Symbolic generating of system equations for 2D regular grid for solving Laplace equation using finite difference method

By Nasser M. Abbasi

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Introduction

When solving $\partial_{x,x} u + \partial_{y,y} u = f(x, y)$ using finite difference method, in order to make it easy to see the internal structure of the A matrix using the standard 5 points Laplacian scheme, the following is a small function which generates the symbolic format of these equations for a given N, the number of grid points on one edge. At the end of this note, the system equations are generated for N=4,5,6,7,8. One can see the form of the A matrix with the dominant diagonal and the corresponding bands. It is mostly a sparse matrix.

The indexing method used is that described in the class.

- **Define U at each grid point**

In[59]:=
```
n = 4;
u = Array[U## &, {n, n}, {0, 0}];
Style[MatrixForm@Reverse@u, 18]
```

Out[61]=

$$\begin{pmatrix} U_{3,0} & U_{3,1} & U_{3,2} & U_{3,3} \\ U_{2,0} & U_{2,1} & U_{2,2} & U_{2,3} \\ U_{1,0} & U_{1,1} & U_{1,2} & U_{1,3} \\ U_{0,0} & U_{0,1} & U_{0,2} & U_{0,3} \end{pmatrix}$$

- **Define the force vector**

In[62]:=
```
f = Array[F## &, {n, n}, {0, 0}];
Style[MatrixForm@Reverse@f, 18]
```

Out[63]=

$$\begin{pmatrix} F_{3,0} & F_{3,1} & F_{3,2} & F_{3,3} \\ F_{2,0} & F_{2,1} & F_{2,2} & F_{2,3} \\ F_{1,0} & F_{1,1} & F_{1,2} & F_{1,3} \\ F_{0,0} & F_{0,1} & F_{0,2} & F_{0,3} \end{pmatrix}$$

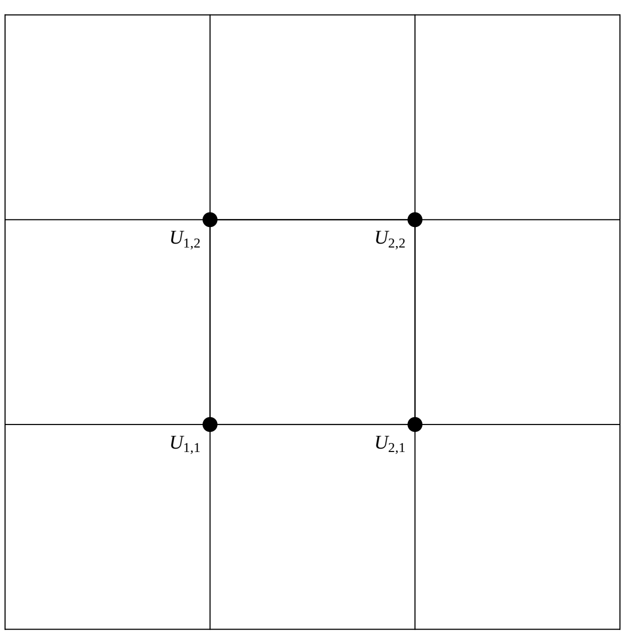
In[64]:=
```
Needs["Notation`"]
Notation[ f_{i_,j_} ⟺ f[[i_ + 1, j_ + 1]] ]
Notation[ u_{i_,j_} ⟺ u[[i_ + 1, j_ + 1]] ]
```

In[67]:=
```
eq[u_, i_, j_, h_] := Module[{},
   (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4 u_{i,j}) / h^2
  ];
```

- **Draw the grid with the unknown above at each point**

In[68]:=
```
nRow = 4; nCol = 4;
makeGrid[nRow_, nCol_, u_] := Module[{i, j},
  internalGrid = Table[{
      Line[{{i - 1, j}, {i + 1, j}}],
      Line[{{i, j - 1}, {i, j + 1}}],
      {PointSize[Large], Point[{i, j}]},
      Text[u_{i,j}, {i, j}, {1.5, 1.5}]
     },
     {j, 1, nCol - 2}, {i, 1, nRow - 2}
    ];
  boundary = {
    Line[{{0, 0}, {nCol - 1, 0}}],
    Line[{{0, 0}, {0, nRow - 1}}],
    Line[{{nCol - 1, nRow - 1}, {nCol - 1, 0}}],
    Line[{{nCol - 1, nRow - 1}, {0, nRow - 1}}]};
  Graphics[boundary ~Join~ internalGrid]
 ]
makeGrid[nRow, nCol, u]
```

