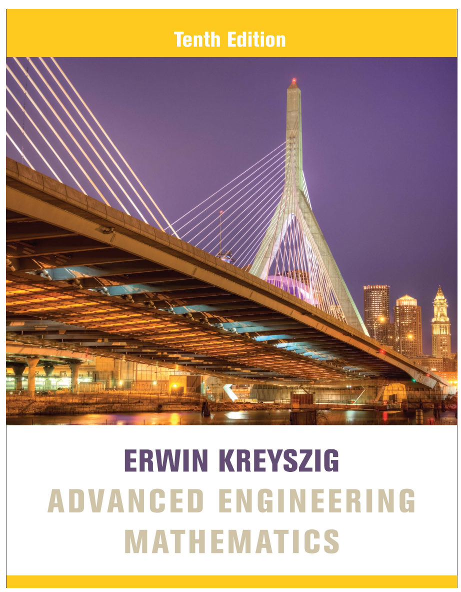
A Solution Manual For

ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011



Nasser M. Abbasi

May 15, 2024

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1 Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174

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1.1 problem 6

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Internal problem ID [5623]
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Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(1+x)\,y'-y=0}$$

With the expansion point for the power series method at $x=0$.

1.1.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y'=f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$\begin{aligned}
y(x)&=y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y''(x_0)+\cdots\\
&=y_0+xf+\frac{x^2}{2}\frac{df}{dx}\bigg|_{x_0,y_0}+\frac{x^3}{3!}\frac{d^2f}{dx^2}\bigg|_{x_0,y_0}+\cdots\\
&=y_0+\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$F_0 = f(x, y) \tag{4}$$

$$F_n = \frac{d}{dx}(F_{n-1})$$

$$= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \tag{5}$$

For example, for $n = 1$ we see that

$$F_1 = \frac{d}{dx}(F_0)$$

$$= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f$$

Which is (1). And when $n = 2$

$$F_2 = \frac{d}{dx}(F_1)$$

$$= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= \frac{y}{1+x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= 0 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= 0 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= 0 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= 0
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= y(0) \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = (1+x)\, y(0) + O\left(x^6\right)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - \frac{y}{1+x} &= 0 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -\frac{1}{1+x} \\ p(x) &= 0 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(1+x)\,y' - y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(1+x)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$a_1 - a_0 = 0$$
$$a_1 = a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+1) a_{n+1} + n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1} = -\frac{a_n(n-1)}{n+1} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = 0$$

For $n = 2$ the recurrence equation gives

$$3a_3 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 0$$

For $n = 3$ the recurrence equation gives

$$4a_4 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 4$ the recurrence equation gives

$$5a_5 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_6 + 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = (1 + x) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = (1 + x) y(0) + O(x^6) \tag{1}$$

$$y = (1 + x) c_1 + O(x^6) \tag{2}$$

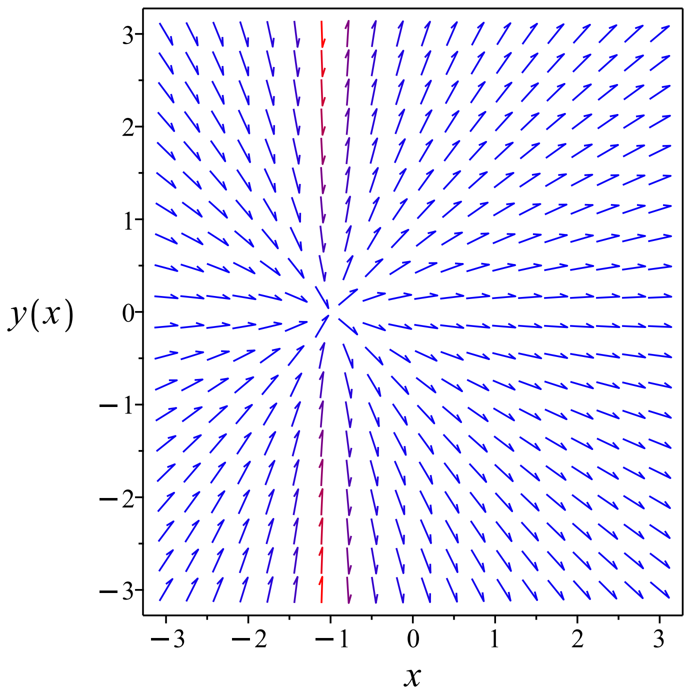

Figure 1: Slope field plot

Verification of solutions

$$y = (1 + x) y(0) + O(x^6)$$

Verified OK.

$$y = (1 + x) c_1 + O(x^6)$$

Verified OK.

1.1.2 Maple step by step solution

Let's solve

$(1+x)\,y' - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{1+x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{1+x}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(1+x) + c_1$

- Solve for y

  $y = \mathrm{e}^{c_1}(1+x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
Order:=6;
dsolve((1+x)*diff(y(x),x)=y(x),y(x),type='series',x=0);
```

$$y(x) = y(0)\,(x+1)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 9

```
AsymptoticDSolveValue[(1+x)*y'[x]==y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1(x+1)$$

1.2 problem 7

1.2.1 Solving as series ode 12
1.2.2 Maple step by step solution 19

Internal problem ID [5624]
Internal file name [`OUTPUT/4872_Sunday_June_05_2022_03_08_51_PM_96409877/index.tex`]

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Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2xy + y' = 0}$$

With the expansion point for the power series method at $x = 0$.

1.2.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) && (4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= -2xy \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \left(4x^2 - 2\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= \left(-8x^3 + 12x\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= 4y\left(4x^4 - 12x^2 + 3\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= -32\left(x^4 - 5x^2 + \frac{15}{4}\right) yx
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= -2y(0) \\
F_2 &= 0 \\
F_3 &= 12y(0) \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) y(0) + O\left(x^6\right)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
2xy + y' &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= 2x \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$
2x \left(\sum_{n=0}^{\infty} a_n x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} 2x^{1+n} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(1+n) a_{1+n} + 2a_{n-1} = 0 \tag{4}$$

Solving for a_{1+n}, gives

$$a_{1+n} = -\frac{2a_{n-1}}{1+n} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -a_0$$

For $n = 2$ the recurrence equation gives

$$3a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 0$$

For $n = 3$ the recurrence equation gives

$$4a_4 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 4$ the recurrence equation gives

$$5a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_6 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{6}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 - a_0 x^2 + \frac{1}{2} a_0 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{2} x^4\right) a_0 + O\left(x^6\right) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) y(0) + O(x^6) \tag{1}$$

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) c_1 + O(x^6) \tag{2}$$

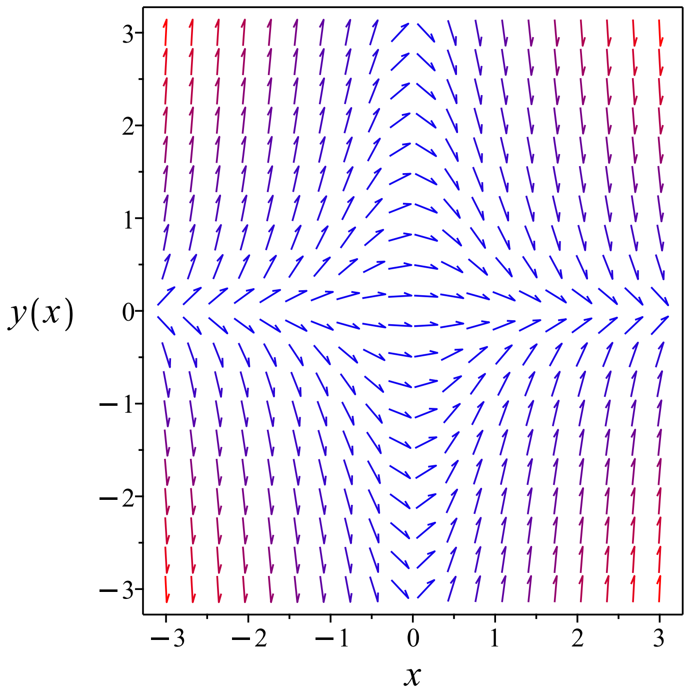

Figure 2: Slope field plot

Verification of solutions

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) y(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) c_1 + O(x^6)$$

Verified OK.

1.2.2 Maple step by step solution

Let's solve

$2xy + y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -2x dx + c_1$

- Evaluate integral

  $\ln(y) = -x^2 + c_1$

- Solve for y

  $y = e^{-x^2 + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
Order:=6;
dsolve(diff(y(x),x)=-2*x*y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{1}{2}x^4\right) y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 20

```
AsymptoticDSolveValue[y'[x]==-2*x*y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^4}{2} - x^2 + 1 \right)$$

1.3 problem 8

1.3.1 Solving as series ode 21
1.3.2 Maple step by step solution 25

Internal problem ID [5625]
Internal file name [`OUTPUT/4873_Sunday_June_05_2022_03_08_52_PM_75155354/index.tex`]

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Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Regular singular point"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xy' - 3y = k}$$

With the expansion point for the power series method at $x = 0$.

1.3.1 Solving as series ode

Writing the ODE as

$$y' + q(x)y = p(x)$$
$$y' - \frac{3y}{x} = \frac{k}{x}$$

Where

$$q(x) = -\frac{3}{x}$$
$$p(x) = \frac{k}{x}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not.

Since $x = 0$ is not an ordinary point, we now check to see if it is a regular singular point. $xq(x) = -3$ has a Taylor series around $x = 0$. Since $x = 0$ is regular singular point, then Frobenius power series is used. Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $y' - \frac{3y}{x} = 0$,and y_p is a particular solution to the inhomogeneous ode. First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \frac{3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)}{x} = 0 \tag{1}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \frac{3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)}{x} = 0 \tag{1}$$

Expanding the second term in (1) gives

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + -3 \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \frac{1}{x} \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n\,=0}^{\infty} \left(-3x^{n+r-1} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n\,=0}^{\infty} \left(-3x^{n+r-1} a_n\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2) this gives

$$(n+r)\, a_n x^{n+r-1} - 3x^{n+r-1} a_n = 0$$

When $n = 0$ the above becomes

$$r a_0 x^{-1+r} - 3x^{-1+r} a_0 = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(x^{-1+m} m - 3x^{-1+m}\right) c_0 = \frac{k}{x}$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-3)\, x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation simplifies to

$$r - 3 = 0$$

Solving for r gives the root of the indicial equation as

$$r = 3$$

We start by finding y_h. From the above we see that there is no recurrence relation since there is only one summation term. Therefore all a_n terms are zero except for a_0. Hence

$$y_h = a_0 x^r$$

Therefore the homogeneous solution is

$$y_h(x) = a_0\left(x^3 + O(x^6)\right)$$

Now we determine the particular solution y_p by solving the balance equation

$$\left(x^{-1+m} m - 3x^{-1+m}\right) c_0 = \frac{k}{x}$$

For c_0 and x. This results in

$$c_0 = -\frac{k}{3}$$
$$m = 0$$

The particular solution is therefore

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+m}$$
$$= \sum_{n=0}^{\infty} c_n x^{n+0}$$

Where in the above $c_0 = -\frac{k}{3}$. The remaining c_n values are found using the same recurrence relation used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. The following are the values of a_n found in terms of the indicial root r. These will be now used to find find c_n by replacing $a_0 = -\frac{k}{3}$ and $r = 0$. The following table gives the a_n values found and the corresponding c_n values which will be used to find the particular solution

n	a_n	c_n
0	$a_0 = 1$	$c_0 = -\frac{k}{3}$

The particular solution is now found using

$$y_p = x^m \sum_{n=0}^{\infty} c_n x^n$$
$$= 1 \sum_{n=0}^{\infty} c_n x^n$$

Using the values found above for c_n into the above sum gives

$$y_p = 1\left(-\frac{k}{3}\right)$$

At $x = 0$ the solution above becomes

$$y = -\frac{k}{3} + O(x^6) + c_1\left(x^3 + O(x^6)\right)$$

Summary
The solution(s) found are the following

$$y = -\frac{k}{3} + O(x^6) + c_1\left(x^3 + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = -\frac{k}{3} + O(x^6) + c_1\left(x^3 + O(x^6)\right)$$

Verified OK.

1.3.2 Maple step by step solution

Let's solve

$y' - \frac{3y}{x} = \frac{k}{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{3y+k} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{3y+k}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\frac{\ln(3y+k)}{3} = \ln(x) + c_1$

- Solve for y

  $y = \frac{e^{3c_1}x^3}{3} - \frac{k}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
Order:=6;
dsolve(x*diff(y(x),x)-3*y(x)=k,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 (1 + \mathrm{O}(x^6)) + \left(-\frac{k}{3} + \mathrm{O}(x^6)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 15

```
AsymptoticDSolveValue[x*y'[x]-3*y[x]==k,y[x],{x,0,5}]
```

$$y(x) \to -\frac{k}{3} + c_1 x^3$$

1.4 problem 9

1.4.1 Maple step by step solution 34

Internal problem ID [5626]
Internal file name [OUTPUT/4874_Sunday_June_05_2022_03_08_54_PM_58810182/index.tex]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__linear__constant__coeff", "second__order__ode__can__be__made__integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} &(1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' &(4)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f &(5)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f &(2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f &(3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= -y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=-\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{3}$$

For $0\leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1)+a_n=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2}a_0 x^2 - \frac{1}{6}a_1 x^3 + \frac{1}{24}a_0 x^4 + \frac{1}{120}a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6) \tag{2}$$

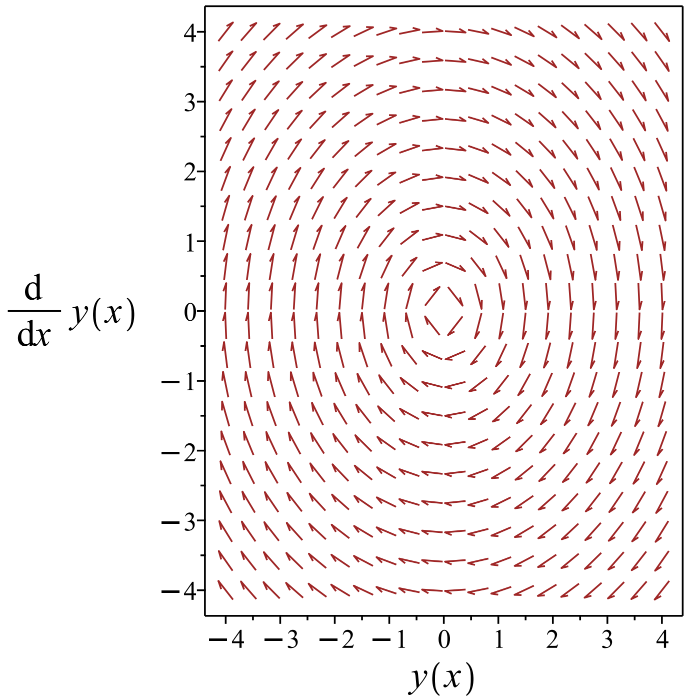

Figure 3: Slope field plot

<u>Verification of solutions</u>

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Verified OK.

1.4.1 Maple step by step solution

Let's solve

$y'' = -y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y = 0$

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = \cos(x) c_1 + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120} - \frac{x^3}{6} + x\right) + c_1\left(\frac{x^4}{24} - \frac{x^2}{2} + 1\right)$$

1.5 problem 10

1.5.1 Maple step by step solution 44

Internal problem ID [5627]
Internal file name [OUTPUT/4875_Sunday_June_05_2022_03_08_55_PM_1437241/index.tex]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (7)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (8)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= y' - xy \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= (1-x)\, y' - (1+x)\, y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= (-1-2x)\, y' + y\left(x^2 - x - 1\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^2 - 3x - 4\right) y' + y\left(2x^2 + 3x - 1\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(3x^2 + 2x - 8\right) y' - y\left(x^3 - 3x^2 - 8x - 3\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= y'(0) \\
F_1 &= -y(0) + y'(0) \\
F_2 &= -y'(0) - y(0) \\
F_3 &= -4y'(0) - y(0) \\
F_4 &= -8y'(0) + 3y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{240}x^6\right) y(0) \\
& + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{30}x^5 - \frac{1}{90}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n$$

$$\sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-(1+n) a_{1+n} x^n\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n\right) + \sum_{n=0}^{\infty} \left(-(1+n) a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_1 = 0$$

$$a_2 = \frac{a_1}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - (1+n)\, a_{1+n} + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n a_{1+n} + a_{1+n} - a_{n-1}}{(n+2)\,(1+n)} \\ &= \frac{a_{1+n}}{n+2} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 2a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6} - \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{24} - \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{30} - \frac{a_0}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_1}{90} + \frac{a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_6 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{1680} + \frac{a_0}{630}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{a_1 x^2}{2} + \left(\frac{a_1}{6} - \frac{a_0}{6}\right) x^3 + \left(-\frac{a_1}{24} - \frac{a_0}{24}\right) x^4 + \left(-\frac{a_1}{30} - \frac{a_0}{120}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5\right) a_0 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{30}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5\right) c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{30}x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{120}x^5+\frac{1}{240}x^6\right)y(0) + \left(x+\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{30}x^5-\frac{1}{90}x^6\right)y'(0)+O(x^6) \tag{1}$$

$$y = \left(1-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{120}x^5\right)c_1+\left(x+\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{30}x^5\right)c_2+O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{120}x^5+\frac{1}{240}x^6\right)y(0) + \left(x+\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{30}x^5-\frac{1}{90}x^6\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{120}x^5\right)c_1+\left(x+\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{30}x^5\right)c_2+O(x^6)$$

Verified OK.

1.5.1 Maple step by step solution

Let's solve

$y'' = y' - xy$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- Shift index using $k->k-1$

  $$x\cdot y=\sum_{k=1}^{\infty}a_{k-1}x^{k}$$

- Convert y' to series expansion

  $$y'=\sum_{k=1}^{\infty}a_k k\,x^{k-1}$$

- Shift index using $k->k+1$

  $$y'=\sum_{k=0}^{\infty}a_{k+1}(k+1)\,x^{k}$$

- Convert y'' to series expansion

  $$y''=\sum_{k=2}^{\infty}a_k k(k-1)\,x^{k-2}$$

- Shift index using $k->k+2$

  $$y''=\sum_{k=0}^{\infty}a_{k+2}(k+2)\,(k+1)\,x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2-a_1+\left(\sum_{k=1}^{\infty}\left(a_{k+2}(k+2)\,(k+1)-a_{k+1}(k+1)+a_{k-1}\right)x^{k}\right)=0$$

- Each term must be 0

  $$2a_2-a_1=0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(k^2+3k+2\right)a_{k+2}-a_{k+1}k+a_{k-1}-a_{k+1}=0$$

- Shift index using $k->k+1$

  $$\left((k+1)^2+3k+5\right)a_{k+3}-a_{k+2}(k+1)+a_k-a_{k+2}=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y=\sum_{k=0}^{\infty}a_k x^{k},a_{k+3}=\frac{ka_{k+2}-a_k+2a_{k+2}}{k^2+5k+6},2a_2-a_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)-diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5\right) y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{30}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[y''[x]-y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{120} - \frac{x^4}{24} - \frac{x^3}{6} + 1\right) + c_2\left(-\frac{x^5}{30} - \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x\right)$$

1.6 problem 11

1.6.1 Maple step by step solution 54

Internal problem ID [5628]
Internal file name [`OUTPUT/4876_Sunday_June_05_2022_03_08_56_PM_17151009/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' + yx^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (10)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (11)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= y' - yx^2 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(-x^2+1\right) y' - x(x+2)\, y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-2x^2-4x+1\right) y' + y(1+x)\left(x^3-x^2-2\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4-3x^2-10x-5\right) y' + 2y\left(x^4+4x^3-\frac{1}{2}x^2-x-1\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(3x^4+12x^3-4x^2-18x-17\right) y' - y\left(x^6-3x^4-18x^3-29x^2+2x+2\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= y'(0) \\
F_1 &= y'(0) \\
F_2 &= y'(0) - 2y(0) \\
F_3 &= -5y'(0) - 2y(0) \\
F_4 &= -17y'(0) - 2y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{12}x^4-\frac{1}{60}x^5-\frac{1}{360}x^6\right) y(0) \\
& + \left(x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{24}x^5-\frac{17}{720}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-(n+1) a_{n+1} x^n\right)$$
$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=0}^{\infty} \left(-(n+1) a_{n+1} x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_1 = 0$$

$$a_2 = \frac{a_1}{2}$$

$n = 1$ gives

$$6a_3 - 2a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$6a_3 - a_1 = 0$$

Or

$$a_3 = \frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\,a_{n+2}(n+1) - (n+1)\,a_{n+1} + a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n a_{n+1} - a_{n-2} + a_{n+1}}{(n+2)\,(n+1)} \\ &= -\frac{a_{n-2}}{(n+2)\,(n+1)} + \frac{a_{n+1}}{n+2} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_3 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{24} - \frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{24} - \frac{a_0}{60}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{17a_1}{720} - \frac{a_0}{360}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{37a_1}{5040} - \frac{a_0}{2520}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \frac{a_1x^2}{2} + \frac{a_1x^3}{6} + \left(\frac{a_1}{24} - \frac{a_0}{12}\right)x^4 + \left(-\frac{a_1}{24} - \frac{a_0}{60}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{12}x^4 - \frac{1}{60}x^5\right)a_0 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{24}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{12}x^4 - \frac{1}{60}x^5\right)c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{24}x^5\right)c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= \left(1-\frac{1}{12}x^{4}-\frac{1}{60}x^{5}-\frac{1}{360}x^{6}\right)y(0) \\
&\quad+\left(x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{24}x^{4}-\frac{1}{24}x^{5}-\frac{17}{720}x^{6}\right)y'(0)+O\left(x^{6}\right) \tag{1}
\end{align*}
$$

$$
y=\left(1-\frac{1}{12}x^{4}-\frac{1}{60}x^{5}\right)c_{1}+\left(x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{24}x^{4}-\frac{1}{24}x^{5}\right)c_{2}+O\left(x^{6}\right) \tag{2}
$$

Verification of solutions

$$
\begin{align*}
y &= \left(1-\frac{1}{12}x^{4}-\frac{1}{60}x^{5}-\frac{1}{360}x^{6}\right)y(0) \\
&\quad+\left(x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{24}x^{4}-\frac{1}{24}x^{5}-\frac{17}{720}x^{6}\right)y'(0)+O\left(x^{6}\right)
\end{align*}
$$

Verified OK.

$$
y=\left(1-\frac{1}{12}x^{4}-\frac{1}{60}x^{5}\right)c_{1}+\left(x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{24}x^{4}-\frac{1}{24}x^{5}\right)c_{2}+O\left(x^{6}\right)
$$

Verified OK.

1.6.1 Maple step by step solution

Let's solve

$y''=y'-yx^{2}$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-y'+yx^{2}=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_{k}x^{k}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^{2}\cdot y$ to series expansion

  $x^{2}\cdot y=\sum_{k=0}^{\infty}a_{k}x^{k+2}$

- Shift index using $k->k-2$

  $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2}x^k$
- Convert y' to series expansion

  $y' = \sum_{k=1}^{\infty} a_k k\, x^{k-1}$
- Shift index using $k->k+1$

  $y' = \sum_{k=0}^{\infty} a_{k+1}(k+1)\, x^k$
- Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
- Shift index using $k->k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - a_1 + (6a_3 - 2a_2)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) + a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - a_1 = 0, 6a_3 - 2a_2 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{a_1}{2}, a_3 = \frac{a_1}{6}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_{k+1}k + a_{k-2} - a_{k+1} = 0$
- Shift index using $k->k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - a_{k+3}(k+2) + a_k - a_{k+3} = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{k a_{k+3} - a_k + 3a_{k+3}}{k^2+7k+12}, a_2 = \frac{a_1}{2}, a_3 = \frac{a_1}{6}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
Order:=6;
dsolve(diff(y(x),x$2)-diff(y(x),x)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{12}x^4 - \frac{1}{60}x^5\right) y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{24}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]-y'[x]+x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{60} - \frac{x^4}{12} + 1\right) + c_2\left(-\frac{x^5}{24} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x\right)$$

1.7 problem 12

1.7.1 Maple step by step solution 64

Internal problem ID [5629]
Internal file name [`OUTPUT/4877_Sunday_June_05_2022_03_08_57_PM_41509662/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(-x^2+1)\, y'' - 2xy' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f\big|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'\big|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (13) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (14) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(-y+xy')}{x^2-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{8(-y+xy')\,x}{(x^2-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{8(-y+xy')\,(5x^2+1)}{(x^2-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{240(-y+xy')\left(x^2+\frac{3}{5}\right)x}{(x^2-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{48(-y+xy')\,(35x^4+42x^2+3)}{(x^2-1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= 0 \\
F_2 &= -8y(0) \\
F_3 &= 0 \\
F_4 &= -144y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6\right) y(0) + xy'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-x^2 + 1)\, y'' - 2xy' + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-x^2 + 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $2 \leq n$, the recurrence equation is

$$-na_n(n-1) + (n+2)\, a_{n+2}(n+1) - 2na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{(n-1)\, a_n}{n+1} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-4a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$-10a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$-18a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{5}$$

For $n = 5$ the recurrence equation gives

$$-28a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - a_0x^2 - \frac{1}{3}a_0x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 - \frac{1}{3}x^4\right)a_0 + a_1x + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 - \frac{1}{3}x^4\right)c_1 + c_2x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6\right)y(0) + xy'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - x^2 - \frac{1}{3}x^4\right)c_1 + c_2x + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6\right)y(0) + xy'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - x^2 - \frac{1}{3}x^4\right)c_1 + c_2x + O(x^6)$$

Verified OK.

1.7.1 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-2xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2xy'}{x^2-1}+\frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^2-1}-\frac{2y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{2}{x^2-1}\right]$

- ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=1$

- ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\,y''+2xy'-2y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)(k+r-1)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+2)(k-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u+1)$

- Revert the change of variables $u = 1 + x$

  $[y = -a_0x]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 - \frac{1}{3}x^4\right) y(0) + D(y)(0)\, x + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 25

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^4}{3} - x^2 + 1\right) + c_2 x$$

1.8 problem 13

1.8.1 Maple step by step solution 74

Internal problem ID [5630]
Internal file name [`OUTPUT/4878_Sunday_June_05_2022_03_08_58_PM_82098902/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (x^2+1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (16)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (17)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4) \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5) \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\left(x^2+1\right) y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -2xy - \left(x^2+1\right) y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -4xy' + y\left(x^4+2x^2-1\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 8xy\left(x^2+1\right) + \left(x^4+2x^2-5\right) y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12\left(x^3+x\right) y' - y\left(x^6+3x^4-27x^2-13\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= -y(0) \\
F_3 &= -5y'(0) \\
F_4 &= 13y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{13}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{24}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -(x^2+1)\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n \\ \sum_{n=0}^{\infty} x^{n+2} a_n &= \sum_{n=2}^{\infty} a_{n-2} x^n \end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n + 2) a_{n+2}(n + 1) + a_{n-2} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{a_{n-2} + a_n}{(n + 2)(n + 1)} \\ &= -\frac{a_n}{(n + 2)(n + 1)} - \frac{a_{n-2}}{(n + 2)(n + 1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_0 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_1 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_2 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{13a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_3 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{5a_1}{1008}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{6} a_1 x^3 - \frac{1}{24} a_0 x^4 - \frac{1}{24} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{24}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{24}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{13}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{24}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{24}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{2}x^2-\frac{1}{24}x^4+\frac{13}{720}x^6\right)y(0)+\left(x-\frac{1}{6}x^3-\frac{1}{24}x^5\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1-\frac{1}{2}x^2-\frac{1}{24}x^4\right)c_1+\left(x-\frac{1}{6}x^3-\frac{1}{24}x^5\right)c_2+O(x^6)$$

Verified OK.

1.8.1 Maple step by step solution

Let's solve

$y'' = -(x^2+1)\,y$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y(-x^2-1)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ◦ Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\,x^{k-2}$

- ◦ Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + a_0 + (6a_3 + a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k + a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $[2a_2 + a_0 = 0, 6a_3 + a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{6}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + a_k + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_{k+2} + a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_{k+2} + a_k}{k^2 + 7k + 12}, a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{6}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+(1+x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{24}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+(1+x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{x^5}{24} - \frac{x^3}{6} + x\right) + c_1\left(-\frac{x^4}{24} - \frac{x^2}{2} + 1\right)$$

1.9 problem 14

1.9.1 Maple step by step solution 84

Internal problem ID [5631]
Internal file name [OUTPUT/4879_Sunday_June_05_2022_03_09_00_PM_36843820/index.tex]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4xy' + (4x^2 - 2)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (19) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (20) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \quad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \quad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \quad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \quad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -4yx^2 + 4xy' + 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -16yx^3 + 12x^2 y' + 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(32x^3 + 48x\right) y' - 48y\left(x^4 + x^2 - \frac{1}{4}\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= -128yx^5 + 80y'x^4 - 320yx^3 + 240x^2 y' + 60y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(192x^5 + 960x^3 + 720x\right) y' - 320y\left(x^6 + \frac{9}{2}x^4 + \frac{9}{4}x^2 - \frac{3}{8}\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 6y'(0) \\
F_2 &= 12y(0) \\
F_3 &= 60y'(0) \\
F_4 &= 120y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6\right) y(0) + \left(x + x^3 + \frac{1}{2}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -4\left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 + 4x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-4n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 4x^{n+2} a_n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n$$
$$\sum_{n=0}^{\infty} 4x^{n+2} a_n = \sum_{n=2}^{\infty} 4a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n\right) + \sum_{n=1}^{\infty} (-4n\, x^n a_n) + \left(\sum_{n=2}^{\infty} 4a_{n-2} x^n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

$n = 1$ gives

$$6a_3 - 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = a_1$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 4na_n + 4a_{n-2} - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{4na_n + 2a_n - 4a_{n-2}}{(n+2)\,(n+1)} \\ &= \frac{2(2n+1)\, a_n}{(n+2)\,(n+1)} - \frac{4a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 10a_2 + 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 14a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{2}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 18a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{6}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 22a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{6}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + a_0 x^2 + a_1 x^3 + \frac{1}{2} a_0 x^4 + \frac{1}{2} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x^2 + \frac{1}{2}x^4\right) a_0 + \left(x + x^3 + \frac{1}{2}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + x^2 + \frac{1}{2}x^4\right) c_1 + \left(x + x^3 + \frac{1}{2}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6\right) y(0) + \left(x + x^3 + \frac{1}{2}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + x^2 + \frac{1}{2}x^4\right) c_1 + \left(x + x^3 + \frac{1}{2}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1+x^{2}+\frac{1}{2}x^{4}+\frac{1}{6}x^{6}\right) y(0)+\left(x+x^{3}+\frac{1}{2}x^{5}\right) y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1+x^{2}+\frac{1}{2}x^{4}\right) c_{1}+\left(x+x^{3}+\frac{1}{2}x^{5}\right) c_{2}+O\left(x^{6}\right)$$

Verified OK.

1.9.1 Maple step by step solution

Let's solve

$y'' = -4yx^{2}+4xy'+2y$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \left(-4x^{2}+2\right) y +4xy'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-4xy'+\left(4x^{2}-2\right) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_{k} x^{k}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^{m}\cdot y$ to series expansion for $m = 0..2$

    $x^{m}\cdot y = \sum_{k=\max(0,-m)}^{\infty} a_{k} x^{k+m}$

  ◦ Shift index using $k->k-m$

    $x^{m}\cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^{k}$

  ◦ Convert $x\cdot y'$ to series expansion

    $x\cdot y' = \sum_{k=0}^{\infty} a_{k} k\, x^{k}$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 - 2a_0 + (6a_3 - 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = a_0, a_3 = a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
Order:=6;
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + x^2 + \frac{1}{2}x^4\right) y(0) + \left(x + x^3 + \frac{1}{2}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[y''[x]-4*x*y'[x]+(4*x^2-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{2} + x^3 + x\right) + c_1\left(\frac{x^4}{2} + x^2 + 1\right)$$

1.10 problem 16

1.10.1 Existence and uniqueness analysis 87
1.10.2 Solving as series ode 88
1.10.3 Maple step by step solution 95

Internal problem ID [5632]
Internal file name [`OUTPUT/4880_Sunday_June_05_2022_03_09_01_PM_43917989/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature", "first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{4y + y' = 1}$$

With initial conditions

$$\left[y(0) = \frac{5}{4}\right]$$

With the expansion point for the power series method at $x = 0$.

1.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 4$$
$$q(x) = 1$$

Hence the ode is

$$4y + y' = 1$$

The domain of $p(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.10.2 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned} F_0 &= f(x,y) \qquad (4) \\ F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)F_0 \qquad (5) \end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right)F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \qquad (6)$$

Hence

$$
\begin{aligned}
F_0 &= -4y + 1 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= 16y - 4 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= -64y + 16 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= 256y - 64 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= -1024y + 256
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = \frac{5}{4}$ gives

$$
\begin{aligned}
F_0 &= -4 \\
F_1 &= 16 \\
F_2 &= -64 \\
F_3 &= 256 \\
F_4 &= -1024
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = \frac{5}{4} - 4x + 8x^2 - \frac{32x^3}{3} + \frac{32x^4}{3} - \frac{128x^5}{15} + O(x^6)
$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$
y(0) = y(0)
$$

Therefore the solution becomes

$$y = \frac{5}{4} - 4x + 8x^2 - \frac{32}{3}x^3 + \frac{32}{3}x^4 - \frac{128}{15}x^5$$

Hence the solution can be written as

$$y = \frac{5}{4} - 4x + 8x^2 - \frac{32x^3}{3} + \frac{32x^4}{3} - \frac{128x^5}{15} + O(x^6)$$

which simplifies to

$$y = \frac{5}{4} - 4x + 8x^2 - \frac{32x^3}{3} + \frac{32x^4}{3} - \frac{128x^5}{15} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ 4y + y' &= 1 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= 4 \\ p(x) &= 1 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$4\left(\sum_{n=0}^{\infty} a_n x^n\right)+\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)=1 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} 4 a_n x^n\right)=1 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+1) a_{n+1} x^n\right)+\left(\sum_{n=0}^{\infty} 4 a_n x^n\right)=1 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$\left((n+1) a_{n+1}+4 a_n\right) x^n=1 \tag{4}$$

For $n=0$ the recurrence equation gives

$$\begin{aligned}
\left(a_1+4 a_0\right) 1 &= 1 \\
a_1+4 a_0 &= 1
\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_1=1-4 a_0$$

For $n=1$ the recurrence equation gives

$$\begin{aligned}
\left(2 a_2+4 a_1\right) x &= 0 \\
2 a_2+4 a_1 &= 0
\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_2 = -2 + 8a_0$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(3a_3 + 4a_2)\,x^2 &= 0\\ 3a_3 + 4a_2 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{8}{3} - \frac{32a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(4a_4 + 4a_3)\,x^3 &= 0\\ 4a_4 + 4a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{8}{3} + \frac{32a_0}{3}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(5a_5 + 4a_4)\,x^4 &= 0\\ 5a_5 + 4a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{32}{15} - \frac{128a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(6a_6 + 4a_5)\,x^5 &= 0\\ 6a_6 + 4a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{64}{45} + \frac{256a_0}{45}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned}y = a_0 + (1 - 4a_0)\,x + (-2 + 8a_0)\,x^2 + \left(\frac{8}{3} - \frac{32a_0}{3}\right)x^3 \\ + \left(-\frac{8}{3} + \frac{32a_0}{3}\right)x^4 + \left(\frac{32}{15} - \frac{128a_0}{15}\right)x^5 + \dots\end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(1 - 4x + 8x^2 - \frac{32}{3}x^3 + \frac{32}{3}x^4 - \frac{128}{15}x^5\right)a_0 + x - 2x^2 + \frac{8x^3}{3} - \frac{8x^4}{3} + \frac{32x^5}{15} + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y(0) = a_0$$

Therefore the solution in Eq(3) now can be written as

$$y = \left(1 - 4x + 8x^2 - \frac{32}{3}x^3 + \frac{32}{3}x^4 - \frac{128}{15}x^5\right)y(0) + x - 2x^2 + \frac{8x^3}{3} - \frac{8x^4}{3} + \frac{32x^5}{15} + O(x^6)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = \frac{5}{4}$$

Therefore the solution becomes

$$y = \frac{5}{4} - 4x + 8x^2 - \frac{32}{3}x^3 + \frac{32}{3}x^4 - \frac{128}{15}x^5$$

Hence the solution can be written as

$$y = \frac{5}{4} - 4x + 8x^2 - \frac{32x^3}{3} + \frac{32x^4}{3} - \frac{128x^5}{15} + O(x^6)$$

which simplifies to

$$y = \frac{5}{4} - 4x + 8x^2 - \frac{32x^3}{3} + \frac{32x^4}{3} - \frac{128x^5}{15} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \frac{5}{4} - 4x + 8x^2 - \frac{32x^3}{3} + \frac{32x^4}{3} - \frac{128x^5}{15} + O(x^6) \tag{1}$$

$$y = \frac{5}{4} - 4x + 8x^2 - \frac{32x^3}{3} + \frac{32x^4}{3} - \frac{128x^5}{15} + O(x^6) \tag{2}$$

Verification of solutions

$$y = \frac{5}{4} - 4x + 8x^2 - \frac{32x^3}{3} + \frac{32x^4}{3} - \frac{128x^5}{15} + O(x^6)$$

Verified OK.

$$y = \frac{5}{4} - 4x + 8x^2 - \frac{32x^3}{3} + \frac{32x^4}{3} - \frac{128x^5}{15} + O(x^6)$$

Verified OK.

1.10.3 Maple step by step solution

Let's solve
$\left[4y + y' = 1, y(0) = \frac{5}{4}\right]$

- Highest derivative means the order of the ODE is 1
  y'
- Separate variables
  $\frac{y'}{-4y+1} = 1$
- Integrate both sides with respect to x
  $\int \frac{y'}{-4y+1} dx = \int 1 dx + c_1$
- Evaluate integral
  $-\frac{\ln(-4y+1)}{4} = x + c_1$

- Solve for y

  $y = -\frac{e^{-4x-4c_1}}{4} + \frac{1}{4}$
- Use initial condition $y(0) = \frac{5}{4}$

  $\frac{5}{4} = -\frac{e^{-4c_1}}{4} + \frac{1}{4}$
- Solve for c_1

  $c_1 = -\frac{\ln(2)}{2} - \frac{I\pi}{4}$
- Substitute $c_1 = -\frac{\ln(2)}{2} - \frac{I\pi}{4}$ into general solution and simplify

  $y = e^{-4x} + \frac{1}{4}$
- Solution to the IVP

  $y = e^{-4x} + \frac{1}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x)+4*y(x)=1,y(0) = 5/4],y(x),type='series',x=0);
```

$$y(x) = \frac{5}{4} - 4x + 8x^2 - \frac{32}{3}x^3 + \frac{32}{3}x^4 - \frac{128}{15}x^5 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 36

```
AsymptoticDSolveValue[{y'[x]+4*y[x]==1,{y[0]==125/100}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{128x^5}{15} + \frac{32x^4}{3} - \frac{32x^3}{3} + 8x^2 - 4x + \frac{5}{4}$$

1.11 problem 17

1.11.1 Existence and uniqueness analysis 97
1.11.2 Maple step by step solution 105

Internal problem ID [5633]
Internal file name [OUTPUT/4881_Sunday_June_05_2022_03_09_02_PM_90872983/index.tex]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3xy' + 2y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

1.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 3x \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 3xy' + 2y = 0$$

The domain of $p(x) = 3x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (23)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (24)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -3xy' - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 9x^2 y' + 6xy - 5y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -27y'x^3 - 18yx^2 + 39xy' + 16y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(81x^4 - 216x^2 + 55\right) y' + \left(54x^3 - 114x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-243x^5 + 1026x^3 - 711x\right) y' + \left(-162x^4 + 594x^2 - 224\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=1$ and $y'(0)=1$ gives

$$
\begin{aligned}
F_0 &= -2 \\
F_1 &= -5 \\
F_2 &= 16 \\
F_3 &= 55 \\
F_4 &= -224
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -x^2 + x + 1 - \frac{5x^3}{6} + \frac{2x^4}{3} + \frac{11x^5}{24} - \frac{14x^6}{45} + O\left(x^6\right)
$$

$$
y = -x^2 + x + 1 - \frac{5x^3}{6} + \frac{2x^4}{3} + \frac{11x^5}{24} - \frac{14x^6}{45} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -3x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 3n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 3n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 3na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(3n+2)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 5a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{5a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 8a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{2a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 11a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{11a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 14a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{14a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 17a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{187a_1}{1008}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{5}{6} a_1 x^3 + \frac{2}{3} a_0 x^4 + \frac{11}{24} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{2}{3} x^4\right) a_0 + \left(x - \frac{5}{6} x^3 + \frac{11}{24} x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{2}{3} x^4\right) c_1 + \left(x - \frac{5}{6} x^3 + \frac{11}{24} x^5\right) c_2 + O\left(x^6\right)$$

$$y = 1 - x^2 + \frac{2x^4}{3} + x - \frac{5x^3}{6} + \frac{11x^5}{24} + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = -x^2 + x + 1 - \frac{5x^3}{6} + \frac{2x^4}{3} + \frac{11x^5}{24} - \frac{14x^6}{45} + O\left(x^6\right) \tag{1}$$

$$y = 1 - x^2 + \frac{2x^4}{3} + x - \frac{5x^3}{6} + \frac{11x^5}{24} + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = -x^2 + x + 1 - \frac{5x^3}{6} + \frac{2x^4}{3} + \frac{11x^5}{24} - \frac{14x^6}{45} + O(x^6)$$

Verified OK.

$$y = 1 - x^2 + \frac{2x^4}{3} + x - \frac{5x^3}{6} + \frac{11x^5}{24} + O(x^6)$$

Verified OK.

1.11.2 Maple step by step solution

Let's solve

$\left[y'' = -3xy' - 2y, y(0) = 1, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 3xy' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(3k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} + 3a_k k + 2a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(3k+2)}{k^2+3k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)+3*x*diff(y(x),x)+2*y(x)=0,y(0) = 1, D(y)(0) = 1],y(x),type='series',x
```

$$y(x) = 1 + x - x^2 - \frac{5}{6}x^3 + \frac{2}{3}x^4 + \frac{11}{24}x^5 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 32

```
AsymptoticDSolveValue[{y''[x]+3*x*y'[x]+2*y[x]==0,{y[0]==1,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{11x^5}{24} + \frac{2x^4}{3} - \frac{5x^3}{6} - x^2 + x + 1$$

1.12 problem 18

1.12.1 Existence and uniqueness analysis 108
1.12.2 Maple step by step solution 116

Internal problem ID [5634]
Internal file name [`OUTPUT/4882_Sunday_June_05_2022_03_09_05_PM_64167886/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(-x^{2}+1\right) y''-2 x y'+30 y=0}$$

With initial conditions

$$\left[y(0)=0, y'(0)=\frac{15}{8}\right]$$

With the expansion point for the power series method at $x=0$.

1.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= -\frac{2x}{-x^{2}+1} \\ q(x) &= \frac{30}{-x^{2}+1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{2xy'}{-x^2+1} + \frac{30y}{-x^2+1} = 0$$

The domain of $p(x) = -\frac{2x}{-x^2+1}$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{30}{-x^2+1}$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{26}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{27}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(xy' - 15y)}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{36x^2 y' - 120xy - 28y'}{\left(x^2 - 1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{24(11y'x^3 - 60yx^2 - 9xy' + 30y)}{\left(x^2 - 1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(2760x^4 - 2880x^2 + 504\right) y' + \left(-13680x^3 + 7920x\right) y}{\left(x^2 - 1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(-30240x^5 + 33600x^3 - 7200x\right) y' + 151200\left(x^4 - \frac{2}{3}x^2 + \frac{1}{21}\right) y}{\left(x^2 - 1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = \frac{15}{8}$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -\frac{105}{2} \\
F_2 &= 0 \\
F_3 &= 945 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \frac{15x}{8} - \frac{35x^3}{4} + \frac{63x^5}{8} + O\left(x^6\right)
$$

$$
y = \frac{15x}{8} - \frac{35x^3}{4} + \frac{63x^5}{8} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(-x^2 + 1)\, y'' - 2xy' + 30y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

$$
y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
$$

Substituting the above back into the ode gives

$$
(-x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \right) - 2x \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) + 30 \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0 \quad (1)
$$

Which simplifies to

$$
\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \right) + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} 30 a_n x^n \right) = 0 \quad (2)
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n \right) + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} 30 a_n x^n \right) = 0 \quad (3)
$$

$n = 0$ gives

$$2a_2 + 30a_0 = 0$$

$$a_2 = -15a_0$$

$n = 1$ gives

$$6a_3 + 28a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{14a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$-na_n(n-1) + (n+2)\,a_{n+2}(n+1) - 2na_n + 30a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 + n - 30)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$24a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 30a_0$$

For $n = 3$ the recurrence equation gives

$$18a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{21a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$10a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -10a_0$$

For $n = 5$ the recurrence equation gives

$$42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 15a_0 x^2 - \frac{14}{3} a_1 x^3 + 30a_0 x^4 + \frac{21}{5} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(30x^4 - 15x^2 + 1\right) a_0 + \left(x - \frac{14}{3}x^3 + \frac{21}{5}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(30x^4 - 15x^2 + 1\right) c_1 + \left(x - \frac{14}{3}x^3 + \frac{21}{5}x^5\right) c_2 + O\left(x^6\right)$$

$$y = \frac{15x}{8} - \frac{35x^3}{4} + \frac{63x^5}{8} + O\left(x^6\right)$$

Summary
The solution(s) found are the following

$$y = \frac{15x}{8} - \frac{35x^3}{4} + \frac{63x^5}{8} + O(x^6) \tag{1}$$

$$y = \frac{15x}{8} - \frac{35x^3}{4} + \frac{63x^5}{8} + O(x^6) \tag{2}$$

Verification of solutions

$$y = \frac{15x}{8} - \frac{35x^3}{4} + \frac{63x^5}{8} + O(x^6)$$

Verified OK.

$$y = \frac{15x}{8} - \frac{35x^3}{4} + \frac{63x^5}{8} + O(x^6)$$

Verified OK.

1.12.2 Maple step by step solution

Let's solve

$\left[(-x^2+1)\, y'' - 2xy' + 30y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = \frac{15}{8}\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2xy'}{x^2-1} + \frac{30y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{30y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{30}{x^2-1}\right]$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 1$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y'' + 2xy' - 30y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 30y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - ○ Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - ○ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

  - ○ Shift index using $k- >k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+6)\,(k+r-5)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+6)(k-5) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+6)(k-5)}{2(k+1)^2}$

- Recursion relation for $r = 0$; series terminates at $k = 5$

  $a_{k+1} = \frac{a_k(k+6)(k-5)}{2(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -15a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{7a_1}{2}$

- Express in terms of a_0

  $a_2 = \frac{105a_0}{2}$

- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{3}$

- Express in terms of a_0

  $a_3 = -70a_0$

- Apply recursion relation for $k = 3$

  $a_4 = -\frac{9a_3}{16}$

- Express in terms of a_0

  $a_4 = \frac{315a_0}{8}$

- Apply recursion relation for $k = 4$

  $a_5 = -\frac{a_4}{5}$

- Express in terms of a_0

  $a_5 = -\frac{63a_0}{8}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 15u + \frac{105}{2}u^2 - 70u^3 + \frac{315}{8}u^4 - \frac{63}{8}u^5\right)$

- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(-\frac{15}{8}x + \frac{35}{4}x^3 - \frac{63}{8}x^5\right)\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
Order:=6;
dsolve([(1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+30*y(x)=0,y(0) = 0, D(y)(0) = 15/8],y(x),typ
```

$$y(x) = \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5 + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 23

```
AsymptoticDSolveValue[{(1-x^2)*y''[x]-2*x*y'[x]+30*y[x]==0,{y[0]==0,y'[0]==1875/1000}},y[x],
```

$$y(x) \to \frac{63x^5}{8} - \frac{35x^3}{4} + \frac{15x}{8}$$

1.13 problem 19

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1.13.2 Solving as series ode 121
1.13.3 Maple step by step solution 128

Internal problem ID [5635]
Internal file name [`OUTPUT/4883_Sunday_June_05_2022_03_09_07_PM_21564007/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.1. page 174
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup", "first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(-2+x)\,y'-xy=0}$$

With initial conditions

$$[y(0)=4]$$

With the expansion point for the power series method at $x=0$.

1.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=-\frac{x}{-2+x}$$
$$q(x)=0$$

Hence the ode is

$$y' - \frac{xy}{-2+x} = 0$$

The domain of $p(x) = -\frac{x}{-2+x}$ is

$$\{x < 2 \vee 2 < x\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

1.13.2 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y''(x_0) + \cdots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned} F_0 &= f(x,y) \qquad (4) \\ F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5) \end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n\big|_{x_0,y_0} \qquad (6)$$

Hence

$$
\begin{aligned}
F_0 &= \frac{xy}{-2+x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \frac{y(x^2-2)}{(-2+x)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= \frac{y(x^2+2x-2)}{(-2+x)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \frac{xy(x+4)}{(-2+x)^2} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= \frac{y(x^2+6x+4)}{(-2+x)^2}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 4$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -2 \\
F_2 &= -2 \\
F_3 &= 0 \\
F_4 &= 4
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = -x^2 + 4 - \frac{x^3}{3} + \frac{x^5}{30} + O(x^6)
$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y = -x^2 + 4 - \frac{1}{3}x^3 + \frac{1}{30}x^5$$

Hence the solution can be written as

$$y = -x^2 + 4 - \frac{x^3}{3} + \frac{x^5}{30} + O(x^6)$$

which simplifies to

$$y = -x^2 + 4 - \frac{x^3}{3} + \frac{x^5}{30} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - \frac{xy}{-2+x} &= 0 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -\frac{x}{-2+x} \\ p(x) &= 0 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(-2 + x)\, y' - xy = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(-2+x)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=1}^{\infty} \left(-2n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} \left(-2n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-2(1+n)\, a_{1+n} x^n\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=0}^{\infty} \left(-2(1+n)\, a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$-2(1+n)\, a_{1+n} + n a_n - a_{n-1} = 0 \tag{4}$$

Solving for a_{1+n}, gives

$$a_{1+n} = \frac{n a_n - a_{n-1}}{2+2n} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$-4a_2 + a_1 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_0}{4}$$

For $n = 2$ the recurrence equation gives

$$-6a_3 + 2a_2 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$-8a_4 + 3a_3 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 4$ the recurrence equation gives

$$-10a_5 + 4a_4 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{120}$$

For $n = 5$ the recurrence equation gives

$$-12a_6 + 5a_5 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{288}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 - \frac{1}{4} a_0 x^2 - \frac{1}{12} a_0 x^3 + \frac{1}{120} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{4} x^2 - \frac{1}{12} x^3 + \frac{1}{120} x^5\right) a_0 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y(0) = a_0$$

Therefore the solution in Eq(3) now can be written as

$$y = \left(1 - \frac{1}{4} x^2 - \frac{1}{12} x^3 + \frac{1}{120} x^5\right) y(0) + O\left(x^6\right)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = 4$$

Therefore the solution becomes

$$y = -x^2 + 4 - \frac{1}{3} x^3 + \frac{1}{30} x^5$$

Hence the solution can be written as

$$y = -x^2 + 4 - \frac{x^3}{3} + \frac{x^5}{30} + O\left(x^6\right)$$

which simplifies to

$$y = -x^2 + 4 - \frac{x^3}{3} + \frac{x^5}{30} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -x^2 + 4 - \frac{x^3}{3} + \frac{x^5}{30} + O(x^6) \tag{1}$$

$$y = -x^2 + 4 - \frac{x^3}{3} + \frac{x^5}{30} + O(x^6) \tag{2}$$

Verification of solutions

$$y = -x^2 + 4 - \frac{x^3}{3} + \frac{x^5}{30} + O(x^6)$$

Verified OK.

$$y = -x^2 + 4 - \frac{x^3}{3} + \frac{x^5}{30} + O(x^6)$$

Verified OK.

1.13.3 Maple step by step solution

Let's solve

$[(-2+x)\,y' - xy = 0, y(0) = 4]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{x}{-2+x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{x}{-2+x}dx + c_1$

- Evaluate integral

  $\ln(y) = x + 2\ln(-2+x) + c_1$

- Solve for y

  $y = e^{x+c_1}(-2+x)^2$

- Use initial condition $y(0) = 4$

$$4 = 4\,e^{c_1}$$

- Solve for c_1

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = e^x(-2 + x)^2$$

- Solution to the IVP

$$y = e^x(-2 + x)^2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=6;
dsolve([(x-2)*diff(y(x),x)=x*y(x),y(0) = 4],y(x),type='series',x=0);
```

$$y(x) = 4 - x^2 - \frac{1}{3}x^3 + \frac{1}{30}x^5 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 24

```
AsymptoticDSolveValue[{(x-2)*y'[x]==x*y[x],{y[0]==4}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{30} - \frac{x^3}{3} - x^2 + 4$$

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2.1 problem 2

Internal problem ID [5636]
Internal file name [`OUTPUT/4884_Sunday_June_05_2022_03_09_08_PM_23824306/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(-2+x)^2 y'' + (x+2) y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f\big|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'\big|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (30) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (31) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \ \vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{xy' + 2y' - y}{(-2+x)^2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(xy' + 2y' - y)(3x - 2)}{(-2+x)^4} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{4(xy' + 2y' - y)\, x(3x - 4)}{(-2+x)^6} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{60((x+2)\, y' - y)\left(x^3 - 2x^2 + \frac{16}{15}\right)}{(-2+x)^8} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{8(xy' + 2y' - y)(45x^4 - 120x^3 + 192x - 112)}{(-2+x)^{10}}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)}{4} - \frac{y'(0)}{2} \\
F_1 &= \frac{y(0)}{8} - \frac{y'(0)}{4} \\
F_2 &= 0 \\
F_3 &= \frac{y'(0)}{2} - \frac{y(0)}{4} \\
F_4 &= -\frac{7y(0)}{8} + \frac{7y'(0)}{4}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$\begin{aligned} y = &\left(1+\frac{1}{8}x^2+\frac{1}{48}x^3-\frac{1}{480}x^5-\frac{7}{5760}x^6\right)y(0) \\ &+\left(x-\frac{1}{4}x^2-\frac{1}{24}x^3+\frac{1}{240}x^5+\frac{7}{2880}x^6\right)y'(0)+O(x^6) \end{aligned}$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2-4x+4)\,y''+(x+2)\,y'-y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$(x^2-4x+4)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+(x+2)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\sum_{n=2}^{\infty}\left(-4n\,x^{n-1}a_n(n-1)\right)+\left(\sum_{n=2}^{\infty} 4n(n-1)\,a_n x^{n-2}\right) \\ &+\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right)+\sum_{n=0}^{\infty}(-a_n x^n)=0 \end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-4n\, x^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-4(n+1)\, a_{n+1} n\, x^n\right)$$

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2} (n+1)\, x^n$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=1}^{\infty} \left(-4(n+1)\, a_{n+1} n\, x^n\right) \\ &+ \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) \\ &+ \left(\sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$8a_2 + 2a_1 - a_0 = 0$$

$$a_2 = \frac{a_0}{8} - \frac{a_1}{4}$$

$n = 1$ gives

$$-4a_2 + 24a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{a_0}{2} + a_1 + 24a_3 = 0$$

Or

$$a_3 = \frac{a_0}{48} - \frac{a_1}{24}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) - 4(n+1)\, a_{n+1} n + 4(n+2)\, a_{n+2}(n+1) + n a_n + 2(n+1)\, a_{n+1} - a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n a_n - 4n a_{n+1} - a_n + 2a_{n+1}}{4\,(n+2)} \\ &= -\frac{(n-1)\, a_n}{4\,(n+2)} - \frac{(-4n+2)\, a_{n+1}}{4\,(n+2)} \end{aligned} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$3a_2 - 18a_3 + 48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$8a_3 - 40a_4 + 80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{480} + \frac{a_1}{240}$$

For $n = 4$ the recurrence equation gives

$$15a_4 - 70a_5 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{5760} + \frac{7a_1}{2880}$$

For $n = 5$ the recurrence equation gives

$$24a_5 - 108a_6 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{13a_0}{26880} + \frac{13a_1}{13440}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(\frac{a_0}{8} - \frac{a_1}{4}\right) x^2 + \left(\frac{a_0}{48} - \frac{a_1}{24}\right) x^3 + \left(-\frac{a_0}{480} + \frac{a_1}{240}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{8}x^2 + \frac{1}{48}x^3 - \frac{1}{480}x^5\right) a_0 + \left(x - \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{240}x^5\right) a_1 + O\left(x^6\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{8}x^2 + \frac{1}{48}x^3 - \frac{1}{480}x^5\right) c_1 + \left(x - \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{240}x^5\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \left(1 + \frac{1}{8}x^2 + \frac{1}{48}x^3 - \frac{1}{480}x^5 - \frac{7}{5760}x^6\right) y(0) \\ &\quad + \left(x - \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{240}x^5 + \frac{7}{2880}x^6\right) y'(0) + O\left(x^6\right) \end{aligned} \quad (1)$$

$$y = \left(1 + \frac{1}{8}x^2 + \frac{1}{48}x^3 - \frac{1}{480}x^5\right) c_1 + \left(x - \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{240}x^5\right) c_2 + O\left(x^6\right) \quad (2)$$

Verification of solutions

$$y = \left(1+\frac{1}{8}x^2+\frac{1}{48}x^3-\frac{1}{480}x^5-\frac{7}{5760}x^6\right)y(0) + \left(x-\frac{1}{4}x^2-\frac{1}{24}x^3+\frac{1}{240}x^5+\frac{7}{2880}x^6\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1+\frac{1}{8}x^2+\frac{1}{48}x^3-\frac{1}{480}x^5\right)c_1+\left(x-\frac{1}{4}x^2-\frac{1}{24}x^3+\frac{1}{240}x^5\right)c_2+O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=6;
dsolve((x-2)^2*diff(y(x),x$2)+(x+2)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{8}x^2+\frac{1}{48}x^3-\frac{1}{480}x^5\right)y(0)+\left(x-\frac{1}{4}x^2-\frac{1}{24}x^3+\frac{1}{240}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[(x-2)^2*y''[x]+(x+2)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(-\frac{x^5}{480}+\frac{x^3}{48}+\frac{x^2}{8}+1\right)+c_2\left(\frac{x^5}{240}-\frac{x^3}{24}-\frac{x^2}{4}+x\right)$$

2.2 problem 3

2.2.1 Maple step by step solution 149

Internal problem ID [5637]
Internal file name [`OUTPUT/4885_Sunday_June_05_2022_03_09_10_PM_98595007/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{xy'' + 2y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = 1$$

Table 14: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + 2\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 14r^3 + 71r^2 + 154r + 120}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -1} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 2(n+r)\, b_n + b_{n-2} = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + 2(n-1)\, b_n + b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-2}}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x}
$$

Verified OK.

2.2.1 Maple step by step solution

Let's solve

$y''x + 2y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + 2y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - ○ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - ○ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - ○ Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

  - ○ Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

  - ○ Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  - ○ Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r)+a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+3+r)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+3)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+2}=-\frac{a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=-\frac{b_k}{(k+2)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 42

```
AsymptoticDSolveValue[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^3}{24} - \frac{x}{2} + \frac{1}{x}\right) + c_2\left(\frac{x^4}{120} - \frac{x^2}{6} + 1\right)$$

2.3 problem 4

2.3.1 Maple step by step solution 164

Internal problem ID [5638]
Internal file name [`OUTPUT/4886_Sunday_June_05_2022_03_09_12_PM_66663530/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{1}{x} \end{aligned}$$

Table 16: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}}{(n+1)n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{(1+r)\,r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)^2\,r\,(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(1+r)^2\,r\,(2+r)^2\,(3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2(3+r)}$	$-\frac{1}{144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2(3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2(3+r)^2(4+r)}$	$\frac{1}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{1}{2880}$
a_5	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$	$-\frac{1}{86400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\big) \\ &= x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_1 \\ &= -\frac{1}{(1+r)\, r}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} -\frac{1}{(1+r)\, r} &= \lim_{r\to 0} -\frac{1}{(1+r)\, r} \\ &= \textit{undefined}\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\frac{d}{dx}y_2(x) = C y_1'(x)\ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$

$$= C y_1'(x)\ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$

$$\frac{d^2}{dx^2}y_2(x) = C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$

$$= C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $xy'' + y = 0$ gives

$$\left(C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left(\left(y_1''(x)\,x + y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x + y_1(x) = 0$$

Eq (7) simplifes to

$$\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)xC + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{n} a_n (n+1)\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n}\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1)\right) + \sum_{n=0}^{\infty} \left(-C\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-C\, x^n a_n\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=1}^{\infty} 2Ca_{n-1}n\, x^{n-1}\right)+\sum_{n=1}^{\infty}\left(-Ca_{n-1}x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n(n-1)\right)+\left(\sum_{n=1}^{\infty} b_{n-1}x^{n-1}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C+1=0$$

Which is solved for C. Solving for C gives

$$C=-1$$

For $n=2$, Eq (2B) gives

$$3Ca_1+b_1+2b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2+\frac{3}{2}=0$$

Solving the above for b_2 gives

$$b_2=-\frac{3}{4}$$

For $n=3$, Eq (2B) gives

$$5Ca_2+b_2+6b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3-\frac{7}{6}=0$$

Solving the above for b_3 gives

$$b_3=\frac{7}{36}$$

For $n=4$, Eq (2B) gives

$$7Ca_3+b_3+12b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_4+\frac{35}{144}=0$$

Solving the above for b_4 gives

$$b_4 = -\frac{35}{1728}$$

For $n = 5$, Eq (2B) gives

$$9Ca_4 + b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$20b_5 - \frac{101}{4320} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{101}{86400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = {} & (-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\right)\ln(x) \\ & + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right) \\ &\quad + c_2\left((-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\right)\ln(x) + 1 \right. \\ &\quad \left. - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right) \\ &\quad + c_2\left(-x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\ln(x) + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} \right. \\ &\quad \left. - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right) + c_2 \left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}+\frac{7x^3}{36}-\frac{35x^4}{1728}+\frac{101x^5}{86400}+O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right) + c_2 \left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}+\frac{7x^3}{36}-\frac{35x^4}{1728}+\frac{101x^5}{86400}+O(x^6)\right)$$

Verified OK.

2.3.1 Maple step by step solution

Let's solve

$y''x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{1}{x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r) + a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{(k+1)k}$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1)k}\right]$$

- Recursion relation for $r = 1$

  $$a_{k+1} = -\frac{a_k}{(k+2)(k+1)}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{(k+2)(k+1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{a_k}{(k+1)k}, b_{k+1} = -\frac{b_k}{(k+2)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
Order:=6;
dsolve(x*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x\left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{144}x^3 + \frac{1}{2880}x^4 - \frac{1}{86400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(\ln\left(x\right)\left(-x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{144}x^4 - \frac{1}{2880}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \frac{101}{86400}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{144}x\left(x^3 - 12x^2 + 72x - 144\right)\log(x) + \frac{-47x^4 + 480x^3 - 2160x^2 + 1728x + 1728}{1728}\right) + c_2\left(\frac{x^5}{2880} - \frac{x^4}{144} + \frac{x^3}{12} - \frac{x^2}{2} + x\right)$$

2.4 problem 5

2.4.1 Maple step by step solution 176

Internal problem ID [5639]
Internal file name [`OUTPUT/4887_Sunday_June_05_2022_03_09_15_PM_74329414/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (1+2x)\,y' + (1+x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (1+2x)\,y' + (1+x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1+2x}{x}$$
$$q(x) = \frac{1+x}{x}$$

Table 18: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+2x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1+x}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (1 + 2x)\, y' + (1 + x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) x + (1 + 2x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (1 + x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-2r - 1}{(1 + r)^2}$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + 2a_{n-1}(n + r - 1) + a_n(n + r) + a_{n-1} + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2na_{n-1} + 2ra_{n-1} + a_{n-2} - a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{-2na_{n-1} - a_{n-2} + a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-1}{(1+r)^2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3r^2 + 6r + 2}{(1+r)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-1}{(1+r)^2}$	-1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-4r^3 - 18r^2 - 22r - 6}{(1+r)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-1}{(1+r)^2}$	-1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$
a_3	$\frac{-4r^3-18r^2-22r-6}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5r^4 + 40r^3 + 105r^2 + 100r + 24}{(1+r)^2 (r+2)^2 (r+3)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-1}{(1+r)^2}$	-1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$
a_3	$\frac{-4r^3-18r^2-22r-6}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{1}{6}$
a_4	$\frac{5r^4+40r^3+105r^2+100r+24}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-6r^5 - 75r^4 - 340r^3 - 675r^2 - 548r - 120}{(1+r)^2 (r+2)^2 (r+3)^2 (4+r)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-1}{(1+r)^2}$	-1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$
a_3	$\frac{-4r^3-18r^2-22r-6}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{1}{6}$
a_4	$\frac{5r^4+40r^3+105r^2+100r+24}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{24}$
a_5	$\frac{-6r^5-75r^4-340r^3-675r^2-548r-120}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2(r+5)^2}$	$-\frac{1}{120}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\
&= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n($
b_0	1	1	N/A since b_n starts from 1	N/
b_1	$\frac{-2r-1}{(1+r)^2}$	-1	$\frac{2r}{(1+r)^3}$	0
b_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$	$\frac{-6r^3-18r^2-14r}{(1+r)^3(r+2)^3}$	0
b_3	$\frac{-4r^3-18r^2-22r-6}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{1}{6}$	$\frac{12\left(r^4+8r^3+\frac{47}{2}r^2+30r+\frac{85}{6}\right)r}{(1+r)^3(r+2)^3(r+3)^3}$	0
b_4	$\frac{5r^4+40r^3+105r^2+100r+24}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{24}$	$-\frac{20\left(r^6+15r^5+\frac{183}{2}r^4+290r^3+\frac{5031}{10}r^2+453r+166\right)r}{(1+r)^3(r+2)^3(r+3)^3(4+r)^3}$	0
b_5	$\frac{-6r^5-75r^4-340r^3-675r^2-548r-120}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2(r+5)^2}$	$-\frac{1}{120}$	$\frac{30r\left(r^8+24r^7+\frac{739}{3}r^6+1410r^5+4915r^4+10668r^3+14063r^2+10290r+\frac{48076}{15}\right)}{(1+r)^3(r+2)^3(r+3)^3(4+r)^3(r+5)^3}$	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned}
$$

Verified OK.

2.4.1 Maple step by step solution

Let's solve

$y''x + (1+2x)\,y' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(1+2x)y'}{x} - \frac{(1+x)y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+2x)y'}{x} + \frac{(1+x)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1+2x}{x}, P_3(x) = \frac{1+x}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + (1+2x)\,y' + (1+x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

$$x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$$

- Convert $x^{m}\cdot y'$ to series expansion for $m=0..1$

  $$x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x\cdot y''$ to series expansion

  $$x\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x\cdot y''=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}r^{2}x^{-1+r}+\left(a_{1}(1+r)^{2}+a_{0}(1+2r)\right)x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)^{2}+a_{k}(2k+2r+1)+a_{k-1}\right)x^{k}\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$r^{2}=0$$

- Values of r that satisfy the indicial equation

  $$r=0$$

- Each term must be 0

  $$a_{1}(1+r)^{2}+a_{0}(1+2r)=0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1)^{2}+2a_{k}k+a_{k}+a_{k-1}=0$$

- Shift index using $k->k+1$

  $$a_{k+2}(k+2)^{2}+2a_{k+1}(k+1)+a_{k+1}+a_{k}=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{2ka_{k+1}+a_{k}+3a_{k+1}}{(k+2)^{2}}$$

- Recursion relation for $r=0$

  $$a_{k+2}=-\frac{2ka_{k+1}+a_{k}+3a_{k+1}}{(k+2)^{2}}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2k a_{k+1} + a_k + 3a_{k+1}}{(k+2)^2}, a_1 + a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(2*x+1)*diff(y(x),x)+(x+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5\right)(c_2 \ln(x) + c_1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 78

```
AsymptoticDSolveValue[x*y''[x]+(2*x+1)*y'[x]+(x+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{120} + \frac{x^4}{24} - \frac{x^3}{6} + \frac{x^2}{2} - x + 1\right) + c_2\left(-\frac{x^5}{120} + \frac{x^4}{24} - \frac{x^3}{6} + \frac{x^2}{2} - x + 1\right)\log(x)$$

2.5 problem 6

Internal problem ID [5640]
Internal file name [`OUTPUT/4888_Sunday_June_05_2022_03_09_18_PM_54282956/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + 2y'x^3 + (x^2 - 2)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y'x^3 + (x^2 - 2)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 2x^2$$
$$q(x) = \frac{x^2 - 2}{x}$$

Table 20: Table $p(x), q(x)$ singularites.

