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MAE 207, Methods of Computer Modeling in Engineering and the Sciences
Isotropic Rectangular Beam Torsion

beam: http://scienceworld.wolfram.com/physics/Beam.html

■ Finite Element Method (FEM) with Triangular Elements

■ Roy's notes

Poisson equation : $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f \to (\nabla u\, v)_{\partial\Omega} - \int_\Omega (\nabla u \nabla v - f\, v)\, d\Omega = 0$

coordinates within triangle : $\xi_1,\ \xi_2$

linearized solution : $\phi \approx \alpha_1 + \alpha_2\,\xi_1 + \alpha_3\,\xi_2$

node 1 : $(\xi_1)_{node1} = \xi_{11} \stackrel{?}{=} 0,\ (\xi_2)_{node1} = \xi_{21} \stackrel{?}{=} 0 \to q_1 \equiv \phi_{node1} = \alpha_1 + \alpha_2\,\xi_{11} + \alpha_3\,\xi_{21} \stackrel{?}{=} \alpha_1$

node 2 : $(\xi_1)_{node2} = \xi_{12} \stackrel{?}{=} 1,\ (\xi_2)_{node2} = \xi_{22} \stackrel{?}{=} 0 \to q_2 \equiv \phi_{node2} = \alpha_1 + \alpha_2\,\xi_{12} + \alpha_3\,\xi_{22} \stackrel{?}{=} \alpha_1 + \alpha_2$

node 3 : $(\xi_1)_{node3} = \xi_{13} \stackrel{?}{=} 0,\ (\xi_2)_{node3} = \xi_{23} \stackrel{?}{=} 1 \to q_3 \equiv \phi_{node3} = \alpha_1 + \alpha_2\,\xi_{13} + \alpha_3\,\xi_{23} \stackrel{?}{=} \alpha_1 + \alpha_3$

$u = q_1\,N_1 + q_2\,N_2 + q_3\,N_3 \stackrel{?}{=} \phi$

$$\nabla N_{i?} \cdot \nabla N_k = \begin{pmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \partial \frac{*}{\partial} * & \partial \frac{*}{\partial} * \\ \partial \frac{*}{\partial} * & \partial \frac{*}{\partial} * \\ \partial \frac{*}{\partial} * & \partial \frac{*}{\partial} * \end{pmatrix}$$

what is N?

■ class notes: 5/1/06

```
Solve[{0 == b x1 + c x2 + d x3, 1 == b x1 + c x2 + d x3, 0 == b x1 + c x2 + d x3}, {x1, x2, x3}]
```

■ Analytical Solution, Nasser's notes

Pandlet

$\boxed{\nabla^2\phi = -2\,G\,K}$, angle at twist : $\frac{d\alpha}{dt}$, $\frac{T}{G\,J}$

$$\phi^{(\alpha,\eta)} = \frac{32\,G\,a^3}{\pi^3} \sum_{n\ odd}^{\infty} \frac{1}{n^3}\,(-1)^{\frac{n-1}{2}} \left(1 - \frac{\text{Cosh}[b]}{\text{Cosh}[b]}\right)$$

Fig1

twist rate : $\frac{d\alpha}{dz} = k\,z$, rectangular beam dimensions : $a \times b$, applied torque : T

modulus of rigidity : $G = \frac{\tau}{\gamma} = \frac{E}{2\,(1+\nu)}$, Young ' s modulus : E,

Poisson ' s ratio : ν, shearing stress : τ, shearing strain : γ

torsion constant : $J = \frac{16}{3} a^3 b \left(1 - \frac{196\, a}{b\, \pi^5} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^5} \mathrm{Tanh}\left[\frac{n \pi b}{2 a}\right]\right) = 2.18498$

Prandtl stress function : Φ, Poisson equation : $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2\, G\, K$, $\Phi_{boundary} = 0$

Timoshenko : $\Phi = \frac{32\, G\, k\, a}{\pi^3} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^3} (-1)^{\frac{n-1}{2}} \left(1 - \frac{\mathrm{Cosh}\left[\frac{n \pi y}{2 a}\right]}{\mathrm{Cosh}\left[\frac{n \pi b}{2 a}\right]}\right) \mathrm{Cos}\left[\frac{n \pi x}{2 a}\right]$,

linear twist : $k = \frac{T}{G\, J}$

shear stress tensor field : $\tau_{yz} = -\frac{\partial \Phi}{\partial x}$, $\tau_{xz} = \frac{\partial \Phi}{\partial y}$, $\tau_{yx} = 0$, $\sigma_x = \sigma_y = \sigma_z = 0$

principle stresses : $\{\sigma_1, \sigma_2, \sigma_3\} = \mathrm{Eigenvalues}\left[\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}\right]$

Von Mises stress : $\sigma_v = \sqrt{((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2) / 2} = (3\, (\tau_{xz}^2 + \tau_{yz}^2))^{1/2}$

strain tensor field (material constitutive relation)