Out[70]=



Generate the discrete equations at each of the internal grid points

In[71]:=
```
n = 4;
Style[MatrixForm[eqs = Flatten@Table[eq[u, i, j, h] == f_{i,j}, {j, n - 2}, {i, n - 2}]], 22]
```

Out[72]=

$$\begin{pmatrix} \frac{U_{0,1}+U_{1,0}-4\,U_{1,1}+U_{1,2}+U_{2,1}}{h^2} == F_{1,1} \\ \frac{U_{1,1}+U_{2,0}-4\,U_{2,1}+U_{2,2}+U_{3,1}}{h^2} == F_{2,1} \\ \frac{U_{0,2}+U_{1,1}-4\,U_{1,2}+U_{1,3}+U_{2,2}}{h^2} == F_{1,2} \\ \frac{U_{1,2}+U_{2,1}-4\,U_{2,2}+U_{2,3}+U_{3,2}}{h^2} == F_{2,2} \end{pmatrix}$$

- **List the unknowns**

In[73]:=
```
Style[MatrixForm[unknowns = Flatten@Table[u_{i,j}, {j, n - 2}, {i, n - 2}]], 22]
```

Out[73]=

$$\begin{pmatrix} U_{1,1} \\ U_{2,1} \\ U_{1,2} \\ U_{2,2} \end{pmatrix}$$

- **Generate the equations of the form AU=F**

In[80]:=
```
unknowns = Flatten@Table[u_{i,j}, {j, n - 2}, {i, n - 2}];
{b, A} = Normal@CoefficientArrays[eqs, unknowns];
Style[Grid[{{1/h^2, MatrixForm[h^2 A], MatrixForm@unknowns, "=", MatrixForm@b}}], 22]
```

Out[82]=

$$\frac{1}{h^2} \begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} U_{1,1} \\ U_{2,1} \\ U_{1,2} \\ U_{2,2} \end{pmatrix} = \begin{pmatrix} -F_{1,1} + \frac{U_{0,1}}{h^2} + \frac{U_{1,0}}{h^2} \\ -F_{2,1} + \frac{U_{2,0}}{h^2} + \frac{U_{3,1}}{h^2} \\ -F_{1,2} + \frac{U_{0,2}}{h^2} + \frac{U_{1,3}}{h^2} \\ -F_{2,2} + \frac{U_{2,3}}{h^2} + \frac{U_{3,2}}{h^2} \end{pmatrix}$$

- **For homegenous Boundary conditions**

In[83]:=
```
Table[u_{0,j} → 0, {j, 0, 3}] ~Join~ Table[u_{i,0} → 0, {i, 0, 3}] ~
  Join ~ Table[u_{3,j} → 0, {j, 0, 3}] ~Join~ Table[u_{i,3} → 0, {i, 0, 3}];
{b0, A} = Normal@CoefficientArrays[eqs, unknowns] /. %;
Style[Grid[{{1/h^2, MatrixForm[h^2 A], MatrixForm@unknowns, "=", MatrixForm@b0}}], 22]
```

Out[85]=

$$\frac{1}{h^2}\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} U_{1,1} \\ U_{2,1} \\ U_{1,2} \\ U_{2,2} \end{pmatrix} = \begin{pmatrix} -F_{1,1} \\ -F_{2,1} \\ -F_{1,2} \\ -F_{2,2} \end{pmatrix}$$

- **set the boundary conditions. Assume left side is U=α,Right side U=β, bottom side U=γ, top side U=η, then the above becomes**

In[86]:=
```
b = b /. Subscript[U, 0, any_] → α; (*left*)
b = b /. Subscript[U, any_, 3] → β; (*right*)
b = b /. Subscript[U, any_, 0] → γ; (*bottom*)
b = b /. Subscript[U, 3, any_] → η;(*top*)
```

- **Display the equations again**

In[90]:=
```
Style[Grid[{{1/h^2, MatrixForm[h^2 A], MatrixForm@unknowns, "=", MatrixForm@b}}], 22]
```