$p(x) = 2x^2$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{x^2-2}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty, -\infty, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y'x^3 + (x^2 - 2)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x^3 + (x^2 - 2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{2+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{2+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{2+n+r} a_n (n+r) = \sum_{n=3}^{\infty} 2a_{n-3}(n+r-3)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{2+n+r} a_n = \sum_{n=3}^{\infty} a_{n-3} x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=3}^{\infty} 2a_{n-3}(n+r-3)\, x^{n+r-1}\right) + \left(\sum_{n=3}^{\infty} a_{n-3} x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{2}{r(1+r)}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{4}{(1+r)^2 r(2+r)}$$

For $3 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-3}(n+r-3) + a_{n-3} - 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2na_{n-3} + 2ra_{n-3} - 5a_{n-3} - 2a_{n-1}}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-2na_{n-3} + 3a_{n-3} + 2a_{n-1}}{(n+1)n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r(1+r)}$	1
a_2	$\frac{4}{(1+r)^2r(2+r)}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-2r^5 - 9r^4 - 14r^3 - 9r^2 - 2r + 8}{(1+r)^2 r(2+r)^2 (3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{7}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r(1+r)}$	1
a_2	$\frac{4}{(1+r)^2 r(2+r)}$	$\frac{1}{3}$
a_3	$\frac{-2r^5-9r^4-14r^3-9r^2-2r+8}{(1+r)^2 r(2+r)^2(3+r)}$	$-\frac{7}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-8r^5 - 56r^4 - 168r^3 - 268r^2 - 220r - 56}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{97}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r(1+r)}$	1
a_2	$\frac{4}{(1+r)^2 r(2+r)}$	$\frac{1}{3}$
a_3	$\frac{-2r^5-9r^4-14r^3-9r^2-2r+8}{(1+r)^2 r(2+r)^2(3+r)}$	$-\frac{7}{36}$
a_4	$\frac{-8r^5-56r^4-168r^3-268r^2-220r-56}{(1+r)^2 r(2+r)^2(3+r)^2(4+r)}$	$-\frac{97}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-24r^5 - 228r^4 - 1000r^3 - 2412r^2 - 3056r - 1552}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{517}{5400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r(1+r)}$	1
a_2	$\frac{4}{(1+r)^2 r(2+r)}$	$\frac{1}{3}$
a_3	$\frac{-2r^5-9r^4-14r^3-9r^2-2r+8}{(1+r)^2 r(2+r)^2(3+r)}$	$-\frac{7}{36}$
a_4	$\frac{-8r^5-56r^4-168r^3-268r^2-220r-56}{(1+r)^2 r(2+r)^2(3+r)^2(4+r)}$	$-\frac{97}{360}$
a_5	$\frac{-24r^5-228r^4-1000r^3-2412r^2-3056r-1552}{(1+r)^2 r(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{517}{5400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 + x + \frac{x^2}{3} - \frac{7x^3}{36} - \frac{97x^4}{360} - \frac{517x^5}{5400} + O\left(x^6\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= \frac{2}{r(1+r)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} \frac{2}{r(1+r)} &= \lim_{r \to 0} \frac{2}{r(1+r)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $xy'' + 2y'x^3 + (x^2-2)\,y = 0$ gives

$$\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&+ 2\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)x^3 \\
&+ (x^2-2)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}
&\left(\left(y_1''(x)\,x + 2y_1'(x)\,x^3 + (x^2-2)\,y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x\right. \\
&\left.+ 2y_1(x)\,x^2\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&+ 2\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x^3 + (x^2-2)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x + 2y_1'(x)\,x^3 + \left(x^2-2\right)y_1(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + 2y_1(x)\,x^2\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + 2\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x^3 + \left(x^2-2\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1}a_n(n+r_1)\right)x + (2x^3-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2 + 2\left(\sum_{n=0}^{\infty} x^{-1+n+r_2}b_n(n+r_2)\right)x^4 + (x^2-2)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{n}a_n(n+1)\right)x + (2x^3-1)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right)C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n}b_n n(n-1)\right)x^2 + 2\left(\sum_{n=0}^{\infty} x^{n-1}b_n n\right)x^4 + (x^2-2)\left(\sum_{n=0}^{\infty} b_n x^{n}\right)x}{x} = 0 \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n} a_{n} (n+1)\right)+\left(\sum_{n=0}^{\infty} 2C\, x^{n+3} a_{n}\right)+\sum_{n=0}^{\infty} \left(-C a_{n} x^{n}\right) \\
&+\left(\sum_{n=0}^{\infty} n\, x^{n-1} b_{n} (n-1)\right)+\left(\sum_{n=0}^{\infty} 2n\, x^{2+n} b_{n}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{2+n} b_{n}\right)+\sum_{n=0}^{\infty} \left(-2 b_{n} x^{n}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n} a_{n} (n+1) &= \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1} \\
\sum_{n=0}^{\infty} 2C\, x^{n+3} a_{n} &= \sum_{n=4}^{\infty} 2C a_{n-4} x^{n-1} \\
\sum_{n=0}^{\infty} \left(-C a_{n} x^{n}\right) &= \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} 2n\, x^{2+n} b_{n} &= \sum_{n=3}^{\infty} 2(n-3)\, b_{n-3} x^{n-1} \\
\sum_{n=0}^{\infty} x^{2+n} b_{n} &= \sum_{n=3}^{\infty} b_{n-3} x^{n-1} \\
\sum_{n=0}^{\infty} \left(-2 b_{n} x^{n}\right) &= \sum_{n=1}^{\infty} \left(-2 b_{n-1} x^{n-1}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right)+\left(\sum_{n=4}^{\infty} 2C a_{n-4} x^{n-1}\right)+\sum_{n\,=1}^{\infty}\left(-C a_{n-1} x^{n-1}\right) \\ +\left(\sum_{n=0}^{\infty} n\, x^{n-1} b_{n}(n-1)\right)+\left(\sum_{n=3}^{\infty} 2(n-3)\, b_{n-3} x^{n-1}\right) \\ +\left(\sum_{n=3}^{\infty} b_{n-3} x^{n-1}\right)+\sum_{n\,=1}^{\infty}\left(-2 b_{n-1} x^{n-1}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C-2=0$$

Which is solved for C. Solving for C gives

$$C=2$$

For $n=2$, Eq (2B) gives

$$3C a_1-2b_1+2b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2+6=0$$

Solving the above for b_2 gives

$$b_2=-3$$

For $n=3$, Eq (2B) gives

$$5C a_2+b_0-2b_2+6b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3+\frac{31}{3}=0$$

Solving the above for b_3 gives

$$b_3=-\frac{31}{18}$$

For $n=4$, Eq (2B) gives

$$(2a_0+7a_3)\, C+3b_1-2b_3+12b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{85}{18} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{85}{216}$$

For $n = 5$, Eq (2B) gives

$$(2a_1 + 9a_4)\, C + 5b_2 - 2b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{4067}{270} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{4067}{5400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 2$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = {} & 2\left(x\left(1 + x + \frac{x^2}{3} - \frac{7x^3}{36} - \frac{97x^4}{360} - \frac{517x^5}{5400} + O(x^6)\right)\right)\ln(x) \\ & + 1 - 3x^2 - \frac{31x^3}{18} - \frac{85x^4}{216} + \frac{4067x^5}{5400} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + x + \frac{x^2}{3} - \frac{7x^3}{36} - \frac{97x^4}{360} - \frac{517x^5}{5400} + O(x^6)\right) \\ &\quad + c_2\left(2\left(x\left(1 + x + \frac{x^2}{3} - \frac{7x^3}{36} - \frac{97x^4}{360} - \frac{517x^5}{5400} + O(x^6)\right)\right)\ln(x) + 1 - 3x^2 \right. \\ &\quad \left. - \frac{31x^3}{18} - \frac{85x^4}{216} + \frac{4067x^5}{5400} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1+x+\frac{x^2}{3}-\frac{7x^3}{36}-\frac{97x^4}{360}-\frac{517x^5}{5400}+O(x^6)\right) \\
&\quad + c_2\left(2x\left(1+x+\frac{x^2}{3}-\frac{7x^3}{36}-\frac{97x^4}{360}-\frac{517x^5}{5400}+O(x^6)\right)\ln(x)+1-3x^2-\frac{31x^3}{18}\right. \\
&\quad \left. -\frac{85x^4}{216}+\frac{4067x^5}{5400}+O(x^6)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1+x+\frac{x^2}{3}-\frac{7x^3}{36}-\frac{97x^4}{360}-\frac{517x^5}{5400}+O(x^6)\right) \\
&\quad + c_2\left(2x\left(1+x+\frac{x^2}{3}-\frac{7x^3}{36}-\frac{97x^4}{360}-\frac{517x^5}{5400}+O(x^6)\right)\ln(x)+1-3x^2\right. \\
&\quad \left. -\frac{31x^3}{18}-\frac{85x^4}{216}+\frac{4067x^5}{5400}+O(x^6)\right)
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y &= c_1 x\left(1+x+\frac{x^2}{3}-\frac{7x^3}{36}-\frac{97x^4}{360}-\frac{517x^5}{5400}+O(x^6)\right) \\
&\quad + c_2\left(2x\left(1+x+\frac{x^2}{3}-\frac{7x^3}{36}-\frac{97x^4}{360}-\frac{517x^5}{5400}+O(x^6)\right)\ln(x)+1-3x^2-\frac{31x^3}{18}\right. \\
&\quad \left. -\frac{85x^4}{216}+\frac{4067x^5}{5400}+O(x^6)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
Order:=6;
dsolve(x*diff(y(x),x$2)+2*x^3*diff(y(x),x)+(x^2-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 + x + \frac{1}{3}x^2 - \frac{7}{36}x^3 - \frac{97}{360}x^4 - \frac{517}{5400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(\ln(x)\left(2x + 2x^2 + \frac{2}{3}x^3 - \frac{7}{18}x^4 - \frac{97}{180}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - 3x^2 - \frac{31}{18}x^3 - \frac{85}{216}x^4 + \frac{4067}{5400}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 83

```
AsymptoticDSolveValue[x*y''[x]+2*x^3*y'[x]+(x^2-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{216}\left(-x^4 - 516x^3 - 1080x^2 - 432x + 216\right) - \frac{1}{18}x\left(7x^3 - 12x^2 - 36x - 36\right)\log(x)\right) + c_2\left(-\frac{97x^5}{360} - \frac{7x^4}{36} + \frac{x^3}{3} + x^2 + x\right)$$

2.6 problem 7

2.6.1 Maple step by step solution 201

Internal problem ID [5641]
Internal file name [`OUTPUT/4889_Sunday_June_05_2022_03_09_22_PM_66828457/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (x - 1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (37) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (38) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -(x-1)\,y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -y - (x-1)\,y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2y' + (x-1)^2\,y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= (x-1)\,((x-1)\,y' + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -(x-1)^3\,y + (6x-6)\,y' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= -y(0) + y'(0) \\
F_2 &= y(0) - 2y'(0) \\
F_3 &= y'(0) - 4y(0) \\
F_4 &= 5y(0) - 6y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5 + \frac{1}{144}x^6\right) y(0) \\
& + \left(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5 - \frac{1}{120}x^6\right) y'(0) + O\big(x^6\big)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -(x-1) \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n$$
$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{-a_{n-1} + a_n}{(n+2)\,(1+n)} \\ &= \frac{a_n}{(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6} + \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12} + \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{30} + \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{144} - \frac{a_1}{120}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{11a_1}{5040} - \frac{a_0}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{a_0 x^2}{2} + \left(-\frac{a_0}{6} + \frac{a_1}{6}\right) x^3 + \left(-\frac{a_1}{12} + \frac{a_0}{24}\right) x^4 + \left(-\frac{a_0}{30} + \frac{a_1}{120}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5\right) a_0 + \left(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5\right) c_1 + \left(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{30}x^5+\frac{1}{144}x^6\right)y(0) + \left(x+\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{1}{120}x^5-\frac{1}{120}x^6\right)y'(0)+O(x^6) \tag{1}$$

$$y = \left(1+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{30}x^5\right)c_1+\left(x+\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{1}{120}x^5\right)c_2+O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{30}x^5+\frac{1}{144}x^6\right)y(0) + \left(x+\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{1}{120}x^5-\frac{1}{120}x^6\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{30}x^5\right)c_1+\left(x+\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{1}{120}x^5\right)c_2+O(x^6)$$

Verified OK.

2.6.1 Maple step by step solution

Let's solve

$y'' = -(x-1)\,y$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (1-x)\,y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 - a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k + a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 - a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - a_k + a_{k-1} = 0$$

- Shift index using $k- > k + 1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{-a_{k+1} + a_k}{k^2 + 5k + 6}, 2a_2 - a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)+(x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{30}x^5\right)y(0) + \left(x+\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{1}{120}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[y''[x]+(x-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120}-\frac{x^4}{12}+\frac{x^3}{6}+x\right)+c_1\left(-\frac{x^5}{30}+\frac{x^4}{24}-\frac{x^3}{6}+\frac{x^2}{2}+1\right)$$

2.7 problem 8

2.7.1 Maple step by step solution 211

Internal problem ID [5642]
Internal file name [`OUTPUT/4890_Sunday_June_05_2022_03_09_23_PM_51342390/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = 1$$

Table 22: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+2)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(4+r)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(4+r)^2}$	$\frac{1}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$	$\frac{2}{(r+2)^3}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(r+2)^2(4+r)^2}$	$\frac{1}{64}$	$\frac{-12-4r}{(r+2)^3(4+r)^3}$	$-\frac{3}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) + c_2\left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) + c_2\left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = c_1&\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \\
&+ c_2\left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y=c_1\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O\left(x^6\right)\right)+c_2\left(\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O\left(x^6\right)\right)\ln\left(x\right)+\frac{x^2}{4}-\frac{3x^4}{128}+O\left(x^6\right)\right)$$

Verified OK.

2.7.1 Maple step by step solution

Let's solve

$y''x+y'+xy=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y'}{x}-y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}+y=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1}{x},P_3(x)=1\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x+y'+xy=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k->k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + a_1(1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1(1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2)^2 + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1) \left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + O(x^6)\right) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + O(x^6)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^4}{64} - \frac{x^2}{4} + 1 \right) + c_2 \left(-\frac{3x^4}{128} + \frac{x^2}{4} + \left(\frac{x^4}{64} - \frac{x^2}{4} + 1 \right) \log(x) \right)$$

2.8 problem 9

2.8.1 Maple step by step solution 225

Internal problem ID [5643]
Internal file name [OUTPUT/4891_Sunday_June_05_2022_03_09_25_PM_35616607/index.tex]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[_Jacobi]
```

$$\boxed{2x(x-1)\,y''-(1+x)\,y'+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^2-2x\right)y''+(-1-x)\,y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{1+x}{2x\,(x-1)}$$
$$q(x)=\frac{1}{2x\,(x-1)}$$

Table 24: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1+x}{2x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{1}{2x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x(x-1)\, y'' + (-1-x)\, y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\, (n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) + (-1-x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r-1} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r-1} a_n(n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-2x^{n+r-1} a_n(n+r)(n+r-1) - (n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$-2x^{-1+r} a_0 r(-1+r) - r a_0 x^{-1+r} = 0$$

Or

$$\left(-2x^{-1+r}r(-1+r) - r\,x^{-1+r}\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(-2r^2 + r\right)x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation becomes

$$-2r^2 + r = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{2} \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(-2r^2 + r\right)x^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-1}(n+r-1)(n+r-2) - 2a_n(n+r)(n+r-1) \\ &\quad - a_{n-1}(n+r-1) - a_n(n+r) + a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(2n^2 + 4nr + 2r^2 - 7n - 7r + 6)}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}(2n^2 - 5n + 3)}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2r^2 - 3r + 1}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-3r+1}{2r^2+3r+1}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2r^3 - 3r^2 + r}{2r^3 + 9r^2 + 13r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-3r+1}{2r^2+3r+1}$	0
a_2	$\frac{2r^3-3r^2+r}{2r^3+9r^2+13r+6}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{2r^3 - 3r^2 + r}{2r^3 + 15r^2 + 37r + 30}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-3r+1}{2r^2+3r+1}$	0
a_2	$\frac{2r^3-3r^2+r}{2r^3+9r^2+13r+6}$	0
a_3	$\frac{2r^3-3r^2+r}{2r^3+15r^2+37r+30}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2r^3 - 3r^2 + r}{2r^3 + 21r^2 + 73r + 84}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-3r+1}{2r^2+3r+1}$	0
a_2	$\frac{2r^3-3r^2+r}{2r^3+9r^2+13r+6}$	0
a_3	$\frac{2r^3-3r^2+r}{2r^3+15r^2+37r+30}$	0
a_4	$\frac{2r^3-3r^2+r}{2r^3+21r^2+73r+84}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{2r^3 - 3r^2 + r}{2r^3 + 27r^2 + 121r + 180}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-3r+1}{2r^2+3r+1}$	0
a_2	$\frac{2r^3-3r^2+r}{2r^3+9r^2+13r+6}$	0
a_3	$\frac{2r^3-3r^2+r}{2r^3+15r^2+37r+30}$	0
a_4	$\frac{2r^3-3r^2+r}{2r^3+21r^2+73r+84}$	0
a_5	$\frac{2r^3-3r^2+r}{2r^3+27r^2+121r+180}$	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\
&= \sqrt{x}\,\left(1 + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&2b_{n-1}(n+r-1)(n+r-2) - 2b_n(n+r)(n+r-1) \\
&- b_{n-1}(n+r-1) - (n+r)\,b_n + b_{n-1} = 0
\end{aligned}
\tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}(2n^2 + 4nr + 2r^2 - 7n - 7r + 6)}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{b_{n-1}(2n^2 - 7n + 6)}{2n^2 - n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2r^2 - 3r + 1}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-3r+1}{2r^2+3r+1}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{2r^3 - 3r^2 + r}{2r^3 + 9r^2 + 13r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-3r+1}{2r^2+3r+1}$	1
b_2	$\frac{2r^3-3r^2+r}{2r^3+9r^2+13r+6}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{2r^3 - 3r^2 + r}{2r^3 + 15r^2 + 37r + 30}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-3r+1}{2r^2+3r+1}$	1
b_2	$\frac{2r^3-3r^2+r}{2r^3+9r^2+13r+6}$	0
b_3	$\frac{2r^3-3r^2+r}{2r^3+15r^2+37r+30}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{2r^3 - 3r^2 + r}{2r^3 + 21r^2 + 73r + 84}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-3r+1}{2r^2+3r+1}$	1
b_2	$\frac{2r^3-3r^2+r}{2r^3+9r^2+13r+6}$	0
b_3	$\frac{2r^3-3r^2+r}{2r^3+15r^2+37r+30}$	0
b_4	$\frac{2r^3-3r^2+r}{2r^3+21r^2+73r+84}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{2r^3 - 3r^2 + r}{2r^3 + 27r^2 + 121r + 180}$$

Which for the root $r = 0$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-3r+1}{2r^2+3r+1}$	1
b_2	$\frac{2r^3-3r^2+r}{2r^3+9r^2+13r+6}$	0
b_3	$\frac{2r^3-3r^2+r}{2r^3+15r^2+37r+30}$	0
b_4	$\frac{2r^3-3r^2+r}{2r^3+21r^2+73r+84}$	0
b_5	$\frac{2r^3-3r^2+r}{2r^3+27r^2+121r+180}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + x + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x}\left(1 + O(x^6)\right) + c_2\left(1 + x + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x}\left(1 + O(x^6)\right) + c_2\left(1 + x + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\left(1 + O(x^6)\right) + c_2\left(1 + x + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x}\left(1 + O(x^6)\right) + c_2\left(1 + x + O(x^6)\right)$$

Verified OK.

2.8.1 Maple step by step solution

Let's solve

$2y''x(x-1)+(-1-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=\frac{(1+x)y'}{2x(x-1)}-\frac{y}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''-\frac{(1+x)y'}{2x(x-1)}+\frac{y}{2x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{1+x}{2x(x-1)},P_3(x)=\frac{1}{2x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$2y''x(x-1)+(-1-x)\,y'+y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{m}\cdot y''$ to series expansion for $m=1..2$

  $$x^{m}\cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^{m}\cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(2k+2r-1)\,(k+r-1)\right) x^{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2a_k(k+r-1)\left(k+r-\frac{1}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r-1)(2k+2r-1)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r=0$; series terminates at $k=1$

  $$a_{k+1} = \frac{a_k(k-1)(2k-1)}{(2k+1)(k+1)}$$

- Apply recursion relation for $k=0$

  $$a_1 = a_0$$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $$y = a_0\cdot(1+x)$$

- Recursion relation for $r=\frac{1}{2}$

  $$a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)k}{(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r=\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot (1+x) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2b_k \left(k-\frac{1}{2}\right) k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 26

```
Order:=6;
dsolve(2*x*(x-1)*diff(y(x),x$2)-(x+1)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1 + O\left(x^6\right)\right) + c_2\left(1 + x + O\left(x^6\right)\right)$$

✓ Solution by Mathematica
Time used: 0.004 (sec). Leaf size: 18

```
AsymptoticDSolveValue[2*x*(x-1)*y''[x]-(x+1)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x} + c_2(x+1)$$

2.9 problem 10

2.9.1 Maple step by step solution 237

Internal problem ID [5644]
Internal file name [`OUTPUT/4892_Sunday_June_05_2022_03_09_28_PM_88180568/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + 2y' + 4xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y' + 4xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = 4$$

Table 26: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 4$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y' + 4xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=2}^{\infty} 4a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + 2a_n(n+r) + 4a_{n-2} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{4a_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
a_n = -\frac{4a_{n-2}}{n(1+n)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{r^2 + 5r + 6}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{r^2+5r+6}$	$-\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{r^2+5r+6}$	$-\frac{2}{3}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{2}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{r^2+5r+6}$	$-\frac{2}{3}$
a_3	0	0
a_4	$\frac{16}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{2}{15}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{r^2+5r+6}$	$-\frac{2}{3}$
a_3	0	0
a_4	$\frac{16}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{2}{15}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 - \frac{2x^2}{3} + \frac{2x^4}{15} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} 0 &= \lim_{r\rightarrow -1} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 2(n+r)\, b_n + 4b_{n-2} = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + 2(n-1)\, b_n + 4b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{4b_{n-2}}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{r^2+5r+6}$	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{r^2+5r+6}$	-2
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{r^2+5r+6}$	-2
b_3	0	0
b_4	$\frac{16}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{2}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{r^2+5r+6}$	-2
b_3	0	0
b_4	$\frac{16}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{2}{3}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - 2x^2 + \frac{2x^4}{3} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 - \frac{2x^2}{3} + \frac{2x^4}{15} + O(x^6)\right) + \frac{c_2 \left(1 - 2x^2 + \frac{2x^4}{3} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - \frac{2x^2}{3} + \frac{2x^4}{15} + O(x^6)\right) + \frac{c_2 \left(1 - 2x^2 + \frac{2x^4}{3} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \left(1 - \frac{2x^2}{3} + \frac{2x^4}{15} + O(x^6)\right) + \frac{c_2 \left(1 - 2x^2 + \frac{2x^4}{3} + O(x^6)\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1 \left(1 - \frac{2x^2}{3} + \frac{2x^4}{15} + O(x^6)\right) + \frac{c_2 \left(1 - 2x^2 + \frac{2x^4}{3} + O(x^6)\right)}{x}
$$

Verified OK.

2.9.1 Maple step by step solution

Let's solve

$y''x + 2y' + 4xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - 4y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + 4y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 4\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + 2y' + 4xy = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k- >k - 1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + 4a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)(2+r)=0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r)+4a_{k-1}=0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+3+r)+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{4a_k}{(k+1)(k+2)}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{4a_k}{(k+1)(k+2)}, 0=0\right]$
- Recursion relation for $r=0$

  $a_{k+2}=-\frac{4a_k}{(k+2)(k+3)}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{4a_k}{(k+2)(k+3)}, 2a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2}=-\frac{4a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=-\frac{4b_k}{(k+2)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+4*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{2}{3}x^2 + \frac{2}{15}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(1 - 2x^2 + \frac{2}{3}x^4 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 40

```
AsymptoticDSolveValue[x*y''[x]+2*y'[x]+4*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{2x^3}{3} - 2x + \frac{1}{x}\right) + c_2\left(\frac{2x^4}{15} - \frac{2x^2}{3} + 1\right)$$

2.10 problem 11

2.10.1 Maple step by step solution 251

Internal problem ID [5645]
Internal file name [`OUTPUT/4893_Sunday_June_05_2022_03_09_30_PM_19903692/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (-2x+2)\,y' + (-2+x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (-2x+2)\,y' + (-2+x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2(x-1)}{x}$$
$$q(x) = \frac{-2+x}{x}$$

Table 28: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2(x-1)}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{-2+x}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-2x + 2)\, y' + (-2 + x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) x + (-2x + 2) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-2 + x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r)+2 r\, x^{-1+r}\right) a_{0}=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r)=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_{1} &= 0 \\ r_{2} &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(1+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_{1}(x) &= x^{r_{1}}\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) \\ y_{2}(x) &= C y_{1}(x) \ln(x)+x^{r_{2}}\left(\sum_{n=0}^{\infty} b_{n} x^{n}\right) \end{aligned}$$

Or

$$\begin{aligned} y_{1}(x) &= \sum_{n=0}^{\infty} a_{n} x^{n} \\ y_{2}(x) &= C y_{1}(x) \ln(x)+\frac{\sum_{n=0}^{\infty} b_{n} x^{n}}{x} \end{aligned}$$

Or

$$\begin{aligned} y_{1}(x) &= \sum_{n=0}^{\infty} a_{n} x^{n} \\ y_{2}(x) &= C y_{1}(x) \ln(x)+\left(\sum_{n=0}^{\infty} b_{n} x^{n-1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{2}{r+2}$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) + 2a_n(n+r) - 2a_{n-1} + a_{n-2} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2na_{n-1} + 2ra_{n-1} - a_{n-2}}{n^2 + 2nr + r^2 + n + r} \quad (4)$$

Which for the root $r = 0$ becomes

$$a_n = \frac{2na_{n-1} - a_{n-2}}{n(1+n)} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r+2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3}{r^2 + 5r + 6}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r+2}$	1
a_2	$\frac{3}{r^2+5r+6}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4}{(4+r)(r+3)(r+2)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r+2}$	1
a_2	$\frac{3}{r^2+5r+6}$	$\frac{1}{2}$
a_3	$\frac{4}{(4+r)(r+3)(r+2)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5}{(r^2+9r+20)(r^2+5r+6)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r+2}$	1
a_2	$\frac{3}{r^2+5r+6}$	$\frac{1}{2}$
a_3	$\frac{4}{(4+r)(r+3)(r+2)}$	$\frac{1}{6}$
a_4	$\frac{5}{(r^2+9r+20)(r^2+5r+6)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{6}{(r+6)(r+5)(4+r)(r+3)(r+2)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r+2}$	1
a_2	$\frac{3}{r^2+5r+6}$	$\frac{1}{2}$
a_3	$\frac{4}{(4+r)(r+3)(r+2)}$	$\frac{1}{6}$
a_4	$\frac{5}{(r^2+9r+20)(r^2+5r+6)}$	$\frac{1}{24}$
a_5	$\frac{6}{(r+6)(r+5)(4+r)(r+3)(r+2)}$	$\frac{1}{120}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= \frac{2}{r+2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{2}{r+2} &= \lim_{r\to -1} \frac{2}{r+2} \\
&= 2
\end{aligned}
$$

The limit is 2. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = \frac{2}{r + 2}$$

For $2 \leq n$ the recursive equation is

$$b_n(n + r)(n + r - 1) - 2b_{n-1}(n + r - 1) + 2(n + r) b_n - 2b_{n-1} + b_{n-2} = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n - 1)(n - 2) - 2b_{n-1}(n - 2) + 2(n - 1) b_n - 2b_{n-1} + b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{2nb_{n-1} + 2rb_{n-1} - b_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{2nb_{n-1} - b_{n-2} - 2b_{n-1}}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{r+2}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{3}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{3}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{r+2}$	2
b_2	$\frac{3}{r^2+5r+6}$	$\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{4}{(r^2 + 7r + 12)(r + 2)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{r+2}$	2
b_2	$\frac{3}{r^2+5r+6}$	$\frac{3}{2}$
b_3	$\frac{4}{(4+r)(r+3)(r+2)}$	$\frac{2}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{5}{(r + 3)(r + 2)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{5}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{r+2}$	2
b_2	$\frac{3}{r^2+5r+6}$	$\frac{3}{2}$
b_3	$\frac{4}{(4+r)(r+3)(r+2)}$	$\frac{2}{3}$
b_4	$\frac{5}{(r+3)(r+2)(r+5)(4+r)}$	$\frac{5}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{6}{(r^2 + 7r + 12)(r + 2)(r^2 + 11r + 30)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{1}{20}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{r+2}$	2
b_2	$\frac{3}{r^2+5r+6}$	$\frac{3}{2}$
b_3	$\frac{4}{(4+r)(r+3)(r+2)}$	$\frac{2}{3}$
b_4	$\frac{5}{(r+3)(r+2)(r+5)(4+r)}$	$\frac{5}{24}$
b_5	$\frac{6}{(r+6)(r+5)(4+r)(r+3)(r+2)}$	$\frac{1}{20}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + 2x + \frac{3x^2}{2} + \frac{2x^3}{3} + \frac{5x^4}{24} + \frac{x^5}{20} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$= c_1\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)+\frac{c_2\left(1 + 2x + \frac{3x^2}{2} + \frac{2x^3}{3} + \frac{5x^4}{24} + \frac{x^5}{20} + O(x^6)\right)}{x}$$

Hence the final solution is

$$y = y_h$$

$$= c_1\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)+\frac{c_2\left(1 + 2x + \frac{3x^2}{2} + \frac{2x^3}{3} + \frac{5x^4}{24} + \frac{x^5}{20} + O(x^6)\right)}{x}$$

Summary

The solution(s) found are the following

$$y = c_1\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right) + \frac{c_2\left(1+2x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{5x^4}{24}+\frac{x^5}{20}+O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right) + \frac{c_2\left(1+2x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{5x^4}{24}+\frac{x^5}{20}+O(x^6)\right)}{x}$$

Verified OK.

2.10.1 Maple step by step solution

Let's solve

$y''x+(-2x+2)\,y'+(-2+x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(-2+x)y}{x}+\frac{2(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2(x-1)y'}{x}+\frac{(-2+x)y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{2(x-1)}{x}, P_3(x)=\frac{-2+x}{x}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=2$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x+(-2x+2)\,y'+(-2+x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m=0..1$

  $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m=0..1$

  $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x\cdot y''$ to series expansion

  $x\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- Shift index using $k->k+1$

  $x\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(1+r)\,x^{-1+r}+(a_1(1+r)\,(2+r)-2a_0(1+r))\,x^r+\left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+2+r)-2a_k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) - 2a_0(1+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+2+r) - 2a_k k - 2a_k r - 2a_k + a_{k-1} = 0$

- Shift index using $k- > k+1$

  $a_{k+2}(k+2+r)(k+3+r) - 2a_{k+1}(k+1) - 2ra_{k+1} - 2a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}+2ra_{k+1}-a_k+4a_{k+1}}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}, 2a_1 - 2a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+2)(k+3)}, 2b_1 - 2b\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(2-2*x)*diff(y(x),x)+(x-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(1 + 2x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{5}{24}x^4 + \frac{1}{20}x^5 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 58

```
AsymptoticDSolveValue[x*y''[x]+(2-2*x)*y'[x]+(x-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{5x^3}{24} + \frac{2x^2}{3} + \frac{3x}{2} + \frac{1}{x} + 2\right) + c_2\left(\frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right)$$

2.11 problem 12

2.11.1 Maple step by step solution 264

Internal problem ID [5646]
Internal file name [`OUTPUT/4894_Sunday_June_05_2022_03_09_33_PM_92840325/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 6xy' + \left(4x^2 + 6\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 6xy' + \left(4x^2 + 6\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{6}{x}$$
$$q(x) = \frac{4x^2 + 6}{x^2}$$

Table 30: Table $p(x), q(x)$ singularites.

$p(x) = \frac{6}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2+6}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + 6xy' + \left(4x^2 + 6\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + 6x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(4x^2 + 6\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{n+r+2} a_n = \sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 6x^{n+r} a_n (n+r) + 6a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 6x^r a_0 r + 6a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 6x^r r + 6x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + 5r + 6\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 5r + 6 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= -2 \\
r_2 &= -3
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(r^2 + 5r + 6\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \frac{\sum_{n=0}^{\infty} a_n x^n}{x^2} \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^3}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n-2} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-3}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + 6a_n(n+r) + 4a_{n-2} + 6a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{n^2 + 2nr + r^2 + 5n + 5r + 6} \tag{4}$$

Which for the root $r = -2$ becomes

$$a_n = -\frac{4a_{n-2}}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{r^2 + 9r + 20}$$

Which for the root $r = -2$ becomes

$$a_2 = -\frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{r^2+9r+20}$	$-\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{r^2+9r+20}$	$-\frac{2}{3}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(r+5)(4+r)(r+7)(r+6)}$$

Which for the root $r = -2$ becomes

$$a_4 = \frac{2}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{r^2+9r+20}$	$-\frac{2}{3}$
a_3	0	0
a_4	$\frac{16}{(r+5)(4+r)(r+7)(r+6)}$	$\frac{2}{15}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{r^2+9r+20}$	$-\frac{2}{3}$
a_3	0	0
a_4	$\frac{16}{(r+5)(4+r)(r+7)(r+6)}$	$\frac{2}{15}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x^2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - \frac{2x^2}{3} + \frac{2x^4}{15} + O(x^6)}{x^2} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} 0 &= \lim_{r \to -3} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-3} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 6b_n(n+r) + 4b_{n-2} + 6b_n = 0 \tag{4}$$

Which for for the root $r = -3$ becomes

$$b_n(n-3)(n-4) + 6b_n(n-3) + 4b_{n-2} + 6b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-2}}{n^2 + 2nr + r^2 + 5n + 5r + 6} \tag{5}$$

Which for the root $r = -3$ becomes

$$b_n = -\frac{4b_{n-2}}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4}{r^2 + 9r + 20}$$

Which for the root $r = -3$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{r^2+9r+20}$	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{r^2+9r+20}$	-2
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(r^2 + 9r + 20)(r^2 + 13r + 42)}$$

Which for the root $r = -3$ becomes

$$b_4 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{r^2+9r+20}$	-2
b_3	0	0
b_4	$\frac{16}{(r+5)(4+r)(r+7)(r+6)}$	$\frac{2}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{r^2+9r+20}$	-2
b_3	0	0
b_4	$\frac{16}{(r+5)(4+r)(r+7)(r+6)}$	$\frac{2}{3}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \frac{1}{x^2}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - 2x^2 + \frac{2x^4}{3} + O(x^6)}{x^3} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\ &= \frac{c_1\left(1-\frac{2x^2}{3}+\frac{2x^4}{15}+O(x^6)\right)}{x^2} + \frac{c_2\left(1-2x^2+\frac{2x^4}{3}+O(x^6)\right)}{x^3}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= \frac{c_1\left(1-\frac{2x^2}{3}+\frac{2x^4}{15}+O(x^6)\right)}{x^2} + \frac{c_2\left(1-2x^2+\frac{2x^4}{3}+O(x^6)\right)}{x^3}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(1-\frac{2x^2}{3}+\frac{2x^4}{15}+O(x^6)\right)}{x^2} + \frac{c_2\left(1-2x^2+\frac{2x^4}{3}+O(x^6)\right)}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(1-\frac{2x^2}{3}+\frac{2x^4}{15}+O(x^6)\right)}{x^2} + \frac{c_2\left(1-2x^2+\frac{2x^4}{3}+O(x^6)\right)}{x^3}$$

Verified OK.

2.11.1 Maple step by step solution

Let's solve

$x^2y'' + 6xy' + (4x^2+6)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(2x^2+3)y}{x^2} - \frac{6y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{6y'}{x} + \frac{2(2x^2+3)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x)=\frac{6}{x}, P_3(x)=\frac{2(2x^2+3)}{x^2}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=6$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=6$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''+6xy'+(4x^2+6)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(2+r)\,x^r+a_1(4+r)\,(3+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r+3)\,(k+r+2)+4a_{k-2}\right)x^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)(2+r)=0$
- Values of r that satisfy the indicial equation

  $r \in \{-3,-2\}$
- Each term must be 0

  $a_1(4+r)(3+r)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)(k+r+2)+4a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+5+r)(k+4+r)+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4a_k}{(k+5+r)(k+4+r)}$
- Recursion relation for $r=-3$

  $a_{k+2}=-\frac{4a_k}{(k+2)(k+1)}$
- Solution for $r=-3$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2}=-\frac{4a_k}{(k+2)(k+1)}, a_1=0\right]$
- Recursion relation for $r=-2$

  $a_{k+2}=-\frac{4a_k}{(k+3)(k+2)}$
- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{4a_k}{(k+3)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-3}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2}=-\frac{4a_k}{(k+1)(k+2)}, a_1=0, b_{k+2}=-\frac{4b_k}{(k+2)(k+3)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+6*x*diff(y(x),x)+(4*x^2+6)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1 - \frac{2}{3}x^2 + \frac{2}{15}x^4 + \mathrm{O}\left(x^6\right)\right)x + c_2\left(1 - 2x^2 + \frac{2}{3}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 38

```
AsymptoticDSolveValue[x^2*y''[x]+6*x*y'[x]+(4*x^2+6)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{x^3} + \frac{2x}{3} - \frac{2}{x}\right) + c_2\left(\frac{2x^2}{15} + \frac{1}{x^2} - \frac{2}{3}\right)$$

2.12 problem 13

2.12.1 Maple step by step solution 276

Internal problem ID [5647]
Internal file name [`OUTPUT/4895_Sunday_June_05_2022_03_09_36_PM_54650723/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (1-2x)\, y' + (x-1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (1-2x)\, y' + (x-1)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x-1}{x}$$
$$q(x) = \frac{x-1}{x}$$

Table 32: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (1 - 2x)\, y' + (x - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) x + (1 - 2x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) &= \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) x^{n+r-1}\right) \\
\sum_{n=0}^{\infty} x^{1+n+r} a_n &= \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} \\
\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) x^{n+r-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0
$$

When $n = 0$ the above becomes

$$
x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0
$$

Or

$$
\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{2r+1}{(1+r)^2}$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) + a_n(n+r) + a_{n-2} - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2na_{n-1} + 2ra_{n-1} - a_{n-2} - a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(2n-1)\, a_{n-1} - a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{(1+r)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3r^2 + 6r + 2}{(1+r)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{(1+r)^2}$	1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4r^3 + 18r^2 + 22r + 6}{(1+r)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{(1+r)^2}$	1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$
a_3	$\frac{4r^3+18r^2+22r+6}{(1+r)^2(r+2)^2(r+3)^2}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5r^4 + 40r^3 + 105r^2 + 100r + 24}{(1+r)^2 (r+2)^2 (r+3)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{(1+r)^2}$	1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$
a_3	$\frac{4r^3+18r^2+22r+6}{(1+r)^2(r+2)^2(r+3)^2}$	$\frac{1}{6}$
a_4	$\frac{5r^4+40r^3+105r^2+100r+24}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{6r^5 + 75r^4 + 340r^3 + 675r^2 + 548r + 120}{(1+r)^2 (r+2)^2 (r+3)^2 (4+r)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{(1+r)^2}$	1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$
a_3	$\frac{4r^3+18r^2+22r+6}{(1+r)^2(r+2)^2(r+3)^2}$	$\frac{1}{6}$
a_4	$\frac{5r^4+40r^3+105r^2+100r+24}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{24}$
a_5	$\frac{6r^5+75r^4+340r^3+675r^2+548r+120}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2(r+5)^2}$	$\frac{1}{120}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{2r+1}{(1+r)^2}$	1	$-\frac{2r}{(1+r)^3}$	0
b_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$	$\frac{-6r^3-18r^2-14r}{(1+r)^3(r+2)^3}$	0
b_3	$\frac{4r^3+18r^2+22r+6}{(1+r)^2(r+2)^2(r+3)^2}$	$\frac{1}{6}$	$-\frac{12\left(r^4+8r^3+\frac{47}{2}r^2+30r+\frac{85}{6}\right)r}{(1+r)^3(r+2)^3(r+3)^3}$	0
b_4	$\frac{5r^4+40r^3+105r^2+100r+24}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{24}$	$-\frac{20\left(r^6+15r^5+\frac{183}{2}r^4+290r^3+\frac{5031}{10}r^2+453r+166\right)r}{(1+r)^3(r+2)^3(r+3)^3(4+r)^3}$	0
b_5	$\frac{6r^5+75r^4+340r^3+675r^2+548r+120}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2(r+5)^2}$	$\frac{1}{120}$	$-\frac{30r\left(r^8+24r^7+\frac{739}{3}r^6+1410r^5+4915r^4+10668r^3+14063r^2+10290r+\frac{48076}{15}\right)}{(1+r)^3(r+2)^3(r+3)^3(4+r)^3(r+5)^3}$	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned}$$

Verified OK.

2.12.1 Maple step by step solution

Let's solve

$y''x + (1-2x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (1-2x)\,y' + (x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right) x^{k}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term must be 0

  $$a_1(1+r)^2 - a_0(1+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1)^2 + (-2k-1)\, a_k + a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$a_{k+2}(k+2)^2 + (-2k-3)\, a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{2k a_{k+1}-a_k+3a_{k+1}}{(k+2)^2}, a_1-a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 43

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(1-2*x)*diff(y(x),x)+(x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=(c_2\ln(x)+c_1)\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5\right)+O(x^6)$$

✓ Solution by Mathematica
Time used: 0.005 (sec). Leaf size: 74

```
AsymptoticDSolveValue[x*y''[x]+(1-2*x)*y'[x]+(x-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1\right)+c_2\left(\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1\right)\log(x)$$

2.13 problem 15

2.13.1 Maple step by step solution 289

Internal problem ID [5648]
Internal file name [`OUTPUT/4896_Sunday_June_05_2022_03_09_38_PM_76748019/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{2x(1-x)\,y''-(1+6x)\,y'-2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^2+2x\right)y''+(-6x-1)\,y'-2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{1+6x}{2x\,(x-1)}$$
$$q(x)=\frac{1}{x\,(x-1)}$$

Table 34: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+6x}{2x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{1}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-2x(x-1)\, y'' + (-6x-1)\, y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) + (-6x-1)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}2x^{n+r-1}a_n(n+r)(n+r-1)\right) \\ +\sum_{n=0}^{\infty}\left(-6x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-(n+r)\,a_n x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-2a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-6x^{n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-6a_{n-1}(n+r-1)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-2a_n x^{n+r}\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty}2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-6a_{n-1}(n+r-1)\,x^{n+r-1}\right) \\ +\sum_{n=0}^{\infty}\left(-(n+r)\,a_n x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1}a_n(n+r)(n+r-1)-(n+r)\,a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$2x^{-1+r}a_0 r(-1+r)-r a_0 x^{-1+r}=0$$

Or

$$
\left(2x^{-1+r} r(-1+r) - r\,x^{-1+r}\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
r\,x^{-1+r}(-3+2r) = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
2r^2 - 3r = 0
$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{3}{2} \\
r_2 &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
r\,x^{-1+r}(-3+2r) = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^n
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&-2a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) \\
&- 6a_{n-1}(n+r-1) - a_n(n+r) - 2a_{n-1} = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2(n+r)\, a_{n-1}}{2n-3+2r} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = \frac{\left(n+\frac{3}{2}\right) a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2+2r}{2r-1}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = \frac{5}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+2r}{2r-1}$	$\frac{5}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2+12r+8}{4r^2-1}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{35}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+2r}{2r-1}$	$\frac{5}{2}$
a_2	$\frac{4r^2+12r+8}{4r^2-1}$	$\frac{35}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^3 + 48r^2 + 88r + 48}{8r^3 + 12r^2 - 2r - 3}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = \frac{105}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+2r}{2r-1}$	$\frac{5}{2}$
a_2	$\frac{4r^2+12r+8}{4r^2-1}$	$\frac{35}{8}$
a_3	$\frac{8r^3+48r^2+88r+48}{8r^3+12r^2-2r-3}$	$\frac{105}{16}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^4 + 160r^3 + 560r^2 + 800r + 384}{16r^4 + 64r^3 + 56r^2 - 16r - 15}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{1155}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+2r}{2r-1}$	$\frac{5}{2}$
a_2	$\frac{4r^2+12r+8}{4r^2-1}$	$\frac{35}{8}$
a_3	$\frac{8r^3+48r^2+88r+48}{8r^3+12r^2-2r-3}$	$\frac{105}{16}$
a_4	$\frac{16r^4+160r^3+560r^2+800r+384}{16r^4+64r^3+56r^2-16r-15}$	$\frac{1155}{128}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32r^5 + 480r^4 + 2720r^3 + 7200r^2 + 8768r + 3840}{32r^5 + 240r^4 + 560r^3 + 360r^2 - 142r - 105}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = \frac{3003}{256}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+2r}{2r-1}$	$\frac{5}{2}$
a_2	$\frac{4r^2+12r+8}{4r^2-1}$	$\frac{35}{8}$
a_3	$\frac{8r^3+48r^2+88r+48}{8r^3+12r^2-2r-3}$	$\frac{105}{16}$
a_4	$\frac{16r^4+160r^3+560r^2+800r+384}{16r^4+64r^3+56r^2-16r-15}$	$\frac{1155}{128}$
a_5	$\frac{32r^5+480r^4+2720r^3+7200r^2+8768r+3840}{32r^5+240r^4+560r^3+360r^2-142r-105}$	$\frac{3003}{256}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{3}{2}}\left(1 + \frac{5x}{2} + \frac{35x^2}{8} + \frac{105x^3}{16} + \frac{1155x^4}{128} + \frac{3003x^5}{256} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}
&-2b_{n-1}(n+r-1)(n+r-2) + 2b_n(n+r)(n+r-1) \\
&- 6b_{n-1}(n+r-1) - (n+r)b_n - 2b_{n-1} = 0
\end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{2(n+r)b_{n-1}}{2n-3+2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{2nb_{n-1}}{2n-3} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2 + 2r}{2r - 1}$$

Which for the root $r = 0$ becomes

$$b_1 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2+2r}{2r-1}$	-2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^2 + 12r + 8}{4r^2 - 1}$$

Which for the root $r = 0$ becomes

$$b_2 = -8$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2+2r}{2r-1}$	-2
b_2	$\frac{4r^2+12r+8}{4r^2-1}$	-8

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8r^3 + 48r^2 + 88r + 48}{8r^3 + 12r^2 - 2r - 3}$$

Which for the root $r = 0$ becomes

$$b_3 = -16$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2+2r}{2r-1}$	-2
b_2	$\frac{4r^2+12r+8}{4r^2-1}$	-8
b_3	$\frac{8r^3+48r^2+88r+48}{8r^3+12r^2-2r-3}$	-16

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r^4 + 160r^3 + 560r^2 + 800r + 384}{16r^4 + 64r^3 + 56r^2 - 16r - 15}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{128}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2+2r}{2r-1}$	-2
b_2	$\frac{4r^2+12r+8}{4r^2-1}$	-8
b_3	$\frac{8r^3+48r^2+88r+48}{8r^3+12r^2-2r-3}$	-16
b_4	$\frac{16r^4+160r^3+560r^2+800r+384}{16r^4+64r^3+56r^2-16r-15}$	$-\frac{128}{5}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32r^5 + 480r^4 + 2720r^3 + 7200r^2 + 8768r + 3840}{32r^5 + 240r^4 + 560r^3 + 360r^2 - 142r - 105}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{256}{7}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2+2r}{2r-1}$	-2
b_2	$\frac{4r^2+12r+8}{4r^2-1}$	-8
b_3	$\frac{8r^3+48r^2+88r+48}{8r^3+12r^2-2r-3}$	-16
b_4	$\frac{16r^4+160r^3+560r^2+800r+384}{16r^4+64r^3+56r^2-16r-15}$	$-\frac{128}{5}$
b_5	$\frac{32r^5+480r^4+2720r^3+7200r^2+8768r+3840}{32r^5+240r^4+560r^3+360r^2-142r-105}$	$-\frac{256}{7}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 - 2x - 8x^2 - 16x^3 - \frac{128x^4}{5} - \frac{256x^5}{7} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{2}} \left(1 + \frac{5x}{2} + \frac{35x^2}{8} + \frac{105x^3}{16} + \frac{1155x^4}{128} + \frac{3003x^5}{256} + O(x^6)\right) \\
&\quad + c_2 \left(1 - 2x - 8x^2 - 16x^3 - \frac{128x^4}{5} - \frac{256x^5}{7} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}} \left(1 + \frac{5x}{2} + \frac{35x^2}{8} + \frac{105x^3}{16} + \frac{1155x^4}{128} + \frac{3003x^5}{256} + O(x^6)\right) \\
&\quad + c_2 \left(1 - 2x - 8x^2 - 16x^3 - \frac{128x^4}{5} - \frac{256x^5}{7} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{3}{2}} \left(1 + \frac{5x}{2} + \frac{35x^2}{8} + \frac{105x^3}{16} + \frac{1155x^4}{128} + \frac{3003x^5}{256} + O(x^6)\right) \\
+ c_2 \left(1 - 2x - 8x^2 - 16x^3 - \frac{128x^4}{5} - \frac{256x^5}{7} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} \left(1 + \frac{5x}{2} + \frac{35x^2}{8} + \frac{105x^3}{16} + \frac{1155x^4}{128} + \frac{3003x^5}{256} + O(x^6)\right) + c_2\left(1 - 2x - 8x^2 - 16x^3 - \frac{128x^4}{5} - \frac{256x^5}{7} + O(x^6)\right)$$

Verified OK.

2.13.1 Maple step by step solution

Let's solve

$-2y''x(x-1) + (-6x-1)\,y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(1+6x)y'}{2x(x-1)} - \frac{y}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+6x)y'}{2x(x-1)} + \frac{y}{x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1+6x}{2x(x-1)}, P_3(x) = \frac{1}{x(x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2y''x(x-1) + (1+6x)\,y' + 2y = 0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m=0..1$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m=1..2$

    $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3+2r)\, x^{-1+r}+\left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+r+1)\,(2k-1+2r)+2a_k(k+r+1)^2\right) x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+r+1)\left(k+r-\frac{1}{2}\right) a_{k+1}+2a_k(k+r+1)^2=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_k(k+r+1)}{2k-1+2r}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{2a_k(k+1)}{2k-1}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{2a_k(k+1)}{2k-1}\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = \frac{2a_k(k+\frac{5}{2})}{2k+2}$

- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k(k+\frac{5}{2})}{2k+2}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{2a_k(k+1)}{2k-1}, b_{k+1} = \frac{2b_k(k+\frac{5}{2})}{2k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 44

```
Order:=6;
dsolve(2*x*(1-x)*diff(y(x),x$2)-(1+6*x)*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{3}{2}}\left(1 + \frac{5}{2}x + \frac{35}{8}x^2 + \frac{105}{16}x^3 + \frac{1155}{128}x^4 + \frac{3003}{256}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(1 - 2x - 8x^2 - 16x^3 - \frac{128}{5}x^4 - \frac{256}{7}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 79

```
AsymptoticDSolveValue[2*x*(1-x)*y''[x]-(1+6*x)*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{256x^5}{7} - \frac{128x^4}{5} - 16x^3 - 8x^2 - 2x + 1\right) + c_1\left(\frac{3003x^5}{256} + \frac{1155x^4}{128} + \frac{105x^3}{16} + \frac{35x^2}{8} + \frac{5x}{2} + 1\right)x^{3/2}$$

2.14 problem 16

2.14.1 Maple step by step solution 303

Internal problem ID [5649]
Internal file name [`OUTPUT/4897_Sunday_June_05_2022_03_09_41_PM_19950706/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[_Jacobi]
```

$$\boxed{x(1-x)\,y''+\left(\frac{1}{2}+2x\right)y'-2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^2+x\right)y''+\left(\frac{1}{2}+2x\right)y'-2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{4x+1}{2x\,(x-1)}$$
$$q(x)=\frac{2}{x\,(x-1)}$$

Table 36: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{4x+1}{2x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{2}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-x(x-1)\, y'' + \left(\frac{1}{2} + 2x\right) y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) + \left(\frac{1}{2} + 2x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) \\ +\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} \frac{(n+r)\, a_n x^{n+r-1}}{2}\right)+\sum_{n=0}^{\infty}\left(-2a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r} a_n (n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)=\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-2a_n x^{n+r}\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty} \frac{(n+r)\, a_n x^{n+r-1}}{2}\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1} x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1)+\frac{(n+r)\, a_n x^{n+r-1}}{2}=0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r)+\frac{r a_0 x^{-1+r}}{2}=0$$

Or

$$\left(x^{-1+r} r(-1+r)+\frac{r\,x^{-1+r}}{2}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}\left(-\frac{1}{2}+r\right) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{2}r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}\left(-\frac{1}{2}+r\right) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &-a_{n-1}(n+r-1)(n+r-2)+a_n(n+r)(n+r-1) \\ &+2a_{n-1}(n+r-1)+\frac{a_n(n+r)}{2}-2a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}(n^2 + 2nr + r^2 - 5n - 5r + 6)}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}(4n^2 - 16n + 15)}{4n^2 + 2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2r^2 - 6r + 4}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-6r+4}{2r^2+3r+1}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4(-1 + r)^2 (-2 + r) r}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-6r+4}{2r^2+3r+1}$	$\frac{1}{2}$
a_2	$\frac{4(-1+r)^2(-2+r)r}{4r^4+20r^3+35r^2+25r+6}$	$-\frac{1}{40}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8(-1+r)^2(-2+r)\,r^2}{8r^5+76r^4+274r^3+461r^2+351r+90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{1}{560}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-6r+4}{2r^2+3r+1}$	$\frac{1}{2}$
a_2	$\frac{4(-1+r)^2(-2+r)r}{4r^4+20r^3+35r^2+25r+6}$	$-\frac{1}{40}$
a_3	$\frac{8(-1+r)^2(-2+r)r^2}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$-\frac{1}{560}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16(-1+r)^2(-2+r)\,r^2(r+1)}{16r^6+240r^5+1432r^4+4296r^3+6697r^2+4959r+1260}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = -\frac{1}{2688}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-6r+4}{2r^2+3r+1}$	$\frac{1}{2}$
a_2	$\frac{4(-1+r)^2(-2+r)r}{4r^4+20r^3+35r^2+25r+6}$	$-\frac{1}{40}$
a_3	$\frac{8(-1+r)^2(-2+r)r^2}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$-\frac{1}{560}$
a_4	$\frac{16(-1+r)^2(-2+r)r^2(r+1)}{16r^6+240r^5+1432r^4+4296r^3+6697r^2+4959r+1260}$	$-\frac{1}{2688}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32(-1+r)^2 r^2(r+1)(r^2-4)}{32r^7+688r^6+6080r^5+28360r^4+74378r^3+107347r^2+76065r+18900}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{1}{8448}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-6r+4}{2r^2+3r+1}$	$\frac{1}{2}$
a_2	$\frac{4(-1+r)^2(-2+r)r}{4r^4+20r^3+35r^2+25r+6}$	$-\frac{1}{40}$
a_3	$\frac{8(-1+r)^2(-2+r)r^2}{8r^5+76r^4+274r^3+461r^2+351r+90}$	$-\frac{1}{560}$
a_4	$\frac{16(-1+r)^2(-2+r)r^2(r+1)}{16r^6+240r^5+1432r^4+4296r^3+6697r^2+4959r+1260}$	$-\frac{1}{2688}$
a_5	$\frac{32(-1+r)^2r^2(r+1)(r^2-4)}{32r^7+688r^6+6080r^5+28360r^4+74378r^3+107347r^2+76065r+18900}$	$-\frac{1}{8448}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x}{2} - \frac{x^2}{40} - \frac{x^3}{560} - \frac{x^4}{2688} - \frac{x^5}{8448} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&-b_{n-1}(n+r-1)(n+r-2)+b_n(n+r)(n+r-1) \\
&+2b_{n-1}(n+r-1)+\frac{(n+r)\,b_n}{2}-2b_{n-1}=0
\end{aligned}
\tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{2b_{n-1}(n^2+2nr+r^2-5n-5r+6)}{2n^2+4nr+2r^2-n-r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{2b_{n-1}(n^2-5n+6)}{n\,(2n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{2r^2-6r+4}{2r^2+3r+1}
$$

Which for the root $r = 0$ becomes

$$
b_1 = 4
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-6r+4}{2r^2+3r+1}$	4

For $n = 2$, using the above recursive equation gives

$$
b_2 = \frac{4(-1+r)^2\,(-2+r)\,r}{4r^4+20r^3+35r^2+25r+6}
$$

Which for the root $r = 0$ becomes

$$
b_2 = 0
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-6r+4}{2r^2+3r+1}$	4
b_2	$\frac{4(-1+r)^2(-2+r)r}{4r^4+20r^3+35r^2+25r+6}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8(-1+r)^2(-2+r)r^2}{8r^5+76r^4+274r^3+461r^2+351r+90}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-6r+4}{2r^2+3r+1}$	4
b_2	$\frac{4(-1+r)^2(-2+r)r}{4r^4+20r^3+35r^2+25r+6}$	0
b_3	$\frac{8(-1+r)^2(-2+r)r^2}{8r^5+76r^4+274r^3+461r^2+351r+90}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16(-1+r)^2(-2+r)r^2(r+1)}{16r^6+240r^5+1432r^4+4296r^3+6697r^2+4959r+1260}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-6r+4}{2r^2+3r+1}$	4
b_2	$\frac{4(-1+r)^2(-2+r)r}{4r^4+20r^3+35r^2+25r+6}$	0
b_3	$\frac{8(-1+r)^2(-2+r)r^2}{8r^5+76r^4+274r^3+461r^2+351r+90}$	0
b_4	$\frac{16(-1+r)^2(-2+r)r^2(r+1)}{16r^6+240r^5+1432r^4+4296r^3+6697r^2+4959r+1260}$	0

For $n=5$, using the above recursive equation gives

$$b_5 = \frac{32(-1+r)^2 r^2 (r+1)(r^2-4)}{32r^7+688r^6+6080r^5+28360r^4+74378r^3+107347r^2+76065r+18900}$$

Which for the root $r=0$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2-6r+4}{2r^2+3r+1}$	4
b_2	$\frac{4(-1+r)^2(-2+r)r}{4r^4+20r^3+35r^2+25r+6}$	0
b_3	$\frac{8(-1+r)^2(-2+r)r^2}{8r^5+76r^4+274r^3+461r^2+351r+90}$	0
b_4	$\frac{16(-1+r)^2(-2+r)r^2(r+1)}{16r^6+240r^5+1432r^4+4296r^3+6697r^2+4959r+1260}$	0
b_5	$\frac{32(-1+r)^2r^2(r+1)(r^2-4)}{32r^7+688r^6+6080r^5+28360r^4+74378r^3+107347r^2+76065r+18900}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + 4x + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x} \left(1 + \frac{x}{2} - \frac{x^2}{40} - \frac{x^3}{560} - \frac{x^4}{2688} - \frac{x^5}{8448} + O(x^6)\right) + c_2 (1 + 4x + O(x^6)) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x} \left(1 + \frac{x}{2} - \frac{x^2}{40} - \frac{x^3}{560} - \frac{x^4}{2688} - \frac{x^5}{8448} + O(x^6)\right) + c_2 (1 + 4x + O(x^6)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x} \left(1 + \frac{x}{2} - \frac{x^2}{40} - \frac{x^3}{560} - \frac{x^4}{2688} - \frac{x^5}{8448} + O(x^6)\right) + c_2 (1 + 4x + O(x^6)) \quad (1)$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1+\frac{x}{2}-\frac{x^2}{40}-\frac{x^3}{560}-\frac{x^4}{2688}-\frac{x^5}{8448}+O(x^6)\right)+c_2(1+4x+O(x^6))$$

Verified OK.

2.14.1 Maple step by step solution

Let's solve

$-y''x(x-1)+\left(\frac{1}{2}+2x\right)y'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{x(x-1)}+\frac{(1+4x)y'}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(1+4x)y'}{2x(x-1)}+\frac{2y}{x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{1+4x}{2x(x-1)}, P_3(x)=\frac{2}{x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2y''x(x-1)+(-4x-1)y'+4y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + 2a_k(k+r-1)\,(k+r-2)\right) x^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2a_k(k+r-1)\,(k+r-2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(k+r-1)(k+r-2)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = \frac{2a_k(k-1)(k-2)}{(2k+1)(k+1)}$$

- Apply recursion relation for $k = 0$

$$a_1 = 4a_0$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y = a_0 \cdot (1 + 4x)$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(2k+2\right)\left(k+\frac{3}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(2k+2\right)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot (1 + 4x) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2b_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(2k+2\right)\left(k+\frac{3}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
Order:=6;
dsolve(x*(1-x)*diff(y(x),x$2)+(1/2+2*x)*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{1}{2}x-\frac{1}{40}x^2-\frac{1}{560}x^3-\frac{1}{2688}x^4-\frac{1}{8448}x^5+O\left(x^6\right)\right)+c_2\left(1+4x+O\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 55

```
AsymptoticDSolveValue[x*(1-x)*y''[x]+(1/2+2*x)*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{x^5}{8448}-\frac{x^4}{2688}-\frac{x^3}{560}-\frac{x^2}{40}+\frac{x}{2}+1\right)+c_2(4x+1)$$

2.15 problem 17

2.15.1 Maple step by step solution 317

Internal problem ID [5650]
Internal file name [`OUTPUT/4898_Sunday_June_05_2022_03_09_45_PM_71058820/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{4xy'' + y' + 8y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4xy'' + y' + 8y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{4x}$$
$$q(x) = \frac{2}{x}$$

Table 38: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{4x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' + y' + 8y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 8\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 8a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 8a_n x^{n+r} = \sum_{n=1}^{\infty} 8a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 8a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$4x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(4x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-3+4r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 3r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{4}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-3+4r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + a_n(n+r) + 8a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{8a_{n-1}}{4n^2 + 8nr + 4r^2 - 3n - 3r} \tag{4}$$

Which for the root $r = \frac{3}{4}$ becomes

$$a_n = -\frac{8a_{n-1}}{n(4n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{8}{4r^2 + 5r + 1}$$

Which for the root $r = \frac{3}{4}$ becomes

$$a_1 = -\frac{8}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8}{4r^2+5r+1}$	$-\frac{8}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{64}{(4r^2 + 5r + 1)(4r^2 + 13r + 10)}$$

Which for the root $r = \frac{3}{4}$ becomes

$$a_2 = \frac{32}{77}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8}{4r^2+5r+1}$	$-\frac{8}{7}$
a_2	$\frac{64}{(4r^2+5r+1)(4r^2+13r+10)}$	$\frac{32}{77}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{512}{(4r^2 + 5r + 1)(4r^2 + 13r + 10)(4r^2 + 21r + 27)}$$

Which for the root $r = \frac{3}{4}$ becomes

$$a_3 = -\frac{256}{3465}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8}{4r^2+5r+1}$	$-\frac{8}{7}$
a_2	$\frac{64}{(4r^2+5r+1)(4r^2+13r+10)}$	$\frac{32}{77}$
a_3	$-\frac{512}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)}$	$-\frac{256}{3465}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4096}{(4r^2 + 5r + 1)(4r^2 + 13r + 10)(4r^2 + 21r + 27)(4r^2 + 29r + 52)}$$

Which for the root $r = \frac{3}{4}$ becomes

$$a_4 = \frac{512}{65835}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8}{4r^2+5r+1}$	$-\frac{8}{7}$
a_2	$\frac{64}{(4r^2+5r+1)(4r^2+13r+10)}$	$\frac{32}{77}$
a_3	$-\frac{512}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)}$	$-\frac{256}{3465}$
a_4	$\frac{4096}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)(4r^2+29r+52)}$	$\frac{512}{65835}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32768}{(4r^2 + 5r + 1)(4r^2 + 13r + 10)(4r^2 + 21r + 27)(4r^2 + 29r + 52)(4r^2 + 37r + 85)}$$

Which for the root $r = \frac{3}{4}$ becomes

$$a_5 = -\frac{4096}{7571025}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{8}{4r^2+5r+1}$	$-\frac{8}{7}$
a_2	$\frac{64}{(4r^2+5r+1)(4r^2+13r+10)}$	$\frac{32}{77}$
a_3	$-\frac{512}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)}$	$-\frac{256}{3465}$
a_4	$\frac{4096}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)(4r^2+29r+52)}$	$\frac{512}{65835}$
a_5	$-\frac{32768}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)(4r^2+29r+52)(4r^2+37r+85)}$	$-\frac{4096}{7571025}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^{\frac{3}{4}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{3}{4}}\left(1 - \frac{8x}{7} + \frac{32x^2}{77} - \frac{256x^3}{3465} + \frac{512x^4}{65835} - \frac{4096x^5}{7571025} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$4b_n(n + r)(n + r - 1) + (n + r)b_n + 8b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{8b_{n-1}}{4n^2 + 8nr + 4r^2 - 3n - 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{8b_{n-1}}{n(4n - 3)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{8}{4r^2 + 5r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = -8$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{8}{4r^2+5r+1}$	-8

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{64}{(4r^2 + 5r + 1)(4r^2 + 13r + 10)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{32}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{8}{4r^2+5r+1}$	-8
b_2	$\frac{64}{(4r^2+5r+1)(4r^2+13r+10)}$	$\frac{32}{5}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{512}{(4r^2 + 5r + 1)(4r^2 + 13r + 10)(4r^2 + 21r + 27)}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{256}{135}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{8}{4r^2+5r+1}$	-8
b_2	$\frac{64}{(4r^2+5r+1)(4r^2+13r+10)}$	$\frac{32}{5}$
b_3	$-\frac{512}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)}$	$-\frac{256}{135}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4096}{(4r^2 + 5r + 1)(4r^2 + 13r + 10)(4r^2 + 21r + 27)(4r^2 + 29r + 52)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{512}{1755}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{8}{4r^2+5r+1}$	-8
b_2	$\frac{64}{(4r^2+5r+1)(4r^2+13r+10)}$	$\frac{32}{5}$
b_3	$-\frac{512}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)}$	$-\frac{256}{135}$
b_4	$\frac{4096}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)(4r^2+29r+52)}$	$\frac{512}{1755}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32768}{(4r^2 + 5r + 1)(4r^2 + 13r + 10)(4r^2 + 21r + 27)(4r^2 + 29r + 52)(4r^2 + 37r + 85)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{4096}{149175}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{8}{4r^2+5r+1}$	-8
b_2	$\frac{64}{(4r^2+5r+1)(4r^2+13r+10)}$	$\frac{32}{5}$
b_3	$-\frac{512}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)}$	$-\frac{256}{135}$
b_4	$\frac{4096}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)(4r^2+29r+52)}$	$\frac{512}{1755}$
b_5	$-\frac{32768}{(4r^2+5r+1)(4r^2+13r+10)(4r^2+21r+27)(4r^2+29r+52)(4r^2+37r+85)}$	$-\frac{4096}{149175}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - 8x + \frac{32x^2}{5} - \frac{256x^3}{135} + \frac{512x^4}{1755} - \frac{4096x^5}{149175} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{4}} \left(1 - \frac{8x}{7} + \frac{32x^2}{77} - \frac{256x^3}{3465} + \frac{512x^4}{65835} - \frac{4096x^5}{7571025} + O(x^6)\right) \\
&\quad + c_2 \left(1 - 8x + \frac{32x^2}{5} - \frac{256x^3}{135} + \frac{512x^4}{1755} - \frac{4096x^5}{149175} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{4}} \left(1 - \frac{8x}{7} + \frac{32x^2}{77} - \frac{256x^3}{3465} + \frac{512x^4}{65835} - \frac{4096x^5}{7571025} + O(x^6)\right) \\
&\quad + c_2 \left(1 - 8x + \frac{32x^2}{5} - \frac{256x^3}{135} + \frac{512x^4}{1755} - \frac{4096x^5}{149175} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{3}{4}} &\left(1 - \frac{8x}{7} + \frac{32x^2}{77} - \frac{256x^3}{3465} + \frac{512x^4}{65835} - \frac{4096x^5}{7571025} + O(x^6)\right) \\
&+ c_2 \left(1 - 8x + \frac{32x^2}{5} - \frac{256x^3}{135} + \frac{512x^4}{1755} - \frac{4096x^5}{149175} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^{\frac{3}{4}} &\left(1 - \frac{8x}{7} + \frac{32x^2}{77} - \frac{256x^3}{3465} + \frac{512x^4}{65835} - \frac{4096x^5}{7571025} + O(x^6)\right) \\
&+ c_2 \left(1 - 8x + \frac{32x^2}{5} - \frac{256x^3}{135} + \frac{512x^4}{1755} - \frac{4096x^5}{149175} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

2.15.1 Maple step by step solution

Let's solve

$4y''x + y' + 8y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{4x} - \frac{2y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{4x} + \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{4x}, P_3(x) = \frac{2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4y''x + y' + 8y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-3+4r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(4k+1+4r) + 8a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-3+4r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{3}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(k+\frac{1}{4}+r\right)(k+1+r)\, a_{k+1} + 8a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{8a_k}{(4k+1+4r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = -\frac{8a_k}{(4k+1)(k+1)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{8a_k}{(4k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{3}{4}$

  $$a_{k+1} = -\frac{8a_k}{(4k+4)\left(k+\frac{7}{4}\right)}$$

- Solution for $r = \frac{3}{4}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = -\frac{8a_k}{(4k+4)\left(k+\frac{7}{4}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1}=-\frac{8a_k}{(4k+1)(k+1)}, b_{k+1}=-\frac{8b_k}{(4k+4)\left(k+\frac{7}{4}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
Order:=6;
dsolve(4*x*diff(y(x),x$2)+diff(y(x),x)+8*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1 x^{\frac{3}{4}}\left(1-\frac{8}{7}x+\frac{32}{77}x^2-\frac{256}{3465}x^3+\frac{512}{65835}x^4-\frac{4096}{7571025}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(1-8x+\frac{32}{5}x^2-\frac{256}{135}x^3+\frac{512}{1755}x^4-\frac{4096}{149175}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 83

```
AsymptoticDSolveValue[4*x*y''[x]+y'[x]+8*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(-\frac{4096x^5}{149175}+\frac{512x^4}{1755}-\frac{256x^3}{135}+\frac{32x^2}{5}-8x+1\right)+c_1 x^{3/4}\left(-\frac{4096x^5}{7571025}+\frac{512x^4}{65835}-\frac{256x^3}{3465}+\frac{32x^2}{77}-\frac{8x}{7}+1\right)$$

2.16 problem 18

2.16.1 Maple step by step solution 328

Internal problem ID [5651]
Internal file name [OUTPUT/4899_Sunday_June_05_2022_03_09_47_PM_26749394/index.tex]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4(t^2-3t+2)\,y''-2y'+y=0}$$

With the expansion point for the power series method at $t=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots\\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{49}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{50}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y-2y'}{4\left(t^2-3t+2\right)}\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= \frac{\left(-2t^2-2t+10\right)y'+y(4t-7)}{8\left(t^2-3t+2\right)^2}\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= \frac{\left(16t^3-28t^2-52t+94\right)y'+\left(-23t^2+81t-73\right)y}{16\left(t^2-3t+2\right)^3}\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \frac{\left(-71t^4+270t^3-104t^2-633t+643\right)y'+88y\left(t^3-\frac{467}{88}t^2+\frac{845}{88}t-\frac{1037}{176}\right)}{16\left(t^2-3t+2\right)^4}\\
F_4 &= \frac{dF_3}{dt}\\
&= \frac{\partial F_3}{\partial t}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= \frac{\left(1488t^5-8466t^4+13692t^3+5550t^2-33312t+22938\right)y'-1689y\left(t^4-\frac{3998}{563}t^3+\frac{10884}{563}t^2-\frac{13391}{563}t+\right.}{64\left(t^2-3t+2\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{y(0)}{8}+\frac{y'(0)}{4}\\
F_1 &= -\frac{7y(0)}{32}+\frac{5y'(0)}{16}\\
F_2 &= -\frac{73y(0)}{128}+\frac{47y'(0)}{64}\\
F_3 &= -\frac{1037y(0)}{512}+\frac{643y'(0)}{256}\\
F_4 &= -\frac{18771y(0)}{2048}+\frac{11469y'(0)}{1024}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{16}t^{2}-\frac{7}{192}t^{3}-\frac{73}{3072}t^{4}-\frac{1037}{61440}t^{5}-\frac{6257}{491520}t^{6}\right) y(0) \\
&+\left(t+\frac{1}{8}t^{2}+\frac{5}{96}t^{3}+\frac{47}{1536}t^{4}+\frac{643}{30720}t^{5}+\frac{3823}{245760}t^{6}\right) y'(0)+O\left(t^{6}\right)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
y''\left(4t^{2}-12t+8\right)+y-2y' = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_{n} t^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} t^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_{n} t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} t^{n-2}\right)\left(4t^{2}-12t+8\right)+\left(\sum_{n=0}^{\infty} a_{n} t^{n}\right)-2\left(\sum_{n=1}^{\infty} n a_{n} t^{n-1}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 4t^{n} a_{n} n(n-1)\right)+\sum_{n=2}^{\infty}\left(-12n\, t^{n-1} a_{n}(n-1)\right) \\
&+\left(\sum_{n=2}^{\infty} 8n(n-1)\, a_{n} t^{n-2}\right)+\sum_{n=1}^{\infty}\left(-2n a_{n} t^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_{n} t^{n}\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^{n} and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-12n\, t^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty}\left(-12(n+1)\, a_{n+1} n\, t^{n}\right)$$

$$\sum_{n=2}^{\infty} 8n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 8(n+2)\, a_{n+2}(n+1)\, t^{n}$$

$$\sum_{n=1}^{\infty}\left(-2n a_n t^{n-1}\right) = \sum_{n=0}^{\infty}\left(-2(n+1)\, a_{n+1} t^{n}\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\begin{gathered}\left(\sum_{n=2}^{\infty} 4t^n a_n n(n-1)\right) + \sum_{n=1}^{\infty}\left(-12(n+1)\, a_{n+1} n\, t^n\right) \\ + \left(\sum_{n=0}^{\infty} 8(n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n=0}^{\infty}\left(-2(n+1)\, a_{n+1} t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0\end{gathered} \tag{3}$$

$n=0$ gives

$$16a_2 - 2a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{16} + \frac{a_1}{8}$$

$n=1$ gives

$$-28a_2 + 48a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{7a_0}{192} + \frac{5a_1}{96}$$

For $2 \leq n$, the recurrence equation is

$$4n a_n (n-1) - 12(n+1)\, a_{n+1} n + 8(n+2)\, a_{n+2}(n+1) - 2(n+1)\, a_{n+1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{4n^2 a_n - 12n^2 a_{n+1} - 4n a_n - 14n a_{n+1} + a_n - 2a_{n+1}}{8\left(n+2\right)\left(n+1\right)} \\
&= -\frac{\left(4n^2-4n+1\right) a_n}{8\left(n+2\right)\left(n+1\right)} - \frac{\left(-12n^2-14n-2\right) a_{n+1}}{8\left(n+2\right)\left(n+1\right)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$9a_2 - 78a_3 + 96a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{73a_0}{3072} + \frac{47a_1}{1536}$$

For $n = 3$ the recurrence equation gives

$$25a_3 - 152a_4 + 160a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{1037a_0}{61440} + \frac{643a_1}{30720}$$

For $n = 4$ the recurrence equation gives

$$49a_4 - 250a_5 + 240a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{6257a_0}{491520} + \frac{3823a_1}{245760}$$

For $n = 5$ the recurrence equation gives

$$81a_5 - 372a_6 + 336a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{137969a_0}{13762560} + \frac{83791a_1}{6881280}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y = a_0 + a_1 t + \left(-\frac{a_0}{16} + \frac{a_1}{8}\right) t^2 + \left(-\frac{7a_0}{192} + \frac{5a_1}{96}\right) t^3 \\
+ \left(-\frac{73a_0}{3072} + \frac{47a_1}{1536}\right) t^4 + \left(-\frac{1037a_0}{61440} + \frac{643a_1}{30720}\right) t^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = \left(1 - \frac{1}{16} t^2 - \frac{7}{192} t^3 - \frac{73}{3072} t^4 - \frac{1037}{61440} t^5\right) a_0 \\
+ \left(t + \frac{1}{8} t^2 + \frac{5}{96} t^3 + \frac{47}{1536} t^4 + \frac{643}{30720} t^5\right) a_1 + O(t^6)
\end{aligned}
\tag{3}
$$

At $t = 0$ the solution above becomes

$$
\begin{aligned}
y = \left(1 - \frac{1}{16} t^2 - \frac{7}{192} t^3 - \frac{73}{3072} t^4 - \frac{1037}{61440} t^5\right) c_1 \\
+ \left(t + \frac{1}{8} t^2 + \frac{5}{96} t^3 + \frac{47}{1536} t^4 + \frac{643}{30720} t^5\right) c_2 + O(t^6)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = \left(1 - \frac{1}{16} t^2 - \frac{7}{192} t^3 - \frac{73}{3072} t^4 - \frac{1037}{61440} t^5 - \frac{6257}{491520} t^6\right) y(0) \\
+ \left(t + \frac{1}{8} t^2 + \frac{5}{96} t^3 + \frac{47}{1536} t^4 + \frac{643}{30720} t^5 + \frac{3823}{245760} t^6\right) y'(0) + O(t^6)
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
y = \left(1 - \frac{1}{16} t^2 - \frac{7}{192} t^3 - \frac{73}{3072} t^4 - \frac{1037}{61440} t^5\right) c_1 \\
+ \left(t + \frac{1}{8} t^2 + \frac{5}{96} t^3 + \frac{47}{1536} t^4 + \frac{643}{30720} t^5\right) c_2 + O(t^6)
\end{aligned}
\tag{2}
$$

Verification of solutions

$$y = \left(1 - \frac{1}{16}t^2 - \frac{7}{192}t^3 - \frac{73}{3072}t^4 - \frac{1037}{61440}t^5 - \frac{6257}{491520}t^6\right) y(0) + \left(t + \frac{1}{8}t^2 + \frac{5}{96}t^3 + \frac{47}{1536}t^4 + \frac{643}{30720}t^5 + \frac{3823}{245760}t^6\right) y'(0) + O(t^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{16}t^2 - \frac{7}{192}t^3 - \frac{73}{3072}t^4 - \frac{1037}{61440}t^5\right) c_1 + \left(t + \frac{1}{8}t^2 + \frac{5}{96}t^3 + \frac{47}{1536}t^4 + \frac{643}{30720}t^5\right) c_2 + O(t^6)$$

Verified OK.

2.16.1 Maple step by step solution

Let's solve

$y''(4t^2 - 12t + 8) + y - 2y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4(t^2-3t+2)} + \frac{y'}{2(t^2-3t+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2(t^2-3t+2)} + \frac{y}{4(t^2-3t+2)} = 0$

☐ Check to see if t_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(t) = -\frac{1}{2(t^2-3t+2)}, P_3(t) = \frac{1}{4(t^2-3t+2)}\right]$

- ∘ $(t-1) \cdot P_2(t)$ is analytic at $t = 1$

  $\left((t-1) \cdot P_2(t)\right)\Big|_{t=1} = \frac{1}{2}$

- ∘ $(t-1)^2 \cdot P_3(t)$ is analytic at $t = 1$

  $\left((t-1)^2 \cdot P_3(t)\right)\Big|_{t=1} = 0$

- ∘ $t = 1$is a regular singular point

Check to see if t_0 is a regular singular point

$t_0 = 1$

- Multiply by denominators

  $y''(4t^2 - 12t + 8) + y - 2y' = 0$

- Change variables using $t = u + 1$ so that the regular singular point is at $u = 0$

  $(4u^2 - 4u)\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $\frac{d}{du}y(u)$ to series expansion

    $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$

  - Shift index using $k->k+1$

    $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(2k+1+2r) + a_k(2k+2r-1)^2\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, \frac{1}{2}\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 - 4\left(k + \frac{1}{2} + r\right) a_{k+1}(k+1+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(2k+2r-1)^2}{2(2k+1+2r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(2k-1)^2}{2(2k+1)(k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(2k-1)^2}{2(2k+1)(k+1)}\right]$

- Revert the change of variables $u = t - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k(t-1)^k, a_{k+1} = \frac{a_k(2k-1)^2}{2(2k+1)(k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k k^2}{(2k+2)(k+\frac{3}{2})}$

- Solution for $r = \frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{(2k+2)(k+\frac{3}{2})}\right]$

- Revert the change of variables $u = t - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k(t-1)^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{(2k+2)(k+\frac{3}{2})}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(t-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(t-1)^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k(2k-1)^2}{2(2k+1)(k+1)}, b_{k+1} = \frac{2b_k k^2}{(2k+2)(k+\frac{3}{2})}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 54

