t}}{3} \\ \frac{(\mathrm{e}^{21t}-1)\,\mathrm{e}^{-6t}}{3} & \frac{(\mathrm{e}^{21t}+2)\,\mathrm{e}^{-6t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2\,\mathrm{e}^{21t}+1)\,\mathrm{e}^{-6t}c_1}{3} + \frac{2(\mathrm{e}^{21t}-1)\,\mathrm{e}^{-6t}c_2}{3} \\ \frac{(\mathrm{e}^{21t}-1)\,\mathrm{e}^{-6t}c_1}{3} + \frac{(\mathrm{e}^{21t}+2)\,\mathrm{e}^{-6t}c_2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{-6t}\left((c_1+c_2)\,\mathrm{e}^{21t}+\frac{c_1}{2}-c_2\right)}{3} \\ \frac{\left((c_1+c_2)\,\mathrm{e}^{21t}-c_1+2c_2\right)\mathrm{e}^{-6t}}{3} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 8-\lambda & 14 \\ 7 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 9\lambda - 90 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -6$$
$$\lambda_2 = 15$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-6	1	real eigenvalue
15	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} - (-6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 14 & 14 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 14 & 14 & 0 \\ 7 & 7 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{cc|c} 14 & 14 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 14 & 14 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 15$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} - (15) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 14 \\ 7 & -14 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -7 & 14 & 0 \\ 7 & -14 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -7 & 14 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -7 & 14 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-6	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
15	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -6 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-6t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}\end{aligned}$$

Since eigenvalue 15 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{15t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{15t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-6t} \\ \mathrm{e}^{-6t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{15t} \\ \mathrm{e}^{15t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\left(-2c_2 \mathrm{e}^{21t} + c_1\right) \mathrm{e}^{-6t} \\ \left(c_2 \mathrm{e}^{21t} + c_1\right) \mathrm{e}^{-6t} \end{bmatrix}$$

The following is the phase plot of the system.

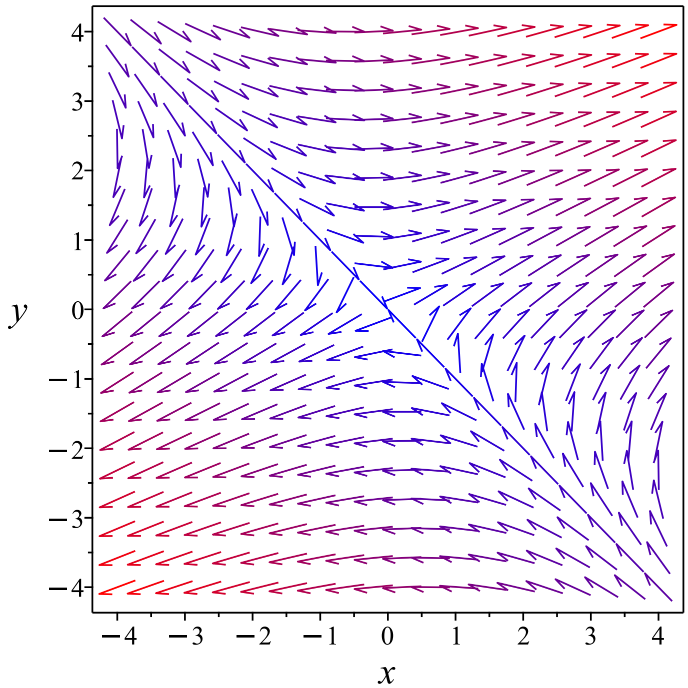


Figure 185: Phase plot

16.1.3 Maple step by step solution

Let's solve

$$[x'(t) = 8x(t) + 14y(t), y'(t) = 7x(t) + y(t)]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 8 & 14 \\ 7 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-6, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[15, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-6, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-6t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[15, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{15t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-6t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{15t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -(-2c_2\mathrm{e}^{21t} + c_1)\,\mathrm{e}^{-6t} \\ (c_2\mathrm{e}^{21t} + c_1)\,\mathrm{e}^{-6t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = -(-2c_2\mathrm{e}^{21t} + c_1)\,\mathrm{e}^{-6t}, y(t) = (c_2\mathrm{e}^{21t} + c_1)\,\mathrm{e}^{-6t}\}$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=8*x(t)+14*y(t),diff(y(t),t)=7*x(t)+y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{-6t}c_1 + c_2\mathrm{e}^{15t}$$
$$y(t) = -\mathrm{e}^{-6t}c_1 + \frac{c_2\mathrm{e}^{15t}}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 71

```
DSolve[{x'[t]==8*x[t]+14*y[t],y'[t]==7*x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{3}e^{-6t}\left(c_1\left(2e^{21t} + 1\right) + 2c_2\left(e^{21t} - 1\right)\right)$$
$$y(t) \to \frac{1}{3}e^{-6t}\left(c_1\left(e^{21t} - 1\right) + c_2\left(e^{21t} + 2\right)\right)$$

16.2 problem 28.2 (ii)

Internal problem ID [12098]
Internal file name [`OUTPUT/10750_Monday_September_11_2023_12_50_21_AM_76006389/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 28, Distinct real eigenvalues. Exercises page 282
Problem number: 28.2 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2x(t)$$
$$y'(t) = -5x(t) - 3y(t)$$

16.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t} & 0 \\ -(e^{5t} - 1)\, e^{-3t} & e^{-3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t} & 0 \\ -(e^{5t}-1)\,e^{-3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 \\ -(e^{5t}-1)\,e^{-3t}c_1 + e^{-3t}c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 \\ -(e^{5t}c_1 - c_1 - c_2)\,e^{-3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & 0 \\ -5 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & 0 \\ -5 & -3-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(2-\lambda)(-3-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-3$$
$$\lambda_2=2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-3$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}2 & 0\\ -5 & -3\end{bmatrix}-(-3)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}5 & 0\\ -5 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}5 & 0 & 0\\ -5 & 0 & 0\end{array}\right]$$

$$R_2=R_2+R_1\Longrightarrow\left[\begin{array}{cc|c}5 & 0 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}5 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 \\ -5 & -3 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -5 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} -5 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-3t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t}\end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_2 \mathrm{e}^{2t} \\ (c_2 \mathrm{e}^{5t} + c_1)\, \mathrm{e}^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

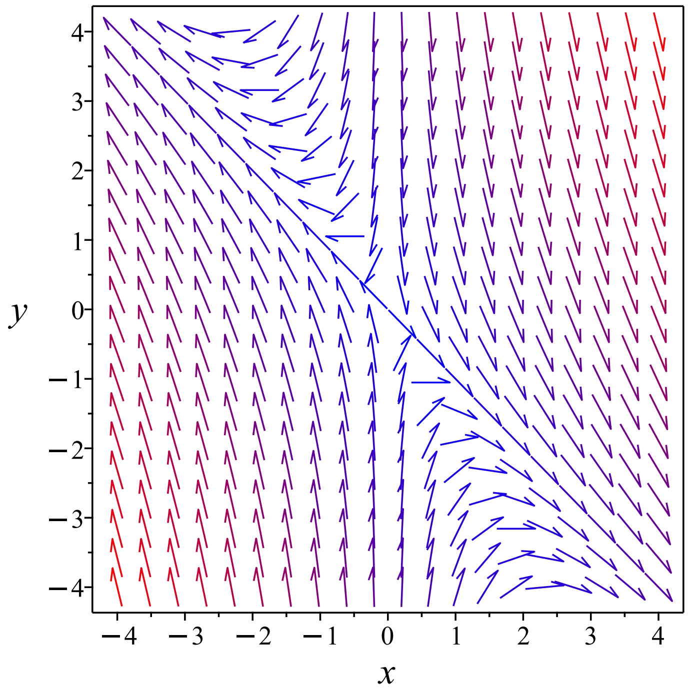


Figure 186: Phase plot

16.2.3 Maple step by step solution

Let's solve

$$[x'(t) = 2x(t), y'(t) = -5x(t) - 3y(t)]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 0 \\ -5 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 0 \\ -5 & -3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 0 \\ -5 & -3 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

  $$\left[\left[-3, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-3, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = e^{-3t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = e^{-3t} c_1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_2 e^{2t} \\ (c_2 e^{5t} + c_1) e^{-3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = -c_2 e^{2t}, y(t) = (c_2 e^{5t} + c_1) e^{-3t}\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve([diff(x(t),t)=2*x(t),diff(y(t),t)=-5*x(t)-3*y(t)],singsol=all)
```

$$x(t) = c_2 e^{2t}$$
$$y(t) = -c_2 e^{2t} + c_1 e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 36

```
DSolve[{x'[t]==2*x[t],y'[t]==-5*x[t]-3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{2t}$$
$$y(t) \to e^{-3t}\left(c_1\left(-e^{5t}\right) + c_1 + c_2\right)$$

16.3 problem 28.2 (iii)

Internal problem ID [12099]
Internal file name [`OUTPUT/10751_Monday_September_11_2023_12_50_22_AM_41435237/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 28, Distinct real eigenvalues. Exercises page 282
Problem number: 28.2 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 11x(t) - 2y(t)$$
$$y'(t) = 3x(t) + 4y(t)$$

16.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{5t}}{5} + \frac{6\,e^{10t}}{5} & -\frac{2\,e^{10t}}{5} + \frac{2\,e^{5t}}{5} \\ \frac{3\,e^{10t}}{5} - \frac{3\,e^{5t}}{5} & \frac{6\,e^{5t}}{5} - \frac{e^{10t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{5t}}{5} + \frac{6\,e^{10t}}{5} & -\frac{2\,e^{10t}}{5} + \frac{2\,e^{5t}}{5} \\ \frac{3\,e^{10t}}{5} - \frac{3\,e^{5t}}{5} & \frac{6\,e^{5t}}{5} - \frac{e^{10t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{5t}}{5} + \frac{6\,e^{10t}}{5}\right) c_1 + \left(-\frac{2\,e^{10t}}{5} + \frac{2\,e^{5t}}{5}\right) c_2 \\ \left(\frac{3\,e^{10t}}{5} - \frac{3\,e^{5t}}{5}\right) c_1 + \left(\frac{6\,e^{5t}}{5} - \frac{e^{10t}}{5}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(6c_1-2c_2)e^{10t}}{5} - \frac{e^{5t}(c_1-2c_2)}{5} \\ \frac{(3c_1-c_2)e^{10t}}{5} - \frac{3\,e^{5t}(c_1-2c_2)}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 11-\lambda & -2 \\ 3 & 4-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 15\lambda + 50 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$
$$\lambda_2 = 10$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
5	1	real eigenvalue
10	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -2 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 6 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 10$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix} - (10) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 3 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$
10	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{5t} \\
&= \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{5t}
\end{aligned}
$$

Since eigenvalue 10 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{10t} \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{10t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{5t}}{3} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,e^{10t} \\ e^{10t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{5t}}{3} + 2c_2 e^{10t} \\ c_1 e^{5t} + c_2 e^{10t} \end{bmatrix}
$$

The following is the phase plot of the system.

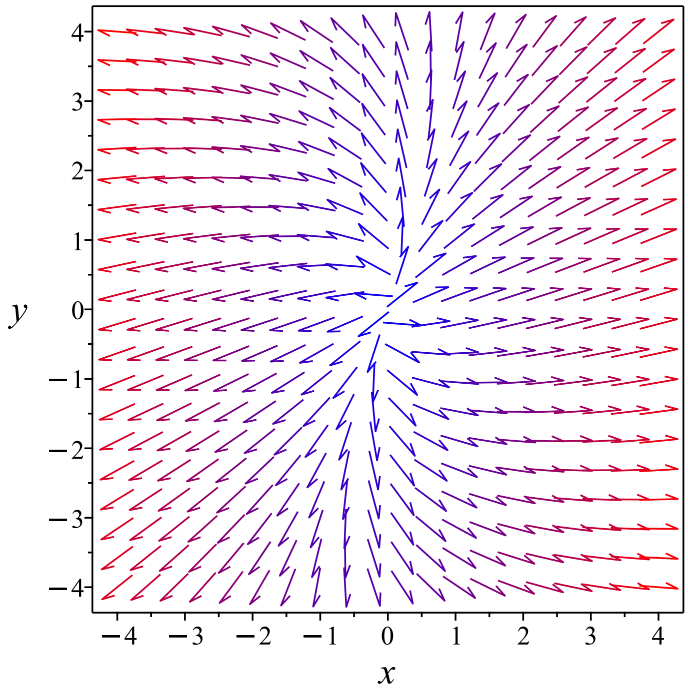


Figure 187: Phase plot

16.3.3 Maple step by step solution

Let's solve

$[x'(t) = 11x(t) - 2y(t), y'(t) = 3x(t) + 4y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[5, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[10, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[5, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{5t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[10, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{10t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_1 \mathrm{e}^{5t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{10t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{5t}}{3} + 2c_2 e^{10t} \\ c_1 e^{5t} + c_2 e^{10t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x(t) = \frac{c_1 e^{5t}}{3} + 2c_2 e^{10t}, y(t) = c_1 e^{5t} + c_2 e^{10t}\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=11*x(t)-2*y(t),diff(y(t),t)=3*x(t)+4*y(t)],singsol=all)
```

