Solving ode's using parametric methods

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November 18, 2022 Compiled on November 18, 2022 at 6:41pm

Contents

1 First order ODE

Let the nonlinear ode be

$$f(x,y,y')=0$$

In parametric methods we let $y'=p$ and $x\equiv x(p)\,,y\equiv y(p)$. If we can isolate $x=G(y,p)$ or $y=G(x,p)$ then we can solve the ode using this method. This method can be simpler than the direct method when the ode is nonlinear. This is really should only be attempted for nonlinear odes. This is learned best by examples. I will add more background theory later.

1.1 Both x,y are present in the ode

1.1.1 Example 1

Solve

$$y(y')^2-4xy'+y=0 \tag{1}$$

Let $y'=p$. The above becomes

$$yp^2-4xp+y=0 \tag{2}$$

Let's see if we can isolate x first. Solving for x gives

$$\begin{aligned} x&=\frac{1}{4p}\left(y+p^2y\right)\\ &=G(y(p)\,,p) \end{aligned} \tag{3}$$

Since G does not depend on x, then we can continoue. Using (3) then (will show how this came about later)

$$
\begin{aligned}
\frac{dy}{dp} &= \frac{p\frac{\partial G}{\partial p}}{1-p\frac{\partial G}{\partial y}} \\
&= \frac{y(p^2-1)}{p\,(3-p^2)}
\end{aligned}
$$

This is separable ode

$$
\begin{aligned}
\frac{dy}{y} &= \frac{(p^2-1)}{p\,(3-p^2)}dp \\
\int \frac{dy}{y} &= \int \frac{(p^2-1)}{p\,(3-p^2)}dp \\
\ln y &= -\frac{1}{3}\ln\left(p\left(p^2-3\right)\right)+c \\
y &= \frac{c_1}{\left(p\,(p^2-3)\right)^{\frac{1}{3}}}
\end{aligned}
$$

Hence the solution to (1) is

$$
\begin{aligned}
x(p) &= \frac{1}{4p}\left(y+p^2y\right) \qquad (4(1)) \\
y(p) &= \frac{c_1}{\left(p\,(p^2-3)\right)^{\frac{1}{3}}} \qquad (4(2))
\end{aligned}
$$

The above is the solution to (1) in parameteric form where the dependecy between y and x is via p. We can stop here. But let see if we can get the solution as $y(x)$ as the normal case is. Eliminating p between 4(1) and 4(2) results in the solution

$$
c^6-64c^3x^3+24c^3xy^2-48x^2y^4+16y^6=0
$$

And the above is the final nonparametric solution. It is an implicit solution.

We might think this method is complicated, but it is actually much simpler than the first method. How would we solve (1) directly? We will starting by solving for y' in (1) which gives

$$
\begin{aligned}
y' &= \frac{2x+\sqrt{4x^2-y^2}}{y} \qquad (5) \\
y' &= \frac{2x-\sqrt{4x^2-y^2}}{y}
\end{aligned}
$$

Starting with the first one above, we notice it is homogeneous ode. Let $u=\frac{y}{x}$ and it becomes

$$
u' = \frac{-u^2+\sqrt{-u^2+4}+2}{ux}
$$

This is separable which results in

$$\int \frac{u}{-u^2 + \sqrt{-u^2 + 4} + 2} du = \int \frac{1}{x} dx$$

The above integrals gives a very complicated antiderivative. After that we have to replace u back by $\frac{y}{x}$ and simplify. We would do the same for the second ode in (5). It is clear here that the parametric method is simpler. But for the parametric method to work, we would have to be able to isolate x or y from (1) and obtain $G(y, p)$ function in the first case or $G(x, p)$ in the second case in order to continue. We also need to be able to elminate p at the end in order to get an explicit $y(x)$ solution, This could proof to be tricky.

In the above example, we isolated x. Let see what happens if we choose to isolate y from (2) instead. Solving for y gives

$$\begin{aligned} yp^2 - 4xp + y &= 0 \\ y(1 + p^2) &= 4xp \\ y &= \frac{4xp}{1 + p^2} \\ &= G(x, p) \end{aligned} \tag{6}$$

Hence this works also. In this case we have

$$\begin{aligned} \frac{dx}{dp} &= \frac{\frac{\partial G}{\partial p}}{p - \frac{\partial G}{\partial x}} \\ &= \frac{-4p^2 x + 4x}{p(p^2 - 3)(p^2 + 1)} \\ &= x\left(\frac{-4p^2 + 4}{p(p^2 - 3)(p^2 + 1)}\right) \end{aligned}$$

This is separable. Solving gives

$$\begin{aligned} x &= c_1 \frac{(1 + p^2)}{p^{\frac{4}{3}} (p^2 - 3)^{\frac{1}{3}}} \\ x^3 &= c_2 \frac{(1 + p^2)^3}{p^4 (p^2 - 3)} \end{aligned} \tag{7}$$

Eliminating p from (6,7) gives the solution

$$x\left(-3x^2 y^4 + y^6 - 256x^3 c_2 + 96xy^2 c_2 + 256c_1^2\right) = 0$$

or

$$-3x^2 y^4 + y^6 - 256x^3 c_2 + 96xy^2 c_2 + 256c_1^2 = 0$$

Which is an implicit solution.

