The problem to solve

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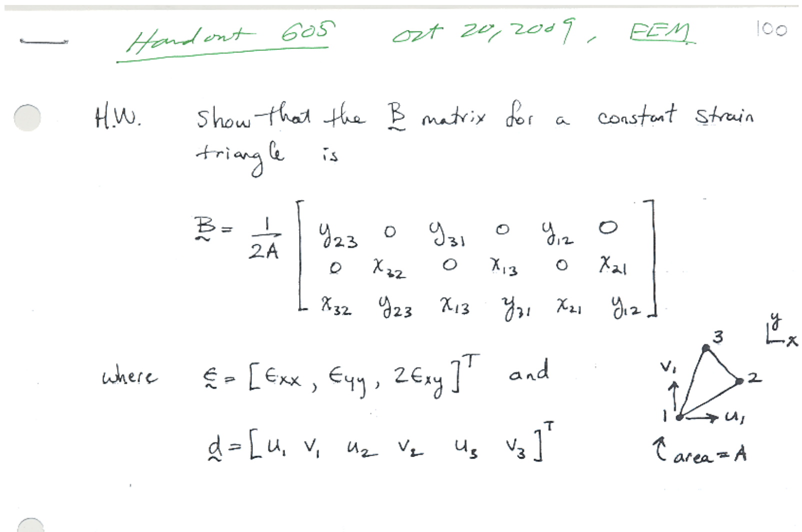

Handout 605 oct 20, 2009, FEM 100

H.W. Show that the $\underset{\sim}{B}$ matrix for a constant strain triangle is

$$\underset{\sim}{B} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

where $\underset{\sim}{\epsilon} = [\epsilon_{xx}, \epsilon_{yy}, 2\epsilon_{xy}]^T$ and

$\underset{\sim}{d} = [u_1\ v_1\ u_2\ v_2\ u_3\ v_3]^T$

In this solution, I start directly by solving for the vector field {*u*, *v*} and starting from the general degrees of freedom, and from it by matrix inversion, find the shape function matrix N (in terms of nodal degrees of freedom). This involves inversting a 6 by 6 matrix. But Ok, I am using a computer. By hand, I would use the method I showed in the analytical note part of this assignment which involves inverting only a 3 by 3 matrix.

```
Needs["Notation`"]
nNodes = 3;
nDegreeOfFreedom = 6;
Symbolize[ u₁ ];
Symbolize[ u₂ ];
Symbolize[ u₃ ];
Symbolize[ v₁ ];
Symbolize[ v₂ ]
Symbolize[ v₃ ];
Symbolize[ a₁ ];
Symbolize[ a₂ ];
Symbolize[ a₃ ];
Symbolize[ a₄ ]
Symbolize[ a₅ ];
Symbolize[ a₆ ];
Symbolize[ x₁ ];
Symbolize[ x₂ ];
Symbolize[ x₃ ]
Symbolize[ y₁ ];
Symbolize[ y₂ ];
Symbolize[ y₃ ]
```

Start by defining the *u* and *v* trial functions (linear polynomials in x and y)

```
nodalDegreesOfFreedom = {u₁, v₁, u₂, v₂, u₃, v₃};
generalDegreesOfFreedom = {a₁, a₂, a₃, a₄, a₅, a₆};
u = a₁ + a₂ x + a₃ y;
v = a₄ + a₅ x + a₆ y;
```

set up the $u = X\,a$ equation

```
{b, xMat} = Normal[CoefficientArrays[{u, v}, generalDegreesOfFreedom]];
Print[ToString[MatrixForm[{{"u"}, {"v"}}], FormatType → TraditionalForm] <> "=",
  ToString[MatrixForm[xMat], FormatType → StandardForm] <>
   ToString[MatrixForm[generalDegreesOfFreedom], FormatType → StandardForm]]
```

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}$$

Now find the shape functions. Start by expression nodal unknowns in terms of nodal coordinates

```
u₁ = u /. {x → x₁, y → y₁}
v₁ = v /. {x → x₁, y → y₁}
u₂ = u /. {x → x₂, y → y₂}
v₂ = v /. {x → x₂, y → y₂}
u₃ = u /. {x → x₃, y → y₃}
v₃ = v /. {x → x₃, y → y₃}
```

$a_1 + a_2 x_1 + a_3 y_1$

$a_4 + a_5 x_1 + a_6 y_1$

$a_1 + a_2 x_2 + a_3 y_2$

$a_4 + a_5 x_2 + a_6 y_2$

$a_1 + a_2 x_3 + a_3 y_3$

$a_4 + a_5 x_3 + a_6 y_3$

Write the $u = A\,a$ equation

```
{b, aMat} = Normal[CoefficientArrays[{u₁, v₁, u₂, v₂, u₃, v₃}, generalDegreesOfFreedom]];

Print[ToString[MatrixForm[Transpose[{{"u₁", "v₁", "u₂", "v₂", "u₃", "v₃"}}]],
      FormatType → TraditionalForm] <> "=",
  ToString[MatrixForm[aMat], FormatType → StandardForm] <>
    ToString[MatrixForm[generalDegreesOfFreedom], FormatType → StandardForm]]
```

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}$$

Find $a = A^{-1}\,u$ from the above by matrix inversion

```
shapeFunctions = xMat.Inverse[aMat];
```

