

■ Notes: <http://bessie.che.uc.edu/tlb/teach/undergrad/chem381f97/math/node7.html>

$$\text{soln} = \int_0^1 \int_0^1 \{ \partial_\xi u, \partial_\eta u \} J^{-T} J^{-1} \begin{pmatrix} \partial_\xi v \\ \partial_\eta v \end{pmatrix} \|J\| d\xi d\eta$$

$$\text{local stiffness matrix : } K_{mn}^j = \int_0^1 \int_0^1 \{ \partial_\xi N_m, \partial_\eta N_m \} J^{-T} J^{-1} \begin{pmatrix} \partial_\xi N_n \\ \partial_\eta N_n \end{pmatrix} \|J\| d\xi d\eta$$

■ Triangle Element

$$\text{shape functions : } N_1 = 1 - \xi - \eta, N_2 = \xi, N_3 = \eta$$

$$\text{trial and test function : } u = \sum_{i=1}^3 q_i N_i, v = \sum_{i=1}^3 p_k N_k \rightarrow \frac{\partial u}{\partial \xi} = -q_1 + q_2, \frac{\partial v}{\partial \xi} = -p_1 + p_2, \frac{\partial u}{\partial \eta} = -q_1 + q_3, \frac{\partial v}{\partial \eta} = -p_1 + p_3$$

$$\bar{x} = \bar{x}_1 + (\bar{x}_2 - \bar{x}_1)\xi + (\bar{x}_3 - \bar{x}_1)\eta \rightarrow x = x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta, y = y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta$$

$$\Delta x_2 \equiv x_2 - x_1, \Delta y_2 \equiv y_2 - y_1, \Delta x_3 \equiv x_3 - x_1, \Delta y_3 \equiv y_3 - y_1$$

$$\text{Jacobian : } J = \begin{pmatrix} \partial_\xi x & \partial_\eta x \\ \partial_\xi y & \partial_\eta y \end{pmatrix} = \begin{pmatrix} \Delta x_2 & \Delta x_3 \\ \Delta y_2 & \Delta y_3 \end{pmatrix}, \{dx, dy\} = J \{d\xi, d\eta\}$$

$$\text{eigenvalues : } \lambda_2 \equiv \Delta x_2^2 + \Delta y_2^2, \lambda_{23} \equiv \Delta x_2 \Delta x_3 + \Delta y_2 \Delta y_3, \lambda_3 \equiv \Delta x_3^2 + \Delta y_3^2$$

$$K_{mn}^j = \frac{1}{\Delta x_2 \Delta y_3 - \Delta x_3 \Delta y_2} \begin{pmatrix} \lambda_2 + \lambda_3 - 2\lambda_{23} & \lambda_{23} - \lambda_3 & \lambda_{23} - \lambda_2 \\ \lambda_{23} - \lambda_3 & \lambda_3 & -\lambda_{23} \\ \lambda_{23} - \lambda_2 & -\lambda_{23} & \lambda_2 \end{pmatrix}$$

■ Quad Element

$$\text{shape functions : } N_1 = (\eta - 1)(\xi - 1), N_2 = \xi(1 - \eta), N_3 = \eta(1 - \xi), N_4 = \xi\eta$$

$$\bar{x} = \bar{x}_1 + (\bar{x}_2 - \bar{x}_1)\xi + (\bar{x}_3 - \bar{x}_1)\eta + (\bar{x}_1 - \bar{x}_2 - \bar{x}_3 + \bar{x}_4)\xi\eta$$

$$\Delta x_2 \equiv x_2 - x_1, \Delta y_2 \equiv y_2 - y_1, \Delta x_3 \equiv x_3 - x_1, \Delta y_3 \equiv y_3 - y_1, \Delta x_a \equiv x_1 - x_2 - x_3 + x_4, \Delta y_a \equiv y_1 - y_2 - y_3 + y_4$$

$$\text{Jacobian : } J = \begin{pmatrix} \partial_\xi x & \partial_\eta x \\ \partial_\xi y & \partial_\eta y \end{pmatrix} = \begin{pmatrix} \Delta x_2 + \eta \Delta x_a & \Delta x_3 + \xi \Delta x_a \\ \Delta y_2 + \eta \Delta y_a & \Delta y_3 + \xi \Delta y_a \end{pmatrix}$$

$$\text{eigenvalues : } \lambda_2 \equiv \Delta x_2^2 + \Delta y_2^2, \lambda_{23} \equiv \Delta x_2 \Delta x_3 + \Delta y_2 \Delta y_3, \lambda_3 \equiv \Delta x_3^2 + \Delta y_3^2$$

