HW 9
Physics 5041 Mathematical Methods for Physics
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Contents

1 Problem 1

Problem Calculate the metric in elliptical coordinates

$$x = \frac{a}{2}\cosh\mu\cos\theta$$
$$y = \frac{a}{2}\sinh\mu\sin\theta$$

Solution

The coordinates in the Cartesian system are $\zeta^1 = x, \zeta^2 = y$ and the coordinates in the other system (Elliptic) are $x^1 = \mu, x^2 = \theta$. The relation between these must be known and invertible also, meaning $\zeta \equiv \zeta(x)$ and $x \equiv x(\zeta)$. This relation is given to use above as

$$\zeta^1 = \frac{a}{2}\cosh\mu\cos\theta$$
$$\zeta^2 = \frac{a}{2}\sinh\mu\sin\theta$$

The first step is to determine the metric tensor g_{ij} for the Polar coordinates. This is given by

$$g_{kl} = \delta_{ij}\frac{\partial\zeta^i}{\partial x^k}\frac{\partial\zeta^j}{\partial x^l}$$

The above using Einstein summation notation.

$$\begin{aligned}
g_{11} &= \frac{\partial\zeta^1}{\partial x^1}\frac{\partial\zeta^1}{\partial x^1} + \frac{\partial\zeta^2}{\partial x^1}\frac{\partial\zeta^2}{\partial x^1}\\
&= \frac{\partial\zeta^1}{\partial\mu}\frac{\partial\zeta^1}{\partial\mu} + \frac{\partial\zeta^2}{\partial\mu}\frac{\partial\zeta^2}{\partial\mu}\\
&= \left(\frac{\partial\zeta^1}{\partial\mu}\right)^2 + \left(\frac{\partial\zeta^2}{\partial\mu}\right)^2\\
&= \left(\frac{a}{2}\sinh\mu\cos\theta\right)^2 + \left(\frac{a}{2}\cosh\mu\sin\theta\right)^2\\
&= \frac{a^2}{4}\left(\sinh^2\mu\cos^2\theta + \cosh^2\mu\sin^2\theta\right)\\
&= \frac{a^2}{4}\left(\left(\cosh^2\mu - 1\right)\cos^2\theta + \cosh^2\mu\left(1 - \cos^2\theta\right)\right)\\
&= \frac{a^2}{4}\left(\cosh^2\mu\cos^2\theta - \cos^2\theta + \cosh^2\mu - \cosh^2\mu\cos^2\theta\right)\\
&= \frac{a^2}{4}\left(\cosh^2\mu - \cos^2\theta\right)
\end{aligned}$$

And

$$\begin{aligned}
g_{12} &= \frac{\partial\zeta^1}{\partial x^1}\frac{\partial\zeta^1}{\partial x^2} + \frac{\partial\zeta^2}{\partial x^1}\frac{\partial\zeta^2}{\partial x^2}\\
&= \frac{\partial\zeta^1}{\partial\mu}\frac{\partial\zeta^1}{\partial\theta} + \frac{\partial\zeta^2}{\partial\mu}\frac{\partial\zeta^2}{\partial\theta}\\
&= \left(\frac{a}{2}\sinh\mu\cos\theta\right)\left(-\frac{a}{2}\cosh\mu\sin\theta\right) + \left(\frac{a}{2}\cosh\mu\sin\theta\right)\left(\frac{a}{2}\sinh\mu\cos\theta\right)\\
&= 0
\end{aligned}$$

The above is as expected since the coordinate system is orthogonal. And

$$\begin{aligned} g_{21} &= \frac{\partial \zeta^1}{\partial x^2}\frac{\partial \zeta^1}{\partial x^1} + \frac{\partial \zeta^2}{\partial x^2}\frac{\partial \zeta^2}{\partial x^1} \\ &= \frac{\partial \zeta^1}{\partial \theta}\frac{\partial \zeta^1}{\partial \mu} + \frac{\partial \zeta^2}{\partial \theta}\frac{\partial \zeta^2}{\partial \mu} \\ &= \left(-\frac{a}{2}\cosh \mu \sin \theta\right)\left(\frac{a}{2}\sinh \mu \cos \theta\right) + \left(\frac{a}{2}\sinh \mu \cos \theta\right)\left(\frac{a}{2}\cosh \mu \sin \theta\right) \\ &= 0 \end{aligned}$$