$\epsilon_x = \frac{1}{E} (\sigma_x - \nu (\sigma_y + \sigma_z)) = 0$, $\epsilon_y = \frac{1}{E} (\sigma_y - \nu (\sigma_x + \sigma_z)) = 0$, $\epsilon_z = \frac{1}{E} (\sigma_z - \nu (\sigma_x + \sigma_y)) = 0$

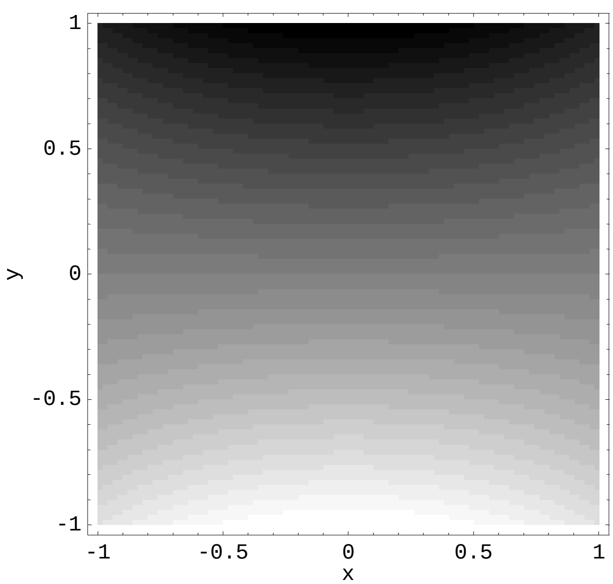
$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0$, $\gamma_{yz} = \frac{\tau_{yz}}{G}$, $\gamma_{xz} = \frac{\tau_{xz}}{G}$

angle of twist : $\alpha = k\, z$

a = b = 2; T = G = 1; J = 2.1849801331564294`; k = $\mathbf{\frac{T}{G\, J}}$**;**

$\mathbf{\Phi} = \frac{32\, G\, k\, a}{\pi^3}\, \mathrm{Sum}\left[\frac{1}{n^3} (-1)^{\frac{n-1}{2}} \left(1 - \frac{\mathrm{Cosh}\left[\frac{n \pi y}{2 a}\right]}{\mathrm{Cosh}\left[\frac{n \pi b}{2 a}\right]}\right) \mathrm{Cos}\left[\frac{n \pi x}{2 a}\right], \{n, 1, 10, 2\}\right]; \tau_{yz} = -\partial_x \Phi; \tau_{xz} = \partial_y \Phi;$

DensityPlot[τ_{xz}, {x, -a / 2, a / 2}, {y, -a / 2, a / 2},
PlotPoints → 100, Mesh → False, FrameLabel → {"x", "y"}];



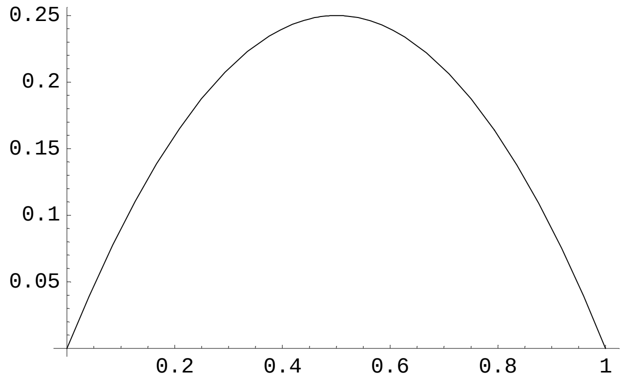
(* http://www.efunda.com/formulae/solid_mechanics/mat_mechanics/strain.cfm *)

$\gamma_{yz} \equiv \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \gamma_{xz} \equiv \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \gamma_{xy} \equiv \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$

x = y = 0; $\left(\frac{1}{2}\left(\left(\frac{x}{a/2}\right)^2 + \left(\frac{y}{b/2}\right)^2\right)\right)^{1/2}$

