Simple detailed worked examples using Gaussian Quadrature method

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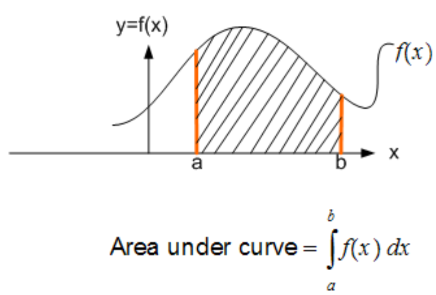
1 Introduction

We seek to find a numerical value for the definite integral of a real valued function of a real variable over a specific range. In other words, to evaluate

$$I = \int_a^b f(x)\ dx$$

Geometrically, this integral represents the area under $f(x)$ from a to b.



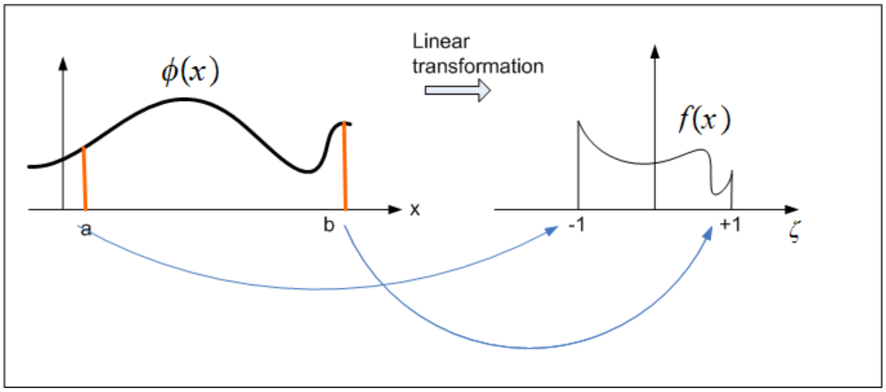
The following are few detailed step-by-step examples showing how to use Gaussian Quadrature (GQ) to solve this problem.

Few points to remember about GQ.

1. There are different versions of GQ depending on the basis polynomials it uses which in turns determines the location of the integration points. We will only use GQ based on Legendre polynomials. The integration points (called x_i) are the roots of the Legendre polynomials.

2. GQ gives an exact answer when the function to be integrated is a polynomial of order $2N-1$ where N is the number of integration points.

3. Since Legendre polynomials are defined over $[-1,1]$, we need to map the range of the function to be integrated to be $[-1,1]$. The actual integration is performed over $[-1,1]$ but the values are mapped back to the original range using a transformation rule. This diagram illustrates this.



See appendix to see how the transformation rule is derived.

4. To be consistent, we follow the book notations and call the user function to be integrated as $\phi(x)$ and the mapped function as $f(x)$.

5. Using GQ, we need to have access to a lookup table to obtain the values of the weights (called Multipliers or α_i in the book) and the integration points x_i. For each different value of N (the numbers of integration points, also called the order of GQ), there will be a specific set of values α_i and x_i to use. Once we have α_i and x_i then the value of the integral is

$$I = \sum_{i}^{N} \alpha_i \; f(x_i)$$

2 The Algorithm

This is an overview of the GQ algorithm. Next we will work out few examples to illustrate.

INPUT: User function $\phi(x)$ and lower integration range a and upper integration range b.