The above is as expected since the coordinate system is orthogonal. It is also because g_{ij} is symmetric and we already found that $g_{12} = 0$. And finally

$$\begin{aligned} g_{22} &= \frac{\partial \zeta^1}{\partial x^2}\frac{\partial \zeta^1}{\partial x^2} + \frac{\partial \zeta^2}{\partial x^2}\frac{\partial \zeta^2}{\partial x^2} \\ &= \frac{\partial \zeta^1}{\partial \theta}\frac{\partial \zeta^1}{\partial \theta} + \frac{\partial \zeta^2}{\partial \theta}\frac{\partial \zeta^2}{\partial \theta} \\ &= \left(\frac{\partial \zeta^1}{\partial \theta}\right)^2 + \left(\frac{\partial \zeta^2}{\partial \theta}\right)^2 \\ &= \left(-\frac{a}{2}\cosh \mu \sin \theta\right)^2 + \left(\frac{a}{2}\sinh \mu \cos \theta\right)^2 \\ &= \frac{a^2}{4}\left(\cosh^2 \mu \sin^2 \theta + \sinh^2 \mu \cos^2 \theta\right) \\ &= \frac{a^2}{4}\left(\cosh^2 \mu \left(1 - \cos^2 \theta\right) + \left(\cosh^2 \mu - 1\right)\cos^2 \theta\right) \\ &= \frac{a^2}{4}\left(\cosh^2 \mu - \cosh^2 \mu \cos^2 \theta + \cosh^2 \mu \cos^2 \theta - \cos^2 \theta\right) \\ &= \frac{a^2}{4}\left(\cosh^2 \mu - \cos^2 \theta\right) \end{aligned}$$

From the above we see that

$$\begin{aligned} g_{ij} &= \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \\ &= \frac{a^2}{4}\begin{pmatrix} \cosh^2 \mu - \cos^2 \theta & 0 \\ 0 & \cosh^2 \mu - \cos^2 \theta \end{pmatrix} \end{aligned}$$

That there are different ways to write the above, and they are all the same. For example, we can write

$$\begin{aligned} g_{ij} &= \frac{a^2}{4}\begin{pmatrix} \left(1 + \sinh^2 \mu\right) - \left(1 - \sin^2 \theta\right) & 0 \\ 0 & \left(1 + \sin^2 \mu\right) - \left(1 - \sin^2 \theta\right) \end{pmatrix} \\ &= \frac{a^2}{4}\begin{pmatrix} \sinh^2 \mu + \sin^2 \theta & 0 \\ 0 & \sinh^2 \mu + \sin^2 \theta \end{pmatrix} \end{aligned}$$

Or we could use the double angle relations $\cos^2 \theta = \frac{1}{2}(1 + \cos(2\theta))$ and $\cosh^2 \mu = \frac{1}{2}(1 + \cosh(2\theta))$ to obtain

$$\begin{aligned} g_{ij} &= \frac{a^2}{4}\begin{pmatrix} \frac{1}{2}(1 + \cosh(2\theta)) - \frac{1}{2}(1 + \cos(2\theta)) & 0 \\ 0 & \frac{1}{2}(1 + \cosh(2\theta)) - \frac{1}{2}(1 + \cos(2\theta)) \end{pmatrix} \\ &= \frac{a^2}{8}\begin{pmatrix} \cosh(2\theta) - \cos(2\theta) & 0 \\ 0 & \cosh(2\theta) - \cos(2\theta) \end{pmatrix} \end{aligned}$$

2 Problem 2

Problem Show that in a general coordinates system $\epsilon_{i_1 \cdots i_N} = g\epsilon^{i_1 \cdots i_N}$ where the covariant form is obtained by lowering the indices on the contravariant form.

Solution

In tensor analysis, contravariant components of a tensor uses upper indices and covariant components uses lower indices. Given a tensor in contravariant form ϵ^i then the covariant form ϵ_i is obtained using

$$\epsilon_i = g_{ij}\epsilon^j$$

Where on the right side the sum is taken over j since it is the repeated index. This operation is called index contracting.