1.1.2 Example 2

Solve

$$y - xy' - y' + (y')^2 = 0 \tag{1}$$

This problem from chapter 7, problem 7. From Boole book, page 137. This is actually a clairaut ode. Let $y' = p$. The above becomes

$$y - xp - p + p^2 = 0 \tag{1}$$

Solving for x gives

$$\begin{aligned} -xp &= \frac{p - p^2}{y} \\ x &= \frac{p-1}{y} \\ &= G(y(p), p) \end{aligned}$$

Since G does not depend on x, then we can continoue. Therefore

$$\begin{aligned} \frac{dy}{dp} &= \frac{p\frac{\partial G}{\partial p}}{1 - p\frac{\partial G}{\partial y}} \\ &= \frac{p\frac{1}{y}}{1 - p\left(\frac{1-p}{y^2}\right)} \\ &= \frac{py}{p^2 + y^2 - p} \end{aligned}$$

This is non-linear ode in y. So this is no better than what we started. Let try to isolate y intead. Solving (1) for y gives

$$\begin{aligned} y &= xp + p - p^2 \\ &= G(x(p), p) \end{aligned}$$

Therefore

$$\frac{dx}{dt} = \frac{\frac{\partial G}{\partial p}}{p - \frac{\partial G}{\partial x}}$$

But $p - \frac{\partial G}{\partial x} = 0$. Hence this method does not work for this ode.

1.2 Only y is present in the ode

1.2.1 Example 1

When only y or x (but not both) are present, we can do the following. Solve

$$y - ay' - \sqrt{1 + (y')^2} = 0 \tag{1}$$

This is problem chapter 7, problem 7. From Boole book, page 137. Let $y' = p$. The above becomes

$$\begin{aligned} y - ap - \sqrt{1 + p^2} &= 0 \\ y &= f(p) \\ &= ap + \sqrt{1 + p^2} \end{aligned} \tag{2}$$

Where $p = \frac{dy}{dx}$. Hence $dx = \frac{1}{p} dy$. But from the above $dy = f'(p)\, dp = \left(a + \frac{p}{\sqrt{1+p^2}}\right) dp$. Hence

$$\begin{aligned} dx &= \frac{1}{p}\left(a + \frac{p}{\sqrt{1+p^2}}\right) dp \\ x &= \int \frac{a}{p} + \frac{1}{\sqrt{1+p^2}} dp \\ &= a \ln p + \operatorname{arcsinh} p + c_1 \end{aligned}$$

Hence

$$\begin{aligned} e^x &= c p^a e^{\operatorname{arcsinh} p} \\ 0 &= c p^a e^{\operatorname{arcsinh}(p) - x} \end{aligned}$$

Therefore p is

$$p = \operatorname{RootOf}\left(c\, \_z^a e^{\operatorname{arcsinh}(\_z) - x}\right)$$

Where $\_z$ is the variable. Hence the solution from (2) becomes

$$\begin{aligned} y &= ap + \sqrt{1 + p^2} \\ &= a \operatorname{RootOf}\left(c\, \_z^a e^{\operatorname{arcsinh}(\_z) - x}\right) + \sqrt{1 + 2 \operatorname{RootOf}\left(c\, \_z^a e^{\operatorname{arcsinh}(\_z) - x}\right)} \end{aligned}$$

1.2.2 Example 2

Solve

$$(y')^2 + 2(y')^3 + y = 0 \tag{1}$$

Let $p = \frac{dy}{dx}$

$$\begin{aligned} y &= p^2 + 2p^3 \\ y &= f(p) \end{aligned} \tag{2}$$

Where $p = \frac{dy}{dx}$. Hence $dx = \frac{1}{p} dy$. But from the above $dy = f'(p)\, dp = (2p + 6p^2)\, dp$. Hence

$$\begin{aligned} dx &= \frac{1}{p}\left(2p + 6p^2\right) dp \\ &= (2 + 6p)\, dp \\ x &= \int (2 + 6p)\, dp \\ &= 2p + 3p^2 + c \end{aligned}$$

Solving for p gives

$$p = \frac{-1 \pm \sqrt{3x + c}}{3}$$

Hence the solution from (2) becomes (for the first root)

$$\begin{aligned} y &= p^2 + 2p^3 \\ &= \left(\frac{-1 + \sqrt{3x + c}}{3}\right)^2 + 2\left(\frac{-1 + \sqrt{3x + c}}{3}\right)^3 \end{aligned}$$

And for the second root

$$\begin{aligned} y &= p^2 + 2p^3 \\ &= \left(\frac{-1 - \sqrt{3x + c}}{3}\right)^2 + 2\left(\frac{-1 - \sqrt{3x + c}}{3}\right)^3 \end{aligned}$$

These methods produce simpler solution if we can solve for p easily in the above.

1.3 Only x is present

1.3.1 Example 1

$$x = 1 + y' + (y')^3$$

Let $y' = p$ therefore

$$\begin{aligned} x &= 1 + p + p^3 \qquad (1) \\ &= f(p) \end{aligned}$$

But $dy = pdx$. But from the above $dx = f'(p)\, dp$. Hence

$$\begin{aligned} dy &= pf'(p)\, dp \\ &= p\left(1 + 3p^2\right) dp \end{aligned}$$

Therefore

$$\begin{aligned} y &= \int p\left(1 + 3p^2\right) dp \\ &= \frac{p^2}{2} + \frac{3}{4}p^4 + c \qquad (2) \end{aligned}$$

p is elminated between (1,2) to obtain the final solution. From (2) there are 4 roots for p. For example, looking at the first root

$$p_1 = \frac{1}{3}\sqrt{-3+3\sqrt{1-12c+12y}}$$

Substituting this in (1) gives one solution to the ode as

$$x = 1 + \left(\frac{1}{3}\sqrt{-3+3\sqrt{1-12c+12y}}\right) + \left(\frac{1}{3}\sqrt{-3+3\sqrt{1-12c+12y}}\right)^3$$

There are 3 more solutions.

2 References

1. Nonlinear ordinary differential equations in transport processes. William F. Ames. Academic press 1968. page 41.
2. Differential equations by George Boole. 1865. page 133.