$$x(t) = e^{5t} c_1 + c_2 e^{10t}$$
$$y(t) = 3\, e^{5t} c_1 + \frac{c_2 e^{10t}}{2}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 95

```
DSolve[{x'[t]==2*x[t]-2*y[t],y'[t]==3*x[t]+4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{5} e^{3t} \left(5c_1 \cos\left(\sqrt{5}t\right) - \sqrt{5}(c_1 + 2c_2) \sin\left(\sqrt{5}t\right)\right)$$
$$y(t) \to \frac{1}{5} e^{3t} \left(5c_2 \cos\left(\sqrt{5}t\right) + \sqrt{5}(3c_1 + c_2) \sin\left(\sqrt{5}t\right)\right)$$

16.4 problem 28.2 (iv)

Internal problem ID [12100]
Internal file name [`OUTPUT/10752_Monday_September_11_2023_12_50_22_AM_18660549/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 28, Distinct real eigenvalues. Exercises page 282
Problem number: 28.2 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = x(t) + 20y(t)$$
$$y'(t) = 40x(t) - 19y(t)$$

16.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 20 \\ 40 & -19 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(2\,e^{60t}+1)e^{-39t}}{3} & \frac{(e^{60t}-1)e^{-39t}}{3} \\ \frac{2(e^{60t}-1)e^{-39t}}{3} & \frac{(e^{60t}+2)e^{-39t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(2\,\mathrm{e}^{60t}+1)\mathrm{e}^{-39t}}{3} & \frac{(\mathrm{e}^{60t}-1)\mathrm{e}^{-39t}}{3} \\ \frac{2(\mathrm{e}^{60t}-1)\mathrm{e}^{-39t}}{3} & \frac{(\mathrm{e}^{60t}+2)\mathrm{e}^{-39t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2\,\mathrm{e}^{60t}+1)\mathrm{e}^{-39t}c_1}{3} + \frac{(\mathrm{e}^{60t}-1)\mathrm{e}^{-39t}c_2}{3} \\ \frac{2(\mathrm{e}^{60t}-1)\mathrm{e}^{-39t}c_1}{3} + \frac{(\mathrm{e}^{60t}+2)\mathrm{e}^{-39t}c_2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-39t}((2c_1+c_2)\mathrm{e}^{60t}+c_1-c_2)}{3} \\ \frac{2\,\mathrm{e}^{-39t}((c_1+\frac{c_2}{2})\mathrm{e}^{60t}-c_1+c_2)}{3} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 20 \\ 40 & -19 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 20 \\ 40 & -19 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 20 \\ 40 & -19-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 18\lambda - 819 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -39$$
$$\lambda_2 = 21$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
21	1	real eigenvalue
−39	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -39$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 20 \\ 40 & -19 \end{bmatrix} - (-39) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 40 & 20 \\ 40 & 20 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 40 & 20 & 0 \\ 40 & 20 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 40 & 20 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 40 & 20 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -\frac{t}{2}\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 21$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 20 \\ 40 & -19 \end{bmatrix} - (21) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -20 & 20 \\ 40 & -40 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -20 & 20 & 0 \\ 40 & -40 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -20 & 20 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -20 & 20 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-39	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
21	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -39 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-39t} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{-39t}\end{aligned}$$

Since eigenvalue 21 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{21t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{21t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\mathrm{e}^{-39t}}{2} \\ \mathrm{e}^{-39t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{21t} \\ \mathrm{e}^{21t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{(-2c_2 \mathrm{e}^{60t} + c_1)\mathrm{e}^{-39t}}{2} \\ (c_2 \mathrm{e}^{60t} + c_1)\, \mathrm{e}^{-39t} \end{bmatrix}$$

The following is the phase plot of the system.

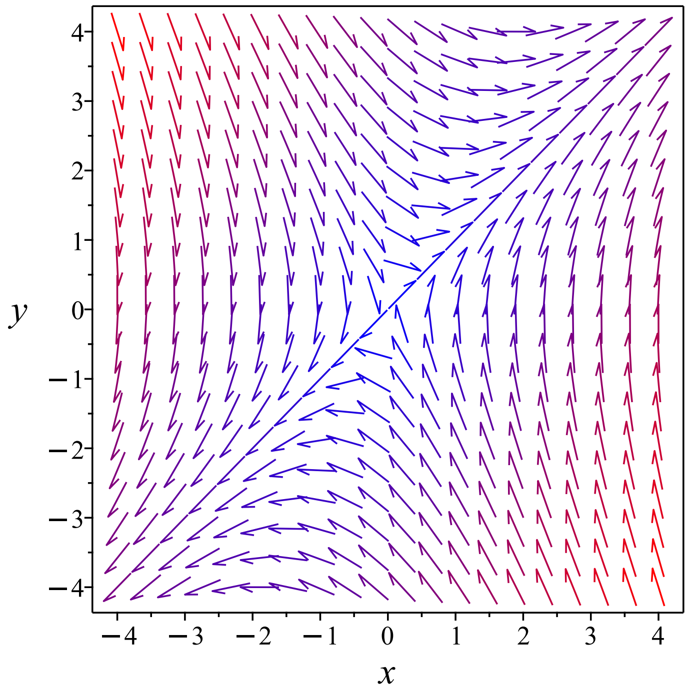


Figure 188: Phase plot

16.4.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + 20y(t), y'(t) = 40x(t) - 19y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 20 \\ 40 & -19 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 20 \\ 40 & -19 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 20 \\ 40 & -19 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-39, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[21, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-39, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{-39t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[21, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{21t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_1 \mathrm{e}^{-39t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{21t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{(-2c_2 e^{60t}+c_1)e^{-39t}}{2} \\ (c_2 e^{60t} + c_1)e^{-39t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x(t) = -\frac{(-2c_2 e^{60t}+c_1)e^{-39t}}{2}, y(t) = (c_2 e^{60t} + c_1)e^{-39t}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=x(t)+20*y(t),diff(y(t),t)=40*x(t)-19*y(t)],singsol=all)
```

$$x(t) = c_1 e^{21t} + c_2 e^{-39t}$$
$$y(t) = c_1 e^{21t} - 2c_2 e^{-39t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 71

```
DSolve[{x'[t]==x[t]+20*y[t],y'[t]==40*x[t]-19*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -
```

$$x(t) \to \frac{1}{3}e^{-39t}\left(c_1\left(2e^{60t}+1\right)+c_2\left(e^{60t}-1\right)\right)$$
$$y(t) \to \frac{1}{3}e^{-39t}\left(2c_1\left(e^{60t}-1\right)+c_2\left(e^{60t}+2\right)\right)$$

16.5 problem 28.6 (iii)

Internal problem ID [12101]
Internal file name [`OUTPUT/10753_Monday_September_11_2023_12_50_23_AM_44456228/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 28, Distinct real eigenvalues. Exercises page 282
Problem number: 28.6 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) + 2y(t) \\
y'(t) &= x(t) - y(t)
\end{aligned}
$$

16.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,\mathrm{e}^{-3t}}{3} + \frac{1}{3} & \frac{2}{3} - \frac{2\,\mathrm{e}^{-3t}}{3} \\ \frac{1}{3} - \frac{\mathrm{e}^{-3t}}{3} & \frac{\mathrm{e}^{-3t}}{3} + \frac{2}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{-3t}}{3}+\frac{1}{3} & \frac{2}{3}-\frac{2\,\mathrm{e}^{-3t}}{3} \\ \frac{1}{3}-\frac{\mathrm{e}^{-3t}}{3} & \frac{\mathrm{e}^{-3t}}{3}+\frac{2}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,\mathrm{e}^{-3t}}{3}+\frac{1}{3}\right)c_1+\left(\frac{2}{3}-\frac{2\,\mathrm{e}^{-3t}}{3}\right)c_2 \\ \left(\frac{1}{3}-\frac{\mathrm{e}^{-3t}}{3}\right)c_1+\left(\frac{\mathrm{e}^{-3t}}{3}+\frac{2}{3}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1-2c_2)\mathrm{e}^{-3t}}{3}+\frac{c_1}{3}+\frac{2c_2}{3} \\ \frac{(-c_1+c_2)\mathrm{e}^{-3t}}{3}+\frac{c_1}{3}+\frac{2c_2}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 2 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 3\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
−3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-3t} \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-3t}\end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{0} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -2\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -2c_1 \mathrm{e}^{-3t} + c_2 \\ c_1 \mathrm{e}^{-3t} + c_2 \end{bmatrix}$$

The following is the phase plot of the system.

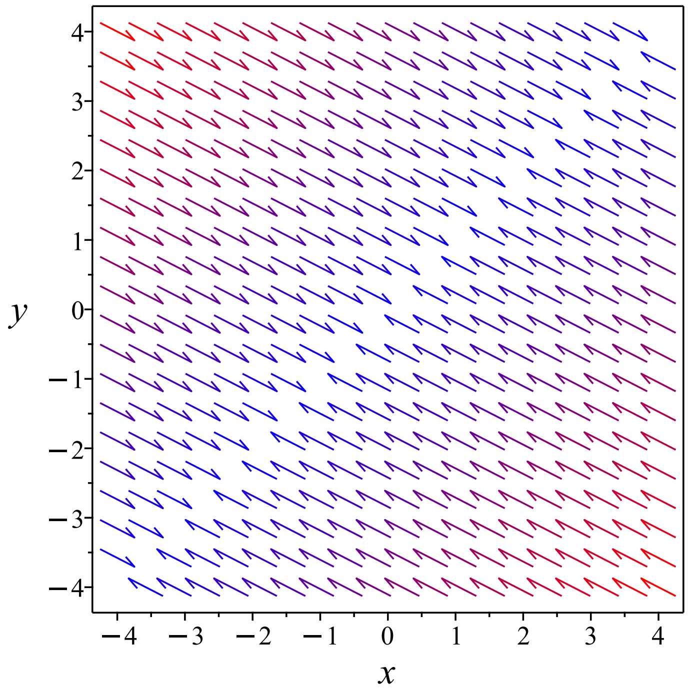


Figure 189: Phase plot

16.5.3 Maple step by step solution

Let's solve

$[x'(t) = -2x(t) + 2y(t), y'(t) = x(t) - y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = \mathrm{e}^{-3t} c_1 \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ c_2 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -2\,\mathrm{e}^{-3t}c_1 + c_2 \\ \mathrm{e}^{-3t}c_1 + c_2 \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = -2\,\mathrm{e}^{-3t}c_1 + c_2, y(t) = \mathrm{e}^{-3t}c_1 + c_2\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve([diff(x(t),t)=-2*x(t)+2*y(t),diff(y(t),t)=x(t)-y(t)],singsol=all)
```

$$x(t) = c_1 + c_2\mathrm{e}^{-3t}$$
$$y(t) = -\frac{c_2\mathrm{e}^{-3t}}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 71