Therefore extending the above to all indices in $\epsilon_{i_1 \cdots i_N}$ results in

$$\epsilon_{i_1 i_2 \cdots i_N} = g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_N j_N} \epsilon^{j_1 j_2 \cdots j_N} \tag{1}$$

But we know that, from page 123 in the Matrices notes, that the determinant of the metric can be written using Levi-Civita tensor as

$$g = \sum_{i_1 i_2 \cdots i_N} g_{1 i_1} g_{2 i_2} \cdots g_{N i_N} \epsilon^{i_1 i_2 \cdots i_N} \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} \epsilon_{123 \cdots N} &= g_{1 i_1} g_{2 i_2} \cdots g_{N i_N} \epsilon^{i_1 i_2 \cdots i_N} \\ &= k\epsilon^{i_1 i_2 \cdots i_N} \end{aligned}$$

Where k is constant, which in the case of $\epsilon_{123 \cdots N}$, this constant is g. Now need to show that the constant is g for all cases of indices in $\epsilon_{i_1 i_2 \cdots i_N}$ and not for the case $\epsilon_{123 \cdots N}$.

Looking at the case of $N = 2$, and let us see what happens if we change the order of the indices.

$$\epsilon_{i_1 i_2} = g_{i_1 j_1} g_{i_2 j_2} \epsilon^{j_1 j_2}$$

And

$$\epsilon_{i_2 i_1} = g_{i_2 j_2} g_{i_1 j_1} \epsilon^{j_2 j_1}$$

But $g_{i_1 j_1} g_{i_2 j_2}$ is the same as $g_{i_2 j_2} g_{i_1 j_1}$. So the ordering of indices does not change the constant k. And since we found that this constant is g from above, therefore we conclude that

$$\epsilon_{i_1 i_2 \cdots i_N} = g\epsilon^{j_1 j_2 \cdots j_N} \tag{3}$$

3 Problem 3

Problem Compute all components of the affine connection in polar coordinates.

Solution

In polar coordinates $x^1 = r, x^2 = \theta$, the relation to the Cartesian coordinates is

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Using

$$\Gamma^i_{jk} = \frac{1}{2} g^{li} \left(\frac{\partial g_{kl}}{\partial x^i} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right) \tag{1}$$

We know that in polar coordinates the metric tensor is $g_{11} = g_{rr} = 1$, and $g_{12} = g_{r\theta} = 0$, and $g_{21} = g_{\theta r} = 0$, and $g_{22} = g_{\theta\theta} = 0$ or in matrix form

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Hence g^{ij} is its inverse

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

Using (1), let $i = r, j = r, k = r$ then

$$\Gamma^r_{rr} = \frac{1}{2} g^{lr} \left(\frac{\partial g_{rl}}{\partial r} + \frac{\partial g_{rl}}{\partial r} - \frac{\partial g_{rr}}{\partial x^l} \right)$$

The sum is now over l, which goes from r, θ since these are the only coordinates. Hence the above becomes

$$\begin{aligned} \Gamma^r_{rr} &= \frac{1}{2} g^{rr} \left(\frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial r} \right) + \frac{1}{2} g^{\theta r} \left(\frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta} \right) \\ &= \frac{1}{2} (1) (0 + 0 - 0) + \frac{1}{2} (0) \left(\frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta} \right) \\ &= 0 \end{aligned} \tag{2}$$

Using (1), let $i = r, j = \theta, k = r$ then

$$\Gamma^r_{\theta r} = \frac{1}{2} g^{lr} \left(\frac{\partial g_{rl}}{\partial r} + \frac{\partial g_{\theta l}}{\partial r} - \frac{\partial g_{\theta r}}{\partial x^l} \right)$$

The sum is now over l, which goes from r, θ since these are the only coordinates. Hence the above becomes

$$\begin{aligned} \Gamma^r_{\theta r} &= \frac{1}{2} g^{rr} \left(\frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{\theta r}}{\partial r} - \frac{\partial g_{\theta r}}{\partial r} \right) + \frac{1}{2} g^{\theta r} \left(\frac{\partial g_{r\theta}}{\partial r} + \frac{\partial g_{\theta\theta}}{\partial r} - \frac{\partial g_{\theta r}}{\partial \theta} \right) \\ &= \frac{1}{2} (1) (0 + 0 - 0) + \frac{1}{2} (0) \left(\frac{\partial g_{r\theta}}{\partial r} + \frac{\partial g_{\theta\theta}}{\partial r} - \frac{\partial g_{\theta r}}{\partial \theta} \right) \\ &= 0 \end{aligned} \tag{3}$$