0

Plot[**g** (**1** - **g**), {**g**, **0**, **1**}];



```
PlotColor[hue_] :=
  RGBColor @@ If[hue == 1, {1, 0, 0}, Module[{x = Mod[4 hue, 1]}, Switch[Mod[Floor[4 hue], 4],
      0, {0, x, 1}, 1, {0, 1, 1 - x}, 2, {x, 1, 0}, 3, {1, 1 - x, 0}]]]];
a = 2.0; b = 2.0; L = 10.0; G = 1; dx = a / 8; dy = b / 8; dz = L / 40; J = 2.1849801331564294`;
Clear[x, y, z, T]; k = T/(G J);
Φ = (32 G k a)/π^3 Sum[1/n^3 (-1)^((n-1)/2) (1 - Cosh[n π y / (2 a)]/Cosh[n π b / (2 a)]) Cos[(n π x)/(2 a)], {n, 1, 10, 2}];
τyz = -∂x Φ; τxz = ∂y Φ; γyz = τyz/G; γxz = τxz/G;
Rotate[{x_, y_, z_}, θ_] := {x Cos[θ] - y Sin[θ], x Sin[θ] + y Cos[θ], z};
animation = {}; scale2 = 10.0; Tmax = scale2; dt = 1.0 / 360;
Prepare[T1_, z1_] := Module[{}, α = k z1 / scale2 /. T → T1;
   scale = ((x^2 + y^2)/((x + γxz / 2)^2 + (y + γyz / 2)^2))^(1/2) /. {x → a / 2, y → b / 2}];
Calculate[T1_, {x1_, y1_, z1_}] :=
  {Rotate[{scale (x + γxz / 2), scale (y + γyz / 2), z1 + scale (x γxz + y γyz) / 2}, α -
      ArcTan[(1 + 0.09576080145730925` T)/(1 - 0.09332536922126035` T)] - π/4], (3 (τxz^2 + τyz^2))^(1/2)} /. {x → x1, y → y1, T → T1};
Do[slices = {}; Show[Graphics3D[{EdgeForm[], Table[T = Tmax (1 - Cos[2 π t]) / 2;
      Prepare[T, z]; slice1 = slice2; slice2 = Map[Calculate[T, {#[[1]], #[[2]], z}] &,
        Flatten[{Table[{x, -b / 2}, {x, -a / 2, a / 2 - dx, dx}], Table[{a / 2, y},
           {y, -b / 2, b / 2 - dy, dy}], Table[{x, b / 2}, {x, a / 2, dx - a / 2, -dx}], Table[
           {-a / 2, y}, {y, b / 2, dy - b / 2, -dy}]}, 1]]; slices = Append[slices, slice2];
      n = Length[slice2]; If[z == 0, Polygon[Map[#[[1]] &, slice2]], Table[
        quad = {slice1[[i]], slice2[[i]], slice2[[Mod[i, n] + 1]], slice1[[Mod[i, n] + 1]]};
        {SurfaceColor[PlotColor[(Plus @@ Map[#[[2]] &, quad] / 4) / 4.632030522701348]],
         Polygon[Map[#[[1]] &, quad]]}, {i, 1, n}]], {z, 0, L, dz}]},
    ViewPoint → {-1, 1, 1}, ViewVertical → {0, 1, 0}, Axes → True,
    AxesLabel → {"x", "y", "z"},
    PlotRange → {{-a, a}, {-b, b}, {0, L}}]];
  animation = Append[animation, slices],
  {t,
   0,
   0.5,
   dt}];
```

```
Zero[x_, n_] := Module[{str = ToString[x]},
   StringJoin @@ Append[Table["0", {n - StringLength[str]}], str]];
POVFormat[x_Real] := ToString[x, CForm]; POVFormat[x_Integer] := ToString[x];
POVFormat[{{x_, y_, z_}, σ_}] := StringJoin["<", POVFormat[x],
   ",", POVFormat[y], ",", POVFormat[z], ",", POVFormat[σ], ">"];
POVFormat[xlist_List] := StringJoin["{", StringJoin @@
    Map[POVFormat[#] <> "," &, Drop[xlist, -1]], POVFormat[Last[xlist]], "}"];
TakeLoop[list_, n_] := Take[Join[list, list], n]; nslice = Length[animation[[1]]];
WriteVar[name_, x_] := WriteString[wf, StringJoin["#declare ", name, "=array[",
    ToString[Length[x]], "][", ToString[Length[x[[1]]]], "] ", POVFormat[x], ";\n"]];
Do[T = Tmax (1 - Cos[2 π t]) / 2; frame = Round[t / dt + 1]; slices = animation[[frame]]; wf =
   OpenWrite[StringJoin["C:/Files/PovRay/Physics/Torsion/", Zero[frame, 3], ".inc"]];
  Do[WriteVar[StringJoin["wall", ToString[iwall]],
    Table[TakeLoop[slices[[islice]], 8 {iwall - 1, iwall} + 1], {islice, 1, nslice}]],
   {iwall, 1, 4}]; Prepare[T, 0]; WriteVar["cap1",
   Table[Calculate[T, {x, y, 0}], {y, -b / 2, b / 2, dy}, {x, -a / 2, a / 2, dx}]];
  Prepare[T, L]; WriteVar["cap2", Table[Calculate[T, {x, y, L}],
    {y, -b / 2, b / 2, dy}, {x, -a / 2, a / 2, dx}]];
  Close[wf], {t, 0, 0.5, dt}];

CopyScaleExport["C:/Files/Animated Gifs/Torsion", 140, "*.bmp"];
CopyScaleExport["C:/Files/Animated Gifs/Torsion", 70, "*.bmp"];

Export["C:/Files/Animated Gifs/Torsion/Torsion.gif", Map[Import[#[[1]]] &, Partition[
    FileNames["*.bmp", "C:/Files/Animated Gifs/Torsion"], 1, 1]], ConversionOptions →
   {"AnimationDisplayTime" → 0, "Loop" → True, "GlobalColorReduction" → True}];
```