```
DSolve[{x'[t]==-2*x[t]+2*y[t],y'[t]==x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tr
```

$$x(t) \to \frac{1}{3}e^{-3t}\left(c_1\left(e^{3t}+2\right)+2c_2\left(e^{3t}-1\right)\right)$$
$$y(t) \to \frac{1}{3}e^{-3t}\left(c_1\left(e^{3t}-1\right)+c_2\left(2e^{3t}+1\right)\right)$$

17 Chapter 29, Complex eigenvalues. Exercises page 292

17.1 problem 29.3 (i)

Internal problem ID [12102]
Internal file name [`OUTPUT/10754_Monday_September_11_2023_12_50_23_AM_30234506/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 29, Complex eigenvalues. Exercises page 292
Problem number: 29.3 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -y(t)$$
$$y'(t) = x(t) - y(t)$$

17.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)+\frac{e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} & -\frac{2\,e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} \\ \frac{2\,e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} & e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)-\frac{e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{-\frac{t}{2}}\left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)+3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)}{3} & -\frac{2\,e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} \\ \frac{2\,e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} & -\frac{\left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)-3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)e^{-\frac{t}{2}}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-\frac{t}{2}}\left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)+3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)}{3} & -\frac{2\,e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} \\ \frac{2\,e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} & -\frac{\left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)-3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)e^{-\frac{t}{2}}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-\frac{t}{2}}\left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)+3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)c_1}{3} - \frac{2\,e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2}{3} \\ \frac{2\,e^{-\frac{t}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_1}{3} - \frac{\left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)-3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)e^{-\frac{t}{2}}c_2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left(\sqrt{3}\,(c_1-2c_2)\sin\left(\frac{\sqrt{3}\,t}{2}\right)+3\cos\left(\frac{\sqrt{3}\,t}{2}\right)c_1\right)e^{-\frac{t}{2}}}{3} \\ \frac{2\,e^{-\frac{t}{2}}\left(\sqrt{3}\,\left(c_1-\frac{c_2}{2}\right)\sin\left(\frac{\sqrt{3}\,t}{2}\right)+\frac{3\cos\left(\frac{\sqrt{3}\,t}{2}\right)c_2}{2}\right)}{3} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-\lambda & -1\\ 1 & -1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	1	complex eigenvalue
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix} - \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{1}{2} + \frac{i\sqrt{3}}{2} & -1\\ 1 & -\frac{1}{2} + \frac{i\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{1}{2} + \frac{i\sqrt{3}}{2} & -1 & 0\\ 1 & -\frac{1}{2} + \frac{i\sqrt{3}}{2} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{2} + \frac{i\sqrt{3}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} + \frac{i\sqrt{3}}{2} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} + \frac{i\sqrt{3}}{2} & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{1+i\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{1+i\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} - \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{3}}{2} & -1 \\ 1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{3}}{2} & -1 & 0 \\ 1 & -\frac{1}{2}-\frac{i\sqrt{3}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{3}}{2} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}-\frac{i\sqrt{3}}{2} & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{i\sqrt{3}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{i\sqrt{3}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	1	1	No	$\left[\begin{array}{c} -\frac{1}{-\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\ 1 \end{array}\right]$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	1	1	No	$\left[\begin{array}{c} -\frac{1}{-\frac{1}{2}-\frac{i\sqrt{3}}{2}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\ e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{3}}{2}} \\ e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\frac{i c_2\left(\sqrt{3}+i\right) e^{-\frac{\left(1+i\sqrt{3}\right)t}{2}}}{2} - \frac{i e^{\frac{\left(i\sqrt{3}-1\right)t}{2}} c_1 \left(i-\sqrt{3}\right)}{2} \\ c_1 e^{\frac{\left(i\sqrt{3}-1\right)t}{2}} + c_2 e^{-\frac{\left(1+i\sqrt{3}\right)t}{2}} \end{array}\right]$$

The following is the phase plot of the system.

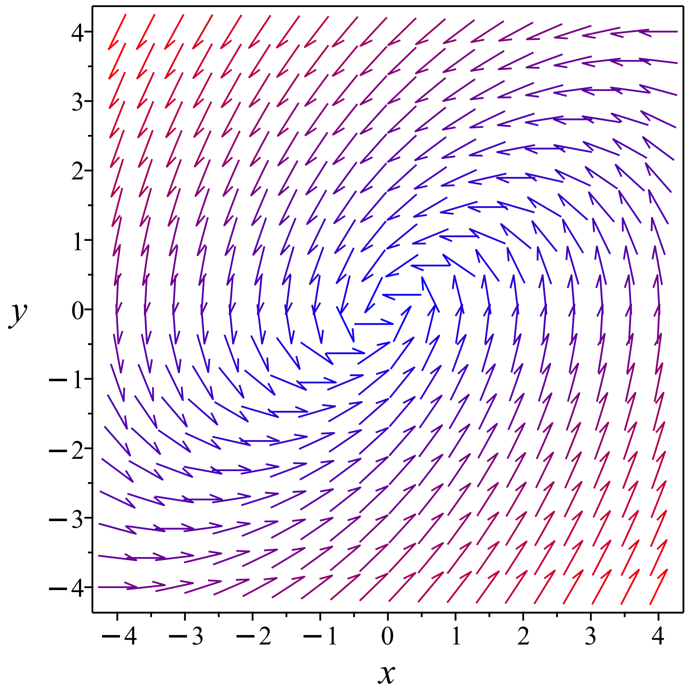


Figure 190: Phase plot

17.1.3 Maple step by step solution

Let's solve

$[x'(t) = -y(t), y'(t) = x(t) - y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} -\frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} -\frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} -\frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)t} \cdot \begin{bmatrix} -\frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-\frac{t}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right) \cdot \begin{bmatrix} -\frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{-\frac{t}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,t}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{x}_1(t) = \mathrm{e}^{-\frac{t}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} - \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \\ \cos\left(\frac{\sqrt{3}\,t}{2}\right) \end{bmatrix}, \overrightarrow{x}_2(t) = \mathrm{e}^{-\frac{t}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,t}{2}\right)\sqrt{3}}{2} - \frac{\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,t}{2}\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-\frac{t}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\, t}{2}\right)}{2}-\frac{\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{2} \\ \cos\left(\frac{\sqrt{3}\, t}{2}\right) \end{array}\right] + c_2 \mathrm{e}^{-\frac{t}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{3}\, t}{2}\right)\sqrt{3}}{2}-\frac{\sin\left(\frac{\sqrt{3}\, t}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\, t}{2}\right) \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\frac{\left(\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\, t}{2}\right)+\sin\left(\frac{\sqrt{3}\, t}{2}\right)\left(\sqrt{3}\, c_1+c_2\right)\right)\mathrm{e}^{-\frac{t}{2}}}{2} \\ \mathrm{e}^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\, t}{2}\right)-c_2\sin\left(\frac{\sqrt{3}\, t}{2}\right)\right) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{\left(\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\, t}{2}\right)+\sin\left(\frac{\sqrt{3}\, t}{2}\right)\left(\sqrt{3}\, c_1+c_2\right)\right)\mathrm{e}^{-\frac{t}{2}}}{2}, y(t) = \mathrm{e}^{-\frac{t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\, t}{2}\right)-c_2\sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 82

```
dsolve([diff(x(t),t)=-y(t),diff(y(t),t)=x(t)-y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{-\frac{t}{2}}\left(\sin\left(\frac{\sqrt{3}\, t}{2}\right)c_1+\cos\left(\frac{\sqrt{3}\, t}{2}\right)c_2\right)$$

$$y(t) = \frac{\mathrm{e}^{-\frac{t}{2}}\left(\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, t}{2}\right)c_2-\sqrt{3}\, \cos\left(\frac{\sqrt{3}\, t}{2}\right)c_1+\sin\left(\frac{\sqrt{3}\, t}{2}\right)c_1+\cos\left(\frac{\sqrt{3}\, t}{2}\right)c_2\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 112

```
DSolve[{x'[t]==-y[t],y'[t]==x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{3}e^{-t/2}\left(3c_1\cos\left(\frac{\sqrt{3}t}{2}\right)+\sqrt{3}(c_1-2c_2)\sin\left(\frac{\sqrt{3}t}{2}\right)\right)$$

$$y(t) \to \frac{1}{3}e^{-t/2}\left(3c_2\cos\left(\frac{\sqrt{3}t}{2}\right)+\sqrt{3}(2c_1-c_2)\sin\left(\frac{\sqrt{3}t}{2}\right)\right)$$

17.2 problem 29.3 (ii)

Internal problem ID [12103]
Internal file name [`OUTPUT/10755_Monday_September_11_2023_12_50_24_AM_84521943/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 29, Complex eigenvalues. Exercises page 292
Problem number: 29.3 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) + 3y(t) \\
y'(t) &= -6x(t) + 4y(t)
\end{aligned}
$$

17.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{t}\cos(3t) - e^{t}\sin(3t) & e^{t}\sin(3t) \\ -2\,e^{t}\sin(3t) & e^{t}\cos(3t) + e^{t}\sin(3t) \end{bmatrix} \\
&= \begin{bmatrix} e^{t}(\cos(3t) - \sin(3t)) & e^{t}\sin(3t) \\ -2\,e^{t}\sin(3t) & e^{t}(\sin(3t) + \cos(3t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^t(\cos(3t)-\sin(3t)) & \mathrm{e}^t\sin(3t) \\ -2\,\mathrm{e}^t\sin(3t) & \mathrm{e}^t(\sin(3t)+\cos(3t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^t(\cos(3t)-\sin(3t))\,c_1 + \mathrm{e}^t\sin(3t)\,c_2 \\ -2\,\mathrm{e}^t\sin(3t)\,c_1 + \mathrm{e}^t(\sin(3t)+\cos(3t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^t((-c_1+c_2)\sin(3t)+c_1\cos(3t)) \\ -2\left(\left(c_1-\frac{c_2}{2}\right)\sin(3t)-\frac{c_2\cos(3t)}{2}\right)\mathrm{e}^t \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 3 \\ -6 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 3 \\ -6 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 10 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 3i$$
$$\lambda_2 = 1 - 3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 + 3i$	1	complex eigenvalue
$1 - 3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 3 \\ -6 & 4 \end{bmatrix} - (1 - 3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 + 3i & 3 \\ -6 & 3 + 3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 + 3i & 3 & 0 \\ -6 & 3 + 3i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 + 3i & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 + 3i & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 + i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 3 \\ -6 & 4 \end{bmatrix} - (1+3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3-3i & 3 \\ -6 & 3-3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3-3i & 3 & 0 \\ -6 & 3-3i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1+i) R_1 \Longrightarrow \left[\begin{array}{cc|c} -3-3i & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3-3i & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$1+3i$	1	1	No	$\begin{bmatrix} \frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$
$1-3i$	1	1	No	$\begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{(1+3i)t} \\ e^{(1+3i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{(1-3i)t} \\ e^{(1-3i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) c_1 e^{(1+3i)t} + \left(\frac{1}{2}+\frac{i}{2}\right) c_2 e^{(1-3i)t} \\ c_1 e^{(1+3i)t} + c_2 e^{(1-3i)t} \end{bmatrix}$$

The following is the phase plot of the system.

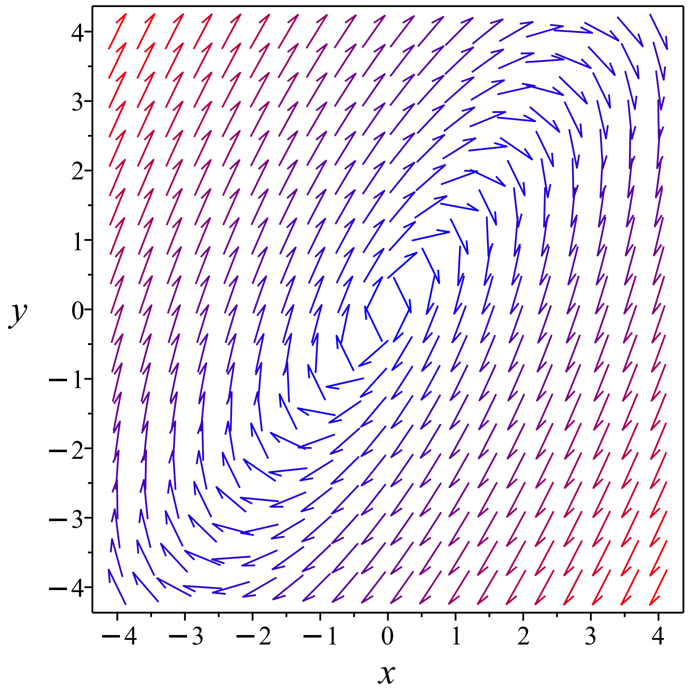


Figure 191: Phase plot

17.2.3 Maple step by step solution

Let's solve

$[x'(t) = -2x(t) + 3y(t), y'(t) = -6x(t) + 4y(t)]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$\vec{x}'(t) = \begin{bmatrix} -2 & 3 \\ -6 & 4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- System to solve

$\vec{x}'(t) = \begin{bmatrix} -2 & 3 \\ -6 & 4 \end{bmatrix} \cdot \vec{x}(t)$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 3 \\ -6 & 4 \end{bmatrix}$$

- Rewrite the system as

  $$\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1-3\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1+3\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[1-3\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{(1-3\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{t} \cdot (\cos(3t) - \mathrm{I}\sin(3t)) \cdot \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

  $$\mathrm{e}^{t} \cdot \begin{bmatrix} \left(\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(3t) - \mathrm{I}\sin(3t)) \\ \cos(3t) - \mathrm{I}\sin(3t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_{1}(t) = \mathrm{e}^{t} \cdot \begin{bmatrix} \frac{\cos(3t)}{2}+\frac{\sin(3t)}{2} \\ \cos(3t) \end{bmatrix}, \overrightarrow{x}_{2}(t) = \mathrm{e}^{t} \cdot \begin{bmatrix} -\frac{\sin(3t)}{2}+\frac{\cos(3t)}{2} \\ -\sin(3t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $$\overrightarrow{x} = c_{1}\overrightarrow{x}_{1}(t) + c_{2}\overrightarrow{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^t \cdot \begin{bmatrix} \frac{\cos(3t)}{2} + \frac{\sin(3t)}{2} \\ \cos(3t) \end{bmatrix} + c_2 e^t \cdot \begin{bmatrix} -\frac{\sin(3t)}{2} + \frac{\cos(3t)}{2} \\ -\sin(3t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{e^t((c_1+c_2)\cos(3t)+\sin(3t)(c_1-c_2))}{2} \\ e^t(c_1\cos(3t) - c_2\sin(3t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{e^t((c_1+c_2)\cos(3t)+\sin(3t)(c_1-c_2))}{2}, y(t) = e^t(c_1\cos(3t) - c_2\sin(3t))\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 53

