Note on solving Clairaut or dAlembert first order ODE's

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1 Introduction

This note is about solving an ODE of the form

$$y(x) = G\left(x, \frac{dy}{dx}\right) \tag{1}$$

Used to solve nonlinear first order ODE's. Let

$$p = \frac{dy}{dx}$$

Then (1) becomes

$$y(x) = G(x, p) \tag{2}$$

Taking derivative w.r.t x gives

$$y'(x) = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial p}\frac{dp}{dx}$$

But $y'(x) = p$ and the above becomes

$$\begin{aligned} p &= \frac{\partial G}{\partial x} + \frac{\partial G}{\partial p}\frac{dp}{dx} \\ p - \frac{\partial G}{\partial x} &= \frac{\partial G}{\partial p}\frac{dp}{dx} \end{aligned} \tag{3}$$

There are two cases to consider here. If $\frac{\partial G}{\partial x} = p$, then this is called Clairaut ODE. It implies the original ODE had the form $y(x) = xp + f(p)$. Where $G \equiv xp + f(p)$ as this is only possibility to get $\frac{\partial G}{\partial x} = p$. Therefore $\frac{\partial G}{\partial x} = p$ is special case. When this happens, then (3) gives $\frac{\partial G}{\partial p}\frac{dp}{dx} = 0$ which means either $\frac{dp}{dx} = 0$ or $\frac{\partial G}{\partial p} = 0$. For the case when $\frac{dp}{dx} = 0$, this implies $p = C_1$ or since $p = \frac{dy}{dx}$ then $y(x) = C_1 x + C_2$. Comparing $y(x) = C_1 x + C_2$ to $y(x) = xp + f(p)$ then $C_2 = f(C_1)$. Hence the solution is $y(x) = C_1 x + f(C_1)$. The second possibility is $\frac{\partial G}{\partial p} = 0$. This is easily solved for p and the second solution is found from $y(x) = G(x, p)$ directly after that.

For the case when $\frac{\partial G}{\partial x} \neq p$. This is now called the d'Alembert ODE. This is harder to solve than Clairaut. To solve d'Alembert, continuing from (3) and solving (3) for $\frac{dp}{dx}$ results in

$$\frac{dp}{dx} = \left(p - \frac{\partial G}{\partial x}\right)\frac{\partial p}{\partial G}$$

(Remembering that $\frac{\partial G}{\partial x} \neq p$ now). Taking x as the dependent variable and p as the independent variable, then solving for $\frac{dx}{dp}$ gives

$$\begin{aligned} \frac{dx}{dp} &= \frac{\partial G}{\partial p}\left(\frac{1}{p - \frac{\partial G}{\partial x}}\right) \\ \frac{dx}{dp}\left(p - \frac{\partial G}{\partial x}\right) &= \frac{\partial G}{\partial p} \end{aligned} \tag{4}$$

This will turn out to be a linear ODE in $x(p)$, where x is the dependent variable and p is the independent variable in (3).

This ODE is now solved for $x(p)$. Then p is solved for from this solution in terms of $x(p)$. This step is the hardest part of this method. Once p is found, then $y(x)$ is found by direct substitution back into (1) because $p = \frac{dy}{dx}$.

To show how this method works, the following ODE's are now solved.

number	ode	transformed	$G(x,p)$	$\frac{\partial G}{\partial x}$	$\frac{\partial G}{\partial p}$	type
1	$(y')^2 - 1 - x - y = 0$	$y = -x + (p^2 - 1)$	$-x + (p^2 - 1)$	-1	$2p$	d'Alembert
2	$yy' - (y')^2 = x$	$y = \frac{1}{p}x + p$	$\frac{1}{p}x + p$	$\frac{1}{p}$	$-\frac{1}{p^2} + 1$	d'Alembert
3	$x(y')^2 - yy' = -1$	$y = xp + \frac{1}{p}$	$xp + \frac{1}{p}$	p	$x - \frac{1}{p^2}$	Clairaut
4	$y = x(y')^2 + y'$	$y = xp^2 + p$	$xp^2 + p$	p^2	$2p + 1$	d'Alembert
5	$y = xy' - (y')^2$	$y = xp - p^2$	$xp - p^2$	p	$x - 2p$	Clairaut
6	$y = xy' - \frac{1}{4}(y')^2$	$y = xp - \frac{1}{4}p^2$	$xp - \frac{1}{4}p^2$	p	$x - \frac{1}{2}p$	Clairaut
7	$y = x(y')^2$	$y = xp^2$	xp^2	p^2	$2xp$	d'Alembert
8	$y = x + (y')^2$	$y = x + p^2$	$x + p^2$	1	$2p$	d'Alembert

Notice in the above table, that the ODE is Clairaut when $\frac{\partial G}{\partial x} = p$ and d'Alembert otherwise.

