HW5, Math 228A Conjugate Gradient Solver

Date due 12/10/2010 UC Davis, California
Fall 2010

Nasser M. Abbasi

Fall 2010 Compiled on January 4, 2020 at 11:15pm [public]

Contents

1 Introduction

Math 228A
Homework 5
Due Friday, 12/10/10, 4:00 P.M.

Homework must be turned in to Arcade before the deadline. You may email him a pdf file or put a hard copy in his mailbox.

Exam week office hours:
Bob, Monday 12-1
Arcade, Tuesday & Wednesday 1:30-2:20

1. Write a program to solve the discrete Poisson equation on the unit square using preconditioned conjugate gradient. Set up a test problem and compare the number of iterations and efficiency of using (i) no preconditioning and (ii) SSOR preconditioning. Run your tests for different grid sizes. How does the number of iterations scale with the number of unknowns as the grid is refined?

   Note that there are two typos in the PCG algorithm in our textbook. See your class notes, another textbook, or the author's webpage for the correct algorithm.

   SSOR preconditioning Symmetric SOR (SSOR) consists of one forward sweep of SOR followed by one backward sweep of SOR. For the discrete Poisson equation, one step of SSOR is

$$u_{i,j}^{k+1/2} = \frac{\omega}{4}(u_{i-1,j}^{k+1/2} + u_{i,j-1}^{k+1/2} + u_{i+1,j}^{k} + u_{i,j+1}^{k} - h^2 f_{i,j}) + (1-\omega)u_{i,j}^{k}$$
$$u_{i,j}^{k+1} = \frac{\omega}{4}(u_{i-1,j}^{k+1/2} + u_{i,j-1}^{k+1/2} + u_{i+1,j}^{k+1} + u_{i,j+1}^{k+1} - h^2 f_{i,j}) + (1-\omega)u_{i,j}^{k+1/2}.$$

   It can be shown that one step of SSOR in matrix form is equivalent to

$$\frac{1}{\omega(2-\omega)}(D-\omega L)D^{-1}(D-\omega U)(\boldsymbol{u}^{k+1} - \boldsymbol{u}^{k}) = \boldsymbol{f},$$

   where $A = D - L - U$.

   For the constant coefficient problem, this suggests the preconditioner.

$$M = (D-\omega L)(D-\omega U).$$

   Note: If you are interested, experiment with incomplete Cholesky factorization preconditioning and multigrid preconditioning. Incomplete Cholesky preconditioning requires that you form the matrix. Vary the amount of fill (in MATLAB use `cholinc` and vary the drop tolerance). Obviously, a factorization with more elements results in fewer iterations of CG, but it is more expensive to compute and to apply the preconditioner. To use MG as a preconditioner, the product $M^{-1}r$ is computed by applying one V-cycle with zero initial guess with right hand side r. If the smoother is symmetric and the number of pre and post smoothing steps are the same, this preconditioner is symmetric positive definite and may be used with CG.

Figure 1: problem description

The test problem used is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on the unit square $[(0,1),(0,1)]$ with zero boundary conditions.

The above problem is solved using the numerical method of conjugate gradient iterative solver. The mesh spacings used is

$$h = \left\{\frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}\right\}$$

and the tolerance used to check for convergence is

$$\varepsilon = 10^{-6}$$

The solver terminates when the mesh norm of the residual becomes smaller than the above quantity using the following check

$$\sqrt{h}\left\|r^{(k)}\right\|_2 < \varepsilon$$

Where in the above, $r^{(k)}$ is the residual at the k^{th} iteration and h is the current value of the mesh spacing.

The reason for using *zero* as the driving force on the RHS of the pde, is to allow the calculation and tracking of the error at each iteration as the exact solution u_{exact} for this problem is now known, which is zero. Now the error at each iteration k to be found using

$$\begin{aligned} e^{(k)} &= \left\| u_{exact} - u^{(k)} \right\| \\ &= \left\| u^{(k)} \right\| \end{aligned}$$

Where in the above $u^{(k)}$ represents the approximate solution at the k^{th} iteration.

A Matlab function *CG.m* was written to implement conjugate gradient solver. One of the parameters this function accepts is the name of the preconditioner to use. The following preconditioners are supported

NONE, SSOR, MultiGrid, IncompleteCholesky

When NONE is specified, then no preconditioning is done.

For each preconditioner, the solver was run to find the solution to the above test problem. The initial guess for the solution $u^{(0)}$ used was generated using Matlab rand() function.

For each preconditioner the following plots were generated

1. Plot of error $\left\| e^{(k)} \right\|$ per iteration k which showed how the rate of error reduction per iteration. The plot was generated in log and linear scale.
2. Plot of the residual $\left\| r^{(k)} \right\|$ per iteration which showed the rate of residual norm reduction per iteration, and also plotted in log and linear scale. The initial residual is defined as $r^{(0)} = f - Au^{(0)}$ and each subsequent iteration, the residual is defined as $r^{(k)} = r^{(k-1)} - \alpha A p^{(k-1)}$. The algorithm below illustrates this in more details.
3. The spectrum of the eigenvalues of A and the spectrum of the eigenvalues of $M^{-1}A$ are plotted using matlab's scatter() command to better see the effect on the condition number value when multiplying A by M^{-1}, where M is the preconditioner matrix.
4. Plot of the final solution found on a 3D mesh plot. The final solution was verified to be close to zero, which is the same as the exact solution.

In addition to the above plots, for each mesh spacing h, the actual result table is printed which tabulates the above values at each iteration. This table was used to generate the above plots. The printed tables also show the ratio of the value of norm of the residual at the current iteration to its value at the previous iteration, similarly for the error norm.

Due to the large size of these tables, the tables for all the spacings and for each solver are available in the appendix.

Conjugate gradient algorithm description

The idea of conjugate gradient is to use preconditioning matrix to speed up the convergence of the conjugate gradient method. The original problem

$$Ax = f$$

is transformed to a new problem

$$M^{-1}Ax = M^{-1}f$$

such that $M^{-1}A$ has a smaller condition number than A. For most iterative solvers, the rate of convergence increases as the condition number of the system matrix A decreases.

The conjugate gradient method works only on symmetric positive definite A matrix, and its speed of convergence is affected by the distribution of the eigenvalues of the A matrix. The estimate of convergence is more accurate if the distribution of eigenvalues is uniform. For the discrete 2D Poisson problem, this is the case, as verified by plots of the spectrum generated below for each case.

The preconditioning is used to modify the spectrum of A so that the eigenvalues of the new system matrix $M^{-1}A$ become more clustered together causing the condition number to become smaller and thus increasing the convergence rate.

The following table was generated to show the effect of preconditioning on lowering the condition number. It shows the condition number for CG (in other words, for the A matrix only), and then the condition number for $M^{-1}A$ for different solvers as mesh spacing is changed. It also shows below the condition number value, in a box, the maximum eigenvalue and the minimum eigenvalue. Notice that in the following table, if one tries to apply the $\frac{|\max\lambda|}{|\min\lambda|}$ to determine the condition number, then the result will not match the condition number as shown. The above formula do not apply in this case, as these are sparse matrices and the condition number was found by estimating its value using the Matlab function condest() and not by applying the above formula.

solver	$h=\frac{1}{16}, N=225$	$h=\frac{1}{32}, N=961$	$h=\frac{1}{64}, N=3969$	$h=\frac{1}{128}, N=16129$
NONE	150, (7.923, 0.0768)	603, (7.9807, 0.01926)	2413, (7.9995, 0.0048)	9655, (7.9987, 0.001205)
SSOR	33, (0.25, 0.018203)	128, (0.25, 0.004748)	511, (0.25, 0.0012)	
IncCholesky $\varepsilon=10^{-2}$	32, (2.365, 0.2279)	108, (2.44, 0.0585)	422, (2.537, 0.0146)	
IncCholesky $\varepsilon=10^{-3}$	48, (2.337, 0.508)	153, (2.526, 0.1603)	373, (2.592, 0.041756)	

This diagram below reflects the above table result to clearly show the reduction of the condition number as a result of preconditioning.

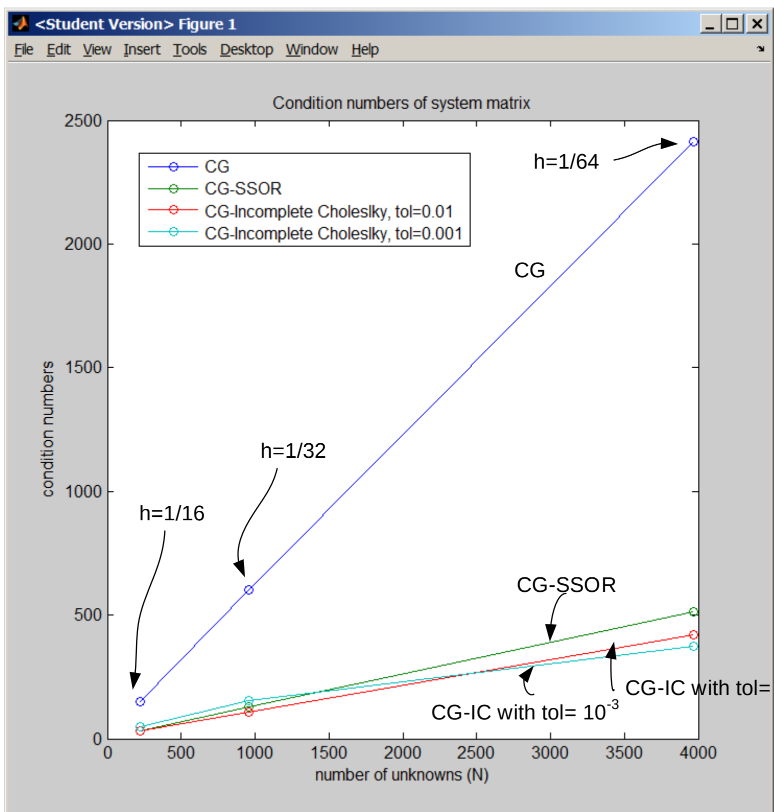


Figure 2: compare condition numbers

CG Algorithm pseudocode

The following is the algorithm used for the implementation of conjugate gradient with preconditioning.

$$
\begin{aligned}
&Input: \; A, f, tol, preconditionSolverName, dropTol\\
&Output: \; \tilde{u} \; approximate \; solution \; to \; Ax = 0\\
\\
&u_0 = \text{rand}() \; (*initial \; guess \; of \; solution \; *)\\
&r_0 = f - Au_0 \; (*initial \; residual*)\\
&z_0 \leftarrow CALL \quad preconditionSolver(r_0, A, preconditionSolverName, dropTol)\\
&p_0 = z_0\\
&FOR \; k = 1,2,3,\cdots\\
&\qquad e_{k-1} = \sqrt{h}\, \|u_{k-1}\|_2 \quad (* \; the \; error \; since \; u_{exact} \; is \; known \; to \; be \; zero*)\\
&\qquad \omega_{k-1} = Ap_{k-1}\\
&\qquad \alpha_{k-1} = \frac{z_{k-1}^T r_{k-1}}{p_{k-1}^T \omega_{k-1}}\\
&\qquad u_k = u_{k-1} + \alpha_{k-1} \; p_{k-1}\\
&\qquad r_k = r_{k-1} - \alpha_{k-1}\omega_{k-1}\\
&\qquad IF \; \left(\sqrt{h}\, \|r_k\|_2\right) < tol \; THEN\\
&\qquad\quad RETURN \; u_k\\
&\qquad END \; IF\\
&\qquad z_k \leftarrow CALL \quad preconditionSolver(r_k, A, preconditionSolverName, dropTol)\\
&\qquad \beta_{k-1} = \frac{z_k^T r_k}{z_{k-1}^T r_{k-1}}\\
&\qquad p_k = z_k + \beta_{k-1} p_{k-1}\\
&END \; FOR
\end{aligned}
$$

The algorithm for the function *preconditionSolver*() is as follows

$$
\begin{aligned}
&Input: \; r, A, preconditionSolverName, dropTol\\
&Output: \; z \; approximate \; solution \; to \; Mz = r\\
&CASE \; preconditionSolverName \; IS\\
&\qquad WHEN \; NONE \; z \leftarrow r \qquad\qquad // \; no \; preconditioning\\
&\qquad WHEN \; MultiGrid\\
&\qquad\quad \mu_1 = \mu_2 = 1 \; (* \; presmoother \; and \; postsmoother*)\\
&\qquad\quad z \leftarrow CALL \; VCYCLE(zeroInitialGuess, r)\\
&\qquad\qquad //VCYCLE \; is \; one \; implemented \; in \; HW4 \; but \; changed \; to \; do\\
&\qquad\qquad //one \; forward \; Gauss-Seidel/red-black \; followed \; by\\
&\qquad\qquad //one \; reverse \; Gauss-Seidel/red-black\\
&\qquad WHEN \quad SSOR\\
&\qquad\quad z \leftarrow CALL \; SOR \; forward \; followed \; by \; SOR \; in \; reverse\\
&\qquad WHEN \; IncompleteCholesky\\
&\qquad\quad R = cholinc(A, dropTol)\\
&\qquad\quad z \leftarrow R\backslash(R^T\backslash r)\\
&END \; CASE\\
&RETURN \; z
\end{aligned}
$$

2 Solvers efficiency and iterations count

In addition to finding the number of iterations needed for convergence by each solver, the problem also asked to compare the efficiency of each solver. This is done by finding the work needed by each solver to converge.

Work needed is defined as

$$Work = NumberOfIterations \; \times \; WorkPerIteration$$

Before determining the work for each solver, the following table lists the *cputime* used by each solver for the different spacings. The cpu time is measured using Matlab cputime function, and measures only the call to CG() and does not include any other calls such as plotting.

preconditioning	$h=\frac{1}{16}$ $N=225$	$h=\frac{1}{32}$ $N=961$	$h=\frac{1}{64}$ $N=3969$	$h=\frac{1}{128}$ $N=16129$
NONE	0.19	0.37	13.48	556.6
Multigrid	0.34	0.6	13.48	566
SSOR	0.23	0.5	13.3	564.6
Incomplete Cholesky $\varepsilon=10^{-2}$	0.16	0.34	13.8	559
Incomplete Cholesky $\varepsilon=10^{-3}$	0.19	0.5	13.3	559

Surprisingly, no appreciable difference can be seen between the different solvers in terms of cpu time. It was expected that NONE would have the largest CPU time as it has the lowest efficiency. This result can be attributed to using small number of N values, which was not large enough in the limit to reflect the difference. One needs to use much larger values of N to see the effect of preconditioning on CPU time difference. Due to memory limitation, this was not possible to implement at this time. Now the work per iteration is analyzed.

2.1 Work per iteration

All solvers perform similar work per iterations except for the step needed to apply the preconditioning to determine z_k. The only difference between not using preconditioning and using one, is in the step to solve for z in $Mz=r$. Using work per iteration as $O(N)$ for the base CG with no preconditioning, then the following can be defined for work per iteration for each solver:

1. No preconditioner is applied: no extra work is needed, as z is the same as r hence $O(N)$
2. *multigrid* : work needed to determine z adds an extra cost of one V cycle. Work for one V cycle was found from HW4 to be $\frac{4}{3}C\times N$ where N is number of unknowns and C is a constant estimated to be $(7(v_1+v_2)+13)$ where v_1,v_2 are the numbers of pre smooth and post smooth operations. These are both *one* in this case. The smoothing is done twice (forward and reverse), hence the above becomes $(2\times 7(v_1+v_2)+13)$, resulting in work per iteration of $\frac{4}{3}(2\times 7(v_1+v_2)+13)N$ or about $55N$. Adding the $O(N)$ from the above, this is still results in $O(N)$.
3. *SSOR* : The cost is twice one SOR step. One step of SOR work is $7N$, where N is number of unknowns, since it takes about 7 flop operations to smooth one grid point, and there are N of these. Hence for SSOR work is twice that or $14N$. As above, this is still an order of N.

2.2 Number of iterations

From lectures notes, it was found that the error rate in conjugate gradient (with no preconditioning) behaves as

$$\|e_k\|_A \leq 2\frac{\sqrt{\Bbbk(A)}-1}{\sqrt{\Bbbk(A)}+1}\|e_0\|_A$$

Where $\Bbbk(A)$ is the condition number of A, it was shown that $\Bbbk(A)=O\left(h^{-1}\right)$ where h is the mesh spacing. Hence, for fixed tolerance, which is the case here, the number of iterations is $O\left(h^{-1}\right)=O\left(N^{1/2}\right)$ where N is number of unknowns.

The results of the numerical experiment done agrees with the above, as shown below, for the case of NONE (which is the case of conjugate gradient with no preconditioning).

The following table was generated from result of running the program. In this table, N is the number of unknowns, $\Bbbk\left(M^{-1}A\right)$ is the condition number of $M^{-1}A$, and $\Bbbk(A)$ is the condition number of A. Since sparse matrices are used, the Matlab function condest() was used to find the condition numbers. In this table I.C. means Incomplete Cholesky

preconditioner	$h=\frac{1}{16}$ $N=225$	$\Bbbk\left(M^{-1}A\right)$	$\Bbbk(A)$	$h=\frac{1}{32}$ $N=961$	$\Bbbk\left(M^{-1}A\right)$	$\Bbbk(A)$
NONE	42	N/A	150	82	N/A	603
Multigrid	4		150	4		603
SSOR	18	33	150	30	128	603
IC $\varepsilon=10^{-2}$	7	32	150	13	108	603
IC $\varepsilon=10^{-3}$	4	48	150	6	153	603

preconditioner	$h=\frac{1}{64}$ $N=3969$	$\Bbbk\left(M^{-1}A\right)$	$\Bbbk\left(A\right)$	$h=\frac{1}{128}$ $N=16129$	$\Bbbk\left(M^{-1}A\right)$	$\Bbbk\left(A\right)$
NONE	157	N/A	2413	291	N/A	9655
Multigrid	4		2413	4		9655
SSOR	56	511	2413	103	Memory problem	9655
IC $\varepsilon=10^{-2}$	22	422	2413	39	Memory problem	9655
IC $\varepsilon=10^{-3}$	8	373	2413	14	Memory problem	9655

The following is a plot that represents the above results.

