Finding the B matrix for constant strain triangle

Nasser M. Abbasi

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1 The problem to solve

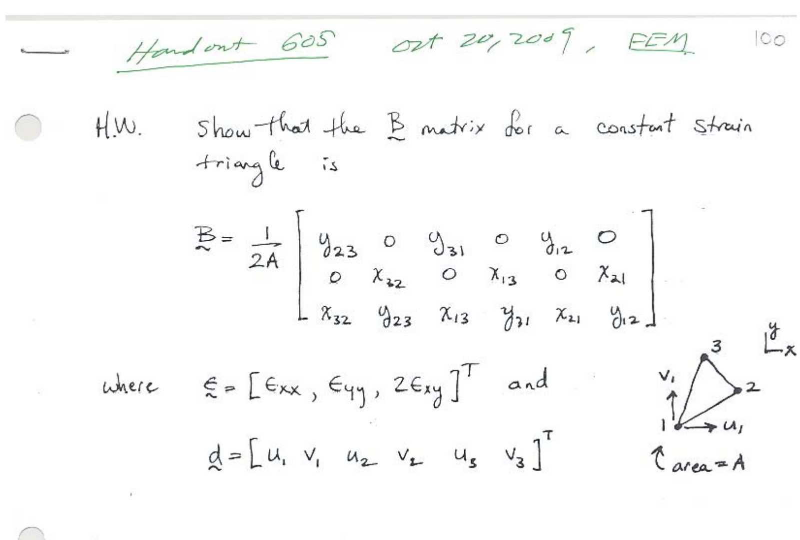


Figure 1: the Problem to solve

2 Analytical derivation

The problem is first solve for scalar field θ with the interpolating polynomial $a_1 + a_2 x + a_3 y$. Writing

$$\theta = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{1}$$

Evaluating the field θ at each node gives

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Hence

$$\begin{aligned} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} &= \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \end{aligned} \tag{2}$$

Where Δ is the determinant $x_1 y_2 - x_2 y_1 - x_1 y_3 + x_3 y_1 + x_2 y_3 - x_3 y_2$. Substituting (2) into (1) gives

$$\begin{aligned} \theta &= \overbrace{\frac{1}{\Delta} \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \\ &= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \end{aligned} \tag{3}$$

Where

$$\begin{aligned} N_1 &= \frac{1}{\Delta}[x_2 y_3 - x_3 y_2 + x(y_2 - y_3) + y(x_3 - x_2)] \\ N_2 &= \frac{1}{\Delta}[x_3 y_1 - x_1 y_3 + x(y_3 - y_1) + y(x_1 - x_3)] \\ N_3 &= \frac{1}{\Delta}[x_1 y_2 - x_2 y_1 + x(y_1 - y_2) + y(x_2 - x_1)] \end{aligned} \tag{4}$$

For constant stress triangle, the field is a vector field. Hence replacing θ with $\begin{bmatrix} u \\ v \end{bmatrix}$ equation (3) becomes

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

From the above

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3 \\ \frac{\partial v}{\partial y} &= \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3 \end{aligned}$$

Hence

$$\begin{aligned} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} &= \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial}{\partial x} u \\ \frac{\partial}{\partial y} v \\ \frac{\partial}{\partial y} u + \frac{\partial}{\partial x} v \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3 \\ \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3 \\ \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3 \end{bmatrix} \\ &= \overbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}}^{B} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} \end{aligned} \tag{5}$$

From (4) all of the $\frac{\partial N_i}{\partial x}, \frac{\partial N_j}{\partial y}$ terms are evaluated. Substituting the result into (5) gives the B matrix

$$\begin{aligned} \frac{\partial N_1}{\partial x} &= \frac{1}{\Delta}(y_2 - y_3) \\ \frac{\partial N_2}{\partial x} &= \frac{1}{\Delta}(y_3 - y_1) \\ \frac{\partial N_3}{\partial x} &= \frac{1}{\Delta}(y_1 - y_2) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N_1}{\partial y} &= \frac{1}{\Delta}(x_3 - x_2) \\ \frac{\partial N_2}{\partial y} &= \frac{1}{\Delta}(x_1 - x_3) \\ \frac{\partial N_3}{\partial y} &= \frac{1}{\Delta}(x_2 - x_1) \end{aligned}$$

Hence B becomes

$$B = \frac{1}{\Delta}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix} \tag{6}$$

Letting $y_i - y_j = y_{ij}$ and $x_i - x_j = x_{ij}$, the above becomes

$$B = \frac{1}{\Delta}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

But the area of triangle is given by

$$A = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix}$$

$$\begin{aligned} 2A &= (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1) \\ &= x_1 y_2 - x_2 y_1 - x_1 y_3 + x_3 y_1 + x_2 y_3 - x_3 y_2 \end{aligned}$$