Output: Numerical value of $\int_a^b \phi(x)\, dx$

Step 1:

```
IF the problem specifies the value N (the number of integration
   points, or the order of GQ) THEN
     GOTO step 2
ELSE
   IF the user function to integrate is a polynomial of order K  THEN
      --- remember that N points can integrate exactly up to 2N-1
      --- polynomials hence we can determine the minimum number of
      --- points needed as follows

      N= (k+1)/2   rounded up.
  ELSE
      STOP we need N
  END IF
END IF
```

Step 2:

```
--- Use lookup table, use N to to find the corresponding values of
--- the weights α_i  (called multipliers   in book) and find the
--- integration points x_i
```

Step 3:

```
Integral = 0    --initialize integral value

FOR I = 1...N DO
        Integral = Integral + QG(I)
END

 FUNCTION  QG(I)
```

$$QG = \alpha_i \left[\frac{b-a}{2} \phi\left(\frac{(b-a)x_i}{2} + \frac{a+b}{2} \right) \right]$$

```
 END QG
```

3 Examples

3.1 example 1

Evaluate $\int_0^{1.5} e^{\frac{-x^2}{2}} dx$, Use $N = 5$

We see that $a = 0$, $b = 1.5, \phi(x) = e^{\frac{-x^2}{2}}$

Answer

step 1: N is given, go to step 2.

step 2: From lookup we see that

$$\begin{aligned} \alpha_1 &= 0.2369269 \\ \alpha_2 &= 0.4786287 \\ \alpha_3 &= 128/225 = 0.568\,89 \\ \alpha_4 &= 0.4786287 \\ \alpha_5 &= 0.2369269 \end{aligned}$$

And

$$\begin{aligned} x_1 &= -0.9061798 \\ x_2 &= -0.5384693 \\ x_3 &= 0 \\ x_4 &= 0.5384693 \\ x_4 &= 0.9061798 \end{aligned}$$

step 3: Evaluate the integral

LOOP over all the points from left to right.

$$\begin{aligned} I_1 &= \alpha_1 \left[\frac{b-a}{2} \phi \left(\frac{(b-a)}{2} x_1 + \frac{a+b}{2} \right) \right] \\ &= 0.2369269 \left[\frac{1.5}{2} \phi \left(\frac{(1.5)(-0.9061798)}{2} + \frac{1.5}{2} \right) \right] \\ &= 0.2369269 \left(0.75 \ \phi(0.07036\,5) \right) \end{aligned}$$

But $\phi(0.07036\,5) = e^{\frac{-(0.07036\,5)^2}{2}} = 0.997\,53$

Hence

$$\begin{aligned} I_1 &= 0.2369269 \ (0.75 \times 0.997\,53) \\ &= 0.177\,26 \end{aligned}$$

Do the next point:

$$
\begin{aligned}
I_2 &= \alpha_2 \left[\frac{b-a}{2} \phi \left(\frac{(b-a)}{2} x_2 + \frac{a+b}{2} \right) \right] \\
&= 0.4786287 \left[\frac{1.5}{2} \phi \left(\frac{(1.5)(-0.5384693)}{2} + \frac{1.5}{2} \right) \right] \\
&= 0.4786287 (0.75 \ \phi (0.346\,15))
\end{aligned}
$$

But $\phi(0.346\,15) = e^{\frac{-(0.346\,15)^2}{2}} = 0.941\,85$

Hence

$$
\begin{aligned}
I_2 &= 0.4786287 (0.75 \ \times 0.941\,85) \\
&= 0.338\,10
\end{aligned}
$$

Do the next point

$$
\begin{aligned}
I_3 &= \alpha_3 \left[\frac{b-a}{2} \phi \left(\frac{(b-a)}{2} x_3 + \frac{a+b}{2} \right) \right] \\
&= 0.568\,89 \left[\frac{1.5}{2} \phi \left(\frac{(1.5)}{2} (0) + \frac{1.5}{2} \right) \right] \\
&= 0.568\,89 (0.75 \phi (0.75))
\end{aligned}
$$

But $\phi(0.75) = e^{\frac{-(0.75)^2}{2}} = 0.754\,84$

Hence

$$
\begin{aligned}
I_3 &= 0.568\,89 (0.75 \ \times 0.754\,84) \\
&= 0.322\,07
\end{aligned}
$$