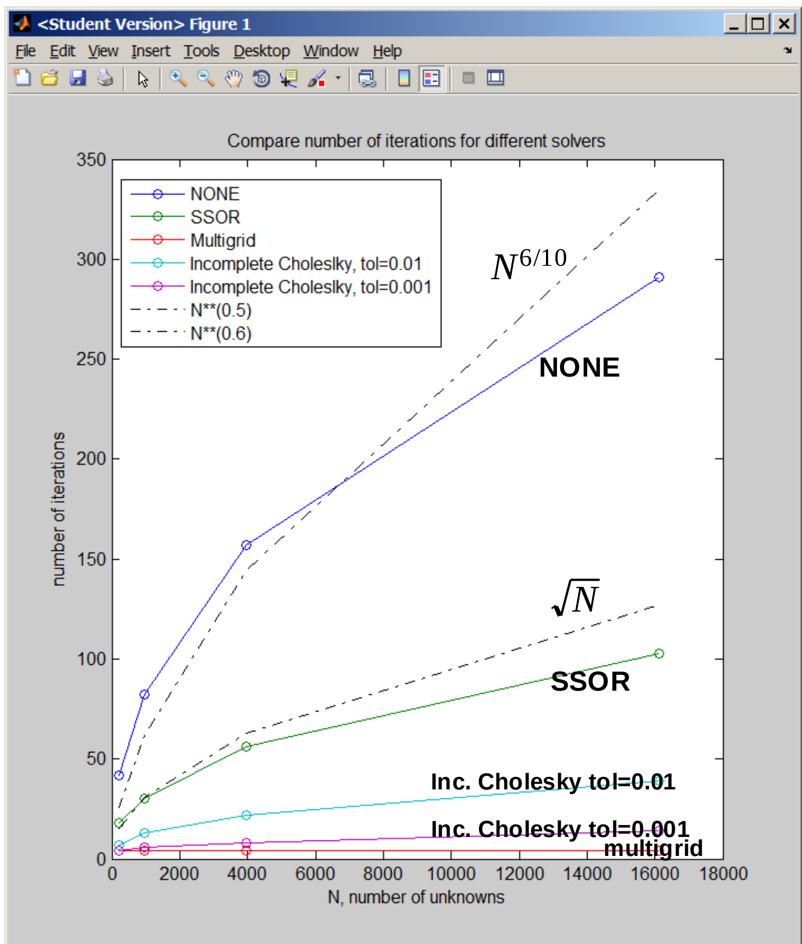


Figure 3: iterations plot

From the above one can see that multigrid has $O(1)$ for the number of iterations. The number of iterations was 4 for all the cases from $h=\frac{1}{16}$ to $h=\frac{1}{128}$. For SSOR, the number of iterations grew sublinear in terms of N, from the above one can estimate this to be $O\left(N^{1/4}\right)$, while for no preconditioning, the number of iterations grew as approximately as $O\left(N^{\frac{1}{2}}\right)$ as predicted by earlier analytical result.

3 Discussion of results and conclusions

The use of preconditioning on A caused a reduction of the number of iterations to convergence to the same fixed tolerance when compared to convergence with no preconditioning. This was due to reduction of the condition number of the system matrix as can be seen in the above table. By reducing the largest eigenvalue, the rate of convergence increased. However, preconditioning also adds an extra cost per iteration. The extra work however, was also of order N and hence the final efficiency was governed by the number of iterations for large N.

Therefore, this is the result of work efficiency for the main solvers, using the formula of

$$\text{Work} = \text{NumberOfIterations} \times \text{WorkPerIteration}$$

1. NONE: $O\left(N^{1/2}\right)\times O(N)=O\left(N^{3/2}\right)$

2. SSOR: $O\left(N^{1/4}\right) \times O(N) = O\left(N^{5/4}\right)$
3. Multigrid: $O(1) \times O(N) = O(N)$

Incomplete Cholesky was not added as it was hard to estimate from the curve above the number of iterations and since depend on the drop tolerance values.

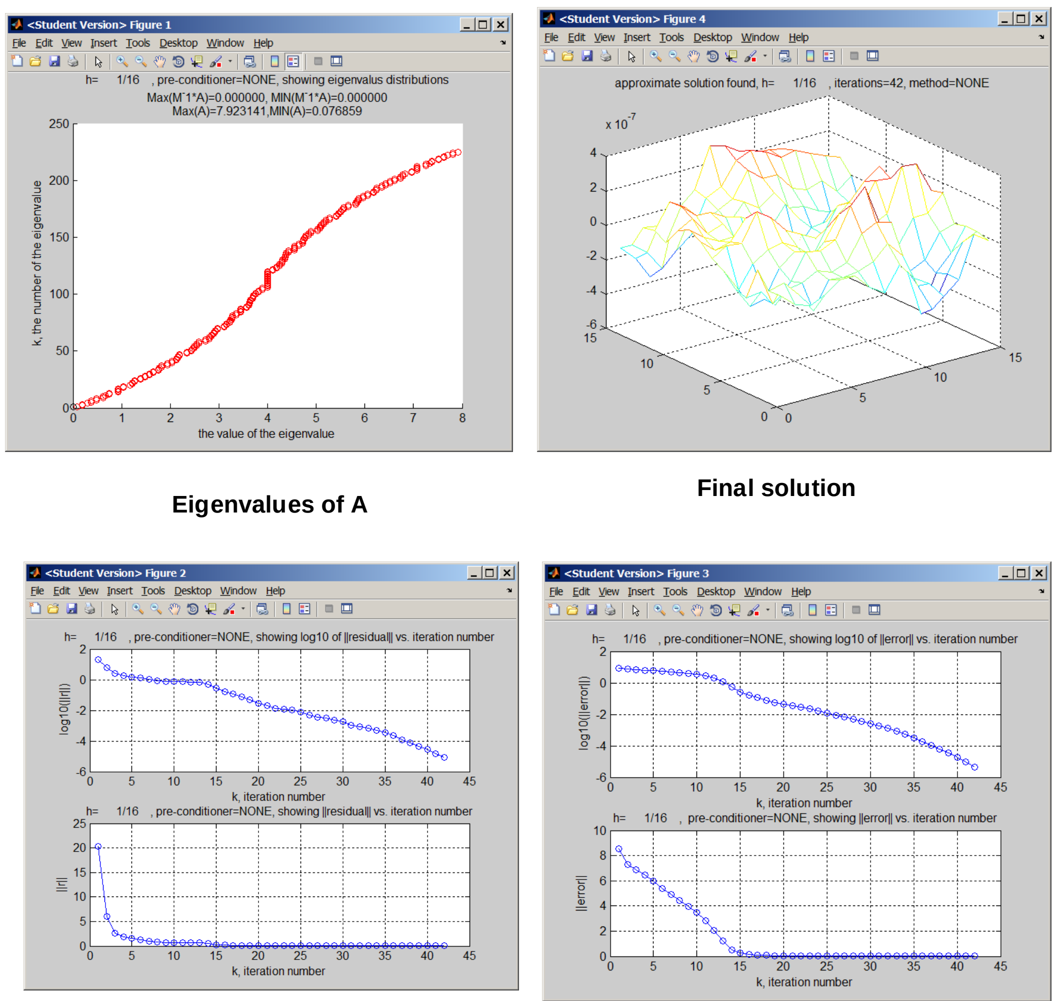
The above analysis showed that **Multigrid is most efficient**, followed by Incomplete Cholesky (but these depend on the tolerance drop term), followed by SSOR, and finally by CG with no preconditioning

In conclusion, using Multigrid for preconditioner for conjugate gradient seems to be the most effective solver.

4 Appendix

4.1 Result for CG with no preconditioner

Plots h=1/16

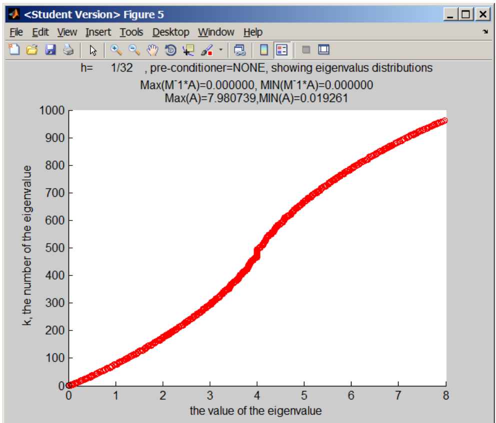


Eigenvalues of A **Final solution**

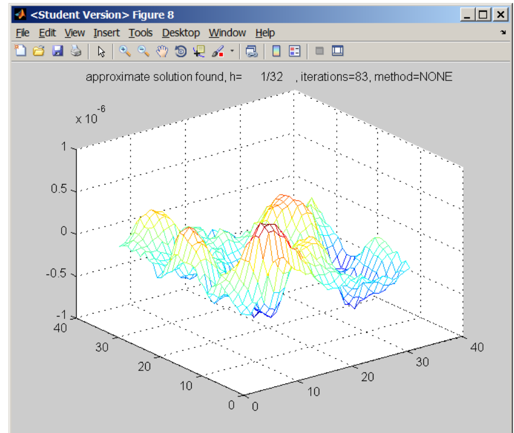
Residual per iteration **ERROR per iteration**