And the determinant Δ was found above to be $x_1 y_2 - x_2 y_1 - x_1 y_3 + x_3 y_1 + x_2 y_3 - x_3 y_2$, hence

$$2A = \Delta$$

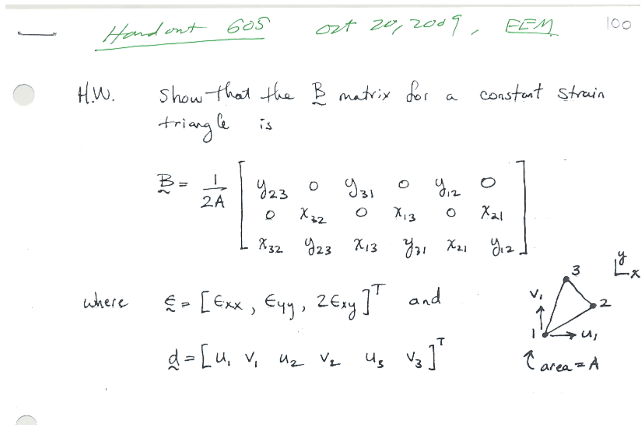
Substituting the above into B found above in equation (6) gives

$$B = \frac{1}{2A}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix}$$

3 Verification using Mathematica

The problem to solve

by Nasser M. Abbasi (oct 2009)



Handout 605 oct 20, 2009, EEM 100

H.W. Show that the $\underset{\sim}{B}$ matrix for a constant strain triangle is

$$\underset{\sim}{B} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

where $\underset{\sim}{\epsilon} = [\epsilon_{xx}, \epsilon_{yy}, 2\epsilon_{xy}]^T$ and

$\underset{\sim}{d} = [u_1 \ v_1 \ u_2 \ v_2 \ u_3 \ v_3]^T$

In this solution, I start directly by solving for the vector field $\{u, v\}$ and starting from the general degrees of freedom, and from it by matrix inversion, find the shape function matrix N (in terms of nodal degrees of freedom). This involves inversting a 6 by 6 matrix. But Ok, I am using a computer. By hand, I would use the method I showed in the analytical note part of this assignment which involves inverting only a 3 by 3 matrix.

```
Needs["Notation`"]
nDegreeOfFreedom = 6;
nNodes = 3;
Symbolize[ u_1 ];
Symbolize[ u_2 ];
Symbolize[ u_3 ];
Symbolize[ v_1 ];
Symbolize[ v_2 ]
Symbolize[ v_3 ];
Symbolize[ a_1 ];
Symbolize[ a_2 ];
Symbolize[ a_3 ];
Symbolize[ a_4 ]
Symbolize[ a_5 ];
Symbolize[ a_6 ];
Symbolize[ x_1 ];
Symbolize[ x_2 ];
Symbolize[ x_3 ]
Symbolize[ y_1 ];
Symbolize[ y_2 ];
Symbolize[ y_3 ]
```

Start by defining the *u* and *v* trial functions (linear polynomials in x and y)

```
nodalDegreesOfFreedom = {u_1, v_1, u_2, v_2, u_3, v_3};
generalDegreesOfFreedom = {a_1, a_2, a_3, a_4, a_5, a_6};
u = a_1 + a_2 x + a_3 y;
v = a_4 + a_5 x + a_6 y;
```

set up the $u = X\,a$ equation

```
{b, xMat} = Normal[CoefficientArrays[{u, v}, generalDegreesOfFreedom]];
Print[ToString[MatrixForm[{{"u"}, {"v"}}], FormatType → TraditionalForm] <> "=",
  ToString[MatrixForm[xMat], FormatType → StandardForm] <>
   ToString[MatrixForm[generalDegreesOfFreedom], FormatType → StandardForm]]
```

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}$$

Now find the shape functions. Start by expression nodal unknowns in terms of nodal coordinates

```
u1 = u /. {x → x1, y → y1}
v1 = v /. {x → x1, y → y1}
u2 = u /. {x → x2, y → y2}
v2 = v /. {x → x2, y → y2}
u3 = u /. {x → x3, y → y3}
v3 = v /. {x → x3, y → y3}
```

$a_1 + a_2 x_1 + a_3 y_1$

$a_4 + a_5 x_1 + a_6 y_1$

$a_1 + a_2 x_2 + a_3 y_2$

$a_4 + a_5 x_2 + a_6 y_2$

$a_1 + a_2 x_3 + a_3 y_3$

$a_4 + a_5 x_3 + a_6 y_3$

Write the $u = A\,a$ equation

```
{b, aMat} = Normal[CoefficientArrays[{u1, v1, u2, v2, u3, v3}, generalDegreesOfFreedom]];

Print[ToString[MatrixForm[Transpose[{{"u1", "v1", "u2", "v2", "u3", "v3"}}]],
         FormatType → TraditionalForm] <> "=",
   ToString[MatrixForm[aMat], FormatType → StandardForm] <>
      ToString[MatrixForm[generalDegreesOfFreedom], FormatType → StandardForm]]
```