Do the next point

$$
\begin{aligned}
I_4 &= \alpha_4 \left[\frac{b-a}{2} \phi \left(\frac{(b-a)}{2} x_4 + \frac{a+b}{2} \right) \right] \\
&= 0.4786287 \left[\frac{1.5}{2} \phi \left(\frac{(1.5)(0.5384693)}{2} + \frac{1.5}{2} \right) \right] \\
&= 0.4786287 (0.75 \ \phi (1.153\,9))
\end{aligned}
$$

But $\phi(1.153\,9) = e^{\frac{-(1.153\,9)^2}{2}} = 0.513\,89$

Hence

$$
\begin{aligned}
I_4 &= 0.4786287(0.75 \times 0.513\,89) \\
&= 0.184\,47
\end{aligned}
$$

Do the next point

$$
\begin{aligned}
I_5 &= \alpha_5 \left[\frac{b-a}{2}\phi\left(\frac{(b-a)}{2}x_5+\frac{a+b}{2}\right)\right] \\
&= 0.2369269\left[\frac{1.5}{2}\phi\left(\frac{(1.5)\,(0.9061798)}{2}+\frac{1.5}{2}\right)\right] \\
&= 0.2369269\,(0.75\ \phi\,(1.\,429\,6))
\end{aligned}
$$

But $\phi\,(1.\,429\,6) = e^{\frac{-(1.\,429\,6)^2}{2}} = 0.359\,92$

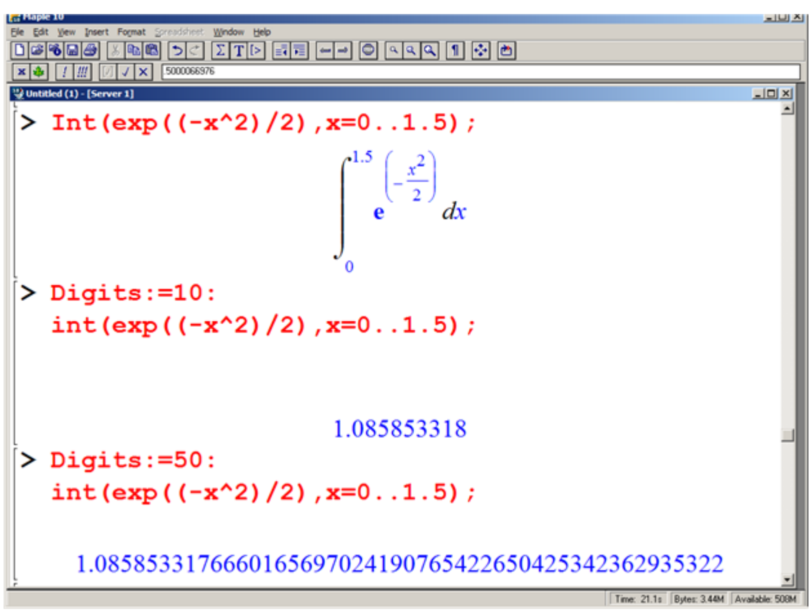
Hence

$$
\begin{aligned}
I_5 &= 0.2369269\,(0.75 \times 0.359\,92) \\
&= 0.063\,95\,6
\end{aligned}
$$

No more points. Add to obtain the final answer

$$
\begin{aligned}
I &= I_1 + I_2 + I_3 + I_4 + I_5 \\
&= 0.177\,26 + 0.338\,10 + 0.322\,07 + 0.184\,47 + 0.063\,95\,6 \\
&= 1.\,085\,9
\end{aligned}
$$

To verify, use say Maple:



3.2 Example 2

Evaluate $\int_{4}^{10} x + 3x^2 + x^5 \ dx$

We see that $a = 4$, $b = 10, \phi(x) = x + 3x^2 + x^5$

Answer:

step 1: N is not given. But since polynomial, we can determine minimum N.