Using (1), now let $i = r, j = \theta, k = \theta$ then

$$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{lr} \left(\frac{\partial g_{\theta l}}{\partial \theta} + \frac{\partial g_{\theta l}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial x^l} \right)$$

The sum is now over l, which goes from r, θ since these are the only coordinates. Hence the above becomes

$$\begin{aligned}\Gamma^r_{\theta\theta} &= \frac{1}{2}g^{rr}\left(\frac{\partial g_{\theta r}}{\partial\theta}+\frac{\partial g_{\theta r}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial r}\right)+\frac{1}{2}g^{\theta r}\left(\frac{\partial g_{\theta\theta}}{\partial\theta}+\frac{\partial g_{\theta\theta}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial\theta}\right)\\ &= \frac{1}{2}(1)\left((0)+(0)-\frac{\partial r^2}{\partial r}\right)+\frac{1}{2}(0)\left(\frac{\partial g_{r\theta}}{\partial r}+\frac{\partial g_{\theta\theta}}{\partial r}-\frac{\partial g_{\theta r}}{\partial\theta}\right)\\ &= \frac{1}{2}(-2r)\\ &= -r \end{aligned}\tag{4}$$

Using (1), now let $i=r, j=r, k=\theta$. Hence we need to find $\Gamma^r_{r\theta}$. But due to symmetry in lower indices, then $\Gamma^r_{r\theta}=\Gamma^r_{\theta r}$ which we found in (3) to be zero. Hence

$$\Gamma^r_{r\theta}=0\tag{5}$$

Using (1), now let $i=\theta, j=r, k=r$ then

$$\Gamma^\theta_{rr}=\frac{1}{2}g^{l\theta}\left(\frac{\partial g_{rl}}{\partial\theta}+\frac{\partial g_{rl}}{\partial r}-\frac{\partial g_{rr}}{\partial x^l}\right)$$

The sum is now over l, which goes from r,θ since these are the only coordinates. Hence the above becomes

$$\begin{aligned}\Gamma^\theta_{rr} &= \frac{1}{2}g^{r\theta}\left(\frac{\partial g_{rr}}{\partial\theta}+\frac{\partial g_{rr}}{\partial r}-\frac{\partial g_{rr}}{\partial r}\right)+\frac{1}{2}g^{\theta\theta}\left(\frac{\partial g_{r\theta}}{\partial\theta}+\frac{\partial g_{r\theta}}{\partial r}-\frac{\partial g_{rr}}{\partial\theta}\right)\\ &= \frac{1}{2}(0)\left(\frac{\partial g_{rr}}{\partial\theta}+\frac{\partial g_{rr}}{\partial r}-\frac{\partial g_{rr}}{\partial r}\right)+\frac{1}{2}\left(\frac{1}{r^2}\right)(0+0-0)\\ &= 0 \end{aligned}\tag{6}$$

Using (1), now let $i=\theta, j=\theta, k=r$ then

$$\Gamma^\theta_{\theta r}=\frac{1}{2}g^{l\theta}\left(\frac{\partial g_{rl}}{\partial\theta}+\frac{\partial g_{\theta l}}{\partial r}-\frac{\partial g_{\theta r}}{\partial x^l}\right)$$

The sum is now over l, which goes from r,θ since these are the only coordinates. Hence the above becomes

$$\begin{aligned}\Gamma^\theta_{\theta r} &= \frac{1}{2}g^{r\theta}\left(\frac{\partial g_{rr}}{\partial\theta}+\frac{\partial g_{\theta r}}{\partial r}-\frac{\partial g_{\theta r}}{\partial r}\right)+\frac{1}{2}g^{\theta\theta}\left(\frac{\partial g_{r\theta}}{\partial\theta}+\frac{\partial g_{\theta\theta}}{\partial r}-\frac{\partial g_{\theta r}}{\partial\theta}\right)\\ &= \frac{1}{2}(0)\left(\frac{\partial g_{rr}}{\partial\theta}+\frac{\partial g_{\theta r}}{\partial r}-\frac{\partial g_{\theta r}}{\partial r}\right)+\frac{1}{2}\frac{1}{r^2}\left(0+\frac{\partial r^2}{\partial r}-0\right)\\ &= \frac{1}{2}\frac{1}{r^2}(2r)\\ &= \frac{1}{r} \end{aligned}\tag{7}$$