1.1 Example 1

To solve $(y')^2 - 1 - x - y = 0$, it is first converted to $y(x) = G(x, y'(x))$ which gives

$$y(x) = -x + \left((y')^2 - 1\right) \tag{1}$$

Using $p = \frac{dy}{dx}$ then (1) becomes

$$\begin{aligned} y(x) &= G(x,p) \\ &= -x + (p^2 - 1) \end{aligned} \tag{2}$$

Hence $\frac{\partial G}{\partial x} = -1$ and $\frac{\partial G}{\partial p} = 2p$. Since $\frac{\partial G}{\partial x} \neq p$ then this is d'Alembert. Therefore equation (4) above is used to solve the ode. (4) becomes

$$\begin{aligned} \frac{dx}{dp}\left(p - \frac{\partial G}{\partial x}\right) &= \frac{\partial G}{\partial p} \\ \frac{dx}{dp}(p+1) &= 2p \\ \frac{dx}{dp} &= \frac{2p}{p+1} \end{aligned}$$

The solution to the above ODE is $x(p) = 2p - 2\ln(p+1) + C_1$. Now p is solved for in terms of x (this is hard step in this algorithm). This might not always be possible and RootOf might have to be used. But in this example, the solution is

$$p = -\operatorname{LambertW}\left(-C_1 e^{-\frac{x}{2}-1}\right) - 1$$

Substituting the above back into (2), since $p = y'(x)$, gives the solution directly

$$y(x) = -x + \left(-\operatorname{LambertW}\left(-C_1 e^{-\frac{x}{2}-1}\right) - 1\right)^2 - 1$$

1.2 Example 2

To solve $x = yy' - (y')^2$, it is first converted to $y(x) = G(x, y'(x))$ which gives

$$\begin{aligned} y(x) &= G\left(x, y'(x)\right) \\ &= \frac{x + (y')^2}{y'} \end{aligned} \tag{1}$$

Using $p = \frac{dy}{dx}$ then (1) becomes

$$\begin{aligned} y(x) &= G(x,p) \\ &= \frac{x}{p} + p \end{aligned} \tag{2}$$

Hence $\frac{\partial G}{\partial x}=\frac{1}{p}$ and $\frac{\partial G}{\partial p}=\frac{x}{p^2}+1$. Since $\frac{\partial G}{\partial x}\neq p$ then this is d'Alembert. Therefore (4) becomes

$$\begin{aligned}
\frac{dx}{dp}\left(p-\frac{\partial G}{\partial x}\right) &= \frac{\partial G}{\partial p}\\
\frac{dx}{dp}\left(p-\frac{1}{p}\right) &= \frac{x}{p^2}+1\\
\frac{dx}{dp}\left(\frac{p^2-1}{p}\right) &= \frac{x}{p^2}+1\\
\frac{dx}{dp} &= \frac{x}{p^2}\left(\frac{p}{p^2-1}\right)+\frac{p}{p^2-1}\\
\frac{dx}{dp}-\frac{x}{p\left(p^2-1\right)} &= \frac{p}{p^2-1}
\end{aligned}$$

This is linear in $x(p)$. Solving the above gives

$$x(p)=\frac{p\sqrt{(p-1)(1+p)}\ln\left(p+\sqrt{p^2-1}\right)}{(1+p)(p-1)}+\frac{pC_1}{\sqrt{(p-1)(1+p)}}$$

Now p has to be solved in terms of x and hence solution is found from (1).

1.3 Example 3

Solving $x\left(y'\right)^2-yy'=-1$. This is written as

$$\begin{aligned}
y &= xy'(x)+\frac{1}{y'(x)}\\
&= G\left(x,y'\right) \qquad (1)
\end{aligned}$$

