Mapping the system function from the s-plane to the z-plane in the presence of multiple order poles.

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Given $H(s)$ of order N with all its poles p_i being distinct, it can be expressed in terms of partial fraction expansion in the form of $H(s) = \sum_{k=1}^{N} \frac{A_k}{s-p_k}$ and the resulting $H(z)$ can be found to be $\sum_{k=1}^{N} \frac{zA_k}{z-e^{p_k T}}$ where T is the sampling period.

In the case when $H(s)$ contains a pole q of order 2, then $H(s)$ can be written as $\left(\sum_{k=1}^{N-2} \frac{A_k}{s-p_k}\right) + \frac{A_q}{(s-q)^2}$ and the resulting $H(z)$ can be found to be $\left(\sum_{k=1}^{N-2} \frac{zA_k}{z-e^{p_k T}}\right) + \frac{Tze^{qT}}{(e^{qT}-z)^2}$.

In the case when $H(s)$ contains a pole q of order 3, then $H(s)$ can be written as $\left(\sum_{k=1}^{N-3} \frac{A_k}{s-p_k}\right) + \frac{A_q}{(s-q)^3}$ and the resulting $H(z)$ can be found to be $\left(\sum_{k=1}^{N-3} \frac{zA_k}{z-e^{p_k T}}\right) + \left(-\frac{e^{2qT}T^2z+e^{qT}T^2z^2}{2(e^{qT}-z)^3}\right)$.

The following table was generated in order to obtain the general formula. This table below shows only the part of $H(z)$ due to the multiple order pole.

n pole order	$H(z)$
2	$\frac{Tze^{qT}}{(e^{qT}-z)^2}$
3	$-\frac{e^{2qT}T^2z+e^{qT}T^2z^2}{2(e^{qT}-z)^3}$
4	$\frac{e^{3qT}T^3z+4e^{2qT}T^3z^2+e^{qT}T^3z^3}{6(e^{qT}-z)^4}$
5	$\frac{-e^{4qT}T^4z-11e^{3qT}T^4z^2-11e^{2qT}T^4z^3-e^{qT}T^4z^4}{24(e^{qT}-z)^5}$
6	$\frac{e^{5qT}T^5z+26e^{4qT}T^5z^2+66e^{3qT}T^5z^3+26e^{2qT}T^5z^4+e^{qT}T^5z^5}{120(e^{qT}-z)^6}$

It is easy to see that the denominator of $H(z)$ has the general form $(n-1)!\left(e^{qT}-z\right)^n$ where n is the pole order, the hard part is to find the general formula for the numerator. The following table is a rewrite of the above table, where only the numerator is show, and e^{qT} was written as A to make it easier to see the general pattern

n pole order	numerator of $H(z)$
2	$(-1)^n (AT) z$
3	$(-1)^n \left[(AT)^2 z - A(Tz)^2\right]$
4	$(-1)^n \left[(AT)^3 z + 4A^2T^3z^2 + A(Tz)^3\right]$
5	$(-1)^n \left[(AT)^4 z - 11A^3T^4z^2 - 11A^2T^4z^3 - A(Tz)^4\right]$
6	$(-1)^n \left[(AT)^5 z + 26A^4T^5z^2 + 66A^3T^5z^3 + 26A^2T^5z^4 + A(Tz)^5\right]$

I am trying to determine the general formula to generate the above. This seems to involve some combination of binomial coefficient. But so far, I did not find the general formula.

1 References

1. Digital signal processing, by Oppenheim and Scafer, page 201
2. Mathematica software version 7