

Butterworth low pass analog filter design

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1 Introduction

This is a detailed review of low pass Butterworth analog filter design. The goal is to generate Butterworth transfer function $H(s)$ from frequency specifications. The following are the four specifications of the design

f_p	The passband corner frequency in Hz
f_s	The stopband corner frequency in Hz
A_p	The attenuation in db at Ω_p
A_s	The attenuation in db at Ω_s

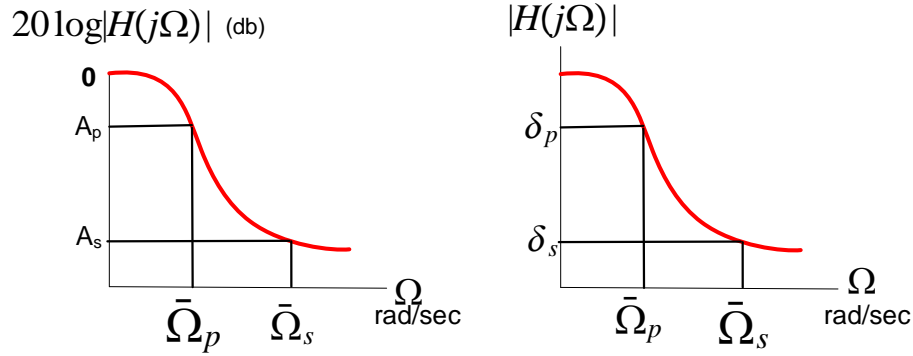


Figure 1: four specifications of the design

The specifications are given in db (the left diagram above) and not in magnitude (right diagram above).

The specifications are given with reference to the transfer function magnitude. The phase is not taken into account in the specifications. Butterworth analog transfer function transfer function magnitude is given by

$$|H(j\Omega)| = \frac{1}{\sqrt{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}}$$

Where Ω_c is the cutoff frequency. This is the frequency at which $|H(j\Omega)| = \frac{1}{\sqrt{2}} = 0.707$.

The goal of the design is to determine N and Ω_c from the specifications. Once N and Ω_c are found, $H(s)$ poles are found. Once the poles are found, then $H(s)$ is now determined.

2 Algorithm diagram

The following diagram outlines the design algorithm

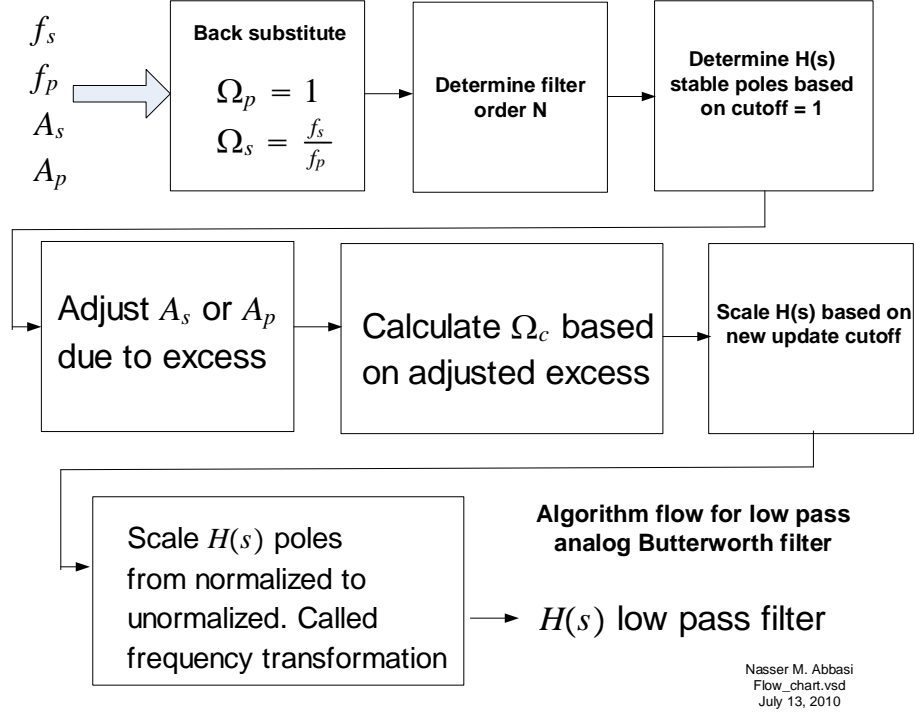


Figure 2: Algorithm diagram

3 Design steps

3.1 backward transformation

Let $\Omega_p = 1$ rad/sec, and let $\Omega_s = \frac{f_s}{f_p}$ rad/sec.