Using (1), now let $i=\theta, j=r, k=\theta$ which finds $\Gamma^\theta_{r\theta}$ but due to symmetry this is the same as $\Gamma^\theta_{\theta r}$ which is found above. Hence

$$\Gamma^\theta_{r\theta}=\frac{1}{r}\tag{8}$$

Using (1), now let $i=\theta, j=\theta, k=\theta$ then

$$\Gamma^\theta_{\theta\theta}=\frac{1}{2}g^{l\theta}\left(\frac{\partial g_{\theta l}}{\partial\theta}+\frac{\partial g_{\theta l}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial x^l}\right)$$

The sum is now over l, which goes from r,θ since these are the only coordinates. Hence the above becomes

$$\begin{aligned}\Gamma^\theta_{\theta\theta} &= \frac{1}{2}g^{r\theta}\left(\frac{\partial g_{\theta r}}{\partial\theta}+\frac{\partial g_{\theta r}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial r}\right)+\frac{1}{2}g^{\theta\theta}\left(\frac{\partial g_{\theta\theta}}{\partial\theta}+\frac{\partial g_{\theta\theta}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial\theta}\right)\\ &= \frac{1}{2}(0)\left(\frac{\partial g_{\theta r}}{\partial\theta}+\frac{\partial g_{\theta r}}{\partial\theta}-\frac{\partial g_{\theta\theta}}{\partial r}\right)+\frac{1}{2}\frac{1}{r^2}(0+0-0)\\ &= 0 \end{aligned}\tag{9}$$

This completes the computation. In summary

$$\Gamma^r_{rr} = 0$$
$$\Gamma^r_{\theta r} = 0$$
$$\Gamma^r_{\theta\theta} = -r$$
$$\Gamma^r_{r\theta} = 0$$
$$\Gamma^\theta_{rr} = 0$$
$$\Gamma^\theta_{\theta r} = \frac{1}{r}$$
$$\Gamma^\theta_{r\theta} = \frac{1}{r}$$
$$\Gamma^\theta_{\theta\theta} = 0$$

4 Problem 4

Problem Calculate the gradient curl and divergence and Laplacian in spherical coordinates using tensor analysis.

Solution

The following coordinates system convention is used

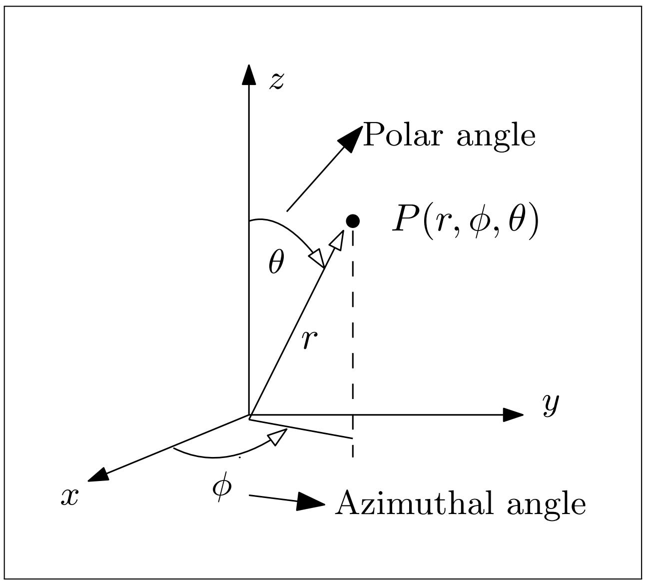

Figure 1: Spherical Coordinates system

4.1 Finding metric tensor g_{ij}

The coordinates in the Cartesian system are $\zeta^1 = x, \zeta^2 = y, \zeta^3 = z$. And the coordinates in the Spherical system are $x^1 = \phi, x^2 = r, x^3 = \theta$. The relation between these is known as (Note that the following depends on convention used for which is θ and which is ϕ. Physics convention as shown in the diagram above is used here).

$$\zeta^1 = r \sin\theta \cos\phi$$
$$\zeta^2 = r \sin\theta \sin\phi$$
$$\zeta^3 = r \cos\theta$$

The first step is to determine the metric tensor g for the Spherical coordinates. This is given by

$$g_{kl} = \delta_{ij} \frac{\partial \zeta^i}{\partial x^k} \frac{\partial \zeta^j}{\partial x^l}$$