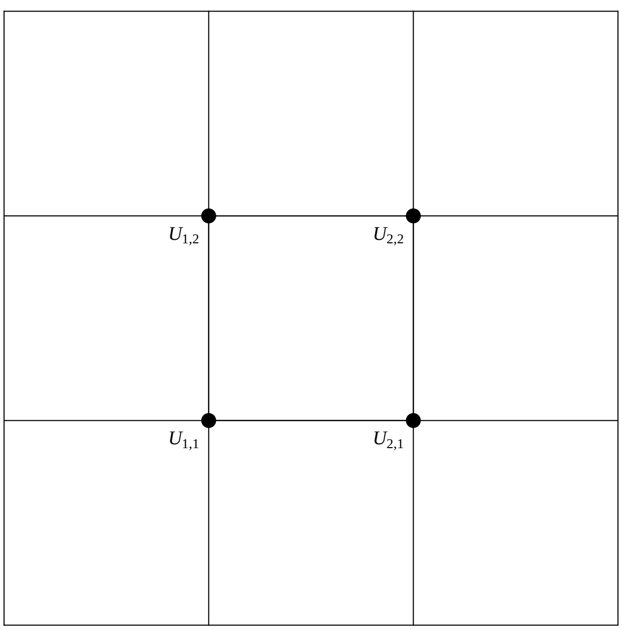
Out[90]=

$$\frac{1}{h^2}\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} U_{1,1} \\ U_{2,1} \\ U_{1,2} \\ U_{2,2} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{h^2} + \frac{\gamma}{h^2} - F_{1,1} \\ \frac{\gamma}{h^2} + \frac{\eta}{h^2} - F_{2,1} \\ \frac{\alpha}{h^2} + \frac{\beta}{h^2} - F_{1,2} \\ \frac{\beta}{h^2} + \frac{\eta}{h^2} - F_{2,2} \end{pmatrix}$$

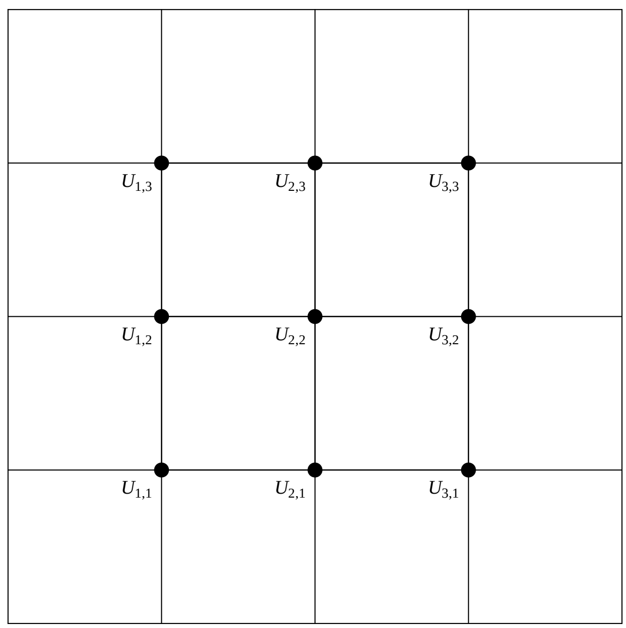
- **Now the system can be solved for the unknowns U , given the force F values.**
- **Below is the system equations generated for N=3,4,5,6,7,8. Put the above code into one function to use it all the time**

In[91]:=
```
generateSystemEquations[n_, U_, F_, h_, α_, β_, γ_, η_] :=
 Module[{m, f, i, j, eqs, unknowns, b, A, u},
  u = Array[U## &, {n, n}, {0, 0}];
  f = Array[F## &, {n, n}, {0, 0}];
  eqs = Flatten@Table[eq[u, i, j, h] == f_{i,j}, {j, n - 2}, {i, n - 2}];
  unknowns = Flatten@Table[u_{i,j}, {j, n - 2}, {i, n - 2}];
  {b, A} = Normal@CoefficientArrays[eqs, unknowns];
  b = b /. Subscript[U, 0, any_] → α; (*left*)
  b = b /. Subscript[U, any_, n - 1] → β; (*right*)
  b = b /. Subscript[U, any_, 0] → γ; (*bottom*)
  b = b /. Subscript[U, n - 1, any_] → η; (*top*)
  Print[makeGrid[n, n, u]];
  Print[Style[Row[{1/h^2, MatrixForm[h^2 A], MatrixForm@unknowns, "=", MatrixForm@b},
     Alignment → Left, ImageSize → Full], 18]]
 ]
```

In[92]:=
```
Clear[U, F, h, α, β, γ, η];
Do[{generateSystemEquations[i, U, F, h, α, β, γ, η],
  Print@Graphics[{Thick, Line[{{0, 0}, {10, 0}}]}]}, {i, 4, 6}]
```