Now find the B matrix from the above N matrix by multiplying the N matrix by the following differetial operartors matrix

```
oper = {{D[#1, x] &, 0 &}, {0 &, D[#1, y] &}, {D[#1, y] &, D[#1, x] &}};
Print[ToString[MatrixForm[oper], FormatType → TraditionalForm]];
```

$$\begin{pmatrix} \frac{\partial \#1}{\partial x}\,\& & 0\,\& \\ 0\,\& & \frac{\partial \#1}{\partial y}\,\& \\ \frac{\partial \#1}{\partial y}\,\& & \frac{\partial \#1}{\partial x}\,\& \end{pmatrix}$$

Now find B = oper * N

```
bMat = Inner[#1[#2] &, oper, shapeFunctions, Plus];
```

```
(bMat = Simplify[Assuming[Element[{y₁, y₂, y₃, x₁, x₂, x₃}, Reals], bMat]]) // MatrixForm
```

$$\begin{pmatrix} \frac{-y_2+y_3}{x_3\,(-y_1+y_2)+x_2\,(y_1-y_3)+x_1\,(-y_2+y_3)} & 0 & \frac{y_1-y_3}{x_2\,y_1-x_3\,y_1-x_1\,y_2+x_3\,y_2+x_1\,y_3-x_2\,y_3} & 0 \\ 0 & \frac{x_2-x_3}{x_2\,y_1-x_3\,y_1-x_1\,y_2+x_3\,y_2+x_1\,y_3-x_2\,y_3} & 0 & \frac{x_1-x_3}{x_3\,(y_1-y_2)+x_1\,(y_2-y_3)+x_2} \\ \frac{x_2-x_3}{x_2\,y_1-x_3\,y_1-x_1\,y_2+x_3\,y_2+x_1\,y_3-x_2\,y_3} & \frac{-y_2+y_3}{x_3\,(-y_1+y_2)+x_2\,(y_1-y_3)+x_1\,(-y_2+y_3)} & \frac{x_1-x_3}{x_3\,(y_1-y_2)+x_1\,(y_2-y_3)+x_2\,(-y_1+y_3)} & \frac{y_1-y_3}{x_2\,y_1-x_3\,y_1-x_1\,y_2+x_3\,y_2+x_1} \end{pmatrix}$$

Factor the determinant term from the above to the outside.

```
den = Denominator[bMat[[1, 1]]] ;
bMat = bMat * den;
Print[ToString[1 / den, FormatType → StandardForm] <>
    ToString[MatrixForm[Simplify[bMat]], FormatType → StandardForm]]
```

$$\frac{1}{x_3\,(-y_1+y_2)+x_2\,(y_1-y_3)+x_1\,(-y_2+y_3)}\begin{pmatrix} -y_2+y_3 & 0 & y_1-y_3 & 0 & -y_1+y_2 & 0 \\ 0 & x_2-x_3 & 0 & -x_1+x_3 & 0 & x_1-x_2 \\ x_2-x_3 & -y_2+y_3 & -x_1+x_3 & y_1-y_3 & x_1-x_2 & -y_1+y_2 \end{pmatrix}$$

But area of triangle is

```
area = (1 / 2) Cross[{x₂ - x₁, y₂ - y₁, 0}, {x₃ - x₁, y₃ - y₁, 0}][[3]]
```

$$\frac{1}{2}\,(-x_2\,y_1+x_3\,y_1+x_1\,y_2-x_3\,y_2-x_1\,y_3+x_2\,y_3)$$

Hence B matrix becomes

```
Panel[Style[ToString[1 / "2 area", FormatType → StandardForm] <>
     ToString[MatrixForm[Simplify[bMat]], FormatType → StandardForm], 18]]
```

$$\frac{1}{2\ \text{area}}\begin{pmatrix} -y_2+y_3 & 0 & y_1-y_3 & 0 & -y_1+y_2 & 0 \\ 0 & x_2-x_3 & 0 & -x_1+x_3 & 0 & x_1-x_2 \\ x_2-x_3 & -y_2+y_3 & -x_1+x_3 & y_1-y_3 & x_1-x_2 & -y_1+y_2 \end{pmatrix}$$

```
finalB = 1/(2 area) bMat;
```

$$\begin{pmatrix} \frac{y_2-y_3}{-x_3\,y_2+x_2\,y_3+y_2\,x_1-y_3\,x_1-x_2\,y_1+x_3\,y_1} & 0 & \frac{y_3-y_1}{-x_3\,y_2+x_2\,y_3+y_2\,x_1-y_3\,x_1-x_2\,y_1+x_3\,y_1} & 0 \\ 0 & \frac{-x_2+x_3}{-x_3\,y_2+x_2\,y_3+y_2\,x_1-y_3\,x_1-x_2\,y_1+x_3\,y_1} & 0 & \frac{-x_3+x_1}{-x_3\,y_2+x_2\,y_3+y_2\,x_1-y_3\,x_1-x_2\,y} \\ \frac{-x_2+x_3}{-x_3\,y_2+x_2\,y_3+y_2\,x_1-y_3\,x_1-x_2\,y_1+x_3\,y_1} & \frac{y_2-y_3}{-x_3\,y_2+x_2\,y_3+y_2\,x_1-y_3\,x_1-x_2\,y_1+x_3\,y_1} & \frac{-x_3+x_1}{-x_3\,y_2+x_2\,y_3+y_2\,x_1-y_3\,x_1-x_2\,y_1+x_3\,y_1} & \frac{y_3-y_1}{-x_3\,y_2+x_2\,y_3+y_2\,x_1-y_3\,x_1-x_2\,y} \end{pmatrix}$$