

# Finding the B matrix for constant strain triangle

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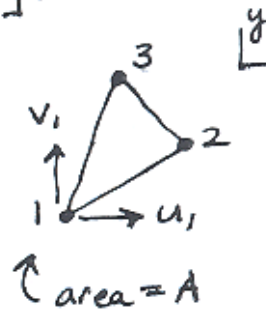
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H.W. Show that the  $B$  matrix for a constant strain triangle is

$$\underline{B} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

where  $\underline{\epsilon} = [\epsilon_{xx}, \epsilon_{yy}, 2\epsilon_{xy}]^T$  and

$$\underline{d} = [u_1, v_1, u_2, v_2, u_3, v_3]^T$$



## 1 Solution

We first solve the problem for scalar field  $\theta$  with the interpolating polynomial  $a_1 + a_2x + a_3y$ , hence we write

$$\theta = [1 \quad x \quad y] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{1}$$

Evaluating the field  $\theta$  at each node, we obtain

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Hence

$$\begin{aligned}
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} &= \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \\
&= \frac{1}{\Delta} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}
\end{aligned} \tag{2}$$

Where  $\Delta$  is the determinant  $x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2$

Substitute (2) into (1) we obtain

$$\begin{aligned}
\theta &= \frac{1}{\Delta} \begin{bmatrix} 1 & x & y \end{bmatrix} \overbrace{\begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}}^{\mathbf{N}} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \\
&= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}
\end{aligned} \tag{3}$$

Where

$$\begin{aligned}
N_1 &= \frac{1}{\Delta} [x_2y_3 - x_3y_2 + x(y_2 - y_3) + y(x_3 - x_2)] \\
N_2 &= \frac{1}{\Delta} [x_3y_1 - x_1y_3 + x(y_3 - y_1) + y(x_1 - x_3)] \\
N_3 &= \frac{1}{\Delta} [x_1y_2 - x_2y_1 + x(y_1 - y_2) + y(x_2 - x_1)]
\end{aligned} \tag{4}$$

Now, for constant stress triangle, the field is a vector field, hence replace  $\theta$  with  $\begin{bmatrix} u \\ v \end{bmatrix}$ , and equation (3) becomes

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

Hence, from the above,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3 \\
\frac{\partial v}{\partial y} &= \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3 \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3
\end{aligned}$$

Hence

$$\begin{aligned}
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} &= \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial}{\partial x} u \\ \frac{\partial}{\partial y} v \\ \frac{\partial}{\partial y} u + \frac{\partial}{\partial x} v \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3 \\ \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3 \\ \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3 \end{bmatrix} \\
&= \overbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}}^B \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} \tag{5}
\end{aligned}$$

From (4) we can evaluate all of the  $\frac{\partial N_i}{\partial x}$ ,  $\frac{\partial N_j}{\partial y}$  terms, then substitute the result into (5) to obtain the B matrix

$$\begin{aligned}
\frac{\partial N_1}{\partial x} &= \frac{1}{\Delta} (y_2 - y_3) \\
\frac{\partial N_2}{\partial x} &= \frac{1}{\Delta} (y_3 - y_1) \\
\frac{\partial N_3}{\partial x} &= \frac{1}{\Delta} (y_1 - y_2)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N_1}{\partial y} &= \frac{1}{\Delta} (x_3 - x_2) \\
\frac{\partial N_2}{\partial y} &= \frac{1}{\Delta} (x_1 - x_3) \\
\frac{\partial N_3}{\partial y} &= \frac{1}{\Delta} (x_2 - x_1)
\end{aligned}$$

Hence  $B$  becomes

$$B = \frac{1}{\Delta} \begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix} \tag{6}$$

Let  $y_i - y_j = y_{ij}$  and  $x_i - x_j = x_{ij}$ , hence the above becomes

$$B = \frac{1}{\Delta} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

But area of triangle is give by

$$\begin{aligned} A &= \frac{1}{2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} \\ 2A &= (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1) \\ &= x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2 \end{aligned}$$

But the determinant  $\Delta$  was found above to be  $x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2$ , hence

$$2A = \Delta$$

Substitute the above into  $B$  found above in equation (6), we obtain

$$B = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix}$$