```
dsolve([diff(x(t),t)=-2*x(t)+3*y(t),diff(y(t),t)=-6*x(t)+4*y(t)],singsol=all)
```

$$x(t) = e^t(c_1\sin(3t) + c_2\cos(3t))$$
$$y(t) = e^t(c_1\cos(3t) + c_2\cos(3t) + c_1\sin(3t) - c_2\sin(3t))$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 56

```
DSolve[{x'[t]==-2*x[t]+3*y[t],y'[t]==-6*x[t]+4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to e^t(c_1\cos(3t) + (c_2 - c_1)\sin(3t))$$
$$y(t) \to e^t(c_2\cos(3t) + (c_2 - 2c_1)\sin(3t))$$

17.3 problem 29.3 (iii)

Internal problem ID [12104]
Internal file name [`OUTPUT/10756_Monday_September_11_2023_12_50_24_AM_28376538/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 29, Complex eigenvalues. Exercises page 292
Problem number: 29.3 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -11x(t) - 2y(t) \\
y'(t) &= 13x(t) - 9y(t)
\end{aligned}
$$

17.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -11 & -2 \\ 13 & -9 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{-10t}\cos(5t) - \frac{e^{-10t}\sin(5t)}{5} & -\frac{2\,e^{-10t}\sin(5t)}{5} \\ \frac{13\,e^{-10t}\sin(5t)}{5} & e^{-10t}\cos(5t) + \frac{e^{-10t}\sin(5t)}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-10t}(5\cos(5t)-\sin(5t))}{5} & -\frac{2\,e^{-10t}\sin(5t)}{5} \\ \frac{13\,e^{-10t}\sin(5t)}{5} & \frac{e^{-10t}(5\cos(5t)+\sin(5t))}{5} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-10t}(5\cos(5t)-\sin(5t))}{5} & -\frac{2\,e^{-10t}\sin(5t)}{5} \\ \frac{13\,e^{-10t}\sin(5t)}{5} & \frac{e^{-10t}(5\cos(5t)+\sin(5t))}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-10t}(5\cos(5t)-\sin(5t))c_1}{5} - \frac{2\,e^{-10t}\sin(5t)c_2}{5} \\ \frac{13\,e^{-10t}\sin(5t)c_1}{5} + \frac{e^{-10t}(5\cos(5t)+\sin(5t))c_2}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-10t}(-c_1-2c_2)\sin(5t)}{5} + e^{-10t}\cos(5t)\,c_1 \\ \frac{e^{-10t}(13c_1+c_2)\sin(5t)}{5} + e^{-10t}\cos(5t)\,c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -11 & -2 \\ 13 & -9 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -11 & -2 \\ 13 & -9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -11-\lambda & -2 \\ 13 & -9-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 20\lambda + 125 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -10 + 5i$$
$$\lambda_2 = -10 - 5i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-10 - 5i$	1	complex eigenvalue
$-10 + 5i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -10 - 5i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -11 & -2 \\ 13 & -9 \end{bmatrix} - (-10 - 5i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1+5i & -2 \\ 13 & 1+5i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+5i & -2 & 0 \\ 13 & 1+5i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{2} + \frac{5i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1+5i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+5i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{13} - \frac{5i}{13}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{13} - \frac{5\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{13} - \frac{5i}{13}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{13} - \frac{5\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{13} - \frac{5i}{13} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{13} - \frac{5\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{13} - \frac{5i}{13} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{13} - \frac{5\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1 - 5i \\ 13 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -10 + 5i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -11 & -2 \\ 13 & -9 \end{bmatrix} - (-10 + 5i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - 5i & -2 \\ 13 & 1 - 5i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 - 5i & -2 & 0 \\ 13 & 1 - 5i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{2} - \frac{5i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1-5i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-5i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{13} + \frac{5i}{13}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{13} + \frac{5\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{13} + \frac{5i}{13}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{13} + \frac{5\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{13} + \frac{5i}{13} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{13} + \frac{5\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{13} + \frac{5i}{13} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{13} + \frac{5\,\mathrm{I}}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1+5i \\ 13 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-10+5i$	1	1	No	$\begin{bmatrix} -\frac{1}{13}+\frac{5i}{13} \\ 1 \end{bmatrix}$
$-10-5i$	1	1	No	$\begin{bmatrix} -\frac{1}{13}-\frac{5i}{13} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{1}{13}+\frac{5i}{13}\right) \mathrm{e}^{(-10+5i)t} \\ \mathrm{e}^{(-10+5i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{1}{13}-\frac{5i}{13}\right) \mathrm{e}^{(-10-5i)t} \\ \mathrm{e}^{(-10-5i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{13}+\frac{5i}{13}\right) c_1\mathrm{e}^{(-10+5i)t} + \left(-\frac{1}{13}-\frac{5i}{13}\right) c_2\mathrm{e}^{(-10-5i)t} \\ c_1\mathrm{e}^{(-10+5i)t} + c_2\mathrm{e}^{(-10-5i)t} \end{bmatrix}$$

The following is the phase plot of the system.

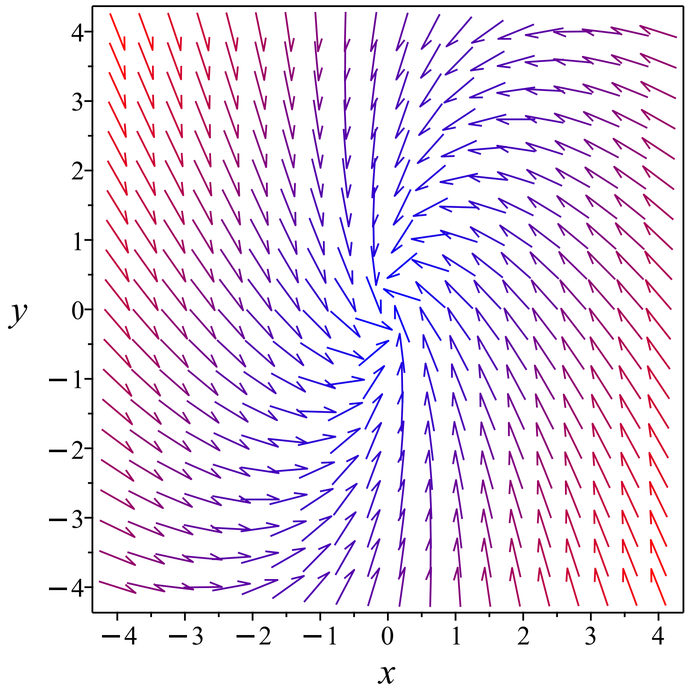


Figure 192: Phase plot

17.3.3 Maple step by step solution

Let's solve

$[x'(t) = -11x(t) - 2y(t), y'(t) = 13x(t) - 9y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -11 & -2 \\ 13 & -9 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -11 & -2 \\ 13 & -9 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -11 & -2 \\ 13 & -9 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-10-5\,\mathrm{I}, \begin{bmatrix} -\frac{1}{13}-\frac{5\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right], \left[-10+5\,\mathrm{I}, \begin{bmatrix} -\frac{1}{13}+\frac{5\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[-10-5\,\mathrm{I}, \begin{bmatrix} -\frac{1}{13}-\frac{5\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{(-10-5\,\mathrm{I})t} \cdot \begin{bmatrix} -\frac{1}{13}-\frac{5\,\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{-10t} \cdot (\cos(5t) - \mathrm{I}\sin(5t)) \cdot \begin{bmatrix} -\frac{1}{13}-\frac{5\,\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Simplify expression

