Finding the B matrix for constant strain triangle

Nasser M. Abbasi

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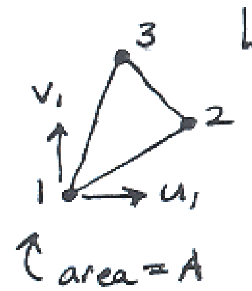
Handout 605 oct 20, 2009, FEM

H.W. Show that the $\underset{\sim}{B}$ matrix for a constant strain triangle is

$$\underset{\sim}{B} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$



where $\underset{\sim}{\epsilon} = [\epsilon_{xx}, \epsilon_{yy}, 2\epsilon_{xy}]^T$ and

$$\underset{\sim}{d} = [u_1 \; v_1 \; u_2 \; v_2 \; u_3 \; v_3]^T$$

1 Solution

The problem is first solve for scalar field θ with the interpolating polynomial $a_1 + a_2x + a_3y$. Writing

$$\theta = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{1}$$

Evaluating the field θ at each node gives

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Hence

$$\begin{aligned} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} &= \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \end{aligned} \tag{2}$$

Where Δ is the determinant $x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2$
Substituting (2) into (1) gives

$$\begin{aligned} \theta &= \overbrace{\frac{1}{\Delta} \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}}^{\mathbf{N}} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \\ &= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \end{aligned} \tag{3}$$

Where

$$\begin{aligned} N_1 &= \frac{1}{\Delta} \left[x_2y_3 - x_3y_2 + x\left(y_2 - y_3\right) + y\left(x_3 - x_2\right) \right] \\ N_2 &= \frac{1}{\Delta} \left[x_3y_1 - x_1y_3 + x\left(y_3 - y_1\right) + y\left(x_1 - x_3\right) \right] \\ N_3 &= \frac{1}{\Delta} \left[x_1y_2 - x_2y_1 + x\left(y_1 - y_2\right) + y\left(x_2 - x_1\right) \right] \end{aligned} \tag{4}$$

For constant stress triangle, the field is a vector field. Hence replacing θ with $\begin{bmatrix} u \\ v \end{bmatrix}$ equation (3) becomes

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

From the above

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial N_1}{\partial x}u_1 + \frac{\partial N_2}{\partial x}u_2 + \frac{\partial N_3}{\partial x}u_3 \\
\frac{\partial v}{\partial y} &= \frac{\partial N_1}{\partial y}v_1 + \frac{\partial N_2}{\partial y}v_2 + \frac{\partial N_3}{\partial y}v_3 \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= \frac{\partial N_1}{\partial y}u_1 + \frac{\partial N_2}{\partial y}u_2 + \frac{\partial N_3}{\partial y}u_3 + \frac{\partial N_1}{\partial x}v_1 + \frac{\partial N_2}{\partial x}v_2 + \frac{\partial N_3}{\partial x}v_3
\end{aligned}
$$

Hence

$$
\begin{aligned}
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} &= \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial}{\partial x}u \\ \frac{\partial}{\partial y}v \\ \frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial N_1}{\partial x}u_1 + \frac{\partial N_2}{\partial x}u_2 + \frac{\partial N_3}{\partial x}u_3 \\ \frac{\partial N_1}{\partial y}v_1 + \frac{\partial N_2}{\partial y}v_2 + \frac{\partial N_3}{\partial y}v_3 \\ \frac{\partial N_1}{\partial y}u_1 + \frac{\partial N_2}{\partial y}u_2 + \frac{\partial N_3}{\partial y}u_3 + \frac{\partial N_1}{\partial x}v_1 + \frac{\partial N_2}{\partial x}v_2 + \frac{\partial N_3}{\partial x}v_3 \end{bmatrix} \\
&= \overbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}}^{B} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}
\end{aligned} \tag{5}
$$

From (4) all of the $\frac{\partial N_i}{\partial x}, \frac{\partial N_j}{\partial y}$ terms are evaluated. Substituting the result into (5) gives the B matrix

$$
\begin{aligned}
\frac{\partial N_1}{\partial x} &= \frac{1}{\Delta}(y_2 - y_3) \\
\frac{\partial N_2}{\partial x} &= \frac{1}{\Delta}(y_3 - y_1) \\
\frac{\partial N_3}{\partial x} &= \frac{1}{\Delta}(y_1 - y_2)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N_1}{\partial y} &= \frac{1}{\Delta}(x_3 - x_2) \\
\frac{\partial N_2}{\partial y} &= \frac{1}{\Delta}(x_1 - x_3) \\
\frac{\partial N_3}{\partial y} &= \frac{1}{\Delta}(x_2 - x_1)
\end{aligned}
$$

Hence B becomes

$$B=\frac{1}{\Delta}\begin{bmatrix} y_2-y_3 & 0 & y_3-y_1 & 0 & y_1-y_2 & 0\\ 0 & x_3-x_2 & 0 & x_1-x_3 & 0 & x_2-x_1\\ x_3-x_2 & y_2-y_3 & x_1-x_3 & y_3-y_1 & x_2-x_1 & y_1-y_2 \end{bmatrix} \tag{6}$$

Letting $y_i - y_j = y_{ij}$ and $x_i - x_j = x_{ij}$, the above becomes

$$\boxed{B=\frac{1}{\Delta}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0\\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21}\\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}}$$

But the area of triangle is give by

$$\begin{aligned} A &= \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_2-x_1 & y_2-y_1 & 0\\ x_3-x_1 & y_3-y_1 & 0 \end{vmatrix}\\ 2A &= (x_2-x_1)(y_3-y_1)-(y_2-y_1)(x_3-x_1)\\ &= x_1y_2-x_2y_1-x_1y_3+x_3y_1+x_2y_3-x_3y_2 \end{aligned}$$

And the determinant Δ was found above to be $x_1y_2-x_2y_1-x_1y_3+x_3y_1+x_2y_3-x_3y_2$, hence

$$2A=\Delta$$

Substituting the above into B found above in equation (6) gives

$$B=\frac{1}{2A}\begin{bmatrix} y_2-y_3 & 0 & y_3-y_1 & 0 & y_1-y_2 & 0\\ 0 & x_3-x_2 & 0 & x_1-x_3 & 0 & x_2-x_1\\ x_3-x_2 & y_2-y_3 & x_1-x_3 & y_3-y_1 & x_2-x_1 & y_1-y_2 \end{bmatrix}$$