```
Order:=6;
dsolve(4*(t^2-3*t+2)*diff(y(t),t$2)-2*diff(y(t),t)+y(t)=0,y(t),type='series',t=0);
```

$$y(t) = \left(1-\frac{1}{16}t^2-\frac{7}{192}t^3-\frac{73}{3072}t^4-\frac{1037}{61440}t^5\right)y(0) + \left(t+\frac{1}{8}t^2+\frac{5}{96}t^3+\frac{47}{1536}t^4+\frac{643}{30720}t^5\right)D(y)(0)+O(t^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 70

```
AsymptoticDSolveValue[4*(t^2-3*t+2)*y''[t]-2*y'[t]+y[t]==0,y[t],{t,0,5}]
```

$$y(t) \to c_1\left(-\frac{1037t^5}{61440}-\frac{73t^4}{3072}-\frac{7t^3}{192}-\frac{t^2}{16}+1\right)+c_2\left(\frac{643t^5}{30720}+\frac{47t^4}{1536}+\frac{5t^3}{96}+\frac{t^2}{8}+t\right)$$

2.17 problem 19

2.17.1 Maple step by step solution 340

Internal problem ID [5652]
Internal file name [`OUTPUT/4900_Sunday_June_05_2022_03_09_49_PM_30337475/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2(t^2-5t+6)\,y''+(2t-3)\,y'-8y=0}$$

With the expansion point for the power series method at $t=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (52)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (53)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2ty' - 3y' - 8y}{2\left(t^2 - 5t + 6\right)} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(24t^2 - 104t + 111\right) y' + \left(-48t + 104\right) y}{4\left(t^2 - 5t + 6\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{30\left(-\frac{23}{10} + t\right)\left(\left(t^2 - \frac{13}{3}t + \frac{37}{8}\right) y' - 2\left(t - \frac{13}{6}\right) y\right)}{\left(t^2 - 5t + 6\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{180\left(t^2 - \frac{23}{5}t + \frac{213}{40}\right)\left(\left(t^2 - \frac{13}{3}t + \frac{37}{8}\right) y' - 2\left(t - \frac{13}{6}\right) y\right)}{\left(t^2 - 5t + 6\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{1260\left(t^3 - \frac{69}{10}t^2 + \frac{639}{40}t - \frac{993}{80}\right)\left(\left(t^2 - \frac{13}{3}t + \frac{37}{8}\right) y' - 2\left(t - \frac{13}{6}\right) y\right)}{\left(t^2 - 5t + 6\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{2y(0)}{3} + \frac{y'(0)}{4} \\
F_1 &= \frac{13y(0)}{18} + \frac{37y'(0)}{48} \\
F_2 &= \frac{299y(0)}{216} + \frac{851y'(0)}{576} \\
F_3 &= \frac{923y(0)}{288} + \frac{2627y'(0)}{768} \\
F_4 &= \frac{30121y(0)}{3456} + \frac{85729y'(0)}{9216}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$\begin{aligned} y = &\left(1+\frac{1}{3}t^2+\frac{13}{108}t^3+\frac{299}{5184}t^4+\frac{923}{34560}t^5+\frac{30121}{2488320}t^6\right)y(0) \\ &+\left(t+\frac{1}{8}t^2+\frac{37}{288}t^3+\frac{851}{13824}t^4+\frac{2627}{92160}t^5+\frac{85729}{6635520}t^6\right)y'(0)+O(t^6) \end{aligned}$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$y''\left(2t^2-10t+12\right)+(2t-3)\,y'-8y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n t^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right)\left(2t^2-10t+12\right)+(2t-3)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)-8\left(\sum_{n=0}^{\infty} a_n t^n\right)=0 \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right)+\sum_{n=2}^{\infty}\left(-10n\,t^{n-1}a_n(n-1)\right)+\left(\sum_{n=2}^{\infty} 12n(n-1)\,a_n t^{n-2}\right) \\ &+\left(\sum_{n=1}^{\infty} 2n a_n t^n\right)+\sum_{n=1}^{\infty}\left(-3n a_n t^{n-1}\right)+\sum_{n=0}^{\infty}\left(-8a_n t^n\right)=0 \end{aligned} \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-10n\,t^{n-1}a_n(n-1)\right) = \sum_{n=1}^{\infty}\left(-10(n+1)\,a_{n+1}n\,t^n\right)$$

$$\sum_{n=2}^{\infty}12n(n-1)\,a_n t^{n-2} = \sum_{n=0}^{\infty}12(n+2)\,a_{n+2}(n+1)\,t^n$$

$$\sum_{n=1}^{\infty}\left(-3na_n t^{n-1}\right) = \sum_{n=0}^{\infty}\left(-3(n+1)\,a_{n+1}t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\begin{aligned}\left(\sum_{n=2}^{\infty}2t^n a_n n(n-1)\right) &+ \sum_{n=1}^{\infty}\left(-10(n+1)\,a_{n+1}n\,t^n\right)\\ &+ \left(\sum_{n=0}^{\infty}12(n+2)\,a_{n+2}(n+1)\,t^n\right) + \left(\sum_{n=1}^{\infty}2na_n t^n\right)\\ &+ \sum_{n=0}^{\infty}\left(-3(n+1)\,a_{n+1}t^n\right) + \sum_{n=0}^{\infty}\left(-8a_n t^n\right) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$24a_2 - 3a_1 - 8a_0 = 0$$

$$a_2 = \frac{a_0}{3} + \frac{a_1}{8}$$

$n = 1$ gives

$$-26a_2 + 72a_3 - 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{13a_0}{108} + \frac{37a_1}{288}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) - 10(n+1)\,a_{n+1}n + 12(n+2)\,a_{n+2}(n+1) + 2na_n - 3(n+1)\,a_{n+1} - 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{2n^2 a_n - 10n^2 a_{n+1} - 13n a_{n+1} - 8a_n - 3a_{n+1}}{12\left(n+2\right)\left(n+1\right)} \\
&= -\frac{\left(2n^2-8\right) a_n}{12\left(n+2\right)\left(n+1\right)} - \frac{\left(-10n^2-13n-3\right) a_{n+1}}{12\left(n+2\right)\left(n+1\right)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$-69a_3 + 144a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{299a_0}{5184} + \frac{851a_1}{13824}$$

For $n = 3$ the recurrence equation gives

$$10a_3 - 132a_4 + 240a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{923a_0}{34560} + \frac{2627a_1}{92160}$$

For $n = 4$ the recurrence equation gives

$$24a_4 - 215a_5 + 360a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{30121a_0}{2488320} + \frac{85729a_1}{6635520}$$

For $n = 5$ the recurrence equation gives

$$42a_5 - 318a_6 + 504a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{161603a_0}{29859840} + \frac{459947a_1}{79626240}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y = a_0 + a_1 t + \left(\frac{a_0}{3} + \frac{a_1}{8}\right) t^2 + \left(\frac{13a_0}{108} + \frac{37a_1}{288}\right) t^3 \\
+ \left(\frac{299a_0}{5184} + \frac{851a_1}{13824}\right) t^4 + \left(\frac{923a_0}{34560} + \frac{2627a_1}{92160}\right) t^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = \left(1 + \frac{1}{3}t^2 + \frac{13}{108}t^3 + \frac{299}{5184}t^4 + \frac{923}{34560}t^5\right) a_0 \\
+ \left(t + \frac{1}{8}t^2 + \frac{37}{288}t^3 + \frac{851}{13824}t^4 + \frac{2627}{92160}t^5\right) a_1 + O(t^6)
\end{aligned}
\tag{3}
$$

At $t = 0$ the solution above becomes

$$
\begin{aligned}
y = \left(1 + \frac{1}{3}t^2 + \frac{13}{108}t^3 + \frac{299}{5184}t^4 + \frac{923}{34560}t^5\right) c_1 \\
+ \left(t + \frac{1}{8}t^2 + \frac{37}{288}t^3 + \frac{851}{13824}t^4 + \frac{2627}{92160}t^5\right) c_2 + O(t^6)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = \left(1 + \frac{1}{3}t^2 + \frac{13}{108}t^3 + \frac{299}{5184}t^4 + \frac{923}{34560}t^5 + \frac{30121}{2488320}t^6\right) y(0) \\
+ \left(t + \frac{1}{8}t^2 + \frac{37}{288}t^3 + \frac{851}{13824}t^4 + \frac{2627}{92160}t^5 + \frac{85729}{6635520}t^6\right) y'(0) + O(t^6)
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
y = \left(1 + \frac{1}{3}t^2 + \frac{13}{108}t^3 + \frac{299}{5184}t^4 + \frac{923}{34560}t^5\right) c_1 \\
+ \left(t + \frac{1}{8}t^2 + \frac{37}{288}t^3 + \frac{851}{13824}t^4 + \frac{2627}{92160}t^5\right) c_2 + O(t^6)
\end{aligned}
\tag{2}
$$

Verification of solutions

$$y = \left(1+\frac{1}{3}t^2+\frac{13}{108}t^3+\frac{299}{5184}t^4+\frac{923}{34560}t^5+\frac{30121}{2488320}t^6\right)y(0) + \left(t+\frac{1}{8}t^2+\frac{37}{288}t^3+\frac{851}{13824}t^4+\frac{2627}{92160}t^5+\frac{85729}{6635520}t^6\right)y'(0)+O(t^6)$$

Verified OK.

$$y = \left(1+\frac{1}{3}t^2+\frac{13}{108}t^3+\frac{299}{5184}t^4+\frac{923}{34560}t^5\right)c_1 + \left(t+\frac{1}{8}t^2+\frac{37}{288}t^3+\frac{851}{13824}t^4+\frac{2627}{92160}t^5\right)c_2+O(t^6)$$

Verified OK.

2.17.1 Maple step by step solution

Let's solve

$y''(2t^2-10t+12)+(2t-3)\,y'-8y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{4y}{t^2-5t+6}-\frac{(2t-3)y'}{2(t^2-5t+6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2t-3)y'}{2(t^2-5t+6)}-\frac{4y}{t^2-5t+6}=0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t)=\frac{2t-3}{2(t^2-5t+6)}, P_3(t)=-\frac{4}{t^2-5t+6}\right]$

  ◦ $(-2+t)\cdot P_2(t)$ is analytic at $t=2$

  $\left((-2+t)\cdot P_2(t)\right)\Big|_{t=2}=-\frac{1}{2}$

  ◦ $(-2+t)^2\cdot P_3(t)$ is analytic at $t=2$

  $\left((-2+t)^2\cdot P_3(t)\right)\Big|_{t=2}=0$

  ◦ $t=2$is a regular singular point

Check to see if t_0 is a regular singular point

$t_0 = 2$

- Multiply by denominators

  $y''(2t^2 - 10t + 12) + (2t - 3)\, y' - 8y = 0$

- Change variables using $t = u + 2$ so that the regular singular point is at $u = 0$

  $(2u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u + 1)\left(\frac{d}{du}y(u)\right) - 8y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\, u^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, u^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3 + 2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k + 1 + r)\,(2k - 1 + 2r) + 2a_k(k + r + 2)\,(k + r - 2)\right) u^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, \frac{3}{2}\}$

- Each term in the series must be 0, giving the recursion relation

  $-2\left(k + r - \frac{1}{2}\right)(k + 1 + r)\, a_{k+1} + 2a_k(k + r + 2)\,(k + r - 2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_k(k+r+2)(k+r-2)}{(2k-1+2r)(k+1+r)}$

- Recursion relation for $r=0$; series terminates at $k=2$

  $a_{k+1}=\frac{2a_k(k+2)(k-2)}{(2k-1)(k+1)}$

- Apply recursion relation for $k=0$

  $a_1=8a_0$

- Apply recursion relation for $k=1$

  $a_2=-3a_1$

- Express in terms of a_0

  $a_2=-24a_0$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot(-24u^2+8u+1)$

- Revert the change of variables $u=-2+t$

  $[y=a_0(-24t^2+104t-111)]$

- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1}=\frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}$

- Solution for $r=\frac{3}{2}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{3}{2}},a_{k+1}=\frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

- Revert the change of variables $u=-2+t$

  $\left[y=\sum_{k=0}^{\infty}a_k(-2+t)^{k+\frac{3}{2}},a_{k+1}=\frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=a_0(-24t^2+104t-111)+\left(\sum_{k=0}^{\infty}b_k(-2+t)^{k+\frac{3}{2}}\right),b_{k+1}=\frac{2b_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve(2*(t^2-5*t+6)*diff(y(t),t$2)+(2*t-3)*diff(y(t),t)-8*y(t)=0,y(t),type='series',t=0);
```

$$y(t) = \left(1+\frac{1}{3}t^2+\frac{13}{108}t^3+\frac{299}{5184}t^4+\frac{923}{34560}t^5\right)y(0) + \left(t+\frac{1}{8}t^2+\frac{37}{288}t^3+\frac{851}{13824}t^4+\frac{2627}{92160}t^5\right)D(y)(0)+O(t^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 70

```
AsymptoticDSolveValue[2*(t^2-5*t+6)*y''[t]+(2*t-3)*y'[t]-8*y[t]==0,y[t],{t,0,5}]
```

$$y(t) \to c_1\left(\frac{923t^5}{34560}+\frac{299t^4}{5184}+\frac{13t^3}{108}+\frac{t^2}{3}+1\right)+c_2\left(\frac{2627t^5}{92160}+\frac{851t^4}{13824}+\frac{37t^3}{288}+\frac{t^2}{8}+t\right)$$

2.18 problem 20

2.18.1 Maple step by step solution 356

Internal problem ID [5653]
Internal file name [`OUTPUT/4901_Sunday_June_05_2022_03_09_50_PM_22191277/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.3. Extended Power Series Method: Frobenius Method page 186
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3t(t+1)\,y''+ty'-y=0}$$

With the expansion point for the power series method at $t=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(3t^2+3t\right)y''+ty'-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(t)y'+q(t)y=0$$

Where

$$p(t)=\frac{1}{3t+3}$$
$$q(t)=-\frac{1}{3t\,(t+1)}$$

Table 42: Table $p(t), q(t)$ singularites.

$p(t) = \frac{1}{3t+3}$	
singularity	type
$t = -1$	"regular"

$q(t) = -\frac{1}{3t(t+1)}$	
singularity	type
$t = -1$	"regular"
$t = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $t = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3t(t+1)\, y'' + ty' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n t^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}$$

Substituting the above back into the ode gives

$$3t(t+1) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}\right) + t\left(\sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n t^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3t^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3t^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} t^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n t^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of t be $n+r-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3t^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)(n+r-2)\, t^{n+r-1}$$

$$\sum_{n=0}^{\infty} t^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, t^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-a_n t^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} t^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)(n+r-2)\, t^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3t^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, t^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} t^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3t^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$3t^{-1+r} a_0 r(-1+r) = 0$$

Or

$$3t^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$3t^{-1+r}r(-1+r) = 0$$

Since the above is true for all t then the indicial equation becomes

$$3r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$3t^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(t) = t^{r_1}\left(\sum_{n=0}^{\infty} a_n t^n\right)$$
$$y_2(t) = Cy_1(t)\ln(t) + t^{r_2}\left(\sum_{n=0}^{\infty} b_n t^n\right)$$

Or

$$y_1(t) = t\left(\sum_{n=0}^{\infty} a_n t^n\right)$$
$$y_2(t) = Cy_1(t)\ln(t) + \left(\sum_{n=0}^{\infty} b_n t^n\right)$$

Or

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^{n+1}$$
$$y_2(t) = Cy_1(t)\ln(t) + \left(\sum_{n=0}^{\infty} b_n t^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$3a_{n-1}(n+r-1)(n+r-2) + 3a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(3n^2 + 6nr + 3r^2 - 8n - 8r + 4)}{3(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}(3n^2 - 2n - 1)}{3(n+1)n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-3r^2 + 2r + 1}{3(1+r)r}$$

Which for the root $r = 1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+2r+1}{3(1+r)r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9r^3 + 6r^2 - 11r - 4}{9(1+r)^2(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+2r+1}{3(1+r)r}$	0
a_2	$\frac{9r^3+6r^2-11r-4}{9(1+r)^2(2+r)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-27r^4 - 81r^3 - 9r^2 + 89r + 28}{27(3+r)(2+r)^2(1+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+2r+1}{3(1+r)r}$	0
a_2	$\frac{9r^3+6r^2-11r-4}{9(1+r)^2(2+r)}$	0
a_3	$\frac{-27r^4-81r^3-9r^2+89r+28}{27(3+r)(2+r)^2(1+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r^5 + 513r^4 + 837r^3 - 177r^2 - 974r - 280}{81(4+r)(1+r)(2+r)(3+r)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+2r+1}{3(1+r)r}$	0
a_2	$\frac{9r^3+6r^2-11r-4}{9(1+r)^2(2+r)}$	0
a_3	$\frac{-27r^4-81r^3-9r^2+89r+28}{27(3+r)(2+r)^2(1+r)}$	0
a_4	$\frac{81r^5+513r^4+837r^3-177r^2-974r-280}{81(4+r)(1+r)(2+r)(3+r)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-243r^6 - 2592r^5 - 9180r^4 - 10350r^3 + 5223r^2 + 13502r + 3640}{243 (5+r) (3+r) (2+r) (1+r) (4+r)^2}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+2r+1}{3(1+r)r}$	0
a_2	$\frac{9r^3+6r^2-11r-4}{9(1+r)^2(2+r)}$	0
a_3	$\frac{-27r^4-81r^3-9r^2+89r+28}{27(3+r)(2+r)^2(1+r)}$	0
a_4	$\frac{81r^5+513r^4+837r^3-177r^2-974r-280}{81(4+r)(1+r)(2+r)(3+r)^2}$	0
a_5	$\frac{-243r^6-2592r^5-9180r^4-10350r^3+5223r^2+13502r+3640}{243(5+r)(3+r)(2+r)(1+r)(4+r)^2}$	0

Using the above table, then the solution $y_1(t)$ is

$$\begin{aligned} y_1(t) &= t\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 \dots\right) \\ &= t\left(1 + O(t^6)\right) \end{aligned}$$

Now the second solution $y_2(t)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-3r^2+2r+1}{3(1+r)r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{-3r^2+2r+1}{3(1+r)r} &= \lim_{r\to 0} \frac{-3r^2+2r+1}{3(1+r)r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(t) = Cy_1(t)\ln(t) + \left(\sum_{n=0}^{\infty} b_n t^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dt} y_2(t) &= Cy_1'(t)\ln(t) + \frac{Cy_1(t)}{t} + \left(\sum_{n=0}^{\infty} \frac{b_n t^{n+r_2}(n+r_2)}{t}\right) \\ &= Cy_1'(t)\ln(t) + \frac{Cy_1(t)}{t} + \left(\sum_{n=0}^{\infty} t^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dt^2} y_2(t) &= Cy_1''(t)\ln(t) + \frac{2Cy_1'(t)}{t} - \frac{Cy_1(t)}{t^2} + \sum_{n=0}^{\infty} \left(\frac{b_n t^{n+r_2}(n+r_2)^2}{t^2} - \frac{b_n t^{n+r_2}(n+r_2)}{t^2}\right) \\ &= Cy_1''(t)\ln(t) + \frac{2Cy_1'(t)}{t} - \frac{Cy_1(t)}{t^2} + \left(\sum_{n=0}^{\infty} t^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $3t(t+1)\,y'' + ty' - y = 0$ gives

$$\begin{aligned} &3t(t+1)\left(Cy_1''(t)\ln(t) + \frac{2Cy_1'(t)}{t} - \frac{Cy_1(t)}{t^2} \right. \\ &\left. + \sum_{n=0}^{\infty} \left(\frac{b_n t^{n+r_2}(n+r_2)^2}{t^2} - \frac{b_n t^{n+r_2}(n+r_2)}{t^2}\right)\right) \\ &+ t\left(Cy_1'(t)\ln(t) + \frac{Cy_1(t)}{t} + \left(\sum_{n=0}^{\infty} \frac{b_n t^{n+r_2}(n+r_2)}{t}\right)\right) \\ &- Cy_1(t)\ln(t) - \left(\sum_{n=0}^{\infty} b_n t^{n+r_2}\right) = 0 \end{aligned}$$

Which can be written as

$$\left(\left(3t(t+1)\, y_1''(t)+y_1'(t)\, t-y_1(t)\right) \ln(t)+3t(t+1) \left(\frac{2y_1'(t)}{t}-\frac{y_1(t)}{t^2}\right)+y_1(t)\right) C \\ + 3t(t+1) \left(\sum_{n=0}^{\infty} \left(\frac{b_n t^{n+r_2} (n+r_2)^2}{t^2} - \frac{b_n t^{n+r_2} (n+r_2)}{t^2}\right)\right) \\ + t \left(\sum_{n=0}^{\infty} \frac{b_n t^{n+r_2} (n+r_2)}{t}\right) - \left(\sum_{n=0}^{\infty} b_n t^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(t)$ is a solution to the ode, then

$$3t(t+1)\, y_1''(t) + y_1'(t)\, t - y_1(t) = 0$$

Eq (7) simplifes to

$$\left(3t(t+1) \left(\frac{2y_1'(t)}{t}-\frac{y_1(t)}{t^2}\right)+y_1(t)\right) C \\ + 3t(t+1) \left(\sum_{n=0}^{\infty} \left(\frac{b_n t^{n+r_2} (n+r_2)^2}{t^2} - \frac{b_n t^{n+r_2} (n+r_2)}{t^2}\right)\right) \\ + t \left(\sum_{n=0}^{\infty} \frac{b_n t^{n+r_2} (n+r_2)}{t}\right) - \left(\sum_{n=0}^{\infty} b_n t^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n t^{n+r_1}$ into the above gives

$$\frac{\left(6t(t+1) \left(\sum_{n=0}^{\infty} t^{-1+n+r_1} a_n (n+r_1)\right) + (-2t-3) \left(\sum_{n=0}^{\infty} a_n t^{n+r_1}\right)\right) C}{t} \\ + \frac{3(t^3+t^2) \left(\sum_{n=0}^{\infty} t^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) + \left(\sum_{n=0}^{\infty} t^{-1+n+r_2} b_n (n+r_2)\right) t^2 - \left(\sum_{n=0}^{\infty} b_n t^{n+r_2}\right) t}{t} \\ = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(6t(t+1) \left(\sum_{n=0}^{\infty} t^n a_n (n+1)\right) + (-2t-3) \left(\sum_{n=0}^{\infty} a_n t^{n+1}\right)\right) C}{t} \\ + \frac{3(t^3+t^2) \left(\sum_{n=0}^{\infty} t^{-2+n} b_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} t^{n-1} b_n n\right) t^2 - \left(\sum_{n=0}^{\infty} b_n t^n\right) t}{t} = 0 \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 6C\, t^{n+1} a_n (n+1)\right) + \left(\sum_{n=0}^{\infty} 6C\, t^{n} a_n (n+1)\right) \\
&+ \sum_{n\,=0}^{\infty} \left(-2C\, t^{n+1} a_n\right) + \sum_{n\,=0}^{\infty} \left(-3C a_n t^n\right) + \left(\sum_{n=0}^{\infty} 3t^n b_n n(n-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 3n\, t^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} t^n b_n n\right) + \sum_{n\,=0}^{\infty} \left(-b_n t^n\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of t be $n-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n\,=0}^{\infty} 6C\, t^{n+1} a_n (n+1) &= \sum_{n=2}^{\infty} 6C a_{-2+n} (n-1)\, t^{n-1} \\
\sum_{n\,=0}^{\infty} 6C\, t^{n} a_n (n+1) &= \sum_{n=1}^{\infty} 6C a_{n-1} n\, t^{n-1} \\
\sum_{n\,=0}^{\infty} \left(-2C\, t^{n+1} a_n\right) &= \sum_{n=2}^{\infty} \left(-2C a_{-2+n} t^{n-1}\right) \\
\sum_{n\,=0}^{\infty} \left(-3C a_n t^n\right) &= \sum_{n=1}^{\infty} \left(-3C a_{n-1} t^{n-1}\right) \\
\sum_{n\,=0}^{\infty} 3t^n b_n n(n-1) &= \sum_{n=1}^{\infty} 3(n-1)\, b_{n-1} (-2+n)\, t^{n-1} \\
\sum_{n\,=0}^{\infty} t^n b_n n &= \sum_{n=1}^{\infty} (n-1)\, b_{n-1} t^{n-1} \\
\sum_{n\,=0}^{\infty} \left(-b_n t^n\right) &= \sum_{n=1}^{\infty} \left(-b_{n-1} t^{n-1}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n-1$.

$$\left(\sum_{n=2}^{\infty} 6C a_{-2+n} (n-1)\, t^{n-1}\right) + \left(\sum_{n=1}^{\infty} 6C a_{n-1} n\, t^{n-1}\right) \\ + \sum_{n=2}^{\infty} \left(-2C a_{-2+n} t^{n-1}\right) + \sum_{n=1}^{\infty} \left(-3C a_{n-1} t^{n-1}\right) \\ + \left(\sum_{n=1}^{\infty} 3(n-1)\, b_{n-1}(-2+n)\, t^{n-1}\right) + \left(\sum_{n=0}^{\infty} 3n\, t^{n-1} b_n (n-1)\right) \\ + \left(\sum_{n=1}^{\infty} (n-1)\, b_{n-1} t^{n-1}\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} t^{n-1}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$3C - 1 = 0$$

Which is solved for C. Solving for C gives

$$C = \frac{1}{3}$$

For $n=2$, Eq (2B) gives

$$(4a_0 + 9a_1)\, C + 6b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{4}{3} + 6b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{2}{9}$$

For $n=3$, Eq (2B) gives

$$(10a_1 + 15a_2)\, C + 7b_2 + 18b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{14}{9} + 18b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{7}{81}$$

For $n = 4$, Eq (2B) gives

$$(16a_2 + 21a_3)\, C + 20b_3 + 36b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{140}{81} + 36b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{35}{729}$$

For $n = 5$, Eq (2B) gives

$$(22a_3 + 27a_4)\, C + 39b_4 + 60b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{455}{243} + 60b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{91}{2916}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(t) = C y_1(t) \ln(t) + \left(\sum_{n=0}^{\infty} b_n t^{n+r_2}\right)$$

Using the above value found for $C = \frac{1}{3}$ and all b_n, then the second solution becomes

$$y_2(t) = \frac{1}{3}\left(t\left(1 + O(t^6)\right)\right) \ln(t) + 1 - \frac{2t^2}{9} + \frac{7t^3}{81} - \frac{35t^4}{729} + \frac{91t^5}{2916} + O(t^6)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(t) &= c_1 y_1(t) + c_2 y_2(t) \\ &= c_1 t\left(1+O(t^6)\right) + c_2\left(\frac{1}{3}\left(t\left(1+O(t^6)\right)\right) \ln(t) + 1 - \frac{2t^2}{9} + \frac{7t^3}{81} - \frac{35t^4}{729} + \frac{91t^5}{2916} + O(t^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 t\left(1 + O(t^6)\right) + c_2\left(\frac{t\left(1 + O(t^6)\right) \ln(t)}{3} + 1 - \frac{2t^2}{9} + \frac{7t^3}{81} - \frac{35t^4}{729} + \frac{91t^5}{2916} + O(t^6)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 t\left(1+O\left(t^6\right)\right)+c_2\left(\frac{t\left(1+O\left(t^6\right)\right)\ln\left(t\right)}{3}+1-\frac{2t^2}{9}+\frac{7t^3}{81}-\frac{35t^4}{729}+\frac{91t^5}{2916}+O\left(t^6\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 t\left(1+O\left(t^6\right)\right)+c_2\left(\frac{t\left(1+O\left(t^6\right)\right)\ln\left(t\right)}{3}+1-\frac{2t^2}{9}+\frac{7t^3}{81}-\frac{35t^4}{729}+\frac{91t^5}{2916}+O\left(t^6\right)\right)$$

Verified OK.

2.18.1 Maple step by step solution

Let's solve

$3t(t+1)\,y''+ty'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{3t(t+1)}-\frac{y'}{3(t+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{3(t+1)}-\frac{y}{3t(t+1)}=0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(t)=\frac{1}{3(t+1)},P_3(t)=-\frac{1}{3t(t+1)}\right]$

  ◦ $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

    $\left((t+1)\cdot P_2(t)\right)\Big|_{t=-1}=\frac{1}{3}$

  ◦ $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

    $\left((t+1)^2\cdot P_3(t)\right)\Big|_{t=-1}=0$

  ◦ $t=-1$is a regular singular point

    Check to see if t_0 is a regular singular point

    $t_0=-1$

- Multiply by denominators

  $3t(t+1)\,y''+ty'-y=0$
- Change variables using $t=u-1$ so that the regular singular point is at $u=0$

  $(3u^2-3u)\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)-y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(-2+3r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(3k+3r+1)+a_k(3k+3r+1)\,(k+r-1)\right)u^{k-}\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-2+3r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{2}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $3((-k-r-1)\,a_{k+1}+a_k(k+r-1))\left(k+r+\frac{1}{3}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-1)}{k+1+r}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)}{k+1}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u + 1)$

- Revert the change of variables $u = t + 1$

  $[y = -a_0 t]$

- Recursion relation for $r = \frac{2}{3}$

  $a_{k+1} = \frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}$

- Solution for $r = \frac{2}{3}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+1} = \frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$

- Revert the change of variables $u = t + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (t+1)^{k+\frac{2}{3}}, a_{k+1} = \frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$

- Combine solutions and rename parameters

  $\left[y = -a_0 t + \left(\sum_{k=0}^{\infty} b_k (t+1)^{k+\frac{2}{3}}\right), b_{k+1} = \frac{b_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
Order:=6;
dsolve(3*t*(1+t)*diff(y(t),t$2)+t*diff(y(t),t)-y(t)=0,y(t),type='series',t=0);
```

$$y(t) = c_1 t\left(1+\mathrm{O}\left(t^6\right)\right)+\left(\frac{1}{3}t+\mathrm{O}\left(t^6\right)\right)\ln\left(t\right)c_2 \\ +\left(1-\frac{1}{3}t-\frac{2}{9}t^2+\frac{7}{81}t^3-\frac{35}{729}t^4+\frac{91}{2916}t^5+\mathrm{O}\left(t^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 43

```
AsymptoticDSolveValue[3*t*(1+t)*y''[t]+t*y'[t]-y[t]==0,y[t],{t,0,5}]
```

$$y(t) \to c_1 \left(\frac{1}{729} \left(-35t^4 + 63t^3 - 162t^2 + 243t + 729 \right) + \frac{1}{3} t \log(t) \right) + c_2 t$$

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3.1 problem 2

3.1.1 Maple step by step solution 371

Internal problem ID [5654]
Internal file name [`OUTPUT/4902_Sunday_June_05_2022_03_09_54_PM_25170467/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.4. Bessels Equation page 195
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{4}{49}\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 - \frac{4}{49}\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{49x^2 - 4}{49x^2}$$

Table 44: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{49x^2-4}{49x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 - \frac{4}{49}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(x^2 - \frac{4}{49}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{gathered}
\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \\
+\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-\frac{4 a_n x^{n+r}}{49}\right)=0
\end{gathered} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_n=\sum_{n=2}^{\infty} a_{n-2} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{gathered}
\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \\
+\left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-\frac{4 a_n x^{n+r}}{49}\right)=0
\end{gathered} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n(n+r)(n+r-1)+x^{n+r} a_n(n+r)-\frac{4 a_n x^{n+r}}{49}=0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r)+x^r a_0 r-\frac{4 a_0 x^r}{49}=0
$$

Or

$$
\left(x^r r(-1+r)+x^r r-\frac{4 x^r}{49}\right) a_0=0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\frac{\left(49 r^2-4\right) x^r}{49}=0
$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{4}{49} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{2}{7}$$
$$r_2 = -\frac{2}{7}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(49r^2 - 4)\, x^r}{49} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{7}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{7}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{2}{7}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - \frac{4a_n}{49} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{49a_{n-2}}{49n^2 + 98nr + 49r^2 - 4} \tag{4}$$

Which for the root $r = \frac{2}{7}$ becomes

$$a_n = -\frac{7a_{n-2}}{n(7n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{7}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{49}{49r^2 + 196r + 192}$$

Which for the root $r = \frac{2}{7}$ becomes

$$a_2 = -\frac{7}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{49}{49r^2+196r+192}$	$-\frac{7}{36}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{49}{49r^2+196r+192}$	$-\frac{7}{36}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2401}{(49r^2 + 196r + 192)(49r^2 + 392r + 780)}$$

Which for the root $r = \frac{2}{7}$ becomes

$$a_4 = \frac{49}{4608}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{49}{49r^2+196r+192}$	$-\frac{7}{36}$
a_3	0	0
a_4	$\frac{2401}{(49r^2+196r+192)(49r^2+392r+780)}$	$\frac{49}{4608}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{49}{49r^2+196r+192}$	$-\frac{7}{36}$
a_3	0	0
a_4	$\frac{2401}{(49r^2+196r+192)(49r^2+392r+780)}$	$\frac{49}{4608}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{2}{7}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{2}{7}}\left(1 - \frac{7x^2}{36} + \frac{49x^4}{4608} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - \frac{4b_n}{49} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{49b_{n-2}}{49n^2 + 98nr + 49r^2 - 4} \tag{4}$$

Which for the root $r = -\frac{2}{7}$ becomes

$$b_n = -\frac{7b_{n-2}}{n(7n-4)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{2}{7}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{49}{49r^2 + 196r + 192}$$

Which for the root $r = -\frac{2}{7}$ becomes

$$b_2 = -\frac{7}{20}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{49}{49r^2+196r+192}$	$-\frac{7}{20}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{49}{49r^2+196r+192}$	$-\frac{7}{20}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{2401}{(49r^2 + 196r + 192)(49r^2 + 392r + 780)}$$

Which for the root $r = -\frac{2}{7}$ becomes

$$b_4 = \frac{49}{1920}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{49}{49r^2+196r+192}$	$-\frac{7}{20}$
b_3	0	0
b_4	$\frac{2401}{(49r^2+196r+192)(49r^2+392r+780)}$	$\frac{49}{1920}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{49}{49r^2+196r+192}$	$-\frac{7}{20}$
b_3	0	0
b_4	$\frac{2401}{(49r^2+196r+192)(49r^2+392r+780)}$	$\frac{49}{1920}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{2}{7}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{7x^2}{20} + \frac{49x^4}{1920} + O(x^6)}{x^{\frac{2}{7}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{2}{7}}\left(1 - \frac{7x^2}{36} + \frac{49x^4}{4608} + O(x^6)\right) + \frac{c_2\left(1 - \frac{7x^2}{20} + \frac{49x^4}{1920} + O(x^6)\right)}{x^{\frac{2}{7}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{2}{7}}\left(1 - \frac{7x^2}{36} + \frac{49x^4}{4608} + O(x^6)\right) + \frac{c_2\left(1 - \frac{7x^2}{20} + \frac{49x^4}{1920} + O(x^6)\right)}{x^{\frac{2}{7}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{2}{7}}\left(1 - \frac{7x^2}{36} + \frac{49x^4}{4608} + O(x^6)\right) + \frac{c_2\left(1 - \frac{7x^2}{20} + \frac{49x^4}{1920} + O(x^6)\right)}{x^{\frac{2}{7}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{2}{7}}\left(1 - \frac{7x^2}{36} + \frac{49x^4}{4608} + O(x^6)\right) + \frac{c_2\left(1 - \frac{7x^2}{20} + \frac{49x^4}{1920} + O(x^6)\right)}{x^{\frac{2}{7}}}
$$

Verified OK.

3.1.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 - \frac{4}{49}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(49x^2-4)y}{49x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(49x^2-4)y}{49x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{49x^2-4}{49x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{4}{49}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $49x^2y'' + 49xy' + (49x^2 - 4) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+7r)\,(-2+7r)\, x^r + a_1(9+7r)\,(5+7r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(7k+7r+2)\,(7k+7r-2) + 49a_k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+7r)\,(-2+7r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{2}{7}, \frac{2}{7}\right\}$

- Each term must be 0

  $a_1(9+7r)\,(5+7r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(7k+7r+2)\,(7k+7r-2) + 49a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(7k+16+7r)\,(7k+12+7r) + 49a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{49a_k}{(7k+16+7r)(7k+12+7r)}$

- Recursion relation for $r = -\frac{2}{7}$

  $a_{k+2} = -\frac{49a_k}{(7k+14)(7k+10)}$

- Solution for $r = -\frac{2}{7}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{2}{7}}, a_{k+2} = -\frac{49a_k}{(7k+14)(7k+10)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{2}{7}$

  $a_{k+2} = -\frac{49a_k}{(7k+18)(7k+14)}$
- Solution for $r = \frac{2}{7}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{2}{7}}, a_{k+2} = -\frac{49a_k}{(7k+18)(7k+14)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{2}{7}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{7}}\right), a_{k+2} = -\frac{49a_k}{(7k+14)(7k+10)}, a_1 = 0, b_{k+2} = -\frac{49b_k}{(7k+18)(7k+14)}, b_1 =\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-4/49)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{4}{7}} \left(1 - \frac{7}{36}x^2 + \frac{49}{4608}x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 - \frac{7}{20}x^2 + \frac{49}{1920}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^{\frac{2}{7}}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-4/49)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x^{2/7} \left(\frac{49x^4}{4608} - \frac{7x^2}{36} + 1 \right) + \frac{c_2 \left(\frac{49x^4}{1920} - \frac{7x^2}{20} + 1 \right)}{x^{2/7}}$$

3.2 problem 3

3.2.1 Maple step by step solution 383

Internal problem ID [5655]
Internal file name [`OUTPUT/4903_Sunday_June_05_2022_03_09_57_PM_70570903/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.4. Bessels Equation page 195
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' + \frac{y}{4} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + \frac{y}{4} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{1}{4x}$$

Table 46: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{4x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + \frac{y}{4} = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \frac{\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)}{4} = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} \frac{a_n x^{n+r}}{4}\right) = 0 \quad (2A)$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \frac{a_n x^{n+r}}{4} = \sum_{n=1}^{\infty} \frac{a_{n-1} x^{n+r-1}}{4}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} \frac{a_{n-1} x^{n+r-1}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + \frac{a_{n-1}}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{4(n^2 + 2nr + r^2)} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}}{4n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{4(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{16\left(r+1\right)^2\left(r+2\right)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{1}{16(r+1)^2(r+2)^2}$	$\frac{1}{64}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{64\left(r+1\right)^2\left(r+2\right)^2\left(r+3\right)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{2304}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{1}{16(r+1)^2(r+2)^2}$	$\frac{1}{64}$
a_3	$-\frac{1}{64(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{2304}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{256 (r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{147456}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{1}{16(r+1)^2(r+2)^2}$	$\frac{1}{64}$
a_3	$-\frac{1}{64(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{2304}$
a_4	$\frac{1}{256(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{147456}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{1024 (r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{14745600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{1}{16(r+1)^2(r+2)^2}$	$\frac{1}{64}$
a_3	$-\frac{1}{64(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{2304}$
a_4	$\frac{1}{256(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{147456}$
a_5	$-\frac{1}{1024(r+1)^2(r+2)^2(r+3)^2(4+r)^2(r+5)^2}$	$-\frac{1}{14745600}$

Using the above table, then the first solution $y_1(x)$ becomes

$$y_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots$$
$$= 1 - \frac{x}{4} + \frac{x^2}{64} - \frac{x^3}{2304} + \frac{x^4}{147456} - \frac{x^5}{14745600} + O(x^6)$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$	$\frac{1}{2(r+1)^3}$	$\frac{1}{2}$
b_2	$\frac{1}{16(r+1)^2(r+2)^2}$	$\frac{1}{64}$	$\frac{-2r-3}{8(r+1)^3(r+2)^3}$	$-\frac{3}{64}$
b_3	$-\frac{1}{64(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{2304}$	$\frac{3r^2+12r+11}{32(r+1)^3(r+2)^3(r+3)^3}$	$\frac{11}{6912}$
b_4	$\frac{1}{256(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{147456}$	$\frac{-2r^3-15r^2-35r-25}{64(r+1)^3(r+2)^3(r+3)^3(4+r)^3}$	$-\frac{25}{884736}$
b_5	$-\frac{1}{1024(r+1)^2(r+2)^2(r+3)^2(4+r)^2(r+5)^2}$	$-\frac{1}{14745600}$	$\frac{5r^4+60r^3+255r^2+450r+274}{512(r+1)^3(r+2)^3(r+3)^3(4+r)^3(r+5)^3}$	$\frac{137}{442368000}$

The above table gives all values of b_n needed. Hence the second solution is

$$y_2(x) = y_1(x) \ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots$$
$$= \left(1 - \frac{x}{4} + \frac{x^2}{64} - \frac{x^3}{2304} + \frac{x^4}{147456} - \frac{x^5}{14745600} + O(x^6)\right) \ln(x)$$
$$+ \frac{x}{2} - \frac{3x^2}{64} + \frac{11x^3}{6912} - \frac{25x^4}{884736} + \frac{137x^5}{442368000} + O(x^6)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right) \\
&+ c_2\left(\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right)\ln(x)+\frac{x}{2}-\frac{3x^2}{64}\right. \\
&\left.+\frac{11x^3}{6912}-\frac{25x^4}{884736}+\frac{137x^5}{442368000}+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right) \\
&+ c_2\left(\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right)\ln(x)+\frac{x}{2}-\frac{3x^2}{64}\right. \\
&\left.+\frac{11x^3}{6912}-\frac{25x^4}{884736}+\frac{137x^5}{442368000}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right) \\
&+ c_2\left(\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right)\ln(x)+\frac{x}{2}-\frac{3x^2}{64}\right. \\
&\left.+\frac{11x^3}{6912}-\frac{25x^4}{884736}+\frac{137x^5}{442368000}+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right) \\
&+ c_2\left(\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right)\ln(x)+\frac{x}{2}-\frac{3x^2}{64}\right. \\
&\left.+\frac{11x^3}{6912}-\frac{25x^4}{884736}+\frac{137x^5}{442368000}+O(x^6)\right)
\end{aligned}
$$

Verified OK.

3.2.1 Maple step by step solution

Let's solve

$y''x + y' + \frac{y}{4} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{y}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{y}{4x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{1}{4x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4y''x + y + 4y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r^2x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(4a_{k+1}(k+1+r)^2 + a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4a_{k+1}(k+1)^2 + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{4(k+1)^2}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{4(k+1)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{4(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+1/4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - \frac{1}{4}x + \frac{1}{64}x^2 - \frac{1}{2304}x^3 + \frac{1}{147456}x^4 - \frac{1}{14745600}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{2}x - \frac{3}{64}x^2 + \frac{11}{6912}x^3 - \frac{25}{884736}x^4 + \frac{137}{442368000}x^5 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 117

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+1/4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{14745600} + \frac{x^4}{147456} - \frac{x^3}{2304} + \frac{x^2}{64} - \frac{x}{4} + 1\right) + c_2\left(\frac{137x^5}{442368000} - \frac{25x^4}{884736} + \frac{11x^3}{6912} - \frac{3x^2}{64} + \left(-\frac{x^5}{14745600} + \frac{x^4}{147456} - \frac{x^3}{2304} + \frac{x^2}{64} - \frac{x}{4} + 1\right)\log(x) + \frac{x}{2}\right)$$

3.3 problem 4

Internal problem ID [5656]
Internal file name [`OUTPUT/4904_Sunday_June_05_2022_03_09_59_PM_47996665/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.4. Bessels Equation page 195
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode_form_A", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(e^{-2x} - \frac{1}{9}\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f\big|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'\big|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (58) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (59) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{(9\,\mathrm{e}^{-2x}-1)\,y}{9} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= (-y'+2y)\,\mathrm{e}^{-2x}+\frac{y'}{9} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{2(-19y+18y')\,\mathrm{e}^{-2x}}{9}+y\,\mathrm{e}^{-4x}+\frac{y}{81} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= (-8y+y')\,\mathrm{e}^{-4x}+\frac{10(8y-11y')\,\mathrm{e}^{-2x}}{9}+\frac{y'}{81} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{(133y-36y')\,\mathrm{e}^{-4x}}{3}+\frac{(-517y+900y')\,\mathrm{e}^{-2x}}{27}-y\,\mathrm{e}^{-6x}+\frac{y}{729}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{8y(0)}{9} \\
F_1 &= 2y(0)-\frac{8y'(0)}{9} \\
F_2 &= -\frac{260y(0)}{81}+4y'(0) \\
F_3 &= \frac{8y(0)}{9}-\frac{908y'(0)}{81} \\
F_4 &= \frac{17632y(0)}{729}+\frac{64y'(0)}{3}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{4}{9}x^{2}+\frac{1}{3}x^{3}-\frac{65}{486}x^{4}+\frac{1}{135}x^{5}+\frac{1102}{32805}x^{6}\right) y(0) \\
&+\left(x-\frac{4}{27}x^{3}+\frac{1}{6}x^{4}-\frac{227}{2430}x^{5}+\frac{4}{135}x^{6}\right) y'(0)+O(x^{6})
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
9y''\mathrm{e}^{2x}+\left(-\mathrm{e}^{2x}+9\right) y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
9\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \mathrm{e}^{2x}+\left(-\mathrm{e}^{2x}+9\right) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Expanding $9\,\mathrm{e}^{2x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
9\,\mathrm{e}^{2x} &= 9+18x+18x^{2}+12x^{3}+6x^{4}+\frac{12}{5}x^{5}+\frac{4}{5}x^{6}+\dots \\
&= 9+18x+18x^{2}+12x^{3}+6x^{4}+\frac{12}{5}x^{5}+\frac{4}{5}x^{6}
\end{aligned}
$$

Expanding $-\mathrm{e}^{2x}+9$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-\mathrm{e}^{2x}+9 &= 8-2x-2x^{2}-\frac{4}{3}x^{3}-\frac{2}{3}x^{4}-\frac{4}{15}x^{5}-\frac{4}{45}x^{6}+\dots \\
&= 8-2x-2x^{2}-\frac{4}{3}x^{3}-\frac{2}{3}x^{4}-\frac{4}{15}x^{5}-\frac{4}{45}x^{6}
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$\left(9+18x+18x^{2}+12x^{3}+6x^{4}+\frac{12}{5}x^{5}+\frac{4}{5}x^{6}\right)\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)$$
$$+\left(8-2x-2x^{2}-\frac{4}{3}x^{3}-\frac{2}{3}x^{4}-\frac{4}{15}x^{5}-\frac{4}{45}x^{6}\right)\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)=0$$

Expanding the first term in (1) gives

$$9\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+18x\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)$$
$$+18x^{2}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+12x^{3}$$
$$\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+6x^{4}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\frac{12x^{5}}{5}$$
$$\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\frac{4x^{6}}{5}\cdot\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)$$
$$+\left(8-2x-2x^{2}-\frac{4}{3}x^{3}-\frac{2}{3}x^{4}-\frac{4}{15}x^{5}-\frac{4}{45}x^{6}\right)\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)=0$$

Expanding the second term in (1) gives

Expression too large to display

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} \frac{4n\, x^{n+4} a_n (n-1)}{5}\right)+\left(\sum_{n=2}^{\infty} \frac{12n\, x^{n+3} a_n (n-1)}{5}\right) \\
&+\left(\sum_{n=2}^{\infty} 6n\, x^{n+2} a_n (n-1)\right)+\left(\sum_{n=2}^{\infty} 12n\, x^{1+n} a_n (n-1)\right) \\
&+\left(\sum_{n=2}^{\infty} 18n a_n x^{n} (n-1)\right)+\left(\sum_{n=2}^{\infty} 18n\, x^{n-1} a_n (n-1)\right) \\
&+\left(\sum_{n=2}^{\infty} 9n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} 8a_n x^{n}\right)+\sum_{n=0}^{\infty}\left(-2x^{1+n} a_n\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-2x^{n+2} a_n\right)+\sum_{n=0}^{\infty}\left(-\frac{4x^{n+3} a_n}{3}\right)+\sum_{n=0}^{\infty}\left(-\frac{2x^{n+4} a_n}{3}\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-\frac{4x^{n+5} a_n}{15}\right)+\sum_{n=0}^{\infty}\left(-\frac{4x^{n+6} a_n}{45}\right)=0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} \frac{4n\, x^{n+4} a_n (n-1)}{5} = \sum_{n=6}^{\infty} \frac{4(n-4)\, a_{n-4} (n-5)\, x^{n}}{5}
$$

$$
\sum_{n=2}^{\infty} \frac{12n\, x^{n+3} a_n (n-1)}{5} = \sum_{n=5}^{\infty} \frac{12(n-3)\, a_{n-3} (n-4)\, x^{n}}{5}
$$

$$
\sum_{n=2}^{\infty} 6n\, x^{n+2} a_n (n-1) = \sum_{n=4}^{\infty} 6(n-2)\, a_{n-2} (n-3)\, x^{n}
$$

$$
\sum_{n=2}^{\infty} 12n\, x^{1+n} a_n (n-1) = \sum_{n=3}^{\infty} 12(n-1)\, a_{n-1} (n-2)\, x^{n}
$$

$$
\sum_{n=2}^{\infty} 18n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 18(1+n)\, a_{1+n} n\, x^{n}
$$

$$
\sum_{n=2}^{\infty} 9n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 9(n+2)\, a_{n+2} (1+n)\, x^{n}
$$

$$\sum_{n=0}^{\infty} \left(-2x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^n\right)$$

$$\sum_{n=0}^{\infty} \left(-2x^{n+2} a_n\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2} x^n\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{4x^{n+3} a_n}{3}\right) = \sum_{n=3}^{\infty} \left(-\frac{4a_{n-3} x^n}{3}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{2x^{n+4} a_n}{3}\right) = \sum_{n=4}^{\infty} \left(-\frac{2a_{n-4} x^n}{3}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{4x^{n+5} a_n}{15}\right) = \sum_{n=5}^{\infty} \left(-\frac{4a_{n-5} x^n}{15}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{4x^{n+6} a_n}{45}\right) = \sum_{n=6}^{\infty} \left(-\frac{4a_{n-6} x^n}{45}\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=6}^{\infty} \frac{4(n-4)\, a_{n-4} (n-5)\, x^n}{5}\right) + \left(\sum_{n=5}^{\infty} \frac{12(n-3)\, a_{n-3} (n-4)\, x^n}{5}\right) \\ &+ \left(\sum_{n=4}^{\infty} 6(n-2)\, a_{n-2} (n-3)\, x^n\right) + \left(\sum_{n=3}^{\infty} 12(n-1)\, a_{n-1} (n-2)\, x^n\right) \\ &+ \left(\sum_{n=2}^{\infty} 18n a_n x^n (n-1)\right) + \left(\sum_{n=1}^{\infty} 18(1+n)\, a_{1+n} n\, x^n\right) \\ &+ \left(\sum_{n=0}^{\infty} 9(n+2)\, a_{n+2} (1+n)\, x^n\right) + \left(\sum_{n=0}^{\infty} 8a_n x^n\right) \\ &+ \sum_{n=1}^{\infty} \left(-2a_{n-1} x^n\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2} x^n\right) + \sum_{n=3}^{\infty} \left(-\frac{4a_{n-3} x^n}{3}\right) \\ &\quad + \sum_{n=4}^{\infty} \left(-\frac{2a_{n-4} x^n}{3}\right) + \sum_{n=5}^{\infty} \left(-\frac{4a_{n-5} x^n}{15}\right) + \sum_{n=6}^{\infty} \left(-\frac{4a_{n-6} x^n}{45}\right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$18a_2 + 8a_0 = 0$$

$$a_2 = -\frac{4a_0}{9}$$

$n = 1$ gives

$$36a_2 + 54a_3 + 8a_1 - 2a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{3} - \frac{4a_1}{27}$$

$n = 2$ gives

$$44a_2 + 108a_3 + 108a_4 - 2a_1 - 2a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{65a_0}{486} + \frac{a_1}{6}$$

$n = 3$ gives

$$22a_2 + 116a_3 + 216a_4 + 180a_5 - 2a_1 - \frac{4a_0}{3} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_0}{135} - \frac{227a_1}{2430}$$

$n = 4$ gives

$$10a_2 + 70a_3 + 224a_4 + 360a_5 + 270a_6 - \frac{4a_1}{3} - \frac{2a_0}{3} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{1102a_0}{32805} + \frac{4a_1}{135}$$

$n = 5$ gives

$$\frac{52a_2}{15} + 34a_3 + 142a_4 + 368a_5 + 540a_6 + 378a_7 - \frac{2a_1}{3} - \frac{4a_0}{15} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{19a_0}{630} + \frac{503a_1}{459270}$$

For $6 \leq n$, the recurrence equation is

$$\begin{aligned} &\frac{4(n-4)\, a_{n-4}(n-5)}{5} + \frac{12(n-3)\, a_{n-3}(n-4)}{5} + 6(n-2)\, a_{n-2}(n-3) \\ &+ 12(n-1)\, a_{n-1}(n-2) + 18na_n(n-1) + 18(1+n)\, a_{1+n}n + 9(n+2)\, a_{n+2}(1+n) \\ &+ 8a_n - 2a_{n-1} - 2a_{n-2} - \frac{4a_{n-3}}{3} - \frac{2a_{n-4}}{3} - \frac{4a_{n-5}}{15} - \frac{4a_{n-6}}{45} = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2(405n^2a_n + 405n^2a_{1+n} + 18n^2a_{n-4} + 54n^2a_{n-3} + 135n^2a_{n-2} + 270n^2a_{n-1} - 405na_n + 405na_{1+n} - 1 \ldots}{4 \ldots}$$

$$\begin{aligned} &= -\frac{2(405n^2 - 405n + 180)\, a_n}{405\,(n+2)\,(1+n)} - \frac{2(405n^2 + 405n)\, a_{1+n}}{405\,(n+2)\,(1+n)} + \frac{4a_{n-6}}{405\,(n+2)\,(1+n)} \\ &+ \frac{4a_{n-5}}{135\,(n+2)\,(1+n)} - \frac{2(18n^2 - 162n + 345)\, a_{n-4}}{405\,(n+2)\,(1+n)} - \frac{2(54n^2 - 378n + 618)\, a_{n-3}}{405\,(n+2)\,(1+n)} \\ &- \frac{2(135n^2 - 675n + 765)\, a_{n-2}}{405\,(n+2)\,(1+n)} - \frac{2(270n^2 - 810n + 495)\, a_{n-1}}{405\,(n+2)\,(1+n)} \end{aligned} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \ldots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{4a_0x^2}{9} + \left(\frac{a_0}{3} - \frac{4a_1}{27}\right)x^3 + \left(-\frac{65a_0}{486} + \frac{a_1}{6}\right)x^4 + \left(\frac{a_0}{135} - \frac{227a_1}{2430}\right)x^5 + \ldots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{4}{9}x^2 + \frac{1}{3}x^3 - \frac{65}{486}x^4 + \frac{1}{135}x^5\right)a_0 + \left(x - \frac{4}{27}x^3 + \frac{1}{6}x^4 - \frac{227}{2430}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{4}{9}x^2 + \frac{1}{3}x^3 - \frac{65}{486}x^4 + \frac{1}{135}x^5\right)c_1 + \left(x - \frac{4}{27}x^3 + \frac{1}{6}x^4 - \frac{227}{2430}x^5\right)c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned} y = &\left(1 - \frac{4}{9}x^2 + \frac{1}{3}x^3 - \frac{65}{486}x^4 + \frac{1}{135}x^5 + \frac{1102}{32805}x^6\right)y(0) \\ &+ \left(x - \frac{4}{27}x^3 + \frac{1}{6}x^4 - \frac{227}{2430}x^5 + \frac{4}{135}x^6\right)y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$y = \left(1 - \frac{4}{9}x^2 + \frac{1}{3}x^3 - \frac{65}{486}x^4 + \frac{1}{135}x^5\right)c_1 + \left(x - \frac{4}{27}x^3 + \frac{1}{6}x^4 - \frac{227}{2430}x^5\right)c_2 + O(x^6) \tag{2}$$

<u>Verification of solutions</u>

$$\begin{aligned} y = &\left(1 - \frac{4}{9}x^2 + \frac{1}{3}x^3 - \frac{65}{486}x^4 + \frac{1}{135}x^5 + \frac{1102}{32805}x^6\right)y(0) \\ &+ \left(x - \frac{4}{27}x^3 + \frac{1}{6}x^4 - \frac{227}{2430}x^5 + \frac{4}{135}x^6\right)y'(0) + O(x^6) \end{aligned}$$

Verified OK.

$$y = \left(1 - \frac{4}{9}x^2 + \frac{1}{3}x^3 - \frac{65}{486}x^4 + \frac{1}{135}x^5\right)c_1 + \left(x - \frac{4}{27}x^3 + \frac{1}{6}x^4 - \frac{227}{2430}x^5\right)c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
   Change of variables used:
      [x = -1/2*ln(t)]
   Linear ODE actually solved:
      (9*t-1)*u(t)+36*t*diff(u(t),t)+36*t^2*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)+(exp(-2*x)-1/9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{4}{9}x^2 + \frac{1}{3}x^3 - \frac{65}{486}x^4 + \frac{1}{135}x^5\right) y(0) + \left(x - \frac{4}{27}x^3 + \frac{1}{6}x^4 - \frac{227}{2430}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[y''[x]+(Exp[-2*x]-1/9)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{227x^5}{2430} + \frac{x^4}{6} - \frac{4x^3}{27} + x\right) + c_1\left(\frac{x^5}{135} - \frac{65x^4}{486} + \frac{x^3}{3} - \frac{4x^2}{9} + 1\right)$$

3.4 problem 6

3.4.1 Maple step by step solution 409

Internal problem ID [5657]
Internal file name [OUTPUT/4905_Sunday_June_05_2022_03_10_01_PM_69803307/index.tex]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.4. Bessels Equation page 195
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \frac{\left(x+\frac{3}{4}\right)y}{4} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(\frac{x}{4} + \frac{3}{16}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned}p(x) &= 0\\ q(x) &= \frac{4x+3}{16x^2}\end{aligned}$$

Table 48: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{4x+3}{16x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(\frac{x}{4} + \frac{3}{16}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(\frac{x}{4} + \frac{3}{16}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} \frac{x^{1+n+r} a_n}{4}\right) + \left(\sum_{n=0}^{\infty} \frac{3a_n x^{n+r}}{16}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \frac{x^{1+n+r} a_n}{4} = \sum_{n=1}^{\infty} \frac{a_{n-1} x^{n+r}}{4}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} \frac{a_{n-1} x^{n+r}}{4}\right) + \left(\sum_{n=0}^{\infty} \frac{3 a_n x^{n+r}}{16}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + \frac{3 a_n x^{n+r}}{16} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + \frac{3 a_0 x^r}{16} = 0$$

Or

$$\left(x^r r(-1+r) + \frac{3x^r}{16}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(16r^2 - 16r + 3)\, x^r}{16} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r + \frac{3}{16} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{4}$$
$$r_2 = \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(16r^2 - 16r + 3)\, x^r}{16} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{4}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{4}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + \frac{a_{n-1}}{4} + \frac{3a_n}{16} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{4a_{n-1}}{16n^2 + 32nr + 16r^2 - 16n - 16r + 3} \tag{4}
$$

Which for the root $r = \frac{3}{4}$ becomes

$$
a_n = -\frac{a_{n-1}}{4n^2 + 2n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = -\frac{4}{16r^2 + 16r + 3}
$$

Which for the root $r = \frac{3}{4}$ becomes

$$a_1 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{16r^2+16r+3}$	$-\frac{1}{6}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{(16r^2 + 16r + 3)(16r^2 + 48r + 35)}$$

Which for the root $r = \frac{3}{4}$ becomes

$$a_2 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{16r^2+16r+3}$	$-\frac{1}{6}$
a_2	$\frac{16}{(16r^2+16r+3)(16r^2+48r+35)}$	$\frac{1}{120}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{(16r^2 + 16r + 3)(16r^2 + 48r + 35)(16r^2 + 80r + 99)}$$

Which for the root $r = \frac{3}{4}$ becomes

$$a_3 = -\frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{16r^2+16r+3}$	$-\frac{1}{6}$
a_2	$\frac{16}{(16r^2+16r+3)(16r^2+48r+35)}$	$\frac{1}{120}$
a_3	$-\frac{64}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)}$	$-\frac{1}{5040}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(16r^2 + 16r + 3)(16r^2 + 48r + 35)(16r^2 + 80r + 99)(16r^2 + 112r + 195)}$$

Which for the root $r = \frac{3}{4}$ becomes

$$a_4 = \frac{1}{362880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{16r^2+16r+3}$	$-\frac{1}{6}$
a_2	$\frac{16}{(16r^2+16r+3)(16r^2+48r+35)}$	$\frac{1}{120}$
a_3	$-\frac{64}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)}$	$-\frac{1}{5040}$
a_4	$\frac{256}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)(16r^2+112r+195)}$	$\frac{1}{362880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{(16r^2 + 16r + 3)(16r^2 + 48r + 35)(16r^2 + 80r + 99)(16r^2 + 112r + 195)(16r^2 + 144r + 323)}$$

Which for the root $r = \frac{3}{4}$ becomes

$$a_5 = -\frac{1}{39916800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{16r^2+16r+3}$	$-\frac{1}{6}$
a_2	$\frac{16}{(16r^2+16r+3)(16r^2+48r+35)}$	$\frac{1}{120}$
a_3	$-\frac{64}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)}$	$-\frac{1}{5040}$
a_4	$\frac{256}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)(16r^2+112r+195)}$	$\frac{1}{362880}$
a_5	$-\frac{1024}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)(16r^2+112r+195)(16r^2+144r+323)}$	$-\frac{1}{39916800}$

Using the above table, then the solution $y_1(x)$ is

$$y_1(x) = x^{\frac{3}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)$$
$$= x^{\frac{3}{4}}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \frac{x^4}{362880} - \frac{x^5}{39916800} + O(x^6)\right)$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n + r)(n + r - 1) + \frac{b_{n-1}}{4} + \frac{3b_n}{16} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{4b_{n-1}}{16n^2 + 32nr + 16r^2 - 16n - 16r + 3} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_n = -\frac{b_{n-1}}{4n^2 - 2n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{4}{16r^2 + 16r + 3}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{16r^2+16r+3}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{16}{(16r^2+16r+3)(16r^2+48r+35)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_2 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{16r^2+16r+3}$	$-\frac{1}{2}$
b_2	$\frac{16}{(16r^2+16r+3)(16r^2+48r+35)}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{64}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_3 = -\frac{1}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{16r^2+16r+3}$	$-\frac{1}{2}$
b_2	$\frac{16}{(16r^2+16r+3)(16r^2+48r+35)}$	$\frac{1}{24}$
b_3	$-\frac{64}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)}$	$-\frac{1}{720}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{256}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)(16r^2+112r+195)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_4 = \frac{1}{40320}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{16r^2+16r+3}$	$-\frac{1}{2}$
b_2	$\frac{16}{(16r^2+16r+3)(16r^2+48r+35)}$	$\frac{1}{24}$
b_3	$-\frac{64}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)}$	$-\frac{1}{720}$
b_4	$\frac{256}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)(16r^2+112r+195)}$	$\frac{1}{40320}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1024}{(16r^2 + 16r + 3)(16r^2 + 48r + 35)(16r^2 + 80r + 99)(16r^2 + 112r + 195)(16r^2 + 144r + 323)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_5 = -\frac{1}{3628800}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{16r^2+16r+3}$	$-\frac{1}{2}$
b_2	$\frac{16}{(16r^2+16r+3)(16r^2+48r+35)}$	$\frac{1}{24}$
b_3	$-\frac{64}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)}$	$-\frac{1}{720}$
b_4	$\frac{256}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)(16r^2+112r+195)}$	$\frac{1}{40320}$
b_5	$-\frac{1024}{(16r^2+16r+3)(16r^2+48r+35)(16r^2+80r+99)(16r^2+112r+195)(16r^2+144r+323)}$	$-\frac{1}{3628800}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{3}{4}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{\frac{1}{4}}\left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \frac{x^4}{40320} - \frac{x^5}{3628800} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{4}} \left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \frac{x^4}{362880} - \frac{x^5}{39916800} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{\frac{1}{4}} \left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \frac{x^4}{40320} - \frac{x^5}{3628800} + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{4}} \left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \frac{x^4}{362880} - \frac{x^5}{39916800} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{\frac{1}{4}} \left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \frac{x^4}{40320} - \frac{x^5}{3628800} + O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{3}{4}} \left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \frac{x^4}{362880} - \frac{x^5}{39916800} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{\frac{1}{4}} \left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \frac{x^4}{40320} - \frac{x^5}{3628800} + O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{3}{4}} \left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \frac{x^4}{362880} - \frac{x^5}{39916800} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{\frac{1}{4}} \left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \frac{x^4}{40320} - \frac{x^5}{3628800} + O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

3.4.1 Maple step by step solution

Let's solve

$x^2y'' + \left(\frac{x}{4} + \frac{3}{16}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y(4x+3)}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y(4x+3)}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{4x+3}{16x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{16}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2y'' + y(4x + 3) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+4r)(-3+4r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k+4r-1)(4k+4r-3) + 4a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+4r)(-3+4r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$16\left(k+r-\frac{3}{4}\right)\left(k+r-\frac{1}{4}\right) a_k + 4a_{k-1} = 0$$

- Shift index using $k- > k+1$

  $$16\left(k+\frac{1}{4}+r\right)\left(k+\frac{3}{4}+r\right) a_{k+1} + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{4a_k}{(4k+1+4r)(4k+3+4r)}$$

- Recursion relation for $r = \frac{1}{4}$

  $$a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}$$

- Solution for $r = \frac{1}{4}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}\right]$$

- Recursion relation for $r = \frac{3}{4}$

  $$a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}$$

- Solution for $r = \frac{3}{4}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}, b_{k+1} = -\frac{4b_k}{(4k+4)(4k+6)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+1/4*(x+3/4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{4}}\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{40320}x^4-\frac{1}{3628800}x^5+O\left(x^6\right)\right) + c_2 x^{\frac{3}{4}}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5040}x^3+\frac{1}{362880}x^4-\frac{1}{39916800}x^5+O\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 90

```
AsymptoticDSolveValue[x^2*y''[x]+1/4*(x+3/4)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \sqrt[4]{x}\left(-\frac{x^5}{3628800}+\frac{x^4}{40320}-\frac{x^3}{720}+\frac{x^2}{24}-\frac{x}{2}+1\right) + c_1 x^{3/4}\left(-\frac{x^5}{39916800}+\frac{x^4}{362880}-\frac{x^3}{5040}+\frac{x^2}{120}-\frac{x}{6}+1\right)$$

3.5 problem 7

3.5.1 Maple step by step solution 421

Internal problem ID [5658]
Internal file name [OUTPUT/4906_Sunday_June_05_2022_03_10_04_PM_48433408/index.tex]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.4. Bessels Equation page 195
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \frac{(x^2 - 1)\, y}{4} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(\frac{x^2}{4} - \frac{1}{4}\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 - 1}{4x^2}$$

Table 50: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(\frac{x^2}{4} - \frac{1}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\frac{x^2}{4} - \frac{1}{4}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+r+2} a_n}{4}\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \frac{x^{n+r+2} a_n}{4} = \sum_{n=2}^{\infty} \frac{a_{n-2} x^{n+r}}{4}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} \frac{a_{n-2} x^{n+r}}{4}\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - \frac{a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - \frac{a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - \frac{x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2-1)\, x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 1)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + \frac{a_{n-2}}{4} - \frac{a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{4n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{4r^2 + 16r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{4r^2+16r+15}$	$-\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{4r^2+16r+15}$	$-\frac{1}{24}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^4 + 192r^3 + 824r^2 + 1488r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{1920}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{4r^2+16r+15}$	$-\frac{1}{24}$
a_3	0	0
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{1920}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{4r^2+16r+15}$	$-\frac{1}{24}$
a_3	0	0
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{1920}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{x^2}{24} + \frac{x^4}{1920} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} 0 &= \lim_{r \to -\frac{1}{2}} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + b_n(n+r) + \frac{b_{n-2}}{4} - \frac{b_n}{4} = 0 \tag{4}
$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$b_n \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) + b_n \left(n - \frac{1}{2}\right) + \frac{b_{n-2}}{4} - \frac{b_n}{4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{b_{n-2}}{4n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{4r^2+16r+15}$	$-\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{4r^2+16r+15}$	$-\frac{1}{8}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{384}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{4r^2+16r+15}$	$-\frac{1}{8}$
b_3	0	0
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{4r^2+16r+15}$	$-\frac{1}{8}$
b_3	0	0
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x^2}{8} + \frac{x^4}{384} + O(x^6)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1 - \frac{x^2}{24} + \frac{x^4}{1920} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{8} + \frac{x^4}{384} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1 - \frac{x^2}{24} + \frac{x^4}{1920} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{8} + \frac{x^4}{384} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \sqrt{x}\left(1 - \frac{x^2}{24} + \frac{x^4}{1920} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{8} + \frac{x^4}{384} + O(x^6)\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \sqrt{x}\left(1 - \frac{x^2}{24} + \frac{x^4}{1920} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{8} + \frac{x^4}{384} + O(x^6)\right)}{\sqrt{x}}
$$

Verified OK.