$$K_{mn}^j = x$$

■ Program

Calculate the x, y coordinates of the nodes :

```

a = b = 2.0; nx = 7; ny = 5;
nodes = Chop[Flatten[Table[{x, y}, {y, -b/2, b/2, b/(ny - 1)}, {x, -a/2, a/2, a/(nx - 1)}], 1]]

{{-1., -1.}, {-0.666667, -1.}, {-0.333333, -1.}, {0, -1.}, {0.333333, -1.}, {0.666667, -1.}, {1., -1.},
{-1., -0.5}, {-0.666667, -0.5}, {-0.333333, -0.5}, {0, -0.5}, {0.333333, -0.5}, {0.666667, -0.5},
{1., -0.5}, {-1., 0}, {-0.666667, 0}, {-0.333333, 0}, {0, 0}, {0.333333, 0}, {0.666667, 0}, {1., 0},
{-1., 0.5}, {-0.666667, 0.5}, {-0.333333, 0.5}, {0, 0.5}, {0.333333, 0.5}, {0.666667, 0.5}, {1., 0.5},
{-1., 1.}, {-0.666667, 1.}, {-0.333333, 1.}, {0, 1.}, {0.333333, 1.}, {0.666667, 1.}, {1., 1.}}

```

Calculate the indices of nodes of the triangle elements :

```

elements = Flatten[Table[{{i, i + 1, i + nx}, {i + 1, i + nx + 1, i + nx}} + (j - 1) nx, {j, 1, ny - 1}, {i, 1, nx - 1}], 2]

{{1, 2, 8}, {2, 9, 8}, {2, 3, 9}, {3, 10, 9}, {3, 4, 10}, {4, 11, 10}, {4, 5, 11}, {5, 12, 11}, {5, 6, 12}, {6, 13, 12}, {6, 7, 13},
{7, 14, 13}, {8, 9, 15}, {9, 16, 15}, {9, 10, 16}, {10, 17, 16}, {10, 11, 17}, {11, 18, 17}, {11, 12, 18}, {12, 19, 18}, {12, 13, 19},
{13, 20, 19}, {13, 14, 20}, {14, 21, 20}, {15, 16, 22}, {16, 23, 22}, {16, 17, 23}, {17, 24, 23}, {17, 18, 24}, {18, 25, 24},
{18, 19, 25}, {19, 26, 25}, {19, 20, 26}, {20, 27, 26}, {20, 21, 27}, {21, 28, 27}, {22, 23, 29}, {23, 30, 29}, {23, 24, 30},
{24, 31, 30}, {24, 25, 31}, {25, 32, 31}, {25, 26, 32}, {26, 33, 32}, {26, 27, 33}, {27, 34, 33}, {27, 28, 34}, {28, 35, 34}}

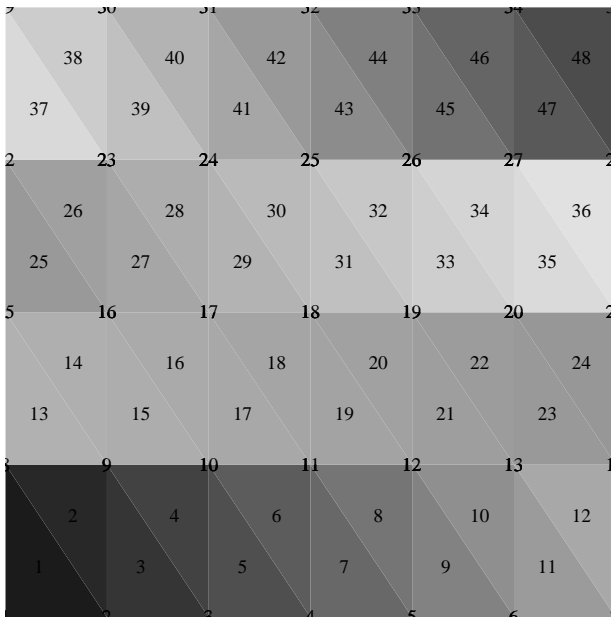
```

Here is a picture to visualize the triangle elements :

```

jmax = Length[elements]; $DefaultFont = {"Times-Roman", 9};
PlotColor[hue_] := Hue[2 (1 - Min[1, Max[0, hue]])/3];
Show[Graphics[Table[plist = Map[nodes[#[#]] &, elements[[j]]; {PlotColor[(j - 1)/(jmax - 1)], Polygon[plist],
RGBColor[0, 0, 0], Text[j, Plus @@ plist/3], Table[Text[elements[[j, i]], plist[[i]], {i, 1, 3}], {j, 1, jmax, 1}],
AspectRatio -> Automatic, PlotRange -> {{-1, 1} a/2, {-1, 1} b/2}];