  $$\mathrm{e}^{-10t} \cdot \begin{bmatrix} \left(-\frac{1}{13}-\frac{5\,\mathrm{I}}{13}\right)(\cos(5t) - \mathrm{I}\sin(5t)) \\ \cos(5t) - \mathrm{I}\sin(5t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\vec{x}_1(t) = \mathrm{e}^{-10t} \cdot \begin{bmatrix} -\frac{\cos(5t)}{13} - \frac{5\sin(5t)}{13} \\ \cos(5t) \end{bmatrix}, \vec{x}_2(t) = \mathrm{e}^{-10t} \cdot \begin{bmatrix} \frac{\sin(5t)}{13} - \frac{5\cos(5t)}{13} \\ -\sin(5t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-10t} \cdot \begin{bmatrix} -\frac{\cos(5t)}{13} - \frac{5\sin(5t)}{13} \\ \cos(5t) \end{bmatrix} + c_2 \mathrm{e}^{-10t} \cdot \begin{bmatrix} \frac{\sin(5t)}{13} - \frac{5\cos(5t)}{13} \\ -\sin(5t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{\mathrm{e}^{-10t}\left((5c_2+c_1)\cos(5t)+5\sin(5t)\left(c_1-\frac{c_2}{5}\right)\right)}{13} \\ \mathrm{e}^{-10t}(c_1\cos(5t) - c_2\sin(5t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{\mathrm{e}^{-10t}\left((5c_2+c_1)\cos(5t)+5\sin(5t)\left(c_1-\frac{c_2}{5}\right)\right)}{13}, y(t) = \mathrm{e}^{-10t}(c_1\cos(5t) - c_2\sin(5t))\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
dsolve([diff(x(t),t)=-11*x(t)-2*y(t),diff(y(t),t)=13*x(t)-9*y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{-10t}(\sin(5t)\,c_1 + c_2\cos(5t))$$
$$y(t) = -\frac{\mathrm{e}^{-10t}(\sin(5t)\,c_1 - 5c_2\sin(5t) + 5\cos(5t)\,c_1 + c_2\cos(5t))}{2}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 69

```
DSolve[{x'[t]==-11*x[t]-2*y[t],y'[t]==13*x[t]-9*y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{5}e^{-10t}(5c_1\cos(5t) - (c_1+2c_2)\sin(5t))$$
$$y(t) \to \frac{1}{5}e^{-10t}(5c_2\cos(5t) + (13c_1+c_2)\sin(5t))$$

17.4 problem 29.3 (iv)

Internal problem ID [12105]
Internal file name [`OUTPUT/10757_Monday_September_11_2023_12_50_25_AM_96493573/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 29, Complex eigenvalues. Exercises page 292
Problem number: 29.3 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 7x(t) - 5y(t)$$
$$y'(t) = 10x(t) - 3y(t)$$

17.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 10 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \begin{bmatrix} \mathrm{e}^{2t}\cos(5t) + \mathrm{e}^{2t}\sin(5t) & -\mathrm{e}^{2t}\sin(5t) \\ 2\,\mathrm{e}^{2t}\sin(5t) & \mathrm{e}^{2t}\cos(5t) - \mathrm{e}^{2t}\sin(5t) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{2t}(\cos(5t) + \sin(5t)) & -\mathrm{e}^{2t}\sin(5t) \\ 2\,\mathrm{e}^{2t}\sin(5t) & \mathrm{e}^{2t}(\cos(5t) - \sin(5t)) \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(\cos(5t)+\sin(5t)) & -e^{2t}\sin(5t) \\ 2\,e^{2t}\sin(5t) & e^{2t}(\cos(5t)-\sin(5t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(\cos(5t)+\sin(5t))\,c_1 - e^{2t}\sin(5t)\,c_2 \\ 2\,e^{2t}\sin(5t)\,c_1 + e^{2t}(\cos(5t)-\sin(5t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (\sin(5t)\,(c_1-c_2)+c_1\cos(5t))\,e^{2t} \\ e^{2t}(2c_1-c_2)\sin(5t)+e^{2t}\cos(5t)\,c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 10 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 7 & -5 \\ 10 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 7-\lambda & -5 \\ 10 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 29 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + 5i$$
$$\lambda_2 = 2 - 5i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2-5i$	1	complex eigenvalue
$2+5i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - 5i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & -5 \\ 10 & -3 \end{bmatrix} - (2-5i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 5+5i & -5 \\ 10 & -5+5i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 5+5i & -5 & 0 \\ 10 & -5+5i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1+i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 5+5i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5+5i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + 5i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & -5 \\ 10 & -3 \end{bmatrix} - (2 + 5i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 - 5i & -5 \\ 10 & -5 - 5i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 5 - 5i & -5 & 0 \\ 10 & -5 - 5i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 5 - 5i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5-5i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2+5i$	1	1	No	$\left[\begin{array}{c}\frac{1}{2}+\frac{i}{2} \\ 1\end{array}\right]$
$2-5i$	1	1	No	$\left[\begin{array}{c}\frac{1}{2}-\frac{i}{2} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} \left(\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(2+5i)t} \\ \mathrm{e}^{(2+5i)t} \end{array}\right] + c_2 \left[\begin{array}{c} \left(\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(2-5i)t} \\ \mathrm{e}^{(2-5i)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \left(\frac{1}{2}+\frac{i}{2}\right) c_1 \mathrm{e}^{(2+5i)t} + \left(\frac{1}{2}-\frac{i}{2}\right) c_2 \mathrm{e}^{(2-5i)t} \\ c_1 \mathrm{e}^{(2+5i)t} + c_2 \mathrm{e}^{(2-5i)t} \end{array}\right]$$

The following is the phase plot of the system.

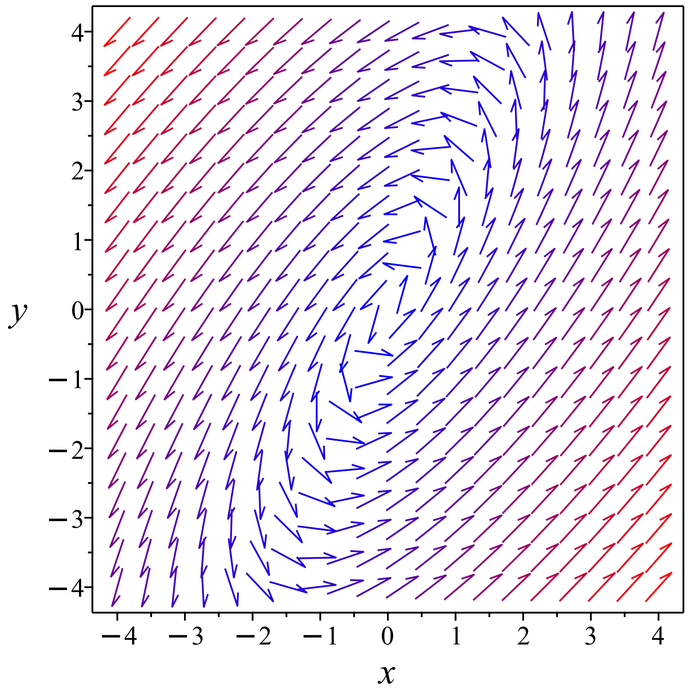


Figure 193: Phase plot

17.4.3 Maple step by step solution

Let's solve

$[x'(t) = 7x(t) - 5y(t), y'(t) = 10x(t) - 3y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 7 & -5 \\ 10 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 7 & -5 \\ 10 & -3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 7 & -5 \\ 10 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2-5\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2+5\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-5\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(2-5\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{2t} \cdot (\cos(5t) - \mathrm{I}\sin(5t)) \cdot \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{2t} \cdot \begin{bmatrix} \left(\frac{1}{2}-\frac{\mathrm{I}}{2}\right)(\cos(5t) - \mathrm{I}\sin(5t)) \\ \cos(5t) - \mathrm{I}\sin(5t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{\cos(5t)}{2} - \frac{\sin(5t)}{2} \\ \cos(5t) \end{bmatrix}, \vec{x}_2(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} -\frac{\sin(5t)}{2} - \frac{\cos(5t)}{2} \\ -\sin(5t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{2t} \cdot \begin{bmatrix} \frac{\cos(5t)}{2} - \frac{\sin(5t)}{2} \\ \cos(5t) \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} -\frac{\sin(5t)}{2} - \frac{\cos(5t)}{2} \\ -\sin(5t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{2t}((c_1-c_2)\cos(5t)-\sin(5t)(c_1+c_2))}{2} \\ e^{2t}(c_1 \cos(5t) - c_2 \sin(5t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{e^{2t}((c_1-c_2)\cos(5t)-\sin(5t)(c_1+c_2))}{2}, y(t) = e^{2t}(c_1 \cos(5t) - c_2 \sin(5t))\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 57

```
dsolve([diff(x(t),t)=7*x(t)-5*y(t),diff(y(t),t)=10*x(t)-3*y(t)],singsol=all)
```

$$x(t) = e^{2t}(\sin(5t)\, c_1 + c_2 \cos(5t))$$
$$y(t) = e^{2t}(\sin(5t)\, c_1 + c_2 \sin(5t) - \cos(5t)\, c_1 + c_2 \cos(5t))$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 62

```
DSolve[{x'[t]==7*x[t]-5*y[t],y'[t]==10*x[t]-3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -
```

$$x(t) \to e^{2t}(c_1 \cos(5t) + (c_1 - c_2)\sin(5t))$$
$$y(t) \to e^{2t}(c_2 \cos(5t) + (2c_1 - c_2)\sin(5t))$$

18 Chapter 30, A repeated real eigenvalue. Exercises page 299

18.1 problem 30.1 (i)

Internal problem ID [12106]
Internal file name [`OUTPUT/10758_Monday_September_11_2023_12_50_26_AM_52766681/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 30, A repeated real eigenvalue. Exercises page 299
Problem number: 30.1 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 5x(t) - 4y(t)$$
$$y'(t) = x(t) + y(t)$$

18.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{3t}(2t+1) & -4t\,\mathrm{e}^{3t} \\ t\,\mathrm{e}^{3t} & \mathrm{e}^{3t}(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t}(2t+1) & -4t\,e^{3t} \\ t\,e^{3t} & e^{3t}(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(2t+1)\,c_1 - 4t\,e^{3t}c_2 \\ t\,e^{3t}c_1 + e^{3t}(1-2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(2tc_1 - 4c_2t + c_1) \\ e^{3t}(tc_1 - 2c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 5-\lambda & -4 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

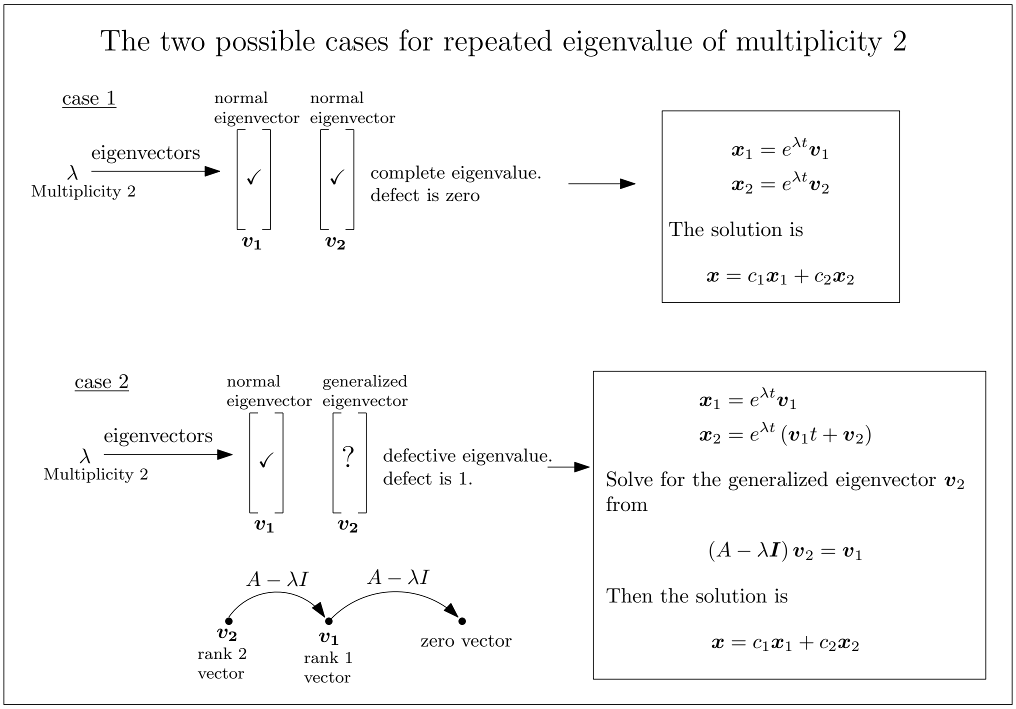


Figure 194: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 3. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^{3t} \\
&= \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \mathrm{e}^{3t} \\
&= \begin{bmatrix} \mathrm{e}^{3t}(2t+3) \\ \mathrm{e}^{3t}(1+t) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{3t}(2t+3) \\ \mathrm{e}^{3t}(1+t) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} ((2t+3)\, c_2 + 2c_1)\, \mathrm{e}^{3t} \\ \mathrm{e}^{3t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

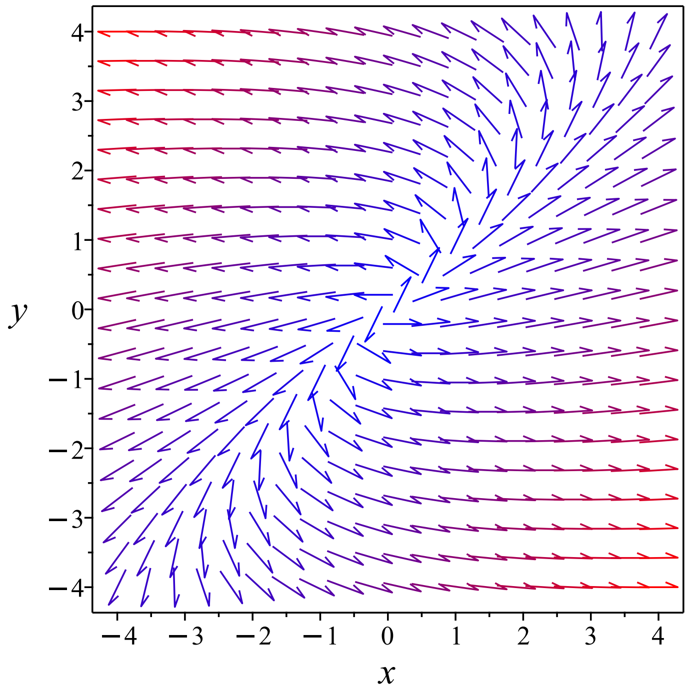


Figure 195: Phase plot

18.1.3 Maple step by step solution

Let's solve

$[x'(t) = 5x(t) - 4y(t), y'(t) = x(t) + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 5 & -4 \\ 1 & 1 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[3, \left[\begin{array}{c} 2 \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[3, \left[\begin{array}{c} 2 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 3

$$\overrightarrow{x}_1(t) = e^{3t} \cdot \left[\begin{array}{c} 2 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

$$\overrightarrow{x}_2(t) = e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = \left(e^{\lambda t} A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = e^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 3

$$\overrightarrow{x}_2(t) = \mathrm{e}^{3t} \cdot \left(t \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{3t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{3t} \cdot \left(t \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{3t}(2c_2 t + 2c_1 + c_2) \\ \mathrm{e}^{3t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = \mathrm{e}^{3t}(2c_2 t + 2c_1 + c_2), y(t) = \mathrm{e}^{3t}(c_2 t + c_1)\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=5*x(t)-4*y(t),diff(y(t),t)=x(t)+y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{3t}(c_2 t + c_1)$$
$$y(t) = \frac{\mathrm{e}^{3t}(2c_2 t + 2c_1 - c_2)}{4}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 45