Order of polynomial is 5 hence we need $N = 3$

step 2: From lookup we see that

$$\begin{aligned} \alpha_1 &= \frac{5}{9} = 0.555\,56 \\ \alpha_2 &= \frac{8}{9} = 0.888\,89 \\ \alpha_3 &= \frac{5}{9} = 0.555\,56 \end{aligned}$$

And

$$\begin{aligned} x_1 &= -0.774\,60 \\ x_2 &= 0 \\ x_3 &= +0.774\,60 \end{aligned}$$

step 3: Evaluate the integral

LOOP over all the points from left to right.

$$\begin{aligned} I_1 &= \alpha_1 \left[\frac{b-a}{2} \phi\left(\frac{(b-a)}{2} x_1 + \frac{a+b}{2} \right) \right] \\ &= 0.555\,56 \left[\frac{6}{2} \phi\left(\frac{6}{2} (-0.774\,60) + \frac{14}{2} \right) \right] \\ &= 0.55556\,(3\ \phi\,(4.\,676\,2)) \end{aligned}$$

But

$$\begin{aligned} \phi\,(4.\,676\,2) &= x + 3x^2 + x^5 \big|_{x=4.\,676\,2} \\ &= 4.\,676\,2 + 3\,(4.\,676\,2)^2 + 4.\,676\,2^5 \\ &= 2306.\,2 \end{aligned}$$

Hence

$$\begin{aligned} I_1 &= 0.555\,56\,(3 \times 2306.\,2) \\ &= 3843.\,7 \end{aligned}$$

Do next point

$$
\begin{aligned}
I_2 &= \alpha_2 \left[\frac{b-a}{2}\phi\left(\frac{(b-a)}{2}x_2 + \frac{a+b}{2}\right)\right] \\
&= 0.888\,89 \left[\frac{6}{2}\phi\left(\frac{6}{2}(0) + \frac{14}{2}\right)\right] \\
&= 0.888\,89\,(3\ \phi(7))
\end{aligned}
$$

But $\phi(7) = x + 3x^2 + x^5|_{x=7} = 7 + 3(7)^2 + 7^5 = 16961$

Hence

$$
\begin{aligned}
I_2 &= 0.888\,89\,(3 \times 16961) \\
&= 45229
\end{aligned}
$$

Do next point

$$
\begin{aligned}
I_3 &= \alpha_3 \left[\frac{b-a}{2}\phi\left(\frac{(b-a)}{2}x_3 + \frac{a+b}{2}\right)\right] \\
&= 0.555\,56 \left[\frac{6}{2}\phi\left(\frac{6}{2}(0.774\,60) + \frac{14}{2}\right)\right] \\
&= 0.555\,56\,(3\ \phi(9.\,323\,8))
\end{aligned}
$$

But

$$
\begin{aligned}
\phi(9.\,323\,8) &= x + 3x^2 + x^5\big|_{x=9.\,323\,8} \\
&= 9.\,323\,8 + 3(9.\,323\,8)^2 + 9.\,323\,8^5 \\
&= 70734
\end{aligned}
$$