3.2 Determine filter order N

$$A_p = -20 \log |H(j\Omega_p)| = -20 \log \frac{1}{\sqrt{1 + \left(\frac{\Omega_p}{\Omega_c}\right)^{2N}}} = 10 \log \left[1 + \left(\frac{1}{\Omega_c}\right)^{2N} \right]$$

Solving for Ω_c from the above gives

$$\Omega_c = \frac{1}{\left(10^{\frac{A_p}{10}} - 1\right)^{\frac{1}{2N}}} \quad (1)$$

Using A_s gives

$$A_s = -20 \log |H(j\Omega_s)| = -20 \log \frac{1}{\sqrt{1 + \left(\frac{\Omega_s}{\Omega_c}\right)^{2N}}} = 10 \log \left[1 + \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} \right]$$

Solving for Ω_s from the above results in

$$\Omega_s = \Omega_c \left(10^{\frac{A_s}{10}} - 1 \right)^{\frac{1}{2N}} \quad (2)$$

Substituting Ω_c found in (1) into the above Ω_s gives

$$\Omega_s = \left[\frac{\left(10^{\frac{A_s}{10}} - 1 \right)}{\left(10^{\frac{A_p}{10}} - 1 \right)} \right]^{\frac{1}{2N}} \quad (3)$$

Solving for N in the above by taking logs gives

$$N = \frac{\log \left[\frac{\left(10^{\frac{A_s}{10}} - 1 \right)}{\left(10^{\frac{A_p}{10}} - 1 \right)} \right]}{2 \log \Omega_s} \quad (4)$$

Since the order of filter is an integer, the above value is rounded upwards to the next integer if it is not an integer. Let this new N be N' to make it clear that this is an updated N from the original N .

3.3 Finding stable poles assuming $\Omega_c = 1$

Since Butterworth magnitude square of the transfer function is

$$|H(s)|^2 = \frac{1}{1 + \left(\frac{s}{j\Omega'_c}\right)^{2N'}}$$

Hence $H(s)$ poles are found by setting the denominator of the above to zero. Setting $\Omega_c = 1$ gives

$$\begin{aligned}
1 + \left(\frac{s}{j}\right)^{2N'} &= 0 \\
\left(\frac{s}{j}\right)^{2N'} &= -1 \\
\left(\frac{s}{j}\right)^{2N'} &= e^{j(\pi+2\pi k)} \\
\frac{s}{j} &= e^{j\left(\frac{\pi+2\pi k}{2N'}\right)} \quad k = 0, 1, 2, \dots, 2N' - 1 \\
s_k &= j e^{j\left(\frac{\pi+2\pi k}{2N'}\right)} \\
s_k &= e^{j\frac{\pi}{2}} e^{j\left(\frac{\pi+2\pi k}{2N'}\right)} \\
s_k &= e^{j\left(\frac{\pi+2\pi k}{2N'} + \frac{\pi}{2}\right)}
\end{aligned} \tag{5}$$

Only the LHS poles are needed, which are located at $k = 0 \dots N' - 1$, because these are the stable poles.

Now that the poles are found, $H(s)$ becomes

$$\begin{aligned}
H(s) &= \frac{1}{\prod_{k=0}^{N'-1} s - s_k} \\
&= \frac{1}{(s - s_0)(s - s_1) \dots (s - s_{N'-1})}
\end{aligned} \tag{6}$$

3.4 Rescaling the poles

Either A_p or A_s have to be adjusted depending on if the excess tolerance is to be assigned to the passband or to the stop band and Ω_c is calculated based on this.