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}$$

Find $a = A^{-1}\,u$ from the above by matrix inversion

```
shapeFunctions = xMat.Inverse[aMat];
```

Now find the B matrix from the above N matrix by multiplying the N matrix by the following differetial operartors matrix

```
oper = {{D[#1, x] &, 0 &}, {0 &, D[#1, y] &}, {D[#1, y] &, D[#1, x] &}};
Print[ToString[MatrixForm[oper], FormatType → TraditionalForm]];
```

$$\begin{pmatrix} \frac{\partial \#1}{\partial x} \,\& & 0\,\& \\ 0\,\& & \frac{\partial \#1}{\partial y} \,\& \\ \frac{\partial \#1}{\partial y} \,\& & \frac{\partial \#1}{\partial x} \,\& \end{pmatrix}$$

Now find B = oper * N

```
bMat = Inner[#1[#2] &, oper, shapeFunctions, Plus];
```

```
(bMat = Simplify[Assuming[Element[{y1, y2, y3, x1, x2, x3}, Reals], bMat]]) // MatrixForm
```

$$\begin{pmatrix} \frac{-y_2+y_3}{x_3(-y_1+y_2)+x_2(y_1-y_3)+x_1(-y_2+y_3)} & 0 & \frac{y_1-y_3}{x_2 y_1-x_3 y_1-x_1 y_2+x_3 y_2+x_1 y_3-x_2 y_3} & 0 \\ 0 & \frac{x_2-x_3}{x_2 y_1-x_3 y_1-x_1 y_2+x_3 y_2+x_1 y_3-x_2 y_3} & 0 & \frac{x_1-x_3}{x_3(y_1-y_2)+x_1(y_2-y_3)+x_2} \\ \frac{x_2-x_3}{x_2 y_1-x_3 y_1-x_1 y_2+x_3 y_2+x_1 y_3-x_2 y_3} & \frac{-y_2+y_3}{x_3(-y_1+y_2)+x_2(y_1-y_3)+x_1(-y_2+y_3)} & \frac{x_1-x_3}{x_3(y_1-y_2)+x_1(y_2-y_3)+x_2(-y_1+y_3)} & \frac{y_1-y_3}{x_2 y_1-x_3 y_1-x_1 y_2+x_3 y_2+x_1} \end{pmatrix}$$

Factor the determinant term from the above to the outside.

```
den = Denominator[bMat[[1, 1]]];
bMat = bMat * den;
Print[ToString[1 / den, FormatType → StandardForm] <>
    ToString[MatrixForm[Simplify[bMat]], FormatType → StandardForm]]
```

$$\frac{1}{x_3(-y_1+y_2)+x_2(y_1-y_3)+x_1(-y_2+y_3)}\begin{pmatrix} -y_2+y_3 & 0 & y_1-y_3 & 0 & -y_1+y_2 & 0 \\ 0 & x_2-x_3 & 0 & -x_1+x_3 & 0 & x_1-x_2 \\ x_2-x_3 & -y_2+y_3 & -x_1+x_3 & y_1-y_3 & x_1-x_2 & -y_1+y_2 \end{pmatrix}$$

But area of triangle is

```
area = (1 / 2) Cross[{x2 - x1, y2 - y1, 0}, {x3 - x1, y3 - y1, 0}][[3]]
```

$$\frac{1}{2}(-x_2 y_1+x_3 y_1+x_1 y_2-x_3 y_2-x_1 y_3+x_2 y_3)$$

Hence B matrix becomes

```
Panel[Style[ToString[1 / "2 area", FormatType → StandardForm] <>
    ToString[MatrixForm[Simplify[bMat]], FormatType → StandardForm], 18]]
```

$$\frac{1}{2\ \text{area}}\begin{pmatrix} -y_2+y_3 & 0 & y_1-y_3 & 0 & -y_1+y_2 & 0 \\ 0 & x_2-x_3 & 0 & -x_1+x_3 & 0 & x_1-x_2 \\ x_2-x_3 & -y_2+y_3 & -x_1+x_3 & y_1-y_3 & x_1-x_2 & -y_1+y_2 \end{pmatrix}$$

```
finalB = 1/(2 area) bMat;
```

$$\begin{pmatrix} \frac{y_2-y_3}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & 0 & \frac{y_3-y_1}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & 0 \\ 0 & \frac{-x_2+x_3}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & 0 & \frac{-x_3+x_1}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y} \\ \frac{-x_2+x_3}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & \frac{y_2-y_3}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & \frac{-x_3+x_1}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & \frac{y_3-y_1}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y} \end{pmatrix}$$