Let $p=\frac{dy}{dx}$ then (1) becomes

$$\begin{aligned}
y(x) &= G(x,p)\\
&= xp+\frac{1}{p} \qquad (2)
\end{aligned}$$

Therefore $\frac{\partial G}{\partial x}=p$ and $\frac{\partial G}{\partial p}=x-\frac{1}{p^2}$. Since $\frac{\partial G}{\partial x}=p$ then this is Clairaut ODE. This is special case. From (3),

$$\frac{\partial G}{\partial p}\frac{dp}{dx}=0$$

First possibility is $\frac{dp}{dx}=0$ which gives $y(x)=C_1x+C_2$ where $C_2=f(C_1)$ and $f(p)=\frac{1}{p}$ in this case looking at (1). Hence $C_1=\frac{1}{C_1}$. Therefore the solution is

$$y_1(x)=C_1x+\frac{1}{C_1}$$

The other solution from considering $\frac{\partial G}{\partial p}=0$ or $x-\frac{1}{p^2}=0$ which implies $p=\pm\frac{1}{\sqrt{x}}$. Substituting this into (2) gives 2 additional solutions

$$\begin{aligned}
y_1(x) &= x\frac{1}{\sqrt{x}}+\sqrt{x}=2\sqrt{x}\\
y_2(x) &= -\left(x\frac{1}{\sqrt{x}}+\sqrt{x}\right)=-2\sqrt{x}
\end{aligned}$$

Therefore, the three solutions are

$$\begin{aligned}
y_1(x) &= C_1x+\frac{1}{C_1}\\
y_2(x) &= 2\sqrt{x}\\
y_3(x) &= -2\sqrt{x}
\end{aligned}$$

The solutions y_2,y_3 are singular solutions, since they can not be obtained from the general solution $y_1(x)=C_1x+\frac{1}{C_1}$ by giving a specific value for C_1.

1.4 Example 4

Solving $y = x(y')^2 + y'$. It is transformed to

$$\begin{aligned} y &= G(x,p) \\ &= xp^2 + p \end{aligned} \tag{1}$$

Hence $\frac{\partial G}{\partial x} = p^2$ and $\frac{\partial G}{\partial p} = 2xp + 1$. Since $\frac{\partial G}{\partial x} \neq p$ then this is d'Alembert. Therefore (4) becomes

$$\begin{aligned} \frac{dx}{dp}\left(p - \frac{\partial G}{\partial x}\right) &= \frac{\partial G}{\partial p} \\ \frac{dx}{dp}\left(p - p^2\right) &= 2xp + 1 \\ \frac{dx}{dp} &= \frac{2xp}{p - p^2} + \frac{1}{p - p^2} \\ \frac{dx}{dp} - x\frac{2}{1-p} &= \frac{1}{p - p^2} \end{aligned}$$

This is linear in $x(p)$. Its solution is

$$x(p) = \frac{-p + \ln(p) + C_1}{(p-1)^2}$$

Now p is solved in terms of x. This gives

$$p = e^{\operatorname{RootOf}\left(-(e^Z)^2 x + 2e^Z x + Z + C_1 - e^Z - x\right)} \tag{2}$$

Hence the solution now found for $y(x)$ from (1)

$$y(x) = xe^{2\operatorname{RootOf}\left(-(e^Z)^2 x + 2e^Z x + Z + C_1 - e^Z - x\right)} + e^{\operatorname{RootOf}\left(-(e^Z)^2 x + 2e^Z x + Z + C_1 - e^Z - x\right)}$$

1.5 Example 5

Solving $y = xy' - (y')^2$. From above, it is transformed to

$$\begin{aligned} y &= G(x,p) \\ &= xp - p^2 \end{aligned} \tag{1}$$

Therefore $\frac{\partial G}{\partial x} = p$ and $\frac{\partial G}{\partial p} = x - 2p$. Since $\frac{\partial G}{\partial x} = p$ then this is Clairaut ODE. This is special case. From (3),

$$\frac{\partial G}{\partial p}\frac{dp}{dx} = 0$$

First possibility is $\frac{dp}{dx} = 0$ which gives $y(x) = C_1 x + C_2$ where $C_2 = f(C_1)$ and $f(p) = -p^2$ in this case looking at (1). Hence $C_2 = -C_1^2$. Therefore the solution is

$$y_1(x) = C_1 x - C_1^2$$

The other solution from considering $\frac{\partial G}{\partial p} = 0$ or $x - 2p = 0$, hence $p = \frac{x}{2}$. Therefore from (1)

$$\begin{aligned} y(x) &= x\left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)^2 \\ &= \frac{1}{4}x^2 \end{aligned}$$

Therefore the solutions are

$$\begin{aligned} y_1(x) &= C_1 x - C_1^2 \\ y_2(x) &= \frac{1}{4}x^2 \end{aligned}$$

The solution $y_2(x) = \frac{1}{4}x^2$ is singular since it can not be obtained from $y_1(x) = C_1 x - C_1^2$.