```
DSolve[{x'[t]==5*x[t]-4*y[t],y'[t]==x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tru
```

$$x(t) \to e^{3t}(2c_1t - 4c_2t + c_1)$$
$$y(t) \to e^{3t}((c_1 - 2c_2)t + c_2)$$

18.2 problem 30.1 (ii)

Internal problem ID [12107]
Internal file name [`OUTPUT/10759_Monday_September_11_2023_12_50_26_AM_80135249/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 30, A repeated real eigenvalue. Exercises page 299
Problem number: 30.1 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -6x(t) + 2y(t) \\
y'(t) &= -2x(t) - 2y(t)
\end{aligned}
$$

18.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{-4t}(1-2t) & 2t\,e^{-4t} \\ -2t\,e^{-4t} & e^{-4t}(2t+1) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-4t}(1-2t) & 2t\,\mathrm{e}^{-4t} \\ -2t\,\mathrm{e}^{-4t} & \mathrm{e}^{-4t}(2t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-4t}(1-2t)\,c_1 + 2t\,\mathrm{e}^{-4t}c_2 \\ -2t\,\mathrm{e}^{-4t}c_1 + \mathrm{e}^{-4t}(2t+1)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-2t) + 2c_2t)\,\mathrm{e}^{-4t} \\ (c_2(2t+1) - 2tc_1)\,\mathrm{e}^{-4t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -6 & 2 \\ -2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -6-\lambda & 2 \\ -2 & -2-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 8\lambda + 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -6 & 2 \\ -2 & -2 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-4	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

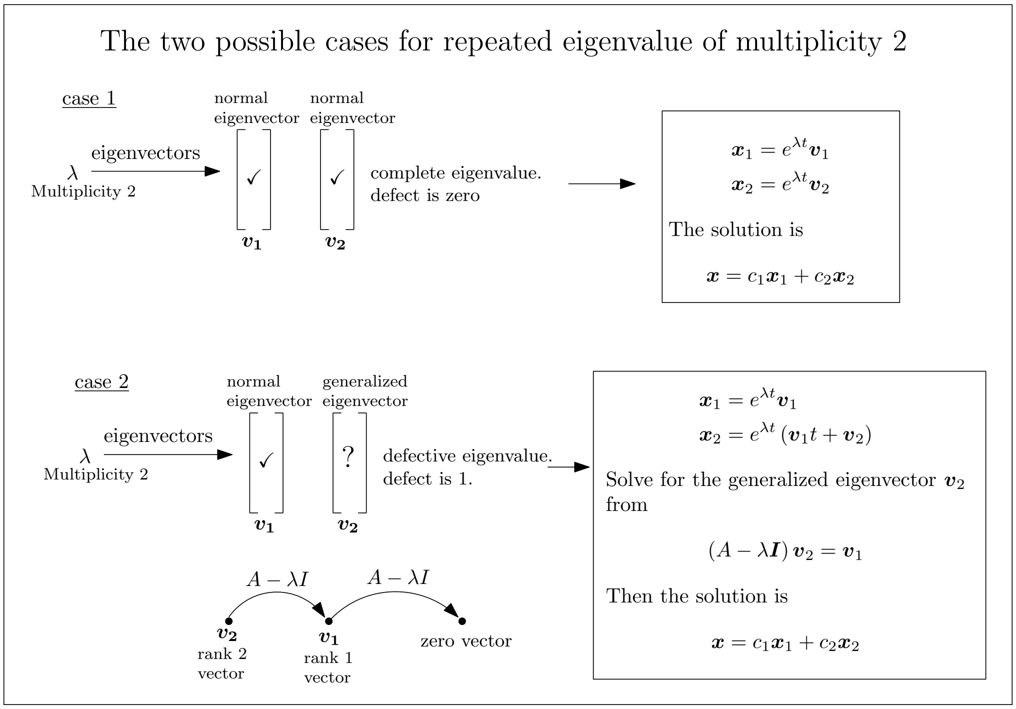


Figure 196: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -6 & 2 \\ -2 & -2 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -4. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t} \\ &= \begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right) e^{-4t} \\ &= \begin{bmatrix} \frac{e^{-4t}(2t+1)}{2} \\ e^{-4t}(1+t) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-4t}\left(t + \frac{1}{2}\right) \\ e^{-4t}(1+t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{-4t}\left(c_1 + c_2 t + \frac{1}{2} c_2\right) \\ e^{-4t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

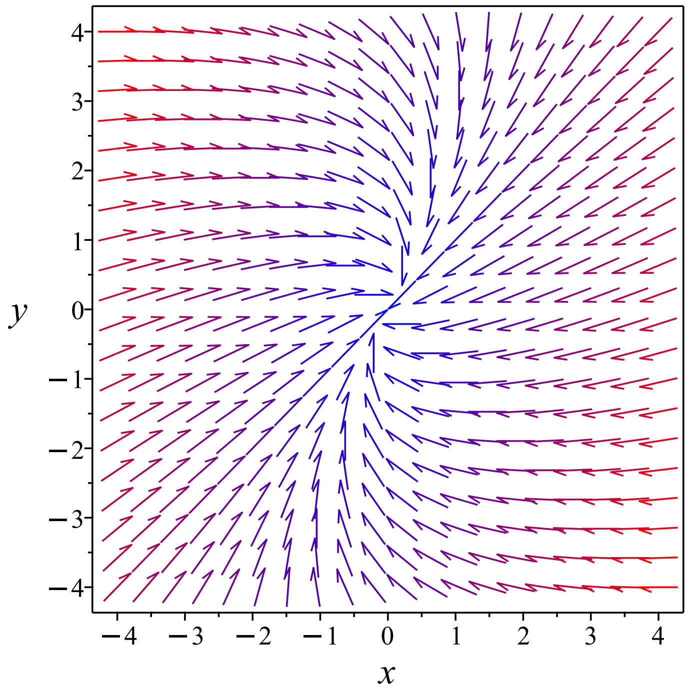


Figure 197: Phase plot

18.2.3 Maple step by step solution

Let's solve

$[x'(t) = -6x(t) + 2y(t), y'(t) = -2x(t) - 2y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -6 & 2 \\ -2 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -6 & 2 \\ -2 & -2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -6 & 2 \\ -2 & -2 \end{bmatrix}$$

- Rewrite the system as

  $$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[-4, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -4

  $$\overrightarrow{x}_1(t) = \mathrm{e}^{-4t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -4$ is the eigenvalue, an

  $$\overrightarrow{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

  $$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -4

$$\left(\begin{bmatrix} -6 & 2 \\ -2 & -2 \end{bmatrix} - (-4) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -4

$$\vec{x}_2(t) = e^{-4t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-4t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-4t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{-4t}\left(c_1 + c_2 t - \frac{1}{2} c_2\right) \\ e^{-4t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = e^{-4t}\left(c_1 + c_2 t - \frac{1}{2} c_2\right), y(t) = e^{-4t}(c_2 t + c_1)\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve([diff(x(t),t)=-6*x(t)+2*y(t),diff(y(t),t)=-2*x(t)-2*y(t)],singsol=all)
```