3.5.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(\frac{x^2}{4} - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)(-1+2r)\, x^r + a_1(3+2r)(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2r-1) + a_{k-2}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

  $$a_1(3+2r)(1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(4k^2 + 8kr + 4r^2 - 1) + a_{k-2} = 0$$

- Shift index using $k- >k+2$

  $$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k}{4k^2+8kr+4r^2+16k+16r+15}$$

- Recursion relation for $r = -\frac{1}{2}$

  $$a_{k+2} = -\frac{a_k}{4k^2+12k+8}$$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = -\frac{a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{b_k}{4k^2+20k+24}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+1/4*(x^2-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1 x\left(1-\frac{1}{24}x^2+\frac{1}{1920}x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(1-\frac{1}{8}x^2+\frac{1}{384}x^4+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 58

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+1/4*(x^2-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^{7/2}}{384}-\frac{x^{3/2}}{8}+\frac{1}{\sqrt{x}}\right)+c_2\left(\frac{x^{9/2}}{1920}-\frac{x^{5/2}}{24}+\sqrt{x}\right)$$

3.6 problem 8

3.6.1 Maple step by step solution 433

Internal problem ID [5659]
Internal file name [OUTPUT/4907_Sunday_June_05_2022_03_10_07_PM_97201833/index.tex]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.4. Bessels Equation page 195
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1+2x)^2 y''+2(1+2x)\,y'+16x(1+x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots \\ &= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (63) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (64) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(8yx^2+2xy'+8xy+y')}{(1+2x)^2}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= -\frac{32\left(x^2+x-\frac{1}{2}\right)\left(\left(\frac{1}{2}+x\right)y'-y\right)}{(1+2x)^3}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= \frac{(128x^3+192x^2-96x-80)\,y'+256y\left(x^4+2x^3+\frac{1}{4}x^2-\frac{3}{4}x+\frac{1}{2}\right)}{(1+2x)^4}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \frac{(512x^5+1280x^4+128x^3-1088x^2+1216x+832)\,y'-1024\left(x^4+2x^3-\frac{3}{2}x^2-\frac{5}{2}x+\frac{19}{16}\right)y}{(1+2x)^5}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= \frac{(-3072x^5-7680x^4+5376x^3+15744x^2-14208x-9984)\,y'-4096\left(x^6+3x^5+\frac{3}{4}x^4-\frac{7}{2}x^3+\frac{21}{4}x^2\right.}{(1+2x)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y'(0)\\
F_1 &= -16y(0)+8y'(0)\\
F_2 &= 128y(0)-80y'(0)\\
F_3 &= -1216y(0)+832y'(0)\\
F_4 &= 14720y(0)-9984y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{8}{3}x^3 + \frac{16}{3}x^4 - \frac{152}{15}x^5 + \frac{184}{9}x^6\right) y(0) \\
&+ \left(x - x^2 + \frac{4}{3}x^3 - \frac{10}{3}x^4 + \frac{104}{15}x^5 - \frac{208}{15}x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(4x^2 + 4x + 1) y'' + (2 + 4x) y' + (16x^2 + 16x) y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(4x^2 + 4x + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + (2 + 4x) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + (16x^2 + 16x) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 4x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 4n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) \\
&+ \left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 4n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 16x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} 16x^{1+n} a_n\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 4(1+n)\, a_{1+n} n\, x^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, x^n$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(1+n)\, a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} 16 x^{n+2} a_n = \sum_{n=2}^{\infty} 16 a_{n-2} x^n$$

$$\sum_{n=0}^{\infty} 16 x^{1+n} a_n = \sum_{n=1}^{\infty} 16 a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} 4x^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 4(1+n)\, a_{1+n} n\, x^n\right) \\ &+ \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, x^n\right) + \left(\sum_{n=0}^{\infty} 2(1+n)\, a_{1+n} x^n\right) \\ &+ \left(\sum_{n=1}^{\infty} 4n a_n x^n\right) + \left(\sum_{n=2}^{\infty} 16 a_{n-2} x^n\right) + \left(\sum_{n=1}^{\infty} 16 a_{n-1} x^n\right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_1 = 0$$

$$a_2 = -a_1$$

$n = 1$ gives

$$12a_2 + 6a_3 + 4a_1 + 16a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{8a_0}{3} + \frac{4a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$\begin{aligned} &4na_n(n-1) + 4(1+n)\, a_{1+n}n + (n+2)\, a_{n+2}(1+n) \\ &+ 2(1+n)\, a_{1+n} + 4na_n + 16a_{n-2} + 16a_{n-1} = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2(2n^2a_n + 2n^2a_{1+n} + 3na_{1+n} + a_{1+n} + 8a_{n-2} + 8a_{n-1})}{(n+2)\,(1+n)} \\ &= -\frac{4n^2a_n}{(n+2)\,(1+n)} - \frac{2(2n^2+3n+1)\, a_{1+n}}{(n+2)\,(1+n)} - \frac{16a_{n-2}}{(n+2)\,(1+n)} - \frac{16a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$16a_2 + 30a_3 + 12a_4 + 16a_0 + 16a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{10a_1}{3} + \frac{16a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$36a_3 + 56a_4 + 20a_5 + 16a_1 + 16a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{152a_0}{15} + \frac{104a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$64a_4 + 90a_5 + 30a_6 + 16a_2 + 16a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{208a_1}{15} + \frac{184a_0}{9}$$

For $n = 5$ the recurrence equation gives

$$100a_5 + 132a_6 + 42a_7 + 16a_3 + 16a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{288a_0}{7} + \frac{8768a_1}{315}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x - a_1 x^2 + \left(-\frac{8a_0}{3} + \frac{4a_1}{3}\right) x^3 \\ + \left(-\frac{10a_1}{3} + \frac{16a_0}{3}\right) x^4 + \left(-\frac{152a_0}{15} + \frac{104a_1}{15}\right) x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{8}{3}x^3 + \frac{16}{3}x^4 - \frac{152}{15}x^5\right) a_0 + \left(x - x^2 + \frac{4}{3}x^3 - \frac{10}{3}x^4 + \frac{104}{15}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{8}{3}x^3 + \frac{16}{3}x^4 - \frac{152}{15}x^5\right) c_1 + \left(x - x^2 + \frac{4}{3}x^3 - \frac{10}{3}x^4 + \frac{104}{15}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = \left(1 - \frac{8}{3}x^3 + \frac{16}{3}x^4 - \frac{152}{15}x^5 + \frac{184}{9}x^6\right) y(0) \\ + \left(x - x^2 + \frac{4}{3}x^3 - \frac{10}{3}x^4 + \frac{104}{15}x^5 - \frac{208}{15}x^6\right) y'(0) + O(x^6) \end{aligned} \quad (1)$$

$$y = \left(1 - \frac{8}{3}x^3 + \frac{16}{3}x^4 - \frac{152}{15}x^5\right) c_1 + \left(x - x^2 + \frac{4}{3}x^3 - \frac{10}{3}x^4 + \frac{104}{15}x^5\right) c_2 + O(x^6) \quad (2)$$

Verification of solutions

$$y = \left(1-\frac{8}{3}x^3+\frac{16}{3}x^4-\frac{152}{15}x^5+\frac{184}{9}x^6\right)y(0) + \left(x-x^2+\frac{4}{3}x^3-\frac{10}{3}x^4+\frac{104}{15}x^5-\frac{208}{15}x^6\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1-\frac{8}{3}x^3+\frac{16}{3}x^4-\frac{152}{15}x^5\right)c_1+\left(x-x^2+\frac{4}{3}x^3-\frac{10}{3}x^4+\frac{104}{15}x^5\right)c_2+O(x^6)$$

Verified OK.

3.6.1 Maple step by step solution

Let's solve

$(4x^2+4x+1)\,y''+(2+4x)\,y'+(16x^2+16x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{16x(1+x)y}{4x^2+4x+1}-\frac{2y'}{1+2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2y'}{1+2x}+\frac{16x(1+x)y}{4x^2+4x+1}=0$

☐ Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2}{1+2x},P_3(x)=\frac{16x(1+x)}{4x^2+4x+1}\right]$

  ◦ $\left(\frac{1}{2}+x\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

    $\left(\left(\frac{1}{2}+x\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}}=1$

  ◦ $\left(\frac{1}{2}+x\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

    $\left(\left(\frac{1}{2}+x\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}}=-1$

  ◦ $x=-\frac{1}{2}$is a regular singular point

    Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

$x_0 = -\frac{1}{2}$

- Multiply by denominators

  $y''(1+2x)\left(4x^2+4x+1\right)+\left(8x^2+8x+2\right)y'+16x(1+x)\left(1+2x\right)y=0$

- Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $8u^3\left(\frac{d^2}{du^2}y(u)\right)+8u^2\left(\frac{d}{du}y(u)\right)+\left(32u^3-8u\right)y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 1..3$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^2 \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

    $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r+1}$

  - Shift index using $k->k-1$

    $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,u^{k+r}$

  - Convert $u^3 \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u^3 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,u^{k+r+1}$

  - Shift index using $k->k-1$

    $u^3 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,(k-2+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $8a_0(1+r)\,(-1+r)\,u^{1+r}+8a_1(2+r)\,r\,u^{2+r}+\left(\sum_{k=3}^{\infty}\left(8a_{k-1}(k+r)\,(k-2+r)+32a_{k-3}\right)u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $8(1+r)\,(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$
- Each term must be 0

  $8a_1(2+r)\, r = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $8a_{k-1}(k+r)\,(k-2+r) + 32a_{k-3} = 0$
- Shift index using $k- >k+3$

  $8a_{k+2}(k+3+r)\,(k+r+1) + 32a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+3+r)(k+r+1)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{4a_k}{(k+2)k}$
- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{4a_k}{(k+2)k}, a_1 = 0\right]$
- Revert the change of variables $u = \frac{1}{2} + x$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(\frac{1}{2} + x\right)^{k-1}, a_{k+2} = -\frac{4a_k}{(k+2)k}, a_1 = 0\right]$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{4a_k}{(k+4)(k+2)}$
- Solution for $r = 1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = -\frac{4a_k}{(k+4)(k+2)}, a_1 = 0\right]$
- Revert the change of variables $u = \frac{1}{2} + x$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(\frac{1}{2} + x\right)^{k+1}, a_{k+2} = -\frac{4a_k}{(k+4)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k \left(\frac{1}{2} + x\right)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k \left(\frac{1}{2} + x\right)^{k+1}\right), a_{k+2} = -\frac{4a_k}{(k+2)k}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(k+4)(k+2)}, b_1\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 49

```
Order:=6;
dsolve((2*x+1)^2*diff(y(x),x$2)+2*(2*x+1)*diff(y(x),x)+16*x*(x+1)*y(x)=0,y(x),type='series',
```

$$y(x) = \left(1 - \frac{8}{3}x^3 + \frac{16}{3}x^4 - \frac{152}{15}x^5\right) y(0) + \left(x - x^2 + \frac{4}{3}x^3 - \frac{10}{3}x^4 + \frac{104}{15}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica
Time used: 0.002 (sec). Leaf size: 61

```
AsymptoticDSolveValue[(2*x+1)^2*y''[x]+2*(2*x+1)*y'[x]+16*x*(x+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{152x^5}{15} + \frac{16x^4}{3} - \frac{8x^3}{3} + 1\right) + c_2\left(\frac{104x^5}{15} - \frac{10x^4}{3} + \frac{4x^3}{3} - x^2 + x\right)$$

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4.7 problem 7 509
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4.9 problem 9 527

4.1 problem 1

4.1.1 Maple step by step solution 447

Internal problem ID [5660]
Internal file name [`OUTPUT/4908_Sunday_June_05_2022_03_10_08_PM_81248895/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.5. Bessel Functions Y(x). General Solution page 200
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[_Bessel]
```

$$\boxed{x^2y'' + xy' + (x^2 - 6)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (x^2 - 6)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 - 6}{x^2}$$

Table 53: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-6}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 - 6\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(x^2 - 6\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-6 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-6 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 6 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 6 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 6 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 6\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 6 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \sqrt{6} \\ r_2 &= -\sqrt{6} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2-6\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2\sqrt{6}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\sqrt{6}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\sqrt{6}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - 6a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 6} \tag{4}$$

Which for the root $r = \sqrt{6}$ becomes

$$a_n = -\frac{a_{n-2}}{n\left(2\sqrt{6}+n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \sqrt{6}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 4r - 2}$$

Which for the root $r = \sqrt{6}$ becomes

$$a_2 = -\frac{1}{4 + 4\sqrt{6}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-2}$	$-\frac{1}{4+4\sqrt{6}}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-2}$	$-\frac{1}{4+4\sqrt{6}}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r^2 + 4r - 2)(r^2 + 8r + 10)}$$

Which for the root $r = \sqrt{6}$ becomes

$$a_4 = \frac{1}{32 \left(1+\sqrt{6}\right) \left(2+\sqrt{6}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-2}$	$-\frac{1}{4+4\sqrt{6}}$
a_3	0	0
a_4	$\frac{1}{(r^2+4r-2)(r^2+8r+10)}$	$\frac{1}{32\left(1+\sqrt{6}\right)\left(2+\sqrt{6}\right)}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-2}$	$-\frac{1}{4+4\sqrt{6}}$
a_3	0	0
a_4	$\frac{1}{(r^2+4r-2)(r^2+8r+10)}$	$\frac{1}{32\left(1+\sqrt{6}\right)\left(2+\sqrt{6}\right)}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\sqrt{6}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\right) \\ &= x^{\sqrt{6}}\left(1 - \frac{x^2}{4+4\sqrt{6}} + \frac{x^4}{32\left(1+\sqrt{6}\right)\left(2+\sqrt{6}\right)} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - 6b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 - 6} \tag{4}$$

Which for the root $r = -\sqrt{6}$ becomes

$$b_n = -\frac{b_{n-2}}{n\left(-2\sqrt{6} + n\right)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\sqrt{6}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 4r - 2}$$

Which for the root $r = -\sqrt{6}$ becomes

$$b_2 = \frac{1}{-4 + 4\sqrt{6}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-2}$	$\frac{1}{-4+4\sqrt{6}}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-2}$	$\frac{1}{-4+4\sqrt{6}}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 4r - 2)(r^2 + 8r + 10)}$$

Which for the root $r = -\sqrt{6}$ becomes

$$b_4 = \frac{1}{32\left(-1 + \sqrt{6}\right)\left(-2 + \sqrt{6}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-2}$	$\frac{1}{-4+4\sqrt{6}}$
b_3	0	0
b_4	$\frac{1}{(r^2+4r-2)(r^2+8r+10)}$	$\frac{1}{32\left(-1+\sqrt{6}\right)\left(-2+\sqrt{6}\right)}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-2}$	$\frac{1}{-4+4\sqrt{6}}$
b_3	0	0
b_4	$\frac{1}{(r^2+4r-2)(r^2+8r+10)}$	$\frac{1}{32\left(-1+\sqrt{6}\right)\left(-2+\sqrt{6}\right)}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\sqrt{6}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{-\sqrt{6}}\left(1 + \frac{x^2}{-4+4\sqrt{6}} + \frac{x^4}{32\left(-1+\sqrt{6}\right)\left(-2+\sqrt{6}\right)} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\sqrt{6}}\left(1 - \frac{x^2}{4+4\sqrt{6}} + \frac{x^4}{32\left(1+\sqrt{6}\right)\left(2+\sqrt{6}\right)} + O(x^6)\right) \\
&\quad + c_2 x^{-\sqrt{6}}\left(1 + \frac{x^2}{-4+4\sqrt{6}} + \frac{x^4}{32\left(-1+\sqrt{6}\right)\left(-2+\sqrt{6}\right)} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\sqrt{6}}\left(1 - \frac{x^2}{4+4\sqrt{6}} + \frac{x^4}{32\left(1+\sqrt{6}\right)\left(2+\sqrt{6}\right)} + O(x^6)\right) \\
&\quad + c_2 x^{-\sqrt{6}}\left(1 + \frac{x^2}{-4+4\sqrt{6}} + \frac{x^4}{32\left(-1+\sqrt{6}\right)\left(-2+\sqrt{6}\right)} + O(x^6)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\sqrt{6}} \left(1-\frac{x^2}{4+4\sqrt{6}}+\frac{x^4}{32\left(1+\sqrt{6}\right)\left(2+\sqrt{6}\right)}+O\left(x^6\right)\right) \\
&\quad + c_2 x^{-\sqrt{6}} \left(1+\frac{x^2}{-4+4\sqrt{6}}+\frac{x^4}{32\left(-1+\sqrt{6}\right)\left(-2+\sqrt{6}\right)}+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\sqrt{6}} \left(1-\frac{x^2}{4+4\sqrt{6}}+\frac{x^4}{32\left(1+\sqrt{6}\right)\left(2+\sqrt{6}\right)}+O\left(x^6\right)\right) \\
&\quad + c_2 x^{-\sqrt{6}} \left(1+\frac{x^2}{-4+4\sqrt{6}}+\frac{x^4}{32\left(-1+\sqrt{6}\right)\left(-2+\sqrt{6}\right)}+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

4.1.1 Maple step by step solution

Let's solve

$x^2y''+xy'+\left(x^2-6\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{\left(x^2-6\right)y}{x^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}+\frac{\left(x^2-6\right)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{x^2-6}{x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -6$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + xy' + (x^2 - 6)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (r^2 - 6)\, x^r + a_1 (r^2 + 2r - 5)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k^2 + 2kr + r^2 - 6) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 - 6 = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\sqrt{6}, -\sqrt{6}\right\}$

- Each term must be 0

  $a_1 (r^2 + 2r - 5) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k^2 + 2kr + r^2 - 6) + a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left((k+2)^2 + 2(k+2)\, r + r^2 - 6\right) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k^2+2kr+r^2+4k+4r-2}$

- Recursion relation for $r = \sqrt{6}$

  $a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{6}+4+4k+4\sqrt{6}}$

- Solution for $r = \sqrt{6}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\sqrt{6}}, a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{6}+4+4k+4\sqrt{6}}, a_1 = 0\right]$

- Recursion relation for $r = -\sqrt{6}$

  $a_{k+2} = -\frac{a_k}{k^2-2k\sqrt{6}+4+4k-4\sqrt{6}}$

- Solution for $r = -\sqrt{6}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\sqrt{6}}, a_{k+2} = -\frac{a_k}{k^2-2k\sqrt{6}+4+4k-4\sqrt{6}}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\sqrt{6}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\sqrt{6}}\right), a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{6}+4+4k+4\sqrt{6}}, a_1 = 0, b_{k+2} = -\frac{b_k}{k^2-2k\sqrt{6}+4+4\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 97

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-6)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{-\sqrt{6}} \left(1 + \frac{1}{-4+4\sqrt{6}} x^2 + \frac{1}{32} \frac{1}{\left(-2+\sqrt{6}\right)\left(-1+\sqrt{6}\right)} x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\sqrt{6}} \left(1 - \frac{1}{4+4\sqrt{6}} x^2 + \frac{1}{32} \frac{1}{\left(2+\sqrt{6}\right)\left(1+\sqrt{6}\right)} x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 210

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-6)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{x^4}{\left(-4-\sqrt{6}+\left(1-\sqrt{6}\right)\left(2-\sqrt{6}\right)\right)\left(-2-\sqrt{6}+\left(3-\sqrt{6}\right)\left(4-\sqrt{6}\right)\right)} - \frac{x^2}{-4-\sqrt{6}+\left(1-\sqrt{6}\right)\left(2-\sqrt{6}\right)} + 1 \right) x^{-\sqrt{6}} + c_1 \left(\frac{x^4}{\left(-4+\sqrt{6}+\left(1+\sqrt{6}\right)\left(2+\sqrt{6}\right)\right)\left(-2+\sqrt{6}+\left(3+\sqrt{6}\right)\left(4+\sqrt{6}\right)\right)} - \frac{x^2}{-4+\sqrt{6}+\left(1+\sqrt{6}\right)\left(2+\sqrt{6}\right)} + 1 \right) x^{\sqrt{6}}$$

4.2 problem 2

4.2.1 Maple step by step solution 463

Internal problem ID [5661]
Internal file name [`OUTPUT/4909_Sunday_June_05_2022_03_10_11_PM_14604385/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.5. Bessel Functions Y(x). General Solution page 200
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{xy'' + 5y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 5y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{5}{x}$$
$$q(x) = 1$$

Table 55: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 5y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + 5\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)(n+r-1) + 5(n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 5r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 5r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(4+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(4+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -4$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(4+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^4}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-4}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + 5a_n(n+r) + a_{n-2} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 + 4n + 4r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
a_n = -\frac{a_{n-2}}{n(n+4)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 8r + 12}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+8r+12}$	$-\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+8r+12}$	$-\frac{1}{12}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+6)(r+2)(r+8)(4+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+8r+12}$	$-\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{(r+6)(r+2)(r+8)(4+r)}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+8r+12}$	$-\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{(r+6)(r+2)(r+8)(4+r)}$	$\frac{1}{384}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{1}{(r+6)(r+2)(r+8)(4+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1}{(r+6)(r+2)(r+8)(4+r)} &= \lim_{r \to -4} \frac{1}{(r+6)(r+2)(r+8)(4+r)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x} \right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2) \right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2) \right) \end{aligned}$$

Substituting these back into the given ode $xy'' + 5y' + xy = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + 5Cy_1'(x)\ln(x) + \frac{5Cy_1(x)}{x} + 5\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + x\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left((y_1(x)\,x + y_1''(x)\,x + 5y_1'(x))\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + \frac{5y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) + 5\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1(x)\,x + y_1''(x)\,x + 5y_1'(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + \frac{5y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) + 5\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} \tag{9}$$
$$+ \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + x^2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) + 5\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x}{x}$$
$$= 0$$

Since $r_1 = 0$ and $r_2 = -4$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n} a_n n\right) x + 4\left(\sum_{n=0}^{\infty} a_n x^{n}\right)\right) C}{x}$$
$$+ \frac{\left(\sum_{n=0}^{\infty} x^{-6+n} b_n (n-4)(-5+n)\right) x^2 + x^2\left(\sum_{n=0}^{\infty} b_n x^{n-4}\right) + 5\left(\sum_{n=0}^{\infty} x^{-5+n} b_n (n-4)\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n n\right) + \left(\sum_{n=0}^{\infty} 4C\, x^{-1+n} a_n\right) + \left(\sum_{n=0}^{\infty} x^{-5+n} b_n (-5+n)(n-4)\right) \tag{2A}$$
$$+ \left(\sum_{n=0}^{\infty} x^{-3+n} b_n\right) + \left(\sum_{n=0}^{\infty} 5x^{-5+n} b_n (n-4)\right) = 0$$

The next step is to make all powers of x be $-5+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-5+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{-1+n} a_n n = \sum_{n=4}^{\infty} 2C(n-4)\, a_{n-4} x^{-5+n}$$

$$\sum_{n=0}^{\infty} 4C\, x^{-1+n} a_n = \sum_{n=4}^{\infty} 4C a_{n-4} x^{-5+n}$$

$$\sum_{n=0}^{\infty} x^{-3+n} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{-5+n}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-5+n$.

$$
\begin{aligned}
&\left(\sum_{n=4}^{\infty} 2C(n-4)\, a_{n-4} x^{-5+n}\right)+\left(\sum_{n=4}^{\infty} 4C a_{n-4} x^{-5+n}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{-5+n} b_n(-5+n)\,(n-4)\right) \\
&+\left(\sum_{n=2}^{\infty} b_{n-2} x^{-5+n}\right)+\left(\sum_{n=0}^{\infty} 5 x^{-5+n} b_n(n-4)\right)=0
\end{aligned}
\tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-3b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1=0$$

Solving the above for b_1 gives

$$b_1=0$$

For $n=2$, Eq (2B) gives

$$-4b_2+b_0=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_2+1=0$$

Solving the above for b_2 gives

$$b_2=\frac{1}{4}$$

For $n=3$, Eq (2B) gives

$$-3b_3+b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3=0$$

Solving the above for b_3 gives

$$b_3=0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + \frac{1}{4} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{16}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{16}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{16}\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right)\ln(x) + \frac{1 + \frac{x^2}{4} + O(x^6)}{x^4}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right) + c_2\left(-\frac{1}{16}\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right)\ln(x) + \frac{1 + \frac{x^2}{4} + O(x^6)}{x^4}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right) \\
&\quad + c_2\left(\left(-\frac{1}{16} + \frac{x^2}{192} - \frac{x^4}{6144} - \frac{O(x^6)}{16}\right)\ln(x) + \frac{1 + \frac{x^2}{4} + O(x^6)}{x^4}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(1-\frac{x^2}{12}+\frac{x^4}{384}+O(x^6)\right) \\
&\quad + c_2\left(\left(-\frac{1}{16}+\frac{x^2}{192}-\frac{x^4}{6144}-\frac{O(x^6)}{16}\right)\ln(x)+\frac{1+\frac{x^2}{4}+O(x^6)}{x^4}\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(1-\frac{x^2}{12}+\frac{x^4}{384}+O(x^6)\right) \\
&\quad + c_2\left(\left(-\frac{1}{16}+\frac{x^2}{192}-\frac{x^4}{6144}-\frac{O(x^6)}{16}\right)\ln(x)+\frac{1+\frac{x^2}{4}+O(x^6)}{x^4}\right)
\end{aligned}
$$

Verified OK.

4.2.1 Maple step by step solution

Let's solve

$y''x+5y'+xy=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{5y'}{x}-y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{5y'}{x}+y=0$

- Simplify ODE

  $x^2y''+5xy'+yx^2=0$

- Make a change of variables

  $y=\frac{u(x)}{x^2}$

- Compute y'

  $y'=-\frac{2u(x)}{x^3}+\frac{u'(x)}{x^2}$

- Compute y''

  $y''=\frac{6u(x)}{x^4}-\frac{4u'(x)}{x^3}+\frac{u''(x)}{x^2}$

- Apply change of variables to the ODE

  $u(x)\,x^2 + u''(x)\,x^2 + u'(x)\,x - 4u(x) = 0$
- ODE is now of the Bessel form
- Solution to Bessel ODE

  $u(x) = c_1 BesselJ(2, x) + c_2 BesselY(2, x)$
- Make the change from y back to y

  $y = \frac{c_1 BesselJ(2,x)+c_2 BesselY(2,x)}{x^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(x*diff(y(x),x$2)+5*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^4 \left(1 - \frac{1}{12}x^2 + \frac{1}{384}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 (\ln\left(x\right)\left(9x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-144 - 36x^2 + \mathrm{O}\left(x^6\right)\right))}{x^4}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 47

```
AsymptoticDSolveValue[x*y''[x]+5*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{x^4}{384} - \frac{x^2}{12} + 1 \right) + c_1 \left(\frac{(x^2+8)^2}{64x^4} - \frac{\log(x)}{16} \right)$$

4.3 problem 3

4.3.1 Maple step by step solution 474

Internal problem ID [5662]
Internal file name [`OUTPUT/4910_Sunday_June_05_2022_03_10_14_PM_63350163/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.5. Bessel Functions Y(x). General Solution page 200
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' + 9xy' + \left(36x^4 - 16\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + 9xy' + \left(36x^4 - 16\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{4x^4 - \frac{16}{9}}{x^2}$$

Table 57: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^4 - \frac{16}{9}}{x^2}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + 9xy' + \left(36x^4 - 16\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) + 9x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + \left(36x^4 - 16\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 36x^{n+r+4} a_n\right) + \sum_{n=0}^{\infty} \left(-16 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 36x^{n+r+4} a_n = \sum_{n=4}^{\infty} 36 a_{n-4} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=4}^{\infty} 36 a_{n-4} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-16 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) + 9x^{n+r} a_n (n+r) - 16 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 9x^r a_0 r - 16 a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 9x^r r - 16x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 16\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 16 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{4}{3}$$
$$r_2 = -\frac{4}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(9r^2 - 16)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{8}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{4}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{4}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = 0$$

For $4 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) + 9a_n(n+r) + 36a_{n-4} - 16a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{36a_{n-4}}{9n^2 + 18nr + 9r^2 - 16} \tag{4}$$

Which for the root $r = \frac{4}{3}$ becomes

$$a_n = -\frac{12a_{n-4}}{n(3n+8)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{4}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = -\frac{36}{9r^2 + 72r + 128}$$

Which for the root $r = \frac{4}{3}$ becomes

$$a_4 = -\frac{3}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{36}{9r^2+72r+128}$	$-\frac{3}{20}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{36}{9r^2+72r+128}$	$-\frac{3}{20}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{4}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{4}{3}}\left(1 - \frac{3x^4}{20} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$b_2 = 0$$

Substituting $n = 3$ in Eq. (2B) gives

$$b_3 = 0$$

For $4 \le n$ the recursive equation is

$$9b_n(n+r)(n+r-1) + 9b_n(n+r) + 36b_{n-4} - 16b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{36b_{n-4}}{9n^2 + 18nr + 9r^2 - 16} \tag{4}$$

Which for the root $r = -\frac{4}{3}$ becomes

$$b_n = -\frac{12b_{n-4}}{n(3n-8)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{4}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = -\frac{36}{9r^2 + 72r + 128}$$

Which for the root $r = -\frac{4}{3}$ becomes

$$b_4 = -\frac{3}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{36}{9r^2+72r+128}$	$-\frac{3}{4}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{36}{9r^2+72r+128}$	$-\frac{3}{4}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{4}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{3x^4}{4} + O(x^6)}{x^{\frac{4}{3}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{4}{3}}\left(1 - \frac{3x^4}{20} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^4}{4} + O(x^6)\right)}{x^{\frac{4}{3}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{4}{3}}\left(1 - \frac{3x^4}{20} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^4}{4} + O(x^6)\right)}{x^{\frac{4}{3}}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{4}{3}}\left(1 - \frac{3x^4}{20} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^4}{4} + O(x^6)\right)}{x^{\frac{4}{3}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{4}{3}}\left(1 - \frac{3x^4}{20} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^4}{4} + O(x^6)\right)}{x^{\frac{4}{3}}}$$

Verified OK.

4.3.1 Maple step by step solution

Let's solve

$9x^2y'' + 9xy' + (36x^4 - 16)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{4(9x^4-4)y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{4(9x^4-4)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4(9x^4-4)}{9x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{16}{9}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2y'' + 9xy' + (36x^4 - 16)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$
- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$
- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(4+3r)\,(-4+3r)\,x^r+a_1(7+3r)\,(-1+3r)\,x^{1+r}+a_2(10+3r)\,(2+3r)\,x^{2+r}+a_3(13+3r)\,(5$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(4+3r)\,(-4+3r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{4}{3},\frac{4}{3}\right\}$
- The coefficients of each power of x must be 0

  $[a_1(7+3r)\,(-1+3r)=0,a_2(10+3r)\,(2+3r)=0,a_3(13+3r)\,(5+3r)=0]$
- Solve for the dependent coefficient(s)

  $\{a_1=0,a_2=0,a_3=0\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r+4)\,(3k+3r-4)+36a_{k-4}=0$
- Shift index using $k->k+4$

  $a_{k+4}(3k+16+3r)\,(3k+8+3r)+36a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+4}=-\frac{36a_k}{(3k+16+3r)(3k+8+3r)}$
- Recursion relation for $r=-\frac{4}{3}$

  $a_{k+4}=-\frac{36a_k}{(3k+12)(3k+4)}$
- Solution for $r=-\frac{4}{3}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{4}{3}},a_{k+4}=-\frac{36a_k}{(3k+12)(3k+4)},a_1=0,a_2=0,a_3=0\right]$

- Recursion relation for $r = \frac{4}{3}$

  $a_{k+4} = -\frac{36a_k}{(3k+20)(3k+12)}$

- Solution for $r = \frac{4}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{4}{3}}, a_{k+4} = -\frac{36a_k}{(3k+20)(3k+12)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{4}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{4}{3}}\right), a_{k+4} = -\frac{36a_k}{(3k+12)(3k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{}{(3k+}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)+9*x*diff(y(x),x)+(36*x^4-16)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{8}{3}} \left(1 - \frac{3}{20}x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 - \frac{3}{4}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^{\frac{4}{3}}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 38

```
AsymptoticDSolveValue[9*x^2*y''[x]+9*x*y'[x]+(36*x^4-16)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(1 - \frac{3x^4}{20}\right) x^{4/3} + \frac{c_2 \left(1 - \frac{3x^4}{4}\right)}{x^{4/3}}$$

4.4 problem 4

4.4.1 Maple step by step solution 485

Internal problem ID [5663]
Internal file name [`OUTPUT/4911_Sunday_June_05_2022_03_10_16_PM_9350364/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.5. Bessel Functions Y(x). General Solution page 200
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (69) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (70) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -xy \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -y - xy' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2y' + yx^2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x(xy' + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -yx^3 + 6xy' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -2y'(0) \\
F_3 &= 0 \\
F_4 &= 4y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x - \frac{1}{12}x^4\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n \\
\sum_{n=0}^{\infty} x^{1+n} a_n &= \sum_{n=1}^{\infty} a_{n-1} x^n
\end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{6}\right) a_0 + \left(x - \frac{1}{12} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6) \tag{2}$$

<u>Verification of solutions</u>

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Verified OK.

4.4.1 Maple step by step solution

Let's solve

$y'' = -xy$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} + a_{k-1} = 0$

- Shift index using $k- >k+1$

$$\left((k+1)^2+3k+5\right)a_{k+3}+a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+3}=-\frac{a_k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{x^3}{6}\right)y(0)+\left(x-\frac{1}{12}x^4\right)D(y)(0)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(x-\frac{x^4}{12}\right)+c_1\left(1-\frac{x^3}{6}\right)$$

4.5 problem 5

4.5.1 Maple step by step solution 495

Internal problem ID [5664]
Internal file name [`OUTPUT/4912_Sunday_June_05_2022_03_10_18_PM_84276166/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.5. Bessel Functions Y(x). General Solution page 200
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{4xy'' + 4y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4xy'' + 4y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{1}{4x}$$

Table 60: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{4x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' + 4y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + 4\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r)(n+r-1) + 4(n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$4x^{-1+r} a_0 r(-1+r) + 4r a_0 x^{-1+r} = 0$$

Or

$$\left(4x^{-1+r} r(-1+r) + 4r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$4x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$4x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 4a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{4(n^2 + 2nr + r^2)} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}}{4n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{4(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{16 (r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{1}{16(r+1)^2(r+2)^2}$	$\frac{1}{64}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{64 (r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{2304}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{1}{16(r+1)^2(r+2)^2}$	$\frac{1}{64}$
a_3	$-\frac{1}{64(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{2304}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{256 (r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{147456}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{1}{16(r+1)^2(r+2)^2}$	$\frac{1}{64}$
a_3	$-\frac{1}{64(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{2304}$
a_4	$\frac{1}{256(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{147456}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{1024 (r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{14745600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{1}{16(r+1)^2(r+2)^2}$	$\frac{1}{64}$
a_3	$-\frac{1}{64(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{2304}$
a_4	$\frac{1}{256(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{147456}$
a_5	$-\frac{1}{1024(r+1)^2(r+2)^2(r+3)^2(4+r)^2(r+5)^2}$	$-\frac{1}{14745600}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\dots \\
&= 1 - \frac{x}{4} + \frac{x^2}{64} - \frac{x^3}{2304} + \frac{x^4}{147456} - \frac{x^5}{14745600} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{1}{4(r+1)^2}$	$-\frac{1}{4}$	$\frac{1}{2(r+1)^3}$	$\frac{1}{2}$
b_2	$\frac{1}{16(r+1)^2(r+2)^2}$	$\frac{1}{64}$	$\frac{-2r-3}{8(r+1)^3(r+2)^3}$	$-\frac{3}{64}$
b_3	$-\frac{1}{64(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{1}{2304}$	$\frac{3r^2+12r+11}{32(r+1)^3(r+2)^3(r+3)^3}$	$\frac{11}{6912}$
b_4	$\frac{1}{256(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{147456}$	$\frac{-2r^3-15r^2-35r-25}{64(r+1)^3(r+2)^3(r+3)^3(4+r)^3}$	$-\frac{25}{884736}$
b_5	$-\frac{1}{1024(r+1)^2(r+2)^2(r+3)^2(4+r)^2(r+5)^2}$	$-\frac{1}{14745600}$	$\frac{5r^4+60r^3+255r^2+450r+274}{512(r+1)^3(r+2)^3(r+3)^3(4+r)^3(r+5)^3}$	$\frac{137}{442368000}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= \left(1 - \frac{x}{4} + \frac{x^2}{64} - \frac{x^3}{2304} + \frac{x^4}{147456} - \frac{x^5}{14745600} + O(x^6)\right)\ln(x) \\
&\quad + \frac{x}{2} - \frac{3x^2}{64} + \frac{11x^3}{6912} - \frac{25x^4}{884736} + \frac{137x^5}{442368000} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right) \\
&\quad + c_2\left(\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right)\ln(x)+\frac{x}{2}-\frac{3x^2}{64}\right. \\
&\quad \left. +\frac{11x^3}{6912}-\frac{25x^4}{884736}+\frac{137x^5}{442368000}+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right) \\
&\quad + c_2\left(\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right)\ln(x)+\frac{x}{2}-\frac{3x^2}{64}\right. \\
&\quad \left. +\frac{11x^3}{6912}-\frac{25x^4}{884736}+\frac{137x^5}{442368000}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right) \\
&\quad + c_2\left(\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right)\ln(x)+\frac{x}{2}-\frac{3x^2}{64}\right. \\
&\quad \left. +\frac{11x^3}{6912}-\frac{25x^4}{884736}+\frac{137x^5}{442368000}+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right) \\
&\quad + c_2\left(\left(1-\frac{x}{4}+\frac{x^2}{64}-\frac{x^3}{2304}+\frac{x^4}{147456}-\frac{x^5}{14745600}+O(x^6)\right)\ln(x)+\frac{x}{2}-\frac{3x^2}{64}\right. \\
&\quad \left. +\frac{11x^3}{6912}-\frac{25x^4}{884736}+\frac{137x^5}{442368000}+O(x^6)\right)
\end{aligned}
$$

Verified OK.

4.5.1 Maple step by step solution

Let's solve

$4y''x + y + 4y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{y}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{y}{4x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{1}{4x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4y''x + y + 4y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- > k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- > k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$4a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(4a_{k+1}(k+1+r)^2 + a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4a_{k+1}(k+1)^2 + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{4(k+1)^2}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{4(k+1)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{4(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
Order:=6;
dsolve(4*x*diff(y(x),x$2)+4*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - \frac{1}{4}x + \frac{1}{64}x^2 - \frac{1}{2304}x^3 + \frac{1}{147456}x^4 - \frac{1}{14745600}x^5 + \mathrm{O}(x^6)\right) + \left(\frac{1}{2}x - \frac{3}{64}x^2 + \frac{11}{6912}x^3 - \frac{25}{884736}x^4 + \frac{137}{442368000}x^5 + \mathrm{O}(x^6)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 117

```
AsymptoticDSolveValue[4*x*y''[x]+4*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{14745600} + \frac{x^4}{147456} - \frac{x^3}{2304} + \frac{x^2}{64} - \frac{x}{4} + 1\right) + c_2\left(\frac{137x^5}{442368000} - \frac{25x^4}{884736} + \frac{11x^3}{6912} - \frac{3x^2}{64} + \left(-\frac{x^5}{14745600} + \frac{x^4}{147456} - \frac{x^3}{2304} + \frac{x^2}{64} - \frac{x}{4} + 1\right)\log(x) + \frac{x}{2}\right)$$

4.6 problem 6

4.6.1 Maple step by step solution 506

Internal problem ID [5665]
Internal file name [`OUTPUT/4913_Sunday_June_05_2022_03_10_20_PM_4755430/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.5. Bessel Functions Y(x). General Solution page 200
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' + 36y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + 36y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{36}{x}$$

Table 62: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{36}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + 36y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 36\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 36 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 36a_n x^{n+r} = \sum_{n=1}^{\infty} 36a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 36a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\,x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + 36a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{36a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{36a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{36}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -36$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{36}{(r+1)^2}$	-36

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1296}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 324$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{36}{(r+1)^2}$	-36
a_2	$\frac{1296}{(r+1)^2(r+2)^2}$	324

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{46656}{(r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -1296$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{36}{(r+1)^2}$	-36
a_2	$\frac{1296}{(r+1)^2(r+2)^2}$	324
a_3	$-\frac{46656}{(r+1)^2(r+2)^2(r+3)^2}$	-1296

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1679616}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = 2916$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{36}{(r+1)^2}$	-36
a_2	$\frac{1296}{(r+1)^2(r+2)^2}$	324
a_3	$-\frac{46656}{(r+1)^2(r+2)^2(r+3)^2}$	-1296
a_4	$\frac{1679616}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	2916

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{60466176}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{104976}{25}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{36}{(r+1)^2}$	-36
a_2	$\frac{1296}{(r+1)^2(r+2)^2}$	324
a_3	$-\frac{46656}{(r+1)^2(r+2)^2(r+3)^2}$	-1296
a_4	$\frac{1679616}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2}$	2916
a_5	$-\frac{60466176}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2(r+5)^2}$	$-\frac{104976}{25}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 2916x^4 - 1296x^3 + 324x^2 - 36x + 1 - \frac{104976x^5}{25} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	b
b_0	1	1	N/A since b_n starts from 1	N
b_1	$-\frac{36}{(r+1)^2}$	-36	$\frac{72}{(r+1)^3}$	7
b_2	$\frac{1296}{(r+1)^2 (r+2)^2}$	324	$\frac{-5184r-7776}{(r+1)^3 (r+2)^3}$	$-$
b_3	$-\frac{46656}{(r+1)^2 (r+2)^2 (r+3)^2}$	-1296	$\frac{279936r^2+1119744r+1026432}{(r+1)^3 (r+2)^3 (r+3)^3}$	4
b_4	$\frac{1679616}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2}$	2916	$-\frac{13436928(r^2+5r+5)\left(r+\frac{5}{2}\right)}{(r+1)^3 (r+2)^3 (r+3)^3 (4+r)^3}$	$-$
b_5	$-\frac{60466176}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2 (r+5)^2}$	$-\frac{104976}{25}$	$\frac{604661760r^4+7255941120r^3+30837749760r^2+54419558400r+33135464448}{(r+1)^3 (r+2)^3 (r+3)^3 (4+r)^3 (r+5)^3}$	2

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(2916x^4 - 1296x^3 + 324x^2 - 36x + 1 - \frac{104976x^5}{25} + O(x^6) \right) \ln(x) \\
&\quad - 12150x^4 + 4752x^3 - 972x^2 + 72x + \frac{2396952x^5}{125} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(2916x^4 - 1296x^3 + 324x^2 - 36x + 1 - \frac{104976x^5}{25} + O(x^6)\right) \\
&\quad + c_2\left(\left(2916x^4 - 1296x^3 + 324x^2 - 36x + 1 - \frac{104976x^5}{25} + O(x^6)\right)\ln(x) \right. \\
&\quad \left. - 12150x^4 + 4752x^3 - 972x^2 + 72x + \frac{2396952x^5}{125} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(2916x^4 - 1296x^3 + 324x^2 - 36x + 1 - \frac{104976x^5}{25} + O(x^6)\right) \\
&\quad + c_2\left(\left(2916x^4 - 1296x^3 + 324x^2 - 36x + 1 - \frac{104976x^5}{25} + O(x^6)\right)\ln(x) - 12150x^4 \right. \\
&\quad \left. + 4752x^3 - 972x^2 + 72x + \frac{2396952x^5}{125} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(2916x^4 - 1296x^3 + 324x^2 - 36x + 1 - \frac{104976x^5}{25} + O(x^6)\right) \\
&\quad + c_2\left(\left(2916x^4 - 1296x^3 + 324x^2 - 36x + 1 - \frac{104976x^5}{25} + O(x^6)\right)\ln(x) \right. \\
&\quad \left. - 12150x^4 + 4752x^3 - 972x^2 + 72x + \frac{2396952x^5}{125} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(2916x^4 - 1296x^3 + 324x^2 - 36x + 1 - \frac{104976x^5}{25} + O(x^6)\right) \\
&\quad + c_2\left(\left(2916x^4 - 1296x^3 + 324x^2 - 36x + 1 - \frac{104976x^5}{25} + O(x^6)\right)\ln(x) \right. \\
&\quad \left. - 12150x^4 + 4752x^3 - 972x^2 + 72x + \frac{2396952x^5}{125} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

4.6.1 Maple step by step solution

Let's solve

$y''x + y' + 36y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{36y}{x} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{36y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{36}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + y' + 36y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + 36 a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1)^2 + 36 a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{36 a_k}{(k+1)^2}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = -\frac{36 a_k}{(k+1)^2}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{36 a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+36*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln (x) + c_1) \left(1 - 36x + 324x^2 - 1296x^3 + 2916x^4 - \frac{104976}{25}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(72x - 972x^2 + 4752x^3 - 12150x^4 + \frac{2396952}{125}x^5 + \mathrm{O}\left(x^6\right)\right) c_2$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 93

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+36*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{104976x^5}{25} + 2916x^4 - 1296x^3 + 324x^2 - 36x + 1\right) + c_2 \left(\frac{2396952x^5}{125} - 12150x^4 + 4752x^3 - 972x^2 + \left(-\frac{104976x^5}{25} + 2916x^4 - 1296x^3 + 324x^2 - 36x + 1\right) \log(x) + 72x\right)$$

4.7 problem 7

4.7.1 Maple step by step solution 516

Internal problem ID [5666]
Internal file name [`OUTPUT/4914_Sunday_June_05_2022_03_10_22_PM_44815352/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.5. Bessel Functions Y(x). General Solution page 200
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + k^2x^2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (74) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (75) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -k^2 x^2 y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -k^2 x(xy' + 2y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= k^2 \left(x^4 k^2 y - 4xy' - 2y\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= k^2 \left(\left(x^4 k^2 - 6\right) y' + 8x^3 k^2 y\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -x^2 k^4 \left(-12xy' + y\left(x^4 k^2 - 30\right)\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -2k^2 y(0) \\
F_3 &= -6y'(0)\, k^2 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) y(0) + \left(x - \frac{1}{20} k^2 x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -k^2 x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} x^{n+2} k^2 a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=0}^{\infty} x^{n+2} k^2 a_n = \sum_{n=2}^{\infty} a_{n-2} k^2 x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right)+\left(\sum_{n=2}^{\infty} a_{n-2} k^2 x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1)+a_{n-2} k^2 = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-2}k^2}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$a_0k^2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0k^2}{12}$$

For $n = 3$ the recurrence equation gives

$$a_1k^2 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1k^2}{20}$$

For $n = 4$ the recurrence equation gives

$$a_2k^2 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$a_3k^2 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x - \frac{1}{12} a_0 k^2 x^4 - \frac{1}{20} a_1 k^2 x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) a_0 + \left(x - \frac{1}{20} k^2 x^5\right) a_1 + O\left(x^6\right) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) c_1 + \left(x - \frac{1}{20} k^2 x^5\right) c_2 + O\left(x^6\right)
$$

Summary

The solution(s) found are the following

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) y(0) + \left(x - \frac{1}{20} k^2 x^5\right) y'(0) + O\left(x^6\right) \tag{1}
$$

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) c_1 + \left(x - \frac{1}{20} k^2 x^5\right) c_2 + O\left(x^6\right) \tag{2}
$$

Verification of solutions

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) y(0) + \left(x - \frac{1}{20} k^2 x^5\right) y'(0) + O\left(x^6\right)
$$

Verified OK.

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) c_1 + \left(x - \frac{1}{20} k^2 x^5\right) c_2 + O\left(x^6\right)
$$

Verified OK.

4.7.1 Maple step by step solution

Let's solve

$y'' = -k^2 x^2 y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + k^2 x^2 y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  - Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

  - Shift index using $k- >k-2$

    $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + k^2 a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + k^2 a_{k-2} = 0$

- Shift index using $k- > k+2$

  $$\left((k+2)^2+3k+8\right) a_{k+4}+k^2 a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{k^2 a_k}{k^2+7k+12}, a_2=0, a_3=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
Order:=6;
dsolve(diff(y(x),x$2)+k^2*x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{k^2 x^4}{12}\right) y(0)+\left(x-\frac{1}{20} k^2 x^5\right) D(y)(0)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[y''[x]+k^2*x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(x-\frac{k^2 x^5}{20}\right)+c_1\left(1-\frac{k^2 x^4}{12}\right)$$

4.8 problem 8

4.8.1 Maple step by step solution 524

Internal problem ID [5667]
Internal file name [`OUTPUT/4915_Sunday_June_05_2022_03_10_23_PM_3980180/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.5. Bessel Functions Y(x). General Solution page 200
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + x^4k^2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (77) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (78) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -x^4 k^2 y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -k^2 x^3 (x y' + 4y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-8 x y' + y\left(x^6 k^2 - 12\right)\right) x^2 k^2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= k^2 x \left(y' k^2 x^7 + 16 x^6 k^2 y - 36 x y' - 24 y\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= k^2 \left(-x^{12} k^4 y + 24 y' k^2 x^7 + 148 x^6 k^2 y - 96 x y' - 24 y\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= -24 k^2 y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^6 k^2}{30}\right) y(0) + x y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x^4 k^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n+4} k^2 a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=0}^{\infty} x^{n+4} k^2 a_n = \sum_{n=4}^{\infty} a_{n-4} k^2 x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=4}^{\infty} a_{n-4} k^2 x^n\right) = 0 \tag{3}$$

For $4 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_{n-4} k^2 = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-4}k^2}{(n+2)(n+1)} \tag{5}$$

For $n = 4$ the recurrence equation gives

$$a_0k^2 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0k^2}{30}$$

For $n = 5$ the recurrence equation gives

$$a_1k^2 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1k^2}{42}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_1x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = a_1x + a_0 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_2x + c_1 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{x^6 k^2}{30}\right) y(0) + x y'(0) + O(x^6) \tag{1}$$

$$y = c_2 x + c_1 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{x^6 k^2}{30}\right) y(0) + x y'(0) + O(x^6)$$

Verified OK.

$$y = c_2 x + c_1 + O(x^6)$$

Verified OK.

4.8.1 Maple step by step solution

Let's solve

$y'' = -x^4 k^2 y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + x^4 k^2 y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^4 \cdot y$ to series expansion

    $x^4 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+4}$

  ◦ Shift index using $k- > k-4$

    $x^4 \cdot y = \sum_{k=4}^{\infty} a_{k-4} x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

  Rewrite ODE with series expansions

  $$20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=4}^{\infty} \left(a_{k+2}(k+2)(k+1) + k^2 a_{k-4}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2) a_{k+2} + k^2 a_{k-4} = 0$$

- Shift index using $k->k+4$

  $$\left((k+4)^2 + 3k + 14\right) a_{k+6} + k^2 a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{k^2 a_k}{k^2 + 11k + 30}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve(diff(y(x),x$2)+k^2*x^4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + D(y)(0)\,x + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 10

```
AsymptoticDSolveValue[y''[x]+k^2*x^4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 x + c_1$$

4.9 problem 9

4.9.1 Maple step by step solution 539

Internal problem ID [5668]
Internal file name [OUTPUT/4916_Sunday_June_05_2022_03_10_25_PM_54572420/index.tex]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. Special Functions. Problem set 5.5. Bessel Functions Y(x). General Solution page 200
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{xy'' - 5y' + xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - 5y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{5}{x}$$
$$q(x) = 1$$

Table 66: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - 5y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x - 5\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-5(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-5(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 5(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 5r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 5r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-6+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-6+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 6$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-6+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^6\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+6}$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 5a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 6n - 6r} \tag{4}$$

Which for the root $r = 6$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+6)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 6$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 - 2r - 8}$$

Which for the root $r = 6$ becomes

$$a_2 = -\frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 - 20r^2 + 64}$$

Which for the root $r = 6$ becomes

$$a_4 = \frac{1}{640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{r^4-20r^2+64}$	$\frac{1}{640}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{r^4-20r^2+64}$	$\frac{1}{640}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)}$$

Which for the root $r = 6$ becomes

$$a_6 = -\frac{1}{46080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{r^4-20r^2+64}$	$\frac{1}{640}$
a_5	0	0
a_6	$-\frac{1}{(r^4-20r^2+64)r(r+6)}$	$-\frac{1}{46080}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^6\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \dots\right) \\
&= x^6\left(1 - \frac{x^2}{16} + \frac{x^4}{640} - \frac{x^6}{46080} + O(x^7)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_6 \\
&= -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)} &= \lim_{r\to 0} -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$
$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$
$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$
$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $xy'' - 5y' + xy = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x - 5Cy_1'(x)\ln(x) - \frac{5Cy_1(x)}{x} - 5\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + x\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(y_1''(x)\,x + y_1(x)\,x - 5y_1'(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - \frac{5y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) - 5\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x + y_1(x)\,x - 5y_1'(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x-\frac{5y_1(x)}{x}\right)C \\ +\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\ +x\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)-5\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x-6\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} \\ +\frac{\left(\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)x^2-5\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x}{x} \\ =0 \tag{9}$$

Since $r_1=6$ and $r_2=0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{5+n}a_n(n+6)\right)x-6\left(\sum_{n=0}^{\infty}a_n x^{n+6}\right)\right)C}{x} \\ +\frac{\left(\left(\sum_{n=0}^{\infty}x^{n-2}b_n n(-1+n)\right)+\left(\sum_{n=0}^{\infty}b_n x^{n}\right)\right)x^2-5\left(\sum_{n=0}^{\infty}x^{-1+n}b_n n\right)x}{x}=0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}2C\,x^{5+n}a_n(n+6)\right)+\sum_{n=0}^{\infty}\left(-6C\,x^{5+n}a_n\right)+\left(\sum_{n=0}^{\infty}n\,x^{-1+n}b_n(-1+n)\right) \\ +\left(\sum_{n=0}^{\infty}x^{1+n}b_n\right)+\sum_{n=0}^{\infty}\left(-5x^{-1+n}b_n n\right)=0 \tag{2A}$$

The next step is to make all powers of x be $-1+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-1+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{5+n} a_n (n+6) = \sum_{n=6}^{\infty} 2C a_{-6+n} n\, x^{-1+n}$$

$$\sum_{n=0}^{\infty} \left(-6C\, x^{5+n} a_n\right) = \sum_{n=6}^{\infty} \left(-6C a_{-6+n} x^{-1+n}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{-1+n}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-1+n$.

$$\begin{aligned}\left(\sum_{n=6}^{\infty} 2C a_{-6+n} n\, x^{-1+n}\right) + \sum_{n=6}^{\infty} \left(-6C a_{-6+n} x^{-1+n}\right) \\ + \left(\sum_{n=0}^{\infty} n\, x^{-1+n} b_n (-1+n)\right) + \left(\sum_{n=2}^{\infty} b_{n-2} x^{-1+n}\right) + \sum_{n=0}^{\infty} \left(-5x^{-1+n} b_n n\right) = 0\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-5b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-5b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n=2$, Eq (2B) gives

$$-8b_2 + b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-8b_2 + 1 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{8}$$

For $n = 3$, Eq (2B) gives

$$-9b_3 + b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-9b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$-8b_4 + b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-8b_4 + \frac{1}{8} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{64}$$

For $n = 5$, Eq (2B) gives

$$-5b_5 + b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

For $n = N$, where $N = 6$ which is the difference between the two roots, we are free to choose $b_6 = 0$. Hence for $n = 6$, Eq (2B) gives

$$6C + \frac{1}{64} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{384}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{384}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{384}\left(x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\right)\ln(x)+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right) \\
&\quad + c_2\left(-\frac{1}{384}\left(x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\right)\ln(x)+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right) \\
&\quad + c_2\left(-\frac{x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\ln(x)}{384}+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right) \\
&\quad + c_2\left(-\frac{x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\ln(x)}{384}+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right) \\
&\quad + c_2\left(-\frac{x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\ln(x)}{384}+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned}$$

Verified OK.

4.9.1 Maple step by step solution

Let's solve

$y''x - 5y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{x} + y = 0$

- Simplify ODE

  $x^2y'' - 5xy' + yx^2 = 0$

- Make a change of variables

  $y = x^3u(x)$

- Compute y'

  $y' = 3x^2u(x) + x^3u'(x)$

- Compute y''

  $y'' = 6xu(x) + 6x^2u'(x) + x^3u''(x)$

- Apply change of variables to the ODE

  $x^2u(x) + u''(x)\,x^2 + u'(x)\,x - 9u(x) = 0$

- ODE is now of the Bessel form
- Solution to Bessel ODE

  $u(x) = c_1 BesselJ(3, x) + c_2 BesselY(3, x)$

- Make the change from y back to y

  $y = (c_1 BesselJ(3, x) + c_2 BesselY(3, x))\,x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*diff(y(x),x$2)-5*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^6 \left(1 - \frac{1}{16}x^2 + \frac{1}{640}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(-86400 - 10800x^2 - 1350x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 44

```
AsymptoticDSolveValue[x*y''[x]-5*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{64} + \frac{x^2}{8} + 1\right) + c_2\left(\frac{x^{10}}{640} - \frac{x^8}{16} + x^6\right)$$

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5.1 problem 11

5.1.1 Maple step by step solution 549

Internal problem ID [5669]
Internal file name [`OUTPUT/4917_Sunday_June_05_2022_03_10_28_PM_25425926/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. REVIEW QUESTIONS. page 201
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (81)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (82)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 16y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 16y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -64y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -4y(0) \\
F_1 &= -4y'(0) \\
F_2 &= 16y(0) \\
F_3 &= 16y'(0) \\
F_4 &= -64y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty}n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}=-4\left(\sum_{n=0}^{\infty}a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty}4a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n\right)+\left(\sum_{n=0}^{\infty}4a_n x^n\right)=0 \tag{3}$$

For $0\leq n$, the recurrence equation is

$$(n+2)\,a_{n+2}(n+1)+4a_n=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{4a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 + 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -2a_0$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{2a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{2a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{4a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{4a_1}{315}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 2a_0 x^2 - \frac{2}{3} a_1 x^3 + \frac{2}{3} a_0 x^4 + \frac{2}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4\right) a_0 + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) c_2 + O(x^6) \tag{2}$$

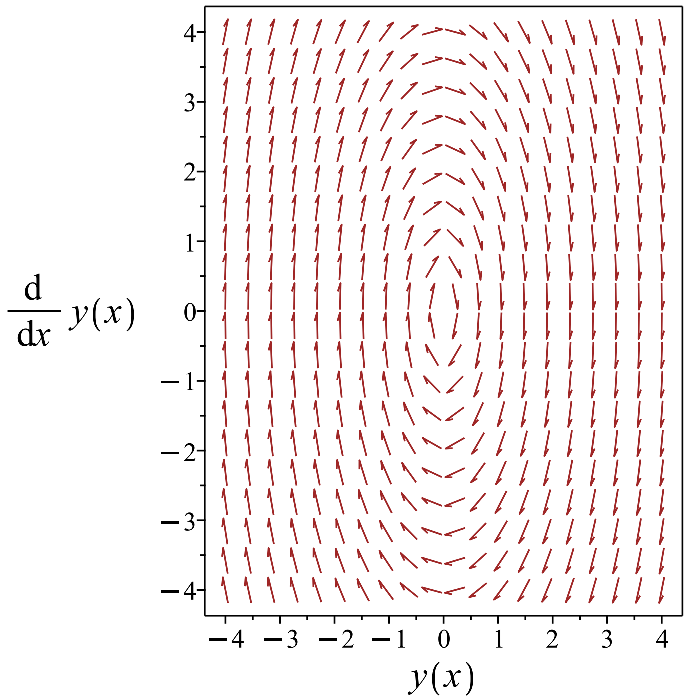

Figure 4: Slope field plot

Verification of solutions

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) c_2 + O(x^6)$$

Verified OK.

5.1.1 Maple step by step solution

Let's solve

$y'' = -4y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 4y = 0$

- Characteristic polynomial of ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(2x) + c_2 \sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - 2x^2 + \frac{2}{3}x^4\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[y''[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{2x^5}{15} - \frac{2x^3}{3} + x \right) + c_1 \left(\frac{2x^4}{3} - 2x^2 + 1 \right)$$

5.2 problem 12

5.2.1 Maple step by step solution 560

Internal problem ID [5670]
Internal file name [`OUTPUT/4918_Sunday_June_05_2022_03_10_29_PM_13387700/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. REVIEW QUESTIONS. page 201
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (1-2x)\, y' + (x-1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (1-2x)\, y' + (x-1)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x-1}{x}$$
$$q(x) = \frac{x-1}{x}$$

Table 69: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (1 - 2x)\,y' + (x - 1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) x + (1 - 2x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} (n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} (n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln (x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{2r + 1}{(1 + r)^2}$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) - 2a_{n-1}(n + r - 1) + a_n(n + r) + a_{n-2} - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2na_{n-1} + 2ra_{n-1} - a_{n-2} - a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(2n-1)\, a_{n-1} - a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{(1+r)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3r^2 + 6r + 2}{(1+r)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{(1+r)^2}$	1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4r^3 + 18r^2 + 22r + 6}{(1+r)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{(1+r)^2}$	1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$
a_3	$\frac{4r^3+18r^2+22r+6}{(1+r)^2(r+2)^2(r+3)^2}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5r^4 + 40r^3 + 105r^2 + 100r + 24}{(1+r)^2 (r+2)^2 (r+3)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{(1+r)^2}$	1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$
a_3	$\frac{4r^3+18r^2+22r+6}{(1+r)^2(r+2)^2(r+3)^2}$	$\frac{1}{6}$
a_4	$\frac{5r^4+40r^3+105r^2+100r+24}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{6r^5 + 75r^4 + 340r^3 + 675r^2 + 548r + 120}{(1+r)^2 (r+2)^2 (r+3)^2 (4+r)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{(1+r)^2}$	1
a_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$
a_3	$\frac{4r^3+18r^2+22r+6}{(1+r)^2(r+2)^2(r+3)^2}$	$\frac{1}{6}$
a_4	$\frac{5r^4+40r^3+105r^2+100r+24}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{24}$
a_5	$\frac{6r^5+75r^4+340r^3+675r^2+548r+120}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2(r+5)^2}$	$\frac{1}{120}$

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{2r+1}{(1+r)^2}$	1	$-\frac{2r}{(1+r)^3}$	0
b_2	$\frac{3r^2+6r+2}{(1+r)^2(r+2)^2}$	$\frac{1}{2}$	$\frac{-6r^3-18r^2-14r}{(1+r)^3(r+2)^3}$	0
b_3	$\frac{4r^3+18r^2+22r+6}{(1+r)^2(r+2)^2(r+3)^2}$	$\frac{1}{6}$	$-\frac{12\left(r^4+8r^3+\frac{47}{2}r^2+30r+\frac{85}{6}\right)r}{(1+r)^3(r+2)^3(r+3)^3}$	0
b_4	$\frac{5r^4+40r^3+105r^2+100r+24}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2}$	$\frac{1}{24}$	$-\frac{20\left(r^6+15r^5+\frac{183}{2}r^4+290r^3+\frac{5031}{10}r^2+453r+166\right)r}{(1+r)^3(r+2)^3(r+3)^3(4+r)^3}$	0
b_5	$\frac{6r^5+75r^4+340r^3+675r^2+548r+120}{(1+r)^2(r+2)^2(r+3)^2(4+r)^2(r+5)^2}$	$\frac{1}{120}$	$-\frac{30\left(r^8+24r^7+\frac{739}{3}r^6+1410r^5+4915r^4+10668r^3+14063r^2+10290r+\frac{48076}{15}\right)r}{(1+r)^3(r+2)^3(r+3)^3(4+r)^3(r+5)^3}$	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) + O(x^6)\right)
\end{aligned}$$

Verified OK.

5.2.1 Maple step by step solution

Let's solve

$y''x + (1-2x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + (1-2x)\,y' + (x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right) x^{k}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term must be 0

$$a_1(1+r)^2 - a_0(1+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + (-2k-1)\, a_k + a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2)^2 + (-2k-3)\, a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(1-2*x)*diff(y(x),x)+(x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 74

```
AsymptoticDSolveValue[x*y''[x]+(1-2*x)*y'[x]+(x-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right) + c_2\left(\frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right)\log(x)$$

5.3 problem 13

5.3.1 Maple step by step solution 571

Internal problem ID [5671]
Internal file name [`OUTPUT/4919_Sunday_June_05_2022_03_10_32_PM_35236915/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. REVIEW QUESTIONS. page 201
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)^2 y''-(x-1) y'-35y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{85}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{86}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{xy' - y' + 35y}{(x-1)^2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(35x - 35)\, y' - 35y}{(x-1)^3} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-70x + 70)\, y' + 1330y}{(x-1)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{1470(x-1)\, y' - 7770y}{(x-1)^5} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-12180x + 12180)\, y' + 90300y}{(x-1)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 35y(0) - y'(0) \\
F_1 &= 35y(0) + 35y'(0) \\
F_2 &= 1330y(0) + 70y'(0) \\
F_3 &= 7770y(0) + 1470y'(0) \\
F_4 &= 90300y(0) + 12180y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1+\frac{35}{2}x^2+\frac{35}{6}x^3+\frac{665}{12}x^4+\frac{259}{4}x^5+\frac{1505}{12}x^6\right)y(0) \\ +\left(x-\frac{1}{2}x^2+\frac{35}{6}x^3+\frac{35}{12}x^4+\frac{49}{4}x^5+\frac{203}{12}x^6\right)y'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(x^2-2x+1\right)y''+(1-x)\,y'-35y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty}na_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(x^2-2x+1\right)\left(\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}\right)+(1-x)\left(\sum_{n=1}^{\infty}na_n x^{n-1}\right)-35\left(\sum_{n=0}^{\infty}a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}x^n a_n n(n-1)\right)+\sum_{n=2}^{\infty}\left(-2n\,x^{n-1}a_n(n-1)\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}\right) \\ +\left(\sum_{n=1}^{\infty}na_n x^{n-1}\right)+\sum_{n=1}^{\infty}\left(-na_n x^n\right)+\sum_{n=0}^{\infty}\left(-35a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-2n\, x^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-2(n+1)\, a_{n+1} n\, x^n\right)$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=1}^{\infty} \left(-2(n+1)\, a_{n+1} n\, x^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n\right) + \sum_{n=1}^{\infty} \left(-n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-35 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 - 35a_0 = 0$$

$$a_2 = \frac{35a_0}{2} - \frac{a_1}{2}$$

$n = 1$ gives

$$-2a_2 + 6a_3 - 36a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{35a_0}{6} + \frac{35a_1}{6}$$

For $2 \le n$, the recurrence equation is

$$n a_n (n-1) - 2(n+1)\, a_{n+1} n + (n+2)\, a_{n+2} (n+1) + (n+1)\, a_{n+1} - n a_n - 35 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n - 2n^2 a_{n+1} - 2n a_n - n a_{n+1} - 35 a_n + a_{n+1}}{(n+2)(n+1)} \\ &= -\frac{(n^2 - 2n - 35)\, a_n}{(n+2)(n+1)} - \frac{(-2n^2 - n + 1)\, a_{n+1}}{(n+2)(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-35a_2 - 9a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{665a_0}{12} + \frac{35a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$-32a_3 - 20a_4 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{259a_0}{4} + \frac{49a_1}{4}$$

For $n = 4$ the recurrence equation gives

$$-27a_4 - 35a_5 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1505a_0}{12} + \frac{203a_1}{12}$$

For $n = 5$ the recurrence equation gives

$$-20a_5 - 54a_6 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{2305a_0}{12} + \frac{331a_1}{12}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x + \left(\frac{35a_0}{2} - \frac{a_1}{2}\right) x^2 + \left(\frac{35a_0}{6} + \frac{35a_1}{6}\right) x^3 \\ + \left(\frac{665a_0}{12} + \frac{35a_1}{12}\right) x^4 + \left(\frac{259a_0}{4} + \frac{49a_1}{4}\right) x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y = &\left(1 + \frac{35}{2}x^2 + \frac{35}{6}x^3 + \frac{665}{12}x^4 + \frac{259}{4}x^5\right) a_0 \\ &+ \left(x - \frac{1}{2}x^2 + \frac{35}{6}x^3 + \frac{35}{12}x^4 + \frac{49}{4}x^5\right) a_1 + O(x^6) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{35}{2}x^2 + \frac{35}{6}x^3 + \frac{665}{12}x^4 + \frac{259}{4}x^5\right) c_1 + \left(x - \frac{1}{2}x^2 + \frac{35}{6}x^3 + \frac{35}{12}x^4 + \frac{49}{4}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 + \frac{35}{2}x^2 + \frac{35}{6}x^3 + \frac{665}{12}x^4 + \frac{259}{4}x^5 + \frac{1505}{12}x^6\right) y(0) \\ &+ \left(x - \frac{1}{2}x^2 + \frac{35}{6}x^3 + \frac{35}{12}x^4 + \frac{49}{4}x^5 + \frac{203}{12}x^6\right) y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$\begin{aligned} y = &\left(1 + \frac{35}{2}x^2 + \frac{35}{6}x^3 + \frac{665}{12}x^4 + \frac{259}{4}x^5\right) c_1 \\ &+ \left(x - \frac{1}{2}x^2 + \frac{35}{6}x^3 + \frac{35}{12}x^4 + \frac{49}{4}x^5\right) c_2 + O(x^6) \end{aligned} \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{35}{2}x^2 + \frac{35}{6}x^3 + \frac{665}{12}x^4 + \frac{259}{4}x^5 + \frac{1505}{12}x^6\right) y(0) + \left(x - \frac{1}{2}x^2 + \frac{35}{6}x^3 + \frac{35}{12}x^4 + \frac{49}{4}x^5 + \frac{203}{12}x^6\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1+\frac{35}{2}x^2+\frac{35}{6}x^3+\frac{665}{12}x^4+\frac{259}{4}x^5\right) c_1+\left(x-\frac{1}{2}x^2+\frac{35}{6}x^3+\frac{35}{12}x^4+\frac{49}{4}x^5\right) c_2+O(x^6)$$

Verified OK.