Figure 4: solver none plots 16

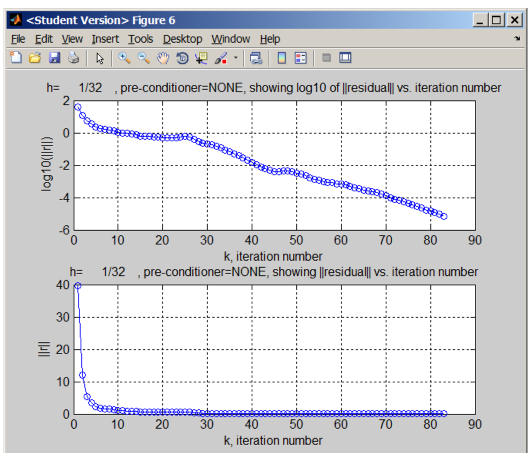
Plots for h=1/32



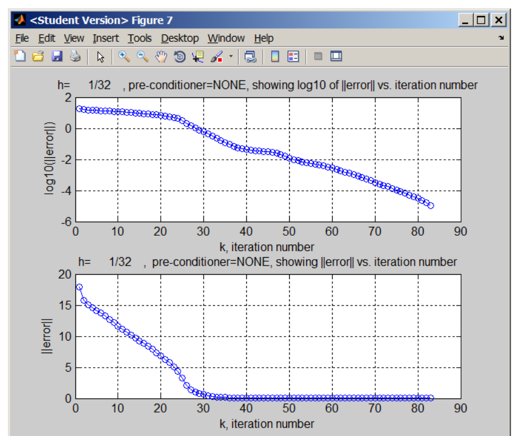
Eigenvalues of A



Final solution



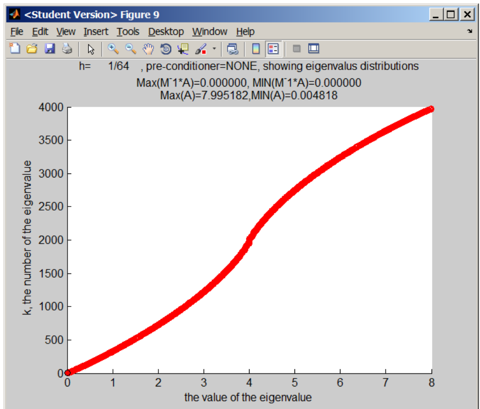
Residual per iteration



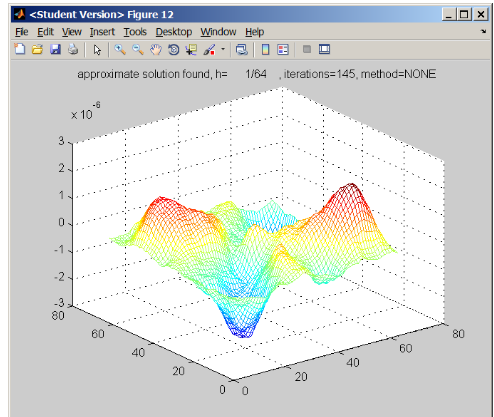
ERROR per iteration

Figure 5: solver none plots 32

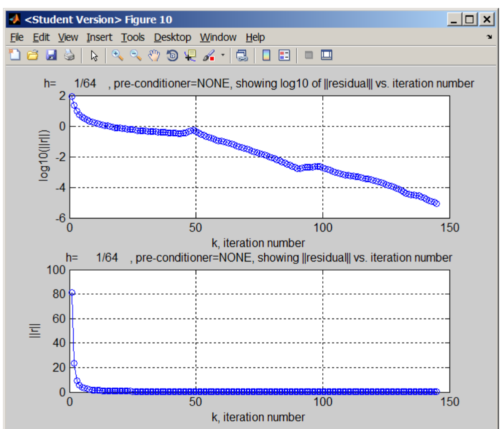
Plot for h=1/64



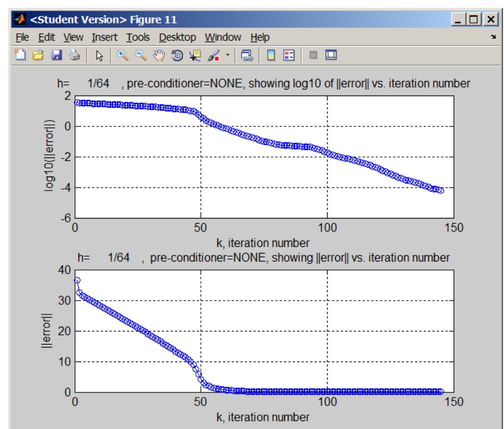
Eigenvalues of A



Final solution



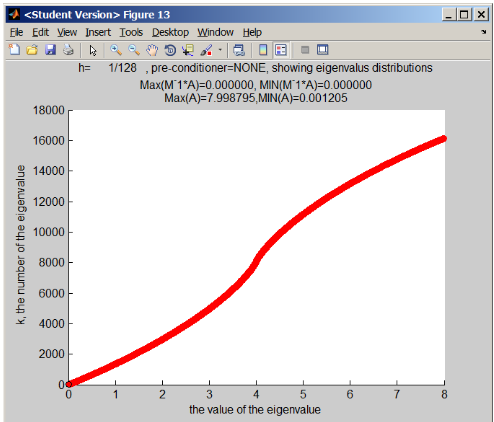
Residual per iteration



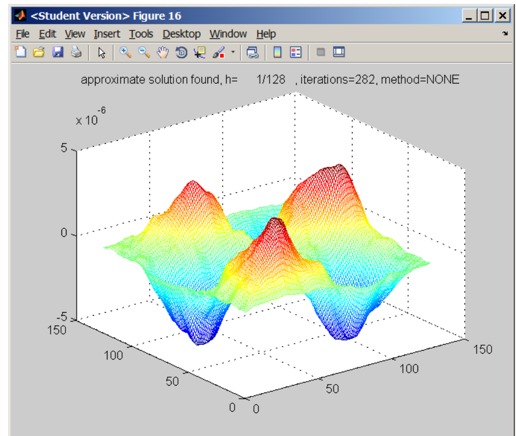
ERROR per iteration

Figure 6: solver none plots 64

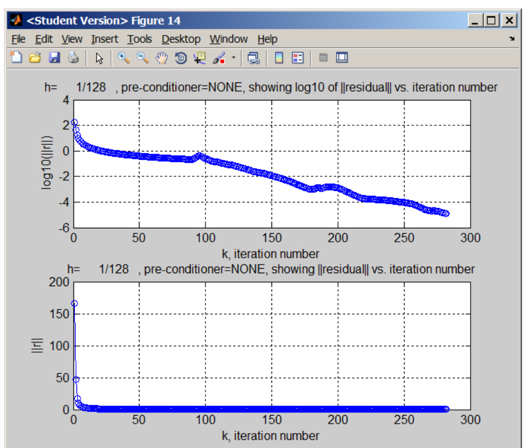
<u>Plots for h=1/128</u>



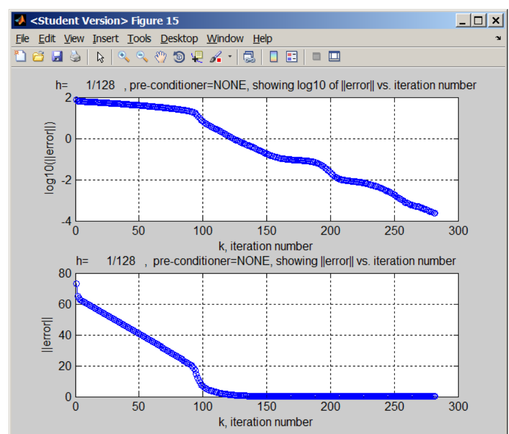
Eigenvalues of A



Final solution



Residual per iteration

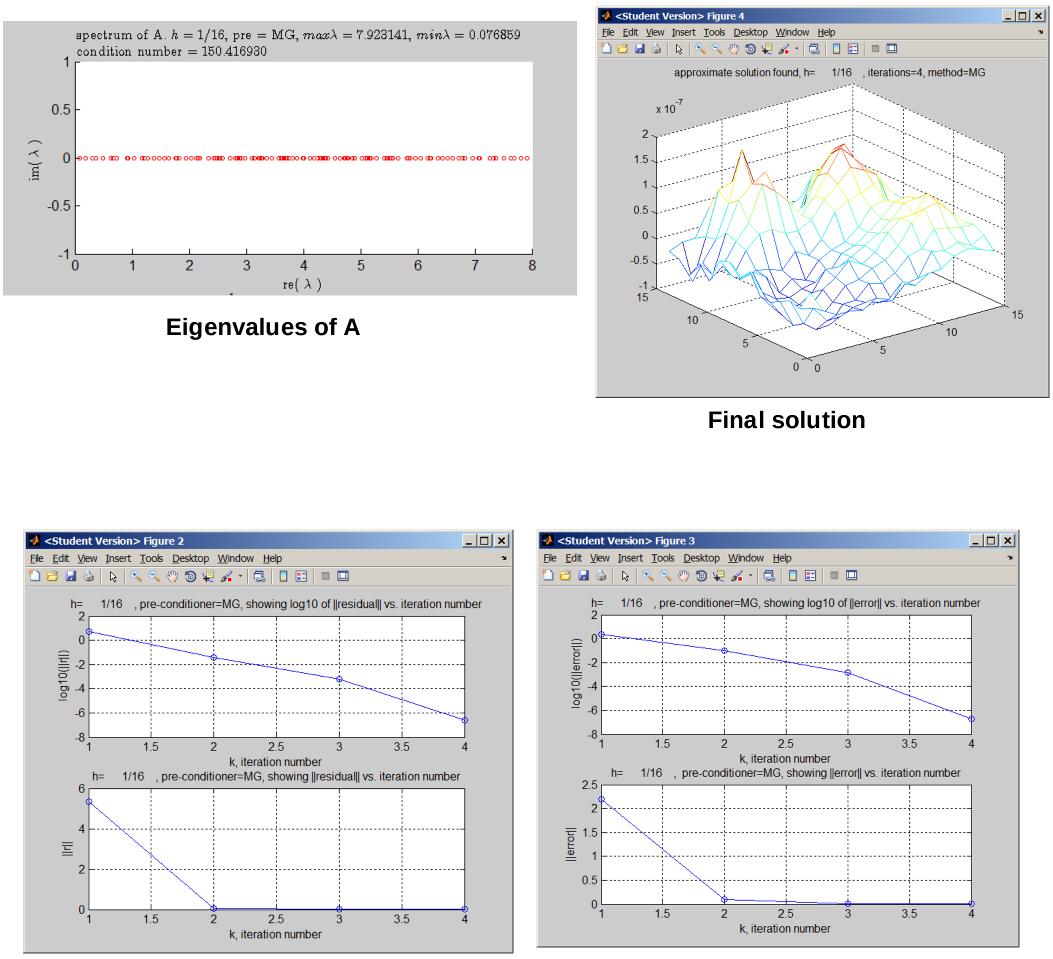


ERROR per iteration

Figure 7: solver none plots 128

4.2 Result for CG with Multigrid preconditioner

Plots for h=1/16



Eigenvalues of A

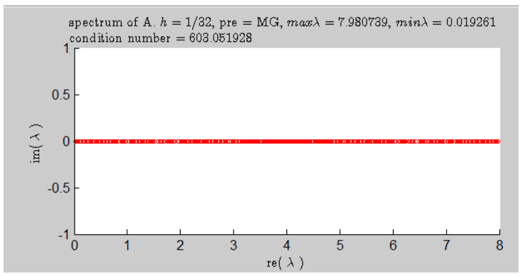
Final solution

Residual per iteration

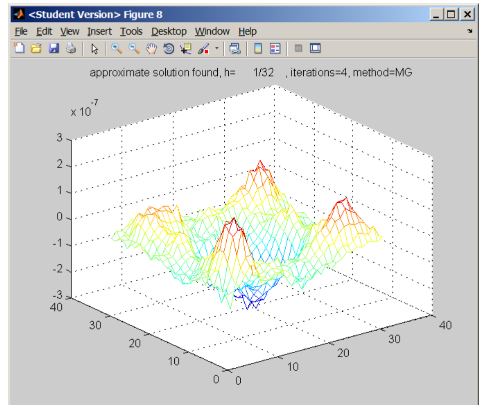
ERROR per iteration

Figure 8: solver MG plots 16

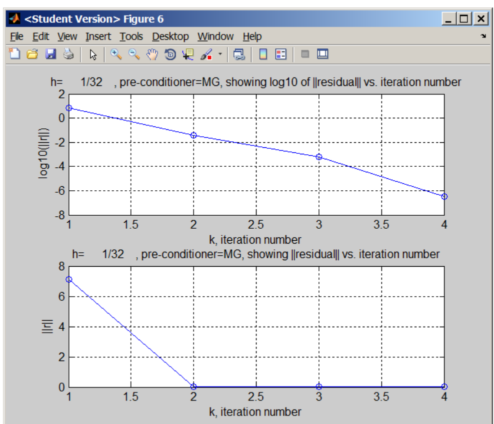
Plots for h1/32



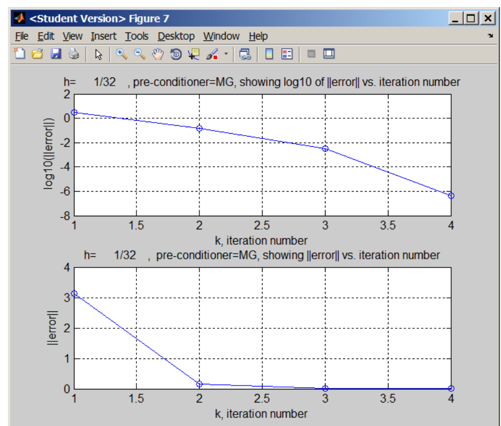
Eigenvalues of A



Final solution



Residual per iteration



ERROR per iteration

Figure 9: solver MG plots 32

Plots for h=164

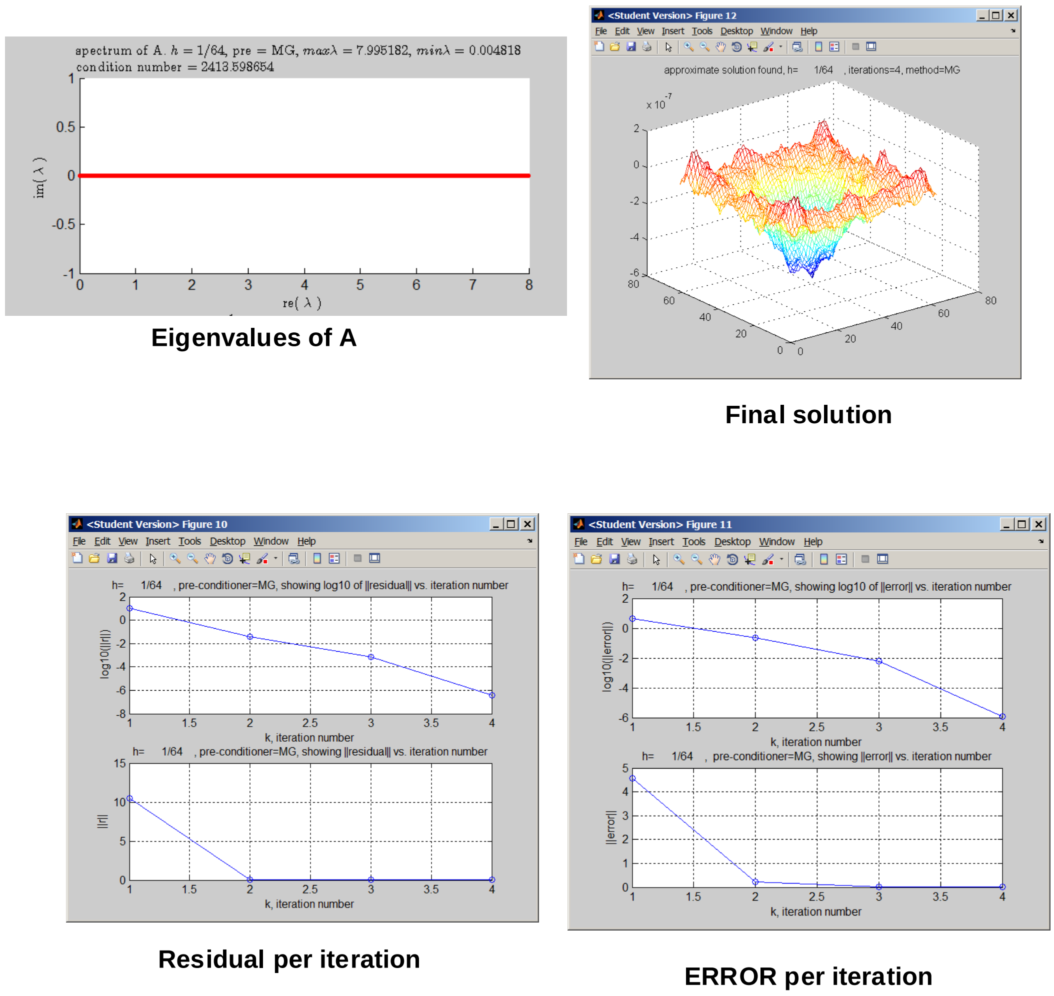
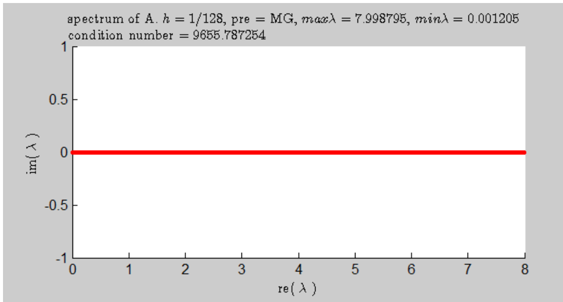
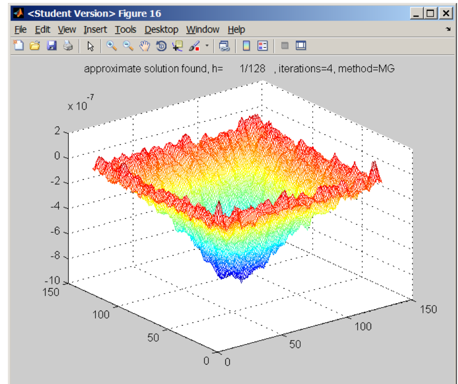