1.6 Example 6

Solving $y = xy' - \frac{1}{4}(y')^2$. From above, it is transformed to

$$\begin{aligned} y &= G(x,p) \\ &= xp - \frac{1}{4}p^2 \end{aligned} \tag{1}$$

Therefore $\frac{\partial G}{\partial x} = p$ and $\frac{\partial G}{\partial p} = x - \frac{1}{2}p$. Since $\frac{\partial G}{\partial x} = p$ then this is Clairaut ODE. This is special case. From (3),

$$\frac{\partial G}{\partial p}\frac{dp}{dx} = 0$$

First possibility is $\frac{dp}{dx} = 0$ which gives $y(x) = C_1 x + C_2$ where $C_2 = f(C_1)$ and $f(p) = -\frac{1}{4}p^2$ in this case looking at (1). Hence $C_2 = -\frac{1}{4}C_1^2$. Therefore the solution is

$$y_1(x) = C_1 x - \frac{1}{4}C_1^2$$

The other solution from considering $\frac{\partial G}{\partial p} = 0$ or $x - \frac{1}{2}p = 0$, hence $p = 2x$. Therefore from (1)

$$\begin{aligned} y(x) &= x(2x) - \frac{1}{4}(2x)^2 \\ &= x^2 \end{aligned}$$

Therefore the solutions are

$$\begin{aligned} y_1(x) &= C_1 x - \frac{1}{4}C_1^2 \\ y_2(x) &= x^2 \end{aligned}$$

The solution $y_2(x) = x^2$ is singular since it can not be obtained from $y_1(x) = C_1 x - \frac{1}{4}C_1^2$.

1.7 Example 7

Solving $y = x(y')^2$. From above, it is transformed to

$$\begin{aligned} y &= G(x,p) \\ &= xp^2 \end{aligned} \tag{1}$$

Hence $\frac{\partial G}{\partial x} = p^2$ and $\frac{\partial G}{\partial p} = 2px$. Since $\frac{\partial G}{\partial x} \neq p$ then this is d'Alembert. Therefore (4) becomes

$$\begin{aligned} \frac{dx}{dp}\left(p - \frac{\partial G}{\partial x}\right) &= \frac{\partial G}{\partial p} \\ \frac{dx}{dp}\left(p - p^2\right) &= 2xp \\ \frac{dx}{dp} &= \frac{2x}{1-p^2} \\ \frac{dx}{dp} - x\frac{2}{1-p} &= 0 \end{aligned}$$

This is linear in $x(p)$. Solving for $x(p)$ gives

$$x(p) = \frac{C_1}{(p-1)^2}$$

Hence solving for p gives

$$\begin{aligned} p_1 &= \frac{x + \sqrt{C_1 x}}{x} \\ p_2 &= \frac{-x + \sqrt{C_1 x}}{x} \end{aligned}$$

From (1) the solutions are

$$\begin{aligned} y_1(x) &= xp_1^2 \\ &= x\left(\frac{x + \sqrt{C_1 x}}{x}\right)^2 \\ &= x + C_1 + 2\sqrt{xC_1} \end{aligned}$$

And

$$
\begin{aligned}
y_2(x) &= x p_1^2 \\
&= x\left(\frac{-x+\sqrt{C_1 x}}{x}\right)^2 \\
&= x + C_1 - 2\sqrt{x C_1}
\end{aligned}
$$

1.8 Example 8

Solving $y = x + (y')^2$. It is transformed to

$$
\begin{aligned}
y &= G(x,p) \\
&= x + p^2 \qquad (1)
\end{aligned}
$$

Hence $\frac{\partial G}{\partial x} = 1$ and $\frac{\partial G}{\partial p} = 2p$. Since $\frac{\partial G}{\partial x} \neq p$ then this is d'Alembert. Therefore (4) becomes

$$
\begin{aligned}
\frac{dx}{dp}\left(p - \frac{\partial G}{\partial x}\right) &= \frac{\partial G}{\partial p} \\
\frac{dx}{dp}(p-1) &= 2p \\
\frac{dx}{dp} &= \frac{2p}{p-1}
\end{aligned}
$$

Hence $x = 2p + 2\ln(p-1) + C_1$. Solving for p in terms of x gives $p = \text{LambertW}\left(C_1 e^{\frac{x}{2}-1}\right) + 1$. Substituting this in (1) gives the solution

$$
y(x) = x + \left(\text{LambertW}\left(C_1 e^{\frac{x}{2}-1}\right) + 1\right)^2
$$

1.9 references

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