If the excess tolerance is to be assigned to the passband, then (3) is solved for A_p and this new found value is called A'_p

$$A'_p = 10 \log \left(\frac{\left(10^{\frac{A_s}{10}} - 1\right)}{\Omega_s^{2N'}} + 1 \right) \tag{6.1}$$

Also Ω_c needs to be determined from (1). Calling this Ω'_c to reflect that this goes with

the updated A'_p and not the original A_p

$$\Omega'_c = \frac{1}{\left(10^{\frac{A'_p}{10}} - 1\right)^{\frac{1}{2N'}}} \quad (6.2)$$

However, if the excess tolerance is to be assigned to the stopband, then (3) is solved for A_s and this new found value is called A'_s

$$A'_s = 10 \log \left(\Omega_s^{2N'} \left(10^{\frac{A_p}{10}} - 1 \right) + 1 \right) \quad (6.3)$$

Ω_c is adjusted to Ω'_c .

From (2), and using the above new value of A'_s gives

$$\Omega'_c = \frac{\Omega_s}{\left(10^{\frac{A'_s}{10}} - 1\right)^{\frac{1}{2N'}}} \quad (6.4)$$

$H(s)$ found above in (6) is now adjusted since that was found for $\Omega_c = 1$ and now an updated Ω'_c is found. To do that s is replaced by $\frac{s}{\Omega'_c}$, hence $H(s)$ becomes

$$\begin{aligned} H(s) &= \frac{1}{\left(\frac{s}{\Omega'_c} - s_0\right) \left(\frac{s}{\Omega'_c} - s_1\right) \cdots \left(\frac{s}{\Omega'_c} - s_{N'-1}\right)} \\ &= \frac{[\Omega'_c]^{(N'-1)}}{(s - \Omega'_c s_0) (s - \Omega'_c s_1) \cdots (s - \Omega'_c s_{N'-1})} \end{aligned} \quad (7)$$

The first part of the design is now complete. N', Ω'_c is found and adjusted A_p or A_s depending on the requirements for excess tolerance. All the parts needed are found to design $H(s)$ by finding its poles. Adjusted values must be used from now on.

3.5 Converting normalized low pass using frequency transformation

The above $H(s)$ found in (7) was designed for frequency $\Omega_p = 1$ and $\Omega_s = \frac{2\pi f_s}{2\pi f_p}$. The above $H(s)$ is called the normalized transfer function. It is a low pass analog filter, which needs to be mapped to a low pass analog filter, but un-normalized based on the actual frequencies specified (Since the above was designed based on using $\Omega_p = 1$).

Adjustment is now made to obtain $H(s)$ for $\Omega_p = 2\pi f_p$ and $\Omega_s = 2\pi f_s$.

To do this, s above is replaced by $\frac{s}{2\pi f_p}$. Equation (7) becomes

$$\begin{aligned}
 H(s) &= \frac{[\Omega'_c]^{(N'-1)}}{\left(\frac{s}{2\pi f_p} - \Omega'_c s_0\right) \left(\frac{s}{2\pi f_p} - \Omega'_c s_1\right) \cdots \left(\frac{s}{2\pi f_p} - \Omega'_c s_{N'-1}\right)} \\
 &= \frac{[\Omega'_c \times 2\pi f_p]^{(N'-1)}}{(s - 2\pi f_p \Omega'_c s_0) (s - 2\pi f_p \Omega'_c s_1) \cdots (s - 2\pi f_p \Omega'_c s_{N'-1})} \quad (8)
 \end{aligned}$$

The zeros of $H(s)$ are located at ∞ and there are N' of them.

When simplifying the denominator above, the complex conjugate terms are multiplied with each others to obtain real coefficients.

4 Example designs

4.1 Example 1

Given

$f_p = 1000$ hz, $f_s = 2000$ hz, $A_p = 1$ db, $A_s = 20$ db, and Excess tolerance at stopband, determine $H(s)$

4.1.1 backward transformation

$$\begin{aligned}
 \Omega_p &= 1 \text{ rad/sec} \\
 \Omega_s &= \frac{f_s}{f_p} = 2 \text{ rad/sec}
 \end{aligned}$$

4.1.2 Determining filter order N

From (4)

$$\begin{aligned}
 N &= \frac{\log \left[\frac{\left(10^{\frac{A_s}{10}} - 1\right)}{\left(10^{\frac{A_p}{10}} - 1\right)} \right]}{2 \log \Omega_s} \\
 &= \frac{\log_{10} \left[\frac{\left(10^{\frac{20}{10}} - 1\right)}{\left(10^{\frac{1}{10}} - 1\right)} \right]}{2 \log_{10} 2} = 4.2894
 \end{aligned}$$