Figure 10: solver MG plots 64

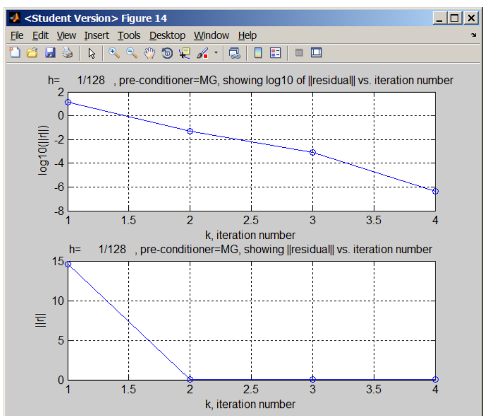
<u>Plots for h=1/128</u>



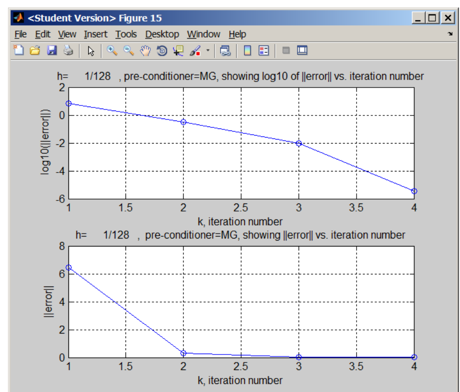
Eigenvalues of A



Final solution



Residual per iteration

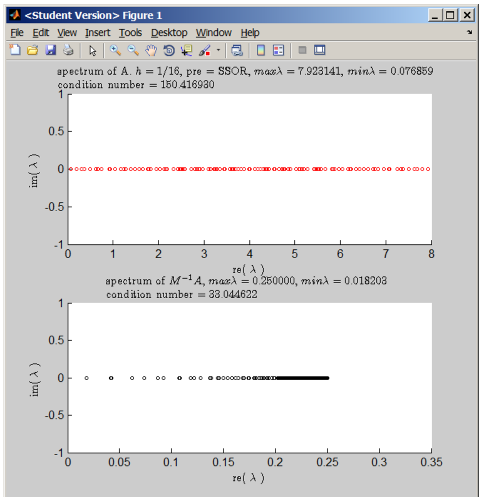


ERROR per iteration

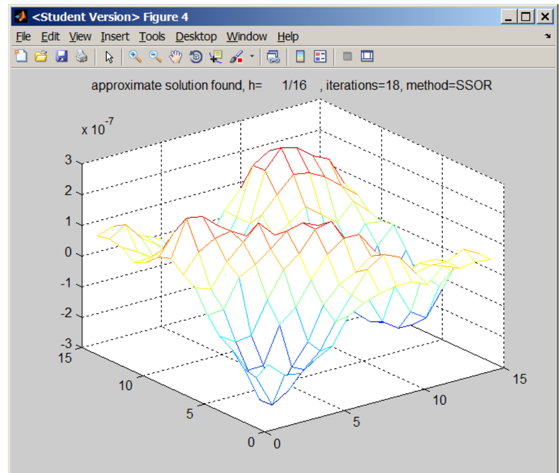
Figure 11: solver MG plots 128

4.3 Result for CG with SSOR preconditioner

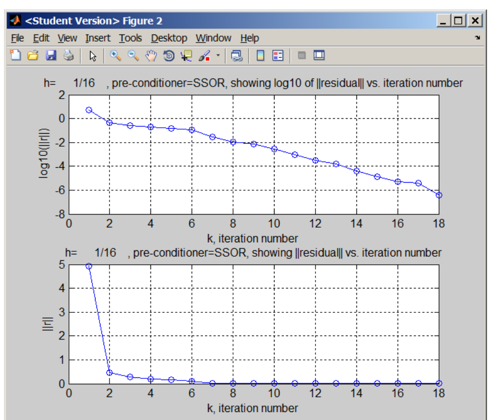
Plots for h=1/16



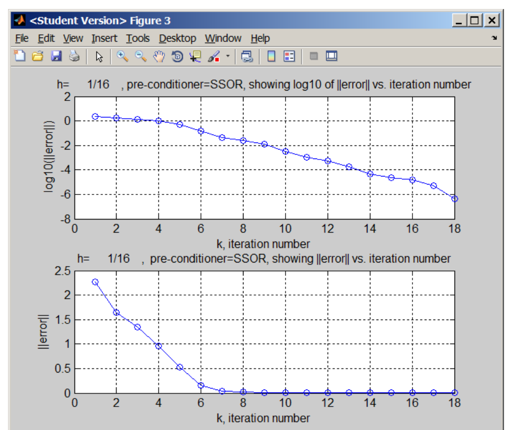
Eigenvalues of A and M^{-1}*A



Final solution



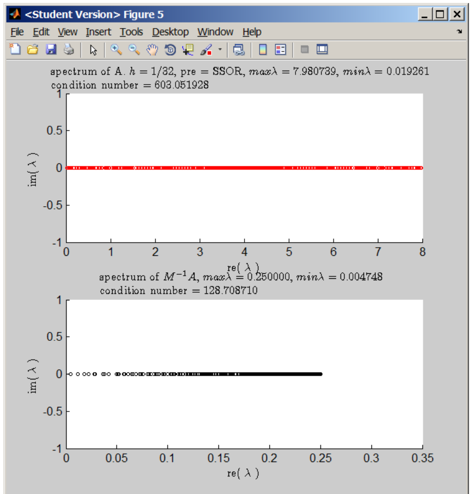
Residual per iteration



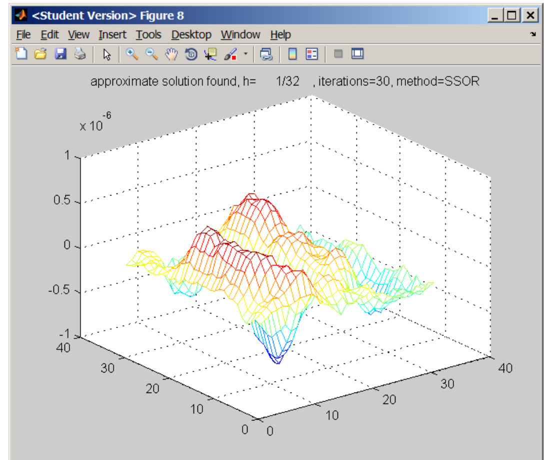
ERROR per iteration

Figure 12: solver SSOR plots 16

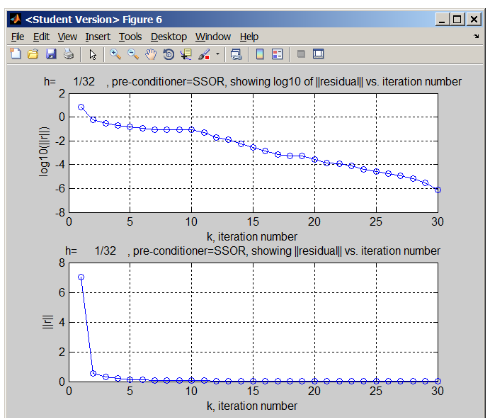
Plots for h=1/32



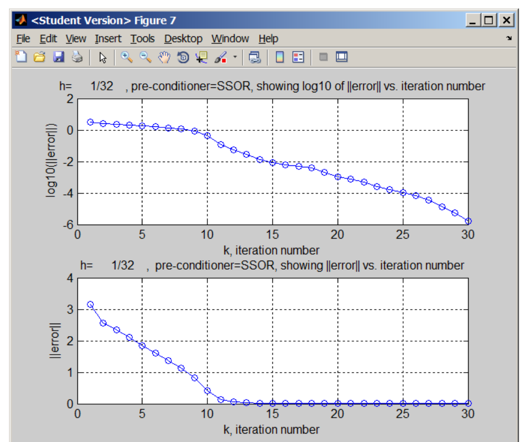
Eigenvalues of A and M^{-1}*A



Final solution



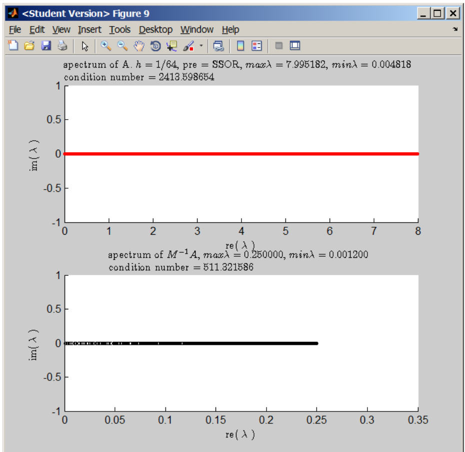
Residual per iteration



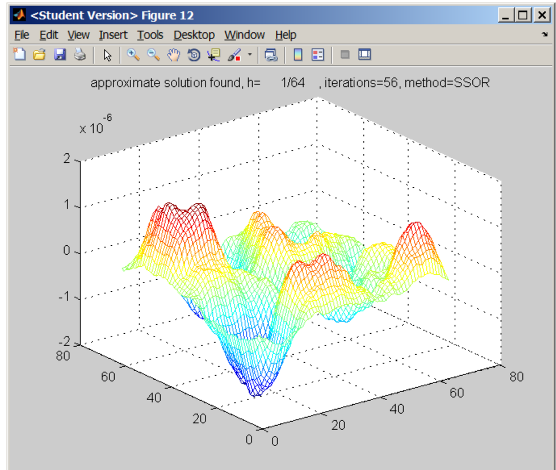
ERROR per iteration

Figure 13: solver SSOR plots 32

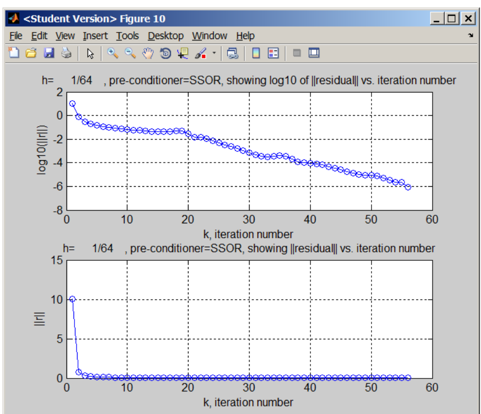
Plots for h=1/64



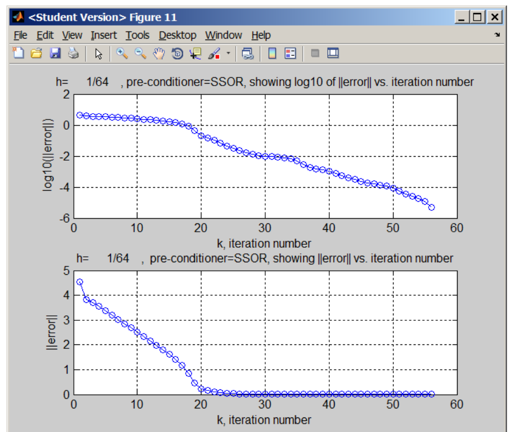
Eigenvalues of A and M^{-1}*A



Final solution



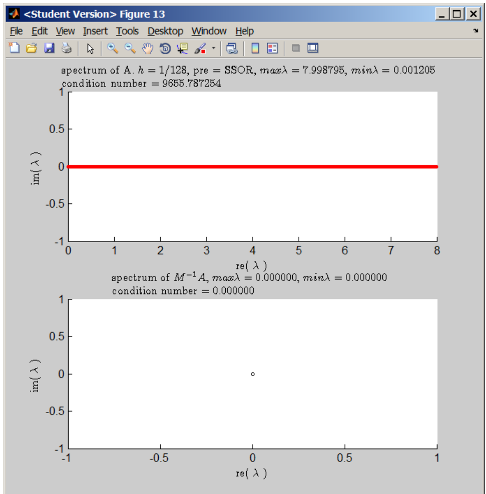
Residual per iteration



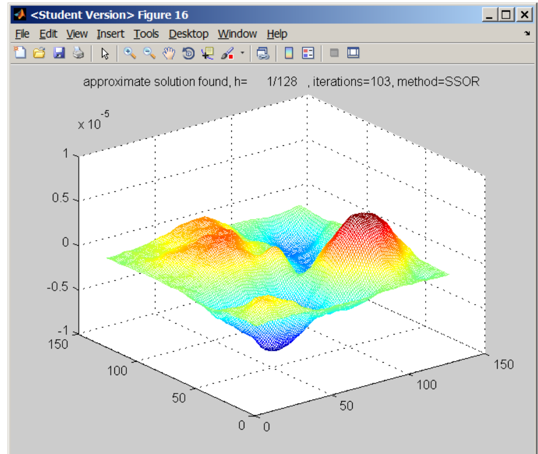
ERROR per iteration

Figure 14: solver SSOR plots 64

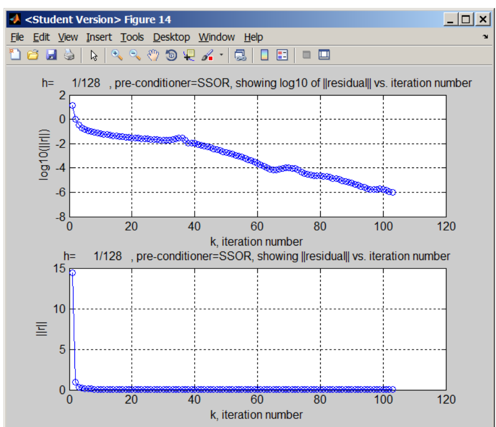
4.4 Plots for h=1/128



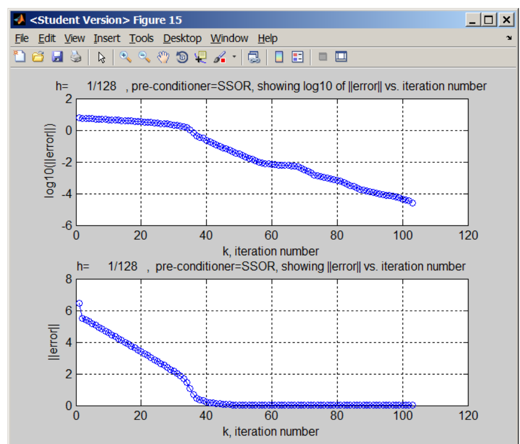
Eigenvalues of A



Final solution



Residual per iteration

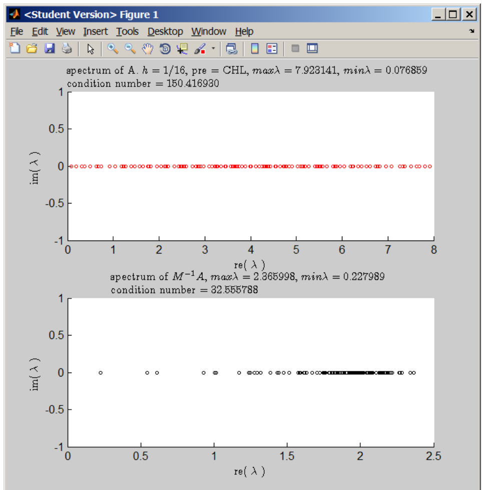


ERROR per iteration

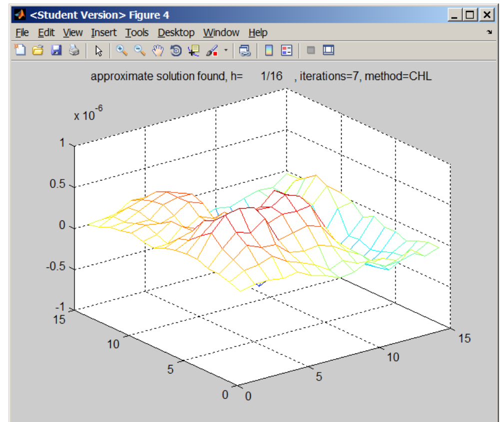
Figure 15: solver SSOR plots 128

4.5 Result for CG with incomplete cholesky preconditioner $\varepsilon = 10^{-2}$

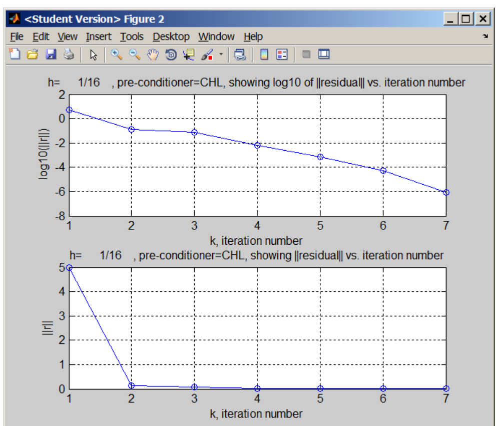
<u>Plots for h=1/16</u>



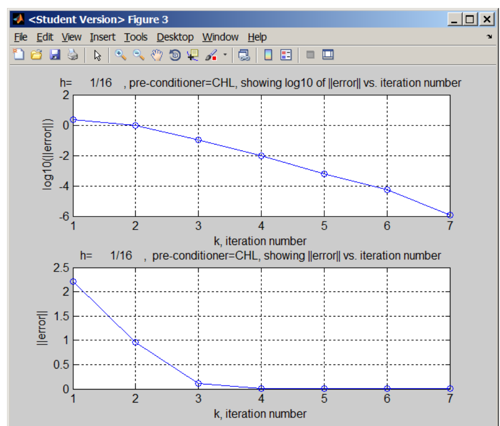
Eigenvalues of A and M^{-1}*A



Final solution



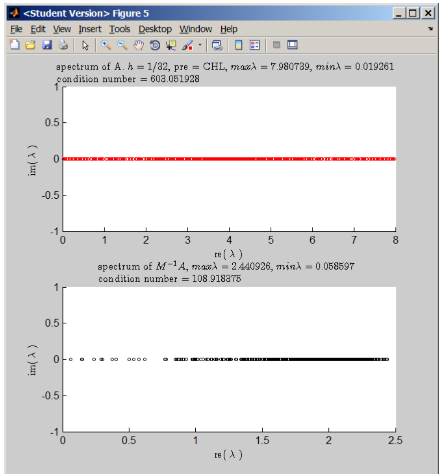
Residual per iteration



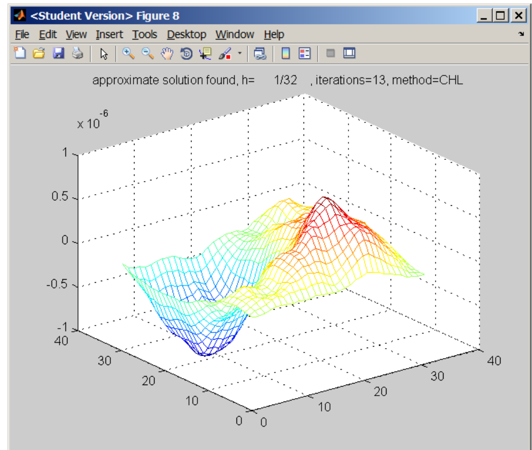
ERROR per iteration

Figure 16: solver incomplete cholesky plots 16

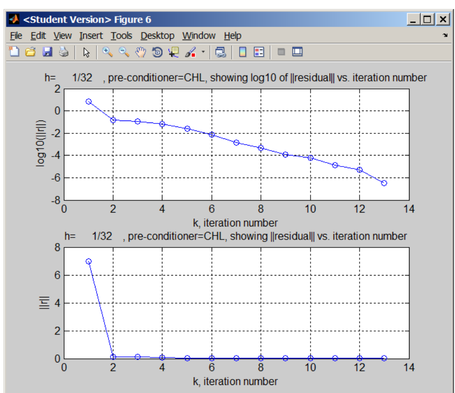
Plots for h=1/32



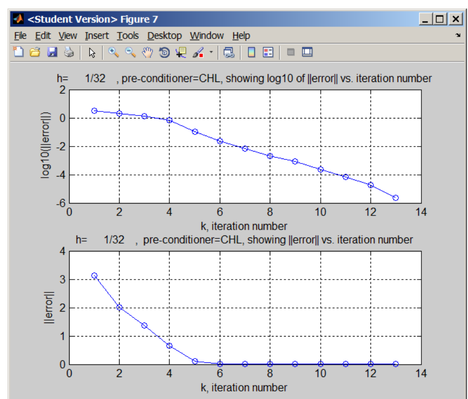
Eigenvalues of A and M^{-1}*A



Final solution



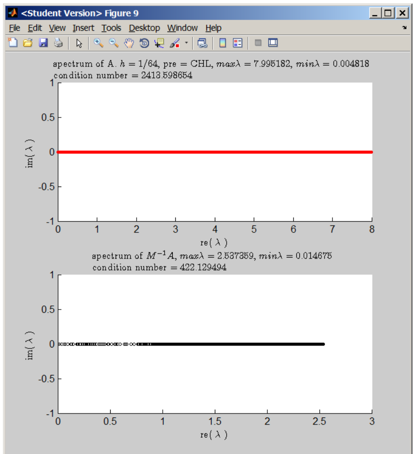
Residual per iteration



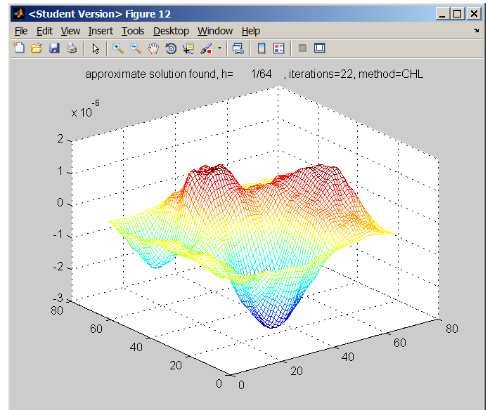
ERROR per iteration

Figure 17: solver incomplete cholesky plots 32

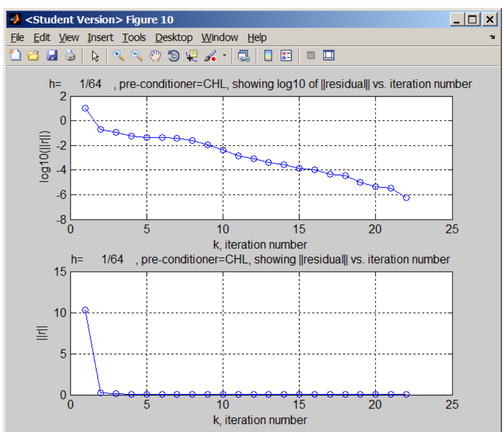
Plots for h=1/64



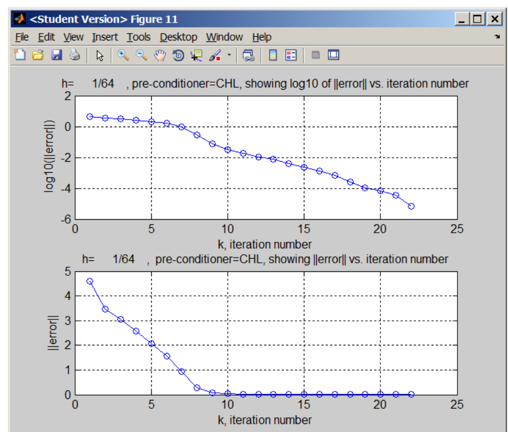
Eigenvalues of A and M^{-1}*A



Final solution



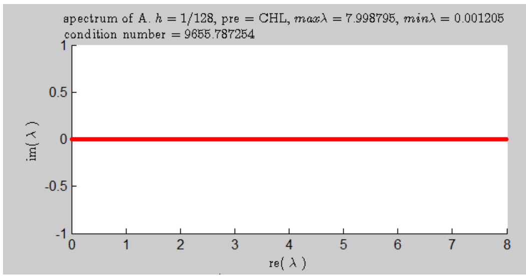
Residual per iteration



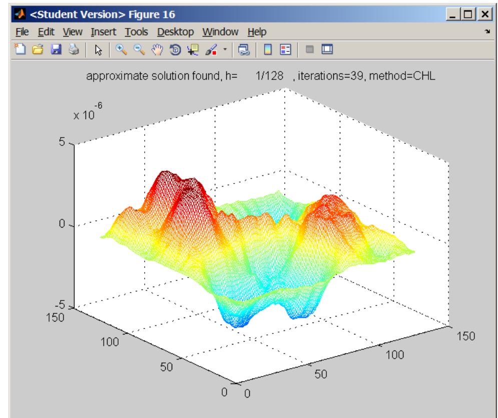
ERROR per iteration

Figure 18: solver incomplete cholesky plots 64

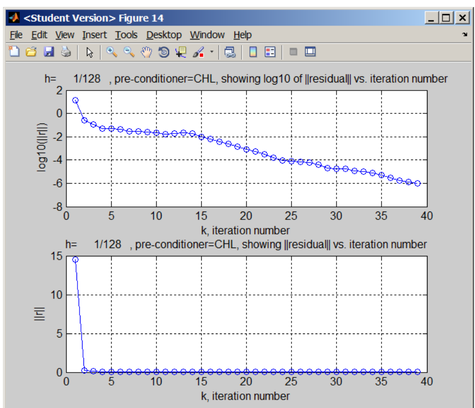
Plots for h=1/128



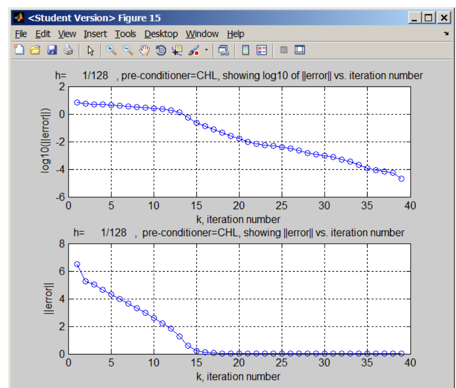
Eigenvalues of A



Final solution



Residual per iteration

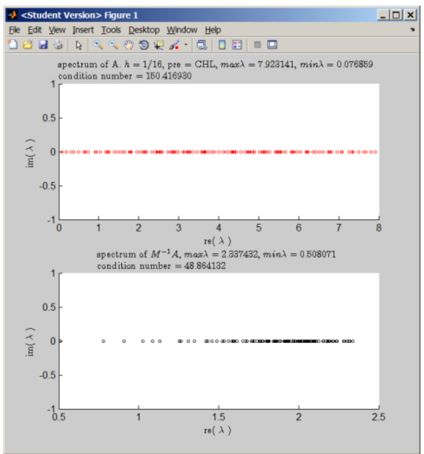


ERROR per iteration

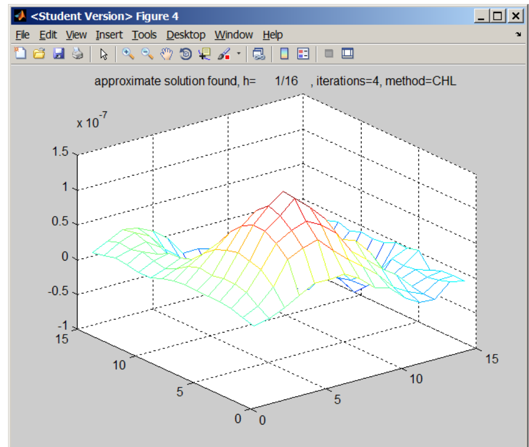
Figure 19: solver incomplete cholesky plots 128