$$x(t) = e^{-4t}(c_2 t + c_1)$$
$$y(t) = \frac{e^{-4t}(2c_2 t + 2c_1 + c_2)}{2}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 46

```
DSolve[{x'[t]==-6*x[t]+2*y[t],y'[t]==-2*x[t]-2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to e^{-4t}(-2c_1t + 2c_2t + c_1)$$
$$y(t) \to e^{-4t}(-2c_1t + 2c_2t + c_2)$$

18.3 problem 30.1 (iii)

Internal problem ID [12108]
Internal file name [`OUTPUT/10760_Monday_September_11_2023_12_50_26_AM_89570837/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 30, A repeated real eigenvalue. Exercises page 299
Problem number: 30.1 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -3x(t) - y(t)$$
$$y'(t) = x(t) - 5y(t)$$

18.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-4t}(1+t) & -t\,\mathrm{e}^{-4t} \\ t\,\mathrm{e}^{-4t} & \mathrm{e}^{-4t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-4t}(1+t) & -t\,e^{-4t} \\ t\,e^{-4t} & e^{-4t}(1-t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-4t}(1+t)\,c_1 - t\,e^{-4t}c_2 \\ t\,e^{-4t}c_1 + e^{-4t}(1-t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-4t}(tc_1 - c_2t + c_1) \\ e^{-4t}(tc_1 - c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -3 & -1 \\ 1 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -3-\lambda & -1 \\ 1 & -5-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 8\lambda + 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -1 \\ 1 & -5 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-4	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

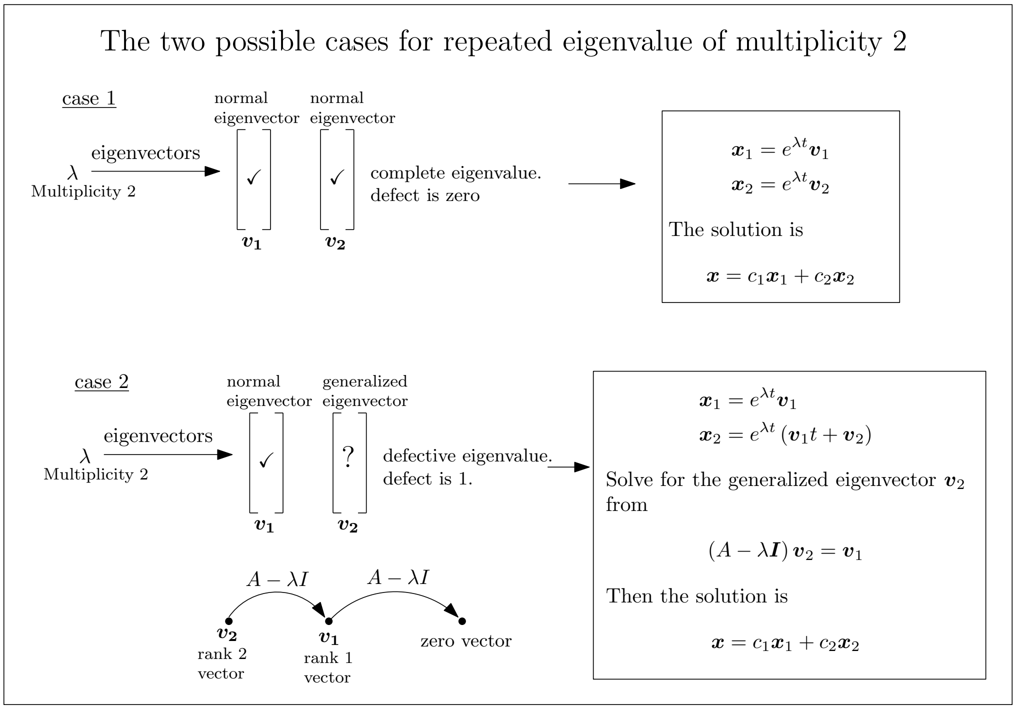


Figure 198: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -3 & -1 \\ 1 & -5 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -4. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{-4t} \\
&= \begin{bmatrix} \mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-4t} \\
&= \begin{bmatrix} \mathrm{e}^{-4t}(t+2) \\ \mathrm{e}^{-4t}(1+t) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-4t}(t+2) \\ \mathrm{e}^{-4t}(1+t) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-4t}((t+2)\, c_2 + c_1) \\ \mathrm{e}^{-4t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

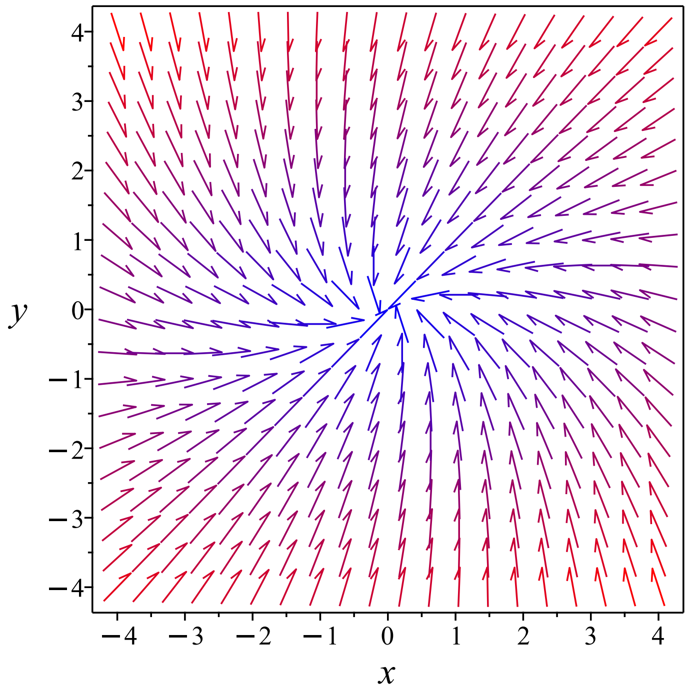


Figure 199: Phase plot

18.3.3 Maple step by step solution

Let's solve

$[x'(t) = -3x(t) - y(t), y'(t) = x(t) - 5y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -3 & -1 \\ 1 & -5 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -3 & -1 \\ 1 & -5 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & -1 \\ 1 & -5 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[-4, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -4

  $$\overrightarrow{x}_1(t) = \mathrm{e}^{-4t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -4$ is the eigenvalue, an

  $\overrightarrow{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $\lambda \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -4

$$\left(\begin{bmatrix} -3 & -1 \\ 1 & -5 \end{bmatrix} - (-4) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -4

$$\vec{x}_2(t) = e^{-4t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-4t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-4t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{-4t}(c_2 t + c_1 + c_2) \\ e^{-4t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = e^{-4t}(c_2 t + c_1 + c_2), y(t) = e^{-4t}(c_2 t + c_1)\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve([diff(x(t),t)=-3*x(t)-y(t),diff(y(t),t)=x(t)-5*y(t)],singsol=all)
```

$$x(t) = e^{-4t}(c_2 t + c_1)$$
$$y(t) = e^{-4t}(c_2 t + c_1 - c_2)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 44

```
DSolve[{x'[t]==-3*x[t]-y[t],y'[t]==x[t]-5*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tr
```

$$x(t) \to e^{-4t}(c_1(t+1) - c_2t)$$
$$y(t) \to e^{-4t}((c_1 - c_2)t + c_2)$$

18.4 problem 30.1 (iv)

Internal problem ID [12109]
Internal file name [`OUTPUT/10761_Monday_September_11_2023_12_50_27_AM_86659631/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 30, A repeated real eigenvalue. Exercises page 299
Problem number: 30.1 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 13x(t)$$
$$y'(t) = 13y(t)$$

18.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{13t} & 0 \\ 0 & e^{13t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \mathrm{e}^{13t} & 0 \\ 0 & \mathrm{e}^{13t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{13t} c_1 \\ \mathrm{e}^{13t} c_2 \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 13 - \lambda & 0 \\ 0 & 13 - \lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(13 - \lambda)(13 - \lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 13$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
13	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 13$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} - (13)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_2\}$ and there are no leading variables. Let $v_1 = t$. Let $v_2 = s$. Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \end{bmatrix}$$
$$= t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
13	2	2	No	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 13 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

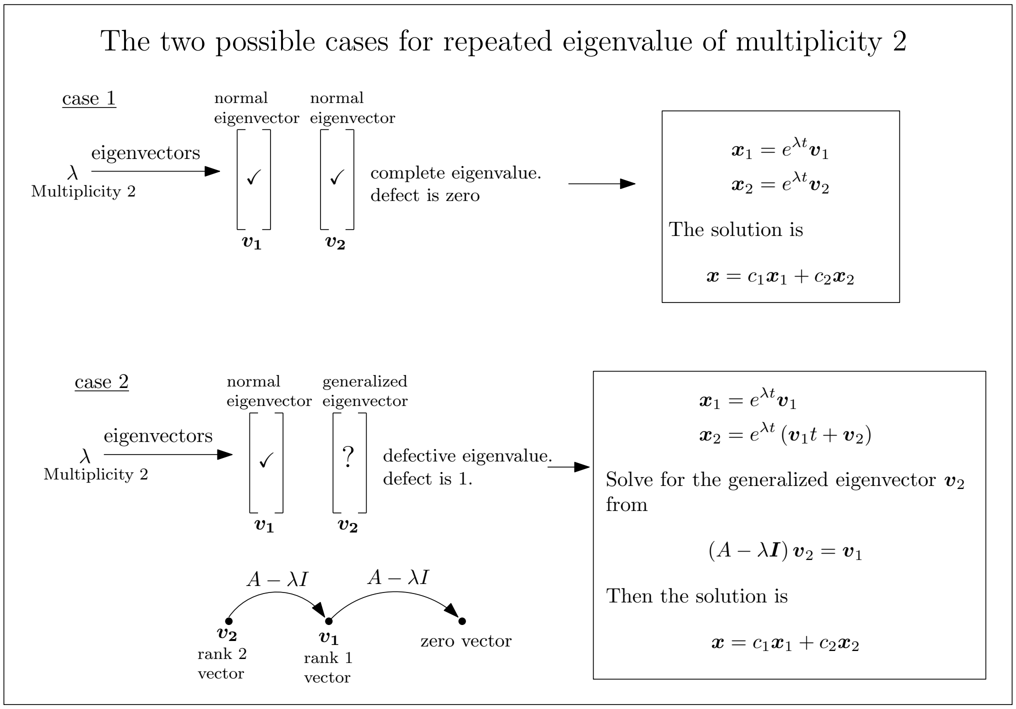


Figure 200: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{13t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{13t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{13t} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{13t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{13t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{13t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{13t} \\ c_2 e^{13t} \end{bmatrix}$$

The following is the phase plot of the system.

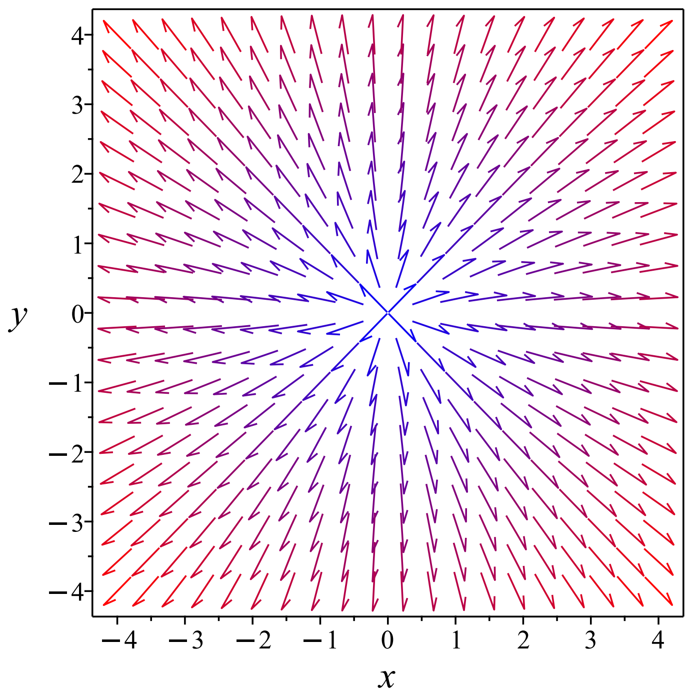


Figure 201: Phase plot

18.4.3 Maple step by step solution

Let's solve

$[x'(t) = 13x(t), y'(t) = 13y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[13, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[13, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[13, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 13

$$\vec{x}_1(t) = \mathrm{e}^{13t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 13$ is the eigenvalue, and

$$\vec{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda t}\vec{v}=\left(\mathrm{e}^{\lambda t}A\right)\cdot\left(t\vec{v}+\vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda t}\vec{v}=\mathrm{e}^{\lambda t}\left(\lambda t\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 13

$$\left(\left[\begin{array}{cc}13 & 0\\ 0 & 13\end{array}\right]-13\cdot\left[\begin{array}{cc}1 & 0\\ 0 & 1\end{array}\right]\right)\cdot\vec{p}=\left[\begin{array}{c}0\\ 1\end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p}=\left[\begin{array}{c}0\\ 0\end{array}\right]$$

- Second solution from eigenvalue 13

$$\vec{x}_2(t)=\mathrm{e}^{13t}\cdot\left(t\cdot\left[\begin{array}{c}0\\ 1\end{array}\right]+\left[\begin{array}{c}0\\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\vec{x}=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x}=c_1\mathrm{e}^{13t}\cdot\left[\begin{array}{c}0\\ 1\end{array}\right]+c_2\mathrm{e}^{13t}\cdot\left(t\cdot\left[\begin{array}{c}0\\ 1\end{array}\right]+\left[\begin{array}{c}0\\ 0\end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t)\\ y(t)\end{array}\right]=\left[\begin{array}{c}0\\ \mathrm{e}^{13t}(c_2t+c_1)\end{array}\right]$$

- Solution to the system of ODEs

$$\{x(t) = 0, y(t) = e^{13t}(c_2t + c_1)\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve([diff(x(t),t)=13*x(t),diff(y(t),t)=13*y(t)],singsol=all)
```

$$x(t) = c_2 e^{13t}$$
$$y(t) = e^{13t} c_1$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 65

```
DSolve[{x'[t]==13*x[t],y'[t]==13*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{13t}$$
$$y(t) \to c_2 e^{13t}$$
$$x(t) \to c_1 e^{13t}$$
$$y(t) \to 0$$
$$x(t) \to 0$$
$$y(t) \to c_2 e^{13t}$$
$$x(t) \to 0$$
$$y(t) \to 0$$

18.5 problem 30.1 (v)

Internal problem ID [12110]
Internal file name [`OUTPUT/10762_Monday_September_11_2023_12_50_27_AM_81599353/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 30, A repeated real eigenvalue. Exercises page 299
Problem number: 30.1 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 7x(t) - 4y(t)$$
$$y'(t) = x(t) + 3y(t)$$