Hence

$$N' = 5$$

4.2 Finding stable poles assuming $\Omega_c = 1$

From (5), and since $N' = 5$

$$\begin{aligned} s_k &= e^{i\left(\frac{\pi+2\pi k}{2N'} + \frac{\pi}{2}\right)} \\ &= e^{i\left(\frac{\pi+2\pi k}{2(5)} + \frac{\pi}{2}\right)} \end{aligned}$$

Find the poles

k	s_k
0	$e^{i\left(\frac{\pi}{10} + \frac{\pi}{2}\right)} = -0.309 + 0.951i$
1	$e^{i\left(\frac{\pi+2\pi}{10} + \frac{\pi}{2}\right)} = -0.809 + 0.587i$
2	$e^{i\left(\frac{\pi+4\pi}{10} + \frac{\pi}{2}\right)} = -1.0$
3	$e^{i\left(\frac{\pi+6\pi}{10} + \frac{\pi}{2}\right)} = -0.809 - 0.587i$
4	$e^{i\left(\frac{\pi+8\pi}{10} + \frac{\pi}{2}\right)} = -0.309 - 0.951i$

Hence from (6)

$$\begin{aligned} H(s) &= \frac{1}{(s - s_0)(s - s_1)(s - s_2)(s - s_3)(s - s_4)} \\ &= \frac{1}{(s - (-0.30902 + 0.95106i))(s - (-0.809 + 0.587i))(s - (-1))} \times \\ &\quad \frac{1}{(s - (-0.809 - 0.587i))(s - (-0.309 - 0.951i))} \end{aligned}$$

4.2.1 Rescale the poles

Excess tolerance is in the stopband, hence from (6.3)

$$\begin{aligned} A'_s &= 10 \log \left(\Omega_s^{2N'} \left(10^{\frac{A_p}{10}} - 1 \right) + 1 \right) \\ &= 10 \log_{10} \left(2^{2(5)} \left(10^{\frac{1}{10}} - 1 \right) + 1 \right) = 24.251 \text{ db} \end{aligned}$$

Hence new Ω'_c is found from (6.4)

$$\begin{aligned} \Omega'_c &= \frac{\Omega_s}{\left(10^{\frac{A'_s}{10}} - 1 \right)^{\frac{1}{2N'}}} \\ &= \frac{2}{\left(10^{\frac{24.251}{10}} - 1 \right)^{\frac{1}{2(5)}}} = 1.144 \end{aligned}$$

Hence the above $H(s)$ becomes (using equation 7 as reference)

$$H(s) = \frac{1.1447^5}{(s - 1.144(-0.309 + 0.951i))(s - 1.144(-0.809 + 0.587i))(s - 1.144(-1))} \times \frac{1}{(s - 1.144(-0.809 - 0.587i))(s - 1.1447(-0.309 - 0.951i))}$$

4.2.2 Convert normalized low pass using frequency transformation

replace s by $\frac{s}{2\pi f_p}$, hence $H(s)$ becomes (using equation 8 as reference), and noting that $2\pi f_p = 2\pi(1000) = 6283.2$

$$H(s) = \frac{(1.144 \times 6283.2)^5}{(s - 1.144 \times 6283.2(-0.309 + 0.951i))} \times \frac{1}{(s - 1.144 \times 6283.2(-0.809 + 0.587i))(s - 1.144 \times 6283.2(-1))} \times \frac{1}{(s - 1.144 \times 6283.2(-0.809 - 0.587i))(s - 1.144 \times 6283.2(-0.309 - 0.951i))}$$

Now multiplying the complex conjugate terms with each others (to remove the complex terms) gives

$$H(s) = \frac{7192.4^5}{(s + 7192.2)(s^2 + 4445.2s + 5.173 \times 10^7)(s^2 + 11638.s + 5.173 \times 10^7)}$$

5 References

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2. Mostafa Shiva, Electrical engineering department, California state university, Fullerton, Lecture notes, handout H.
3. John Proakis, Dimitris Manolakis, digital signal processing, 3rd edition