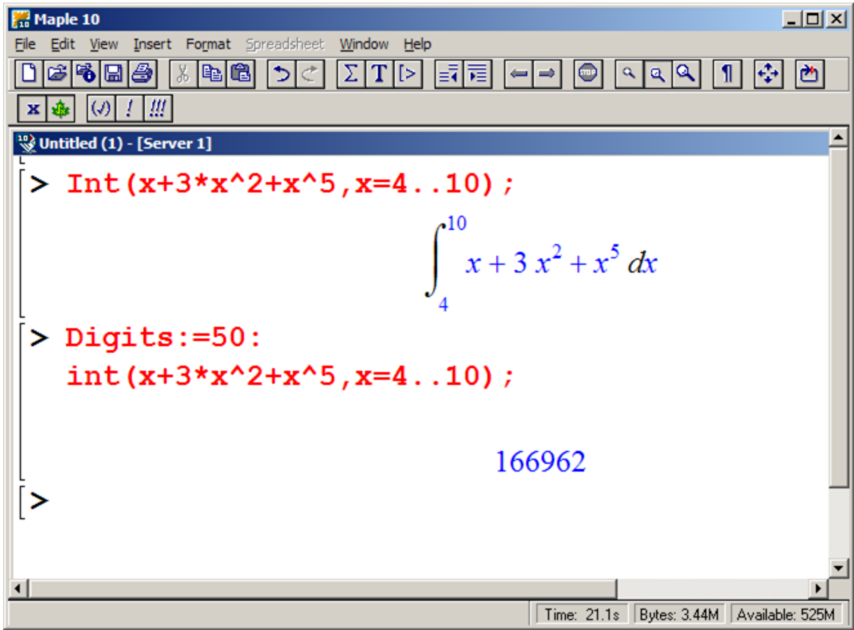
Hence

$$
\begin{aligned}
I_3 &= 0.55556\,(3 \times 70734) \\
&= 1178\,90
\end{aligned}
$$

Hence the answer is

$$
\begin{aligned}
I &= 3843.\,7 + 45229 + 1178\,90 \\
&= 1669\,60
\end{aligned}
$$

Verify


Maple 10
File Edit View Insert Format Spreadsheet Window Help
Untitled (1) - [Server 1]
> Int(x+3*x^2+x^5,x=4..10);
$\int_4^{10} x + 3\,x^2 + x^5\,dx$
> Digits:=50:
int(x+3*x^2+x^5,x=4..10);
166962
>
Time: 21.1s
Bytes: 3.44M
Available: 525M

3.3 Example 3

Evaluate $\int_{-1}^{1} x + 3x^2 + x^5 \; dx$

We see that $a = -1$, $b = 1, \phi(x) = x + 3x^2 + x^5$

Answer:

step 1: N is not given. But since polynomial, we can determine minimum N.

Order of polynomial is 5 hence we need $N = 3$

step 2: From lookup we see that

$$\begin{aligned} \alpha_1 &= \frac{5}{9} = 0.555\,56 \\ \alpha_2 &= \frac{8}{9} = 0.888\,89 \\ \alpha_3 &= \frac{5}{9} = 0.555\,56 \end{aligned}$$

And

$$\begin{aligned} x_1 &= -0.774\,60 \\ x_2 &= 0 \\ x_3 &= +0.774\,60 \end{aligned}$$

step 3: Evaluate the integral

LOOP over all the points from left to right.

$$\begin{aligned} I_1 &= \alpha_1 \left[\frac{b-a}{2} \phi \left(\frac{(b-a)}{2} x_1 + \frac{a+b}{2} \right) \right] \\ &= 0.555\,56 \left[\frac{2}{2} \phi \left(\frac{2}{2} (-0.774\,60) + \frac{0}{2} \right) \right] \\ &= 0.55556 \left(\phi(-0.774\,6) \right) \end{aligned}$$

But

$$\begin{aligned} \phi(-0.774\,6) &= x + 3x^2 + x^5 \big|_{x=-0.774\,6} \\ &= -0.774\,6 + 3(-0.774\,6)^2 + -0.774\,6^5 \\ &= 0.746\,55 \end{aligned}$$

Hence

$$\begin{aligned} I_1 &= 0.55556\,(0.746\,55) \\ &= 0.414\,75 \end{aligned}$$

Do next point

$$
\begin{aligned}
I_2 &= \alpha_2 \left[\frac{b-a}{2}\phi\left(\frac{(b-a)}{2}x_2 + \frac{a+b}{2}\right)\right] \\
&= 0.888\,89 \left[\frac{2}{2}\phi\left(\frac{2}{2}(0) + \frac{0}{2}\right)\right] \\
&= 0.888\,89\,(\phi(0))
\end{aligned}
$$