5.3.1 Maple step by step solution

Let's solve

$(x^2 - 2x + 1)\, y'' + (1 - x)\, y' - 35y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{35y}{x^2-2x+1} + \frac{y'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x-1} - \frac{35y}{x^2-2x+1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{1}{x-1}, P_3(x) = -\frac{35}{x^2-2x+1}\right]$

- ◦ $(x - 1) \cdot P_2(x)$ is analytic at $x = 1$

  $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

- ◦ $(x - 1)^2 \cdot P_3(x)$ is analytic at $x = 1$

  $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = -35$

- ◦ $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $(x-1)\, y''(x^2-2x+1) + (-x^2+2x-1)\, y' + (-35x+35)\, y = 0$

- Change variables using $x = u+1$ so that the regular singular point is at $u = 0$

  $u^3\left(\frac{d^2}{du^2}y(u)\right) - u^2\left(\frac{d}{du}y(u)\right) - 35uy(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite DE with series expansions

  - Convert $u \cdot y(u)$ to series expansion

    $u \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+1}$

  - Shift index using $k->k-1$

    $u \cdot y(u) = \sum_{k=1}^{\infty} a_{k-1} u^{k+r}$

  - Convert $u^2 \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

    $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r+1}$

  - Shift index using $k->k-1$

    $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, u^{k+r}$

  - Convert $u^3 \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u^3 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, u^{k+r+1}$

  - Shift index using $k->k-1$

    $u^3 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,(k-2+r)\, u^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=1}^{\infty} a_{k-1}(k+4+r)\,(k-8+r)\, u^{k+r} = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

$a_{k-1}(k+4)(k-8)=0$

- Shift index using $k- >k+1$

  $a_k(k+5)(k-7)=0$

- Recursion relation that defines series solution to ODE

  $a_k=0$

- Recursion relation for $r=0$

  $a_k=0$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_k=0\right]$

- Revert the change of variables $u=x-1$

  $\left[y=\sum_{k=0}^{\infty} a_k(x-1)^k, a_k=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
Order:=6;
dsolve((x-1)^2*diff(y(x),x$2)-(x-1)*diff(y(x),x)-35*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1+\frac{35}{2}x^2+\frac{35}{6}x^3+\frac{665}{12}x^4+\frac{259}{4}x^5\right)y(0)+\left(x-\frac{1}{2}x^2+\frac{35}{6}x^3+\frac{35}{12}x^4+\frac{49}{4}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 70

```
AsymptoticDSolveValue[(x-1)^2*y''[x]-(x-1)*y'[x]-35*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{259x^5}{4} + \frac{665x^4}{12} + \frac{35x^3}{6} + \frac{35x^2}{2} + 1\right) + c_2\left(\frac{49x^5}{4} + \frac{35x^4}{12} + \frac{35x^3}{6} - \frac{x^2}{2} + x\right)$$

5.4 problem 14

5.4.1 Maple step by step solution 583

Internal problem ID [5672]
Internal file name [`OUTPUT/4920_Sunday_June_05_2022_03_10_33_PM_38014976/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. REVIEW QUESTIONS. page 201
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{16(1+x)^2 y'' + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (88)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (89)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= -\frac{3y}{16\left(1+x\right)^2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(-3x-3\right) y' + 6y}{16\left(1+x\right)^3} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-\frac{279y}{256} + \frac{3(1+x)y'}{4}}{\left(1+x\right)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{1080y + \left(-855x-855\right) y'}{256\left(1+x\right)^5} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{-\frac{83835y}{4096} + \frac{1125(1+x)y'}{64}}{\left(1+x\right)^6}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -\frac{3y(0)}{16} \\
F_1 &= \frac{3y(0)}{8} - \frac{3y'(0)}{16} \\
F_2 &= -\frac{279y(0)}{256} + \frac{3y'(0)}{4} \\
F_3 &= \frac{135y(0)}{32} - \frac{855y'(0)}{256} \\
F_4 &= -\frac{83835y(0)}{4096} + \frac{1125y'(0)}{64}
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{3}{32}x^2+\frac{1}{16}x^3-\frac{93}{2048}x^4+\frac{9}{256}x^5-\frac{1863}{65536}x^6\right)y(0) \\
&+\left(x-\frac{1}{32}x^3+\frac{1}{32}x^4-\frac{57}{2048}x^5+\frac{25}{1024}x^6\right)y'(0)+O(x^6)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(16x^2+32x+16)\,y''+3y=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(16x^2+32x+16)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+3\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 16x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} 32n\,x^{n-1}a_n(n-1)\right) \\
&+\left(\sum_{n=2}^{\infty} 16n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} 3a_n x^n\right)=0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} 32n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 32(n+1)\, a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} 16n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 16(n+2)\, a_{n+2} (n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 16x^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 32(n+1)\, a_{n+1} n\, x^n\right) + \left(\sum_{n=0}^{\infty} 16(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$32a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{32}$$

$n = 1$ gives

$$64a_2 + 96a_3 + 3a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{16} - \frac{a_1}{32}$$

For $2 \leq n$, the recurrence equation is

$$16na_n(n-1) + 32(n+1)\, a_{n+1} n + 16(n+2)\, a_{n+2}(n+1) + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{16n^2 a_n + 32n^2 a_{n+1} - 16na_n + 32na_{n+1} + 3a_n}{16\,(n+2)\,(n+1)} \\ &= -\frac{(16n^2 - 16n + 3)\, a_n}{16\,(n+2)\,(n+1)} - \frac{(32n^2 + 32n)\, a_{n+1}}{16\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$35a_2 + 192a_3 + 192a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{93a_0}{2048} + \frac{a_1}{32}$$

For $n = 3$ the recurrence equation gives

$$99a_3 + 384a_4 + 320a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{9a_0}{256} - \frac{57a_1}{2048}$$

For $n = 4$ the recurrence equation gives

$$195a_4 + 640a_5 + 480a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1863a_0}{65536} + \frac{25a_1}{1024}$$

For $n = 5$ the recurrence equation gives

$$323a_5 + 960a_6 + 672a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{777a_0}{32768} - \frac{1409a_1}{65536}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{3a_0 x^2}{32} + \left(\frac{a_0}{16} - \frac{a_1}{32}\right) x^3 + \left(-\frac{93a_0}{2048} + \frac{a_1}{32}\right) x^4 + \left(\frac{9a_0}{256} - \frac{57a_1}{2048}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$\begin{aligned} y = {} & \left(1 - \frac{3}{32}x^2 + \frac{1}{16}x^3 - \frac{93}{2048}x^4 + \frac{9}{256}x^5\right) a_0 \\ & + \left(x - \frac{1}{32}x^3 + \frac{1}{32}x^4 - \frac{57}{2048}x^5\right) a_1 + O(x^6) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{32}x^2 + \frac{1}{16}x^3 - \frac{93}{2048}x^4 + \frac{9}{256}x^5\right) c_1 + \left(x - \frac{1}{32}x^3 + \frac{1}{32}x^4 - \frac{57}{2048}x^5\right) c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 - \frac{3}{32}x^2 + \frac{1}{16}x^3 - \frac{93}{2048}x^4 + \frac{9}{256}x^5 - \frac{1863}{65536}x^6\right) y(0) \\ & + \left(x - \frac{1}{32}x^3 + \frac{1}{32}x^4 - \frac{57}{2048}x^5 + \frac{25}{1024}x^6\right) y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$\begin{aligned} y = {} & \left(1 - \frac{3}{32}x^2 + \frac{1}{16}x^3 - \frac{93}{2048}x^4 + \frac{9}{256}x^5\right) c_1 \\ & + \left(x - \frac{1}{32}x^3 + \frac{1}{32}x^4 - \frac{57}{2048}x^5\right) c_2 + O(x^6) \end{aligned} \tag{2}$$

<u>Verification of solutions</u>

$$\begin{aligned} y = {} & \left(1 - \frac{3}{32}x^2 + \frac{1}{16}x^3 - \frac{93}{2048}x^4 + \frac{9}{256}x^5 - \frac{1863}{65536}x^6\right) y(0) \\ & + \left(x - \frac{1}{32}x^3 + \frac{1}{32}x^4 - \frac{57}{2048}x^5 + \frac{25}{1024}x^6\right) y'(0) + O(x^6) \end{aligned}$$

Verified OK.

$$y = \left(1 - \frac{3}{32}x^2 + \frac{1}{16}x^3 - \frac{93}{2048}x^4 + \frac{9}{256}x^5\right) c_1 + \left(x - \frac{1}{32}x^3 + \frac{1}{32}x^4 - \frac{57}{2048}x^5\right) c_2 + O(x^6)$$

Verified OK.

5.4.1 Maple step by step solution

Let's solve

$(16x^2 + 32x + 16) y'' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{16(x^2+2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y}{16(x^2+2x+1)} = 0$

□ Check to see if $x_0 = -1$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{3}{16(x^2+2x+1)}\right]$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 0$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = \frac{3}{16}$

  - $x = -1$is a regular singular point

  Check to see if $x_0 = -1$ is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(16x^2 + 32x + 16) y'' + 3y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $16u^2\left(\frac{d^2}{du^2}y(u)\right) + 3y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite DE with series expansions

- Convert $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $$u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)u^{k+r}$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} a_k(4k+4r-1)(4k+4r-3)u^{k+r} = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k-1)(4k-3) = 0$

- Recursion relation that defines series solution to ODE

  $a_k = 0$

- Recursion relation for $r = 0$

  $a_k = 0$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_k = 0\right]$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_k = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(16*(x+1)^2*diff(y(x),x$2)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{32}x^2 + \frac{1}{16}x^3 - \frac{93}{2048}x^4 + \frac{9}{256}x^5\right) y(0) + \left(x - \frac{1}{32}x^3 + \frac{1}{32}x^4 - \frac{57}{2048}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[16*(x+1)^2*y''[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{57x^5}{2048} + \frac{x^4}{32} - \frac{x^3}{32} + x\right) + c_1\left(\frac{9x^5}{256} - \frac{93x^4}{2048} + \frac{x^3}{16} - \frac{3x^2}{32} + 1\right)$$

5.5 problem 15

5.5.1 Maple step by step solution 595

Internal problem ID [5673]
Internal file name [OUTPUT/4921_Sunday_June_05_2022_03_10_34_PM_45195600/index.tex]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. REVIEW QUESTIONS. page 201
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[_Bessel]
```

$$\boxed{x^2y'' + xy' + (x^2 - 5)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (x^2 - 5)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 - 5}{x^2}$$

Table 73: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-5}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + (x^2 - 5)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (x^2-5)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-5 a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-5 a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 5 a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) + x^r a_0 r - 5 a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) + x^r r - 5 x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2 - 5\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
r^2 - 5 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \sqrt{5}$$
$$r_2 = -\sqrt{5}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 5)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2\sqrt{5}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\sqrt{5}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\sqrt{5}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - 5a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 5} \tag{4}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_n = -\frac{a_{n-2}}{n\,(2\sqrt{5} + n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \sqrt{5}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 4r - 1}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_2 = -\frac{1}{4 + 4\sqrt{5}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r^2 + 4r - 1)(r^2 + 8r + 11)}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_4 = \frac{1}{32\left(\sqrt{5}+1\right)\left(2+\sqrt{5}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$
a_3	0	0
a_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32\left(\sqrt{5}+1\right)\left(2+\sqrt{5}\right)}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+4r-1}$	$-\frac{1}{4+4\sqrt{5}}$
a_3	0	0
a_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32\left(\sqrt{5}+1\right)\left(2+\sqrt{5}\right)}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\sqrt{5}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\sqrt{5}}\left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}+1\right)\left(2+\sqrt{5}\right)} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - 5b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 - 5} \tag{4}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_n = -\frac{b_{n-2}}{n\left(-2\sqrt{5} + n\right)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\sqrt{5}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 4r - 1}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_2 = \frac{1}{-4 + 4\sqrt{5}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 4r - 1)(r^2 + 8r + 11)}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_4 = \frac{1}{32\left(\sqrt{5} - 1\right)\left(-2 + \sqrt{5}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$
b_3	0	0
b_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+4r-1}$	$\frac{1}{-4+4\sqrt{5}}$
b_3	0	0
b_4	$\frac{1}{(r^2+4r-1)(r^2+8r+11)}$	$\frac{1}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\sqrt{5}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{-\sqrt{5}}\left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\sqrt{5}}\left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}+1\right)\left(2+\sqrt{5}\right)} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{-\sqrt{5}}\left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\sqrt{5}}\left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}+1\right)\left(2+\sqrt{5}\right)} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{-\sqrt{5}}\left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\sqrt{5}} \left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}+1\right)\left(2+\sqrt{5}\right)} + O(x^6)\right) + c_2 x^{-\sqrt{5}} \left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\sqrt{5}} \left(1 - \frac{x^2}{4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}+1\right)\left(2+\sqrt{5}\right)} + O(x^6)\right) + c_2 x^{-\sqrt{5}} \left(1 + \frac{x^2}{-4+4\sqrt{5}} + \frac{x^4}{32\left(\sqrt{5}-1\right)\left(-2+\sqrt{5}\right)} + O(x^6)\right)$$

Verified OK.

5.5.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + (x^2-5)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-5)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-5)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-5}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -5$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x y' + (x^2 - 5)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- > k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(r^2 - 5)\, x^r + a_1(r^2 + 2r - 4)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k^2 + 2kr + r^2 - 5) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 - 5 = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\sqrt{5}, -\sqrt{5}\right\}$

- Each term must be 0

  $a_1(r^2 + 2r - 4) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k^2 + 2kr + r^2 - 5) + a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left((k+2)^2 + 2(k+2)\, r + r^2 - 5\right) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k^2+2kr+r^2+4k+4r-1}$

- Recursion relation for $r = \sqrt{5}$

  $a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}$

- Solution for $r = \sqrt{5}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}}, a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}, a_1 = 0\right]$

- Recursion relation for $r = -\sqrt{5}$

  $a_{k+2} = -\frac{a_k}{k^2-2k\sqrt{5}+4+4k-4\sqrt{5}}$

- Solution for $r = -\sqrt{5}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\sqrt{5}}, a_{k+2} = -\frac{a_k}{k^2-2k\sqrt{5}+4+4k-4\sqrt{5}}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\sqrt{5}}\right), a_{k+2} = -\frac{a_k}{k^2+2k\sqrt{5}+4+4k+4\sqrt{5}}, a_1 = 0, b_{k+2} = -\frac{b_k}{k^2-2k\sqrt{5}+4+4}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 97

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-5)*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x^{-\sqrt{5}} \left(1 + \frac{1}{-4+4\sqrt{5}} x^2 + \frac{1}{32} \frac{1}{\left(-2+\sqrt{5}\right)\left(\sqrt{5}-1\right)} x^4 + \mathrm{O}\left(x^6\right)\right) \\
& + c_2 x^{\sqrt{5}} \left(1 - \frac{1}{4+4\sqrt{5}} x^2 + \frac{1}{32} \frac{1}{\left(\sqrt{5}+2\right)\left(\sqrt{5}+1\right)} x^4 + \mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 210

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-5)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^4}{\left(-3-\sqrt{5}+\left(1-\sqrt{5}\right)\left(2-\sqrt{5}\right)\right)\left(-1-\sqrt{5}+\left(3-\sqrt{5}\right)\left(4-\sqrt{5}\right)\right)} - \frac{x^2}{-3-\sqrt{5}+\left(1-\sqrt{5}\right)\left(2-\sqrt{5}\right)}+1\right)x^{-\sqrt{5}} + c_1\left(\frac{x^4}{\left(-3+\sqrt{5}+\left(1+\sqrt{5}\right)\left(2+\sqrt{5}\right)\right)\left(-1+\sqrt{5}+\left(3+\sqrt{5}\right)\left(4+\sqrt{5}\right)\right)} - \frac{x^2}{-3+\sqrt{5}+\left(1+\sqrt{5}\right)\left(2+\sqrt{5}\right)}+1\right)x^{\sqrt{5}}$$

5.6 problem 16

5.6.1 Maple step by step solution 609

Internal problem ID [5674]
Internal file name [`OUTPUT/4922_Sunday_June_05_2022_03_10_37_PM_25431950/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. REVIEW QUESTIONS. page 201
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 2y'x^3 + \left(x^2 - 2\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 2y'x^3 + \left(x^2 - 2\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 2x$$
$$q(x) = \frac{x^2 - 2}{x^2}$$

Table 75: Table $p(x), q(x)$ singularites.

$p(x) = 2x$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{x^2-2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty, -\infty, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + 2y' x^3 + \left(x^2 - 2\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 2\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x^3 + \left(x^2 - 2\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2 x^{2+n+r} a_n (n+r)\right) \\ +\left(\sum_{n=0}^{\infty} x^{2+n+r} a_n\right)+\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2 x^{2+n+r} a_n (n+r)=\sum_{n=2}^{\infty} 2 a_{n-2}(n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{2+n+r} a_n=\sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 2 a_{n-2}(n+r-2) x^{n+r}\right) \\ +\left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1)-2 a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)-2 a_0 x^r=0$$

Or

$$\left(x^r r(-1+r)-2 x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-r-2\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-2}(n+r-2) + a_{n-2} - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(2n+2r-3)}{n^2+2nr+r^2-n-r-2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{(-2n-1)\, a_{n-2}}{n\,(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-2r-1}{r\,(r+3)}$$

Which for the root $r = 2$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2r-1}{r(r+3)}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2r-1}{r(r+3)}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 + 12r + 5}{r(r+3)(r+5)(r+2)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{9}{56}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2r-1}{r(r+3)}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{4r^2+12r+5}{r(r+3)(r+5)(r+2)}$	$\frac{9}{56}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2r-1}{r(r+3)}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{4r^2+12r+5}{r(r+3)(r+5)(r+2)}$	$\frac{9}{56}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2 \left(1 - \frac{x^2}{2} + \frac{9x^4}{56} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} 0 &= \lim_{r \to -1} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n + r)(n + r - 1) + 2b_{n-2}(n + r - 2) + b_{n-2} - 2b_n = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n - 1)(n - 2) + 2b_{n-2}(n - 3) + b_{n-2} - 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}(2n+2r-3)}{n^2+2nr+r^2-n-r-2} \tag{5}$$

Which for the root $r=-1$ becomes

$$b_n = -\frac{b_{n-2}(2n-5)}{n^2-3n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r=-1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n=2$, using the above recursive equation gives

$$b_2 = -\frac{1+2r}{r(r+3)}$$

Which for the root $r=-1$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2r-1}{r(r+3)}$	$-\frac{1}{2}$

For $n=3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2r-1}{r(r+3)}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^2 + 12r + 5}{r(r+3)(r^2 + 7r + 10)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{3}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2r-1}{r(r+3)}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{4r^2+12r+5}{r(r+3)(r+5)(r+2)}$	$\frac{3}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2r-1}{r(r+3)}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{4r^2+12r+5}{r(r+3)(r+5)(r+2)}$	$\frac{3}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^2 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 - \frac{x^2}{2} + \frac{9x^4}{56} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - \frac{x^2}{2} + \frac{9x^4}{56} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \left(1 - \frac{x^2}{2} + \frac{9x^4}{56} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(1 - \frac{x^2}{2} + \frac{9x^4}{56} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x}$$

Verified OK.

5.6.1 Maple step by step solution

Let's solve

$x^2 y'' + 2y' x^3 + (x^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -2xy' - \frac{(x^2-2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2xy' + \frac{(x^2-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = 2x, P_3(x) = \frac{x^2-2}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + 2y' x^3 + (x^2 - 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^3 \cdot y'$ to series expansion

  $x^3 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+2}$

- Shift index using $k- >k - 2$

  $x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2} (k-2+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\, (k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$$a_0(1+r)(-2+r)x^r + a_1(2+r)(-1+r)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)(k-2+r) + a_{k-2}(2k-3\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term must be 0

  $a_1(2+r)(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k-2+r) + 2a_{k-2}\left(k-\frac{3}{2}+r\right) = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+3+r)(k+r) + 2a_k\left(k+\frac{1}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(2k+2r+1)}{(k+3+r)(k+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k(2k-1)}{(k+2)(k-1)}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(2k-1)}{(k+2)(k-1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k(2k+5)}{(k+5)(k+2)}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k(2k+5)}{(k+5)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k(2k-1)}{(k+2)(k-1)}, a_1 = 0, b_{k+2} = -\frac{b_k(2k+5)}{(k+5)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+2*x^3*diff(y(x),x)+(x^2-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 - \frac{1}{2}x^2 + \frac{9}{56}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(12 - 6x^2 + \frac{9}{2}x^4 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 44

```
AsymptoticDSolveValue[x^2*y''[x]+2*x^3*y'[x]+(x^2-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{3x^3}{8} - \frac{x}{2} + \frac{1}{x}\right) + c_2\left(\frac{9x^6}{56} - \frac{x^4}{2} + x^2\right)$$

5.7 problem 17

5.7.1 Maple step by step solution 624

Internal problem ID [5675]
Internal file name [`OUTPUT/4923_Sunday_June_05_2022_03_10_39_PM_80628939/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. REVIEW QUESTIONS. page 201
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{xy'' - (1 + x)\, y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (-1 - x)\, y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1 + x}{x}$$
$$q(x) = \frac{1}{x}$$

Table 77: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1+x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-1 - x)\, y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + (-1-x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n+r} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{a_{n-1}}{n+2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{1+r}$$

Which for the root $r = 2$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)(2+r)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{3}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(2+r)(3+r)(1+r)}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{1}{60}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{3}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(2+r)(3+r)(1+r)}$	$\frac{1}{60}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(3+r)(1+r)(2+r)(4+r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{3}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(2+r)(3+r)(1+r)}$	$\frac{1}{60}$
a_4	$\frac{1}{(3+r)(1+r)(2+r)(4+r)}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(1+r)(2+r)(4+r)(3+r)(5+r)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{1}{2520}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	$\frac{1}{3}$
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(2+r)(3+r)(1+r)}$	$\frac{1}{60}$
a_4	$\frac{1}{(3+r)(1+r)(2+r)(4+r)}$	$\frac{1}{360}$
a_5	$\frac{1}{(1+r)(2+r)(4+r)(3+r)(5+r)}$	$\frac{1}{2520}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= x^2 \left(1 + \frac{x}{3} + \frac{x^2}{12} + \frac{x^3}{60} + \frac{x^4}{360} + \frac{x^5}{2520} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{1}{(1+r)(2+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1}{(1+r)(2+r)} &= \lim_{r \to 0} \frac{1}{(1+r)(2+r)} \\ &= \frac{1}{2} \end{aligned}$$

The limit is $\frac{1}{2}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^n
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) - (n+r)b_n + b_{n-1} = 0 \tag{4}
$$

Which for for the root $r = 0$ becomes

$$
b_n n(n-1) - b_{n-1}(n-1) - nb_n + b_{n-1} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = \frac{b_{n-1}}{n+r} \tag{5}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{b_{n-1}}{n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{1}{1+r}
$$

Which for the root $r = 0$ becomes

$$
b_1 = 1
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{(2+r)(3+r)(1+r)}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(1+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(3+r)(1+r)(2+r)(4+r)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(1+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(3+r)(1+r)(2+r)(4+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(1+r)(2+r)(4+r)(3+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{1+r}$	1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(1+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(3+r)(1+r)(2+r)(4+r)}$	$\frac{1}{24}$
b_5	$\frac{1}{(1+r)(2+r)(4+r)(3+r)(5+r)}$	$\frac{1}{120}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 + \frac{x}{3} + \frac{x^2}{12} + \frac{x^3}{60} + \frac{x^4}{360} + \frac{x^5}{2520} + O(x^6)\right) + c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \right. \\
&\qquad \left. + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 + \frac{x}{3} + \frac{x^2}{12} + \frac{x^3}{60} + \frac{x^4}{360} + \frac{x^5}{2520} + O(x^6)\right) + c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \right. \\
&\qquad \left. + O(x^6)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^2 &\left(1 + \frac{x}{3} + \frac{x^2}{12} + \frac{x^3}{60} + \frac{x^4}{360} + \frac{x^5}{2520} + O(x^6)\right) \\
&+ c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)
\end{aligned}
\tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y = c_1 x^2 &\left(1 + \frac{x}{3} + \frac{x^2}{12} + \frac{x^3}{60} + \frac{x^4}{360} + \frac{x^5}{2520} + O(x^6)\right) \\
&+ c_2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

5.7.1 Maple step by step solution

Let's solve

$y''x+(-1-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x}+\frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(1+x)y'}{x}+\frac{y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{1+x}{x},P_3(x)=\frac{1}{x}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $y''x+(-1-x)\,y'+y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y'$ to series expansion for $m=0..1$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_{k+1}(k+1+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=6;
dsolve(x*diff(y(x),x$2)-(x+1)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 + \frac{1}{3}x + \frac{1}{12}x^2 + \frac{1}{60}x^3 + \frac{1}{360}x^4 + \frac{1}{2520}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(-2 - 2x - x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 - \frac{1}{60}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 66

```
AsymptoticDSolveValue[x*y''[x]-(x+1)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right) + c_2\left(\frac{x^6}{360} + \frac{x^5}{60} + \frac{x^4}{12} + \frac{x^3}{3} + x^2\right)$$

5.8 problem 18

5.8.1 Maple step by step solution 635

Internal problem ID [5676]
Internal file name [`OUTPUT/4924_Sunday_June_05_2022_03_10_42_PM_99217/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. REVIEW QUESTIONS. page 201
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + 3y' + 4y x^3 = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 3y' + 4y x^3 = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = 4x^2$$

Table 79: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 4x^2$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 3y' + 4yx^3 = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + 3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x^3 = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4x^{3+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{3+n+r} a_n = \sum_{n=4}^{\infty} 4a_{n-4} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=4}^{\infty} 4a_{n-4} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r) (n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r} (2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r} (2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

Substituting $n = 2$ in Eq. (2B) gives

$$
a_2 = 0
$$

Substituting $n = 3$ in Eq. (2B) gives

$$
a_3 = 0
$$

For $4 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + 3a_n(n+r) + 4a_{n-4} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{4a_{n-4}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{4a_{n-4}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = -\frac{4}{r^2 + 10r + 24}$$

Which for the root $r = 0$ becomes

$$a_4 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{4}{r^2+10r+24}$	$-\frac{1}{6}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{4}{r^2+10r+24}$	$-\frac{1}{6}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 - \frac{x^4}{6} + O(x^6)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} 0 &= \lim_{r \to -2} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-2}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = 0$$

Substituting $n = 3$ in Eq(3) gives

$$b_3 = 0$$

For $4 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 3(n+r)b_n + 4b_{n-4} = 0 \tag{4}$$

Which for for the root $r = -2$ becomes

$$b_n(n-2)(n-3) + 3(n-2)b_n + 4b_{n-4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-4}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{5}$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{4b_{n-4}}{n^2 - 2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = -\frac{4}{r^2 + 10r + 24}$$

Which for the root $r = -2$ becomes

$$b_4 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{4}{r^2+10r+24}$	$-\frac{1}{2}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{4}{r^2+10r+24}$	$-\frac{1}{2}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x^4}{2} + O(x^6)}{x^2}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - \frac{x^4}{6} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^4}{2} + O(x^6)\right)}{x^2}
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-\frac{x^4}{6}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^4}{2}+O(x^6)\right)}{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1-\frac{x^4}{6}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^4}{2}+O(x^6)\right)}{x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1-\frac{x^4}{6}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^4}{2}+O(x^6)\right)}{x^2}
$$

Verified OK.

5.8.1 Maple step by step solution

Let's solve

$y''x + 3y' + 4yx^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - 4yx^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} + 4yx^2 = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + 3y' + 4yx^3 = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - Convert $x^3 \cdot y$ to series expansion

    $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

  - Shift index using $k- >k-3$

    $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - Shift index using $k- >k+1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - Shift index using $k- >k+1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)(3+r)=0, a_2(2+r)(4+r)=0, a_3(3+r)(5+r)=0]$

- Solve for the dependent coefficient(s)

  $\{a_1=0, a_2=0, a_3=0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r+3)+4a_{k-3}=0$

- Shift index using $k->k+3$

  $a_{k+4}(k+4+r)(k+6+r)+4a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+4}=-\frac{4a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r=-2$

  $a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}$

- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0\right]$

- Recursion relation for $r=0$

  $a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{4b_k}{(k+4)(k+6)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
Order:=6;
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+4*x^3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{1}{6}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(-2 + x^4 + \mathrm{O}\left(x^6\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 30

```
AsymptoticDSolveValue[x*y''[x]+3*y'[x]+4*x^3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(1 - \frac{x^4}{6}\right) + c_1\left(\frac{1}{x^2} - \frac{x^2}{2}\right)$$

5.9 problem 19

5.9.1 Maple step by step solution 651

Internal problem ID [5677]
Internal file name [`OUTPUT/4925_Sunday_June_05_2022_03_10_45_PM_67524443/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. REVIEW QUESTIONS. page 201
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + \frac{y}{4x} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + \frac{y}{4x} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{1}{4x}$$

Table 81: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{4x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$4x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$4x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$4x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$4x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$

$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n \right)$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{4(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}}{4(n+1)n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{4(1+r)r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(1+r)r}$	$-\frac{1}{8}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{16(1+r)^2 r(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(1+r)r}$	$-\frac{1}{8}$
a_2	$\frac{1}{16(1+r)^2 r(2+r)}$	$\frac{1}{192}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{64(1+r)^2 r(2+r)^2 (3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{9216}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(1+r)r}$	$-\frac{1}{8}$
a_2	$\frac{1}{16(1+r)^2r(2+r)}$	$\frac{1}{192}$
a_3	$-\frac{1}{64(1+r)^2r(2+r)^2(3+r)}$	$-\frac{1}{9216}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{256 (1+r)^2 r (2+r)^2 (3+r)^2 (4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{737280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(1+r)r}$	$-\frac{1}{8}$
a_2	$\frac{1}{16(1+r)^2r(2+r)}$	$\frac{1}{192}$
a_3	$-\frac{1}{64(1+r)^2r(2+r)^2(3+r)}$	$-\frac{1}{9216}$
a_4	$\frac{1}{256(1+r)^2r(2+r)^2(3+r)^2(4+r)}$	$\frac{1}{737280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{1024 (1+r)^2 r (2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{88473600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4(1+r)r}$	$-\frac{1}{8}$
a_2	$\frac{1}{16(1+r)^2r(2+r)}$	$\frac{1}{192}$
a_3	$-\frac{1}{64(1+r)^2r(2+r)^2(3+r)}$	$-\frac{1}{9216}$
a_4	$\frac{1}{256(1+r)^2r(2+r)^2(3+r)^2(4+r)}$	$\frac{1}{737280}$
a_5	$-\frac{1}{1024(1+r)^2r(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{1}{88473600}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x}{8} + \frac{x^2}{192} - \frac{x^3}{9216} + \frac{x^4}{737280} - \frac{x^5}{88473600} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= -\frac{1}{4\left(1+r\right)r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \rightarrow r_2} -\frac{1}{4\left(1+r\right)r} &= \lim_{r \rightarrow 0} -\frac{1}{4\left(1+r\right)r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln\left(x\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$

$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$

$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$

$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $4xy'' + y = 0$ gives

$$4\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left((4y_1''(x)\,x + y_1(x))\ln(x) + 4\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x\right)C + 4\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$4y_1''(x)\,x + y_1(x) = 0$$

Eq (7) simplifes to

$$4\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)xC + 4\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{8\left(\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - \frac{\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)}{2}\right) C}{x} + 4\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{8\left(\left(\sum_{n=0}^{\infty} x^{n} a_n (n+1)\right) x - \frac{\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)}{2}\right) C}{x} + 4\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x + \left(\sum_{n=0}^{\infty} b_n x^{n}\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 8C\, x^n a_n (n+1)\right) + \sum_{n=0}^{\infty} (-4C\, x^n a_n) + \left(\sum_{n=0}^{\infty} 4n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 8C\, x^n a_n (n+1) = \sum_{n=1}^{\infty} 8C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} (-4C\, x^n a_n) = \sum_{n=1}^{\infty} \left(-4C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=1}^{\infty} 8C a_{n-1} n\, x^{n-1}\right)+\sum_{n\,=1}^{\infty}\left(-4C a_{n-1} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 4n\, x^{n-1} b_n(n-1)\right)+\left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$4C+1=0$$

Which is solved for C. Solving for C gives

$$C=-\frac{1}{4}$$

For $n=2$, Eq (2B) gives

$$12C a_1+b_1+8b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_2+\frac{3}{8}=0$$

Solving the above for b_2 gives

$$b_2=-\frac{3}{64}$$

For $n=3$, Eq (2B) gives

$$20C a_2+b_2+24b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$24b_3-\frac{7}{96}=0$$

Solving the above for b_3 gives

$$b_3=\frac{7}{2304}$$

For $n=4$, Eq (2B) gives

$$28C a_3+b_3+48b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$48b_4 + \frac{35}{9216} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{35}{442368}$$

For $n = 5$, Eq (2B) gives

$$36Ca_4 + b_4 + 80b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$80b_5 - \frac{101}{1105920} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{101}{88473600}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{4}$ and all b_n, then the second solution becomes

$$\begin{aligned}y_2(x) = &-\frac{1}{4}\left(x\left(1 - \frac{x}{8} + \frac{x^2}{192} - \frac{x^3}{9216} + \frac{x^4}{737280} - \frac{x^5}{88473600} + O(x^6)\right)\right)\ln(x)\\ &+ 1 - \frac{3x^2}{64} + \frac{7x^3}{2304} - \frac{35x^4}{442368} + \frac{101x^5}{88473600} + O(x^6)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\ &= c_1 x\left(1 - \frac{x}{8} + \frac{x^2}{192} - \frac{x^3}{9216} + \frac{x^4}{737280} - \frac{x^5}{88473600} + O(x^6)\right)\\ &\quad + c_2\left(-\frac{1}{4}\left(x\left(1 - \frac{x}{8} + \frac{x^2}{192} - \frac{x^3}{9216} + \frac{x^4}{737280} - \frac{x^5}{88473600} + O(x^6)\right)\right)\ln(x)\right.\\ &\quad \left. + 1 - \frac{3x^2}{64} + \frac{7x^3}{2304} - \frac{35x^4}{442368} + \frac{101x^5}{88473600} + O(x^6)\right)\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1-\frac{x}{8}+\frac{x^2}{192}-\frac{x^3}{9216}+\frac{x^4}{737280}-\frac{x^5}{88473600}+O\left(x^6\right)\right) \\
&\quad + c_2\left(-\frac{x\left(1-\frac{x}{8}+\frac{x^2}{192}-\frac{x^3}{9216}+\frac{x^4}{737280}-\frac{x^5}{88473600}+O\left(x^6\right)\right)\ln\left(x\right)}{4}+1-\frac{3x^2}{64}+\frac{7x^3}{2304}\right. \\
&\quad \left. -\frac{35x^4}{442368}+\frac{101x^5}{88473600}+O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1-\frac{x}{8}+\frac{x^2}{192}-\frac{x^3}{9216}+\frac{x^4}{737280}-\frac{x^5}{88473600}+O\left(x^6\right)\right) \\
&\quad + c_2\left(-\frac{x\left(1-\frac{x}{8}+\frac{x^2}{192}-\frac{x^3}{9216}+\frac{x^4}{737280}-\frac{x^5}{88473600}+O\left(x^6\right)\right)\ln\left(x\right)}{4}+1-\frac{3x^2}{64}\right. \\
&\quad \left. +\frac{7x^3}{2304}-\frac{35x^4}{442368}+\frac{101x^5}{88473600}+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1-\frac{x}{8}+\frac{x^2}{192}-\frac{x^3}{9216}+\frac{x^4}{737280}-\frac{x^5}{88473600}+O\left(x^6\right)\right) \\
&\quad + c_2\left(-\frac{x\left(1-\frac{x}{8}+\frac{x^2}{192}-\frac{x^3}{9216}+\frac{x^4}{737280}-\frac{x^5}{88473600}+O\left(x^6\right)\right)\ln\left(x\right)}{4}+1-\frac{3x^2}{64}+\frac{7x^3}{2304}\right. \\
&\quad \left. -\frac{35x^4}{442368}+\frac{101x^5}{88473600}+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

5.9.1 Maple step by step solution

Let's solve

$4y''x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y}{4x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{1}{4x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4y''x + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^{k+r-1}$

- ◦ Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$4a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(4a_{k+1}(k+1+r)(k+r) + a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $4a_{k+1}(k+1+r)(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{4(k+1+r)(k+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{4(k+1)k}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{4(k+1)k}\right]$
- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{4(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{4(k+2)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{a_k}{4(k+1)k}, b_{k+1} = -\frac{b_k}{4(k+2)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 58

```
Order:=6;
dsolve(diff(y(x),x$2)+1/(4*x)*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x \left(1 - \frac{1}{8}x + \frac{1}{192}x^2 - \frac{1}{9216}x^3 + \frac{1}{737280}x^4 - \frac{1}{88473600}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + c_2 \left(\ln(x)\left(-\frac{1}{4}x + \frac{1}{32}x^2 - \frac{1}{768}x^3 + \frac{1}{36864}x^4 - \frac{1}{2949120}x^5 + \mathrm{O}\left(x^6\right)\right)\right. \\
& \left. + \left(1 - \frac{3}{64}x^2 + \frac{7}{2304}x^3 - \frac{35}{442368}x^4 + \frac{101}{88473600}x^5 + \mathrm{O}\left(x^6\right)\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 85

```
AsymptoticDSolveValue[y''[x]+1/(4*x)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \rightarrow c_1 \bigg(& \frac{x(x^3 - 48x^2 + 1152x - 9216)\log(x)}{36864} \\
& + \frac{-47x^4 + 1920x^3 - 34560x^2 + 110592x + 442368}{442368} \bigg) \\
& + c_2 \left(\frac{x^5}{737280} - \frac{x^4}{9216} + \frac{x^3}{192} - \frac{x^2}{8} + x \right)
\end{aligned}
$$

5.10 problem 20

5.10.1 Maple step by step solution 662

Internal problem ID [5678]
Internal file name [`OUTPUT/4926_Sunday_June_05_2022_03_10_48_PM_2520066/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 5. Series Solutions of ODEs. REVIEW QUESTIONS. page 201
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + y' - xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -1$$

Table 83: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' - xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r) (n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+2)^2 (4+r)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(4+r)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(4+r)^2}$	$\frac{1}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{1}{(r+2)^2}$	$\frac{1}{4}$	$-\frac{2}{(r+2)^3}$	$-\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(r+2)^2(4+r)^2}$	$\frac{1}{64}$	$\frac{-12-4r}{(r+2)^3(4+r)^3}$	$-\frac{3}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= \left(1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) - \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(1+\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)+c_2\left(\left(1+\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)\ln(x)-\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1+\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)+c_2\left(\left(1+\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)\ln(x)-\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1&\left(1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \\
&+ c_2\left(\left(1 + \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) - \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\left(1+\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)+c_2\left(\left(1+\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)\ln(x)-\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)$$

Verified OK.

5.10.1 Maple step by step solution

Let's solve

$y''x + y' - xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + y' - xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

◦ Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

◦ Shift index using $k- > k - 1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

◦ Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

◦ Shift index using $k- > k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

◦ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

◦ Shift index using $k- > k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1(1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term must be 0

  $a_1(1+r)^2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - a_{k-1} = 0$

- Shift index using $k- > k + 1$

  $a_{k+2}(k+2)^2 - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+2)^2}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{(k+2)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \mathrm{O}(x^6)\right) + \left(-\frac{1}{4}x^2 - \frac{3}{128}x^4 + \mathrm{O}(x^6)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x*y''[x]+y'[x]-x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{64}+\frac{x^2}{4}+1\right)+c_2\left(-\frac{3x^4}{128}-\frac{x^2}{4}+\left(\frac{x^4}{64}+\frac{x^2}{4}+1\right)\log(x)\right)$$

6 Chapter 6. Laplace Transforms. Problem set 6.2, page 216

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6.4 problem 4 681
6.5 problem 5 687
6.6 problem 6 692
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6.1 problem 1

6.1.1 Existence and uniqueness analysis 667
6.1.2 Solving as laplace ode 668
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Internal problem ID [5679]
Internal file name [`OUTPUT/4927_Sunday_June_05_2022_03_10_50_PM_62015448/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + \frac{26y}{5} = \frac{97 \sin(2t)}{5}}$$

With initial conditions

$$[y(0) = 0]$$

6.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{26}{5}$$
$$q(t) = \frac{97 \sin(2t)}{5}$$

Hence the ode is

$$y' + \frac{26y}{5} = \frac{97\sin(2t)}{5}$$

The domain of $p(t) = \frac{26}{5}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{97\sin(2t)}{5}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + \frac{26Y(s)}{5} = \frac{194}{5(s^2+4)} \tag{1}$$

Replacing initial condition gives

$$sY(s) + \frac{26Y(s)}{5} = \frac{194}{5(s^2+4)}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{194}{(s^2+4)(5s+26)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{5}{4\left(s+\frac{26}{5}\right)} + \frac{-\frac{5}{8}-\frac{13i}{8}}{s-2i} + \frac{-\frac{5}{8}+\frac{13i}{8}}{s+2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{5}{4\left(s+\frac{26}{5}\right)}\right) = \frac{5\,\mathrm{e}^{-\frac{26t}{5}}}{4}$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{5}{8}-\frac{13i}{8}}{s-2i}\right) = \left(-\frac{5}{8}-\frac{13i}{8}\right)\mathrm{e}^{2it}$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{5}{8}+\frac{13i}{8}}{s+2i}\right) = \left(-\frac{5}{8}+\frac{13i}{8}\right)\mathrm{e}^{-2it}$$

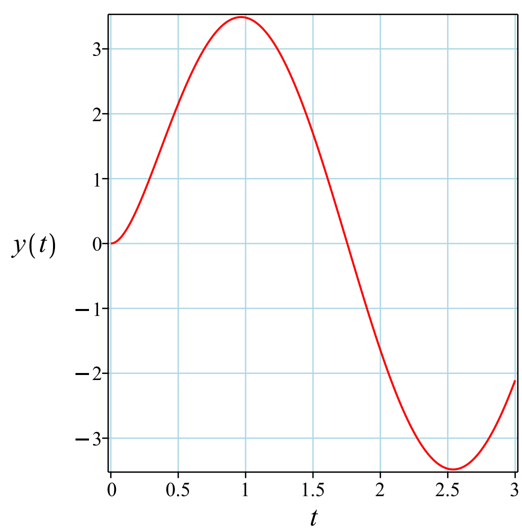
Adding the above results and simplifying gives

$$y = \frac{5\,\mathrm{e}^{-\frac{26t}{5}}}{4} - \frac{5\cos(2t)}{4} + \frac{13\sin(2t)}{4}$$

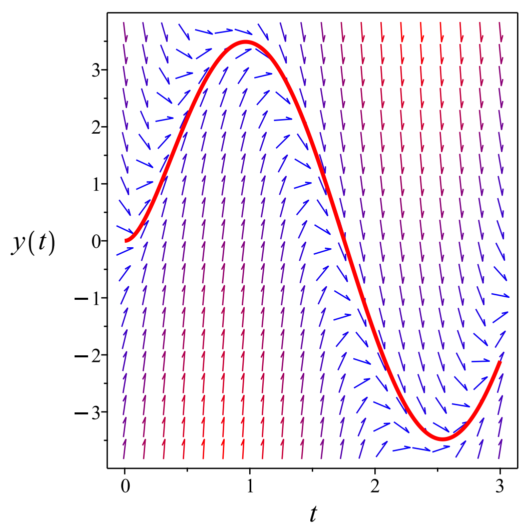
Summary

The solution(s) found are the following

$$y = \frac{5\,\mathrm{e}^{-\frac{26t}{5}}}{4} - \frac{5\cos(2t)}{4} + \frac{13\sin(2t)}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{5\,\mathrm{e}^{-\frac{26t}{5}}}{4} - \frac{5\cos(2t)}{4} + \frac{13\sin(2t)}{4}$$

Verified OK.

6.1.3 Maple step by step solution

Let's solve

$\left[y' + \frac{26y}{5} = \frac{97\sin(2t)}{5}, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{26y}{5} + \frac{97\sin(2t)}{5}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{26y}{5} = \frac{97\sin(2t)}{5}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{26y}{5}\right) = \frac{97\mu(t)\sin(2t)}{5}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{26y}{5}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{26\mu(t)}{5}$

- Solve to find the integrating factor

  $\mu(t) = e^{\frac{26t}{5}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \frac{97\mu(t)\sin(2t)}{5} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \frac{97\mu(t)\sin(2t)}{5} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{97\mu(t)\sin(2t)}{5} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{\frac{26t}{5}}$

  $y = \frac{\int \frac{97\, e^{\frac{26t}{5}}\sin(2t)}{5} dt + c_1}{e^{\frac{26t}{5}}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{5\, e^{\frac{26t}{5}}\cos(2t)}{4} + \frac{13\, e^{\frac{26t}{5}}\sin(2t)}{4} + c_1}{e^{\frac{26t}{5}}}$

- Simplify

  $y = \frac{13\sin(2t)}{4} - \frac{5\cos(2t)}{4} + c_1 e^{-\frac{26t}{5}}$
- Use initial condition $y(0) = 0$

  $0 = -\frac{5}{4} + c_1$
- Solve for c_1

  $c_1 = \frac{5}{4}$
- Substitute $c_1 = \frac{5}{4}$ into general solution and simplify

  $y = \frac{5\,e^{-\frac{26t}{5}}}{4} - \frac{5\cos(2t)}{4} + \frac{13\sin(2t)}{4}$
- Solution to the IVP

  $y = \frac{5\,e^{-\frac{26t}{5}}}{4} - \frac{5\cos(2t)}{4} + \frac{13\sin(2t)}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.891 (sec). Leaf size: 23

```
dsolve([diff(y(t),t)+52/10*y(t)=194/10*sin(2*t),y(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{5\,e^{-\frac{26t}{5}}}{4} - \frac{5\cos(2t)}{4} + \frac{13\sin(2t)}{4}$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 31

```
DSolve[{y'[t]+52/10*y[t]==194/10*Sin[2*t],{y[0]==0}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to \frac{1}{4}\left(5e^{-26t/5} + 13\sin(2t) - 5\cos(2t)\right)$$

6.2 problem 2

6.2.1 Existence and uniqueness analysis 672
6.2.2 Solving as laplace ode 673
6.2.3 Maple step by step solution 674

Internal problem ID [5680]
Internal file name [`OUTPUT/4928_Sunday_June_05_2022_03_10_51_PM_30685895/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{2y + y' = 0}$$

With initial conditions

$$\left[y(0) = \frac{3}{2}\right]$$

6.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 2$$
$$q(t) = 0$$

Hence the ode is

$$2y + y' = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

6.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$2Y(s) + sY(s) - y(0) = 0 \tag{1}$$

Replacing initial condition gives

$$2Y(s) + sY(s) - \frac{3}{2} = 0$$

Solving for $Y(s)$ gives

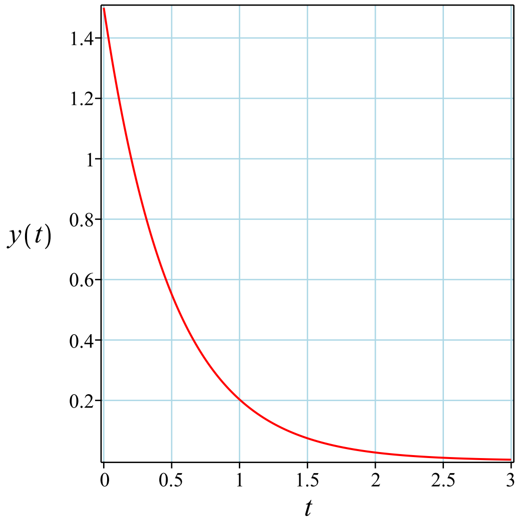
$$Y(s) = \frac{3}{2\,(2+s)}$$

Taking the inverse Laplace transform gives

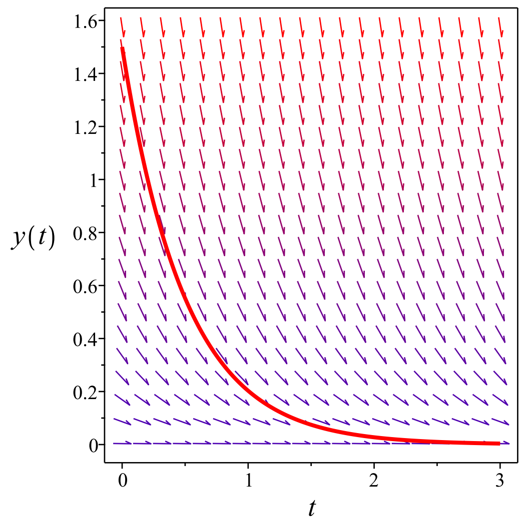
$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{3}{2\,(2+s)}\right) \\ &= \frac{3\,\mathrm{e}^{-2t}}{2} \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{3\,\mathrm{e}^{-2t}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3\,e^{-2t}}{2}$$

Verified OK.

6.2.3 Maple step by step solution

Let's solve

$\left[2y + y' = 0, y(0) = \frac{3}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -2$

- Integrate both sides with respect to t

  $\int \frac{y'}{y} dt = \int (-2)\, dt + c_1$

- Evaluate integral

  $\ln(y) = -2t + c_1$

- Solve for y

$y = \mathrm{e}^{-2t+c_1}$

- Use initial condition $y(0) = \frac{3}{2}$

  $\frac{3}{2} = \mathrm{e}^{c_1}$

- Solve for c_1

  $c_1 = \ln\left(\frac{3}{2}\right)$

- Substitute $c_1 = \ln\left(\frac{3}{2}\right)$ into general solution and simplify

  $y = \frac{3\,\mathrm{e}^{-2t}}{2}$

- Solution to the IVP

  $y = \frac{3\,\mathrm{e}^{-2t}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.688 (sec). Leaf size: 10

```
dsolve([diff(y(t),t)+2*y(t)=0,y(0) = 3/2],y(t), singsol=all)
```

$$y(t) = \frac{3\,\mathrm{e}^{-2t}}{2}$$

✓ Solution by Mathematica
Time used: 0.038 (sec). Leaf size: 31

```
DSolve[{y'[t]+52/10*y[t]==194/10*Sin[2*t],{y[0]==15/10}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to \frac{1}{4}\left(11e^{-26t/5} + 13\sin(2t) - 5\cos(2t)\right)$$

6.3 problem 3

6.3.1 Existence and uniqueness analysis 676
6.3.2 Maple step by step solution 679

Internal problem ID [5681]
Internal file name [`OUTPUT/4929_Sunday_June_05_2022_03_10_52_PM_84049704/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 6y = 0}$$

With initial conditions

$$[y(0) = 11, y'(0) = 28]$$

6.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = -1$$
$$q(t) = -6$$
$$F = 0$$

Hence the ode is

$$y'' - y' - 6y = 0$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) - 6Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 11 \\ y'(0) &= 28 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 17 - 11s - sY(s) - 6Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{11s + 17}{s^2 - s - 6}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s+2} + \frac{10}{s-3}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) &= \mathrm{e}^{-2t} \\ \mathcal{L}^{-1}\left(\frac{10}{s-3}\right) &= 10\,\mathrm{e}^{3t} \end{aligned}$$

Adding the above results and simplifying gives

$$y = 10\,e^{3t} + e^{-2t}$$

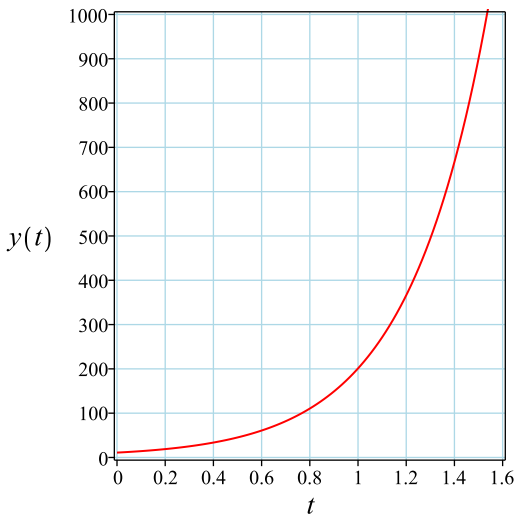
Simplifying the solution gives

$$y = \left(10\,e^{5t} + 1\right) e^{-2t}$$

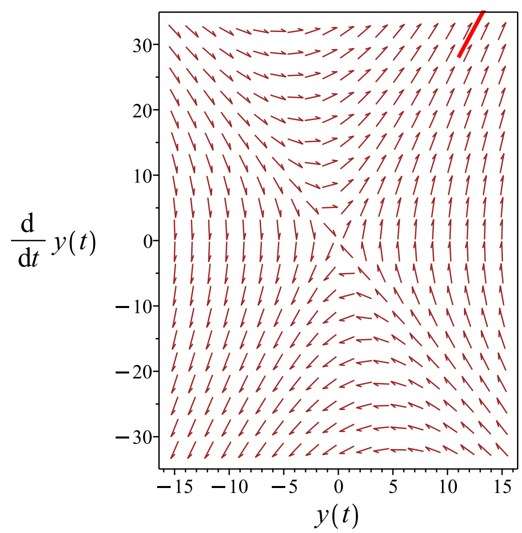
Summary

The solution(s) found are the following

$$y = \left(10\,e^{5t} + 1\right) e^{-2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \left(10\,e^{5t} + 1\right) e^{-2t}$$

Verified OK.

6.3.2 Maple step by step solution

Let's solve

$\left[y'' - y' - 6y = 0, y(0) = 11, y'\Big|_{\{t=0\}} = 28\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{-2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{-2t} + e^{3t} c_2$

□ Check validity of solution $y = c_1 e^{-2t} + e^{3t} c_2$

  - Use initial condition $y(0) = 11$

    $11 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -2c_1 e^{-2t} + 3\,e^{3t} c_2$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 28$

    $28 = -2c_1 + 3c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 10\}$

- Substitute constant values into general solution and simplify

  $$y = (10\,e^{5t} + 1)\,e^{-2t}$$

- Solution to the IVP

  $$y = (10\,e^{5t} + 1)\,e^{-2t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.797 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$2)-diff(y(t),t)-6*y(t)=0,y(0) = 11, D(y)(0) = 28],y(t), singsol=all)
```

$$y(t) = \left(10\,e^{5t} + 1\right)e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[{y''[t]-y'[t]-6*y[t]==0,{y[0]==11,y'[0]==28}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to e^{-2t} + 10e^{3t}$$

6.4 problem 4

6.4.1 Existence and uniqueness analysis 681
6.4.2 Maple step by step solution 684

Internal problem ID [5682]
Internal file name [`OUTPUT/4930_Sunday_June_05_2022_03_10_54_PM_15329181/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 9y = 10\,\mathrm{e}^{-t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

6.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 9 \\ F &= 10\,\mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$y'' + 9y = 10\,\mathrm{e}^{-t}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 10\,\mathrm{e}^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{10}{s+1} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 9Y(s) = \frac{10}{s+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{10}{(s+1)(s^2+9)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-\frac{1}{2} - \frac{i}{6}}{s-3i} + \frac{-\frac{1}{2} + \frac{i}{6}}{s+3i} + \frac{1}{s+1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{2}-\frac{i}{6}}{s-3i}\right) = \left(-\frac{1}{2}-\frac{i}{6}\right) e^{3it}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{2}+\frac{i}{6}}{s+3i}\right) = \left(-\frac{1}{2}+\frac{i}{6}\right) e^{-3it}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$$

Adding the above results and simplifying gives

$$y = -\cos(3t) + \frac{\sin(3t)}{3} + e^{-t}$$

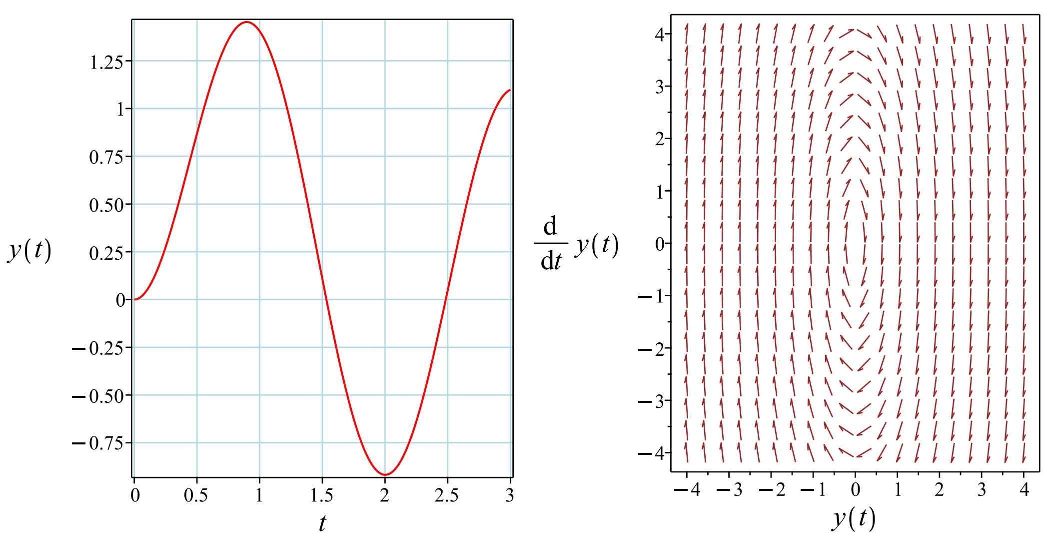
Simplifying the solution gives

$$y = -\cos(3t) + \frac{\sin(3t)}{3} + e^{-t}$$

Summary

The solution(s) found are the following

$$y = -\cos(3t) + \frac{\sin(3t)}{3} + e^{-t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\cos(3t) + \frac{\sin(3t)}{3} + e^{-t}$$

Verified OK.

6.4.2 Maple step by step solution

Let's solve

$\left[y'' + 9y = 10\,e^{-t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 10\,e^{-t}\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 3$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{10\cos(3t)\left(\int \sin(3t)\mathrm{e}^{-t}dt\right)}{3} + \frac{10\sin(3t)\left(\int \cos(3t)\mathrm{e}^{-t}dt\right)}{3}$

- Compute integrals

  $y_p(t) = \mathrm{e}^{-t}$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(3t) + c_2\sin(3t) + \mathrm{e}^{-t}$

☐ Check validity of solution $y = c_1\cos(3t) + c_2\sin(3t) + \mathrm{e}^{-t}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + 1$

- Compute derivative of the solution

  $y' = -3c_1\sin(3t) + 3c_2\cos(3t) - \mathrm{e}^{-t}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -1 + 3c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -1, c_2 = \frac{1}{3}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\cos(3t) + \frac{\sin(3t)}{3} + \mathrm{e}^{-t}$

- Solution to the IVP

  $y = -\cos(3t) + \frac{\sin(3t)}{3} + \mathrm{e}^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.891 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)+9*y(t)=10*exp(-t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\cos(3t) + \frac{\sin(3t)}{3} + e^{-t}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 25

```
DSolve[{y''[t]+9*y[t]==10*Exp[-t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to e^{-t} + \frac{1}{3}\sin(3t) - \cos(3t)$$

6.5 problem 5

6.5.1 Existence and uniqueness analysis 687
6.5.2 Maple step by step solution 690

Internal problem ID [5683]
Internal file name [`OUTPUT/4931_Sunday_June_05_2022_03_10_55_PM_72200753/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - \frac{y}{4} = 0}$$

With initial conditions

$$[y(0) = 12, y'(0) = 0]$$

6.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 0$$
$$q(t) = -\frac{1}{4}$$
$$F = 0$$

Hence the ode is

$$y'' - \frac{y}{4} = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -\frac{1}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - \frac{Y(s)}{4} = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 12\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 12s - \frac{Y(s)}{4} = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{48s}{4s^2 - 1}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{6}{s + \frac{1}{2}} + \frac{6}{s - \frac{1}{2}}$$

The inverse Laplace of each term above is now found, which gives

$$
\mathcal{L}^{-1}\left(\frac{6}{s+\frac{1}{2}}\right) = 6\,\mathrm{e}^{-\frac{t}{2}}
$$

$$
\mathcal{L}^{-1}\left(\frac{6}{s-\frac{1}{2}}\right) = 6\,\mathrm{e}^{\frac{t}{2}}
$$

Adding the above results and simplifying gives

$$
y = 12\cosh\left(\frac{t}{2}\right)
$$

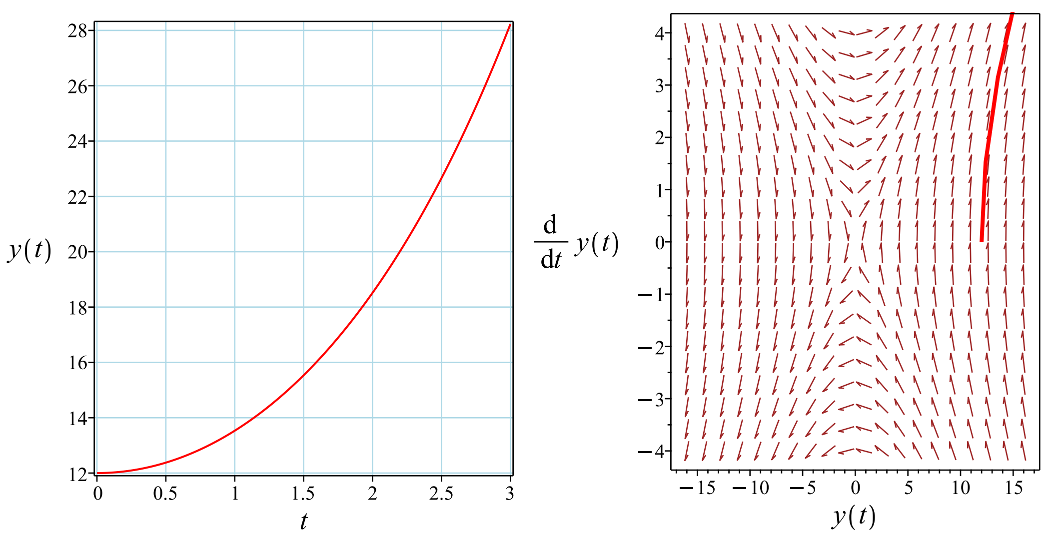
Simplifying the solution gives

$$
y = 12\cosh\left(\frac{t}{2}\right)
$$

Summary

The solution(s) found are the following

$$
y = 12\cosh\left(\frac{t}{2}\right) \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$
y = 12\cosh\left(\frac{t}{2}\right)
$$

Verified OK.

6.5.2 Maple step by step solution

Let's solve

$\left[y'' - \frac{y}{4} = 0, y(0) = 12, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)(2r+1)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2}, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-\frac{t}{2}}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{\frac{t}{2}}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-\frac{t}{2}} + c_2 \mathrm{e}^{\frac{t}{2}}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-\frac{t}{2}} + c_2 \mathrm{e}^{\frac{t}{2}}$

  ◦ Use initial condition $y(0) = 12$

    $12 = c_1 + c_2$

  ◦ Compute derivative of the solution

    $y' = -\frac{c_1 \mathrm{e}^{-\frac{t}{2}}}{2} + \frac{c_2 \mathrm{e}^{\frac{t}{2}}}{2}$

  ◦ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -\frac{c_1}{2} + \frac{c_2}{2}$

  ◦ Solve for c_1 and c_2

    $\{c_1 = 6, c_2 = 6\}$

- Substitute constant values into general solution and simplify

  $y = 6\,\mathrm{e}^{-\frac{t}{2}} + 6\,\mathrm{e}^{\frac{t}{2}}$

- Solution to the IVP

  $y = 6\,\mathrm{e}^{-\frac{t}{2}} + 6\,\mathrm{e}^{\frac{t}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.781 (sec). Leaf size: 10

```
dsolve([diff(y(t),t$2)-1/4*y(t)=0,y(0) = 12, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = 12\cosh\left(\frac{t}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 19

```
DSolve[{y''[t]-1/4*y[t]==0,{y[0]==12,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 6e^{-t/2}(e^t + 1)$$

6.6 problem 6

6.6.1 Existence and uniqueness analysis 692
6.6.2 Maple step by step solution 695

Internal problem ID [5684]
Internal file name [`OUTPUT/4932_Sunday_June_05_2022_03_10_56_PM_44958753/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 6y' + 5y = 29\cos(2t)}$$

With initial conditions

$$\left[y(0) = \frac{16}{5}, y'(0) = \frac{31}{5}\right]$$

6.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -6 \\ q(t) &= 5 \\ F &= 29\cos(2t) \end{aligned}$$

Hence the ode is

$$y'' - 6y' + 5y = 29\cos(2t)$$

The domain of $p(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 29\cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 6sY(s) + 6y(0) + 5Y(s) = \frac{29s}{s^2+4} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= \frac{16}{5} \\ y'(0) &= \frac{31}{5}\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 13 - \frac{16s}{5} - 6sY(s) + 5Y(s) = \frac{29s}{s^2+4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{16s^3 - 65s^2 + 209s - 260}{5(s^2+4)(s^2-6s+5)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{1}{10}+\frac{6i}{5}}{s-2i}+\frac{\frac{1}{10}-\frac{6i}{5}}{s+2i}+\frac{1}{s-1}+\frac{2}{s-5}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{\frac{1}{10}+\frac{6i}{5}}{s-2i}\right) &= \left(\frac{1}{10}+\frac{6i}{5}\right)\mathrm{e}^{2it} \\
\mathcal{L}^{-1}\left(\frac{\frac{1}{10}-\frac{6i}{5}}{s+2i}\right) &= \left(\frac{1}{10}-\frac{6i}{5}\right)\mathrm{e}^{-2it} \\
\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) &= \mathrm{e}^{t} \\
\mathcal{L}^{-1}\left(\frac{2}{s-5}\right) &= 2\,\mathrm{e}^{5t}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = 2\,\mathrm{e}^{5t}+\mathrm{e}^{t}+\frac{\cos(2t)}{5}-\frac{12\sin(2t)}{5}$$

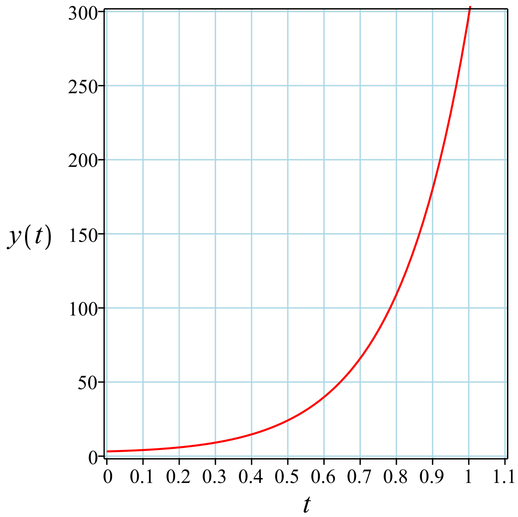
Simplifying the solution gives

$$y = 2\,\mathrm{e}^{5t}+\mathrm{e}^{t}+\frac{\cos(2t)}{5}-\frac{12\sin(2t)}{5}$$

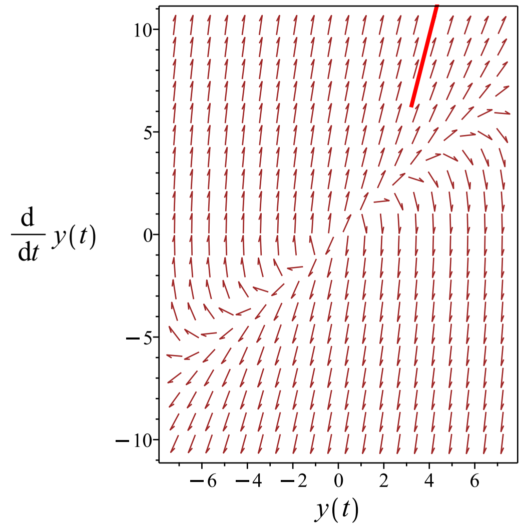
Summary

The solution(s) found are the following

$$y = 2\,\mathrm{e}^{5t}+\mathrm{e}^{t}+\frac{\cos(2t)}{5}-\frac{12\sin(2t)}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 2\,e^{5t} + e^{t} + \frac{\cos(2t)}{5} - \frac{12\sin(2t)}{5}$$

Verified OK.