```



Calculate the local stiffness matrices :

my calculations (does this agree with Roy?): $\lambda_2 = \Delta x_2^2 + \Delta y_2^2$, $\lambda_{23} = \Delta x_2 \Delta y_2 + \Delta x_3 \Delta y_3$, $\lambda_3 = \Delta x_3^2 + \Delta y_3^2$, $\lambda_a = \Delta x_3^2 - \Delta y_2^2$

$$K_{\text{local}} = \frac{1}{\Delta x_2 \Delta y_3 - \Delta x_3 \Delta y_2} \begin{pmatrix} (\Delta x_2 - \Delta y_2)^2 + (\Delta x_3 - \Delta y_3)^2 & \Delta x_2 \Delta y_2 - \Delta y_2^2 + \Delta x_3 \Delta y_3 - \Delta y_3^2 & -\Delta x_2^2 - \Delta x_3^2 + \Delta x_2 \Delta y_2 + \Delta x_3 \Delta y_3 \\ \Delta x_2 \Delta y_2 - \Delta y_2^2 + \Delta x_3 \Delta y_3 - \Delta y_3^2 & \Delta y_2^2 + \Delta y_3^2 & -\Delta x_2 \Delta y_2 - \Delta x_3 \Delta y_3 \\ -\Delta x_2^2 - \Delta x_3^2 + \Delta x_2 \Delta y_2 + \Delta x_3 \Delta y_3 & -\Delta x_2 \Delta y_2 - \Delta x_3 \Delta y_3 & \Delta x_2^2 + \Delta x_3^2 \end{pmatrix}$$

$$= \frac{1}{\Delta x_2 \Delta y_3 - \Delta x_3 \Delta y_2} \begin{pmatrix} \lambda_2 + \lambda_3 - 2\lambda_{23} & \lambda_{23} - \lambda_3 + \lambda_a & \lambda_{23} - \lambda_2 - \lambda_a \\ \lambda_{23} - \lambda_3 + \lambda_a & \lambda_3 - \lambda_a & -\lambda_{23} \\ \lambda_{23} - \lambda_2 - \lambda_a & -\lambda_{23} & \lambda_2 + \lambda_a \end{pmatrix}$$

Klocal =

Map[Module[{}, Do[{ Δx_i , Δy_i } = nodes[#[[i]]] - nodes[#[[1]]], {i, 2, 3}]; $\lambda_2 = \Delta x_2^2 + \Delta y_2^2$; $\lambda_{23} = \Delta x_2 \Delta x_3 + \Delta y_2 \Delta y_3$;

$\lambda_3 = \Delta x_3^2 + \Delta y_3^2$; $\mathbf{d} = \Delta x_2 \Delta y_3 - \Delta x_3 \Delta y_2$; Chop[$\frac{1}{\mathbf{d}}$ $\begin{pmatrix} \lambda_2 + \lambda_3 - 2\lambda_{23} & \lambda_{23} - \lambda_3 & \lambda_{23} - \lambda_2 \\ \lambda_{23} - \lambda_3 & \lambda_3 & -\lambda_{23} \\ \lambda_{23} - \lambda_2 & -\lambda_{23} & \lambda_2 \end{pmatrix}$]] &, elements];

Map[

MatrixForm,

Klocal]