18.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{5t}(2t+1) & -4t\,\mathrm{e}^{5t} \\ t\,\mathrm{e}^{5t} & \mathrm{e}^{5t}(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{5t}(2t+1) & -4t\,e^{5t} \\ t\,e^{5t} & e^{5t}(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{5t}(2t+1)\,c_1 - 4t\,e^{5t}c_2 \\ t\,e^{5t}c_1 + e^{5t}(1-2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{5t}(2tc_1 - 4c_2t + c_1) \\ e^{5t}(tc_1 - 2c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 7 & -4 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 7-\lambda & -4 \\ 1 & 3-\lambda \end{bmatrix} \right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 10\lambda + 25 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & -4 \\ 1 & 3 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

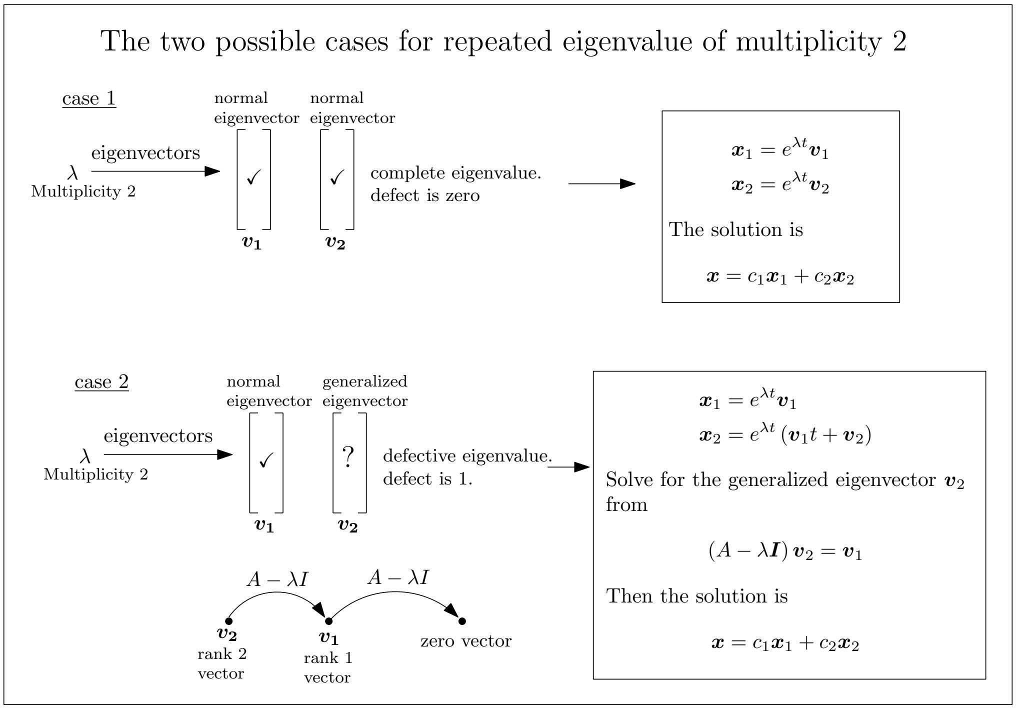


Figure 202: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 7 & -4 \\ 1 & 3 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 5. Therefore the two basis solution associated with this eigenvalue are

$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t} = \begin{bmatrix} 2\,e^{5t} \\ e^{5t} \end{bmatrix}$$

And

$$\vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) e^{5t} = \begin{bmatrix} e^{5t}(2t+3) \\ e^{5t}(1+t) \end{bmatrix}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,e^{5t} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} e^{5t}(2t+3) \\ e^{5t}(1+t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} ((2t+3)\, c_2 + 2c_1)\, e^{5t} \\ e^{5t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

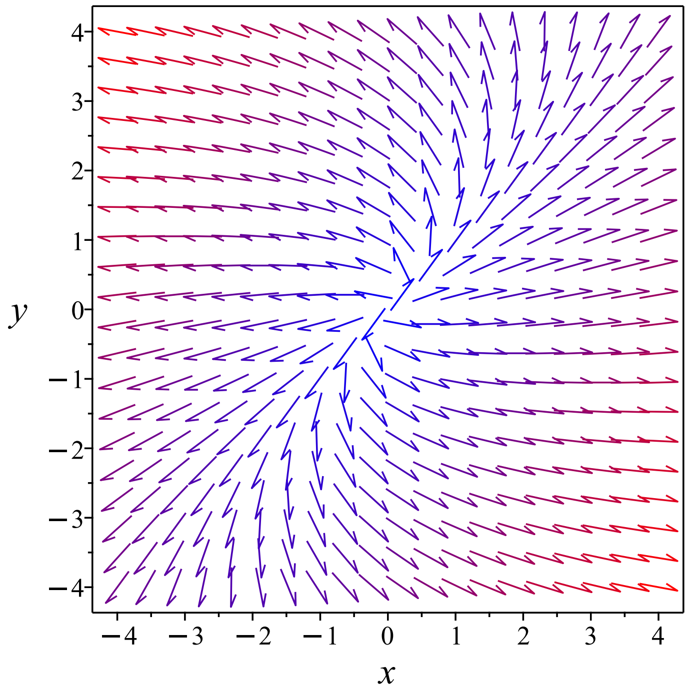


Figure 203: Phase plot

18.5.3 Maple step by step solution

Let's solve

$[x'(t) = 7x(t) - 4y(t), y'(t) = x(t) + 3y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 7 & -4 \\ 1 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 7 & -4 \\ 1 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 7 & -4 \\ 1 & 3 \end{bmatrix}$$

- Rewrite the system as

  $$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[5, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[5, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 5

  $$\overrightarrow{x}_1(t) = \mathrm{e}^{5t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 5$ is the eigenvalue, and

  $$\overrightarrow{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

  $$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 5

$$\left(\begin{bmatrix} 7 & -4 \\ 1 & 3 \end{bmatrix} - 5 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 5

$$\vec{x}_2(t) = e^{5t} \cdot \left(t \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{5t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{5t} \cdot \left(t \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{5t}(2c_2 t + 2c_1 + c_2) \\ e^{5t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = e^{5t}(2c_2 t + 2c_1 + c_2), y(t) = e^{5t}(c_2 t + c_1)\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=7*x(t)-4*y(t),diff(y(t),t)=x(t)+3*y(t)],singsol=all)
```

$$x(t) = e^{5t}(c_2 t + c_1)$$
$$y(t) = \frac{e^{5t}(2c_2 t + 2c_1 - c_2)}{4}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 45

```
DSolve[{x'[t]==7*x[t]-4*y[t],y'[t]==x[t]+3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to e^{5t}(2c_1t - 4c_2t + c_1)$$
$$y(t) \to e^{5t}((c_1 - 2c_2)t + c_2)$$

18.6 problem 30.5 (iii)

Internal problem ID [12111]
Internal file name [`OUTPUT/10763_Monday_September_11_2023_12_50_27_AM_21214228/index.tex`]

Book: AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS by JAMES C. ROBINSON. Cambridge University Press 2004
Section: Chapter 30, A repeated real eigenvalue. Exercises page 299
Problem number: 30.5 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -x(t) + y(t)$$
$$y'(t) = -x(t) + y(t)$$

18.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (1-t)\, c_1 + tc_2 \\ -tc_1 + (1+t)\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-c_1 + c_2)\, t + c_1 \\ (-c_1 + c_2)\, t + c_2 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

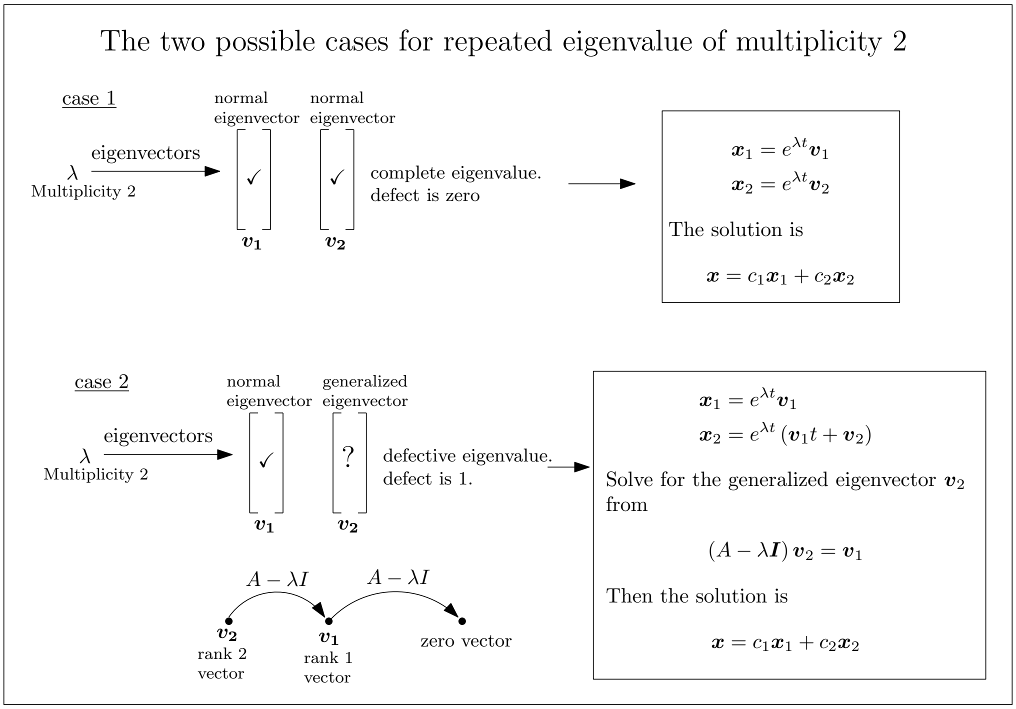


Figure 204: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) 1 \\
&= \begin{bmatrix} t \\ 1+t \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1+t \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 t + c_1 \\ c_2 t + c_1 + c_2 \end{bmatrix}
$$

The following is the phase plot of the system.

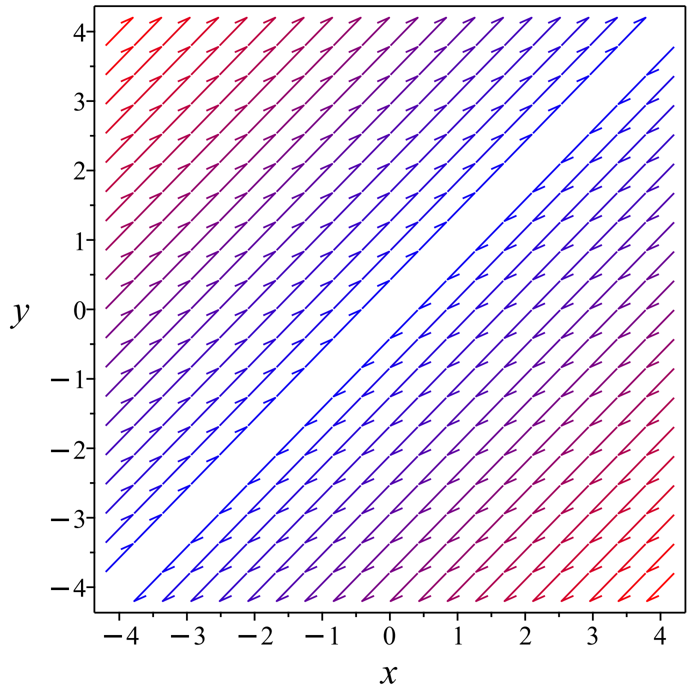


Figure 205: Phase plot

18.6.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + y(t), y'(t) = -x(t) + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = c_1, y(t) = c_1\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve([diff(x(t),t)=-x(t)+y(t),diff(y(t),t)=-x(t)+y(t)],singsol=all)
```

$$x(t) = c_1t + c_2$$
$$y(t) = c_1t + c_1 + c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 32

```
DSolve[{x'[t]==-x[t]+y[t],y'[t]==-x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1(-t) + c_2t + c_1$$
$$y(t) \to (c_2 - c_1)t + c_2$$