6.6.2 Maple step by step solution

Let's solve

$\left[y'' - 6y' + 5y = 29\cos(2t), y(0) = \frac{16}{5}, y'\Big|_{\{t=0\}} = \frac{31}{5}\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 5 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-5) = 0$

- Roots of the characteristic polynomial

  $r = (1, 5)$

- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^t$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{5t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{5t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 29\cos(2t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^t & \mathrm{e}^{5t} \\ \mathrm{e}^t & 5\,\mathrm{e}^{5t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 4\,\mathrm{e}^{6t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{29\,\mathrm{e}^t \left(\int \cos(2t)\mathrm{e}^{-t} dt\right)}{4} + \frac{29\,\mathrm{e}^{5t}\left(\int \cos(2t)\mathrm{e}^{-5t} dt\right)}{4}$

- ○ Compute integrals

  $y_p(t) = \frac{\cos(2t)}{5} - \frac{12\sin(2t)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{5t} + \frac{\cos(2t)}{5} - \frac{12\sin(2t)}{5}$

☐ Check validity of solution $y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{5t} + \frac{\cos(2t)}{5} - \frac{12\sin(2t)}{5}$

- ○ Use initial condition $y(0) = \frac{16}{5}$

  $\frac{16}{5} = c_1 + c_2 + \frac{1}{5}$

- ○ Compute derivative of the solution

  $y' = c_1 \mathrm{e}^t + 5c_2 \mathrm{e}^{5t} - \frac{2\sin(2t)}{5} - \frac{24\cos(2t)}{5}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = \frac{31}{5}$

  $\frac{31}{5} = c_1 + 5c_2 - \frac{24}{5}$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 2\}$
- Substitute constant values into general solution and simplify

  $y = 2\,e^{5t} + e^t + \frac{\cos(2t)}{5} - \frac{12\sin(2t)}{5}$

- Solution to the IVP

  $y = 2\,e^{5t} + e^t + \frac{\cos(2t)}{5} - \frac{12\sin(2t)}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.079 (sec). Leaf size: 25

```
dsolve([diff(y(t),t$2)-6*diff(y(t),t)+5*y(t)=29*cos(2*t),y(0) = 16/5, D(y)(0) = 31/5],y(t),
```

$$y(t) = \frac{\cos(2t)}{5} - \frac{12\sin(2t)}{5} + e^t + 2\,e^{5t}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 32

```
DSolve[{y''[t]-6*y'[t]+5*y[t]==29*Cos[2*t],{y[0]==32/10,y'[0]==62/10}},y[t],t,IncludeSingula
```

$$y(t) \to e^t + 2e^{5t} - \frac{12}{5}\sin(2t) + \frac{1}{5}\cos(2t)$$

6.7 problem 7

6.7.1 Existence and uniqueness analysis 698
6.7.2 Maple step by step solution 701

Internal problem ID [5685]
Internal file name [`OUTPUT/4933_Sunday_June_05_2022_03_10_57_PM_13379888/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 7y' + 12y = 21\,e^{3t}}$$

With initial conditions

$$\left[y(0) = \frac{7}{2}, y'(0) = -10\right]$$

6.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 7 \\ q(t) &= 12 \\ F &= 21\,e^{3t} \end{aligned}$$

Hence the ode is

$$y'' + 7y' + 12y = 21\,e^{3t}$$

The domain of $p(t) = 7$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 12$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 21\,e^{3t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 7sY(s) - 7y(0) + 12Y(s) = \frac{21}{s-3} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= \frac{7}{2}\\ y'(0) &= -10\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - \frac{29}{2} - \frac{7s}{2} + 7sY(s) + 12Y(s) = \frac{21}{s-3}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{7s^2 + 8s - 45}{2\,(s-3)\,(s^2 + 7s + 12)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{2s-6} + \frac{1}{2s+6} + \frac{5}{2(s+4)}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{2s-6}\right) = \frac{e^{3t}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{1}{2s+6}\right) = \frac{e^{-3t}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{5}{2(s+4)}\right) = \frac{5\,e^{-4t}}{2}$$

Adding the above results and simplifying gives

$$y = \frac{5\,e^{-4t}}{2} + \cosh(3t)$$

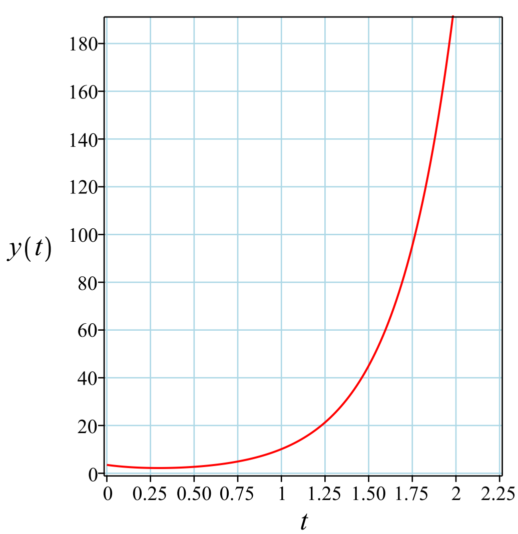
Simplifying the solution gives

$$y = \frac{5\,e^{-4t}}{2} + \cosh(3t)$$

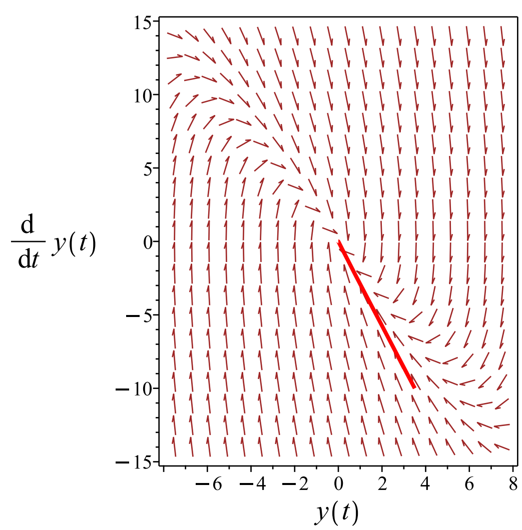
Summary

The solution(s) found are the following

$$y = \frac{5\,e^{-4t}}{2} + \cosh(3t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{5\,\mathrm{e}^{-4t}}{2} + \cosh(3t)$$

Verified OK.

6.7.2 Maple step by step solution

Let's solve

$$\left[y'' + 7y' + 12y = 21\,\mathrm{e}^{3t}, y(0) = \frac{7}{2}, y'\Big|_{\{t=0\}} = -10\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 + 7r + 12 = 0$
- Factor the characteristic polynomial
  $(r+4)(r+3) = 0$
- Roots of the characteristic polynomial
  $r = (-4, -3)$
- 1st solution of the homogeneous ODE
  $y_1(t) = \mathrm{e}^{-4t}$
- 2nd solution of the homogeneous ODE
  $y_2(t) = \mathrm{e}^{-3t}$
- General solution of the ODE
  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE
  $y = c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-3t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE
  - Use variation of parameters to find y_p here $f(t)$ is the forcing function
    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 21\,\mathrm{e}^{3t}\right]$
  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{-3t} \\ -4\,e^{-4t} & -3\,e^{-3t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-7t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -21\,e^{-4t}\left(\int e^{7t}dt\right) + 21\,e^{-3t}\left(\int e^{6t}dt\right)$

- Compute integrals

  $y_p(t) = \frac{e^{3t}}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1e^{-4t} + c_2e^{-3t} + \frac{e^{3t}}{2}$

☐ Check validity of solution $y = c_1e^{-4t} + c_2e^{-3t} + \frac{e^{3t}}{2}$

- Use initial condition $y(0) = \frac{7}{2}$

  $\frac{7}{2} = c_1 + c_2 + \frac{1}{2}$

- Compute derivative of the solution

  $y' = -4c_1e^{-4t} - 3c_2e^{-3t} + \frac{3\,e^{3t}}{2}$

- Use the initial condition $y'\Big|_{\{t=0\}} = -10$

  $-10 = -4c_1 - 3c_2 + \frac{3}{2}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{5}{2}, c_2 = \frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{(e^{7t}+e^{t}+5)e^{-4t}}{2}$

- Solution to the IVP

  $y = \frac{(e^{7t}+e^{t}+5)e^{-4t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.797 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$2)+7*diff(y(t),t)+12*y(t)=21*exp(3*t),y(0) = 7/2, D(y)(0) = -10],y(t), s
```

$$y(t) = \frac{5\,\mathrm{e}^{-4t}}{2} + \cosh\left(3t\right)$$

✓ Solution by Mathematica
Time used: 0.019 (sec). Leaf size: 28

```
DSolve[{y''[t]+7*y'[t]+12*y[t]==21*Exp[3*t],{y[0]==32/10,y'[0]==62/10}},y[t],t,IncludeSingul
```

$$y(t) \to \frac{1}{10}e^{-4t}\left(155e^{t} + 5e^{7t} - 128\right)$$

6.8 problem 8

6.8.1 Existence and uniqueness analysis 704
6.8.2 Maple step by step solution 707

Internal problem ID [5686]
Internal file name [`OUTPUT/4934_Sunday_June_05_2022_03_10_59_PM_77286167/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 4y = 0}$$

With initial conditions

$$\left[y(0) = \frac{81}{10}, y'(0) = \frac{39}{10}\right]$$

6.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 4y = 0$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4sY(s) + 4y(0) + 4Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= \frac{81}{10} \\
y'(0) &= \frac{39}{10}
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + \frac{57}{2} - \frac{81s}{10} - 4sY(s) + 4Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{-\frac{57}{2} + \frac{81s}{10}}{s^2 - 4s + 4}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{123}{10\,(s-2)^2} + \frac{81}{10\,(s-2)}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{123}{10\left(s-2\right)^{2}}\right) = -\frac{123t\,\mathrm{e}^{2t}}{10}$$
$$\mathcal{L}^{-1}\left(\frac{81}{10\left(s-2\right)}\right) = \frac{81\,\mathrm{e}^{2t}}{10}$$

Adding the above results and simplifying gives

$$y = -\frac{3(41t-27)\,\mathrm{e}^{2t}}{10}$$

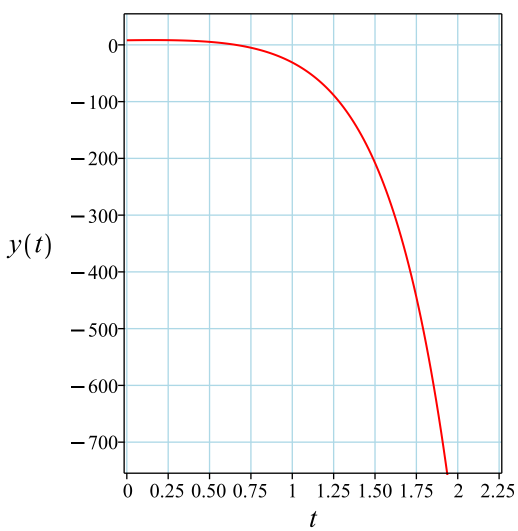
Simplifying the solution gives

$$y = -\frac{3(41t-27)\,\mathrm{e}^{2t}}{10}$$

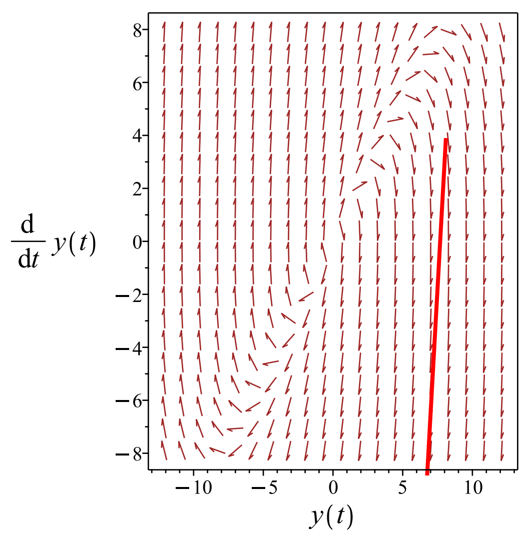
Summary

The solution(s) found are the following

$$y = -\frac{3(41t-27)\,\mathrm{e}^{2t}}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3(41t-27)\,\mathrm{e}^{2t}}{10}$$

Verified OK.

6.8.2 Maple step by step solution

Let's solve

$\left[y'' - 4y' + 4y = 0, y(0) = \frac{81}{10}, y'\Big|_{\{t=0\}} = \frac{39}{10}\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the ODE

  $y_1(t) = e^{2t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{2t} + c_2 t\,e^{2t}$

□ Check validity of solution $y = c_1 e^{2t} + c_2 t e^{2t}$

  ○ Use initial condition $y(0) = \frac{81}{10}$

    $\frac{81}{10} = c_1$

  ○ Compute derivative of the solution

    $y' = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t\,e^{2t}$

  ○ Use the initial condition $y'\Big|_{\{t=0\}} = \frac{39}{10}$

    $\frac{39}{10} = 2c_1 + c_2$

  ○ Solve for c_1 and c_2

    $\left\{c_1 = \frac{81}{10}, c_2 = -\frac{123}{10}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{3(41t-27)e^{2t}}{10}$

- Solution to the IVP

  $y = -\frac{3(41t-27)e^{2t}}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.719 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$2)-4*diff(y(t),t)+4*y(t)=0,y(0) = 81/10, D(y)(0) = 39/10],y(t), singsol=
```

$$y(t) = -\frac{3(41t-27)\,e^{2t}}{10}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 19

```
DSolve[{y''[t]-4*y'[t]+4*y[t]==0,{y[0]==81/10,y'[0]==39/10}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to -\frac{3}{10}e^{2t}(41t-27)$$

6.9 problem 9

6.9.1 Existence and uniqueness analysis 709
6.9.2 Maple step by step solution 712

Internal problem ID [5687]
Internal file name [`OUTPUT/4935_Sunday_June_05_2022_03_11_00_PM_11693725/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y' + 3y = 6t - 8}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

6.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 3 \\ F &= 6t - 8 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 3y = 6t - 8$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 6t - 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4sY(s) + 4y(0) + 3Y(s) = \frac{6}{s^2} - \frac{8}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4sY(s) + 3Y(s) = \frac{6}{s^2} - \frac{8}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{2(-3 + 4s)}{s^2 (s^2 - 4s + 3)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{2}{s^2} - \frac{1}{s - 3} + \frac{1}{s - 1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{2}{s^2}\right) = 2t$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s-3}\right) = -\mathrm{e}^{3t}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = \mathrm{e}^{t}$$

Adding the above results and simplifying gives

$$y = 2t - 2\,\mathrm{e}^{2t}\sinh(t)$$

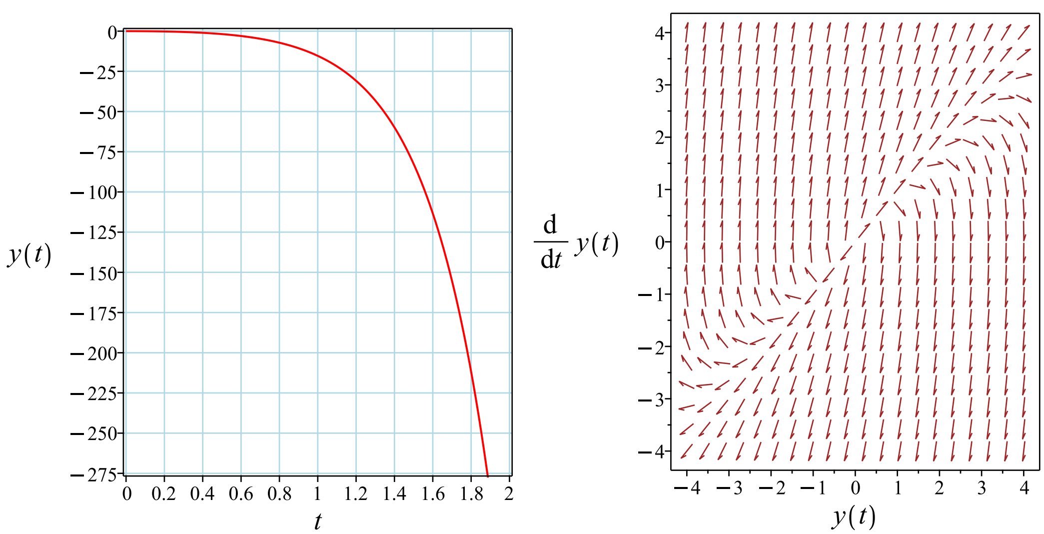
Simplifying the solution gives

$$y = 2t - 2\,\mathrm{e}^{2t}\sinh(t)$$

Summary

The solution(s) found are the following

$$y = 2t - 2\,\mathrm{e}^{2t}\sinh(t) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2t - 2\,\mathrm{e}^{2t}\sinh(t)$$

Verified OK.

6.9.2 Maple step by step solution

Let's solve

$$\left[y'' - 4y' + 3y = 6t - 8, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (1, 3)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^t$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^t + e^{3t} c_2 + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 6t - 8\right]$$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(t), y_2(t)) = \begin{bmatrix} e^t & e^{3t} \\ e^t & 3 e^{3t} \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2 e^{4t}$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = -e^t \left(\int (3t-4)\, e^{-t} dt\right) + e^{3t} \left(\int (3t-4)\, e^{-3t} dt\right)$

- Compute integrals

  $y_p(t) = 2t$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^t + e^{3t} c_2 + 2t$

□ Check validity of solution $y = c_1 e^t + e^{3t} c_2 + 2t$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

- Compute derivative of the solution

  $y' = c_1 e^t + 3\, e^{3t} c_2 + 2$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_1 + 3c_2 + 2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = -1\}$

- Substitute constant values into general solution and simplify

  $y = e^t - e^{3t} + 2t$

- Solution to the IVP

  $y = e^t - e^{3t} + 2t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.672 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$2)-4*diff(y(t),t)+3*y(t)=6*t-8,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = 2t - 2\,e^{2t}\sinh(t)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 19

```
DSolve[{y''[t]-4*y'[t]+3*y[t]==6*t-8,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to 2t + e^{t} - e^{3t}$$

6.10 problem 10

6.10.1 Existence and uniqueness analysis 715
6.10.2 Maple step by step solution 718

Internal problem ID [5688]
Internal file name [`OUTPUT/4936_Sunday_June_05_2022_03_11_01_PM_71765021/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \frac{y}{25} = \frac{t^2}{50}}$$

With initial conditions

$$[y(0) = -25, y'(0) = 0]$$

6.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 0$$
$$q(t) = \frac{1}{25}$$
$$F = \frac{t^2}{50}$$

Hence the ode is

$$y'' + \frac{y}{25} = \frac{t^2}{50}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{1}{25}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{t^2}{50}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + \frac{Y(s)}{25} = \frac{1}{25s^3} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -25\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 25s + \frac{Y(s)}{25} = \frac{1}{25s^3}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{25s^2 - 1}{s^3}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{25}{s} + \frac{1}{s^3}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{25}{s}\right) = -25$$
$$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}$$

Adding the above results and simplifying gives

$$y = -25 + \frac{t^2}{2}$$

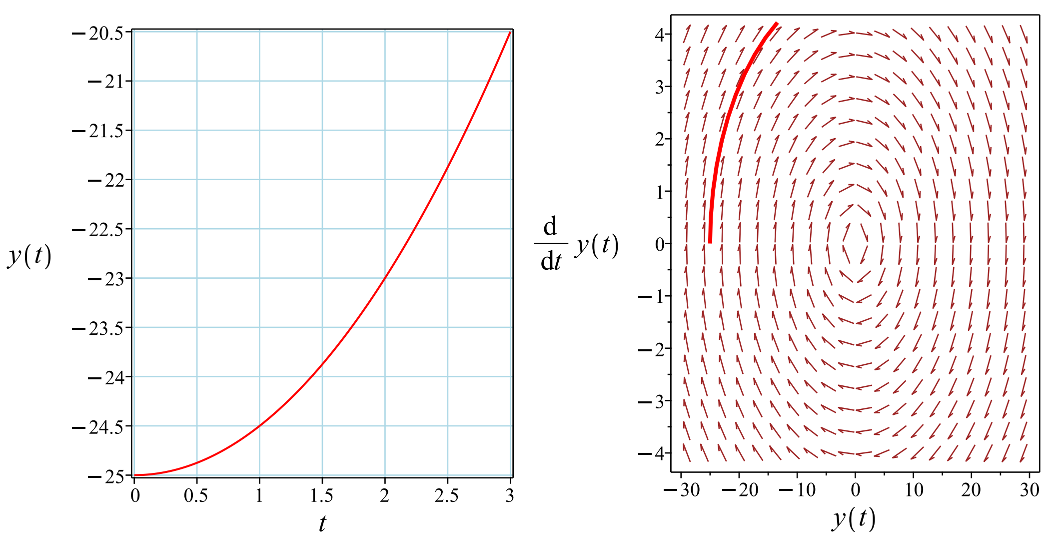
Simplifying the solution gives

$$y = -25 + \frac{t^2}{2}$$

Summary

The solution(s) found are the following

$$y = -25 + \frac{t^2}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -25 + \frac{t^2}{2}$$

Verified OK.

6.10.2 Maple step by step solution

Let's solve

$\left[y'' + \frac{y}{25} = \frac{t^2}{50}, y(0) = -25, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{1}{25} = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-\frac{4}{25}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{\mathrm{I}}{5}, \frac{\mathrm{I}}{5}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos\left(\frac{t}{5}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin\left(\frac{t}{5}\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos\left(\frac{t}{5}\right) + c_2 \sin\left(\frac{t}{5}\right) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \frac{t^2}{50}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos\left(\frac{t}{5}\right) & \sin\left(\frac{t}{5}\right) \\ -\frac{\sin\left(\frac{t}{5}\right)}{5} & \frac{\cos\left(\frac{t}{5}\right)}{5} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = \frac{1}{5}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos\left(\frac{t}{5}\right)\left(\int \sin\left(\frac{t}{5}\right)t^2 dt\right)}{10} + \frac{\sin\left(\frac{t}{5}\right)\left(\int \cos\left(\frac{t}{5}\right)t^2 dt\right)}{10}$

- Compute integrals

  $y_p(t) = -25 + \frac{t^2}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos\left(\frac{t}{5}\right) + c_2 \sin\left(\frac{t}{5}\right) - 25 + \frac{t^2}{2}$

☐ Check validity of solution $y = c_1 \cos\left(\frac{t}{5}\right) + c_2 \sin\left(\frac{t}{5}\right) - 25 + \frac{t^2}{2}$

- Use initial condition $y(0) = -25$

  $-25 = c_1 - 25$

- Compute derivative of the solution

  $y' = -\frac{c_1 \sin\left(\frac{t}{5}\right)}{5} + \frac{c_2 \cos\left(\frac{t}{5}\right)}{5} + t$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = \frac{c_2}{5}$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = -25 + \frac{t^2}{2}$

- Solution to the IVP

  $y = -25 + \frac{t^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.672 (sec). Leaf size: 11

```
dsolve([diff(y(t),t$2)+4/100*y(t)=2/100*t^2,y(0) = -25, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{t^2}{2} - 25$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 14

```
DSolve[{y''[t]+4/100*y[t]==2/100*t^2,{y[0]==-25,y'[0]==0}},y[t],t,IncludeSingularSolutions -
```

$$y(t) \to \frac{1}{2}(t^2 - 50)$$

6.11 problem 11

6.11.1 Existence and uniqueness analysis 721
6.11.2 Maple step by step solution 724

Internal problem ID [5689]
Internal file name [`OUTPUT/4937_Sunday_June_05_2022_03_11_03_PM_24778006/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + \frac{9y}{4} = 9t^3 + 64}$$

With initial conditions

$$\left[y(0) = 1, y'(0) = \frac{63}{2}\right]$$

6.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= \frac{9}{4} \\ F &= 9t^3 + 64 \end{aligned}$$

Hence the ode is

$$y'' + 3y' + \frac{9y}{4} = 9t^3 + 64$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{9}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 9t^3 + 64$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) + \frac{9Y(s)}{4} = \frac{54}{s^4} + \frac{64}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= \frac{63}{2}\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - \frac{69}{2} - s + 3sY(s) + \frac{9Y(s)}{4} = \frac{54}{s^4} + \frac{64}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4s^5 + 138s^4 + 256s^3 + 216}{s^4\left(4s^2 + 12s + 9\right)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{32}{s^3} + \frac{24}{s^4} + \frac{32}{s^2} + \frac{1}{\left(s+\frac{3}{2}\right)^2} + \frac{1}{s+\frac{3}{2}}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{32}{s^3}\right) = -16t^2$$
$$\mathcal{L}^{-1}\left(\frac{24}{s^4}\right) = 4t^3$$
$$\mathcal{L}^{-1}\left(\frac{32}{s^2}\right) = 32t$$
$$\mathcal{L}^{-1}\left(\frac{1}{\left(s+\frac{3}{2}\right)^2}\right) = t\,\mathrm{e}^{-\frac{3t}{2}}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s+\frac{3}{2}}\right) = \mathrm{e}^{-\frac{3t}{2}}$$

Adding the above results and simplifying gives

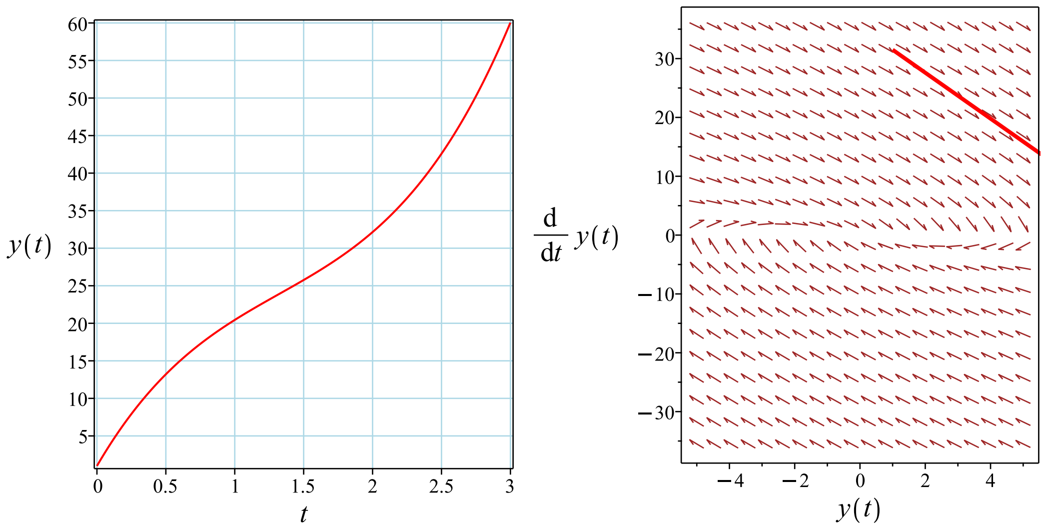
$$y = 4t^3 - 16t^2 + 32t + \mathrm{e}^{-\frac{3t}{2}}(t+1)$$

Simplifying the solution gives

$$y = 4t^3 + t\,\mathrm{e}^{-\frac{3t}{2}} - 16t^2 + \mathrm{e}^{-\frac{3t}{2}} + 32t$$

Summary
The solution(s) found are the following

$$y = 4t^3 + t\,\mathrm{e}^{-\frac{3t}{2}} - 16t^2 + \mathrm{e}^{-\frac{3t}{2}} + 32t \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 4t^3 + t\,e^{-\frac{3t}{2}} - 16t^2 + e^{-\frac{3t}{2}} + 32t$$

Verified OK.

6.11.2 Maple step by step solution

Let's solve

$\left[y'' + 3y' + \frac{9y}{4} = 9t^3 + 64, y(0) = 1, y'\Big|_{\{t=0\}} = \frac{63}{2}\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + \frac{9}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+3)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = -\frac{3}{2}$

- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{-\frac{3t}{2}}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-\frac{3t}{2}}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-\frac{3t}{2}} + c_2 t\,\mathrm{e}^{-\frac{3t}{2}} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 9t^3 + 64\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-\frac{3t}{2}} & t\,\mathrm{e}^{-\frac{3t}{2}} \\ -\frac{3\,\mathrm{e}^{-\frac{3t}{2}}}{2} & \mathrm{e}^{-\frac{3t}{2}} - \frac{3t\,\mathrm{e}^{-\frac{3t}{2}}}{2} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{-3t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \mathrm{e}^{-\frac{3t}{2}}\left(-\left(\int (9t^4 + 64t)\,\mathrm{e}^{\frac{3t}{2}}dt\right) + \left(\int \mathrm{e}^{\frac{3t}{2}}(9t^3 + 64)\,dt\right)t\right)$

- ○ Compute integrals

  $y_p(t) = 4t(t^2 - 4t + 8)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-\frac{3t}{2}} + c_2 t\,\mathrm{e}^{-\frac{3t}{2}} + 4t(t^2 - 4t + 8)$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-\frac{3t}{2}} + c_2 t \mathrm{e}^{-\frac{3t}{2}} + 4t(t^2 - 4t + 8)$

- ○ Use initial condition $y(0) = 1$

  $1 = c_1$

- ○ Compute derivative of the solution

  $y' = -\frac{3c_1 \mathrm{e}^{-\frac{3t}{2}}}{2} + c_2 \mathrm{e}^{-\frac{3t}{2}} - \frac{3c_2 t\,\mathrm{e}^{-\frac{3t}{2}}}{2} + 4t^2 - 16t + 32 + 4t(2t - 4)$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = \frac{63}{2}$

  $\frac{63}{2} = -\frac{3c_1}{2} + c_2 + 32$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = 4t^3 + t\,\mathrm{e}^{-\frac{3t}{2}} - 16t^2 + \mathrm{e}^{-\frac{3t}{2}} + 32t$

- Solution to the IVP

  $y = 4t^3 + t\,\mathrm{e}^{-\frac{3t}{2}} - 16t^2 + \mathrm{e}^{-\frac{3t}{2}} + 32t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.703 (sec). Leaf size: 26

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+225/100*y(t)=9*t^3+64,y(0) = 1, D(y)(0) = 63/2],y(t),
```

$$y(t) = 4t^3 + \mathrm{e}^{-\frac{3t}{2}}t - 16t^2 + \mathrm{e}^{-\frac{3t}{2}} + 32t$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 28

```
DSolve[{y''[t]+3*y'[t]+225/100*y[t]==9*t^3+64,{y[0]==1,y'[0]==315/10}},y[t],t,IncludeSingula
```

$$y(t) \to 4t\left(t^2 - 4t + 8\right) + e^{-3t/2}(t + 1)$$

6.12 problem 12

6.12.1 Existence and uniqueness analysis 727
6.12.2 Maple step by step solution 730

Internal problem ID [5690]
Internal file name [`OUTPUT/4938_Sunday_June_05_2022_03_11_04_PM_64268551/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' - 3y = 0}$$

With initial conditions

$$[y(4) = -3, y'(4) = -17]$$

6.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = -2$$
$$q(t) = -3$$
$$F = 0$$

Hence the ode is

$$y'' - 2y' - 3y = 0$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 4$ is inside this domain. The domain of $q(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 4$ is also inside this domain. Hence solution exists and is unique.

Since both initial conditions are not at zero, then let

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) - 3Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - c_2 - sc_1 - 2sY(s) + 2c_1 - 3Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{sc_1 - 2c_1 + c_2}{s^2 - 2s - 3}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{c_1}{4} + \frac{c_2}{4}}{s - 3} + \frac{\frac{3c_1}{4} - \frac{c_2}{4}}{s + 1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{c_1}{4}+\frac{c_2}{4}}{s-3}\right)=\frac{(c_1+c_2)\,\mathrm{e}^{3t}}{4}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{3c_1}{4}-\frac{c_2}{4}}{s+1}\right)=\frac{(3c_1-c_2)\,\mathrm{e}^{-t}}{4}$$

Adding the above results and simplifying gives

$$y=\frac{\mathrm{e}^{t}(2c_1\cosh(2t)+\sinh(2t)(-c_1+c_2))}{2}$$

Since both initial conditions given are not at zero, then we need to setup two equations to solve for c_1, c_1. At $t=4$ the first equation becomes, using the above solution

$$-3=\frac{\mathrm{e}^{4}(2c_1\cosh(8)+\sinh(8)(-c_1+c_2))}{2}$$

And taking derivative of the solution and evaluating at $t=4$ gives the second equation as

$$-17=\frac{\mathrm{e}^{4}(2c_1\cosh(8)+\sinh(8)(-c_1+c_2))}{2}+\frac{\mathrm{e}^{4}(4c_1\sinh(8)+2\cosh(8)(-c_1+c_2))}{2}$$

Solving gives

$$c_1=-\frac{\mathrm{e}^{-4}(-7\sinh(8)+3\cosh(8))}{\cosh(8)^2-\sinh(8)^2}$$
$$c_2=-\frac{(17\cosh(8)-13\sinh(8))\,\mathrm{e}^{-4}}{\cosh(8)^2-\sinh(8)^2}$$

Subtituting these in the solution obtained above gives

$$y=\frac{\mathrm{e}^{t}\left(-\frac{2\,\mathrm{e}^{-4}(-7\sinh(8)+3\cosh(8))\cosh(2t)}{\cosh(8)^2-\sinh(8)^2}+\sinh(2t)\left(\frac{\mathrm{e}^{-4}(-7\sinh(8)+3\cosh(8))}{\cosh(8)^2-\sinh(8)^2}-\frac{(17\cosh(8)-13\sinh(8))\mathrm{e}^{-4}}{\cosh(8)^2-\sinh(8)^2}\right)\right)}{2}$$
$$=-3\,\mathrm{e}^{t-4}\left(\left(\cosh(8)-\frac{7\sinh(8)}{3}\right)\cosh(2t)+\frac{7\left(\cosh(8)-\frac{3\sinh(8)}{7}\right)\sinh(2t)}{3}\right)$$

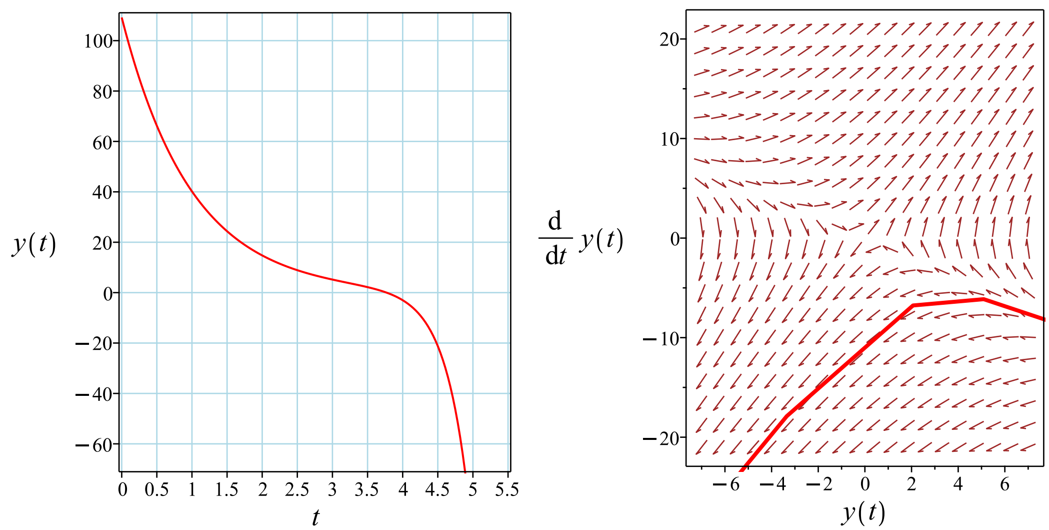
Simplifying the solution gives

$$y=-3\,\mathrm{e}^{t-4}\left(\left(\cosh(8)-\frac{7\sinh(8)}{3}\right)\cosh(2t)+\frac{7\left(\cosh(8)-\frac{3\sinh(8)}{7}\right)\sinh(2t)}{3}\right)$$

Summary

The solution(s) found are the following

$$y = -3\,\mathrm{e}^{t-4}\left(\left(\cosh(8) - \frac{7\sinh(8)}{3}\right)\cosh(2t) + \frac{7\left(\cosh(8) - \frac{3\sinh(8)}{7}\right)\sinh(2t)}{3}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -3\,\mathrm{e}^{t-4}\left(\left(\cosh(8) - \frac{7\sinh(8)}{3}\right)\cosh(2t) + \frac{7\left(\cosh(8) - \frac{3\sinh(8)}{7}\right)\sinh(2t)}{3}\right)$$

Verified OK.

6.12.2 Maple step by step solution

Let's solve

$$\left[y'' - 2y' - 3y = 0, y(4) = -3, y'\Big|_{\{t=4\}} = -17\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

$r^2 - 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 3)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = \mathrm{e}^{-t} c_1 + \mathrm{e}^{3t} c_2$

☐ Check validity of solution $y = \mathrm{e}^{-t} c_1 + \mathrm{e}^{3t} c_2$

  ◦ Use initial condition $y(4) = -3$

    $-3 = \mathrm{e}^{-4} c_1 + \mathrm{e}^{12} c_2$

  ◦ Compute derivative of the solution

    $y' = -\mathrm{e}^{-t} c_1 + 3\,\mathrm{e}^{3t} c_2$

  ◦ Use the initial condition $y'\Big|_{\{t=4\}} = -17$

    $-17 = -\mathrm{e}^{-4} c_1 + 3\,\mathrm{e}^{12} c_2$

  ◦ Solve for c_1 and c_2

    $\left\{c_1 = \frac{2}{\mathrm{e}^{-4}}, c_2 = -\frac{5}{\mathrm{e}^{12}}\right\}$

  ◦ Substitute constant values into general solution and simplify

    $y = 2\,\mathrm{e}^{-t+4} - 5\,\mathrm{e}^{3t-12}$

- Solution to the IVP

  $y = 2\,\mathrm{e}^{-t+4} - 5\,\mathrm{e}^{3t-12}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.672 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)-3*y(t)=0,y(4) = -3, D(y)(4) = -17],y(t), singsol=all)
```

$$y(t) = -5\,\mathrm{e}^{3t-12} + 2\,\mathrm{e}^{-t+4}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 24

```
DSolve[{y''[t]-2*y'[t]-3*y[t]==0,{y[4]==-3,y'[4]==-17}},y[t],t,IncludeSingularSolutions -> T
```

$$y(t) \to 2e^{4-t} - 5e^{3(t-4)}$$

6.13 problem 13

6.13.1 Existence and uniqueness analysis 733
6.13.2 Solving as laplace ode 734
6.13.3 Maple step by step solution 735

Internal problem ID [5691]
Internal file name [`OUTPUT/4939_Sunday_June_05_2022_03_11_05_PM_14214297/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 6y = 0}$$

With initial conditions

$$[y(-1) = 4]$$

6.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -6$$
$$q(t) = 0$$

Hence the ode is

$$y' - 6y = 0$$

The domain of $p(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = -1$ is inside this domain. Hence solution exists and is unique.

6.13.2 Solving as laplace ode

Since initial condition is not at zero, then let

$$y(0) = c_1$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - 6Y(s) = 0 \tag{1}$$

Replacing initial condition gives

$$sY(s) - c_1 - 6Y(s) = 0$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{c_1}{s-6}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{c_1}{s-6}\right) \\ &= c_1 e^{6t} \end{aligned}$$

The constant c_1 is determined from the given initial condition $y(0) = c_1$ using the solution found above. This results in

$$4 = c_1 e^{-6}$$

Solving gives

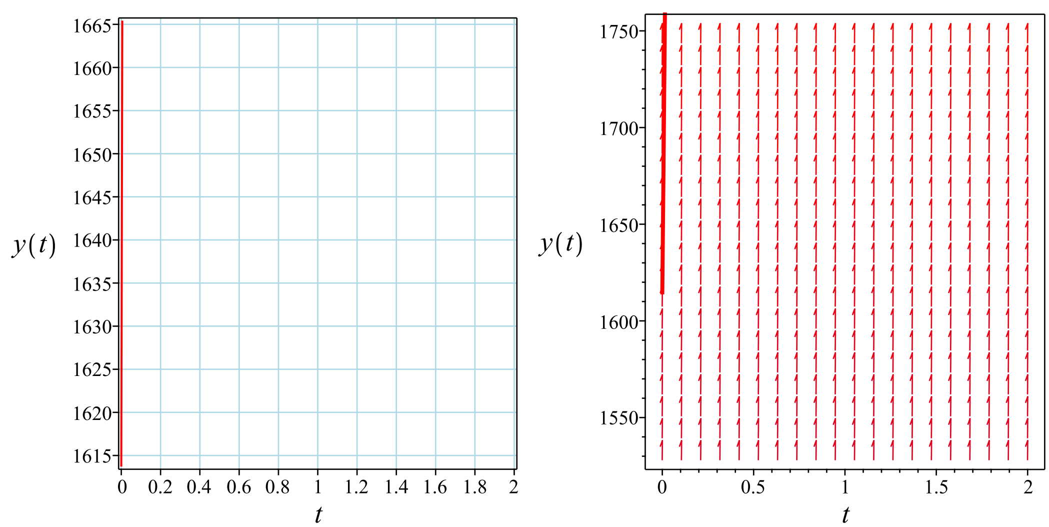
$$c_1 = 4\,\mathrm{e}^{6}$$

Hence the solution now becomes

$$y = 4\,\mathrm{e}^{6}\mathrm{e}^{6t}$$

Summary
The solution(s) found are the following

$$y = 4\,\mathrm{e}^{6}\mathrm{e}^{6t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 4\,\mathrm{e}^{6}\mathrm{e}^{6t}$$

Verified OK.

6.13.3 Maple step by step solution

Let's solve

$[y' - 6y = 0, y(-1) = 4]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 6$

- Integrate both sides with respect to t

  $\int \frac{y'}{y} dt = \int 6dt + c_1$

- Evaluate integral

  $\ln(y) = 6t + c_1$

- Solve for y

  $y = e^{6t+c_1}$

- Use initial condition $y(-1) = 4$

  $4 = e^{-6+c_1}$

- Solve for c_1

  $c_1 = 6 + 2\ln(2)$

- Substitute $c_1 = 6 + 2\ln(2)$ into general solution and simplify

  $y = 4\,e^{6t+6}$

- Solution to the IVP

  $y = 4\,e^{6t+6}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.625 (sec). Leaf size: 12

```
dsolve([diff(y(t),t)-6*y(t)=0,y(-1) = 4],y(t), singsol=all)
```

$$y(t) = 4\,\mathrm{e}^{6t+6}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 14

```
DSolve[{y'[t]-6*y[t]==0,{y[-1]==4}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 4e^{6t+6}$$

6.14 problem 14

6.14.1 Existence and uniqueness analysis 738
6.14.2 Maple step by step solution 742

Internal problem ID [5692]
Internal file name [`OUTPUT/4940_Sunday_June_05_2022_03_11_07_PM_29162432/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' + 5y = 50t - 100}$$

With initial conditions

$$[y(2) = -4, y'(2) = 14]$$

6.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 5 \\ F &= 50t - 100 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 5y = 50t - 100$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 2$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 2$ is also inside this domain. The domain of $F = 50t - 100$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 2$ is also inside this domain. Hence solution exists and is unique.

Since both initial conditions are not at zero, then let

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2 Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 5Y(s) = \frac{50}{s^2} - \frac{100}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) - c_2 - sc_1 + 2sY(s) - 2c_1 + 5Y(s) = \frac{50}{s^2} - \frac{100}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{c_1 s^3 + 2c_1 s^2 + c_2 s^2 - 100s + 50}{s^2 (s^2 + 2s + 5)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{(-1+2i)\left(-\frac{c_1}{8} - \frac{c_2}{8} - \frac{7}{4}\right) + \frac{3c_1}{8} - \frac{c_2}{8} + \frac{41}{4}}{s+1-2i} + \frac{(-1-2i)\left(-\frac{c_1}{8} - \frac{c_2}{8} - \frac{7}{4}\right) + \frac{3c_1}{8} - \frac{c_2}{8} + \frac{41}{4}}{s+1+2i} - \frac{24}{s} + \frac{10}{s^2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{(-1+2i)\left(-\frac{c_1}{8} - \frac{c_2}{8} - \frac{7}{4}\right) + \frac{3c_1}{8} - \frac{c_2}{8} + \frac{41}{4}}{s+1-2i}\right) = \frac{e^{(-1+2i)t}(48 - 14i - ic_2 + (2-i)\, c_1)}{4}$$

$$\mathcal{L}^{-1}\left(\frac{(-1-2i)\left(-\frac{c_1}{8} - \frac{c_2}{8} - \frac{7}{4}\right) + \frac{3c_1}{8} - \frac{c_2}{8} + \frac{41}{4}}{s+1+2i}\right) = \frac{e^{(-1-2i)t}(48 + 14i + ic_2 + (2+i)\, c_1)}{4}$$

$$\mathcal{L}^{-1}\left(-\frac{24}{s}\right) = -24$$

$$\mathcal{L}^{-1}\left(\frac{10}{s^2}\right) = 10t$$

Adding the above results and simplifying gives

$$y = -24 + 10t + \frac{(2\cos(2t)\,(c_1 + 24) + \sin(2t)\,(c_1 + c_2 + 14))\, e^{-t}}{2}$$

Since both initial conditions given are not at zero, then we need to setup two equations to solve for c_1, c_1. At $t = 2$ the first equation becomes, using the above solution

$$-4 = -4 + \frac{(2\cos(4)\,(c_1 + 24) + \sin(4)\,(c_1 + c_2 + 14))\, e^{-2}}{2}$$

And taking derivative of the solution and evaluating at $t = 2$ gives the second equation as

$$14 = 10 + \frac{(-4\sin(4)\,(c_1 + 24) + 2\cos(4)\,(c_1 + c_2 + 14))\, e^{-2}}{2} - \frac{(2\cos(4)\,(c_1 + 24) + \sin(4)\,(c_1 + c_2 + 14}{2}$$

Solving gives

$$c_1 = -\frac{2\, e^2 \left(12\cos(4)^2 e^{-2} + 12\sin(4)^2 e^{-2} + \sin(4)\right)}{\cos(4)^2 + \sin(4)^2}$$

$$c_2 = \frac{2\left(5\cos(4)^2 e^{-2} + 5\sin(4)^2 e^{-2} + 2\cos(4) + \sin(4)\right) e^2}{\cos(4)^2 + \sin(4)^2}$$

Subtituting these in the solution obtained above gives

$$y = -24 + 10t + \frac{\left(2\cos(2t)\left(-\frac{2\,\mathrm{e}^{2}\left(12\cos(4)^{2}\mathrm{e}^{-2}+12\sin(4)^{2}\mathrm{e}^{-2}+\sin(4)\right)}{\cos(4)^{2}+\sin(4)^{2}}+24\right)+\sin(2t)\left(-\frac{2\,\mathrm{e}^{2}\left(12\cos(4)^{2}\mathrm{e}^{-2}+12\sin(\right.}{\cos(4)^{2}+\sin(4}\right.}{2}$$

$$= 2\cos(4)\,\mathrm{e}^{2-t}\sin(2t) - 2\,\mathrm{e}^{2-t}\sin(4)\cos(2t) + 10t - 24$$

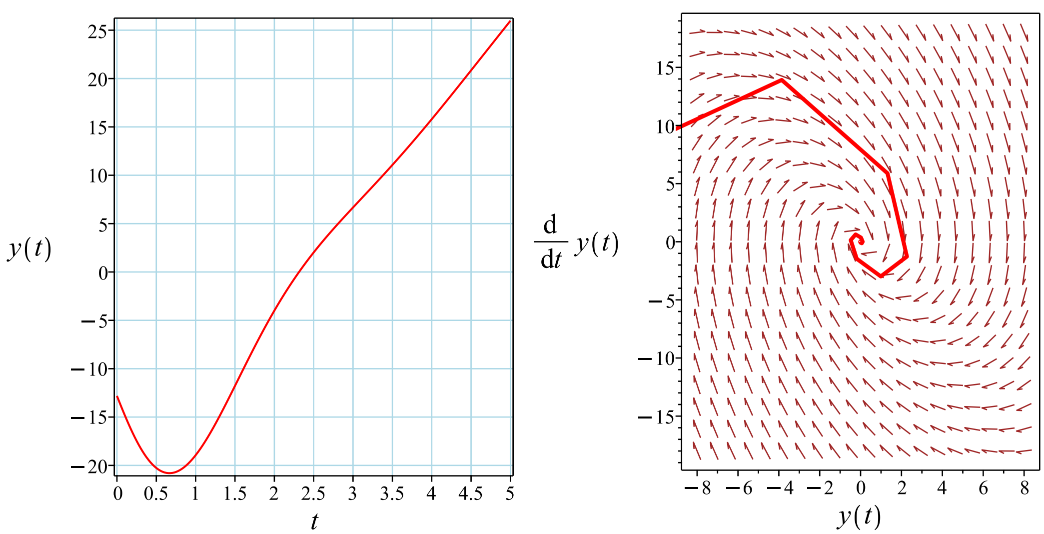
Simplifying the solution gives

$$y = 2\cos(4)\,\mathrm{e}^{2-t}\sin(2t) - 2\,\mathrm{e}^{2-t}\sin(4)\cos(2t) + 10t - 24$$

Summary

The solution(s) found are the following

$$y = 2\cos(4)\,\mathrm{e}^{2-t}\sin(2t) - 2\,\mathrm{e}^{2-t}\sin(4)\cos(2t) + 10t - 24 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 2\cos(4)\,\mathrm{e}^{2-t}\sin(2t) - 2\,\mathrm{e}^{2-t}\sin(4)\cos(2t) + 10t - 24$$

Verified OK.

6.14.2 Maple step by step solution

Let's solve

$\left[y'' + 2y' + 5y = 50t - 100, y(2) = -4, y'\Big|_{\{t=2\}} = 14\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - 2\,\mathrm{I}, -1 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)\,\mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)\,\mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t)\,\mathrm{e}^{-t} + c_2 \sin(2t)\,\mathrm{e}^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 50t - 100\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t)\,\mathrm{e}^{-t} & \sin(2t)\,\mathrm{e}^{-t} \\ -2\sin(2t)\,\mathrm{e}^{-t} - \cos(2t)\,\mathrm{e}^{-t} & 2\cos(2t)\,\mathrm{e}^{-t} - \sin(2t)\,\mathrm{e}^{-t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2\,\mathrm{e}^{-2t}$

  - Substitute functions into equation for $y_p(t)$

$$y_p(t) = -25\,\mathrm{e}^{-t}\left(\cos(2t)\left(\int(-2+t)\sin(2t)\,\mathrm{e}^{t}dt\right) - \sin(2t)\left(\int(-2+t)\cos(2t)\,\mathrm{e}^{t}dt\right)\right)$$

- Compute integrals

$$y_p(t) = -24 + 10t$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(2t)\,\mathrm{e}^{-t} + c_2\sin(2t)\,\mathrm{e}^{-t} - 24 + 10t$$

☐ Check validity of solution $y = c_1\cos(2t)\,\mathrm{e}^{-t} + c_2\sin(2t)\,\mathrm{e}^{-t} - 24 + 10t$

- Use initial condition $y(2) = -4$

$$-4 = c_1\cos(4)\,\mathrm{e}^{-2} + c_2\sin(4)\,\mathrm{e}^{-2} - 4$$

- Compute derivative of the solution

$$y' = -2c_1\sin(2t)\,\mathrm{e}^{-t} - c_1\cos(2t)\,\mathrm{e}^{-t} + 2c_2\cos(2t)\,\mathrm{e}^{-t} - c_2\sin(2t)\,\mathrm{e}^{-t} + 10$$

- Use the initial condition $y'\Big|_{\{t=2\}} = 14$

$$14 = -2c_1\sin(4)\,\mathrm{e}^{-2} - c_1\cos(4)\,\mathrm{e}^{-2} + 2c_2\cos(4)\,\mathrm{e}^{-2} - c_2\sin(4)\,\mathrm{e}^{-2} + 10$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{2\sin(4)}{\mathrm{e}^{-2}\left(\cos(4)^2+\sin(4)^2\right)}, c_2 = \frac{2\cos(4)}{\mathrm{e}^{-2}\left(\cos(4)^2+\sin(4)^2\right)}\right\}$$

- Substitute constant values into general solution and simplify

$$y = 2\cos(4)\,\mathrm{e}^{2-t}\sin(2t) - 2\,\mathrm{e}^{2-t}\sin(4)\cos(2t) + 10t - 24$$

- Solution to the IVP

$$y = 2\cos(4)\,\mathrm{e}^{2-t}\sin(2t) - 2\,\mathrm{e}^{2-t}\sin(4)\cos(2t) + 10t - 24$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.672 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+5*y(t)=50*t-100,y(2) = -4, D(y)(2) = 14],y(t), singsol
```

$$y(t) = 2\sin(2t-4)\,\mathrm{e}^{-t+2} - 24 + 10t$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 25

```
DSolve[{y''[t]+2*y'[t]+5*y[t]==50*t-100,{y[2]==-4,y'[2]==14}},y[t],t,IncludeSingularSolution
```

$$y(t) \to 10t - 2e^{2-t}\sin(4-2t) - 24$$

6.15 problem 15

6.15.1 Existence and uniqueness analysis 745
6.15.2 Maple step by step solution 749

Internal problem ID [5693]
Internal file name [`OUTPUT/4941_Sunday_June_05_2022_03_11_08_PM_53618701/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.2, page 216
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y' - 4y = 6\,e^{2t-3}}$$

With initial conditions

$$\left[y\left(\frac{3}{2}\right) = 4, y'\left(\frac{3}{2}\right) = 5\right]$$

6.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= -4 \\ F &= 6\,e^{2t-3} \end{aligned}$$

Hence the ode is

$$y'' + 3y' - 4y = 6\,\mathrm{e}^{2t-3}$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = \frac{3}{2}$ is inside this domain. The domain of $q(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = \frac{3}{2}$ is also inside this domain. The domain of $F = 6\,\mathrm{e}^{2t-3}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = \frac{3}{2}$ is also inside this domain. Hence solution exists and is unique.

Since both initial conditions are not at zero, then let

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) - 4Y(s) = \frac{6\,\mathrm{e}^{-3}}{s-2} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - c_2 - sc_1 + 3sY(s) - 3c_1 - 4Y(s) = \frac{6\,e^{-3}}{s-2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2c_1 + sc_1 + c_2s + 6\,e^{-3} - 6c_1 - 2c_2}{(s-2)\,(s^2+3s-4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{4c_1}{5} + \frac{c_2}{5} - \frac{6\,e^{-3}}{5}}{s-1} + \frac{e^{-3}}{s-2} + \frac{\frac{c_1}{5} - \frac{c_2}{5} + \frac{e^{-3}}{5}}{s+4}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{4c_1}{5} + \frac{c_2}{5} - \frac{6\,e^{-3}}{5}}{s-1}\right) = \frac{e^t(4c_1 + c_2 - 6\,e^{-3})}{5}$$
$$\mathcal{L}^{-1}\left(\frac{e^{-3}}{s-2}\right) = e^{2t-3}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{c_1}{5} - \frac{c_2}{5} + \frac{e^{-3}}{5}}{s+4}\right) = \frac{(c_1 - c_2 + e^{-3})\,e^{-4t}}{5}$$

Adding the above results and simplifying gives

$$y = e^{2t-3} + \frac{e^t(4c_1 + c_2 - 6\,e^{-3})}{5} + \frac{(c_1 - c_2 + e^{-3})\,e^{-4t}}{5}$$

Since both initial conditions given are not at zero, then we need to setup two equations to solve for c_1, c_1. At $t = \frac{3}{2}$ the first equation becomes, using the above solution

$$4 = 1 + \frac{e^{\frac{3}{2}}(4c_1 + c_2 - 6\,e^{-3})}{5} + \frac{(c_1 - c_2 + e^{-3})\,e^{-6}}{5}$$

And taking derivative of the solution and evaluating at $t = \frac{3}{2}$ gives the second equation as

$$5 = 2 + \frac{e^{\frac{3}{2}}(4c_1 + c_2 - 6\,e^{-3})}{5} - \frac{4(c_1 - c_2 + e^{-3})\,e^{-6}}{5}$$

Solving gives

$$c_1 = \left(e^{\frac{3}{2}}e^{-3} + 3\right)e^{-\frac{3}{2}}$$
$$c_2 = e^{-\frac{3}{2}}\left(2\,e^{\frac{3}{2}}e^{-3} + 3\right)$$

Subtituting these in the solution obtained above gives

$$y = \mathrm{e}^{2t-3} + \frac{\mathrm{e}^{t}\left(4\left(\mathrm{e}^{\frac{3}{2}}\mathrm{e}^{-3}+3\right)\mathrm{e}^{-\frac{3}{2}} + \mathrm{e}^{-\frac{3}{2}}\left(2\,\mathrm{e}^{\frac{3}{2}}\mathrm{e}^{-3}+3\right) - 6\,\mathrm{e}^{-3}\right)}{5} + \frac{\left(\left(\mathrm{e}^{\frac{3}{2}}\mathrm{e}^{-3}+3\right)\mathrm{e}^{-\frac{3}{2}} - \mathrm{e}^{-\frac{3}{2}}\left(2\,\mathrm{e}^{\frac{3}{2}}\mathrm{e}^{-3}+3\right)}{5}$$

$$= \mathrm{e}^{2t-3} + 3\,\mathrm{e}^{t-\frac{3}{2}}$$

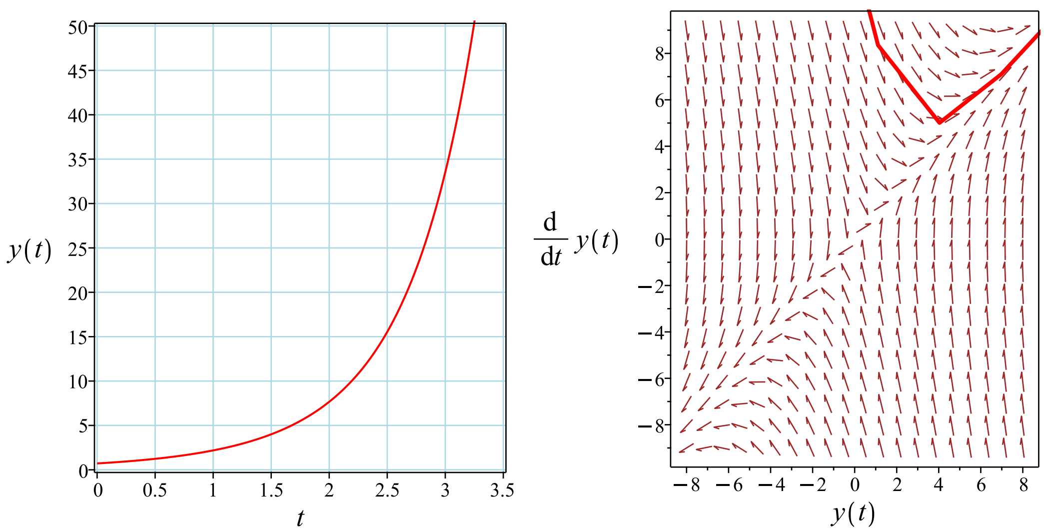
Simplifying the solution gives

$$y = \mathrm{e}^{2t-3} + 3\,\mathrm{e}^{t-\frac{3}{2}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2t-3} + 3\,\mathrm{e}^{t-\frac{3}{2}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{2t-3} + 3\,\mathrm{e}^{t-\frac{3}{2}}$$

Verified OK.

6.15.2 Maple step by step solution

Let's solve

$\left[y'' + 3y' - 4y = 6\,e^{2t-3}, y\left(\frac{3}{2}\right) = 4, y'\Big|_{\left\{t=\frac{3}{2}\right\}} = 5\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r - 4 = 0$

- Factor the characteristic polynomial

  $(r + 4)(r - 1) = 0$

- Roots of the characteristic polynomial

  $r = (-4, 1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4t} + c_2 e^{t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 6\,e^{2t-3}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{t} \\ -4\,e^{-4t} & e^{t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 5\,e^{-3t}$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = \frac{6\left(\mathrm{e}^{5t}\left(\int \mathrm{e}^{t-3}dt\right)-\left(\int \mathrm{e}^{6t-3}dt\right)\right)\mathrm{e}^{-4t}}{5}$

- Compute integrals

  $y_p(t) = \mathrm{e}^{2t-3}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-4t} + c_2\mathrm{e}^{t} + \mathrm{e}^{2t-3}$

☐ Check validity of solution $y = c_1\mathrm{e}^{-4t} + c_2\mathrm{e}^{t} + \mathrm{e}^{2t-3}$

- Use initial condition $y\left(\frac{3}{2}\right) = 4$

  $4 = c_1\mathrm{e}^{-6} + c_2\mathrm{e}^{\frac{3}{2}} + 1$

- Compute derivative of the solution

  $y' = -4c_1\mathrm{e}^{-4t} + c_2\mathrm{e}^{t} + 2\,\mathrm{e}^{2t-3}$

- Use the initial condition $y'\Big|_{\left\{t=\frac{3}{2}\right\}} = 5$

  $5 = -4c_1\mathrm{e}^{-6} + c_2\mathrm{e}^{\frac{3}{2}} + 2$

- Solve for c_1 and c_2

  $\left\{c_1 = 0, c_2 = \frac{3}{\mathrm{e}^{\frac{3}{2}}}\right\}$

- Substitute constant values into general solution and simplify

  $y = \mathrm{e}^{2t-3} + 3\,\mathrm{e}^{t-\frac{3}{2}}$

- Solution to the IVP

  $y = \mathrm{e}^{2t-3} + 3\,\mathrm{e}^{t-\frac{3}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.812 (sec). Leaf size: 17

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)-4*y(t)=6*exp(2*t-3),y(3/2) = 4, D(y)(3/2) = 5],y(t), s
```

$$y(t) = e^{2t-3} + 3\,e^{t-\frac{3}{2}}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 22

```
DSolve[{y''[t]+3*y'[t]-4*y[t]==6*Exp[2*t-3],{y[15/10]==4,y'[15/10]==5}},y[t],t,IncludeSingul
```

$$y(t) \to 3e^{t-\frac{3}{2}} + e^{2t-3}$$

7 Chapter 6. Laplace Transforms. Problem set 6.3, page 224

7.1 problem 18 . . . 753
7.2 problem 19 . . . 758
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7.1 problem 18

7.1.1 Existence and uniqueness analysis 753
7.1.2 Maple step by step solution 755

Internal problem ID [5694]
Internal file name [`OUTPUT/4942_Sunday_June_05_2022_03_11_10_PM_38793469/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.3, page 224
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{9y'' - 6y' + y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 1]$$

7.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -\frac{2}{3} \\ q(t) &= \frac{1}{9} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{2y'}{3} + \frac{y}{9} = 0$$

The domain of $p(t) = -\frac{2}{3}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{1}{9}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$9s^2Y(s) - 9y'(0) - 9sy(0) - 6sY(s) + 6y(0) + Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 3\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$9s^2Y(s) + 9 - 27s - 6sY(s) + Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{9}{3s - 1}$$

Taking inverse Laplace transform gives

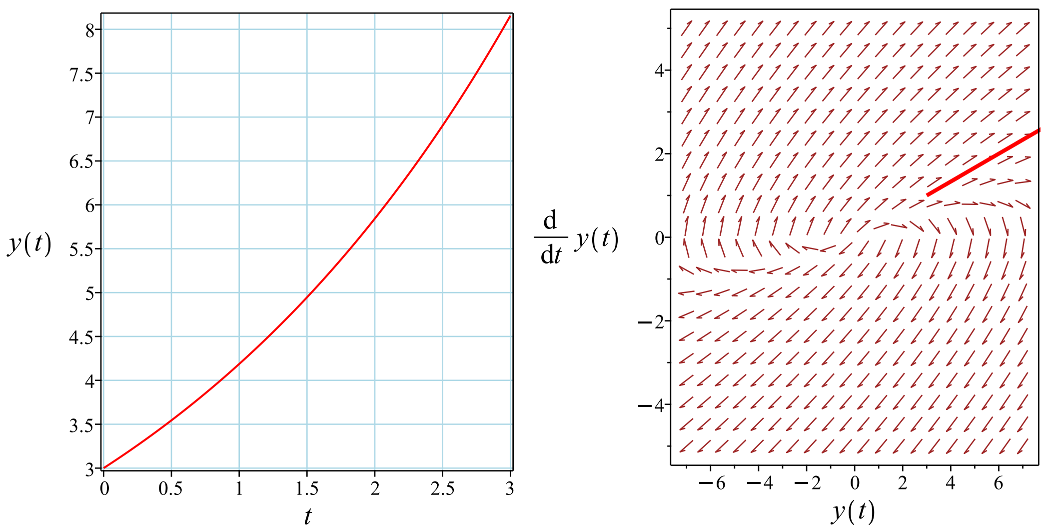
$$\mathcal{L}^{-1}\left(\frac{9}{3s - 1}\right) = 3\,\mathrm{e}^{\frac{t}{3}}$$

Simplifying the solution gives

$$y = 3\,e^{\frac{t}{3}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 3\,e^{\frac{t}{3}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = 3\,e^{\frac{t}{3}}$$

Verified OK.

7.1.2 Maple step by step solution

Let's solve

$$\left[9y'' - 6y' + y = 0, y(0) = 3, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{2y'}{3} - \frac{y}{9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{3} + \frac{y}{9} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{2}{3}r + \frac{1}{9} = 0$

- Factor the characteristic polynomial

  $\frac{(3r-1)^2}{9} = 0$

- Root of the characteristic polynomial

  $r = \frac{1}{3}$

- 1st solution of the ODE

  $y_1(t) = e^{\frac{t}{3}}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\, e^{\frac{t}{3}}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{\frac{t}{3}} + c_2 t\, e^{\frac{t}{3}}$

□ Check validity of solution $y = c_1 e^{\frac{t}{3}} + c_2 t e^{\frac{t}{3}}$

  - Use initial condition $y(0) = 3$

    $3 = c_1$

  - Compute derivative of the solution

    $y' = \frac{c_1 e^{\frac{t}{3}}}{3} + c_2 e^{\frac{t}{3}} + \frac{c_2 t\, e^{\frac{t}{3}}}{3}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = \frac{c_1}{3} + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

    $y = 3\, e^{\frac{t}{3}}$

- Solution to the IVP

$$y = 3\,\mathrm{e}^{\frac{t}{3}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.719 (sec). Leaf size: 10

```
dsolve([9*diff(y(t),t$2)-6*diff(y(t),t)+y(t)=0,y(0) = 3, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = 3\,\mathrm{e}^{\frac{t}{3}}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 14

```
DSolve[{9*y''[t]-6*y'[t]+y[t]==0,{y[0]==3,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to 3e^{t/3}$$

7.2 problem 19

7.2.1 Existence and uniqueness analysis 758
7.2.2 Maple step by step solution 761

Internal problem ID [5695]
Internal file name [`OUTPUT/4943_Sunday_June_05_2022_03_11_11_PM_55865171/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.3, page 224
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 8y = \mathrm{e}^{-3t} - \mathrm{e}^{-5t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

7.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 8 \\ F &= \mathrm{e}^{-3t} - \mathrm{e}^{-5t} \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 8y = \mathrm{e}^{-3t} - \mathrm{e}^{-5t}$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \mathrm{e}^{-3t} - \mathrm{e}^{-5t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 6sY(s) - 6y(0) + 8Y(s) = \frac{2}{(s+3)(s+5)} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 6sY(s) + 8Y(s) = \frac{2}{(s+3)(s+5)}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2}{(s+3)(s+5)(s^2+6s+8)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s+3} - \frac{1}{3(s+5)} + \frac{1}{3s+6} + \frac{1}{s+4}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{s+3}\right) = -\mathrm{e}^{-3t}$$

$$\mathcal{L}^{-1}\left(-\frac{1}{3\,(s+5)}\right) = -\frac{\mathrm{e}^{-5t}}{3}$$

$$\mathcal{L}^{-1}\left(\frac{1}{3s+6}\right) = \frac{\mathrm{e}^{-2t}}{3}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) = \mathrm{e}^{-4t}$$

Adding the above results and simplifying gives

$$y = \mathrm{e}^{-4t} - \frac{\mathrm{e}^{-5t}}{3} - \mathrm{e}^{-3t} + \frac{\mathrm{e}^{-2t}}{3}$$

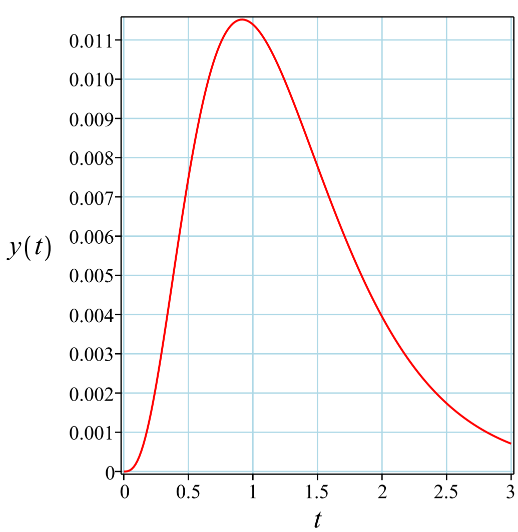
Simplifying the solution gives

$$y = \mathrm{e}^{-4t} - \frac{\mathrm{e}^{-5t}}{3} - \mathrm{e}^{-3t} + \frac{\mathrm{e}^{-2t}}{3}$$

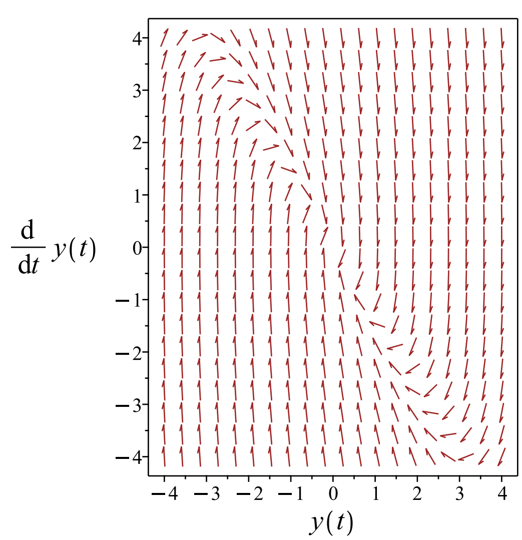
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-4t} - \frac{\mathrm{e}^{-5t}}{3} - \mathrm{e}^{-3t} + \frac{\mathrm{e}^{-2t}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-4t} - \frac{e^{-5t}}{3} - e^{-3t} + \frac{e^{-2t}}{3}$$

Verified OK.

7.2.2 Maple step by step solution

Let's solve

$$\left[y'' + 6y' + 8y = e^{-3t} - e^{-5t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 8 = 0$

- Factor the characteristic polynomial

  $(r + 4)(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = e^{-3t} - e^{-5t}\right]$$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{-2t} \\ -4\,e^{-4t} & -2\,e^{-2t} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{-6t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{e^{-4t}\left(\int (e^{t} - e^{-t})\,dt\right)}{2} + \frac{e^{-2t}\left(\int (e^{2t} - 1)\,e^{-3t}\,dt\right)}{2}$

- ○ Compute integrals

  $y_p(t) = -\frac{e^{-5t}}{3} - e^{-3t}$

- ● Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} - \frac{e^{-5t}}{3} - e^{-3t}$

□ Check validity of solution $y = c_1 e^{-4t} + c_2 e^{-2t} - \frac{e^{-5t}}{3} - e^{-3t}$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - \frac{4}{3}$

- ○ Compute derivative of the solution

  $y' = -4c_1 e^{-4t} - 2c_2 e^{-2t} + \frac{5\,e^{-5t}}{3} + 3\,e^{-3t}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -4c_1 - 2c_2 + \frac{14}{3}$

- ○ Solve for c_1 and c_2

  $\left\{c_1 = 1, c_2 = \frac{1}{3}\right\}$

- ○ Substitute constant values into general solution and simplify

  $y = e^{-4t} - \frac{e^{-5t}}{3} - e^{-3t} + \frac{e^{-2t}}{3}$

- ● Solution to the IVP

  $y = e^{-4t} - \frac{e^{-5t}}{3} - e^{-3t} + \frac{e^{-2t}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.812 (sec). Leaf size: 27

```
dsolve([diff(y(t),t$2)+6*diff(y(t),t)+8*y(t)=exp(-3*t)-exp(-5*t),y(0) = 0, D(y)(0) = 0],y(t)
```

$$y(t) = \mathrm{e}^{-4t} - \mathrm{e}^{-3t} - \frac{\mathrm{e}^{-5t}}{3} + \frac{\mathrm{e}^{-2t}}{3}$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 21

```
DSolve[{y''[t]+6*y'[t]+8*y[t]==Exp[-3*t]-Exp[-5*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingula
```

$$y(t) \to \frac{1}{3}e^{-5t}(e^t - 1)^3$$

7.3 problem 20

7.3.1 Existence and uniqueness analysis 764
7.3.2 Maple step by step solution 767

Internal problem ID [5696]
Internal file name [`OUTPUT/4944_Sunday_June_05_2022_03_11_12_PM_3379136/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.3, page 224
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 10y' + 24y = 144t^2}$$

With initial conditions

$$\left[y(0) = \frac{19}{12}, y'(0) = -5\right]$$

7.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 10 \\ q(t) &= 24 \\ F &= 144t^2 \end{aligned}$$

Hence the ode is

$$y'' + 10y' + 24y = 144t^2$$

The domain of $p(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 24$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 144t^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 10sY(s) - 10y(0) + 24Y(s) = \frac{288}{s^3} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= \frac{19}{12}\\ y'(0) &= -5\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - \frac{65}{6} - \frac{19s}{12} + 10sY(s) + 24Y(s) = \frac{288}{s^3}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{19s^2 - 60s + 144}{12s^3}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{19}{12s} - \frac{5}{s^2} + \frac{12}{s^3}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{19}{12s}\right) = \frac{19}{12}$$
$$\mathcal{L}^{-1}\left(-\frac{5}{s^2}\right) = -5t$$
$$\mathcal{L}^{-1}\left(\frac{12}{s^3}\right) = 6t^2$$

Adding the above results and simplifying gives

$$y = 6t^2 - 5t + \frac{19}{12}$$

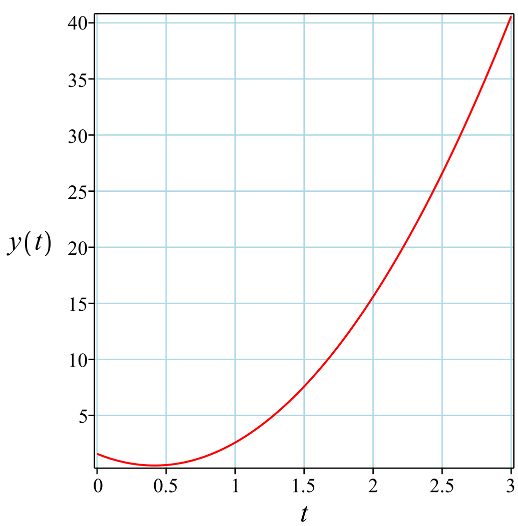
Simplifying the solution gives

$$y = 6t^2 - 5t + \frac{19}{12}$$

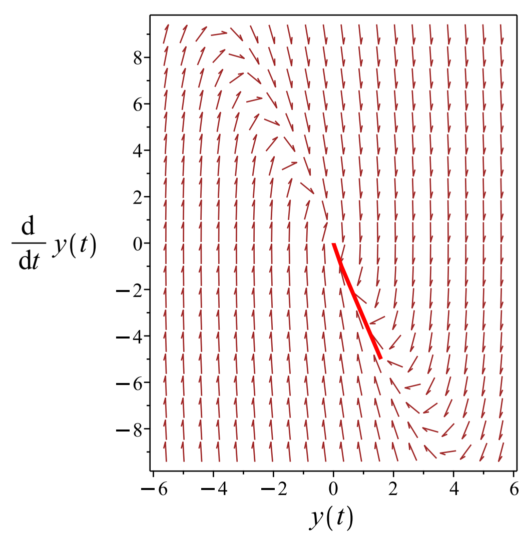
Summary

The solution(s) found are the following

$$y = 6t^2 - 5t + \frac{19}{12} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 6t^2 - 5t + \frac{19}{12}$$

Verified OK.