5 Result for CG with incomplete cholesky preconditioner $\varepsilon = 10^{-3}$

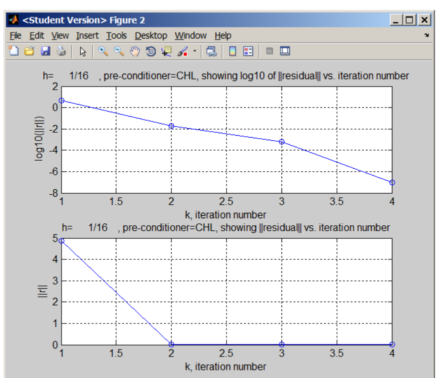
Plots for h=1/16



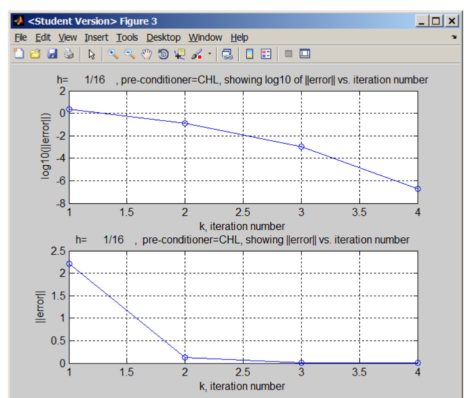
Eigenvalues of A and M^{-1}*A



Final solution



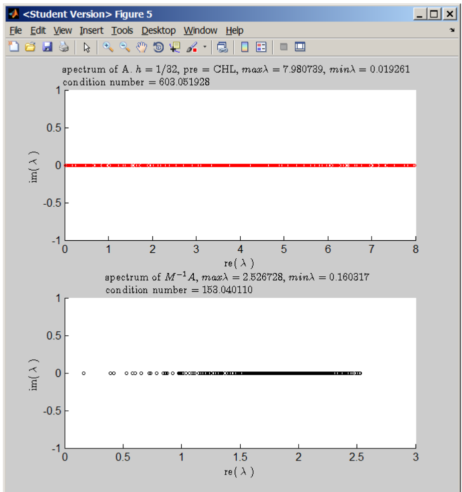
Residual per iteration



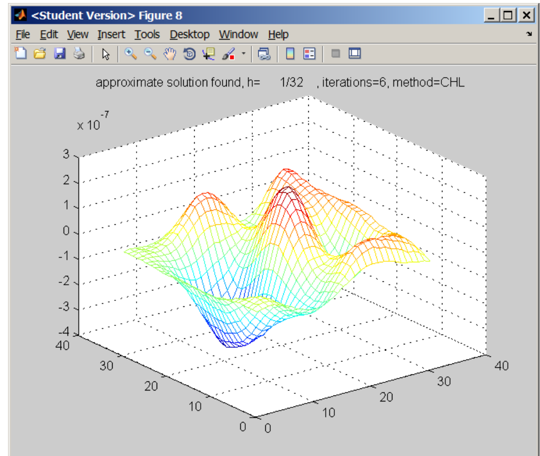
ERROR per iteration

Figure 20: solver CG with incomplete cholesky preconditioner 16

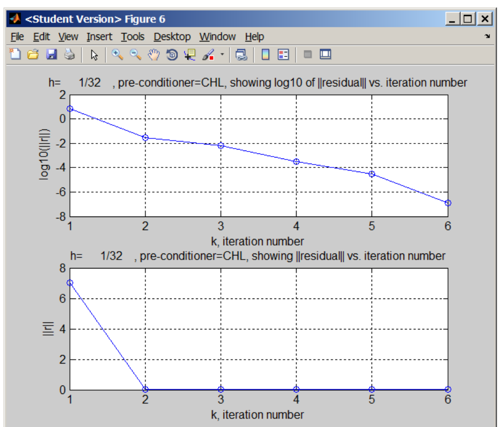
underlinePlots for h=1/32



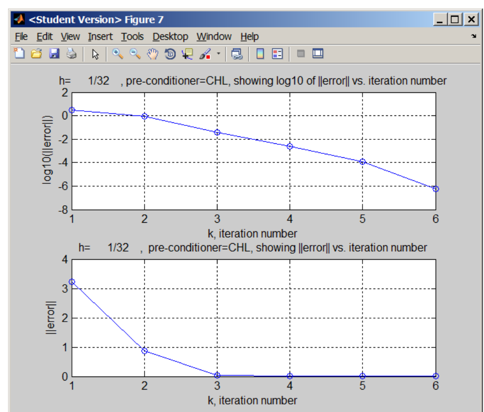
Eigenvalues of A and M^{-1}*A



Final solution



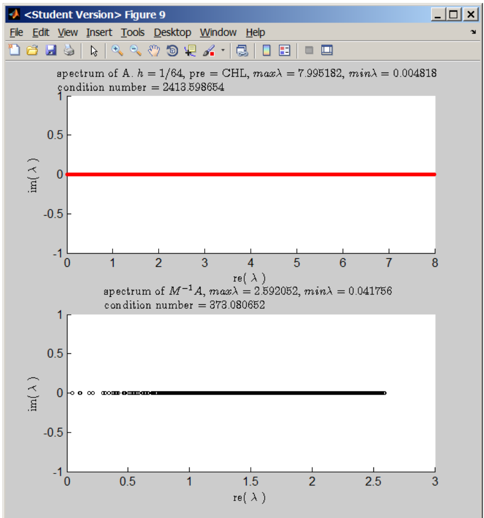
Residual per iteration



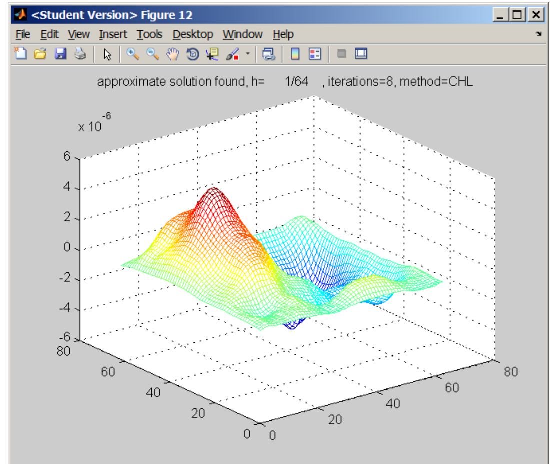
ERROR per iteration

Figure 21: solver CG with incomplete cholesky preconditioner 32

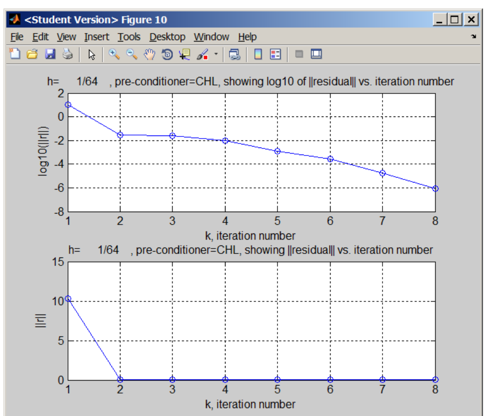
Plots for $h = \frac{1}{64}$



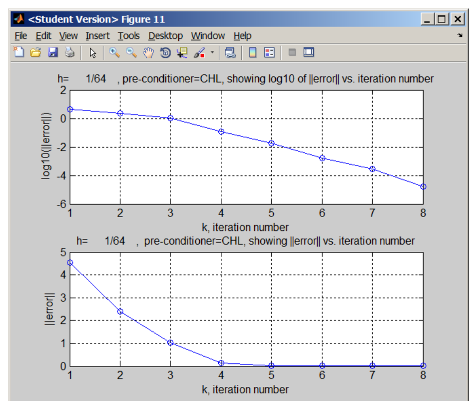
Eigenvalues of A and M^{-1}*A



Final solution



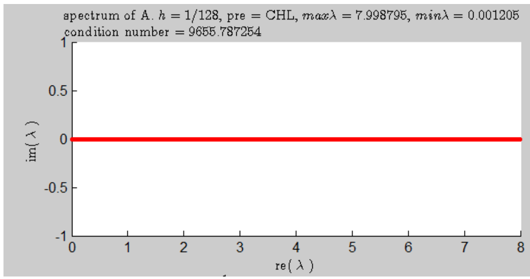
Residual per iteration



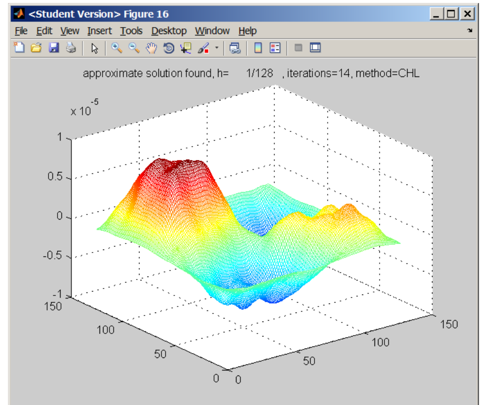
ERROR per iteration

Figure 22: solver CG with incomplete cholesky preconditioner 64

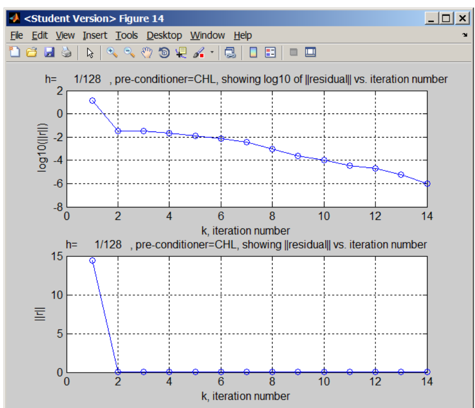
Plots for h=1/128



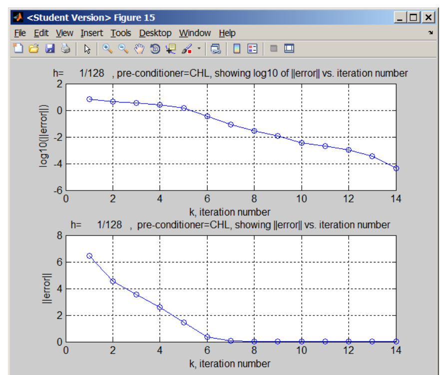
Eigenvalues of A



Final solution



Residual per iteration



ERROR per iteration

Figure 23: solver CG with incomplete cholesky preconditioner 128

5.1 References

1. R. J. LeVeque. Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems. SIAM, 2007.

5.2 Computation Tables

table for CG with no preconditioner

table for CG with no preconditioner $h = \frac{1}{16}$

Tolerance=10^{-6} method=NONE, Iterations = 42, condition number A=150.4167

```
           k          |e|         ratio           |r|          ratio
           1    2.1830924     0.0000000     4.7210582      0.0000000
           2    1.8864972     0.8641399     1.4184825      0.3004586
           3    1.7843339     0.9458449     0.6424897      0.4529416
           4    1.6861080     0.9449509     0.4396146      0.6842360
           5    1.5849468     0.9400032     0.3336348      0.7589256
           6    1.4553992     0.9182637     0.3163522      0.9481991
           7    1.2957415     0.8902998     0.2736152      0.8649068
           8    1.1409611     0.8805468     0.2420287      0.8845587
           9    0.9977301     0.8744646     0.2038050      0.8420696
          10    0.8499980     0.8519318     0.2051285      1.0064937
          11    0.6651595     0.7825425     0.1974297      0.9624684
          12    0.4864479     0.7313251     0.1747116      0.8849309
          13    0.2960128     0.6085191     0.1779989      1.0188155
          14    0.1451740     0.4904313     0.1199788      0.6740421
          15    0.0935858     0.6446460     0.0678391      0.5654261
          16    0.0756963     0.8088434     0.0408147      0.6016397
          17    0.0635298     0.8392727     0.0298095      0.7303614
          18    0.0529339     0.8332136     0.0218161      0.7318495
          19    0.0429735     0.8118324     0.0195722      0.8971481
          20    0.0322133     0.7496102     0.0163238      0.8340275
          21    0.0237202     0.7363457     0.0127609      0.7817369
          22    0.0157525     0.6640959     0.0107769      0.8445245
          23    0.0090001     0.5713465     0.0081136      0.7528675
          24    0.0049041     0.5448943     0.0056646      0.6981578
          25    0.0029273     0.5968981     0.0032193      0.5683292
          26    0.0021383     0.7304863     0.0017564      0.5455764
          27    0.0016007     0.7485962     0.0012297      0.7001280
          28    0.0011635     0.7268656     0.0008660      0.7042553
          29    0.0008013     0.6886615     0.0006405      0.7396406
          30    0.0005365     0.6695259     0.0005187      0.8097440
          31    0.0003581     0.6674555     0.0002751      0.5304362
          32    0.0002692     0.7517878     0.0001920      0.6979471
          33    0.0001902     0.7066626     0.0001491      0.7762729
          34    0.0001158     0.6085937     0.0001143      0.7665146
          35    0.0000648     0.5601207     0.0000746      0.6531095
          36    0.0000377     0.5807215     0.0000446      0.5983047
          37    0.0000238     0.6316540     0.0000268      0.5993857
          38    0.0000137     0.5773472     0.0000177      0.6619224
          39    0.0000077     0.5599609     0.0000104      0.5863875
          40    0.0000043     0.5545265     0.0000060      0.5767504
          41    0.0000023     0.5432272     0.0000034      0.5745691
          42    0.0000005     0.2307583     0.0000009      0.2689011
```

Table CG with no preconditioner for $h = \frac{1}{32}$

Tolerance=10^{-6} method=NONE, Iterations =82, condition number A=603

```
k                  |e|         ratio           |r|        ratio
           1    3.1348156     0.0000000     7.2393385      0.0000000
           2    2.7330660     0.8718427     2.0520496      0.2834582
           3    2.6175107     0.9577196     0.9150415      0.4459159
           4    2.5284570     0.9659777     0.6101126      0.6667595
           5    2.4421554     0.9658679     0.4386752      0.7190070
           6    2.3586462     0.9658051     0.3380251      0.7705588
           7    2.2766645     0.9652421     0.2853482      0.8441627
           8    2.1887867     0.9614006     0.2476061      0.8677331
           9    2.1039473     0.9612391     0.2079568      0.8398696
          10    2.0254995     0.9627140     0.1877177      0.9026765
```

11	1.9387591	0.9571758	0.1699200	0.9051889
12	1.8539303	0.9562459	0.1545855	0.9097544
13	1.7682926	0.9538075	0.1400435	0.9059291
14	1.6834581	0.9520246	0.1278646	0.9130349
15	1.6022568	0.9517652	0.1160407	0.9075285
16	1.5217100	0.9497291	0.1094328	0.9430547
17	1.4395506	0.9460085	0.1027356	0.9388016
18	1.3562155	0.9421104	0.0942611	0.9175109
19	1.2759187	0.9407934	0.0898504	0.9532076
20	1.1866336	0.9300229	0.0872327	0.9708661
21	1.0934573	0.9214784	0.0826619	0.9476025
22	1.0024063	0.9167311	0.0788741	0.9541777
23	0.8992085	0.8970499	0.0830311	1.0527040
24	0.7631293	0.8486678	0.0970541	1.1688879
25	0.5530580	0.7247239	0.1109983	1.1436745
26	0.3359324	0.6074089	0.0961082	0.8658534
27	0.2191717	0.6524281	0.0639449	0.6653430
28	0.1626895	0.7422922	0.0484804	0.7581592
29	0.1228014	0.7548210	0.0407252	0.8400342
30	0.0917790	0.7473771	0.0335094	0.8228158
31	0.0700724	0.7634914	0.0273614	0.8165289
32	0.0540805	0.7717795	0.0214580	0.7842445
33	0.0432646	0.8000041	0.0163945	0.7640291
34	0.0364310	0.8420503	0.0126036	0.7687676
35	0.0309924	0.8507149	0.0103112	0.8181153
36	0.0268480	0.8662776	0.0080396	0.7796959
37	0.0236146	0.8795683	0.0066579	0.8281358
38	0.0209965	0.8891317	0.0052897	0.7945024
39	0.0191928	0.9140930	0.0040810	0.7714996
40	0.0177298	0.9237770	0.0033815	0.8285992
41	0.0164277	0.9265582	0.0027478	0.8126066
42	0.0154290	0.9392033	0.0020770	0.7558581
43	0.0144978	0.9396484	0.0018150	0.8738906
44	0.0134250	0.9260010	0.0016909	0.9316005
45	0.0121487	0.9049280	0.0017116	1.0122194
46	0.0103132	0.8489185	0.0019431	1.1352723
47	0.0076740	0.7440911	0.0020798	1.0703591
48	0.0052118	0.6791481	0.0017795	0.8556085
49	0.0036674	0.7036863	0.0012802	0.7194204
50	0.0028794	0.7851283	0.0009362	0.7313235
51	0.0023421	0.8133830	0.0007764	0.8292528
52	0.0019119	0.8163410	0.0006295	0.8108465
53	0.0016151	0.8447753	0.0004538	0.7207790
54	0.0014025	0.8683421	0.0003797	0.8368191
55	0.0011871	0.8464182	0.0003408	0.8975492
56	0.0009760	0.8221407	0.0002934	0.8609202
57	0.0007699	0.7888788	0.0002736	0.9323579
58	0.0005719	0.7428557	0.0002359	0.8624696
59	0.0004284	0.7489784	0.0001812	0.7678420
60	0.0003281	0.7659409	0.0001493	0.8242995
61	0.0002531	0.7714986	0.0001175	0.7865263
62	0.0002060	0.8136716	0.0000861	0.7334368
63	0.0001797	0.8726498	0.0000546	0.6339584
64	0.0001644	0.9147896	0.0000414	0.7578420
65	0.0001500	0.9121801	0.0000324	0.7822544
66	0.0001359	0.9062892	0.0000282	0.8704977
67	0.0001196	0.8799064	0.0000263	0.9342924
68	0.0001024	0.8564000	0.0000246	0.9353246
69	0.0000828	0.8079831	0.0000233	0.9453090
70	0.0000620	0.7488319	0.0000226	0.9703960
71	0.0000452	0.7287425	0.0000167	0.7383479
72	0.0000366	0.8101289	0.0000122	0.7285159
73	0.0000299	0.8162470	0.0000101	0.8335502
74	0.0000239	0.8015867	0.0000085	0.8430243
75	0.0000186	0.7787371	0.0000074	0.8676639
76	0.0000140	0.7525703	0.0000060	0.8134688
77	0.0000104	0.7435247	0.0000050	0.8225143
78	0.0000076	0.7246112	0.0000039	0.7889123
79	0.0000055	0.7232358	0.0000030	0.7583253
80	0.0000040	0.7287703	0.0000023	0.7618901
81	0.0000029	0.7237151	0.0000017	0.7706604
82	0.0000015	0.5224390	0.0000010	0.5519726