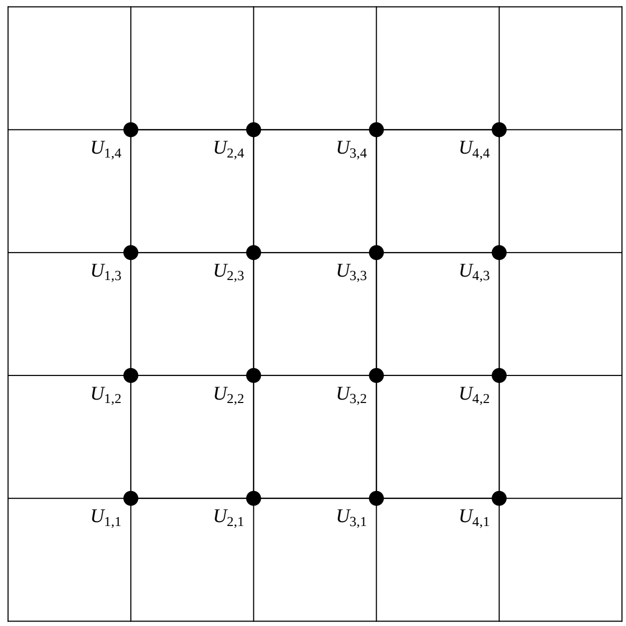


$$\frac{1}{h^2}\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix}\begin{pmatrix} U_{1,1} \\ U_{2,1} \\ U_{1,2} \\ U_{2,2} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{h^2} + \frac{\gamma}{h^2} - F_{1,1} \\ \frac{\gamma}{h^2} + \frac{\eta}{h^2} - F_{2,1} \\ \frac{\alpha}{h^2} + \frac{\beta}{h^2} - F_{1,2} \\ \frac{\beta}{h^2} + \frac{\eta}{h^2} - F_{2,2} \end{pmatrix}$$



$$\frac{1}{h^2}\begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix}$$

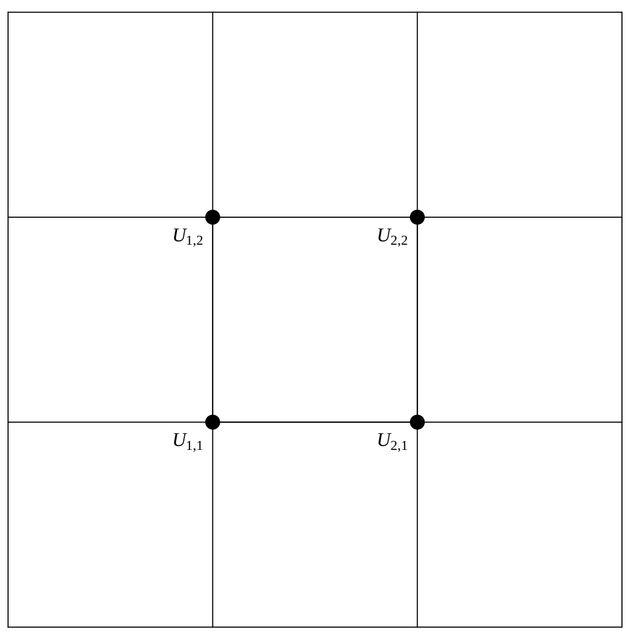
$$\begin{pmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{1,2} \\ U_{2,2} \\ U_{3,2} \\ U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{h^2} + \frac{\gamma}{h^2} - F_{1,1} \\ \frac{\gamma}{h^2} - F_{2,1} \\ \frac{\gamma}{h^2} + \frac{\eta}{h^2} - F_{3,1} \\ \frac{\alpha}{h^2} - F_{1,2} \\ -F_{2,2} \\ \frac{\eta}{h^2} - F_{3,2} \\ \frac{\alpha}{h^2} + \frac{\beta}{h^2} - F_{1,3} \\ \frac{\beta}{h^2} - F_{2,3} \\ \frac{\beta}{h^2} + \frac{\eta}{h^2} - F_{3,3} \end{pmatrix}$$