7.3.2 Maple step by step solution

Let's solve

$\left[y'' + 10y' + 24y = 144t^2, y(0) = \frac{19}{12}, y'\Big|_{\{t=0\}} = -5\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 10r + 24 = 0$
- Factor the characteristic polynomial

  $(r+6)(r+4) = 0$
- Roots of the characteristic polynomial

  $r = (-6, -4)$
- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-6t}$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-4t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-6t} + c_2 \mathrm{e}^{-4t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 144t^2\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t)\,,y_2(t))=\left[\begin{array}{cc} \mathrm{e}^{-6t} & \mathrm{e}^{-4t} \\ -6\,\mathrm{e}^{-6t} & -4\,\mathrm{e}^{-4t} \end{array}\right]$$

- Compute Wronskian

  $W(y_1(t)\,,y_2(t))=2\,\mathrm{e}^{-10t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t)=-72\,\mathrm{e}^{-6t}\left(\int t^2\mathrm{e}^{6t}dt\right)+72\,\mathrm{e}^{-4t}\left(\int t^2\mathrm{e}^{4t}dt\right)$

- Compute integrals

  $y_p(t)=6t^2-5t+\frac{19}{12}$

- Substitute particular solution into general solution to ODE

  $y=c_1\mathrm{e}^{-6t}+c_2\mathrm{e}^{-4t}+6t^2-5t+\frac{19}{12}$

☐ Check validity of solution $y=c_1\mathrm{e}^{-6t}+c_2\mathrm{e}^{-4t}+6t^2-5t+\frac{19}{12}$

- Use initial condition $y(0)=\frac{19}{12}$

  $\frac{19}{12}=c_1+c_2+\frac{19}{12}$

- Compute derivative of the solution

  $y'=-6c_1\mathrm{e}^{-6t}-4c_2\mathrm{e}^{-4t}+12t-5$

- Use the initial condition $y'\Big|_{\{t=0\}}=-5$

  $-5=-6c_1-4c_2-5$

- Solve for c_1 and c_2

  $\{c_1=0,c_2=0\}$

- Substitute constant values into general solution and simplify

  $y=6t^2-5t+\frac{19}{12}$

- Solution to the IVP

  $y=6t^2-5t+\frac{19}{12}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.687 (sec). Leaf size: 14

```
dsolve([diff(y(t),t$2)+10*diff(y(t),t)+24*y(t)=144*t^2,y(0) = 19/12, D(y)(0) = -5],y(t), sin
```

$$y(t) = 6t^2 - 5t + \frac{19}{12}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 17

```
DSolve[{y''[t]+10*y'[t]+24*y[t]==144*t^2,{y[0]==19/12,y'[0]==-5}},y[t],t,IncludeSingularSolu
```

$$y(t) \to 6t^2 - 5t + \frac{19}{12}$$

7.4 problem 21

7.4.1 Existence and uniqueness analysis 770
7.4.2 Maple step by step solution 773

Internal problem ID [5697]
Internal file name [`OUTPUT/4945_Sunday_June_05_2022_03_11_14_PM_59146011/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.3, page 224
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + 9y = \begin{cases} 8\sin(t) & 0 < t < \pi \\ 0 & \pi < t \end{cases}$$

With initial conditions

$$[y(0) = 0, y'(0) = 4]$$

7.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 0$$
$$q(t) = 9$$
$$F = \begin{cases} 0 & t \leq 0 \\ 8\sin(t) & t < \pi \\ 0 & \pi \leq t \end{cases}$$

Hence the ode is

$$y'' + 9y = \begin{cases} 0 & t \le 0 \\ 8 \sin(t) & t < \pi \\ 0 & \pi \le t \end{cases}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t \le 0 \\ 8 \sin(t) & t < \pi \\ 0 & \pi \le t \end{cases}$ is

$$\{0 \le t \le \pi, \pi \le t \le \infty, -\infty \le t \le 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{8 + 8\,\mathrm{e}^{-\pi s}}{s^2 + 1} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 4 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4 + 9Y(s) = \frac{8 + 8\,\mathrm{e}^{-\pi s}}{s^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4s^2 + 8\,\mathrm{e}^{-\pi s} + 12}{(s^2 + 1)(s^2 + 9)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{4s^2 + 8\,\mathrm{e}^{-\pi s} + 12}{(s^2 + 1)(s^2 + 9)}\right) \\ &= \sin(3t) + \sin(t) - \frac{4\,\mathrm{Heaviside}(t - \pi)\sin(t)^3}{3} \end{aligned}$$

Hence the final solution is

$$y = \sin(3t) + \sin(t) - \frac{4\,\mathrm{Heaviside}(t - \pi)\sin(t)^3}{3}$$

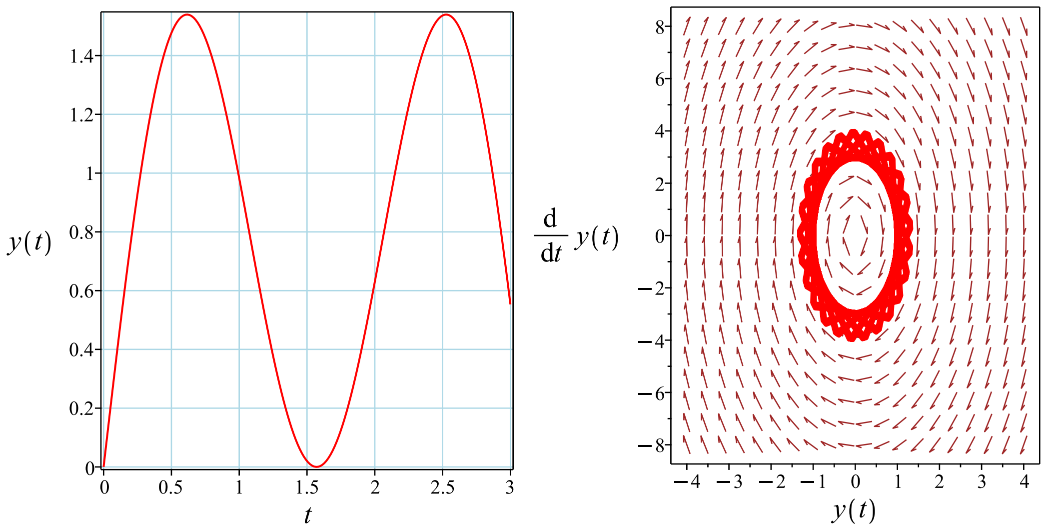
Simplifying the solution gives

$$y = -\frac{4\,\mathrm{Heaviside}(t - \pi)\sin(t)^3}{3} + 4\sin(t)\cos(t)^2$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{4\,\mathrm{Heaviside}(t - \pi)\sin(t)^3}{3} + 4\sin(t)\cos(t)^2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{4\,\text{Heaviside}(t-\pi)\sin(t)^3}{3} + 4\sin(t)\cos(t)^2$$

Verified OK.

7.4.2 Maple step by step solution

Let's solve

$$\left[y'' + 9y = \begin{cases} 0 & t \leq 0 \\ 8\sin(t) & t < \pi \\ 0 & \pi \leq t \end{cases}, y(0) = 0, y'\Big|_{\{t=0\}} = 4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \begin{cases} 0 & t \le 0 \\ 8\sin(t) & t < \pi \\ 0 & \pi \le t \end{cases}\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(t), y_2(t)) = 3$$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\cos(3t)\left(\int \left(\begin{cases} 0 & t \le 0 \\ \frac{8\sin(3t)\sin(t)}{3} & t < \pi \\ 0 & \pi \le t \end{cases}\right) dt\right) + \sin(3t)\left(\int \left(\begin{cases} 0 & t \le 0 \\ \frac{8\cos(3t)\sin(t)}{3} & t < \pi \\ 0 & \pi \le t \end{cases}\right) d$$

- Compute integrals

$$y_p(t) = \begin{cases} 0 & t \le 0 \\ \frac{4\sin(t)^3}{3} & t \le \pi \\ 0 & \pi < t \end{cases}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(3t) + c_2 \sin(3t) + \left\{\begin{array}{cc} 0 & t \leq 0 \\ \frac{4\sin(t)^3}{3} & t \leq \pi \\ 0 & \pi < t \end{array}\right.$$

☐ Check validity of solution $y = c_1 \cos(3t) + c_2 \sin(3t) + \left\{\begin{array}{cc} 0 & t \leq 0 \\ \frac{4\sin(t)^3}{3} & t \leq \pi \\ 0 & \pi < t \end{array}\right.$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $$y' = -3c_1 \sin(3t) + 3c_2 \cos(3t) + \left\{\begin{array}{cc} 0 & t \leq 0 \\ 4\sin(t)^2 \cos(t) & t \leq \pi \\ 0 & \pi < t \end{array}\right.$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 4$

  $4 = 3c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = 0, c_2 = \frac{4}{3}\right\}$

- Substitute constant values into general solution and simplify

  $$y = \frac{4\sin(3t)}{3} + \left\{\begin{array}{cc} 0 & t \leq 0 \\ \frac{4\sin(t)^3}{3} & t \leq \pi \\ 0 & \pi < t \end{array}\right.$$

- Solution to the IVP

  $$y = \frac{4\sin(3t)}{3} + \left\{\begin{array}{cc} 0 & t \leq 0 \\ \frac{4\sin(t)^3}{3} & t \leq \pi \\ 0 & \pi < t \end{array}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.281 (sec). Leaf size: 31

```
dsolve([diff(y(t),t$2)+9*y(t)=piecewise(0<t and t<Pi,8*sin(t),t>Pi,0),y(0) = 0, D(y)(0) = 4]
```

$$y(t) = 4 \left(\begin{cases} \sin(t)\cos(t)^2 & t < \pi \\ \frac{\sin(3t)}{3} & \pi \leq t \end{cases} \right)$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 30

```
DSolve[{y''[t]+9*y[t]==Piecewise[{{8*Sin[t],0<t<Pi},{0,t>Pi}}],{y[0]==0,y'[0]==4}},y[t],t,In
```

$$y(t) \to \begin{cases} \frac{4}{3}\sin(3t) & t > \pi \lor t \leq 0 \\ \sin(t) + \sin(3t) & \text{True} \end{cases}$$

7.5 problem 22

7.5.1 Existence and uniqueness analysis 777
7.5.2 Maple step by step solution 781

Internal problem ID [5698]
Internal file name [`OUTPUT/4946_Sunday_June_05_2022_03_11_18_PM_59599965/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.3, page 224
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + 3y' + 2y = \begin{cases} 4t & 0 < t < 1 \\ 8 & 1 < t \end{cases}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

7.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$
\begin{aligned}
p(t) &= 3 \\
q(t) &= 2 \\
F &= 4\left(\left\{\begin{array}{cc}0 & t \le 0 \\ t & t < 1 \\ 0 & t = 1 \\ 2 & 1 < t\end{array}\right.\right)
\end{aligned}
$$

Hence the ode is

$$
y'' + 3y' + 2y = 4\left(\left\{\begin{array}{cc}0 & t \le 0 \\ t & t < 1 \\ 0 & t = 1 \\ 2 & 1 < t\end{array}\right.\right)
$$

The domain of $p(t) = 3$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 4\left(\left\{\begin{array}{cc}0 & t \le 0 \\ t & t < 1 \\ 0 & t = 1 \\ 2 & 1 < t\end{array}\right.\right)$ is

$$
\{t < 1 \lor 1 < t\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) + 2Y(s) = \text{laplace}\left(\left\{\begin{array}{cc} 0 & t \leq 0 \\ 4t & t < 1 \\ 0 & t = 1 \\ 8 & 1 < t \end{array}\right., t, s\right) \quad (1)$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3sY(s) + 2Y(s) = \text{laplace}\left(\left\{\begin{array}{cc} 0 & t \leq 0 \\ 4t & t < 1 \\ 0 & t = 1 \\ 8 & 1 < t \end{array}\right., t, s\right)$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\text{laplace}\left(\left\{\begin{array}{cc} 0 & t \leq 0 \\ 4t & t < 1 \\ 0 & t = 1 \\ 8 & 1 < t \end{array}\right., t, s\right)}{s^2 + 3s + 2}$$

Taking the inverse Laplace transform gives

$$y = \mathcal{L}^{-1}(Y(s))$$
$$= \mathcal{L}^{-1}\left(\frac{\text{laplace}\left(\left(\begin{cases} 0 & t \le 0 \\ 4t & t < 1 \\ 0 & t = 1 \\ 8 & 1 < t \end{cases}, t, s\right)\right)}{s^2+3s+2}\right)$$
$$= (2t-7)\,\text{Heaviside}\,(-t+1) + \left(3\,\mathrm{e}^{-2t+2} - 8\,\mathrm{e}^{-t+1}\right)\text{Heaviside}\,(t-1) + 4\,\mathrm{e}^{-t} + 4 - \mathrm{e}^{-2t}$$

Hence the final solution is

$$y = (2t-7)\,\text{Heaviside}\,(-t+1) + \left(3\,\mathrm{e}^{-2t+2} - 8\,\mathrm{e}^{-t+1}\right)\text{Heaviside}\,(t-1) + 4\,\mathrm{e}^{-t} + 4 - \mathrm{e}^{-2t}$$

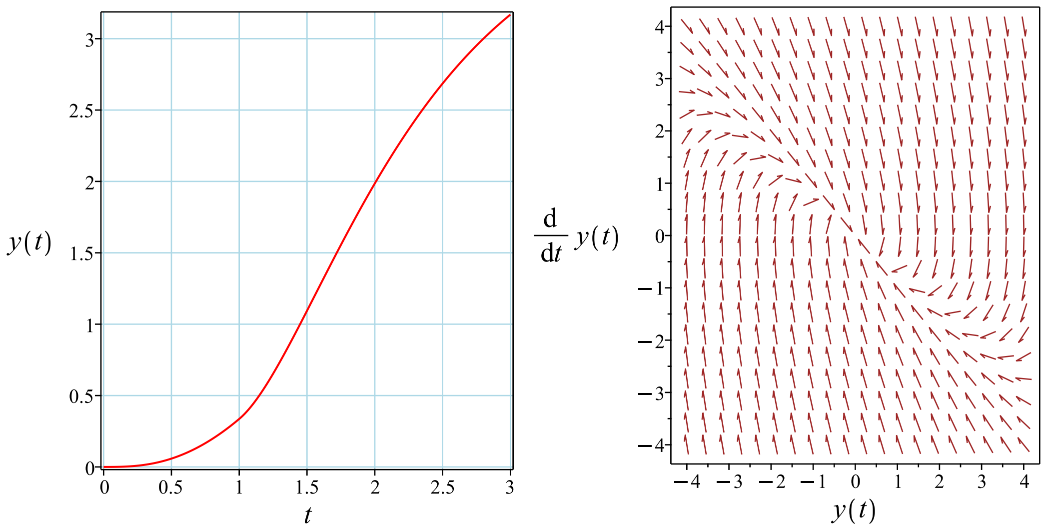
Simplifying the solution gives

$$\begin{aligned} y = {} & 3\,\text{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2} - 8\,\text{Heaviside}\,(t-1)\,\mathrm{e}^{-t+1} \\ & + (-2t+7)\,\text{Heaviside}\,(t-1) + 2t - \mathrm{e}^{-2t} + 4\,\mathrm{e}^{-t} - 3 \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & 3\,\text{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2} - 8\,\text{Heaviside}\,(t-1)\,\mathrm{e}^{-t+1} \\ & + (-2t+7)\,\text{Heaviside}\,(t-1) + 2t - \mathrm{e}^{-2t} + 4\,\mathrm{e}^{-t} - 3 \end{aligned} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$
\begin{aligned}
y = {} & 3\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2} - 8\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-t+1} \\
& + (-2t+7)\,\mathrm{Heaviside}\,(t-1) + 2t - \mathrm{e}^{-2t} + 4\,\mathrm{e}^{-t} - 3
\end{aligned}
$$

Verified OK.

7.5.2 Maple step by step solution

Let's solve

$$
\left[y'' + 3y' + 2y = \left\{\begin{array}{cc} 0 & t \leq 0 \\ 4t & t < 1 \\ 0 & t = 1 \\ 8 & 1 < t \end{array}\right., y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]
$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$
- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \begin{cases} 0 & t \le 0 \\ 4t & t < 1 \\ 0 & t = 1 \\ 8 & 1 < t \end{cases} \right]$$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{-t} \\ -2\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} \end{bmatrix}$$

  - Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{-3t}$

  - Substitute functions into equation for $y_p(t)$

$$y_p(t) = -4\,\mathrm{e}^{-2t}\left(\int \mathrm{e}^{2t}\left(\begin{cases} 0 & t \le 0 \\ t & t < 1 \\ 0 & t = 1 \\ 2 & 1 < t \end{cases}\right) dt\right) + 4\,\mathrm{e}^{-t}\left(\int \mathrm{e}^{t}\left(\begin{cases} 0 & t \le 0 \\ t & t < 1 \\ 0 & t = 1 \\ 2 & 1 < t \end{cases}\right) dt\right)$$

  - Compute integrals

$$y_p(t) = \begin{cases} 0 & t \leq 0 \\ -3+4\,\mathrm{e}^{-t}-\mathrm{e}^{-2t}+2t & t \leq 1 \\ 4+4\,\mathrm{e}^{-t}-\mathrm{e}^{-2t}+3\,\mathrm{e}^{-2t+2}-8\,\mathrm{e}^{-t+1} & 1 < t \end{cases}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{-t}+\begin{cases} 0 & t \leq 0 \\ -3+4\,\mathrm{e}^{-t}-\mathrm{e}^{-2t}+2t & t \leq 1 \\ 4+4\,\mathrm{e}^{-t}-\mathrm{e}^{-2t}+3\,\mathrm{e}^{-2t+2}-8\,\mathrm{e}^{-t+1} & 1 < t \end{cases}$$

□ Check validity of solution $y = c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{-t}+\begin{cases} 0 & t \leq 0 \\ -3+4\mathrm{e}^{-t}-\mathrm{e}^{-2t}+2t & t \leq 1 \\ 4+4\mathrm{e}^{-t}-\mathrm{e}^{-2t}+3\mathrm{e}^{-2t+2}-8\mathrm{e}^{-t+1} & 1 < t \end{cases}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2$
  - Compute derivative of the solution

    $$y' = -2c_1\mathrm{e}^{-2t}-c_2\mathrm{e}^{-t}+\begin{cases} 0 & t \leq 0 \\ -4\,\mathrm{e}^{-t}+2\,\mathrm{e}^{-2t}+2 & t \leq 1 \\ -4\,\mathrm{e}^{-t}+2\,\mathrm{e}^{-2t}-6\,\mathrm{e}^{-2t+2}+8\,\mathrm{e}^{-t+1} & 1 < t \end{cases}$$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -2c_1 - c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$
  - Substitute constant values into general solution and simplify

    $$y = \begin{cases} 0 & t \leq 0 \\ -3+4\,\mathrm{e}^{-t}-\mathrm{e}^{-2t}+2t & t \leq 1 \\ 4+4\,\mathrm{e}^{-t}-\mathrm{e}^{-2t}+3\,\mathrm{e}^{-2t+2}-8\,\mathrm{e}^{-t+1} & 1 < t \end{cases}$$

- Solution to the IVP

$$
y = \begin{cases} 0 & t \leq 0 \\ -3+4\,\mathrm{e}^{-t}-\mathrm{e}^{-2t}+2t & t \leq 1 \\ 4+4\,\mathrm{e}^{-t}-\mathrm{e}^{-2t}+3\,\mathrm{e}^{-2t+2}-8\,\mathrm{e}^{-t+1} & 1 < t \end{cases}
$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 1.156 (sec). Leaf size: 71

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=piecewise(0<t and t<1,4*t,t>1,8),y(0) = 0, D(y)
```

$$
y(t) = \begin{cases} 2t-\mathrm{e}^{-2t}-3+4\,\mathrm{e}^{-t} & t < 1 \\ -\mathrm{e}^{-2}+1+4\,\mathrm{e}^{-1} & t = 1 \\ 3\,\mathrm{e}^{-2t+2}-8\,\mathrm{e}^{1-t}-\mathrm{e}^{-2t}+4+4\,\mathrm{e}^{-t} & 1 < t \end{cases}
$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 70

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==Piecewise[{{4*t,0<t<1},{8,t>1}}],{y[0]==0,y'[0]==0}},y[t],t,I
```

$$
y(t) \to \left\{ \begin{array}{cc} 0 & t \leq 0 \\ 2t-e^{-2t}+4e^{-t}-3 & 0 < t \leq 1 \\ e^{-2t}\left(-1+3e^2+4e^t+4e^{2t}-8e^{t+1}\right) & \text{True} \end{array} \right.
$$

7.6 problem 23

7.6.1 Existence and uniqueness analysis 785
7.6.2 Maple step by step solution 788

Internal problem ID [5699]
Internal file name [`OUTPUT/4947_Sunday_June_05_2022_03_11_22_PM_35045146/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.3, page 224
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + y' - 2y = \begin{cases} 3\sin(t) - \cos(t) & 0 < t < 2\pi \\ 3\sin(2t) - \cos(2t) & 2\pi < t \end{cases}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

7.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$
\begin{aligned}
p(t) &= 1 \\
q(t) &= -2 \\
F &= \begin{cases} 0 & t \le 0 \\ 3\sin(t) - \cos(t) & t < 2\pi \\ 0 & t = 2\pi \\ 3\sin(2t) - \cos(2t) & 2\pi < t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y'' + y' - 2y = \begin{cases} 0 & t \le 0 \\ 3\sin(t) - \cos(t) & t < 2\pi \\ 0 & t = 2\pi \\ 3\sin(2t) - \cos(2t) & 2\pi < t \end{cases}
$$

The domain of $p(t) = 1$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -2$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t \le 0 \\ 3\sin(t) - \cos(t) & t < 2\pi \\ 0 & t = 2\pi \\ 3\sin(2t) - \cos(2t) & 2\pi < t \end{cases}$
is

$$
\{0 \le t \le 2\pi, 2\pi \le t \le \infty, -\infty \le t \le 0\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + sY(s) - y(0) - 2Y(s) = \frac{3 - s + \frac{3\,\mathrm{e}^{-2\pi s}(s+2)(s-1)}{s^2+4}}{s^2+1} \tag{1}$$

But the initial conditions are

$$y(0) = 1$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - s + sY(s) - 2Y(s) = \frac{3 - s + \frac{3\,\mathrm{e}^{-2\pi s}(s+2)(s-1)}{s^2+4}}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^4 - s^3 + 3\,\mathrm{e}^{-2\pi s}s + 6s^2 - 3\,\mathrm{e}^{-2\pi s} - 4s + 8}{(s-1)\,(s^2+1)\,(s^2+4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{s^4 - s^3 + 3\,\mathrm{e}^{-2\pi s}s + 6s^2 - 3\,\mathrm{e}^{-2\pi s} - 4s + 8}{(s-1)\,(s^2+1)\,(s^2+4)}\right) \\
&= -\sin(t) + \mathrm{e}^t + \frac{\mathrm{Heaviside}(t-2\pi)\,(2\sin(t) - \sin(2t))}{2}
\end{aligned}$$

Hence the final solution is

$$y = -\sin(t) + \mathrm{e}^t + \frac{\mathrm{Heaviside}(t-2\pi)\,(2\sin(t) - \sin(2t))}{2}$$

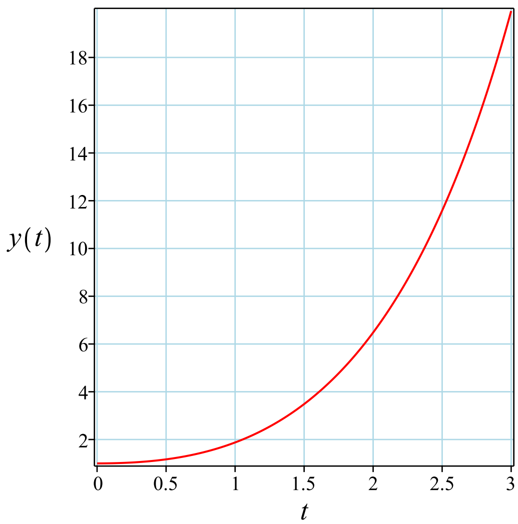
Simplifying the solution gives

$$y = -\sin(t)\,(\cos(t) - 1)\,\mathrm{Heaviside}(t-2\pi) + \mathrm{e}^t - \sin(t)$$

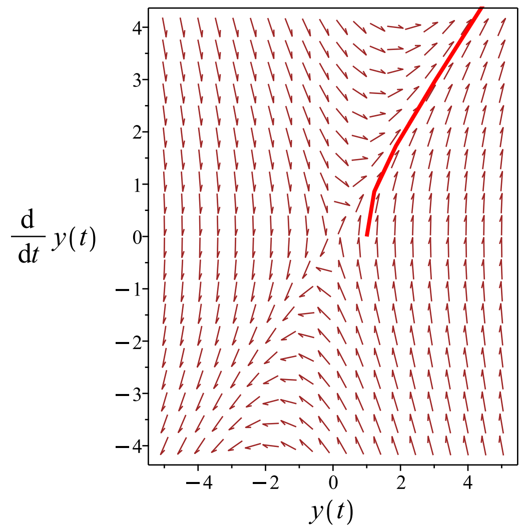
Summary

The solution(s) found are the following

$$y = -\sin(t)\,(\cos(t) - 1)\,\mathrm{Heaviside}(t-2\pi) + \mathrm{e}^t - \sin(t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\sin(t)(\cos(t) - 1)\,\text{Heaviside}(t - 2\pi) + e^{t} - \sin(t)$$

Verified OK.

7.6.2 Maple step by step solution

Let's solve

$$\left[y'' + y' - 2y = \begin{cases} 0 & t \le 0 \\ 3\sin(t) - \cos(t) & t < 2\pi \\ 0 & t = 2\pi \\ 3\sin(2t) - \cos(2t) & 2\pi < t \end{cases}, y(0) = 1, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$
- Factor the characteristic polynomial

  $(r + 2)(r - 1) = 0$
- Roots of the characteristic polynomial

$r=(-2,1)$

- 1st solution of the homogeneous ODE

  $y_1(t)=e^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t)=e^{t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1e^{-2t}+c_2e^{t}+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t)=-y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\begin{cases}0 & t \\ 3\sin(t)-\cos(t) & t < \\ 0 & t = \\ 3\sin(2t)-\cos(2t) & 2\pi\end{cases}\right.$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(t),y_2(t))=\begin{bmatrix} e^{-2t} & e^{t} \\ -2e^{-2t} & e^{t}\end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t),y_2(t))=3e^{-t}$

- Substitute functions into equation for $y_p(t)$

$$y_p(t)=-\frac{\left(-e^{3t}\left(\int\left(\begin{cases}0 & t\leq 0\\ 3\sin(t)-\cos(t) & t<2\pi\\ 0 & t=2\pi\\ 3\sin(2t)-\cos(2t) & 2\pi<t\end{cases}\right)e^{-t}dt\right)+\int\left(\begin{cases}0 & t\leq 0\\ 3\sin(t)-\cos(t) & t<2\pi\\ 0 & t=2\pi\\ 3\sin(2t)-\cos(2t) & 2\pi<t\end{cases}\right)e^{2t}\right.}{3}$$

- Compute integrals

$$y_p(t) = \frac{e^{-2t}\left(\begin{cases} 0 & t \le 0 \\ e^{3t} - 3\sin(t)\,e^{2t} - 1 & t \le 2\pi \\ -\frac{3\sin(2t)e^{2t}}{2} + e^{3t} - 1 & 2\pi < t \end{cases}\right)}{3}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-2t} + c_2 e^{t} + \frac{e^{-2t}\left(\begin{cases} 0 & t \le 0 \\ e^{3t} - 3\sin(t)\,e^{2t} - 1 & t \le 2\pi \\ -\frac{3\sin(2t)e^{2t}}{2} + e^{3t} - 1 & 2\pi < t \end{cases}\right)}{3}$$

□ Check validity of solution $y = c_1 e^{-2t} + c_2 e^{t} + \frac{e^{-2t}\left(\begin{cases} 0 & t \le 0 \\ e^{3t} - 3\sin(t)\,e^{2t} - 1 & t \le 2\pi \\ -\frac{3\sin(2t)e^{2t}}{2} + e^{3t} - 1 & 2\pi < t \end{cases}\right)}{3}$

- ○ Use initial condition $y(0) = 1$

  $1 = c_1 + c_2$

- ○ Compute derivative of the solution

$$y' = -2c_1 e^{-2t} + c_2 e^{t} - \frac{2\,e^{-2t}\left(\begin{cases} 0 & t \le 0 \\ e^{3t} - 3\sin(t)\,e^{2t} - 1 & t \le 2\pi \\ -\frac{3\sin(2t)e^{2t}}{2} + e^{3t} - 1 & 2\pi < t \end{cases}\right)}{3} + \frac{e^{-2t}\left(\begin{cases} 0 \\ 3\,e^{3t} - 3\,e^{2t}\cos(t) - \\ -3\cos(2t)\,e^{2t} - 3\sin(\end{cases}\right.}{3}$$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -2c_1 + c_2$

- ○ Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{3}, c_2 = \frac{2}{3}\right\}$

- ○ Substitute constant values into general solution and simplify

$$y = \begin{cases} \frac{(2\,e^{3t}+1)e^{-2t}}{3} & t \le 0 \\ -\sin(t) + e^{t} & t \le 2\pi \\ e^{t} - \frac{\sin(2t)}{2} & 2\pi < t \end{cases}$$

- Solution to the IVP

$$y = \begin{cases} \frac{(2\,\mathrm{e}^{3t}+1)\,\mathrm{e}^{-2t}}{3} & t \le 0 \\ -\sin(t) + \mathrm{e}^{t} & t \le 2\pi \\ \mathrm{e}^{t} - \frac{\sin(2t)}{2} & 2\pi < t \end{cases}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.344 (sec). Leaf size: 31

```
dsolve([diff(y(t),t$2)+diff(y(t),t)-2*y(t)=piecewise(0<t and t<2*Pi,3*sin(t)-cos(t),t>2*Pi,3
```

$$y(t) = \mathrm{e}^{t} - \left(\begin{cases} \sin(t) & t < 2\pi \\ \frac{\sin(2t)}{2} & 2\pi \le t \end{cases}\right)$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 55

```
DSolve[{y''[t]+y'[t]-2*y[t]==Piecewise[{{3*Sin[t]-Cos[t],0<t<2*Pi},{3*Sin[2*t]-Cos[2*t],t>2*
```

$$y(t) \to \left\{ \begin{array}{cc} \frac{e^{-2t}}{3} + \frac{2e^{t}}{3} & t \le 0 \\ e^{t} - \sin(t) & 0 < t \le 2\pi \\ e^{t} - \cos(t)\sin(t) & \text{True} \end{array} \right.$$

7.7 problem 24

7.7.1 Existence and uniqueness analysis 792
7.7.2 Maple step by step solution 795

Internal problem ID [5700]
Internal file name [`OUTPUT/4948_Sunday_June_05_2022_03_11_28_PM_98332360/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.3, page 224
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t \end{cases}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

7.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 3$$
$$q(t) = 2$$
$$F = \begin{cases} 0 & t \leq 0 \\ 1 & t < 1 \\ 0 & 1 \leq t \end{cases}$$

Hence the ode is

$$
y'' + 3y' + 2y = \begin{cases} 0 & t \leq 0 \\ 1 & t < 1 \\ 0 & 1 \leq t \end{cases}
$$

The domain of $p(t) = 3$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t \leq 0 \\ 1 & t < 1 \\ 0 & 1 \leq t \end{cases}$ is

$$
\{0 \leq t \leq 1, 1 \leq t \leq \infty, -\infty \leq t \leq 0\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$
s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) + 2Y(s) = \frac{1 - \mathrm{e}^{-s}}{s} \tag{1}
$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= 0
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1 - \mathrm{e}^{-s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-1 + \mathrm{e}^{-s}}{s\,(s^2 + 3s + 2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{-1 + \mathrm{e}^{-s}}{s\,(s^2 + 3s + 2)}\right) \\
&= \frac{\mathrm{Heaviside}\,(-t+1)}{2} - \mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{2} + \frac{(-\mathrm{e}^{-2t+2} + 2\,\mathrm{e}^{-t+1})\,\mathrm{Heaviside}\,(t-1)}{2}
\end{aligned}$$

Hence the final solution is

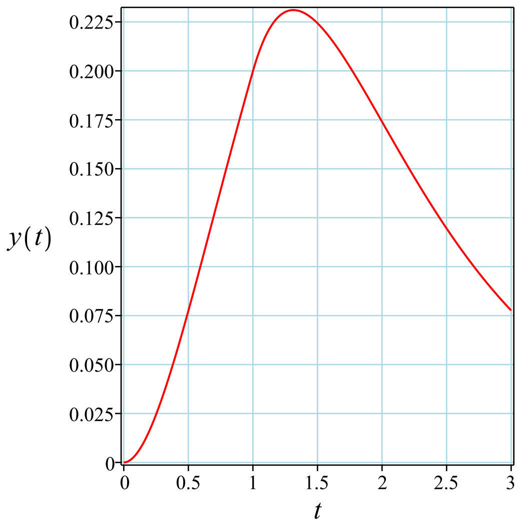
$$y = \frac{\mathrm{Heaviside}\,(-t+1)}{2} - \mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{2} + \frac{(-\mathrm{e}^{-2t+2} + 2\,\mathrm{e}^{-t+1})\,\mathrm{Heaviside}\,(t-1)}{2}$$

Simplifying the solution gives

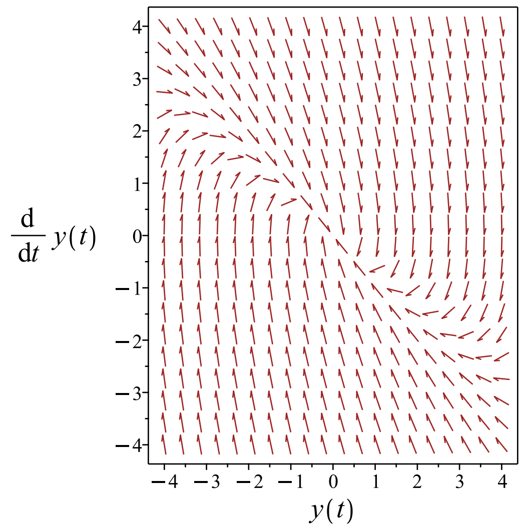
$$y = \frac{1}{2} - \frac{\mathrm{Heaviside}\,(t-1)}{2} - \mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{2} - \frac{\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2}}{2} + \mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-t+1}$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned}
y = {} & \frac{1}{2} - \frac{\mathrm{Heaviside}\,(t-1)}{2} - \mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{2} \\
& - \frac{\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2}}{2} + \mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-t+1}
\end{aligned} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{2} - \frac{\text{Heaviside}\,(t-1)}{2} - e^{-t} + \frac{e^{-2t}}{2} - \frac{\text{Heaviside}\,(t-1)\,e^{-2t+2}}{2} + \text{Heaviside}\,(t-1)\,e^{-t+1}$$

Verified OK.

7.7.2 Maple step by step solution

Let's solve

$$\left[y'' + 3y' + 2y = \begin{cases} 0 & t \leq 0 \\ 1 & t < 1 \\ 0 & 1 \leq t \end{cases}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$
- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$
- Roots of the characteristic polynomial

$r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t < 1 \\ 0 & 1 \leq t \end{cases}\right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{-t} \\ -2\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{-3t}$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\mathrm{e}^{-2t}\left(\int \left(\begin{cases} 0 & t \leq 0 \\ \mathrm{e}^{2t} & t < 1 \\ 0 & 1 \leq t \end{cases}\right) dt\right) + \mathrm{e}^{-t}\left(\int \left(\begin{cases} 0 & t \leq 0 \\ \mathrm{e}^{t} & t < 1 \\ 0 & 1 \leq t \end{cases}\right) dt\right)$$

- ○ Compute integrals

$$y_p(t) = -\frac{\left(\begin{cases} 0 & t \leq 0 \\ 2\,\mathrm{e}^{-t} - \mathrm{e}^{-2t} - 1 & t \leq 1 \\ 2\,\mathrm{e}^{-t} - \mathrm{e}^{-2t} + \mathrm{e}^{-2t+2} - 2\,\mathrm{e}^{-t+1} & 1 < t \end{cases}\right)}{2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-2t} + c_2 e^{-t} - \frac{\left(\begin{cases} 0 & t \le 0 \\ 2\,e^{-t} - e^{-2t} - 1 & t \le 1 \\ 2\,e^{-t} - e^{-2t} + e^{-2t+2} - 2\,e^{-t+1} & 1 < t \end{cases}\right)}{2}$$

☐ Check validity of solution $y = c_1 e^{-2t} + c_2 e^{-t} - \frac{\left(\begin{cases} 0 & t \le 0 \\ 2e^{-t} - e^{-2t} - 1 & t \le 1 \\ 2e^{-t} - e^{-2t} + e^{-2t+2} - 2e^{-t+1} & 1 < t \end{cases}\right)}{2}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

- Compute derivative of the solution

  $$y' = -2c_1 e^{-2t} - c_2 e^{-t} - \frac{\left(\begin{cases} 0 & t \le 0 \\ -2\,e^{-t} + 2\,e^{-2t} & t \le 1 \\ -2\,e^{-t} + 2\,e^{-2t} - 2\,e^{-2t+2} + 2\,e^{-t+1} & 1 < t \end{cases}\right)}{2}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -2c_1 - c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $$y = -\frac{\left(\begin{cases} 0 & t \le 0 \\ 2\,e^{-t} - e^{-2t} - 1 & t \le 1 \\ 2\,e^{-t} - e^{-2t} + e^{-2t+2} - 2\,e^{-t+1} & 1 < t \end{cases}\right)}{2}$$

- Solution to the IVP

$$y = -\frac{\left(\begin{cases} 0 & t \le 0 \\ 2\,e^{-t} - e^{-2t} - 1 & t \le 1 \\ 2\,e^{-t} - e^{-2t} + e^{-2t+2} - 2\,e^{-t+1} & 1 < t \end{cases}\right)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 1.062 (sec). Leaf size: 65

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=piecewise(0<t and t<1,1,t>1,0),y(0) = 0, D(y)(0
```

$$y(t) = \frac{\left(\begin{cases} 1-2\,\mathrm{e}^{-t}+\mathrm{e}^{-2t} & t<1 \\ -2\,\mathrm{e}^{-1}+\mathrm{e}^{-2}+2 & t=1 \\ 2\,\mathrm{e}^{1-t}-\mathrm{e}^{-2t+2}-2\,\mathrm{e}^{-t}+\mathrm{e}^{-2t} & 1<t \end{cases}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 57

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==Piecewise[{{1,0<t<1},{0,t>1}}],{y[0]==0,y'[0]==0}},y[t],t,Inc
```

$$y(t) \to \begin{cases} 0 & t \leq 0 \\ \frac{1}{2}e^{-2t}(-1+e^t)^2 & 0<t\leq 1 \\ \frac{1}{2}(-1+e)e^{-2t}(-1-e+2e^t) & \text{True} \end{cases}$$

7.8 problem 25

7.8.1 Existence and uniqueness analysis 799
7.8.2 Maple step by step solution 802

Internal problem ID [5701]
Internal file name [`OUTPUT/4949_Sunday_June_05_2022_03_11_32_PM_75529370/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.3, page 224
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \begin{cases} t & 0 < t < 1 \\ 0 & 1 < t \end{cases}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

7.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \begin{cases} 0 & t \leq 0 \\ t & t < 1 \\ 0 & 1 \leq t \end{cases} \end{aligned}$$

Hence the ode is

$$y'' + y = \begin{cases} 0 & t \leq 0 \\ t & t < 1 \\ 0 & 1 \leq t \end{cases}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t \leq 0 \\ t & t < 1 \\ 0 & 1 \leq t \end{cases}$ is

$$\{t < 1 \vee 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \frac{-(s+1)\,e^{-s} + 1}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + Y(s) = \frac{-(s+1)\,\mathrm{e}^{-s} + 1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{\mathrm{e}^{-s}s + \mathrm{e}^{-s} - 1}{s^2\,(s^2+1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{\mathrm{e}^{-s}s + \mathrm{e}^{-s} - 1}{s^2\,(s^2+1)}\right) \\
&= -\sin(t) + t - \text{Heaviside}\,(t-1)\left(2\sin\left(\frac{t}{2} - \frac{1}{2}\right)^2 - \sin(t-1) + t - 1\right)
\end{aligned}$$

Hence the final solution is

$$y = -\sin(t) + t - \text{Heaviside}\,(t-1)\left(2\sin\left(\frac{t}{2} - \frac{1}{2}\right)^2 - \sin(t-1) + t - 1\right)$$

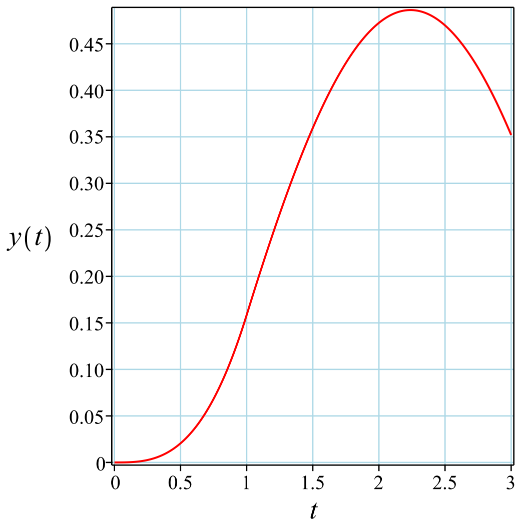
Simplifying the solution gives

$$y = -\sin(t) + t + (\cos(t-1) + \sin(t-1) - t)\,\text{Heaviside}\,(t-1)$$

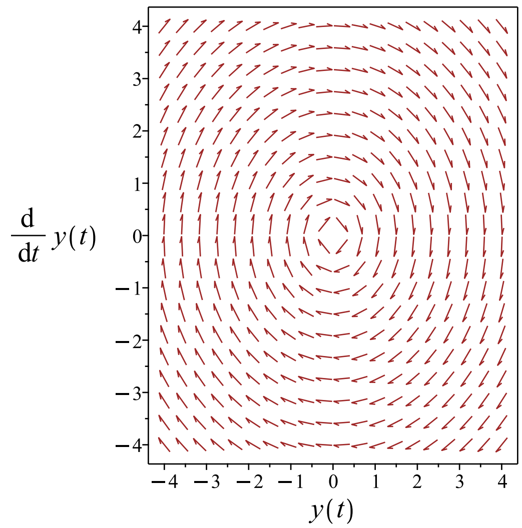
Summary

The solution(s) found are the following

$$y = -\sin(t) + t + (\cos(t-1) + \sin(t-1) - t)\,\text{Heaviside}\,(t-1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\sin(t) + t + (\cos(t-1) + \sin(t-1) - t)\,\text{Heaviside}(t-1)$$

Verified OK.

7.8.2 Maple step by step solution

Let's solve

$$\left[y'' + y = \begin{cases} 0 & t \leq 0 \\ t & t < 1 \\ 0 & 1 \leq t \end{cases}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \begin{cases} 0 & t \leq 0 \\ t & t < 1 \\ 0 & 1 \leq t \end{cases}\right]$$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 1$

- ○ Substitute functions into equation for $y_p(t)$

  $$y_p(t) = -\cos(t)\left(\int \left(\begin{cases} 0 & t \leq 0 \\ \sin(t)\, t & t < 1 \\ 0 & 1 \leq t \end{cases}\right) dt\right) + \sin(t)\left(\int \left(\begin{cases} 0 & t \leq 0 \\ \cos(t)\, t & t < 1 \\ 0 & 1 \leq t \end{cases}\right) dt\right)$$

- ○ Compute integrals

  $$y_p(t) = \begin{cases} 0 & t \leq 0 \\ -\sin(t) + t & t \leq 1 \\ (\cos(1) - \sin(1))\cos(t) + (\sin(1) + \cos(1) - 1)\sin(t) & 1 < t \end{cases}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(t) + c_2 \sin(t) + \left\{\begin{array}{cc} 0 & t \leq 0 \\ -\sin(t) + t & t \leq 1 \\ (\cos(1) - \sin(1))\cos(t) + (\sin(1) + \cos(1) - 1)\sin(t) & 1 < t \end{array}\right.$$

☐ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) + \left\{\begin{array}{c} 0 \\ -\sin(t) + t \\ (\cos(1) - \sin(1))\cos(t) + (\sin(1) + \text{co} \end{array}\right.$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $$y' = -c_1 \sin(t) + c_2 \cos(t) + \left\{\begin{array}{cc} 0 & t \leq 0 \\ -\cos(t) + 1 & t \leq 1 \\ -(\cos(1) - \sin(1))\sin(t) + (\sin(1) + \cos(1) - 1)\cos(t) & 1 < t \end{array}\right.$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $$y = \left\{\begin{array}{cc} 0 & t \leq 0 \\ -\sin(t) + t & t \leq 1 \\ (\cos(1) - \sin(1))\cos(t) + (\sin(1) + \cos(1) - 1)\sin(t) & 1 < t \end{array}\right.$$

- Solution to the IVP

  $$y = \left\{\begin{array}{cc} 0 & t \leq 0 \\ -\sin(t) + t & t \leq 1 \\ (\cos(1) - \sin(1))\cos(t) + (\sin(1) + \cos(1) - 1)\sin(t) & 1 < t \end{array}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.61 (sec). Leaf size: 37

```
dsolve([diff(y(t),t$2)+y(t)=piecewise(0<t and t<1,t,t>1,0),y(0) = 0, D(y)(0) = 0],y(t), sing
```

$$y(t) = -\sin(t) + \left(\begin{cases} t & t < 1 \\ \sin(t-1) + \cos(t-1) & 1 \leq t \end{cases}\right)$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 44

```
DSolve[{y''[t]+y[t]==Piecewise[{{t,0<t<1},{0,t>1}}],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingul
```

$$y(t) \to \begin{cases} t - \sin(t) & 0 < t \leq 1 \\ \cos(1-t) - \sin(1-t) - \sin(t) & t > 1 \end{cases}$$

7.9 problem 26

7.9.1 Existence and uniqueness analysis 806

Internal problem ID [5702]
Internal file name [`OUTPUT/4950_Sunday_June_05_2022_03_14_28_PM_35172071/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.3, page 224
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 2y' + 5y = \begin{cases} 10\sin(t) & 0 < t < 2\pi \\ 0 & 2\pi < t \end{cases}}$$

With initial conditions

$$[y(\pi) = 1, y'(\pi) = 2\,e^{-\pi} - 2]$$

7.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 5 \\ F &= \begin{cases} 0 & t \le 0 \\ 10\sin(t) & t < 2\pi \\ 0 & 2\pi \le t \end{cases} \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 5y = \begin{cases} 0 & t \le 0 \\ 10\sin(t) & t < 2\pi \\ 0 & 2\pi \le t \end{cases}$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = \pi$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = \pi$ is also inside this domain. The domain of $F = \begin{cases} 0 & t \le 0 \\ 10\sin(t) & t < 2\pi \\ 0 & 2\pi \le t \end{cases}$ is

$$\{0 \le t \le 2\pi, 2\pi \le t \le \infty, -\infty \le t \le 0\}$$

And the point $t_0 = \pi$ is also inside this domain. Hence solution exists and is unique. Since both initial conditions are not at zero, then let

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 5Y(s) = \frac{10 - 10\,\mathrm{e}^{-2\pi s}}{s^2 + 1} \tag{1}$$

But the initial conditions are

$$y(0) = c_1$$
$$y'(0) = c_2$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) - c_2 - sc_1 + 2sY(s) - 2c_1 + 5Y(s) = \frac{10 - 10\,\mathrm{e}^{-2\pi s}}{s^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-c_1 s^3 - 2c_1 s^2 - c_2 s^2 - sc_1 + 10\,\mathrm{e}^{-2\pi s} - 2c_1 - c_2 - 10}{(s^2+1)(s^2+2s+5)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-c_1 s^3 - 2c_1 s^2 - c_2 s^2 - sc_1 + 10\,\mathrm{e}^{-2\pi s} - 2c_1 - c_2 - 10}{(s^2+1)(s^2+2s+5)}\right) \\ &= \frac{(-2\cos(2t) + \sin(2t))\,\text{Heaviside}(t-2\pi)\,\mathrm{e}^{2\pi - t}}{2} + \frac{(2\cos(2t)(c_1+1) + \sin(2t)(-1+c_1+c_2))\,\mathrm{e}^{-t}}{2} + \end{aligned}$$

Since both initial conditions given are not at zero, then we need to setup two equations to solve for c_1, c_1. At $t = \pi$ the first equation becomes, using the above solution

$$1 = 1 + \frac{(2+2c_1)\,\mathrm{e}^{-\pi}}{2}$$

And taking derivative of the solution and evaluating at $t = \pi$ gives the second equation as

$$2\,\mathrm{e}^{-\pi} - 2 = -2 + \frac{(-2+2c_1+2c_2)\,\mathrm{e}^{-\pi}}{2} - \frac{(2+2c_1)\,\mathrm{e}^{-\pi}}{2}$$

Solving gives

$$c_1 = -1$$
$$c_2 = 4$$

Subtituting these in the solution obtained above gives

$$\begin{aligned} y &= \frac{(-2\cos(2t) + \sin(2t))\,\text{Heaviside}(t-2\pi)\,\mathrm{e}^{2\pi - t}}{2} + \sin(2t)\,\mathrm{e}^{-t} + (-\cos(t) + 2\sin(t))\,\text{Heaviside}(2\pi - \\ &= \left(-2\cos(t)^2 + \cos(t)\sin(t) + 1\right)\text{Heaviside}(t-2\pi)\,\mathrm{e}^{2\pi-t} + (\cos(t) - 2\sin(t))\,\text{Heaviside}(t-2\pi) + 2 \end{aligned}$$

Hence the final solution is

$$y = \left(-2\cos(t)^2 + \cos(t)\sin(t) + 1\right)\text{Heaviside}(t - 2\pi)\,e^{2\pi - t} + (\cos(t) - 2\sin(t))\,\text{Heaviside}(t - 2\pi) + 2\sin(t)\cos(t)\,e^{-t} - \cos(t) + 2\sin(t)$$

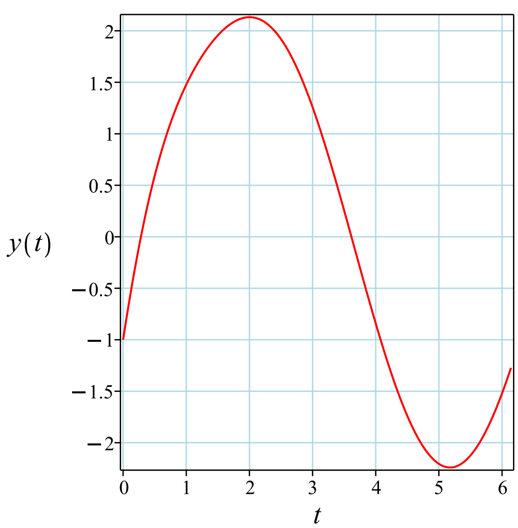
Simplifying the solution gives

$$y = \left(-2\cos(t)^2 + \cos(t)\sin(t) + 1\right)\text{Heaviside}(t - 2\pi)\,e^{2\pi - t} + (\cos(t) - 2\sin(t))\,\text{Heaviside}(t - 2\pi) + 2\sin(t)\cos(t)\,e^{-t} - \cos(t) + 2\sin(t)$$

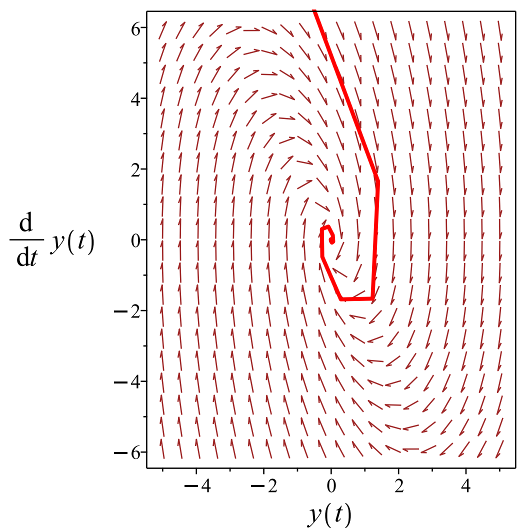
<u>Summary</u>

The solution(s) found are the following

$$y = \left(-2\cos(t)^2 + \cos(t)\sin(t) + 1\right)\text{Heaviside}(t - 2\pi)\,e^{2\pi - t} + (\cos(t) - 2\sin(t))\,\text{Heaviside}(t - 2\pi) + 2\sin(t)\cos(t)\,e^{-t} - \cos(t) + 2\sin(t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \left(-2\cos(t)^2 + \cos(t)\sin(t) + 1\right)\text{Heaviside}(t - 2\pi)\,e^{2\pi - t} + (\cos(t) - 2\sin(t))\,\text{Heaviside}(t - 2\pi) + 2\sin(t)\cos(t)\,e^{-t} - \cos(t) + 2\sin(t)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.343 (sec). Leaf size: 70

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+5*y(t)=piecewise(0<t and t<2*Pi,10*sin(t),t>2*Pi,0),y(
```

$$y(t) = \begin{cases} \sin(2t)\,\mathrm{e}^{-t} - \cos(t) + 2\sin(t) & t < 2\pi \\ -2 & t = 2\pi \\ \sin(2t)\,\mathrm{e}^{-t} + \frac{(-2\cos(2t)+\sin(2t))\,\mathrm{e}^{2\pi - t}}{2} & 2\pi < t \end{cases}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 94

```
DSolve[{y''[t]+2*y'[t]+5*y[t]==Piecewise[{{10*Sin[t],0<t<2*Pi},{0,t>2*Pi}}],{y[Pi]==1,y'[Pi]
```

$$y(t) \to \begin{cases} \frac{1}{2}e^{-t}(3\sin(2t) - 2\cos(2t)) & t \leq 0 \\ -\cos(t) + 2\sin(t) + e^{-t}\sin(2t) & 0 < t \leq 2\pi \\ \frac{1}{2}e^{-t}((2+e^{2\pi})\sin(2t) - 2e^{2\pi}\cos(2t)) & \text{True} \end{cases}$$

7.10 problem 27

7.10.1 Existence and uniqueness analysis 811
7.10.2 Maple step by step solution 815

Internal problem ID [5703]
Internal file name [`OUTPUT/4951_Sunday_June_05_2022_03_14_35_PM_63270777/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.3, page 224
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \left\{\begin{array}{cc} 8t^2 & 0 < t < 5 \\ 0 & 5 < t \end{array}\right.}$$

With initial conditions

$$[y(1) = 1 + \cos(2), y'(1) = 4 - 2\sin(2)]$$

7.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 0$$
$$q(t) = 4$$
$$F = \left\{\begin{array}{cc} 0 & t \le 0 \\ 8t^2 & t < 5 \\ 0 & 5 \le t \end{array}\right.$$

Hence the ode is

$$y'' + 4y = \begin{cases} 0 & t \leq 0 \\ 8t^2 & t < 5 \\ 0 & 5 \leq t \end{cases}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. The domain of $F = \begin{cases} 0 & t \leq 0 \\ 8t^2 & t < 5 \\ 0 & 5 \leq t \end{cases}$ is

$$\{t < 5 \vee 5 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

Since both initial conditions are not at zero, then let

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{16 - 8(25s^2 + 10s + 2)\,\mathrm{e}^{-5s}}{s^3} \tag{1}$$

But the initial conditions are

$$y(0) = c_1$$
$$y'(0) = c_2$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) - c_2 - s c_1 + 4Y(s) = \frac{16 - 8(25s^2 + 10s + 2)\,\mathrm{e}^{-5s}}{s^3}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-s^4 c_1 - c_2 s^3 + 200\,\mathrm{e}^{-5s} s^2 + 80\,\mathrm{e}^{-5s} s + 16\,\mathrm{e}^{-5s} - 16}{s^3\,(s^2 + 4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{-s^4 c_1 - c_2 s^3 + 200\,\mathrm{e}^{-5s} s^2 + 80\,\mathrm{e}^{-5s} s + 16\,\mathrm{e}^{-5s} - 16}{s^3\,(s^2 + 4)}\right) \\
&= -1 + \frac{c_2 \sin(2t)}{2} + 2t^2 + \cos(2t)\,(c_1 + 1) - \mathrm{Heaviside}\,(t - 5)\left(100 \sin(t - 5)^2 + 2t^2 + \cos(-10 + 2t)\right.
\end{aligned}$$

Since both initial conditions given are not at zero, then we need to setup two equations to solve for c_1, c_1. At $t = 1$ the first equation becomes, using the above solution

$$1 + \cos(2) = 1 + \frac{c_2 \sin(2)}{2} + \cos(2)\,(c_1 + 1)$$

And taking derivative of the solution and evaluating at $t = 1$ gives the second equation as

$$4 - 2\sin(2) = c_2 \cos(2) + 4 - 2\sin(2)\,(c_1 + 1)$$

Solving gives

$$c_1 = 0$$
$$c_2 = 0$$

Subtituting these in the solution obtained above gives

$$\begin{aligned}
y &= -1 + 2t^2 + \cos(2t) - \mathrm{Heaviside}\,(t - 5)\left(100 \sin(t - 5)^2 + 2t^2 + \cos(-10 + 2t) - 10 \sin(-10 + 2t) -\right. \\
&= -2\,\mathrm{Heaviside}\,(t - 5)\,t^2 + 2t^2 + \mathrm{Heaviside}\,(t - 5) - 1 + 49\,\mathrm{Heaviside}\,(t - 5) \cos(-10 + 2t) + 10\,\mathrm{Heavi}
\end{aligned}$$

Hence the final solution is

$$y = -2\,\text{Heaviside}(t-5)\,t^2 + 2t^2 + \text{Heaviside}(t-5) - 1 + 49\,\text{Heaviside}(t-5)\cos(-10+2t) + 10\,\text{Heaviside}(t-5)\sin(-10+2t) + \cos(2t)$$

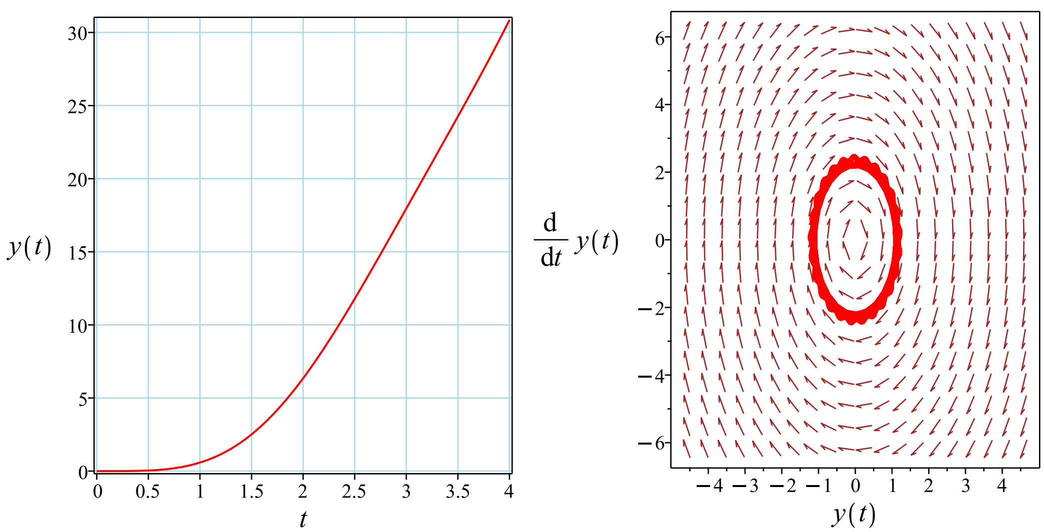
Simplifying the solution gives

$$y = -2\,\text{Heaviside}(t-5)\,t^2 + 2t^2 + \text{Heaviside}(t-5) - 1 + 49\,\text{Heaviside}(t-5)\cos(-10+2t) + 10\,\text{Heaviside}(t-5)\sin(-10+2t) + \cos(2t)$$

Summary

The solution(s) found are the following

$$y = -2\,\text{Heaviside}(t-5)\,t^2 + 2t^2 + \text{Heaviside}(t-5) - 1 + 49\,\text{Heaviside}(t-5)\cos(-10+2t) + 10\,\text{Heaviside}(t-5)\sin(-10+2t) + \cos(2t) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -2\,\text{Heaviside}(t-5)\,t^2 + 2t^2 + \text{Heaviside}(t-5) - 1 + 49\,\text{Heaviside}(t-5)\cos(-10+2t) + 10\,\text{Heaviside}(t-5)\sin(-10+2t) + \cos(2t)$$

Verified OK.

7.10.2 Maple step by step solution

Let's solve

$$\left[y''+4y=\left\{\begin{array}{cc}0 & t\le 0\\ 8t^2 & t<5\\ 0 & 5\le t\end{array}\right., y(1)=1+\cos(2), y'\Big|_{\{t=1\}}=4-2\sin(2)\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+4=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r=(-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(2t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(2t)+c_2\sin(2t)+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\left\{\begin{array}{cc}0 & t\le 0\\ 8t^2 & t<5\\ 0 & 5\le t\end{array}\right.\right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\cos(2t)\left(\int\left(\left(\begin{cases} 0 & t \leq 0 \\ 4\sin(2t)\,t^2 & t < 5 \\ 0 & 5 \leq t \end{cases}\right) dt\right) + \sin(2t)\left(\int\left(\left(\begin{cases} 0 & t \leq 0 \\ 4\cos(2t)\,t^2 & t < 5 \\ 0 & 5 \leq t \end{cases}\right) d$$

- Compute integrals

$$y_p(t) = \begin{cases} 0 & t \leq 0 \\ -1 + 2t^2 + \cos(2t) & t \leq 5 \\ (49\sin(10) + 10\cos(10))\sin(2t) + 49\cos(2t)\left(\cos(10) - \frac{10\sin(10)}{49} + \frac{1}{49}\right) & 5 < t \end{cases}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(2t) + c_2\sin(2t) + \begin{cases} 0 \\ -1 + 2t^2 + \cos(2t) \\ (49\sin(10) + 10\cos(10))\sin(2t) + 49\cos(2t)\left(\cos(10) - \frac{10\sin(1}{49}\right. \end{cases}$$

☐ Check validity of solution $y = c_1\cos(2t) + c_2\sin(2t) + \begin{cases} 0 \\ -1 + 2t^2 + \\ (49\sin(10) + 10\cos(10))\sin(2t) + 4 \end{cases}$

- Use initial condition $y(1) = 1 + \cos(2)$

  $1 + \cos(2) = c_1\cos(2) + c_2\sin(2) + 1 + \cos(2)$

- Compute derivative of the solution

$$y' = -2c_1\sin(2t) + 2c_2\cos(2t) + \begin{cases} 0 \\ 4t - 2\sin(2t) \\ 2(49\sin(10) + 10\cos(10))\cos(2t) - 98\sin(2t)\left(\cos(10) -\right. \end{cases}$$

- Use the initial condition $y'\Big|_{\{t=1\}} = 4 - 2\sin(2)$

$$4-2\sin(2) = -2c_1\sin(2)+2c_2\cos(2)+4-2\sin(2)$$

- Solve for c_1 and c_2

$$\{c_1 = 0, c_2 = 0\}$$

- Substitute constant values into general solution and simplify

$$y = \begin{cases} 0 & t \le 0 \\ -1+2t^2+\cos(2t) & t \le 5 \\ (49\sin(10)+10\cos(10))\sin(2t)+49\cos(2t)\left(\cos(10)-\frac{10\sin(10)}{49}+\frac{1}{49}\right) & 5 < t \end{cases}$$

- Solution to the IVP

$$y = \begin{cases} 0 & t \le 0 \\ -1+2t^2+\cos(2t) & t \le 5 \\ (49\sin(10)+10\cos(10))\sin(2t)+49\cos(2t)\left(\cos(10)-\frac{10\sin(10)}{49}+\frac{1}{49}\right) & 5 < t \end{cases}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.844 (sec). Leaf size: 87

```
dsolve([diff(y(t),t$2)+4*y(t)=piecewise(0<t and t<5,8*t^2,t>5,0),y(1) = 1+cos(2), D(y)(1) =
```

$$y(t) = \cos(2t) + \left(\begin{cases} 2t^2-1 & t < 5 \\ 10\sin(2t-10)+49\cos(2t-10) & 5 \le t \end{cases}\right)$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 51

```
DSolve[{y''[t]+4*y[t]==Piecewise[{{8*t^2,0<t<5},{0,t>5}}],{y[1]==1+Cos[2],y'[1]==4-2*Sin[2]}
```

$$y(t) \to \begin{cases} 2t^2 + \cos(2t) - 1 & 0 < t \leq 5 \\ 49\cos(2(t-5)) + \cos(2t) - 10\sin(10-2t) & t > 5 \end{cases}$$

8 Chapter 6. Laplace Transforms. Problem set 6.4, page 230

8.1 problem 3 820
8.2 problem 4 826
8.3 problem 5 832
8.4 problem 6 838
8.5 problem 7 844
8.6 problem 8 850
8.7 problem 9 856
8.8 problem 10 862
8.9 problem 11 869
8.10 problem 12 875

8.1 problem 3

8.1.1 Existence and uniqueness analysis 820
8.1.2 Maple step by step solution 823

Internal problem ID [5704]
Internal file name [`OUTPUT/4952_Sunday_June_05_2022_03_14_40_PM_37812013/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.4, page 230
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \delta(t - \pi)}$$

With initial conditions

$$[y(0) = 8, y'(0) = 0]$$

8.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 0$$
$$q(t) = 4$$
$$F = \delta(t - \pi)$$

Hence the ode is

$$y'' + 4y = \delta(t - \pi)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - \pi)$ is

$$\{t < \pi \lor \pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \mathrm{e}^{-\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 8\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 8s + 4Y(s) = \mathrm{e}^{-\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-\pi s} + 8s}{s^2 + 4}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-\pi s} + 8s}{s^2 + 4}\right) \\
&= \frac{\text{Heaviside}\,(t - \pi) \sin(2t)}{2} + 8\cos(2t)
\end{aligned}
$$

Hence the final solution is

$$
y = \frac{\text{Heaviside}\,(t - \pi) \sin(2t)}{2} + 8\cos(2t)
$$

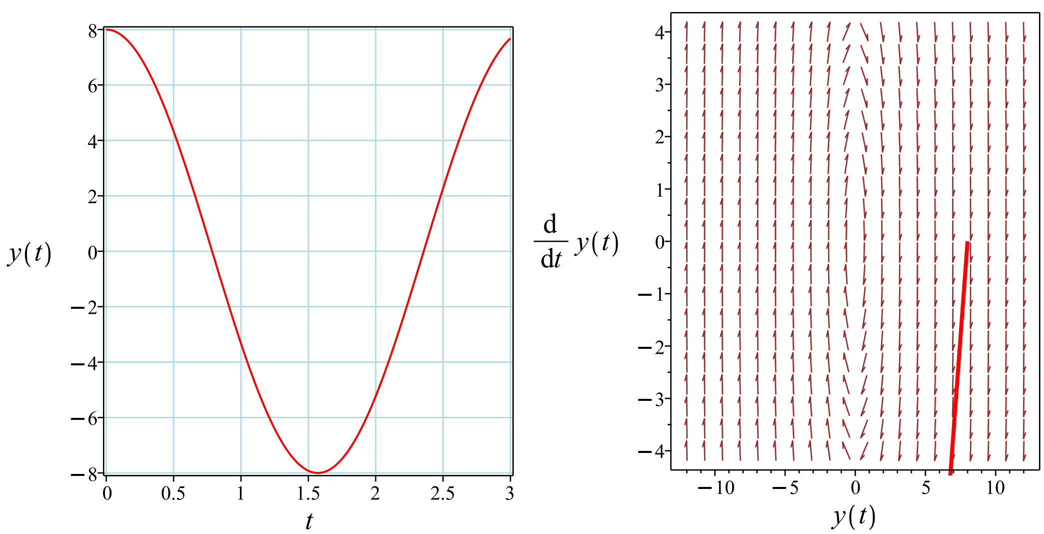
Simplifying the solution gives

$$
y = \frac{\text{Heaviside}\,(t - \pi) \sin(2t)}{2} + 8\cos(2t)
$$

Summary

The solution(s) found are the following

$$
y = \frac{\text{Heaviside}\,(t - \pi) \sin(2t)}{2} + 8\cos(2t) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = \frac{\text{Heaviside}\,(t - \pi) \sin(2t)}{2} + 8\cos(2t)
$$

Verified OK.

8.1.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y = Dirac(t - \pi), y(0) = 8, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = Dirac(t - \pi)\right]$$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = \frac{\sin(2t)(\int Dirac(t-\pi)dt)}{2}$

- Compute integrals

  $y_p(t) = \frac{Heaviside(t-\pi)\sin(2t)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{Heaviside(t-\pi)\sin(2t)}{2}$

- □ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{Heaviside(t-\pi)\sin(2t)}{2}$
  - Use initial condition $y(0) = 8$

    $8 = c_1$
  - Compute derivative of the solution

    $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \frac{Dirac(t-\pi)\sin(2t)}{2} + Heaviside(t-\pi)\cos(2t)$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = 2c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 8, c_2 = 0\}$
  - Substitute constant values into general solution and simplify

    $y = \frac{Heaviside(t-\pi)\sin(2t)}{2} + 8\cos(2t)$

- Solution to the IVP

  $y = \frac{Heaviside(t-\pi)\sin(2t)}{2} + 8\cos(2t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.797 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+4*y(t)=Dirac(t-Pi),y(0) = 8, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{\text{Heaviside}\,(t-\pi)\sin\,(2t)}{2} + 8\cos\,(2t)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 23

```
DSolve[{y''[t]+4*y[t]==DiracDelta[t-Pi],{y[0]==8,y'[0]==0}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to \theta(t-\pi)\sin(t)\cos(t) + 8\cos(2t)$$

8.2 problem 4

8.2.1 Existence and uniqueness analysis 826
8.2.2 Maple step by step solution 829

Internal problem ID [5705]
Internal file name [`OUTPUT/4953_Sunday_June_05_2022_03_14_42_PM_93725491/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.4, page 230
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 16y = 4\delta(t - 3\pi)}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0]$$

8.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 16 \\ F &= 4\delta(t - 3\pi) \end{aligned}$$

Hence the ode is

$$y'' + 16y = 4\delta(t - 3\pi)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 16$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 4\delta(t - 3\pi)$ is

$$\{t < 3\pi \vee 3\pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2 Y(s) - y'(0) - sy(0) + 16Y(s) = 4\,\mathrm{e}^{-3\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) - 2s + 16Y(s) = 4\,\mathrm{e}^{-3\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4\,\mathrm{e}^{-3\pi s} + 2s}{s^2 + 16}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{4\,\mathrm{e}^{-3\pi s}+2s}{s^{2}+16}\right) \\ &= \operatorname{Heaviside}\left(t-3\pi\right)\sin\left(4t\right)+2\cos\left(4t\right)\end{aligned}$$

Hence the final solution is

$$y = \operatorname{Heaviside}\left(t-3\pi\right)\sin\left(4t\right)+2\cos\left(4t\right)$$

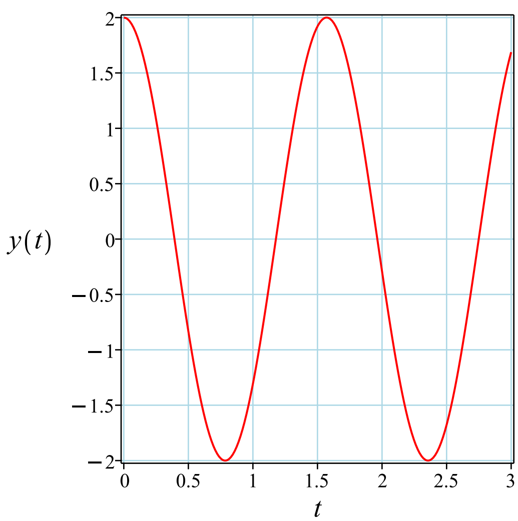
Simplifying the solution gives

$$y = \operatorname{Heaviside}\left(t-3\pi\right)\sin\left(4t\right)+2\cos\left(4t\right)$$

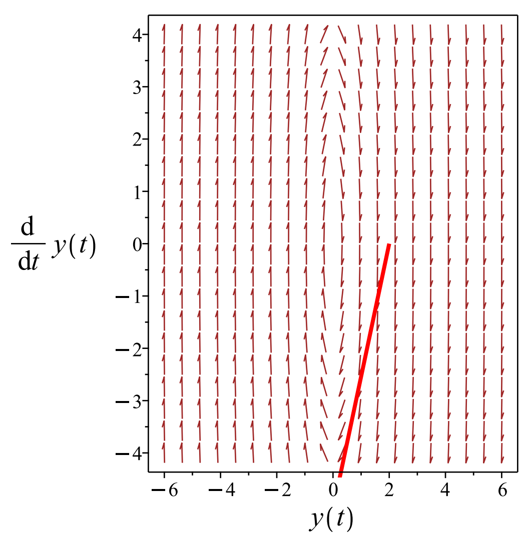
Summary

The solution(s) found are the following

$$y = \operatorname{Heaviside}\left(t-3\pi\right)\sin\left(4t\right)+2\cos\left(4t\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \operatorname{Heaviside}\left(t-3\pi\right)\sin\left(4t\right)+2\cos\left(4t\right)$$

Verified OK.