Since the coordinate system are orthogonal, g_{kl} will be diagonal. Hence only g_{11}, g_{22}, g_{33} are non zero.

$$\begin{aligned}
g_{11} &= g_{\phi\phi} \\
&= \frac{\partial \zeta^1}{\partial x^1}\frac{\partial \zeta^1}{\partial x^1} + \frac{\partial \zeta^2}{\partial x^1}\frac{\partial \zeta^2}{\partial x^1} + \frac{\partial \zeta^3}{\partial x^1}\frac{\partial \zeta^3}{\partial x^1} \\
&= \frac{\partial \zeta^1}{\partial \phi}\frac{\partial \zeta^1}{\partial \phi} + \frac{\partial \zeta^2}{\partial \phi}\frac{\partial \zeta^2}{\partial \phi} + \frac{\partial \zeta^3}{\partial \phi}\frac{\partial \zeta^3}{\partial \phi} \\
&= \left(\frac{\partial \zeta^1}{\partial \phi}\right)^2 + \left(\frac{\partial \zeta^2}{\partial \phi}\right)^2 + \left(\frac{\partial \zeta^3}{\partial \phi}\right)^2 \\
&= \left(-r \sin\theta \sin\phi\right)^2 + \left(r \sin\theta \cos\phi\right)^2 + (0)^2 \\
&= r^2 \sin^2\theta \sin^2\phi + r^2 \sin^2\theta \cos^2\phi \\
&= r^2 \sin^2\theta \left(\sin^2\phi + \cos^2\phi\right) \\
&= r^2 \sin^2\theta
\end{aligned}$$

And

$$\begin{aligned}
g_{22} &= g_{rr} \\
&= \frac{\partial \zeta^1}{\partial x^2}\frac{\partial \zeta^1}{\partial x^2} + \frac{\partial \zeta^2}{\partial x^2}\frac{\partial \zeta^2}{\partial x^2} + \frac{\partial \zeta^3}{\partial x^2}\frac{\partial \zeta^3}{\partial x^2} \\
&= \frac{\partial \zeta^1}{\partial r}\frac{\partial \zeta^1}{\partial r} + \frac{\partial \zeta^2}{\partial r}\frac{\partial \zeta^2}{\partial r} + \frac{\partial \zeta^3}{\partial r}\frac{\partial \zeta^3}{\partial r} \\
&= \left(\frac{\partial \zeta^1}{\partial r}\right)^2 + \left(\frac{\partial \zeta^2}{\partial r}\right)^2 + \left(\frac{\partial \zeta^3}{\partial r}\right)^2 \\
&= \left(\sin\theta\cos\phi\right)^2 + \left(\sin\theta\sin\phi\right)^2 + (\cos\theta)^2 \\
&= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta \\
&= \sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta \\
&= \sin^2\theta + \cos^2\theta \\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
g_{33} &= g_{\theta\theta} \\
&= \frac{\partial \zeta^1}{\partial x^3}\frac{\partial \zeta^1}{\partial x^3} + \frac{\partial \zeta^2}{\partial x^3}\frac{\partial \zeta^2}{\partial x^3} + \frac{\partial \zeta^3}{\partial x^3}\frac{\partial \zeta^3}{\partial x^3} \\
&= \frac{\partial \zeta^1}{\partial \theta}\frac{\partial \zeta^1}{\partial \theta} + \frac{\partial \zeta^2}{\partial \theta}\frac{\partial \zeta^2}{\partial \theta} + \frac{\partial \zeta^3}{\partial \theta}\frac{\partial \zeta^3}{\partial \theta} \\
&= \left(\frac{\partial \zeta^1}{\partial \theta}\right)^2 + \left(\frac{\partial \zeta^2}{\partial \theta}\right)^2 + \left(\frac{\partial \zeta^3}{\partial \theta}\right)^2 \\
&= \left(r\cos\theta\cos\phi\right)^2 + \left(r\cos\theta\sin\phi\right)^2 + (-r\sin\theta)^2 \\
&= r^2\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right) + r^2\sin^2\theta \\
&= r^2\cos^2\theta + r^2\sin^2\theta \\
&= r^2
\end{aligned}$$

