Finding angle of departure for rolling disk on semicyliner

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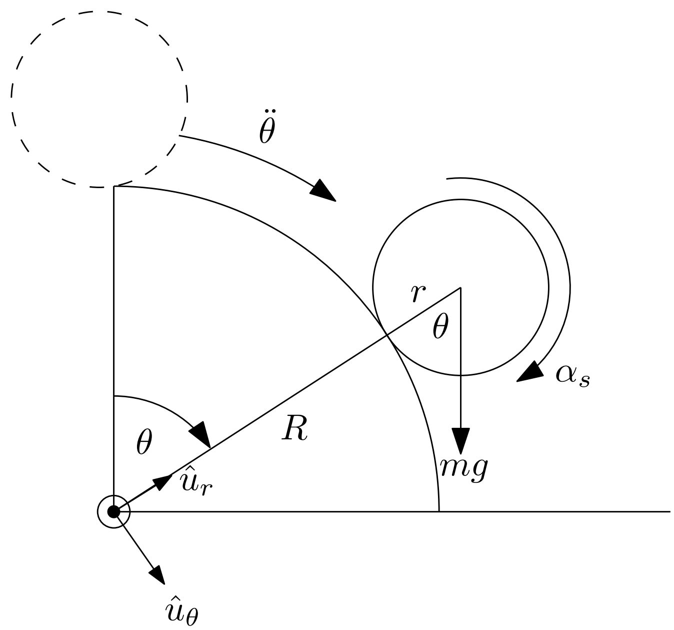
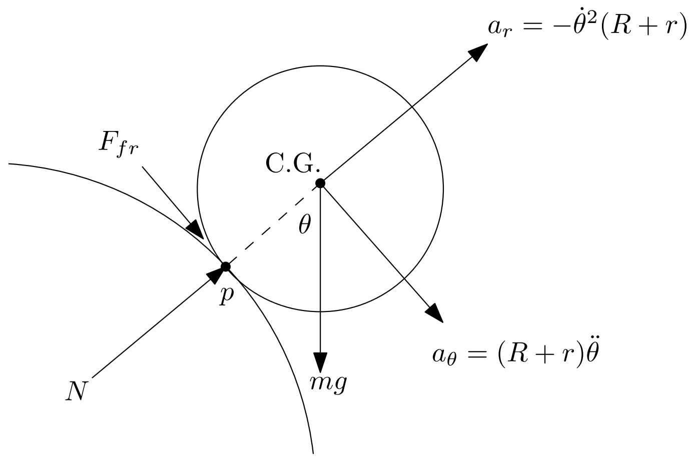
December 4, 2017 Compiled on September 10, 2023 at 9:09am [public]

A small sphere of mass m starts to roll with no slip on top of semicylinder. The problem is to determine at what angle θ the small sphere will depart the surface of the semicylinder.



The free body diagram for the sphere is



Resolving forces along the normal N gives

$$\begin{aligned} N - mg\cos\theta &= ma_r \\ &= -m\dot{\theta}^2(R+r) \end{aligned}$$

Hence

$$N = m\big(g\cos\theta - \dot{\theta}^2(R+r)\big) \tag{1}$$

To find when $N=0$, we need to find θ. Taking moments around point p where sphere is on contact with the cylinder (this way we do not have to solve for F, the friction). Using anti-clock wise as positive then

$$mg\sin\theta = I_{cg}\alpha_s + ma_\theta r \tag{2}$$

Notice that we had to add $ma_\theta r$, which is the moment around p due to inertia acceleration of the sphere, since the point we are taking moment about (point p) is not fixed and it is not the C.G. In (2) α_s is the angular acceleration of the sphere around its mass center. Not to confuse this with $\ddot{\theta}$ of the whole sphere around the center of the semicylinder itself.

Now, since the sphere rolls without slip, then

$$a_\theta = r\alpha_s$$

And since $I_{cg} = \frac{2}{5}mr^2$, then (2) becomes

$$\begin{aligned} rmg\sin\theta &= \frac{2}{5}mr^2\frac{a_\theta}{r} + ma_\theta r \\ rg\sin\theta &= \frac{2}{5}ra_\theta + a_\theta r \\ g\sin\theta &= \frac{7}{5}a_\theta \end{aligned}$$

But $a_\theta = (R+r)\ddot{\theta}$, therefore the above becomes

$$g\sin\theta = \frac{7}{5}(R+r)\ddot{\theta} \tag{3}$$

Let $\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$. Hence (3) becomes

$$g\sin\theta d\theta = \dot{\theta}\frac{7}{5}(R+r)\,d\dot{\theta}$$

Integrating (The sphere starts rolling with zero initial velocity)

$$\begin{aligned}
\int_0^{\theta_{slip}} g\sin\theta d\theta &= \int_0^{\dot{\theta}_{slip}} \dot{\theta}\frac{7}{5}(R+r)\,d\dot{\theta} \\
-g(\cos\theta)_0^{\theta_{slip}} &= \frac{7}{10}(R+r)\,\dot{\theta}_{slip}^2 \\
g(1-\cos\theta_{slip}) &= \frac{7}{10}(R+r)\,\dot{\theta}_{slip}^2 \\
\dot{\theta}_{slip}^2 &= \frac{10}{7}\frac{g(1-\cos\theta_{slip})}{(R+r)}
\end{aligned}$$

Using the above expression for $\dot{\theta}^2$ in (1) gives

$$\begin{aligned}
N &= m\left(g\cos\theta - \left(\frac{10}{7}\frac{g(1-\cos\theta)}{(R+r)}\right)(R+r)\right) \\
&= m\left(g\cos\theta - \frac{10}{7}g(1-\cos\theta)\right)
\end{aligned}$$

This is zero when

$$\begin{aligned}
\cos\theta - \frac{10}{7}(1-\cos\theta) &= 0 \\
\cos\theta - \frac{10}{7} + \frac{10}{7}\cos\theta &= 0 \\
\frac{17}{7}\cos\theta &= \frac{10}{7} \\
\cos\theta &= \frac{10}{17}
\end{aligned}$$

The first solution for this is

$$\theta_{slip} = 53.9681^0$$

This is the angle from the vertical when the sphere will depart the surface of the cylinder.