8.2.2 Maple step by step solution

Let's solve

$\left[y'' + 16y = 4Dirac(t - 3\pi), y(0) = 2, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 16 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-4\,\mathrm{I}, 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(4t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(4t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(4t) + c_2 \sin(4t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 4Dirac(t - 3\pi)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(4t) & \sin(4t) \\ -4\sin(4t) & 4\cos(4t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 4$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = \sin(4t)\left(\int \mathit{Dirac}(t-3\pi)\,dt\right)$

- Compute integrals

  $y_p(t) = \mathit{Heaviside}(t-3\pi)\sin(4t)$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(4t) + c_2\sin(4t) + \mathit{Heaviside}(t-3\pi)\sin(4t)$

□ Check validity of solution $y = c_1\cos(4t) + c_2\sin(4t) + \mathit{Heaviside}(t-3\pi)\sin(4t)$

- Use initial condition $y(0) = 2$

  $2 = c_1$

- Compute derivative of the solution

  $y' = -4c_1\sin(4t) + 4c_2\cos(4t) + \mathit{Dirac}(t-3\pi)\sin(4t) + 4\mathit{Heaviside}(t-3\pi)\cos(4t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 4c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = \mathit{Heaviside}(t-3\pi)\sin(4t) + 2\cos(4t)$

- Solution to the IVP

  $y = \mathit{Heaviside}(t-3\pi)\sin(4t) + 2\cos(4t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.0 (sec). Leaf size: 22

```
dsolve([diff(y(t),t$2)+16*y(t)=4*Dirac(t-3*Pi),y(0) = 2, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \text{Heaviside}\,(t - 3\pi)\sin(4t) + 2\cos(4t)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 23

```
DSolve[{y''[t]+16*y[t]==4*DiracDelta[t-3*Pi],{y[0]==2,y'[0]==0}},y[t],t,IncludeSingularSolut
```

$$y(t) \to \theta(t - 3\pi)\sin(4t) + 2\cos(4t)$$

8.3 problem 5

8.3.1 Existence and uniqueness analysis 832
8.3.2 Maple step by step solution 835

Internal problem ID [5706]
Internal file name [`OUTPUT/4954_Sunday_June_05_2022_03_14_45_PM_37094733/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.4, page 230
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \delta(t - \pi) - \delta(t - 2\pi)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

8.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= \delta(t - \pi) - \delta(t - 2\pi) \end{aligned}$$

Hence the ode is

$$y'' + y = \delta(t - \pi) - \delta(t - 2\pi)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - \pi) - \delta(t - 2\pi)$ is

$$\{\pi \leq t \leq 2\pi, 2\pi \leq t \leq \infty, -\infty \leq t \leq \pi\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + Y(s) = \mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s} + 1}{s^2 + 1}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-\pi s} - e^{-2\pi s} + 1}{s^2 + 1}\right) \\
&= \sin(t)\,(-\operatorname{Heaviside}(t-\pi) + \operatorname{Heaviside}(2\pi - t))
\end{aligned}
$$

Hence the final solution is

$$
y = \sin(t)\,(-\operatorname{Heaviside}(t-\pi) + \operatorname{Heaviside}(2\pi - t))
$$

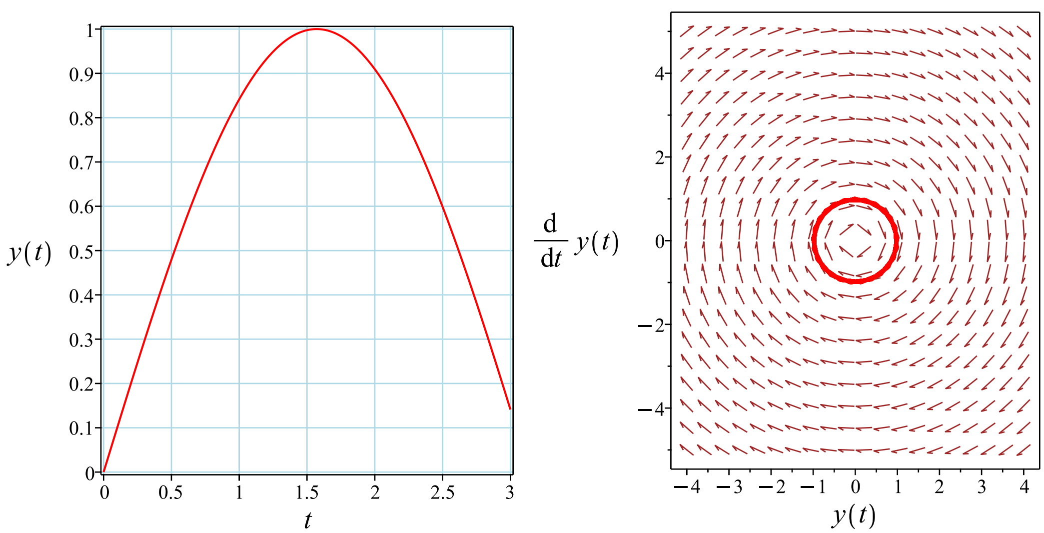
Simplifying the solution gives

$$
y = \sin(t)\,(1 - \operatorname{Heaviside}(t-\pi) - \operatorname{Heaviside}(t-2\pi))
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \sin(t)\,(1 - \operatorname{Heaviside}(t-\pi) - \operatorname{Heaviside}(t-2\pi)) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$
y = \sin(t)\,(1 - \operatorname{Heaviside}(t-\pi) - \operatorname{Heaviside}(t-2\pi))
$$

Verified OK.

8.3.2 Maple step by step solution

Let's solve

$\left[y'' + y = Dirac(t-\pi) - Dirac(t-2\pi), y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = Dirac(t-\pi) - Dirac(t-2\right.$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 1$

- ○ Substitute functions into equation for $y_p(t)$

$y_p(t) = \sin(t) \left(\int \left(-Dirac(t-\pi) - Dirac(t-2\pi)\right) dt\right)$

- Compute integrals

  $y_p(t) = \sin(t) \left(-Heaviside(t-\pi) - Heaviside(t-2\pi)\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + \sin(t) \left(-Heaviside(t-\pi) - Heaviside(t-2\pi)\right)$

□ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) + \sin(t) \left(-Heaviside(t-\pi) - Heaviside(t-\right.$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = -c_1 \sin(t) + c_2 \cos(t) + \cos(t) \left(-Heaviside(t-\pi) - Heaviside(t-2\pi)\right) + \sin(t) \left(-Dirac(\right.$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = \sin(t) \left(1 - Heaviside(t-\pi) - Heaviside(t-2\pi)\right)$

- Solution to the IVP

  $y = \sin(t) \left(1 - Heaviside(t-\pi) - Heaviside(t-2\pi)\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.844 (sec). Leaf size: 24

```
dsolve([diff(y(t),t$2)+y(t)=Dirac(t-Pi)-Dirac(t-2*Pi),y(0) = 0, D(y)(0) = 1],y(t), singsol=a
```

$$y(t) = \sin(t)(1 - \text{Heaviside}(t - 2\pi) - \text{Heaviside}(t - \pi))$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 23

```
DSolve[{y''[t]+y[t]==DiracDelta[t-Pi]-DiracDelta[t-2*Pi],{y[0]==0,y'[0]==1}},y[t],t,IncludeS
```

$$y(t) \to -((\theta(t - 2\pi) + \theta(t - \pi) - 1)\sin(t))$$

8.4 problem 6

8.4.1 Existence and uniqueness analysis 838
8.4.2 Maple step by step solution 841

Internal problem ID [5707]
Internal file name [`OUTPUT/4955_Sunday_June_05_2022_03_14_47_PM_41430436/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.4, page 230
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 5y = \delta(t-1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 3]$$

8.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 5 \\ F &= \delta(t-1) \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 5y = \delta(t-1)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t-1)$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 5Y(s) = \mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 3\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 + 4sY(s) + 5Y(s) = \mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-s} + 3}{s^2 + 4s + 5}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-s} + 3}{s^2 + 4s + 5}\right) \\ &= \text{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2}\sin\,(t-1) + 3\,\mathrm{e}^{-2t}\sin\,(t)\end{aligned}$$

Hence the final solution is

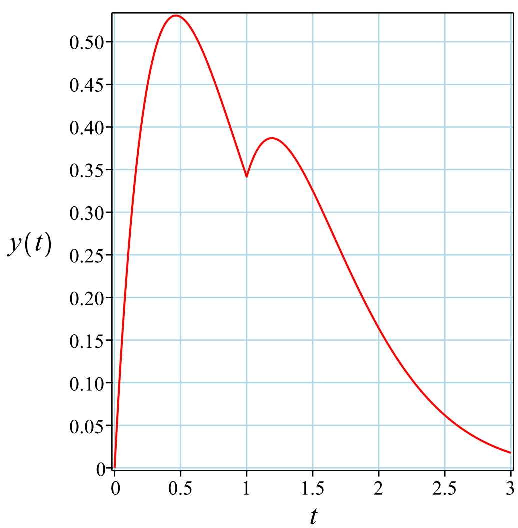
$$y = \text{Heaviside}\left(t-1\right) \mathrm{e}^{-2t+2} \sin\left(t-1\right)+3\,\mathrm{e}^{-2t} \sin\left(t\right)$$

Simplifying the solution gives

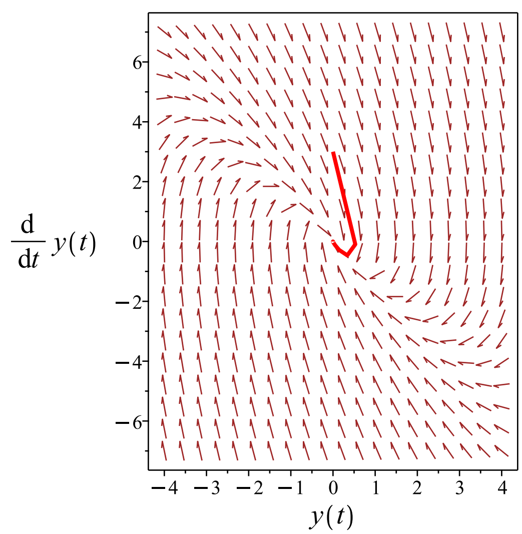
$$y = \text{Heaviside}\left(t-1\right) \mathrm{e}^{-2t+2} \sin\left(t-1\right)+3\,\mathrm{e}^{-2t} \sin\left(t\right)$$

Summary
The solution(s) found are the following

$$y = \text{Heaviside}\left(t-1\right) \mathrm{e}^{-2t+2} \sin\left(t-1\right)+3\,\mathrm{e}^{-2t} \sin\left(t\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \text{Heaviside}\left(t-1\right) \mathrm{e}^{-2t+2} \sin\left(t-1\right)+3\,\mathrm{e}^{-2t} \sin\left(t\right)$$

Verified OK.

8.4.2 Maple step by step solution

Let's solve

$\left[y'' + 4y' + 5y = Dirac(t-1), y(0) = 0, y'\Big|_{\{t=0\}} = 3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - \mathrm{I}, -2 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t} \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t} \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(t)\,\mathrm{e}^{-2t} c_1 + \sin(t)\,\mathrm{e}^{-2t} c_2 + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = Dirac(t-1)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t}\cos(t) & \mathrm{e}^{-2t}\sin(t) \\ -2\,\mathrm{e}^{-2t}\cos(t) - \mathrm{e}^{-2t}\sin(t) & -2\,\mathrm{e}^{-2t}\sin(t) + \mathrm{e}^{-2t}\cos(t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-4t}$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = (\sin(t)\cos(1) - \cos(t)\sin(1)) \left(\int \mathit{Dirac}(t-1)\,dt\right) e^{-2t+2}$

- Compute integrals

  $y_p(t) = \mathit{Heaviside}(t-1)\, e^{-2t+2}(\sin(t)\cos(1) - \cos(t)\sin(1))$

- Substitute particular solution into general solution to ODE

  $y = \cos(t)\, e^{-2t} c_1 + \sin(t)\, e^{-2t} c_2 + \mathit{Heaviside}(t-1)\, e^{-2t+2}(\sin(t)\cos(1) - \cos(t)\sin(1))$

□ Check validity of solution $y = \cos(t)\, e^{-2t} c_1 + \sin(t)\, e^{-2t} c_2 + \mathit{Heaviside}(t-1)\, e^{-2t+2}(\sin(t)\cos($

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = -\sin(t)\, e^{-2t} c_1 - 2\cos(t)\, e^{-2t} c_1 + \cos(t)\, e^{-2t} c_2 - 2\sin(t)\, e^{-2t} c_2 + \mathit{Dirac}(t-1)\, e^{-2t+2}(\sin(t$

- Use the initial condition $y'\Big|_{\{t=0\}} = 3$

  $3 = -2c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 3\}$

- Substitute constant values into general solution and simplify

  $y = \mathit{Heaviside}(t-1)\, e^{-2t+2}(\sin(t)\cos(1) - \cos(t)\sin(1)) + 3\, e^{-2t}\sin(t)$

- Solution to the IVP

  $y = \mathit{Heaviside}(t-1)\, e^{-2t+2}(\sin(t)\cos(1) - \cos(t)\sin(1)) + 3\, e^{-2t}\sin(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.859 (sec). Leaf size: 28

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+5*y(t)=Dirac(t-1),y(0) = 0, D(y)(0) = 3],y(t), singsol
```

$$y(t) = \text{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2}\sin\,(t-1) + 3\,\mathrm{e}^{-2t}\sin\,(t)$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 31

```
DSolve[{y''[t]+4*y'[t]+5*y[t]==DiracDelta[t-1],{y[0]==0,y'[0]==3}},y[t],t,IncludeSingularSol
```

$$y(t) \to e^{-2t}\left(3\sin(t) - e^2\theta(t-1)\sin(1-t)\right)$$

8.5 problem 7

8.5.1 Existence and uniqueness analysis 844
8.5.2 Maple step by step solution 847

Internal problem ID [5708]
Internal file name [`OUTPUT/4956_Sunday_June_05_2022_03_14_51_PM_21833028/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.4, page 230
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4y'' + 24y' + 37y = 17\,\mathrm{e}^{-t} + \delta\left(t - \frac{1}{2}\right)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

8.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= \frac{37}{4} \\ F &= \frac{17\,\mathrm{e}^{-t}}{4} + \frac{\delta\left(t - \frac{1}{2}\right)}{4} \end{aligned}$$

Hence the ode is

$$y'' + 6y' + \frac{37y}{4} = \frac{17\,\mathrm{e}^{-t}}{4} + \frac{\delta\left(t - \frac{1}{2}\right)}{4}$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{37}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{17\,\mathrm{e}^{-t}}{4} + \frac{\delta\left(t - \frac{1}{2}\right)}{4}$ is

$$\left\{t < \frac{1}{2} \vee \frac{1}{2} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$4s^2Y(s) - 4y'(0) - 4sy(0) + 24sY(s) - 24y(0) + 37Y(s) = \frac{17}{s+1} + \mathrm{e}^{-\frac{s}{2}} \tag{1}$$

But the initial conditions are

$$y(0) = 1$$
$$y'(0) = 1$$

Substituting these initial conditions in above in Eq (1) gives

$$4s^2Y(s) - 28 - 4s + 24sY(s) + 37Y(s) = \frac{17}{s+1} + \mathrm{e}^{-\frac{s}{2}}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-\frac{s}{2}} s + 4s^2 + e^{-\frac{s}{2}} + 32s + 45}{(s+1)(4s^2 + 24s + 37)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-\frac{s}{2}} s + 4s^2 + e^{-\frac{s}{2}} + 32s + 45}{(s+1)(4s^2 + 24s + 37)}\right) \\ &= \frac{\text{Heaviside}\left(t - \frac{1}{2}\right) e^{-3t + \frac{3}{2}} \sin\left(\frac{t}{2} - \frac{1}{4}\right)}{2} + 4\,e^{-3t} \sin\left(\frac{t}{2}\right) + e^{-t} \end{aligned}$$

Hence the final solution is

$$y = \frac{\text{Heaviside}\left(t - \frac{1}{2}\right) e^{-3t + \frac{3}{2}} \sin\left(\frac{t}{2} - \frac{1}{4}\right)}{2} + 4\,e^{-3t} \sin\left(\frac{t}{2}\right) + e^{-t}$$

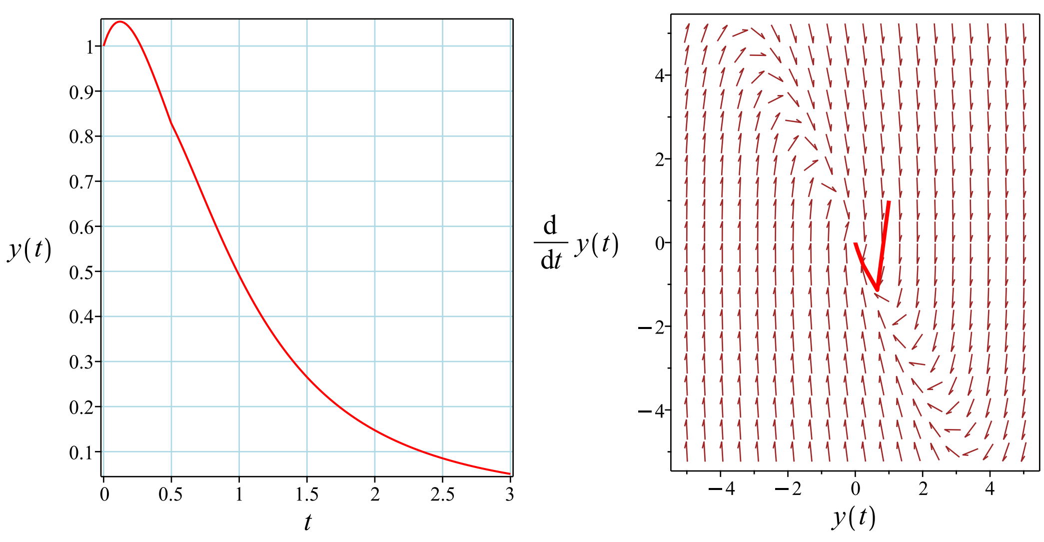
Simplifying the solution gives

$$y = \frac{\text{Heaviside}\left(t - \frac{1}{2}\right) e^{-3t + \frac{3}{2}} \sin\left(\frac{t}{2} - \frac{1}{4}\right)}{2} + 4\,e^{-3t} \sin\left(\frac{t}{2}\right) + e^{-t}$$

Summary

The solution(s) found are the following

$$y = \frac{\text{Heaviside}\left(t - \frac{1}{2}\right) e^{-3t + \frac{3}{2}} \sin\left(\frac{t}{2} - \frac{1}{4}\right)}{2} + 4\,e^{-3t} \sin\left(\frac{t}{2}\right) + e^{-t} \qquad (1)$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\text{Heaviside}\left(t-\frac{1}{2}\right) e^{-3t+\frac{3}{2}} \sin\left(\frac{t}{2}-\frac{1}{4}\right)}{2} + 4\,e^{-3t} \sin\left(\frac{t}{2}\right) + e^{-t}$$

Verified OK.

8.5.2 Maple step by step solution

Let's solve

$$\left[4y'' + 24y' + 37y = 17\,e^{-t} + \mathit{Dirac}\left(t-\frac{1}{2}\right), y(0) = 1, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -6y' - \frac{37y}{4} + \frac{17\,e^{-t}}{4} + \frac{\mathit{Dirac}\left(t-\frac{1}{2}\right)}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 6y' + \frac{37y}{4} = \frac{17\,e^{-t}}{4} + \frac{\mathit{Dirac}\left(t-\frac{1}{2}\right)}{4}$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + \frac{37}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-1})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-3-\frac{I}{2}, -3+\frac{I}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-3t} \cos\left(\frac{t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-3t} \sin\left(\frac{t}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3t} \cos\left(\frac{t}{2}\right) + c_2 e^{-3t} \sin\left(\frac{t}{2}\right) + y_p(t)$

- ☐ Find a particular solution $y_p(t)$ of the ODE
  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \frac{17\,\mathrm{e}^{-t}}{4} + \frac{Dirac\left(t-\frac{1}{2}\right)}{4}\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t}\cos\left(\frac{t}{2}\right) & \mathrm{e}^{-3t}\sin\left(\frac{t}{2}\right) \\ -3\,\mathrm{e}^{-3t}\cos\left(\frac{t}{2}\right) - \frac{\mathrm{e}^{-3t}\sin\left(\frac{t}{2}\right)}{2} & -3\,\mathrm{e}^{-3t}\sin\left(\frac{t}{2}\right) + \frac{\mathrm{e}^{-3t}\cos\left(\frac{t}{2}\right)}{2} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \frac{\mathrm{e}^{-6t}}{2}$
  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{\mathrm{e}^{-3t}\left(\cos\left(\frac{t}{2}\right)\left(\int\left(\sin\left(\frac{1}{4}\right)\mathrm{e}^{\frac{3}{2}} Dirac\left(t-\frac{1}{2}\right) + 17\sin\left(\frac{t}{2}\right)\mathrm{e}^{2t}\right)dt\right) - \sin\left(\frac{t}{2}\right)\left(\int\left(\cos\left(\frac{1}{4}\right)\mathrm{e}^{\frac{3}{2}} Dirac\left(t-\frac{1}{2}\right) + 17\cos\left(\frac{t}{2}\right)\mathrm{e}^{2t}\right)dt\right)\right)}{2}$
  - Compute integrals

    $y_p(t) = \frac{\mathrm{e}^{-3t}\left(\mathrm{e}^{\frac{3}{2}}\left(\sin\left(\frac{t}{2}\right)\cos\left(\frac{1}{4}\right) - \cos\left(\frac{t}{2}\right)\sin\left(\frac{1}{4}\right)\right) Heaviside\left(t-\frac{1}{2}\right) + 2\,\mathrm{e}^{2t}\right)}{2}$
- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-3t}\cos\left(\frac{t}{2}\right) + c_2\mathrm{e}^{-3t}\sin\left(\frac{t}{2}\right) + \frac{\mathrm{e}^{-3t}\left(\mathrm{e}^{\frac{3}{2}}\left(\sin\left(\frac{t}{2}\right)\cos\left(\frac{1}{4}\right) - \cos\left(\frac{t}{2}\right)\sin\left(\frac{1}{4}\right)\right) Heaviside\left(t-\frac{1}{2}\right) + 2\,\mathrm{e}^{2t}\right)}{2}$
- ☐ Check validity of solution $y = c_1\mathrm{e}^{-3t}\cos\left(\frac{t}{2}\right) + c_2\mathrm{e}^{-3t}\sin\left(\frac{t}{2}\right) + \frac{\mathrm{e}^{-3t}\left(\mathrm{e}^{\frac{3}{2}}\left(\sin\left(\frac{t}{2}\right)\cos\left(\frac{1}{4}\right) - \cos\left(\frac{t}{2}\right)\sin\left(\frac{1}{4}\right)\right) Hea\right.}{2}$
  - Use initial condition $y(0) = 1$

    $1 = c_1 + 1$
  - Compute derivative of the solution

    $y' = -3c_1\mathrm{e}^{-3t}\cos\left(\frac{t}{2}\right) - \frac{c_1\mathrm{e}^{-3t}\sin\left(\frac{t}{2}\right)}{2} - 3c_2\mathrm{e}^{-3t}\sin\left(\frac{t}{2}\right) + \frac{c_2\mathrm{e}^{-3t}\cos\left(\frac{t}{2}\right)}{2} - \frac{3\,\mathrm{e}^{-3t}\left(\mathrm{e}^{\frac{3}{2}}\left(\sin\left(\frac{t}{2}\right)\cos\left(\frac{1}{4}\right) - \cos\left(\frac{t}{2}\right)\sin\right.\right.}{2}$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = -3c_1 - 1 + \frac{c_2}{2}$
  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 4\}$
  - Substitute constant values into general solution and simplify

    $y = \frac{\left(\mathrm{e}^{\frac{3}{2}}\left(\sin\left(\frac{t}{2}\right)\cos\left(\frac{1}{4}\right) - \cos\left(\frac{t}{2}\right)\sin\left(\frac{1}{4}\right)\right) Heaviside\left(t-\frac{1}{2}\right) + 2\,\mathrm{e}^{2t} + 8\sin\left(\frac{t}{2}\right)\right)\mathrm{e}^{-3t}}{2}$

- Solution to the IVP

$$y = \frac{\left(e^{\frac{3}{2}} \left(\sin\left(\frac{t}{2}\right)\cos\left(\frac{1}{4}\right)-\cos\left(\frac{t}{2}\right)\sin\left(\frac{1}{4}\right)\right) \mathit{Heaviside}\left(t-\frac{1}{2}\right)+2\,e^{2t}+8\sin\left(\frac{t}{2}\right)\right)e^{-3t}}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.281 (sec). Leaf size: 37

```
dsolve([4*diff(y(t),t$2)+24*diff(y(t),t)+37*y(t)=17*exp(-t)+Dirac(t-1/2),y(0) = 1, D(y)(0) =
```

$$y(t) = \frac{\mathrm{Heaviside}\left(t-\frac{1}{2}\right)e^{-3t+\frac{3}{2}}\sin\left(-\frac{1}{4}+\frac{t}{2}\right)}{2} + 4\,e^{-3t}\sin\left(\frac{t}{2}\right) + e^{-t}$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 63

```
DSolve[{4*y''[t]+24*y'[t]+27*y[t]==17*Exp[-t]+DiracDelta[t-1/2],{y[0]==1,y'[0]==1}},y[t],t,I
```

$$y(t) \to \frac{1}{84}e^{-9t/2}\left(7e^{3/4}\left(e^{3t}-e^{3/2}\right)\theta(2t-1)+12\left(-7e^{3t}+17e^{7t/2}-3\right)\right)$$

8.6 problem 8

8.6.1 Existence and uniqueness analysis 850
8.6.2 Maple step by step solution 853

Internal problem ID [5709]
Internal file name [`OUTPUT/4957_Sunday_June_05_2022_03_14_56_PM_50845792/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.4, page 230
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = 10\sin(t) + 10\delta(t-1)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

8.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= 2 \\ F &= 10\sin(t) + 10\delta(t-1) \end{aligned}$$

Hence the ode is

$$y'' + 3y' + 2y = 10\sin(t) + 10\delta(t-1)$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 10\sin(t) + 10\delta(t-1)$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) + 2Y(s) = \frac{10}{s^2+1} + 10\,\mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 2 - s + 3sY(s) + 2Y(s) = \frac{10}{s^2+1} + 10\,\mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{10\,\mathrm{e}^{-s}s^2 + s^3 + 2s^2 + 10\,\mathrm{e}^{-s} + s + 12}{(s^2+1)(s^2+3s+2)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{10\,\mathrm{e}^{-s} s^2 + s^3 + 2s^2 + 10\,\mathrm{e}^{-s} + s + 12}{(s^2+1)(s^2+3s+2)}\right) \\
&= -3\cos(t) + \sin(t) + 6\,\mathrm{e}^{-t} - 2\,\mathrm{e}^{-2t} + 10\left(\mathrm{e}^{-t+1} - \mathrm{e}^{-2t+2}\right) \mathrm{Heaviside}(t-1)
\end{aligned}
$$

Hence the final solution is

$$
y = -3\cos(t) + \sin(t) + 6\,\mathrm{e}^{-t} - 2\,\mathrm{e}^{-2t} + 10\left(\mathrm{e}^{-t+1} - \mathrm{e}^{-2t+2}\right) \mathrm{Heaviside}(t-1)
$$

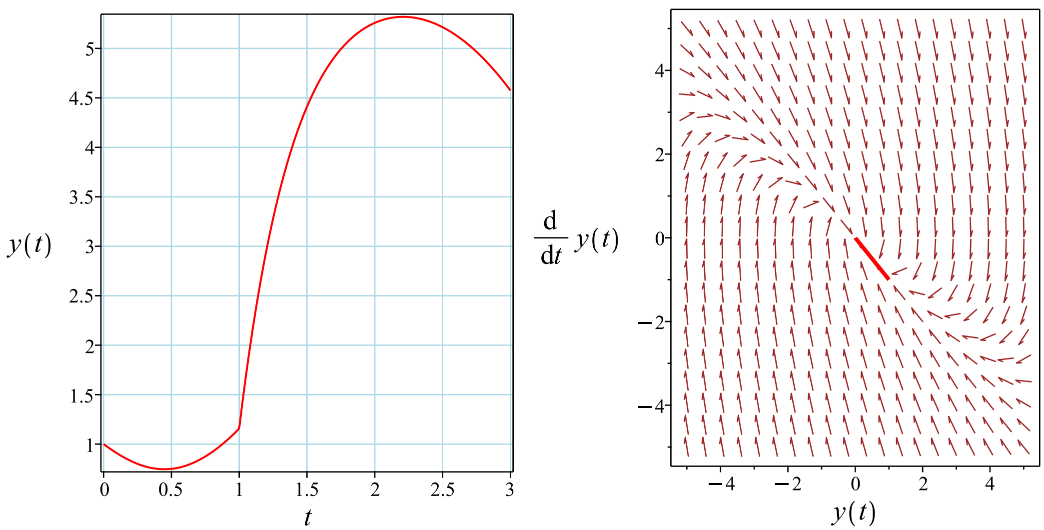
Simplifying the solution gives

$$
y = 10\,\mathrm{Heaviside}(t-1)\,\mathrm{e}^{-t+1} - 10\,\mathrm{Heaviside}(t-1)\,\mathrm{e}^{-2t+2} + \sin(t) - 2\,\mathrm{e}^{-2t} + 6\,\mathrm{e}^{-t} - 3\cos(t)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & 10\,\mathrm{Heaviside}(t-1)\,\mathrm{e}^{-t+1} - 10\,\mathrm{Heaviside}(t-1)\,\mathrm{e}^{-2t+2} \\
& + \sin(t) - 2\,\mathrm{e}^{-2t} + 6\,\mathrm{e}^{-t} - 3\cos(t)
\end{aligned} \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$
\begin{aligned}
y = {} & 10\,\mathrm{Heaviside}(t-1)\,\mathrm{e}^{-t+1} - 10\,\mathrm{Heaviside}(t-1)\,\mathrm{e}^{-2t+2} + \sin(t) - 2\,\mathrm{e}^{-2t} + 6\,\mathrm{e}^{-t} \\
& - 3\cos(t)
\end{aligned}
$$

Verified OK.

8.6.2 Maple step by step solution

Let's solve

$$\left[y'' + 3y' + 2y = 10\sin(t) + 10\mathit{Dirac}(t-1), y(0) = 1, y'\Big|_{\{t=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 10\sin(t) + 10\mathit{Dirac}(t-1)\right]$$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{-t} \\ -2\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{-3t}$

- ○ Substitute functions into equation for $y_p(t)$

$y_p(t) = -10\,\mathrm{e}^{-2t}\left(\int \left(Dirac(t-1)\,\mathrm{e}^{2} + \sin(t)\,\mathrm{e}^{2t}\right) dt\right) + 10\,\mathrm{e}^{-t}\left(\int \left(\sin(t) + Dirac(t-1)\right) \mathrm{e}^{t} dt\right)$

- Compute integrals

  $y_p(t) = -3\cos(t) + \sin(t) - 10Heaviside(t-1)\,\mathrm{e}^{-2t+2} + 10Heaviside(t-1)\,\mathrm{e}^{-t+1}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{-t} - 3\cos(t) + \sin(t) - 10Heaviside(t-1)\,\mathrm{e}^{-2t+2} + 10Heaviside(t-1)\,\mathrm{e}^{-t+1}$

☐ Check validity of solution $y = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{-t} - 3\cos(t) + \sin(t) - 10Heaviside(t-1)\,\mathrm{e}^{-2t+2} + 1$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2 - 3$

- Compute derivative of the solution

  $y' = -2c_1\mathrm{e}^{-2t} - c_2\mathrm{e}^{-t} + 3\sin(t) + \cos(t) - 10Dirac(t-1)\,\mathrm{e}^{-2t+2} + 20Heaviside(t-1)\,\mathrm{e}^{-2t+2} +$

- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = -2c_1 - c_2 + 1$

- Solve for c_1 and c_2

  $\{c_1 = -2, c_2 = 6\}$

- Substitute constant values into general solution and simplify

  $y = 10Heaviside(t-1)\,\mathrm{e}^{-t+1} - 10Heaviside(t-1)\,\mathrm{e}^{-2t+2} + \sin(t) - 2\,\mathrm{e}^{-2t} + 6\,\mathrm{e}^{-t} - 3\cos(t)$

- Solution to the IVP

  $y = 10Heaviside(t-1)\,\mathrm{e}^{-t+1} - 10Heaviside(t-1)\,\mathrm{e}^{-2t+2} + \sin(t) - 2\,\mathrm{e}^{-2t} + 6\,\mathrm{e}^{-t} - 3\cos(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.984 (sec). Leaf size: 44

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=10*(sin(t)+Dirac(t-1)),y(0) = 1, D(y)(0) = -1],
```

$$y(t) = -10\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2} + 10\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{1-t} - 2\,\mathrm{e}^{-2t} + \sin(t) - 3\cos(t) + 6\,\mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.165 (sec). Leaf size: 46

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==10*(Sin[t]+DiracDelta[t-1]),{y[0]==1,y'[0]==-1}},y[t],t,Inclu
```

$$y(t) \to 10e^{1-2t}\left(e^t - e\right)\theta(t-1) - 2e^{-2t} + 6e^{-t} + \sin(t) - 3\cos(t)$$

8.7 problem 9

8.7.1 Existence and uniqueness analysis 856
8.7.2 Maple step by step solution 859

Internal problem ID [5710]
Internal file name [`OUTPUT/4958_Sunday_June_05_2022_03_15_00_PM_82102922/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.4, page 230
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 5y = (1 - \text{Heaviside}\left(-10 + t\right))\,\mathrm{e}^{t} - \mathrm{e}^{10}\delta(-10 + t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

8.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 5 \\ F &= -\mathrm{e}^{t}\,\text{Heaviside}\left(-10 + t\right) - \mathrm{e}^{10}\delta(-10 + t) + \mathrm{e}^{t} \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 5y = -\mathrm{e}^{t}\,\text{Heaviside}\left(-10 + t\right) - \mathrm{e}^{10}\delta(-10 + t) + \mathrm{e}^{t}$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -e^t \operatorname{Heaviside}(-10+t) - e^{10}\delta(-10+t) + e^t$ is

$$\{t < 10 \lor 10 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 5Y(s) = \frac{-e^{-10s+10}s + 1}{s-1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + 4sY(s) + 5Y(s) = \frac{-e^{-10s+10}s + 1}{s-1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s(e^{-10s+10} - 1)}{(s-1)(s^2+4s+5)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{s(\mathrm{e}^{-10s+10}-1)}{(s-1)(s^2+4s+5)}\right) \\
&= \frac{\mathrm{e}^{t}\,\mathrm{Heaviside}(10-t)}{10} + \frac{\mathrm{e}^{30-2t}\,\mathrm{Heaviside}(-10+t)\left(\cos(-10+t)-7\sin(-10+t)\right)}{10} + \frac{(-\cos(t)+7\,\mathrm{s}}{10}
\end{aligned}
$$

Hence the final solution is

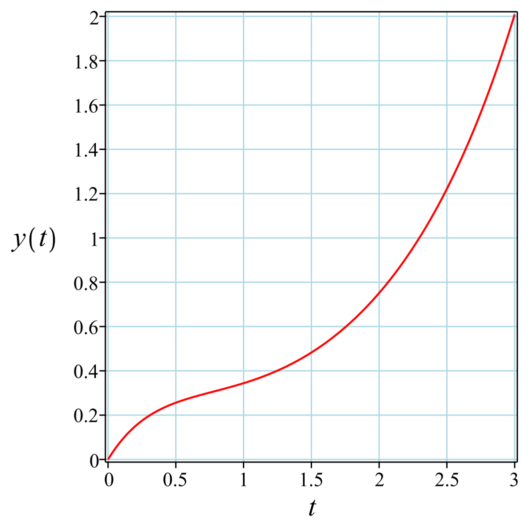
$$
y = \frac{\mathrm{e}^{t}\,\mathrm{Heaviside}(10-t)}{10} + \frac{\mathrm{e}^{30-2t}\,\mathrm{Heaviside}(-10+t)\left(\cos(-10+t)-7\sin(-10+t)\right)}{10} + \frac{(-\cos(t)+7\sin(t))\,\mathrm{e}^{-2t}}{10}
$$

Simplifying the solution gives

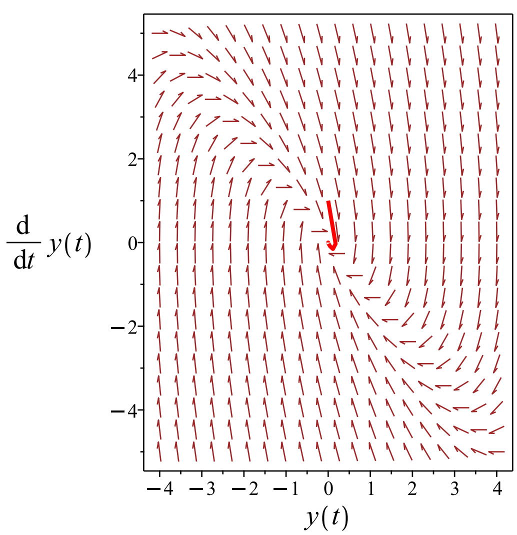
$$
y = \frac{((-\mathrm{e}^{3t}+((\cos(10)+7\sin(10))\cos(t)+(-7\cos(10)+\sin(10))\sin(t))\,\mathrm{e}^{30})\,\mathrm{Heaviside}(-10+t)-\cos(}{10}
$$

Summary
The solution(s) found are the following

$$
y = \frac{((-\mathrm{e}^{3t}+((\cos(10)+7\sin(10))\cos(t)+(-7\cos(10)+\sin(10))\sin(t))\,\mathrm{e}^{30})\,\mathrm{Heaviside}(-10+t)-\cos(}{10} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{((-e^{3t} + ((\cos(10) + 7\sin(10))\cos(t) + (-7\cos(10) + \sin(10))\sin(t))e^{30})\,\text{Heaviside}(-10+t) - \cos(}{10}$$

Verified OK.

8.7.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 5y = -e^t Heaviside(-10+t) - e^{10} Dirac(-10+t) + e^t, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - \mathrm{I}, -2 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-2t}\cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-2t}\sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(t)e^{-2t}c_1 + \sin(t)e^{-2t}c_2 + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = -e^t Heaviside(-10+t) - e\right.$$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t}\cos(t) & e^{-2t}\sin(t) \\ -2\,e^{-2t}\cos(t) - e^{-2t}\sin(t) & -2\,e^{-2t}\sin(t) + e^{-2t}\cos(t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-4t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = e^{-2t}\left(\left(\int \sin(t)\,(e^{2t+10}\mathit{Dirac}(-10+t) + e^{3t}(-1+\mathit{Heaviside}(-10+t)))\,dt\right)\cos(t) - \left(\int \text{co}\right.\right.$

- Compute integrals

  $y_p(t) = \frac{((-e^{3t}+((\cos(t)-7\sin(t))\cos(10)+(7\cos(t)+\sin(t))\sin(10))e^{30})\mathit{Heaviside}(-10+t)+e^{3t})e^{-2t}}{10}$

- Substitute particular solution into general solution to ODE

  $y = \cos(t)\,e^{-2t}c_1 + \sin(t)\,e^{-2t}c_2 + \frac{((-e^{3t}+((\cos(t)-7\sin(t))\cos(10)+(7\cos(t)+\sin(t))\sin(10))e^{30})\mathit{Heaviside}(-10+t)+}{10}$

☐ Check validity of solution $y = \cos(t)\,e^{-2t}c_1 + \sin(t)\,e^{-2t}c_2 + \frac{((-e^{3t}+((\cos(t)-7\sin(t))\cos(10)+(7\cos(t)+\text{si}}{10}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + \frac{1}{10}$

- Compute derivative of the solution

  $y' = -\sin(t)\,e^{-2t}c_1 - 2\cos(t)\,e^{-2t}c_1 + \cos(t)\,e^{-2t}c_2 - 2\sin(t)\,e^{-2t}c_2 + \frac{((-3\,e^{3t}+((-7\cos(t)-\sin(t))\cos(}{}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = \frac{1}{10} - 2c_1 + c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{10}, c_2 = \frac{7}{10}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{((-e^{3t}+((\cos(10)+7\sin(10))\cos(t)+(-7\cos(10)+\sin(10))\sin(t))e^{30})\mathit{Heaviside}(-10+t)-\cos(t)+7\sin(t)+e^{3t})e^{-2t}}{10}$

- Solution to the IVP

  $y = \frac{((-e^{3t}+((\cos(10)+7\sin(10))\cos(t)+(-7\cos(10)+\sin(10))\sin(t))e^{30})\mathit{Heaviside}(-10+t)-\cos(t)+7\sin(t)+e^{3t})e^{-2t}}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.328 (sec). Leaf size: 53

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+5*y(t)=(1-Heaviside(t-10))*exp(t)-exp(10)*Dirac(t-10),
```

$$y(t) = \frac{e^{-2t}((-e^{3t} + ((-7\cos(10) + \sin(10))\sin(t) + (\cos(10) + 7\sin(10))\cos(t))e^{30})\operatorname{Heaviside}(t-10) - \mathrm{co}}{10}$$

✓ Solution by Mathematica

Time used: 0.571 (sec). Leaf size: 94

```
DSolve[{y''[t]+4*y'[t]+5*y[t]==(1-UnitStep[t-10])*Exp[t]-Exp[10]*DiracDelta[t-10],{y[0]==0,y
```

$$y(t) \to \frac{1}{10}e^{-2t}\left(10e^{30}\theta(t-10)\sin(10-t)+\theta(10-t)\left(e^{3t}+3e^{30}\sin(10-t)-e^{30}\cos(10-t)\right) - 3e^{30}\sin(10-t) + 7\sin(t) + e^{30}\cos(10-t) - \cos(t)\right)$$

8.8 problem 10

8.8.1 Existence and uniqueness analysis 862
8.8.2 Maple step by step solution 865

Internal problem ID [5711]
Internal file name [`OUTPUT/4959_Sunday_June_05_2022_03_15_06_PM_67783294/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.4, page 230
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + 5y' + 6y = \delta\left(t - \frac{\pi}{2}\right) + \cos(t)\,\mathrm{Heaviside}(t - \pi)$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

8.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= 6 \\ F &= \delta\left(t - \frac{\pi}{2}\right) + \cos(t)\,\mathrm{Heaviside}(t - \pi) \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 6y = \delta\left(t - \frac{\pi}{2}\right) + \cos(t) \operatorname{Heaviside}(t - \pi)$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta\left(t - \frac{\pi}{2}\right) + \cos(t) \operatorname{Heaviside}(t - \pi)$ is

$$\left\{\pi \le t \le \frac{\pi}{2}, \frac{\pi}{2} \le t \le \infty, -\infty \le t \le \pi\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 5sY(s) - 5y(0) + 6Y(s) = e^{-\frac{\pi s}{2}} - \frac{e^{-\pi s}s}{s^2 + 1} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 5sY(s) + 6Y(s) = e^{-\frac{\pi s}{2}} - \frac{e^{-\pi s}s}{s^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-\frac{\pi s}{2}} s^2 - e^{-\pi s} s + e^{-\frac{\pi s}{2}}}{(s^2+1)(s^2+5s+6)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-\frac{\pi s}{2}} s^2 - e^{-\pi s} s + e^{-\frac{\pi s}{2}}}{(s^2+1)(s^2+5s+6)}\right) \\ &= \frac{\text{Heaviside}(t-\pi)\left(\cos(t) + 4\,e^{2\pi-2t} - 3\,e^{3\pi-3t} + \sin(t)\right)}{10} + \left(-e^{-3t+\frac{3\pi}{2}} + e^{-2t+\pi}\right)\text{Heaviside}\left(t - \frac{\pi}{2}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y =& \frac{\text{Heaviside}(t-\pi)\left(\cos(t) + 4\,e^{2\pi-2t} - 3\,e^{3\pi-3t} + \sin(t)\right)}{10} \\ &+ \left(-e^{-3t+\frac{3\pi}{2}} + e^{-2t+\pi}\right)\text{Heaviside}\left(t - \frac{\pi}{2}\right) \end{aligned}$$

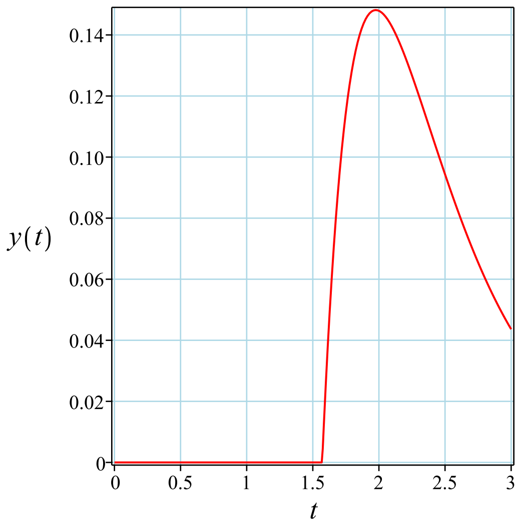
Simplifying the solution gives

$$\begin{aligned} y =& -\text{Heaviside}\left(t-\frac{\pi}{2}\right) e^{-3t+\frac{3\pi}{2}} + \frac{2\,\text{Heaviside}(t-\pi)\,e^{2\pi-2t}}{5} - \frac{3\,\text{Heaviside}(t-\pi)\,e^{3\pi-3t}}{10} \\ &+ \text{Heaviside}\left(t-\frac{\pi}{2}\right) e^{-2t+\pi} + \frac{\text{Heaviside}(t-\pi)\left(\sin(t)+\cos(t)\right)}{10} \end{aligned}$$

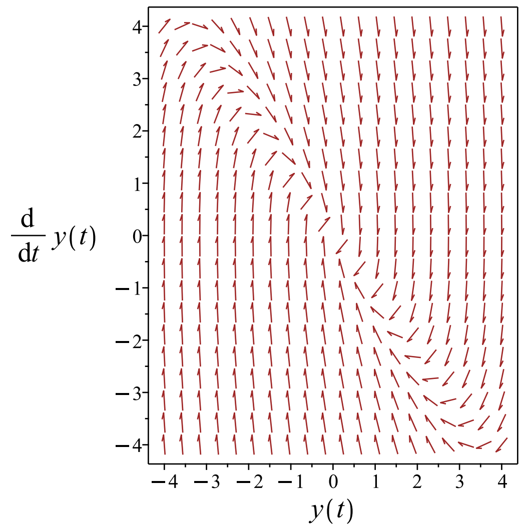
Summary

The solution(s) found are the following

$$\begin{aligned} y =& -\text{Heaviside}\left(t-\frac{\pi}{2}\right) e^{-3t+\frac{3\pi}{2}} + \frac{2\,\text{Heaviside}(t-\pi)\,e^{2\pi-2t}}{5} \\ &- \frac{3\,\text{Heaviside}(t-\pi)\,e^{3\pi-3t}}{10} + \text{Heaviside}\left(t-\frac{\pi}{2}\right) e^{-2t+\pi} \\ &+ \frac{\text{Heaviside}(t-\pi)\left(\sin(t)+\cos(t)\right)}{10} \end{aligned} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\text{Heaviside}\left(t-\frac{\pi}{2}\right)e^{-3t+\frac{3\pi}{2}} + \frac{2\,\text{Heaviside}\,(t-\pi)\,e^{2\pi-2t}}{5} - \frac{3\,\text{Heaviside}\,(t-\pi)\,e^{3\pi-3t}}{10} + \text{Heaviside}\left(t-\frac{\pi}{2}\right)e^{-2t+\pi} + \frac{\text{Heaviside}\,(t-\pi)\,(\sin(t)+\cos(t))}{10}$$

Verified OK.

8.8.2 Maple step by step solution

Let's solve

$$\left[y''+5y'+6y = Dirac\left(t-\frac{\pi}{2}\right)+\cos(t)\,Heaviside(t-\pi)\,, y(0)=0, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+5r+6=0$

- Factor the characteristic polynomial

  $(r+3)(r+2)=0$

- Roots of the characteristic polynomial

  $r=(-3,-2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-3t}$
- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-2t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3t} + c_2 e^{-2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = Dirac\left(t - \frac{\pi}{2}\right) + \cos(t)\, Hea\right.$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-3t} & e^{-2t} \\ -3\,e^{-3t} & -2\,e^{-2t} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(t), y_2(t)) = e^{-5t}$
  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -e^{-3t}\left(\int \left(Dirac\left(t - \frac{\pi}{2}\right) e^{\frac{3\pi}{2}} + \cos(t)\, Heaviside(t - \pi)\, e^{3t}\right) dt\right) + e^{-2t}\left(\int \left(Dirac\left(t - \frac{\pi}{2}\right) e^{\pi}\right.\right.$
  - Compute integrals

    $y_p(t) = -Heaviside\left(t - \frac{\pi}{2}\right) e^{-3t + \frac{3\pi}{2}} + \frac{2 Heaviside(t-\pi) e^{2\pi - 2t}}{5} - \frac{3 Heaviside(t-\pi) e^{3\pi - 3t}}{10} + Heaviside\left(t - \frac{\pi}{2}\right)$
- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-3t} + c_2 e^{-2t} - Heaviside\left(t - \frac{\pi}{2}\right) e^{-3t + \frac{3\pi}{2}} + \frac{2 Heaviside(t-\pi) e^{2\pi - 2t}}{5} - \frac{3 Heaviside(t-\pi) e^{3\pi - 3t}}{10} + Heav$

☐ Check validity of solution $y = c_1 e^{-3t} + c_2 e^{-2t} - Heaviside\left(t - \frac{\pi}{2}\right) e^{-3t + \frac{3\pi}{2}} + \frac{2 Heaviside(t-\pi) e^{2\pi - 2t}}{5} -$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2$
  - Compute derivative of the solution

    $y' = -3 c_1 e^{-3t} - 2 c_2 e^{-2t} - Dirac\left(t - \frac{\pi}{2}\right) e^{-3t + \frac{3\pi}{2}} + 3 Heaviside\left(t - \frac{\pi}{2}\right) e^{-3t + \frac{3\pi}{2}} + \frac{2 Dirac(t-\pi) e^{2\pi - 2t}}{5} -$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$0 = -3c_1 - 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = -Heaviside\left(t - \frac{\pi}{2}\right) e^{-3t + \frac{3\pi}{2}} + \frac{2Heaviside(t-\pi)e^{2\pi-2t}}{5} - \frac{3Heaviside(t-\pi)e^{3\pi-3t}}{10} + Heaviside\left(t - \frac{\pi}{2}\right) e^{-2}$

- Solution to the IVP

  $y = -Heaviside\left(t - \frac{\pi}{2}\right) e^{-3t + \frac{3\pi}{2}} + \frac{2Heaviside(t-\pi)e^{2\pi-2t}}{5} - \frac{3Heaviside(t-\pi)e^{3\pi-3t}}{10} + Heaviside\left(t - \frac{\pi}{2}\right) e^{-2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.047 (sec). Leaf size: 62

```
dsolve([diff(y(t),t$2)+5*diff(y(t),t)+6*y(t)=Dirac(t-1/2*Pi)+Heaviside(t-Pi)*cos(t),y(0) = 0
```

$$y(t) = -\operatorname{Heaviside}\left(t - \frac{\pi}{2}\right) e^{-3t + \frac{3\pi}{2}} - \frac{3 \operatorname{Heaviside}(t-\pi) e^{-3t+3\pi}}{10} + \frac{2 \operatorname{Heaviside}(t-\pi) e^{-2t+2\pi}}{5} + \operatorname{Heaviside}\left(t - \frac{\pi}{2}\right) e^{-2t+\pi} + \frac{\operatorname{Heaviside}(t-\pi)(\cos(t) + \sin(t))}{10}$$

✓ Solution by Mathematica

Time used: 0.511 (sec). Leaf size: 85

```
DSolve[{y''[t]+5*y'[t]+6*y[t]==DiracDelta[t-1/2*Pi]+UnitStep[t-Pi]*Cos[t],{y[0]==0,y'[0]==0}
```

$$y(t) \to \frac{1}{10}e^{-3t}\left((\theta(\pi-t)-1)\left(-4e^{t+2\pi}-e^{3t}\sin(t)-e^{3t}\cos(t)+3e^{3\pi}\right) - 10e^{\pi}\left(e^{\pi/2}-e^{t}\right)\theta(2t-\pi)\right)$$

8.9 problem 11

8.9.1 Existence and uniqueness analysis 869
8.9.2 Maple step by step solution 872

Internal problem ID [5712]
Internal file name [`OUTPUT/4960_Sunday_June_05_2022_03_15_11_PM_45893023/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.4, page 230
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 5y' + 6y = \text{Heaviside}\,(t-1) + \delta(-2+t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

8.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= 6 \\ F &= \text{Heaviside}\,(t-1) + \delta(-2+t) \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 6y = \text{Heaviside}\,(t-1) + \delta(-2+t)$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \text{Heaviside}\,(t-1) + \delta(-2+t)$ is

$$\{1 \leq t \leq 2, 2 \leq t \leq \infty, -\infty \leq t \leq 1\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 5sY(s) - 5y(0) + 6Y(s) = \frac{\mathrm{e}^{-s}}{s} + \mathrm{e}^{-2s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + 5sY(s) + 6Y(s) = \frac{\mathrm{e}^{-s}}{s} + \mathrm{e}^{-2s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-2s}s + \mathrm{e}^{-s} + s}{s\,(s^2 + 5s + 6)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-2s}s + e^{-s} + s}{s\left(s^2+5s+6\right)}\right) \\
&= -e^{-3t} + e^{-2t} + \text{Heaviside}\left(-2+t\right)\left(-e^{6-3t} + e^{-2t+4}\right) + \frac{\text{Heaviside}\left(t-1\right)\left(1+2\,e^{-3t+3}-3\,e^{-2t+2}\right)}{6}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y = {} & -e^{-3t} + e^{-2t} + \text{Heaviside}\left(-2+t\right)\left(-e^{6-3t} + e^{-2t+4}\right) \\
& + \frac{\text{Heaviside}\left(t-1\right)\left(1+2\,e^{-3t+3}-3\,e^{-2t+2}\right)}{6}
\end{aligned}
$$

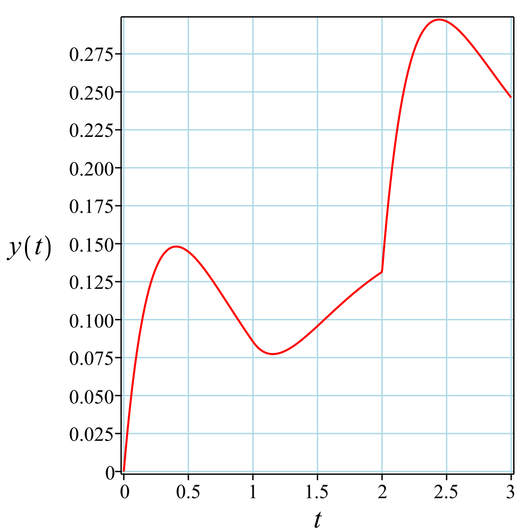
Simplifying the solution gives

$$
\begin{aligned}
y = {} & -e^{-3t} + e^{-2t} - \text{Heaviside}\left(-2+t\right)e^{6-3t} + \text{Heaviside}\left(-2+t\right)e^{-2t+4} \\
& + \frac{\text{Heaviside}\left(t-1\right)e^{-3t+3}}{3} - \frac{\text{Heaviside}\left(t-1\right)e^{-2t+2}}{2} + \frac{\text{Heaviside}\left(t-1\right)}{6}
\end{aligned}
$$

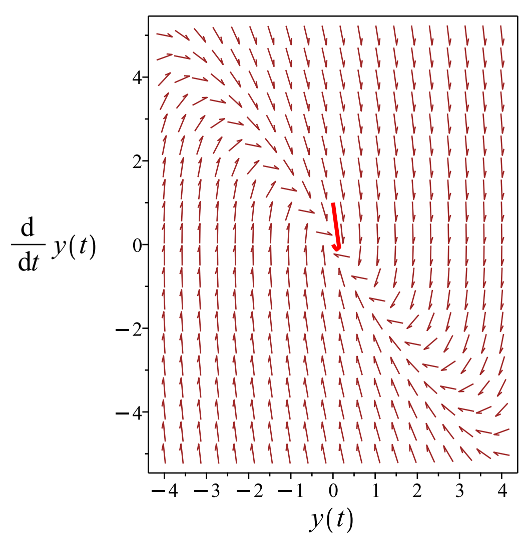
Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & -e^{-3t} + e^{-2t} - \text{Heaviside}\left(-2+t\right)e^{6-3t} + \text{Heaviside}\left(-2+t\right)e^{-2t+4} \\
& + \frac{\text{Heaviside}\left(t-1\right)e^{-3t+3}}{3} - \frac{\text{Heaviside}\left(t-1\right)e^{-2t+2}}{2} + \frac{\text{Heaviside}\left(t-1\right)}{6}
\end{aligned} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\mathrm{e}^{-3t} + \mathrm{e}^{-2t} - \text{Heaviside}\left(-2+t\right)\mathrm{e}^{6-3t} + \text{Heaviside}\left(-2+t\right)\mathrm{e}^{-2t+4} + \frac{\text{Heaviside}\left(t-1\right)\mathrm{e}^{-3t+3}}{3} - \frac{\text{Heaviside}\left(t-1\right)\mathrm{e}^{-2t+2}}{2} + \frac{\text{Heaviside}\left(t-1\right)}{6}$$

Verified OK.

8.9.2 Maple step by step solution

Let's solve

$$\left[y''+5y'+6y = \mathit{Heaviside}(t-1)+\mathit{Dirac}(-2+t)\,, y(0)=0, y'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2+5r+6=0$
- Factor the characteristic polynomial

  $(r+3)(r+2)=0$
- Roots of the characteristic polynomial

  $r=(-3,-2)$
- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-3t}$
- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{-2t}$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-3t}+c_2\mathrm{e}^{-2t}+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\mathit{Heaviside}(t-1)+\mathit{Dirac}(-\right.$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-3t} & e^{-2t} \\ -3\,e^{-3t} & -2\,e^{-2t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-5t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -e^{-3t}\left(\int (Dirac(-2+t)\,e^6 + Heaviside(t-1)\,e^{3t})\,dt\right) + e^{-2t}\left(\int (Dirac(-2+t)\,e^4 + Hea$

- Compute integrals

  $y_p(t) = -Heaviside(-2+t)\,e^{6-3t} + \frac{Heaviside(t-1)}{6} + \frac{Heaviside(t-1)e^{-3t+3}}{3} + Heaviside(-2+t)\,e^{-2t+4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-3t} + c_2 e^{-2t} - Heaviside(-2+t)\,e^{6-3t} + \frac{Heaviside(t-1)}{6} + \frac{Heaviside(t-1)e^{-3t+3}}{3} + Heaviside(-2$

□ Check validity of solution $y = c_1 e^{-3t} + c_2 e^{-2t} - Heaviside(-2+t)\,e^{6-3t} + \frac{Heaviside(t-1)}{6} + \frac{Heavisid}{}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

- Compute derivative of the solution

  $y' = -3c_1 e^{-3t} - 2c_2 e^{-2t} - Dirac(-2+t)\,e^{6-3t} + 3Heaviside(-2+t)\,e^{6-3t} + \frac{Dirac(t-1)}{6} + \frac{Dirac(t-1}{3}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = -3c_1 - 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = -e^{-3t} + e^{-2t} - Heaviside(-2+t)\,e^{6-3t} + Heaviside(-2+t)\,e^{-2t+4} + \frac{Heaviside(t-1)e^{-3t+3}}{3} - \frac{He}{}$

- Solution to the IVP

  $y = -e^{-3t} + e^{-2t} - Heaviside(-2+t)\,e^{6-3t} + Heaviside(-2+t)\,e^{-2t+4} + \frac{Heaviside(t-1)e^{-3t+3}}{3} - \frac{He}{}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.937 (sec). Leaf size: 59

```
dsolve([diff(y(t),t$2)+5*diff(y(t),t)+6*y(t)=Heaviside(t-1)+Dirac(t-2),y(0) = 0, D(y)(0) = 1
```

$$y(t) = -\mathrm{e}^{-3t} + \mathrm{e}^{-2t} + \mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{-2t+4} - \mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{-3t+6} - \frac{\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2}}{2} + \frac{\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-3t+3}}{3} + \frac{\mathrm{Heaviside}\,(t-1)}{6}$$

✓ Solution by Mathematica
Time used: 0.208 (sec). Leaf size: 80

```
DSolve[{y''[t]+5*y'[t]+6*y[t]==UnitStep[t-1]+DiracDelta[t-2],{y[0]==0,y'[0]==1}},y[t],t,Incl
```

$$y(t) \to \frac{1}{6}e^{-3t}\left(6e^4\left(e^t-e^2\right)\theta(t-2) - \left(\left(e^t+2e\right)\left(e-e^t\right)^2\theta(1-t)\right) + 6e^t + e^{3t} - 3e^{t+2} + 2e^3 - 6\right)$$

8.10 problem 12

8.10.1 Existence and uniqueness analysis 875
8.10.2 Maple step by step solution 878

Internal problem ID [5713]
Internal file name [`OUTPUT/4961_Sunday_June_05_2022_03_15_16_PM_24343582/index.tex`]

Book: ADVANCED ENGINEERING MATHEMATICS. ERWIN KREYSZIG, HERBERT KREYSZIG, EDWARD J. NORMINTON. 10th edition. John Wiley USA. 2011
Section: Chapter 6. Laplace Transforms. Problem set 6.4, page 230
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 5y = 25t - 100\delta(t - \pi)}$$

With initial conditions

$$[y(0) = -2, y'(0) = 5]$$

8.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 5 \\ F &= 25t - 100\delta(t - \pi) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 5y = 25t - 100\delta(t - \pi)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 25t - 100\delta(t - \pi)$ is

$$\{t < \pi \lor \pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 5Y(s) = \frac{25}{s^2} - 100\,\mathrm{e}^{-\pi s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -2\\ y'(0) &= 5\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + 2s + 2sY(s) + 5Y(s) = \frac{25}{s^2} - 100\,\mathrm{e}^{-\pi s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{100\,\mathrm{e}^{-\pi s}s^2 + 2s^3 - s^2 - 25}{s^2\,(s^2 + 2s + 5)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{100\,\mathrm{e}^{-\pi s} s^2 + 2s^3 - s^2 - 25}{s^2\,(s^2 + 2s + 5)}\right) \\
&= -50\,\mathrm{Heaviside}\,(t-\pi)\,\mathrm{e}^{\pi - t}\sin{(2t)} - 2 + 5t
\end{aligned}
$$

Hence the final solution is

$$
y = -50\,\mathrm{Heaviside}\,(t-\pi)\,\mathrm{e}^{\pi - t}\sin{(2t)} - 2 + 5t
$$

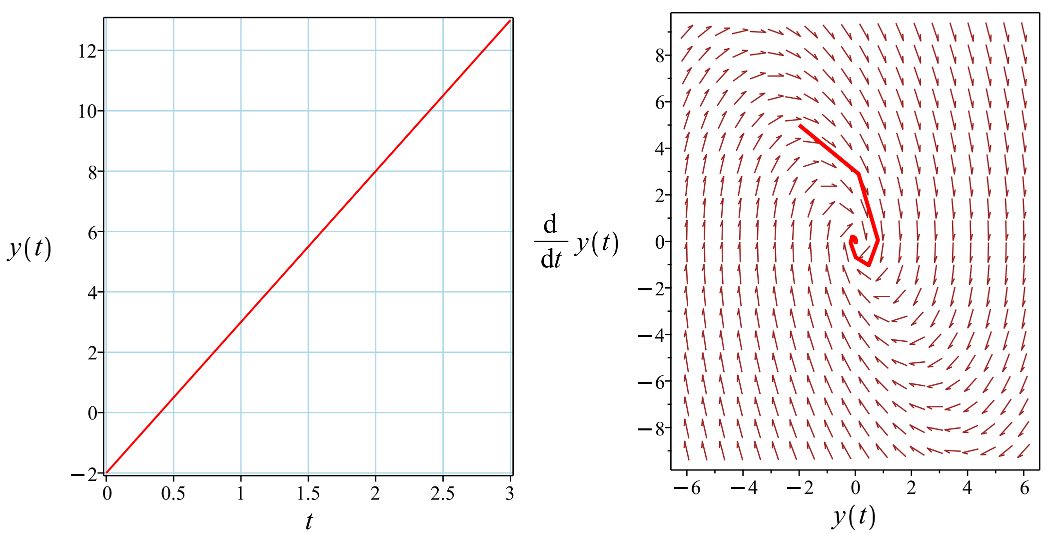
Simplifying the solution gives

$$
y = -50\,\mathrm{Heaviside}\,(t-\pi)\,\mathrm{e}^{\pi - t}\sin{(2t)} - 2 + 5t
$$

Summary

The solution(s) found are the following

$$
y = -50\,\mathrm{Heaviside}\,(t-\pi)\,\mathrm{e}^{\pi - t}\sin{(2t)} - 2 + 5t \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = -50\,\mathrm{Heaviside}\,(t-\pi)\,\mathrm{e}^{\pi - t}\sin{(2t)} - 2 + 5t
$$

Verified OK.

8.10.2 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + 5y = 25t - 100 Dirac(t-\pi), y(0) = -2, y'\Big|_{\{t=0\}} = 5\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-1-2\,\mathrm{I}, -1+2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)\,\mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)\,\mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t)\,\mathrm{e}^{-t} + c_2 \sin(2t)\,\mathrm{e}^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 25t - 100 Dirac(t-\pi)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t)\,\mathrm{e}^{-t} & \sin(2t)\,\mathrm{e}^{-t} \\ -2\sin(2t)\,\mathrm{e}^{-t} - \cos(2t)\,\mathrm{e}^{-t} & 2\cos(2t)\,\mathrm{e}^{-t} - \sin(2t)\,\mathrm{e}^{-t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2\,\mathrm{e}^{-2t}$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = -\frac{25\,\mathrm{e}^{-t}\left(\cos(2t)\left(\int t\sin(2t)\mathrm{e}^{t}dt\right)-\sin(2t)\left(\int \left(-4 Dirac(t-\pi)\mathrm{e}^{\pi}+t\cos(2t)\mathrm{e}^{t}\right)dt\right)\right)}{2}$

- Compute integrals

  $y_p(t) = -50 Heaviside(t-\pi)\,\mathrm{e}^{\pi-t}\sin(2t) - 2 + 5t$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(2t)\,\mathrm{e}^{-t} + c_2\sin(2t)\,\mathrm{e}^{-t} - 50 Heaviside(t-\pi)\,\mathrm{e}^{\pi-t}\sin(2t) - 2 + 5t$

☐ Check validity of solution $y = c_1\cos(2t)\,\mathrm{e}^{-t} + c_2\sin(2t)\,\mathrm{e}^{-t} - 50 Heaviside(t-\pi)\,\mathrm{e}^{\pi-t}\sin(2t) -$

- Use initial condition $y(0) = -2$

  $-2 = -2 + c_1$

- Compute derivative of the solution

  $y' = -2c_1\sin(2t)\,\mathrm{e}^{-t} - c_1\cos(2t)\,\mathrm{e}^{-t} + 2c_2\cos(2t)\,\mathrm{e}^{-t} - c_2\sin(2t)\,\mathrm{e}^{-t} - 50 Dirac(t-\pi)\,\mathrm{e}^{\pi-t}\sin$

- Use the initial condition $y'\Big|_{\{t=0\}} = 5$

  $5 = 5 - c_1 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = -50 Heaviside(t-\pi)\,\mathrm{e}^{\pi-t}\sin(2t) - 2 + 5t$

- Solution to the IVP

  $y = -50 Heaviside(t-\pi)\,\mathrm{e}^{\pi-t}\sin(2t) - 2 + 5t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.875 (sec). Leaf size: 27

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+5*y(t)=25*t-100*Dirac(t-Pi),y(0) = -2, D(y)(0) = 5],y(
```

$$y(t) = -50\,\mathrm{Heaviside}\,(t-\pi)\sin(2t)\,\mathrm{e}^{\pi-t}+5t-2$$

✓ Solution by Mathematica

Time used: 0.271 (sec). Leaf size: 29

```
DSolve[{y''[t]+2*y'[t]+5*y[t]==25*t-100*DiracDelta[t-Pi],{y[0]==-2,y'[0]==5}},y[t],t,Include
```

$$y(t)\to -50e^{\pi-t}\theta(t-\pi)\sin(2t)+5t-2$$