Hence ds^2 in Spherical coordinates is

$$\begin{aligned}
ds^2 &= g_{kl}dx^k dx^l \\
&= g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2\right)^2 + g_{33}\left(dx^3\right)^2 \\
&= g_{11}\left(d\phi\right)^2 + g_{22}(dr)^2 + g_{33}(d\theta)^2 \\
&= r^2\sin^2\theta\left(d\phi\right)^2 + (dr)^2 + r^2(d\theta)^2
\end{aligned}$$

From the above we see that, using the order ϕ, r, θ for the rows and columns

$$\begin{aligned}
g_{ij} &= \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \\
&= \begin{pmatrix} r^2\sin^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix}
\end{aligned}$$

Therefore the determinant is $g = r^4\sin^2\theta$ and h_i are given by the square root of the diagonal elements of g_{ij}

$$\begin{aligned}
h_1 &= r\sin\theta \qquad \text{(A)} \\
h_2 &= 1 \\
h_3 &= r
\end{aligned}$$

4.2 Finding Gradient

$$\nabla = \left(\frac{1}{h_1}\frac{\partial}{\partial x^1}, \frac{1}{h_2}\frac{\partial}{\partial x^2}, \frac{1}{h_3}\frac{\partial}{\partial x^3}\right)$$

Where h_i are given in (A) and $x^1 = \phi, x^2 = r, x^3 = \theta$. Therefore

$$\nabla = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}\right)$$

Hence given a function scalar $f\left(\phi, r, \theta\right)$ then

$$\nabla f = \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{e}_\phi + \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{e}_\theta$$

4.3 Finding Curl

Using h_i in (A) and $x^1 = \phi, x^2 = r, x^3 = \theta$ then

$$\left(\vec{\nabla}\times\vec{V}\right)_1 = \frac{1}{h_2h_3}\left(\frac{\partial}{\partial x^2}\left(h_3V_3\right) - \frac{\partial}{\partial x^3}\left(h_2V_2\right)\right)$$
$$\left(\vec{\nabla}\times\vec{v}\right)_\phi = \frac{1}{r}\left(\frac{\partial\left(rV_\theta\right)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right)$$

And

$$\left(\vec{\nabla}\times\vec{V}\right)_2 = \frac{1}{h_3h_1}\left(\frac{\partial}{\partial x^3}\left(h_1V_1\right) - \frac{\partial}{\partial x^1}\left(h_3V_3\right)\right)$$
$$\left(\vec{\nabla}\times\vec{V}\right)_r = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}\left(r\sin\theta V_\phi\right) - \frac{\partial}{\partial\phi}\left(rV_\theta\right)\right)$$
$$= \frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta V_\phi\right)}{\partial\theta} - \frac{\partial V_\theta}{\partial\phi}\right)$$

And

$$\left(\vec{\nabla}\times\vec{V}\right)_3 = \frac{1}{h_1h_2}\left(\frac{\partial}{\partial x^1}\left(h_2V_2\right) - \frac{\partial}{\partial x^2}\left(h_1V_1\right)\right)$$
$$\left(\vec{\nabla}\times\vec{V}\right)_\theta = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\phi}\left(V_r\right) - \frac{\partial}{\partial r}\left(r\sin\theta V_\phi\right)\right)$$
$$= \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial\left(rV_\phi\right)}{\partial r}\right)$$

Therefore given a vector $\vec{V}$, its curl is

$$\vec{\nabla}\times\vec{V} = \frac{1}{r}\left(\frac{\partial\left(rV_\theta\right)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right)\hat{e}_\phi + \frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta V_\phi\right)}{\partial\theta} - \frac{\partial V_\theta}{\partial\phi}\right)\hat{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial\left(rV_\phi\right)}{\partial r}\right)\hat{e}_\theta$$

4.4 Finding Divergence

$$\nabla\cdot V = \nabla_i V^i = \frac{\partial}{\partial x^i}V^i + \Gamma^i_{ij}V^j \tag{1}$$

Where $\Gamma^i_{ij} = \frac{1}{2}g^{li}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right) = \frac{1}{2}g^{li}\left(\frac{\partial g_{il}}{\partial x^j}\right)$ which simplifies to as shown in class notes page 143 to hence above becomes

$$\Gamma^i_{ij} = \frac{1}{\sqrt{g}}\frac{\partial}{x^j}\left(\sqrt{g}\right)$$

Hence (1) becomes

$$\begin{aligned}\nabla \cdot V &= \frac{\partial}{\partial x^i} V^i + \frac{1}{\sqrt{g}} \frac{\partial}{x^j} \left(\sqrt{g}\right) V^j \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{x^i} \left(\sqrt{g} V^i\right)\end{aligned}$$