But $\phi(7) = x + 3x^2 + x^5|_{x=0} = 0$

Hence

$$
\begin{aligned}
I_2 &= 0.888\,89\,(0) \\
&= 0
\end{aligned}
$$

Do next point

$$
\begin{aligned}
I_3 &= \alpha_3 \left[\frac{b-a}{2}\phi\left(\frac{(b-a)}{2}x_3 + \frac{a+b}{2}\right)\right] \\
&= 0.555\,56 \left[\frac{2}{2}\phi\left(\frac{2}{2}(0.774\,60) + \frac{0}{2}\right)\right] \\
&= 0.555\,56\,(\phi(0.774\,60))
\end{aligned}
$$

But

$$
\begin{aligned}
\phi(0.774\,60) &= x + 3x^2 + x^5\big|_{x=0.774\,60} \\
&= 0.774\,60 + 3\,(0.774\,60)^2 + 0.774\,60^5 \\
&= 2.\,853\,5
\end{aligned}
$$

Hence

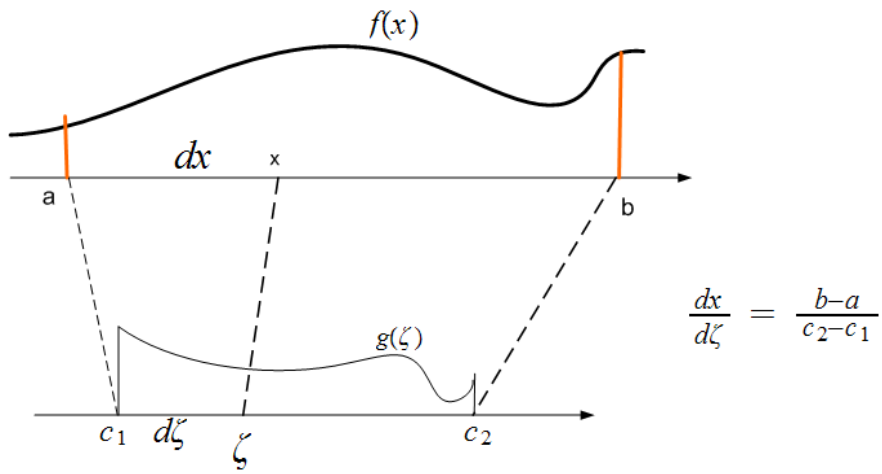
$$
\begin{aligned}
I_3 &= 0.55556\,(2.\,853\,5) \\
&= 1.\,585\,3
\end{aligned}
$$

Hence the answer is

$$
\begin{aligned}
I &= 0.414\,75 + 0 + 1.\,585\,3 \\
&= 2.\,000\,1
\end{aligned}
$$

4 Appendix

An easy way to find how the function changes when we change the range is to align the ranges over each others and take the ratio between them as the scale factor. This diagram shows this for a general case where we map $f(x)$ defined over $[a, b]$ to a new range defined over $[c_1, c_2]$



We see from the diagram that

$$\zeta = c_1 + d\zeta$$

But

$$\frac{dx}{d\zeta} = \frac{b-a}{c_2 - c_1} \tag{1}$$

The above is called the Jacobian of the transformation.

Now, From the diagram we see that

$$dx = x - a$$

and

$$d\zeta = \zeta - c_1$$

Hence (1) becomes

$$\frac{x-a}{\zeta - c_1} = \frac{b-a}{c_2 - c_1}$$

$$x = \frac{b-a}{c_2 - c_1}(\zeta - c_1) + a$$

For the specific case when $c_1 = -1$ and $c_2 = +1$ the above expressions become

$$
\begin{aligned}
x &= \frac{b-a}{2}(\zeta+1)+a \\
&= \frac{(b-a)}{2}\zeta+\frac{(a+b)}{2}
\end{aligned}
$$

Which is the mapping used in the Gaussian Quadrature method.

It interesting to see the effect of this transformation on the shape of some functions. Below I plotted some functions under this transformation. The left plots are the original functions plotted over some range, in this case $[4,10]$ and the left side plots show the new shape (the function $g(\zeta)$) over the new range $[-1,1]$