$$\begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix},$$

$$\begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix},$$

$$\begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix},$$

$$\begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix},$$

$$\begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix},$$

$$\begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix},$$

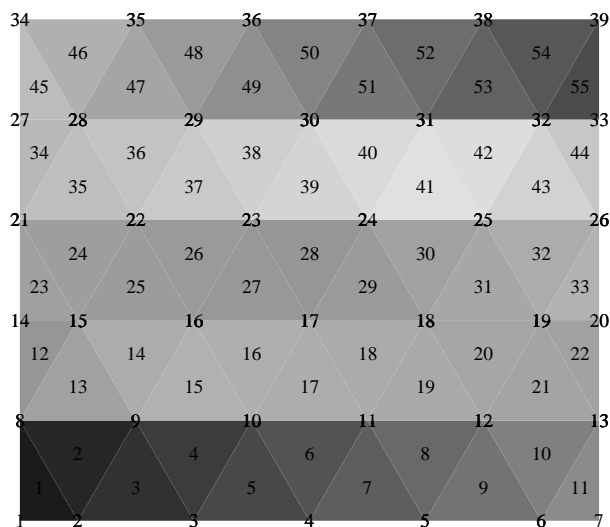
■ Honey Comb

```

nx = 50; ny = Round[2 nx Tan[ $\pi$  / 6]]; dx = 1.0 / (nx - 2);
nodes = Flatten[Table[Table[{Min[1, Max[-1, x]], dx (j - 0.5 (ny + 1)) / Tan[ $\pi$  / 6],
{x, -(1 + dx Mod[j, 2]), 1 + dx, 2 dx}], {j, 1, ny}], 1];
elements = Flatten[Table[Table[i + (j - 1) nx - Floor[j / 2] + Join[If[Mod[j, 2] == 1 || i > 1, {{0, 1, nx}}, {}],
If[Mod[j, 2] == 0 || i < nx - 1, {{1, nx + 1, nx}}, {}], {i, 1, nx - 1}], {j, 1, ny - 1}], 2];
jmax = Length[elements]; PlotColor[hue_] := Hue[2 (1 - Min[1, Max[0, hue]]) / 3];

$DefaultFont = {"Times-Roman", 9};
Show[Graphics[Table[plist = Map[nodes[#] &, elements[j]];
{PlotColor[(j - 1) / (jmax - 1)], Polygon[plist], RGBColor[0, 0, 0], Text[j, Plus @@ plist / 3],
Table[Text[elements[j, i], plist[i]], {i, 1, 3}], {j, 1, jmax, 1}], AspectRatio -> Automatic, PlotRange -> All];

```



Klocal =

Map[Module[{}, Do[{ Δx_i , Δy_i } = nodes[[#][i]] - nodes[[#][1]], {i, 2, 3}]; $\lambda_2 = \Delta x_2^2 + \Delta y_2^2$; $\lambda_{23} = \Delta x_2 \Delta x_3 + \Delta y_2 \Delta y_3$;

$\lambda_3 = \Delta x_3^2 + \Delta y_3^2$; $d = \Delta x_2 \Delta y_3 - \Delta x_3 \Delta y_2$; Chop[$\frac{1}{d} \begin{pmatrix} \lambda_2 + \lambda_3 - 2\lambda_{23} & \lambda_{23} - \lambda_3 & \lambda_{23} - \lambda_2 \\ \lambda_{23} - \lambda_3 & \lambda_3 & -\lambda_{23} \\ \lambda_{23} - \lambda_2 & -\lambda_{23} & \lambda_2 \end{pmatrix}$] &, elements];

nxy = Length[nodes]; Kglobal = Table[0, {nxy}, {nxy}];

Do[element = elements[[j]]; Do[Kglobal[element[[m]], element[[n]]] += Klocal[[j, m, n]], {m, 1, 3}, {n, 1, 3}], {j, 1, jmax}];

Q = Table[0, {nxy}]; Map[Module[{}, Do[{ Δx_i , Δy_i } = nodes[[#][i]] - nodes[[#][1]], {i, 2, 3}];

Do[Q[[#][n]] += ($\Delta x_2 \Delta y_3 - \Delta x_3 \Delta y_2$)/6, {n, 1, 3}] &, elements];

ymax = Max[Map[#][2] &, nodes];

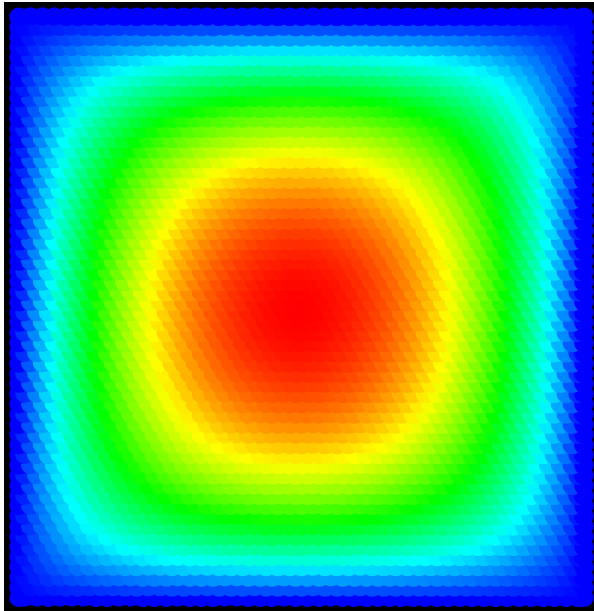
Do[{x, y} = nodes[[i]];