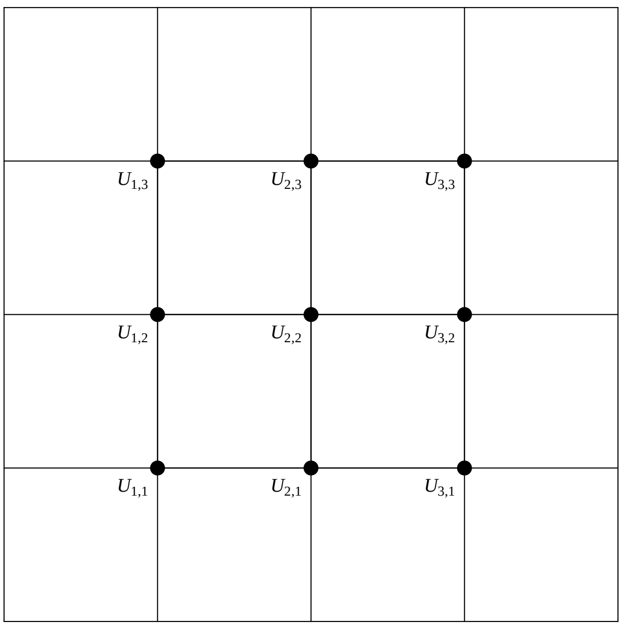
$$\frac{1}{h^2}\left(\begin{array}{ccccccccccccc}
-4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right.$$

- **For homogenous boundary conditions**

In[94]:=
```
generateSystemEquations[n_, U_, F_, h_, α_, β_, γ_, η_] :=
 Module[{m, f, i, j, eqs, unknowns, b, A, u},
  u = Array[U## &, {n, n}, {0, 0}];
  f = Array[F## &, {n, n}, {0, 0}];
  eqs = Flatten@Table[eq[u, i, j, h] == f_{i,j}, {j, n - 2}, {i, n - 2}];
  unknowns = Flatten@Table[u_{i,j}, {j, n - 2}, {i, n - 2}];
  {b, A} = Normal@CoefficientArrays[eqs, unknowns];
  b = b /. Subscript[U, 0, any_] → 0; (*left*)
  b = b /. Subscript[U, any_, n - 1] → 0; (*right*)
  b = b /. Subscript[U, any_, 0] → 0; (*bottom*)
  b = b /. Subscript[U, n - 1, any_] → 0; (*top*)
  Print[makeGrid[n, n, u]];
  Print[Style[Row[{1/h^2, MatrixForm[h^2 A], MatrixForm@unknowns, "=", MatrixForm@b},
     Alignment → Left, ImageSize → Full], 18]]
 ]
Do[{generateSystemEquations[i, U, F, h, α, β, γ, η],
  Print@Graphics[{Thick, Line[{{0, 0}, {10, 0}}]}]}, {i, 4, 6}]
```

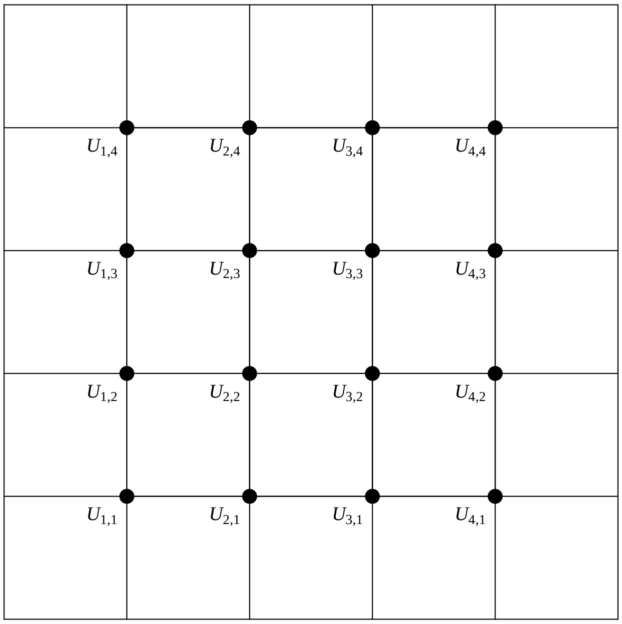


$$\frac{1}{h^2}\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} U_{1,1} \\ U_{2,1} \\ U_{1,2} \\ U_{2,2} \end{pmatrix} = \begin{pmatrix} -F_{1,1} \\ -F_{2,1} \\ -F_{1,2} \\ -F_{2,2} \end{pmatrix}$$



$$\frac{1}{h^2}\begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix}$$

$$\begin{pmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{1,2} \\ U_{2,2} \\ U_{3,2} \\ U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{pmatrix} = \begin{pmatrix} -F_{1,1} \\ -F_{2,1} \\ -F_{3,1} \\ -F_{1,2} \\ -F_{2,2} \\ -F_{3,2} \\ -F_{1,3} \\ -F_{2,3} \\ -F_{3,3} \end{pmatrix}$$



$$\frac{1}{h^2}\left(\begin{array}{ccccccccccccc}
-4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right.$$