Table for CG with no preconditioner $h = \frac{1}{64}$

Tolerance=10^{-6} method=NONE, Iterations = 157, condition number A=2413

k	|e|	ratio	|r|	ratio
1	4.5618175	0.0000000	10.2041977	0.0000000
2	4.0484423	0.8874626	2.9223637	0.2863884
3	3.9296490	0.9706570	1.1769802	0.4027494
4	3.8598809	0.9822457	0.7112536	0.6043038
5	3.7995456	0.9843686	0.4805222	0.6755989
6	3.7431529	0.9851580	0.3718902	0.7739293
7	3.6863983	0.9848378	0.2978303	0.8008555
8	3.6292174	0.9844887	0.2541673	0.8533964
9	3.5685722	0.9832897	0.2226181	0.8758725
10	3.5069581	0.9827342	0.2001179	0.8989292
11	3.4429589	0.9817508	0.1826321	0.9126225
12	3.3782370	0.9812017	0.1713830	0.9384053
13	3.3146909	0.9811896	0.1522840	0.8885599
14	3.2564054	0.9824160	0.1328174	0.8721691
15	3.2008849	0.9829504	0.1224334	0.9218176
16	3.1455152	0.9827018	0.1137015	0.9286800
17	3.0882964	0.9818094	0.1060581	0.9327769
18	3.0308365	0.9813943	0.0996429	0.9395123
19	2.9722551	0.9806716	0.0956821	0.9602499
20	2.9114870	0.9795549	0.0903023	0.9437739
21	2.8528462	0.9798588	0.0851183	0.9425936
22	2.7941605	0.9794291	0.0812703	0.9547925
23	2.7330511	0.9781296	0.0777886	0.9571584
24	2.6717953	0.9775870	0.0762112	0.9797225
25	2.6080786	0.9761521	0.0724646	0.9508390
26	2.5466310	0.9764395	0.0682339	0.9416164
27	2.4869367	0.9765595	0.0647979	0.9496439
28	2.4272004	0.9759800	0.0620220	0.9571603
29	2.3688169	0.9759461	0.0598978	0.9657516
30	2.3065758	0.9737248	0.0596884	0.9965033
31	2.2416075	0.9718334	0.0576136	0.9652406
32	2.1770400	0.9711959	0.0558695	0.9697274
33	2.1148235	0.9714215	0.0530152	0.9489108
34	2.0513467	0.9699848	0.0526473	0.9930604
35	1.9830861	0.9667240	0.0520339	0.9883494
36	1.9164540	0.9663998	0.0501573	0.9639341
37	1.8482870	0.9644306	0.0495531	0.9879541
38	1.7802122	0.9631687	0.0470089	0.9486572
39	1.7180330	0.9650720	0.0434971	0.9252956
40	1.6579511	0.9650286	0.0428035	0.9840528
41	1.5950758	0.9620765	0.0419171	0.9792931
42	1.5314235	0.9600945	0.0407596	0.9723845
43	1.4665547	0.9576415	0.0403056	0.9888615
44	1.3990502	0.9539707	0.0399968	0.9923390
45	1.3251017	0.9471438	0.0414126	1.0353981
46	1.2399275	0.9357225	0.0446116	1.0772483
47	1.1274063	0.9092518	0.0518707	1.1627170
48	0.9608210	0.8522402	0.0624979	1.2048781
49	0.7395289	0.7696844	0.0669797	1.0717122
50	0.5283032	0.7143780	0.0585964	0.8748381
51	0.3950155	0.7477062	0.0442497	0.7551608
52	0.3199062	0.8098571	0.0353741	0.7994194
53	0.2652143	0.8290380	0.0311022	0.8792360
54	0.2243041	0.8457467	0.0261245	0.8399568
55	0.1942000	0.8657888	0.0227046	0.8690920
56	0.1684643	0.8674782	0.0210588	0.9275151
57	0.1453235	0.8626370	0.0187644	0.8910446
58	0.1265923	0.8711071	0.0167577	0.8930590
59	0.1097734	0.8671408	0.0154345	0.9210396
60	0.0947833	0.8634453	0.0138393	0.8966502
61	0.0823391	0.8687086	0.0123950	0.8956317
62	0.0713607	0.8666691	0.0111566	0.9000953
63	0.0615239	0.8621533	0.0099908	0.8954998
64	0.0531284	0.8635401	0.0089677	0.8976025
65	0.0460486	0.8667428	0.0078037	0.8701995

66	0.0401432	0.8717562	0.0068652	0.8797327
67	0.0349347	0.8702530	0.0061224	0.8918082
68	0.0305072	0.8732635	0.0053390	0.8720462
69	0.0267317	0.8762420	0.0047623	0.8919769
70	0.0234572	0.8775033	0.0041094	0.8628988
71	0.0207112	0.8829387	0.0036173	0.8802590
72	0.0181979	0.8786512	0.0032920	0.9100613
73	0.0159488	0.8764068	0.0028970	0.8800222
74	0.0140658	0.8819366	0.0025342	0.8747771
75	0.0125049	0.8890299	0.0021879	0.8633171
76	0.0111985	0.8955273	0.0019189	0.8770658
77	0.0101148	0.9032251	0.0016804	0.8757329
78	0.0091865	0.9082280	0.0015038	0.8948967
79	0.0084158	0.9160976	0.0013190	0.8770684
80	0.0078196	0.9291617	0.0011071	0.8393543
81	0.0073758	0.9432389	0.0009256	0.8360613
82	0.0070385	0.9542812	0.0007869	0.8502076
83	0.0067629	0.9608446	0.0006653	0.8453718
84	0.0065430	0.9674723	0.0005542	0.8330332
85	0.0063606	0.9721362	0.0004631	0.8356598
86	0.0062041	0.9753943	0.0003904	0.8429988
87	0.0060628	0.9772213	0.0003301	0.8455336
88	0.0059316	0.9783588	0.0002793	0.8462168
89	0.0058038	0.9784528	0.0002434	0.8714097
90	0.0056707	0.9770673	0.0002200	0.9036392
91	0.0055215	0.9736957	0.0002182	0.9922456
92	0.0053270	0.9647725	0.0002329	1.0671837
93	0.0050613	0.9501202	0.0002563	1.1003357
94	0.0047075	0.9300953	0.0002754	1.0744267
95	0.0043063	0.9147708	0.0002692	0.9778062
96	0.0039172	0.9096464	0.0002592	0.9626214
97	0.0035245	0.8997507	0.0002627	1.0135967
98	0.0030928	0.8774993	0.0002717	1.0343359
99	0.0026357	0.8522177	0.0002652	0.9760185
100	0.0022317	0.8467214	0.0002384	0.8988006
101	0.0019165	0.8587692	0.0002124	0.8912480
102	0.0016658	0.8691736	0.0001896	0.8925872
103	0.0014748	0.8853198	0.0001562	0.8237657
104	0.0013425	0.9102889	0.0001341	0.8584851
105	0.0012338	0.9190797	0.0001229	0.9163269
106	0.0011364	0.9210448	0.0001110	0.9029318
107	0.0010529	0.9264833	0.0000981	0.8837443
108	0.0009822	0.9329029	0.0000884	0.9015361
109	0.0009173	0.9338643	0.0000805	0.9108633
110	0.0008605	0.9381542	0.0000706	0.8763963
111	0.0008106	0.9420035	0.0000648	0.9185117
112	0.0007612	0.9390822	0.0000609	0.9395127
113	0.0007143	0.9383693	0.0000557	0.9139756
114	0.0006696	0.9374446	0.0000519	0.9329107
115	0.0006260	0.9348144	0.0000481	0.9264502
116	0.0005854	0.9351858	0.0000443	0.9199678
117	0.0005460	0.9326180	0.0000424	0.9576760
118	0.0005044	0.9239461	0.0000413	0.9738142
119	0.0004627	0.9172921	0.0000391	0.9473491
120	0.0004208	0.9094667	0.0000381	0.9752708
121	0.0003784	0.8992550	0.0000372	0.9763676
122	0.0003372	0.8911407	0.0000350	0.9403920
123	0.0002996	0.8885168	0.0000322	0.9205549
124	0.0002665	0.8895264	0.0000296	0.9186267
125	0.0002382	0.8936247	0.0000267	0.9014817
126	0.0002145	0.9005476	0.0000235	0.8810575
127	0.0001949	0.9087432	0.0000205	0.8702270
128	0.0001793	0.9195943	0.0000191	0.9333716
129	0.0001645	0.9175205	0.0000165	0.8627026
130	0.0001529	0.9299627	0.0000145	0.8810031
131	0.0001428	0.9334836	0.0000126	0.8674787
132	0.0001338	0.9370194	0.0000113	0.8986161
133	0.0001252	0.9354832	0.0000105	0.9248925
134	0.0001161	0.9280017	0.0000101	0.9641272
135	0.0001061	0.9132398	0.0000104	1.0276371
136	0.0000941	0.8872630	0.0000108	1.0403204
137	0.0000809	0.8591660	0.0000106	0.9825534

138	0.0000692	0.8553820	0.0000095	0.9003622
139	0.0000600	0.8674816	0.0000080	0.8360480
140	0.0000535	0.8925543	0.0000066	0.8225625
141	0.0000486	0.9079807	0.0000057	0.8611308
142	0.0000443	0.9113730	0.0000052	0.9135594
143	0.0000402	0.9066825	0.0000048	0.9354110
144	0.0000361	0.8975802	0.0000045	0.9304066
145	0.0000323	0.8956690	0.0000041	0.9034467
146	0.0000288	0.8917449	0.0000038	0.9321324
147	0.0000255	0.8865527	0.0000034	0.9105714
148	0.0000226	0.8841495	0.0000031	0.9002286
149	0.0000200	0.8862625	0.0000028	0.8885652
150	0.0000177	0.8849574	0.0000025	0.8993699
151	0.0000155	0.8766179	0.0000024	0.9491487
152	0.0000133	0.8572703	0.0000022	0.9393268
153	0.0000113	0.8482148	0.0000020	0.9135275
154	0.0000095	0.8451107	0.0000018	0.8915801
155	0.0000081	0.8501849	0.0000015	0.8470707
156	0.0000070	0.8664622	0.0000013	0.8602920
157	0.0000054	0.7724505	0.0000010	0.7331119

Table for CG with no preconditioner $h = \frac{1}{128}$

Tolerance=10^{-6} method=NONE, Iterations = 291, condition number A=9655

k	|e|	ratio	|r|	ratio
1	6.5216900	0.0000000	14.4577169	0.0000000
2	5.8211800	0.8925877	4.0764931	0.2819597
3	5.6842162	0.9764715	1.5799015	0.3875639
4	5.6198883	0.9886831	0.8968116	0.5676377
5	5.5711714	0.9913313	0.5778805	0.6443723
6	5.5285647	0.9923523	0.4303215	0.7446548
7	5.4869080	0.9924652	0.3341897	0.7766047
8	5.4468461	0.9926986	0.2785535	0.8335192
9	5.4044559	0.9922175	0.2424388	0.8703491
10	5.3616029	0.9920708	0.2146822	0.8855108
11	5.3191442	0.9920810	0.1862262	0.8674507
12	5.2782868	0.9923188	0.1677396	0.9007302
13	5.2363954	0.9920634	0.1524839	0.9090513
14	5.1941353	0.9919295	0.1420818	0.9317825
15	5.1513426	0.9917614	0.1293917	0.9106846
16	5.1099665	0.9919679	0.1195224	0.9237248
17	5.0683715	0.9918600	0.1119117	0.9363248
18	5.0255870	0.9915585	0.1066056	0.9525867
19	4.9819254	0.9913121	0.0994276	0.9326674
20	4.9398031	0.9915450	0.0934835	0.9402165
21	4.8972751	0.9913908	0.0895284	0.9576918
22	4.8541439	0.9911928	0.0849488	0.9488477
23	4.8107070	0.9910516	0.0811183	0.9549077
24	4.7673483	0.9909871	0.0782140	0.9641979
25	4.7234798	0.9907981	0.0742769	0.9496615
26	4.6807711	0.9909582	0.0714271	0.9616324
27	4.6372353	0.9906990	0.0684405	0.9581873
28	4.5948580	0.9908615	0.0651097	0.9513333
29	4.5530996	0.9909119	0.0621499	0.9545408
30	4.5112658	0.9908120	0.0601683	0.9681159
31	4.4687574	0.9905773	0.0584916	0.9721339
32	4.4262843	0.9904955	0.0566503	0.9685198
33	4.3834625	0.9903256	0.0547127	0.9657966
34	4.3405454	0.9902093	0.0529867	0.9684541
35	4.2983731	0.9902841	0.0507715	0.9581929
36	4.2562271	0.9901949	0.0496163	0.9772479
37	4.2138748	0.9900493	0.0481310	0.9700646
38	4.1708118	0.9897807	0.0472816	0.9823505
39	4.1266023	0.9894003	0.0466718	0.9871047
40	4.0823351	0.9892727	0.0454224	0.9732297
41	4.0380267	0.9891463	0.0441588	0.9721808
42	3.9946964	0.9892694	0.0426145	0.9650276
43	3.9522103	0.9893644	0.0409349	0.9605877
44	3.9106277	0.9894786	0.0398300	0.9730069
45	3.8684976	0.9892268	0.0392197	0.9846780
46	3.8257317	0.9889451	0.0385564	0.9830880

47	3.7830507	0.9888437	0.0372870	0.9670762
48	3.7403946	0.9887244	0.0366904	0.9839996
49	3.6974958	0.9885310	0.0359622	0.9801539
50	3.6549289	0.9884876	0.0352283	0.9795926
51	3.6113640	0.9880805	0.0345719	0.9813650
52	3.5675354	0.9878637	0.0340535	0.9850064
53	3.5231205	0.9875503	0.0336276	0.9874943
54	3.4787130	0.9873954	0.0329058	0.9785343
55	3.4345867	0.9873153	0.0323069	0.9817996
56	3.3901581	0.9870643	0.0316810	0.9806276
57	3.3461400	0.9870159	0.0306893	0.9686978
58	3.3036285	0.9872954	0.0298158	0.9715352
59	3.2612089	0.9871597	0.0294984	0.9893541
60	3.2175113	0.9866008	0.0293399	0.9946276
61	3.1732557	0.9862454	0.0288105	0.9819557
62	3.1292108	0.9861200	0.0282986	0.9822332
63	3.0851351	0.9859147	0.0277461	0.9804746
64	3.0412129	0.9857633	0.0274147	0.9880581
65	2.9964828	0.9852920	0.0267788	0.9768022
66	2.9522621	0.9852425	0.0265317	0.9907736
67	2.9083386	0.9851221	0.0258464	0.9741706
68	2.8648102	0.9850332	0.0254640	0.9852068
69	2.8199937	0.9843562	0.0255037	1.0015562
70	2.7742200	0.9837682	0.0252014	0.9881474
71	2.7287776	0.9836197	0.0245978	0.9760481
72	2.6844169	0.9837434	0.0237590	0.9659003
73	2.6412743	0.9839285	0.0235033	0.9892381
74	2.5968246	0.9831711	0.0234332	0.9970183
75	2.5517313	0.9826352	0.0230106	0.9819663
76	2.5070630	0.9824949	0.0225744	0.9810401
77	2.4633246	0.9825539	0.0220316	0.9759583
78	2.4188791	0.9819571	0.0219368	0.9956962
79	2.3736882	0.9813174	0.0216703	0.9878504
80	2.3279918	0.9807488	0.0217378	1.0031152
81	2.2808231	0.9797385	0.0215495	0.9913383
82	2.2336755	0.9793287	0.0212204	0.9847292
83	2.1862533	0.9787694	0.0208521	0.9826415
84	2.1387752	0.9782833	0.0206835	0.9919146
85	2.0909984	0.9776616	0.0203045	0.9816770
86	2.0438264	0.9774405	0.0199742	0.9837320
87	1.9962370	0.9767155	0.0198473	0.9936479
88	1.9483597	0.9760162	0.0195917	0.9871222
89	1.8995855	0.9749666	0.0195412	0.9974214
90	1.8476324	0.9726503	0.0202739	1.0374967
91	1.7887809	0.9681476	0.0215423	1.0625600
92	1.7179957	0.9604282	0.0239573	1.1121081
93	1.6232429	0.9448469	0.0281514	1.1750654
94	1.4854374	0.9151048	0.0338549	1.2025994
95	1.2901182	0.8685107	0.0389859	1.1515605
96	1.0644065	0.8250457	0.0390577	1.0018411
97	0.8691133	0.8165238	0.0342085	0.8758440
98	0.7330221	0.8434137	0.0280872	0.8210594
99	0.6386873	0.8713069	0.0245142	0.8727891
100	0.5637403	0.8826547	0.0221816	0.9048483
101	0.5029967	0.8922489	0.0199448	0.8991569
102	0.4540556	0.9027011	0.0177812	0.8915224
103	0.4150739	0.9141476	0.0160900	0.9048868
104	0.3813251	0.9186921	0.0152002	0.9446998
105	0.3512848	0.9212212	0.0140495	0.9243007
106	0.3250302	0.9252614	0.0132617	0.9439243
107	0.3006703	0.9250533	0.0127023	0.9578196
108	0.2782780	0.9255254	0.0118652	0.9340934
109	0.2583980	0.9285605	0.0111782	0.9421066
110	0.2399280	0.9285211	0.0107487	0.9615730
111	0.2224847	0.9272978	0.0101863	0.9476738
112	0.2067140	0.9291155	0.0095446	0.9370026
113	0.1922750	0.9301500	0.0090068	0.9436612
114	0.1789377	0.9306343	0.0085970	0.9544985
115	0.1659538	0.9274389	0.0084277	0.9803081
116	0.1532174	0.9232532	0.0081486	0.9668770
117	0.1413960	0.9228457	0.0076599	0.9400255
118	0.1307287	0.9245573	0.0071955	0.9393847