If[Abs[x] == 1 || Abs[y] == ymax, Kglobal[[i]] = Table[If[j == i, 1, 0], {j, 1, nxy}]; Q[[i]] = 0], {i, 1, nxy}];

q = LinearSolve[Kglobal, Q]; q1 = Min[q]; q2 = Max[q];

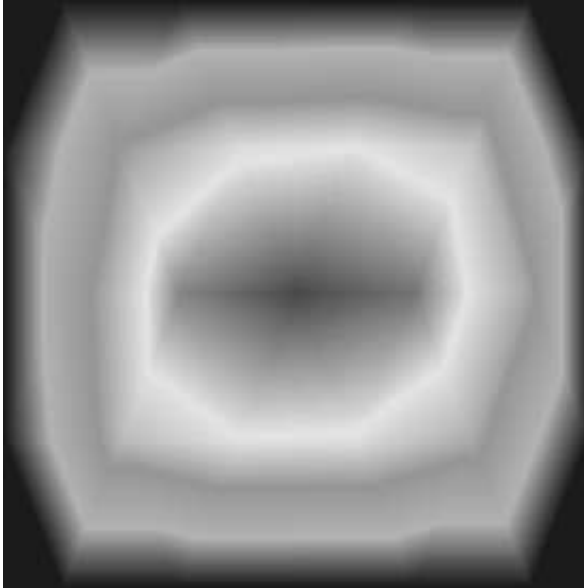
Show[Graphics[{PointSize[1.5 dx], Table[{PlotColor[(q[[i]] - q1)/(q2 - q1)], Point[nodes[[i]]], {i, 1, nxy}}],

AspectRatio -> Automatic, Background -> RGBColor[0, 0, 0]]];



(* Interpolated plot, runtime: 2.3 minutes *)

```
Interpolate[x_, y_] :=
Module[{j = 1, ξ = 2, η = 0}, While[!(0 ≤ ξ ≤ 1 && 0 ≤ η ≤ 1 && ξ + η ≤ 1), element = elements[[j]];
  {x1, y1} = nodes[element[[1]]; Do[{Δxi, Δyi} = nodes[element[[i]]] - {x1, y1}, {i, 2, 3}]; d = Δx2 Δy3 - Δx3 Δy2;
  {ξ, η} = ((x - x1) {Δy3, -Δy2} + (y - y1) {-Δx3, Δx2}) / d; j++; q[element][1 - ξ - η, ξ, η];
ymax = 0.5 dx (ny - 1) / Tan[π / 6];
DensityPlot[Interpolate[x, y], {x, -1, 1}, {y, -ymax, 0.999 ymax},
  PlotPoints → 275, Mesh → False, Frame → False, ColorFunction → PlotColor];
```



■ Unstructured Grid

```
nodes =
  Transpose[Import["C:/School/Engineering/ME207 - Computer Modeling/Unstructured Grid/mesh_p.txt", "Table"]];
elements = Transpose[Round[Delete[
  Import["C:/School/Engineering/ME207 - Computer Modeling/Unstructured Grid/mesh_t.txt", "Table"], 4]]];
jmax = Length[elements]; xlist = Map[#[[1]] &, nodes]; ylist = Map[#[[2]] &, nodes];
{x1, x2} = {Min[xlist], Max[xlist]}; {y1, y2} = {Min[ylist], Max[ylist]};
nxy = Length[nodes]; Q = Table[0, {nxy}];
Klocal = Map[Module[{}, Do[{Δxi, Δyi} = nodes[#[[i]]] - nodes[#[[1]]], {i, 2, 3}]; λ2 = Δx22 + Δy22;
  λ23 = Δx2 Δx3 + Δy2 Δy3; λ3 = Δx32 + Δy32; d = Δx2 Δy3 - Δx3 Δy2; Do[Q[#[[n]]] += d / 6, {n, 1, 3}];
  Chop[1/d (
    {λ2 + λ3 - 2 λ23, λ23 - λ3, λ23 - λ2,
     λ23 - λ3, λ3, -λ23,
     λ23 - λ2, -λ23, λ2}
  )] &, elements]; nxy = Length[nodes];
Kglobal = Table[0, {nxy}, {nxy}];
Do[element = elements[[j]];
  Do[Kglobal[element[[m]], element[[n]]] += Klocal[j, m, n], {m, 1, 3}, {n, 1, 3}], {j, 1, jmax}];
Do[{x, y} = nodes[[i]]; If[x == x1 || x == x2 || y == y1 || y == y2, Kglobal[[i]] = Table[If[j == i, 1, 0], {j, 1, nxy}]; Q[[i]] = 0],
  {i, 1, nxy}]; q = Chop[LinearSolve[Kglobal, Q];
```