Using the covariant form the above becomes

$$\nabla \cdot V = \frac{1}{\sqrt{g}} \frac{\partial}{x^i} \left(\frac{\sqrt{g}}{\sqrt{g_{ii}}} V_i\right)$$

Where in class notes h_i is used in place of $\sqrt{g_{ii}}$, but it is it the same.

The sum is over i. From above, the spherical coordinates are $x^1 = \phi, x^2 = r, x^3 = \theta$. And $g = r^4 \sin^2 \theta$. Hence the above becomes after expanding

$$\begin{aligned}\nabla . V &= \frac{1}{\sqrt{r^4 \sin^2 \theta}} \left(\frac{\partial}{\partial \phi} \left(\frac{\sqrt{r^4 \sin^2 \theta}}{\sqrt{g_{\phi\phi}}} V_\phi \right) + \frac{\partial}{\partial r} \left(\frac{\sqrt{r^4 \sin^2 \theta}}{\sqrt{g_{rr}}} V_r \right) + \frac{\partial}{\partial \theta} \left(\frac{\sqrt{r^4 \sin^2 \theta}}{\sqrt{g_{\theta\theta}}} V_\theta \right) \right) \\ &= \frac{1}{r^2 \sin \theta} \left(\frac{\partial}{\partial \phi} \left(\frac{r^2 \sin \theta}{r \sin \theta} V_\phi \right) + \frac{\partial}{\partial r} \left(\frac{r^2 \sin \theta}{1} V_r \right) + \frac{\partial}{\partial \theta} \left(\frac{r^2 \sin \theta}{r} V_\theta \right) \right) \\ &= \frac{1}{r^2 \sin \theta} \left(\frac{\partial}{\partial \phi} \left(r V_\phi \right) + \frac{\partial}{\partial r} \left(r^2 \sin \theta V_r \right) + \frac{\partial}{\partial \theta} \left(\sin \theta V_\theta \right) \right) \\ &= \frac{\partial}{\partial \phi} \left(\frac{1}{r \sin \theta} V_\phi \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 V_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta V_\theta \right)\end{aligned}$$

4.5 Finding Laplacian

The Laplacian is given by

$$\nabla^2 = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i} \left(\frac{\sqrt{\det(g)}}{g_{ii}} \frac{\partial}{\partial x^i} \right)$$

Hence

$$\begin{aligned}\nabla^2 &= \frac{1}{\sqrt{r^4 \sin^2 \theta}} \frac{\partial}{\partial x_1} \left(\frac{\sqrt{r^4 \sin^2 \theta}}{g_{11}} \frac{\partial}{\partial x^1} \right) + \frac{1}{\sqrt{r^4 \sin^2 \theta}} \frac{\partial}{\partial x_2} \left(\frac{\sqrt{r^4 \sin^2 \theta}}{g_{22}} \frac{\partial}{\partial x^2} \right) + \frac{1}{\sqrt{r^4 \sin^2 \theta}} \frac{\partial}{\partial x_3} \left(\frac{\sqrt{r^4 \sin^2 \theta}}{g_{33}} \frac{\partial}{\partial x^3} \right) \\ &= \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{r^2 \sin \theta}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left(\frac{r^2 \sin \theta}{1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\frac{r^2 \sin \theta}{r^2} \frac{\partial}{\partial \theta} \right) \\ &= \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) \\ &= \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2} \left(2r \frac{\partial}{\partial r} + r^2 \frac{\partial^2}{\partial r^2} \right) + \frac{1}{r^2 \sin \theta} \left(\cos \theta \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial^2}{\partial \theta^2} \right) \\ &= \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \theta^2} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}\end{aligned}$$

Therefore

$$\begin{aligned}\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \left(\frac{\cos \theta}{\sin \theta} \frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial \theta^2} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \\ &= u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \left(\frac{\cos \theta}{\sin \theta} u_\theta + u_{\theta\theta} \right) + \frac{1}{r^2 \sin^2 \theta} u_{\phi\phi}\end{aligned}$$