Using Mason rule to obtain the transfer function from state space description

Nasser M. Abbasi

September 21, 2015 Compiled on January 30, 2024 at 1:56am

Given the observable form

$$x' = \begin{pmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + [\gamma] u$$

Show that the transfer function is $\frac{Y}{U} = \gamma + \frac{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}$ using Mason rule.

The transfer function can ofcourse be found using $\frac{Y}{S} = C(sI - A)^{-1} B + D$ which gives

$$\begin{aligned} \frac{Y}{U} &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 1 & s & 0 \\ 0 & 0 & 1 & s \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{pmatrix} \right)^{-1} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} + \gamma \\ &= \gamma + \frac{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0} \end{aligned}$$

But we want to use Mason rule here. The first step is to write the equations so that the nodes variable are on the left side of the equation. The node variables are the states x_1, x_2, x_3, x_4. From the matrix equations we obtain

$$\begin{aligned} x_1' &= -\alpha_0 x_4 + \beta_0 u \\ x_2' &= x_1 - \alpha_1 x_4 + \beta_1 u \\ x_3' &= x_2 - \alpha_2 x_4 + \beta_2 u \\ x_4' &= x_3 - \alpha_3 x_4 + \beta_3 u \end{aligned}$$

Solving for x now, and in the process we change x' to sx by taking Laplace transforms of all variables, and we add the output equation as well

$$
\begin{aligned}
X_4 &= -\frac{s}{\alpha_0}X_1 + \frac{\beta_0}{\alpha_0}U \\
X_1 &= sX_2 + \alpha_1 X_4 - \beta_1 U \\
X_2 &= sX_3 + \alpha_2 X_4 - \beta_2 U \\
X_3 &= (s + \alpha_3) X_4 - \beta_3 U \\
Y &= X_4 + \gamma U
\end{aligned}
$$

We now draw the Mason diagram, putting U on the left most node and Y on the right most node (the input and output). Here is the result

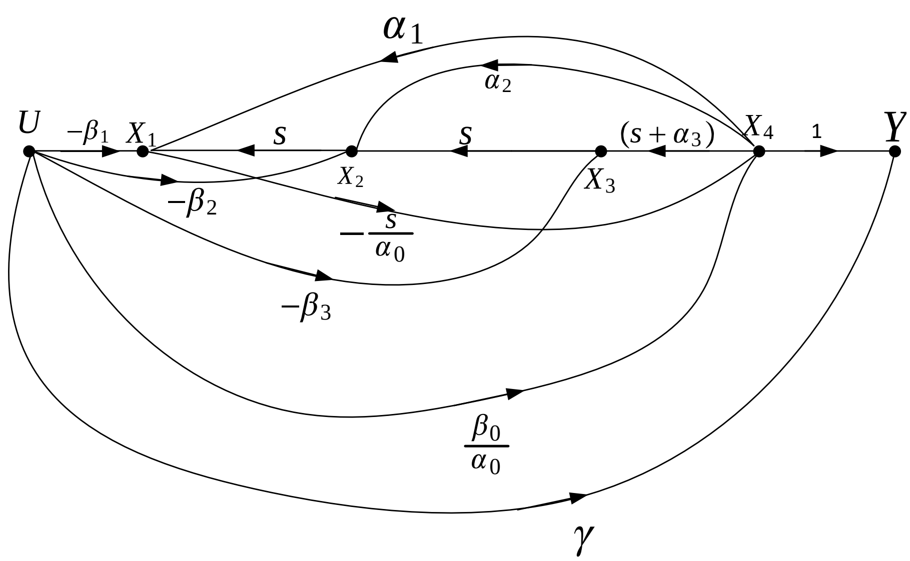

Figure 1: Mason graph

The forward paths from U to Y are

$$
\begin{aligned}
F_1 &= \frac{\beta_0}{\alpha_0} \\
F_2 &= \gamma \\
F_3 &= (-\beta_2)(s)\left(-\frac{s}{\alpha_0}\right) \\
F_4 &= (-\beta_1)\left(-\frac{s}{\alpha_0}\right) \\
F_5 &= (-\beta_3)(s)(s)\left(-\frac{s}{\alpha_0}\right)
\end{aligned}
$$

Now we find all the loops. There are only three loops. A loop is one that starts from a node and returns back to it without visiting a node more than once.

$$\begin{aligned}
L_1 &= \left(-\frac{s}{\alpha_0}\right)(\alpha_2)(s) \\
L_2 &= \left(-\frac{s}{\alpha_0}\right)(s+\alpha_3)(s)(s) \\
L_3 &= \left(-\frac{s}{\alpha_0}\right)(\alpha_1)
\end{aligned}$$

We now need to calculate the associated Δ_k for each of the above forward loops. Δ_k is found by removing F_k from the graph and then calculating the main mason Δ of what is left in the graph. When remove F_1 no loops remain, hence $\Delta_1 = 1$. When removing F_2 all the loops remain, hence $\Delta_2 = 1 - (L_1 + L_2 + L_3)$, and when removing F_3, F_4, F_5 no loops remain, hence $\Delta_3 = \Delta_4 = \Delta_5 = 1$. Therefore

$$\frac{Y}{U} = \frac{F_1\Delta_1 + F_2\Delta_2 + F_3\Delta_3 + F_4\Delta_4 + F_5\Delta_5}{1 - (L_1 + L_2 + L_3)}$$

There is no other combinations of loops. The above becomes

$$\begin{aligned}
\frac{Y}{U} &= \frac{F_1 + \gamma(1 - (L_1 + L_2 + L_3)) + F_3 + F_4 + F_5}{1 - (L_1 + L_2 + L_3)} \\
&= \gamma + \frac{F_1 + F_3 + F_4 + F_5}{1 - (L_1 + L_2 + L_3)} \\
&= \gamma + \frac{\frac{\beta_0}{\alpha_0} + (-\beta_2)(s)\left(-\frac{s}{\alpha_0}\right) + (-\beta_1)\left(-\frac{s}{\alpha_0}\right) + (-\beta_3)(s)(s)\left(-\frac{s}{\alpha_0}\right)}{1 - \left(\left(-\frac{s}{\alpha_0}\right)(\alpha_2)(s) + \left(-\frac{s}{\alpha_0}\right)(s+\alpha_3)(s)(s) + \left(-\frac{s}{\alpha_0}\right)(\alpha_1)\right)} \\
&= \gamma + \frac{\frac{\beta_0}{\alpha_0} + \beta_2\frac{s^2}{\alpha_0} + \beta_1\frac{s}{\alpha_0} + \beta_3\frac{s^3}{\alpha_0}}{1 - \left(-\frac{s^2}{\alpha_0}\alpha_2 - \frac{s^2}{\alpha_0}(s+\alpha_3) - \frac{s}{\alpha_0}\alpha_1\right)} \\
&= \gamma + \frac{\frac{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{\alpha_0}}{1 + \frac{s^3 + s^2\alpha_3 + s^2\alpha_2 + s\alpha_1}{\alpha_0}}
\end{aligned}$$

Hence

$$G(s) = \frac{Y}{U} = \gamma + \frac{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{s^3 + s^2\alpha_3 + s^2\alpha_2 + s\alpha_1 + \alpha_0}$$