119	0.1209090	0.9248852	0.0067734	0.9413369
120	0.1120288	0.9265547	0.0063259	0.9339284
121	0.1039599	0.9279741	0.0059769	0.9448358
122	0.0964983	0.9282266	0.0056345	0.9427016
123	0.0896672	0.9292102	0.0053026	0.9411039
124	0.0833125	0.9291304	0.0049828	0.9396944
125	0.0776324	0.9318209	0.0046406	0.9313226
126	0.0724280	0.9329618	0.0043473	0.9367799
127	0.0676853	0.9345182	0.0040083	0.9220193
128	0.0635380	0.9387263	0.0036864	0.9197089
129	0.0597311	0.9400845	0.0034773	0.9432675
130	0.0560418	0.9382361	0.0033230	0.9556383
131	0.0525073	0.9369301	0.0031496	0.9478249
132	0.0491855	0.9367361	0.0029710	0.9432938
133	0.0460666	0.9365890	0.0027967	0.9413153
134	0.0431344	0.9363485	0.0026568	0.9499861
135	0.0403771	0.9360763	0.0024800	0.9334519
136	0.0378184	0.9366311	0.0023383	0.9428438
137	0.0354510	0.9374012	0.0021922	0.9375558
138	0.0332424	0.9376994	0.0020377	0.9294867
139	0.0312191	0.9391363	0.0019028	0.9338130
140	0.0293648	0.9406007	0.0017822	0.9366054
141	0.0275654	0.9387233	0.0017140	0.9617729
142	0.0258209	0.9367154	0.0016189	0.9444659
143	0.0242216	0.9380629	0.0015084	0.9317937
144	0.0227180	0.9379219	0.0014412	0.9554332
145	0.0212899	0.9371386	0.0013528	0.9386613
146	0.0199738	0.9381832	0.0012813	0.9471259
147	0.0187271	0.9375794	0.0012066	0.9417288
148	0.0175963	0.9396166	0.0011173	0.9259961
149	0.0165930	0.9429837	0.0010360	0.9271850
150	0.0156685	0.9442818	0.0009697	0.9359917
151	0.0148093	0.9451670	0.0009143	0.9429451
152	0.0140387	0.9479640	0.0008446	0.9236841
153	0.0133481	0.9508081	0.0007828	0.9269097
154	0.0127077	0.9520237	0.0007435	0.9498209
155	0.0121136	0.9532515	0.0007005	0.9420813
156	0.0115697	0.9551005	0.0006583	0.9397314
157	0.0110828	0.9579138	0.0006133	0.9317308
158	0.0106451	0.9605075	0.0005721	0.9327397
159	0.0102507	0.9629485	0.0005351	0.9353541
160	0.0099019	0.9659712	0.0005003	0.9348993
161	0.0095977	0.9692748	0.0004553	0.9101756
162	0.0093373	0.9728771	0.0004165	0.9146442
163	0.0091078	0.9754214	0.0003875	0.9305895
164	0.0089051	0.9777442	0.0003573	0.9218909
165	0.0087271	0.9800029	0.0003282	0.9185575
166	0.0085684	0.9818138	0.0003015	0.9188234
167	0.0084309	0.9839583	0.0002719	0.9016052
168	0.0083109	0.9857609	0.0002449	0.9008619
169	0.0082058	0.9873564	0.0002206	0.9008608
170	0.0081135	0.9887523	0.0001955	0.8858579
171	0.0080298	0.9896858	0.0001764	0.9025415
172	0.0079505	0.9901290	0.0001610	0.9127841
173	0.0078748	0.9904714	0.0001459	0.9060863
174	0.0078020	0.9907636	0.0001330	0.9115998
175	0.0077322	0.9910514	0.0001201	0.9027629
176	0.0076646	0.9912555	0.0001102	0.9181004
177	0.0075980	0.9913082	0.0000994	0.9021361
178	0.0075343	0.9916206	0.0000903	0.9080043
179	0.0074697	0.9914240	0.0000843	0.9338806
180	0.0073991	0.9905492	0.0000836	0.9916163
181	0.0073144	0.9885474	0.0000873	1.0443408
182	0.0072065	0.9852455	0.0000944	1.0807980
183	0.0070661	0.9805186	0.0001041	1.1030488
184	0.0068861	0.9745304	0.0001130	1.0857231
185	0.0066684	0.9683929	0.0001181	1.0445829
186	0.0064350	0.9649952	0.0001170	0.9906712
187	0.0062072	0.9645942	0.0001121	0.9587357
188	0.0059900	0.9650072	0.0001088	0.9702163
189	0.0057630	0.9621089	0.0001137	1.0446340
190	0.0054891	0.9524693	0.0001251	1.1007868

191	0.0051459	0.9374773	0.0001347	1.0766640
192	0.0047715	0.9272480	0.0001344	0.9978854
193	0.0044069	0.9235887	0.0001313	0.9767634
194	0.0040493	0.9188441	0.0001316	1.0022126
195	0.0036749	0.9075451	0.0001353	1.0284495
196	0.0032688	0.8895021	0.0001380	1.0193791
197	0.0028661	0.8768005	0.0001344	0.9740698
198	0.0024955	0.8707062	0.0001284	0.9558213
199	0.0021550	0.8635418	0.0001229	0.9570292
200	0.0018567	0.8615758	0.0001137	0.9250923
201	0.0016166	0.8706679	0.0001009	0.8877237
202	0.0014368	0.8887817	0.0000897	0.8888228
203	0.0012963	0.9022463	0.0000810	0.9033006
204	0.0011838	0.9131675	0.0000720	0.8884030
205	0.0010994	0.9286959	0.0000635	0.8813002
206	0.0010326	0.9392711	0.0000578	0.9112359
207	0.0009772	0.9463790	0.0000524	0.9068811
208	0.0009315	0.9532468	0.0000471	0.8981476
209	0.0008948	0.9606123	0.0000418	0.8876419
210	0.0008645	0.9660678	0.0000379	0.9073230
211	0.0008375	0.9687479	0.0000348	0.9164967
212	0.0008140	0.9720156	0.0000308	0.8855352
213	0.0007936	0.9749646	0.0000281	0.9143270
214	0.0007742	0.9755578	0.0000264	0.9377285
215	0.0007555	0.9758417	0.0000243	0.9211895
216	0.0007382	0.9770402	0.0000225	0.9238694
217	0.0007210	0.9766651	0.0000216	0.9599043
218	0.0007031	0.9752798	0.0000205	0.9529276
219	0.0006856	0.9749951	0.0000197	0.9592020
220	0.0006669	0.9727617	0.0000196	0.9934542
221	0.0006474	0.9706996	0.0000188	0.9599738
222	0.0006279	0.9699750	0.0000182	0.9670444
223	0.0006080	0.9682281	0.0000178	0.9778561
224	0.0005876	0.9665155	0.0000170	0.9586794
225	0.0005677	0.9661052	0.0000167	0.9784560
226	0.0005461	0.9619817	0.0000169	1.0115654
227	0.0005230	0.9576032	0.0000169	1.0017682
228	0.0004988	0.9538113	0.0000169	0.9996731
229	0.0004738	0.9498157	0.0000167	0.9877224
230	0.0004487	0.9470574	0.0000163	0.9743509
231	0.0004240	0.9450280	0.0000160	0.9872214
232	0.0003992	0.9414172	0.0000157	0.9805466
233	0.0003747	0.9386238	0.0000154	0.9774359
234	0.0003508	0.9362417	0.0000150	0.9738462
235	0.0003281	0.9354138	0.0000143	0.9557997
236	0.0003072	0.9361931	0.0000135	0.9455793
237	0.0002887	0.9398553	0.0000125	0.9270502
238	0.0002722	0.9427474	0.0000119	0.9470597
239	0.0002566	0.9425761	0.0000115	0.9644387
240	0.0002423	0.9445275	0.0000106	0.9283254
241	0.0002296	0.9476198	0.0000099	0.9320668
242	0.0002183	0.9506384	0.0000093	0.9384921
243	0.0002078	0.9516694	0.0000088	0.9457131
244	0.0001979	0.9526251	0.0000083	0.9406033
245	0.0001890	0.9550989	0.0000078	0.9394368
246	0.0001807	0.9558598	0.0000074	0.9484184
247	0.0001727	0.9559939	0.0000072	0.9728821
248	0.0001647	0.9537430	0.0000070	0.9768819
249	0.0001571	0.9536549	0.0000067	0.9498763
250	0.0001499	0.9544458	0.0000064	0.9553444
251	0.0001433	0.9558753	0.0000060	0.9370424
252	0.0001374	0.9585791	0.0000056	0.9316401
253	0.0001320	0.9606996	0.0000052	0.9306494
254	0.0001272	0.9637858	0.0000048	0.9190719
255	0.0001228	0.9655622	0.0000044	0.9347832
256	0.0001188	0.9668738	0.0000044	0.9940088
257	0.0001143	0.9625490	0.0000043	0.9723134
258	0.0001102	0.9637233	0.0000041	0.9537761
259	0.0001063	0.9649779	0.0000038	0.9226164
260	0.0001030	0.9687202	0.0000035	0.9165359
261	0.0001000	0.9705944	0.0000032	0.9210341
262	0.0000972	0.9725780	0.0000030	0.9262509

263	0.0000947	0.9740029	0.0000027	0.9186089
264	0.0000923	0.9752465	0.0000025	0.9217604
265	0.0000902	0.9762555	0.0000023	0.9262124
266	0.0000880	0.9765195	0.0000022	0.9434152
267	0.0000859	0.9755254	0.0000021	0.9638640
268	0.0000836	0.9736855	0.0000021	0.9837715
269	0.0000812	0.9707770	0.0000021	0.9930902
270	0.0000785	0.9675646	0.0000021	1.0173215
271	0.0000755	0.9611090	0.0000022	1.0450837
272	0.0000719	0.9527471	0.0000023	1.0645178
273	0.0000678	0.9424429	0.0000024	1.0398748
274	0.0000632	0.9323311	0.0000025	1.0201026
275	0.0000585	0.9260217	0.0000024	0.9839147
276	0.0000541	0.9241385	0.0000023	0.9540136
277	0.0000500	0.9240218	0.0000022	0.9530567
278	0.0000462	0.9245567	0.0000021	0.9527424
279	0.0000426	0.9230466	0.0000020	0.9612610
280	0.0000394	0.9231135	0.0000019	0.9543941
281	0.0000364	0.9245357	0.0000018	0.9222267
282	0.0000339	0.9301992	0.0000016	0.9061832
283	0.0000317	0.9376877	0.0000015	0.9005013
284	0.0000299	0.9433252	0.0000013	0.9244644
285	0.0000283	0.9433824	0.0000013	0.9573062
286	0.0000266	0.9404446	0.0000013	0.9814851
287	0.0000249	0.9364105	0.0000012	0.9666827
288	0.0000232	0.9340297	0.0000012	0.9701371
289	0.0000216	0.9308285	0.0000012	0.9738573
290	0.0000200	0.9257747	0.0000011	0.9852032
291	0.0000171	0.8523176	0.0000010	0.8703063

tables for CG with Multigrid preconditioner

h = 1/16 **eps**=0.000001 method=MG
k=4, **cond** A=150.416930

k	|e|	ratio	|r|	ratio
1	2.1998630	0.0000000	5.3253083	0.0000000
2	0.0956633	0.0434860	0.0363016	0.0068168
3	0.0014271	0.0149181	0.0006556	0.0180607
4	0.0000002	0.0001388	0.0000002	0.0003796

h = 1/32 **eps**=0.000001 method=MG
k=4, **cond** A=603.051928

k	|e|	ratio	|r|	ratio
1	3.1204692	0.0000000	7.1238671	0.0000000
2	0.1461191	0.0468260	0.0360066	0.0050544
3	0.0031325	0.0214382	0.0006083	0.0168931
4	0.0000004	0.0001403	0.0000003	0.0005447

h = 1/64 ,n=65 **eps**=0.000001 method=MG
k=4, **cond** A=2413.598654

k	|e|	ratio	|r|	ratio
1	4.5764854	0.0000000	10.4620417	0.0000000
2	0.2180254	0.0476404	0.0388724	0.0037156
3	0.0058026	0.0266142	0.0006767	0.0174076
4	0.0000012	0.0002040	0.0000004	0.0005660

h = 1/128 ,n=129 **eps**=0.000001 method=MG
k=4, **cond** A=9655.787254

k	|e|	ratio	|r|	ratio
1	6.4563811	0.0000000	14.6329466	0.0000000
2	0.3174234	0.0491643	0.0482766	0.0032992
3	0.0094526	0.0297790	0.0008327	0.0172495
4	0.0000034	0.0003597	0.0000004	0.0005216

Tables for CG with SSOR preconditioner

Tables for CG with SSOR preconditioner h=1/16

k	|e|	ratio	|r|	ratio

1	2.2693639	0.0000000	4.9322226	0.0000000
2	1.6468398	0.7256835	0.4605221	0.0933701
3	1.3513665	0.8205816	0.2730178	0.5928442
4	0.9606275	0.7108564	0.1964313	0.7194818
5	0.5252730	0.5468019	0.1561994	0.7951855
6	0.1534885	0.2922070	0.1084332	0.6941974
7	0.0416077	0.2710804	0.0275143	0.2537445
8	0.0262239	0.6302655	0.0103129	0.3748198
9	0.0123709	0.4717420	0.0068234	0.6616409
10	0.0033437	0.2702897	0.0029164	0.4274017
11	0.0011039	0.3301359	0.0008844	0.3032476
12	0.0005418	0.4908182	0.0003122	0.3530697
13	0.0001777	0.3279708	0.0001683	0.5389642
14	0.0000471	0.2650202	0.0000421	0.2501429
15	0.0000242	0.5133066	0.0000142	0.3362731
16	0.0000147	0.6099509	0.0000054	0.3833065
17	0.0000053	0.3600554	0.0000040	0.7457782
18	0.0000004	0.0818492	0.0000004	0.0934337

Tables for CG with SSOR preconditioner h=1/32

k	|e|	ratio	|r|	ratio
1	3.1617307	0.0000000	7.0150453	0.0000000
2	2.5617890	0.8102490	0.5539254	0.0789625
3	2.3542794	0.9189982	0.2962589	0.5348354
4	2.1065890	0.8947914	0.1986559	0.6705483
5	1.8518713	0.8790853	0.1422168	0.7158951
6	1.6156673	0.8724512	0.1133796	0.7972310
7	1.3789468	0.8534844	0.0910121	0.8027200
8	1.1345409	0.8227590	0.0843711	0.9270315
9	0.8224095	0.7248831	0.0847173	1.0041038
10	0.4180835	0.5083641	0.0825150	0.9740034
11	0.1235549	0.2955268	0.0482795	0.5850996
12	0.0551450	0.4463198	0.0203644	0.4218024
13	0.0278858	0.5056810	0.0122853	0.6032745
14	0.0128799	0.4618815	0.0057640	0.4691758
15	0.0081751	0.6347162	0.0026993	0.4683099
16	0.0058303	0.7131826	0.0013998	0.5185673
17	0.0047541	0.8154064	0.0006696	0.4783424
18	0.0038683	0.8136734	0.0005167	0.7716602
19	0.0021487	0.5554644	0.0005517	1.0677534
20	0.0011032	0.5134135	0.0002821	0.5114078
21	0.0007772	0.7045585	0.0001444	0.5119507
22	0.0005067	0.6519440	0.0001267	0.8768386
23	0.0002475	0.4884356	0.0000763	0.6020652
24	0.0001620	0.6545659	0.0000398	0.5221399
25	0.0001077	0.6648438	0.0000249	0.6246745
26	0.0000674	0.6260178	0.0000172	0.6916799
27	0.0000335	0.4974052	0.0000121	0.7029690
28	0.0000133	0.3976940	0.0000067	0.5550799
29	0.0000054	0.4034871	0.0000029	0.4265622
30	0.0000016	0.2990445	0.0000007	0.2473623

Tables for CG with SSOR preconditioner h=1/64

k	|e|	ratio	|r|	ratio
1	4.5490436	0.0000000	10.0541553	0.0000000
2	3.8362026	0.8432987	0.7280693	0.0724148
3	3.7003001	0.9645737	0.3053137	0.4193471
4	3.5423025	0.9573014	0.1926717	0.6310615
5	3.3783165	0.9537064	0.1437188	0.7459256
6	3.1995620	0.9470877	0.1180847	0.8216375
7	3.0218250	0.9444496	0.1001557	0.8481680
8	2.8564678	0.9452790	0.0828611	0.8273226
9	2.6921619	0.9424794	0.0706543	0.8526838
10	2.5198220	0.9359846	0.0632266	0.8948726
11	2.3462801	0.9311293	0.0583966	0.9236077
12	2.1638926	0.9222652	0.0533587	0.9137299
13	1.9814273	0.9156773	0.0489931	0.9181837
14	1.8029129	0.9099062	0.0433065	0.8839308
15	1.6149689	0.8957553	0.0422280	0.9750970
16	1.4112614	0.8738629	0.0412639	0.9771686
17	1.1749024	0.8325193	0.0429641	1.0412016