(* 17 minutes *)

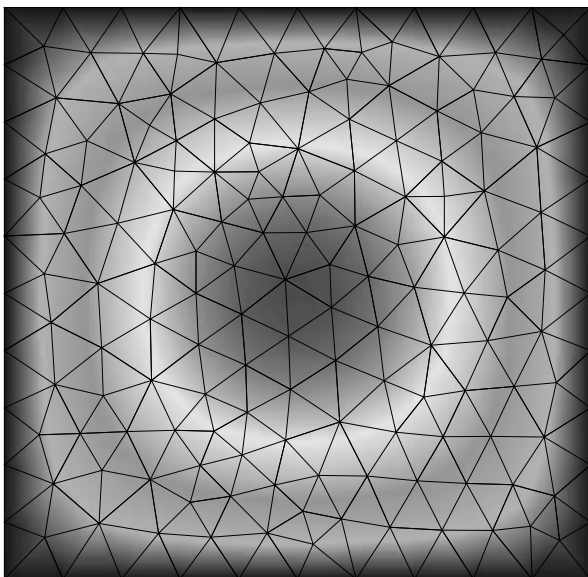
Interpolate[x_, y_] :=

```
Module[{j = 1, xi = 2, eta = 0, x1, y1}, While[!(0 ≤ xi ≤ 1 && 0 ≤ eta ≤ 1 && xi + eta ≤ 1), element = elements[[j]];
  {x1, y1} = nodes[element[[1]]; Do[{Δxi, Δyi} = nodes[element[[i]]] - {x1, y1}, {i, 2, 3}]; d = Δx2 Δy3 - Δx3 Δy2;
  {xi, eta} = ((x - x1) {Δy3, -Δy2} + (y - y1) {-Δx3, Δx2}) / d; j++; q[element][{1 - xi - eta, xi, eta}];
```

DensityPlot[Interpolate[x, y], {x, x1, x2}, {y, y1, y2}, PlotPoints → 275, AspectRatio → Automatic,

```
Mesh → False, Frame → False, ColorFunction → (Hue[2 (1 - #) / 3] &),
```

```
Epilog → Table[{Line[Map[nodes[#] &, elements[[j]]]}, {j, 1, jmax}]]];
```



```

n = 275; image = Table[0, {n}, {n}];
Interpolate[x_, y_] := Module[{x1, y1}, {x1, y1} = plist[[1]]; Do[{Δxi, Δyi} = plist[[i]] - {x1, y1}, {i, 2, 3}];
  {ξ, η} = ((x - x1) {Δy3, -Δy2} + (y - y1) {-Δx3, Δx2}) / (Δx2 Δy3 - Δx3 Δy2); q[element].{1 - ξ - η, ξ, η};
Do[element = elements[[j]]; plist = nodes[element]; xlist = Map[#[[1]] &, plist]; ylist = Map[#[[2]] &, plist];
  xIntersect[{{x1_, y1_}, {x2_, y2_}}] := x1 + If[y1 == y2, 0, (y - y1) (x2 - x1) / (y2 - y1)];
  Do[y = y1 + (y2 - y1) (i - 1) / (n - 1); jlist =
    Floor[n (Select[Map[xIntersect, Partition[plist, 2, 1, 1]], (Min[xlist] ≤ # ≤ Max[xlist]) &] - x1) / (x2 - x1) + 1];
    If[jlist != {}, Do[x = x1 + (x2 - x1) (jj - 1) / (n - 1);
      image[Max[1, Min[n, i]], Max[1, Min[n, jj]]] = Interpolate[x, y, {jj, Min[jlist], Max[jlist]}],
      {i, Floor[n (Min[ylist] - y1) / (y2 - y1)] + 1, Floor[n (Max[ylist] - y1) / (y2 - y1)] + 1}], {j, 1, jmax}];
ListDensityPlot[image, AspectRatio → Automatic, Mesh → False, Frame → False, ColorFunction → (Hue[2 (1 - #) / 3] &),
  Epilog → Table[{Line[Map[n (nodes[[#]] + 1) / 2 &, elements[[j]]]}, {j, 1, jmax}]];

```