18	0.8528075	0.7258539	0.0493421	1.1484509
19	0.4518308	0.5298157	0.0484776	0.9824796
20	0.2132808	0.4720368	0.0278557	0.5746100
21	0.1501244	0.7038816	0.0146585	0.5262282
22	0.1111903	0.7406542	0.0140617	0.9592857
23	0.0662040	0.5954116	0.0111642	0.7939485
24	0.0437040	0.6601420	0.0069109	0.6190224
25	0.0303526	0.6945044	0.0050944	0.7371500
26	0.0221190	0.7287351	0.0033023	0.6482254
27	0.0165263	0.7471507	0.0024898	0.7539705
28	0.0129979	0.7865014	0.0016588	0.6662198
29	0.0110448	0.8497327	0.0011217	0.6762264
30	0.0098942	0.8958298	0.0007244	0.6457952
31	0.0091291	0.9226731	0.0004742	0.6545639
32	0.0084946	0.9304946	0.0003511	0.7405437
33	0.0076879	0.9050339	0.0003237	0.9219407
34	0.0065529	0.8523626	0.0003485	1.0763733
35	0.0047519	0.7251674	0.0003915	1.1234523
36	0.0028376	0.5971538	0.0003422	0.8742031
37	0.0018887	0.6656051	0.0002004	0.5855785
38	0.0015408	0.8157954	0.0001243	0.6204067
39	0.0012973	0.8419573	0.0001008	0.8108109
40	0.0010583	0.8157939	0.0000853	0.8463996
41	0.0007990	0.7549702	0.0000787	0.9219365
42	0.0005529	0.6919610	0.0000666	0.8468646
43	0.0003932	0.7111390	0.0000455	0.6826150
44	0.0003029	0.7704465	0.0000339	0.7445688
45	0.0002333	0.7702420	0.0000257	0.7580561
46	0.0001916	0.8211070	0.0000175	0.6821221
47	0.0001600	0.8349544	0.0000136	0.7744449
48	0.0001358	0.8490333	0.0000095	0.7032869
49	0.0001119	0.8239944	0.0000090	0.9472819
50	0.0000825	0.7372094	0.0000083	0.9171382
51	0.0000541	0.6554525	0.0000073	0.8784722
52	0.0000349	0.6460945	0.0000048	0.6616077
53	0.0000248	0.7089250	0.0000034	0.7101718
54	0.0000182	0.7361531	0.0000023	0.6772298
55	0.0000124	0.6811877	0.0000021	0.9093606
56	0.0000048	0.3868446	0.0000009	0.4194420

Tables for CG with SSOR preconditioner h=1/128

k	|e|	ratio	|r|	ratio
1	6.4457756	0.0000000	14.4540779	0.0000000
2	5.5139993	0.8554439	0.9798537	0.0677908
3	5.4088913	0.9809380	0.3403982	0.3473970
4	5.3026393	0.9803560	0.2034492	0.5976800
5	5.1853961	0.9778897	0.1483529	0.7291887
6	5.0654954	0.9768772	0.1202717	0.8107138
7	4.9451401	0.9762402	0.1012563	0.8418966
8	4.8245020	0.9756047	0.0858063	0.8474169
9	4.7066959	0.9755817	0.0722637	0.8421718
10	4.5874237	0.9746590	0.0649999	0.8994827
11	4.4629279	0.9728615	0.0605125	0.9309626
12	4.3391818	0.9722724	0.0545367	0.9012460
13	4.2186913	0.9722320	0.0488740	0.8961687
14	4.1021192	0.9723677	0.0440151	0.9005828
15	3.9854067	0.9715483	0.0412976	0.9382589
16	3.8680259	0.9705473	0.0382945	0.9272820
17	3.7513422	0.9698338	0.0359069	0.9376523
18	3.6295795	0.9675415	0.0344123	0.9583746
19	3.5086283	0.9666763	0.0316764	0.9204971
20	3.3903746	0.9662963	0.0299359	0.9450520
21	3.2691604	0.9642475	0.0290988	0.9720394
22	3.1436705	0.9616140	0.0277443	0.9534507
23	3.0186217	0.9602220	0.0262770	0.9471139
24	2.8957417	0.9592927	0.0248096	0.9441578
25	2.7732455	0.9576978	0.0238982	0.9632612
26	2.6506297	0.9557862	0.0223533	0.9353568
27	2.5323184	0.9553648	0.0216356	0.9678923
28	2.4045866	0.9495594	0.0212703	0.9831139
29	2.2772037	0.9470250	0.0200174	0.9410971

30	2.1483420	0.9434123	0.0196727	0.9827790
31	2.0149086	0.9378901	0.0192478	0.9784036
32	1.8686594	0.9274165	0.0199859	1.0383474
33	1.6895903	0.9041724	0.0217554	1.0885392
34	1.4442876	0.8548153	0.0258210	1.1868754
35	1.0931431	0.7568736	0.0302817	1.1727536
36	0.6837108	0.6254540	0.0283683	0.9368158
37	0.4418863	0.6463058	0.0184521	0.6504463
38	0.3518847	0.7963243	0.0114809	0.6221984
39	0.3006824	0.8544912	0.0105466	0.9186222
40	0.2372194	0.7889368	0.0114586	1.0864744
41	0.1840178	0.7757283	0.0085694	0.7478555
42	0.1528291	0.8305127	0.0068315	0.7971989
43	0.1237482	0.8097158	0.0067367	0.9861302
44	0.0985025	0.7959918	0.0053549	0.7948776
45	0.0801615	0.8138015	0.0046556	0.8694177
46	0.0645647	0.8054324	0.0038580	0.8286741
47	0.0538188	0.8335634	0.0030348	0.7866214
48	0.0443159	0.8234283	0.0027050	0.8913447
49	0.0369617	0.8340510	0.0021314	0.7879483
50	0.0306704	0.8297872	0.0018818	0.8828969
51	0.0251646	0.8204853	0.0015780	0.8385403
52	0.0206586	0.8209407	0.0013350	0.8459954
53	0.0170433	0.8249991	0.0010816	0.8101736
54	0.0140243	0.8228592	0.0009494	0.8778358
55	0.0116435	0.8302376	0.0007679	0.8087564
56	0.0099038	0.8505901	0.0006226	0.8107796
57	0.0087211	0.8805767	0.0004813	0.7731395
58	0.0078802	0.9035759	0.0003957	0.8221169
59	0.0072607	0.9213899	0.0003211	0.8113631
60	0.0068350	0.9413714	0.0002460	0.7660613
61	0.0065434	0.9573309	0.0001854	0.7536923
62	0.0063343	0.9680457	0.0001386	0.7475188
63	0.0061677	0.9736986	0.0001019	0.7352154
64	0.0060189	0.9758793	0.0000781	0.7662600
65	0.0058564	0.9730031	0.0000697	0.8929152
66	0.0056418	0.9633473	0.0000711	1.0197906
67	0.0053083	0.9408880	0.0000812	1.1420576
68	0.0047907	0.9024899	0.0000954	1.1756096
69	0.0040810	0.8518580	0.0001007	1.0553830
70	0.0033234	0.8143729	0.0000974	0.9667177
71	0.0026225	0.7891023	0.0000930	0.9553683
72	0.0019626	0.7483464	0.0000878	0.9441616
73	0.0014856	0.7569965	0.0000692	0.7874535
74	0.0012467	0.8391488	0.0000479	0.6925092
75	0.0011161	0.8952418	0.0000366	0.7631700
76	0.0010177	0.9118797	0.0000315	0.8615686
77	0.0009285	0.9123426	0.0000269	0.8534135
78	0.0008523	0.9179524	0.0000225	0.8376764
79	0.0007739	0.9080042	0.0000224	0.9961889
80	0.0006799	0.8785639	0.0000216	0.9629637
81	0.0005881	0.8648625	0.0000201	0.9290428
82	0.0004882	0.8301801	0.0000208	1.0351272
83	0.0003913	0.8015060	0.0000176	0.8481698
84	0.0003242	0.8285988	0.0000142	0.8060784
85	0.0002702	0.8334427	0.0000129	0.9107954
86	0.0002246	0.8311510	0.0000109	0.8432222
87	0.0001916	0.8532194	0.0000089	0.8125177
88	0.0001647	0.8592511	0.0000077	0.8643326
89	0.0001439	0.8739769	0.0000063	0.8266711
90	0.0001263	0.8773506	0.0000057	0.8934908
91	0.0001118	0.8853776	0.0000047	0.8270530
92	0.0001006	0.8995960	0.0000040	0.8506773
93	0.0000920	0.9145468	0.0000031	0.7799253
94	0.0000856	0.9305279	0.0000025	0.8191927
95	0.0000800	0.9350646	0.0000021	0.8149292
96	0.0000752	0.9394217	0.0000017	0.8363557
97	0.0000698	0.9282142	0.0000017	0.9814936
98	0.0000632	0.9062999	0.0000017	1.0138043
99	0.0000547	0.8654208	0.0000018	1.0641346
100	0.0000461	0.8418280	0.0000016	0.8914509
101	0.0000391	0.8481096	0.0000014	0.8598060

102	0.0000338	0.8637455	0.0000011	0.8164981
103	0.0000254	0.7524275	0.0000009	0.8216432

5.3 Table CG with incomplete cholesky

Table CG with incomplete cholesky preconditioner $\epsilon = 10^{-2}$, $h = \frac{1}{16}$

k	|e|	ratio	|r|	ratio
1	2.2117568	0.0000000	4.9847338	0.0000000
2	0.9605500	0.4342928	0.1236319	0.0248021
3	0.1101739	0.1146987	0.0740761	0.5991666
4	0.0098178	0.0891114	0.0064913	0.0876300
5	0.0006397	0.0651529	0.0007141	0.1100128
6	0.0000566	0.0885013	0.0000495	0.0693702
7	0.0000012	0.0210676	0.0000009	0.0178494

Table CG with incomplete cholesky preconditioner $\epsilon = 10^{-2}$, $h = \frac{1}{32}$

k	|e|	ratio	|r|	ratio
1	3.1411335	0.0000000	6.9640117	0.0000000
2	2.0214097	0.6435287	0.1420103	0.0203920
3	1.3623817	0.6739760	0.1149900	0.8097304
4	0.6575397	0.4826399	0.0682371	0.5934171
5	0.1011955	0.1539003	0.0247907	0.3633033
6	0.0225210	0.2225494	0.0069472	0.2802356
7	0.0067767	0.3009055	0.0015084	0.2171254
8	0.0019548	0.2884659	0.0004569	0.3028904
9	0.0008448	0.4321526	0.0001217	0.2664539
10	0.0002326	0.2753055	0.0000595	0.4889375
11	0.0000681	0.2927750	0.0000127	0.2125976
12	0.0000188	0.2754409	0.0000052	0.4094558
13	0.0000022	0.1195568	0.0000003	0.0640489

Table CG with incomplete cholesky preconditioner $\epsilon = 10^{-2}$, $h = \frac{1}{64}$

k	|e|	ratio	|r|	ratio
1	4.6047158	0.0000000	10.3310633	0.0000000
2	3.4648962	0.7524669	0.1937203	0.0187512
3	3.0608082	0.8833766	0.1211410	0.6253400
4	2.5757427	0.8415237	0.0563599	0.4652418
5	2.0650650	0.8017358	0.0413553	0.7337726
6	1.5529332	0.7520021	0.0446463	1.0795781
7	0.9346100	0.6018353	0.0369208	0.8269618
8	0.2831630	0.3029745	0.0265590	0.7193503
9	0.0732691	0.2587522	0.0112939	0.4252372
10	0.0321756	0.4391431	0.0039872	0.3530414
11	0.0178107	0.5535469	0.0013842	0.3471708
12	0.0110120	0.6182812	0.0008543	0.6171309
13	0.0073396	0.6665038	0.0004321	0.5057645
14	0.0039218	0.5343410	0.0002636	0.6101306
15	0.0023472	0.5984915	0.0001345	0.5102407
16	0.0013514	0.5757587	0.0001004	0.7467383
17	0.0007088	0.5244614	0.0000461	0.4586731
18	0.0002407	0.3396233	0.0000342	0.7426365
19	0.0001089	0.4524637	0.0000103	0.3000293
20	0.0000698	0.6407427	0.0000045	0.4406339
21	0.0000343	0.4918772	0.0000034	0.7510210
22	0.0000068	0.1977073	0.0000005	0.1584427

Table CG with incomplete cholesky preconditioner $\epsilon = 10^{-2}$, $h = \frac{1}{128}$

k	|e|	ratio	|r|	ratio
1	6.4885732	0.0000000	14.5067213	0.0000000
2	5.2612798	0.8108531	0.2641937	0.0182118
3	5.0111119	0.9524511	0.1164715	0.4408564
4	4.6620201	0.9303364	0.0500341	0.4295820
5	4.3222195	0.9271130	0.0496910	0.9931436
6	3.9989939	0.9252177	0.0423171	0.8516054
7	3.6528553	0.9134436	0.0289715	0.6846290
8	3.2996211	0.9032991	0.0270517	0.9337335
9	2.9620134	0.8976829	0.0260842	0.9642352
10	2.6075430	0.8803279	0.0207816	0.7967142

```
                11        2.2325951        0.8562064        0.0176972        0.8515802
                12        1.8177887        0.8142044        0.0202955        1.1468181
                13        1.2782021        0.7031632        0.0227175        1.1193341
                14        0.5725682        0.4479481        0.0191123        0.8413068
                15        0.2181351        0.3809767        0.0101596        0.5315716
                16        0.1256071        0.5758222        0.0059499        0.5856484
                17        0.0759987        0.6050512        0.0036252        0.6092867
                18        0.0443639        0.5837452        0.0024628        0.6793574
                19        0.0240991        0.5432141        0.0014339        0.5822123
                20        0.0156123        0.6478383        0.0008604        0.6000823
                21        0.0094219        0.6034940        0.0005069        0.5891044
                22        0.0066705        0.7079759        0.0002955        0.5830443
                23        0.0054586        0.8183167        0.0001677        0.5673221
                24        0.0047031        0.8615927        0.0000873        0.5205250
                25        0.0040218        0.8551456        0.0000745        0.8537794
                26        0.0030834        0.7666791        0.0000688        0.9227595
                27        0.0021837        0.7082150        0.0000606        0.8814875
                28        0.0014407        0.6597450        0.0000421        0.6948456
                29        0.0011682        0.8108344        0.0000212        0.5036729
                30        0.0009944        0.8512118        0.0000176        0.8296109
                31        0.0007347        0.7388340        0.0000182        1.0336879
                32        0.0005166        0.7031852        0.0000114        0.6264216
                33        0.0003559        0.6888947        0.0000107        0.9380414
                34        0.0002062        0.5794708        0.0000081        0.7584751
                35        0.0001239        0.6006305        0.0000052        0.6377109
                36        0.0000837        0.6759086        0.0000032        0.6095393
                37        0.0000659        0.7871051        0.0000017        0.5390215
                38        0.0000539        0.8184570        0.0000013        0.7405635
                39        0.0000202        0.3752176        0.0000009        0.7459009
```

Tables for CG with incomplete cholesky preconditioner $\epsilon = 10^{-3}$

```
h =      1/16    ,n=17   eps=0.000001   method=CHL
                 k              |e|            ratio              |r|            ratio
                 1        2.2113015        0.0000000        4.8698707        0.0000000
                 2        0.1268927        0.0573837        0.0196604        0.0040372
                 3        0.0011025        0.0086886        0.0005919        0.0301065
                 4        0.0000002        0.0001684        0.0000001        0.0001565

h =      1/32    ,n=33   eps=0.000001   method=CHL
                 k              |e|            ratio              |r|            ratio
                 1        3.2189095        0.0000000        7.0357482        0.0000000
                 2        0.8589679        0.2668506        0.0306470        0.0043559
                 3        0.0363666        0.0423375        0.0064895        0.2117505
                 4        0.0025347        0.0696990        0.0003268        0.0503545
                 5        0.0001276        0.0503368        0.0000319        0.0976751
                 6        0.0000006        0.0047450        0.0000001        0.0037884
h =      1/64    ,n=65   eps=0.000001   method=CHL
                 k              |e|            ratio              |r|            ratio
                 1        4.5281055        0.0000000       10.2891303        0.0000000
                 2        2.3962171        0.5291876        0.0305161        0.0029659
                 3        1.0190972        0.4252942        0.0239474        0.7847488
                 4        0.1236226        0.1213060        0.0102118        0.4264245
                 5        0.0186393        0.1507755        0.0012096        0.1184530
                 6        0.0016236        0.0871065        0.0002558        0.2114490
                 7        0.0002756        0.1697321        0.0000182        0.0713307
                 8        0.0000163        0.0592016        0.0000009        0.0502623
h =      1/128   ,n=129  eps=0.000001   method=CHL
                 k              |e|            ratio              |r|            ratio
                 1        6.4658650        0.0000000       14.3821767        0.0000000
                 2        4.5500543        0.7037039        0.0349350        0.0024290
                 3        3.5640524        0.7832989        0.0316566        0.9061590
                 4        2.5832848        0.7248167        0.0229909        0.7262586
                 5        1.4701895        0.5691163        0.0119903        0.5215250
                 6        0.3578766        0.2434221        0.0077001        0.6421915
                 7        0.0850789        0.2377325        0.0036799        0.4779069
                 8        0.0274688        0.3228626        0.0009809        0.2665421
                 9        0.0121215        0.4412808        0.0002445        0.2492235
                10        0.0034751        0.2866930        0.0001041        0.4258668
                11        0.0019516        0.5615987        0.0000341        0.3280329
                12        0.0010720        0.5492590        0.0000211        0.6186532
                13        0.0003595        0.3353743        0.0000058        0.2766025
```

```
            14     0.0000439     0.1220343     0.0000010     0.1636738
```