Point Collocation solution for u'''' + u = 1

A Mathemematica based solution to u''''+u=1 using the point collocation method.

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Boundary conditions are u(0)=u(1)=u''(0)=u''(1)=0

First Obtain the approximate numerical solution using Collocation Method

```
Remove["Global`*"]
<< Graphics`
nBasis = 5;  nPoints = 5;
trial[x_] := Sum[a_n (x (x - 1))^n, {n, 1, nBasis}]
residual[x_] := trial''''[x] + trial[x] - 1
A = Table[residual[x] /. x -> {n/(2 nPoints - 1)}, {n, 1, nPoints - 1}] ;
A = Flatten[Append[A, {2 a_1 +  2 a_2}]];
sol = Flatten[Solve[A == 0, Table[a_n, {n, 5}]]] ;
```

```
Print["Coefficients of trial solution are "]
sol // N
```

```
Coefficients of trial solution are
```

$\{a_{1.} \to -0.0412493,\ a_{2.} \to 0.0412493,\ a_{3.} \to 0.000147289,\ a_{4.} \to -0.0000245233,\ a_{5.} \to -3.28259\times 10^{-8}\}$

```
numericalSolution[x_] := Sum[sol[[n, 2]] (x (x - 1))^n, {n, 1, nBasis}]
numericalSolutionDiff4[x_] := D[numericalSolution[x], {x, 4}]
```

Now get the analytical solution using DSolve and compare to the numerica solution

```
eq = u''''[x] + u[x] == 1;
bc = {u[0] == 0, u[1] == 0, u''[0] == 0, u''[1] == 0};
DSolve[{eq, u[0] == 0, u[1] == 0, u''[0] == 0, u''[1] == 0}, u[x], x];
mmaAnalyticSol = (u[x] /. %)[[1]];
mmaAnalyticMoment := D[mmaAnalyticSol, {x, 2}];
```

```
a = 1/Sqrt[2];
d = 2 E^-a Cos[a] + E^(-2 a) + 1;
textAnalyticSol[x_] := 1 + ((-E^-a (Cos[a] + E^-a))/d) E^(a x) Cos[a x] +
  ((-E^-a Sin[a])/d) E^(a x) Sin[a x] + ((-(E^-a Cos[a] + 1))/d) E^(-a x) Cos[a x] + ((-E^-a Sin[a])/d) E^(-a x) Sin[a x]

textDispErr[x_] := Abs[(numericalSolution[x] - textAnalyticSol[x])/textAnalyticSol[x]]
textAnalyticMoment[x_] := textAnalyticSol''[x]
textMomentErr[x_] := Abs[(numericalSolution''[x] - textAnalyticSol''[x])/textAnalyticMoment[.5]]

pp1 = Plot[textDispErr[x], {x, 0, 1}, PlotStyle -> {Red},
  PlotLabel -> "textbook Displacment Error %", DisplayFunction -> Identity]
pp2 = Plot[textMomentErr[x], {x, 0, 1}, PlotLabel -> "textbook Moment Error %",
  PlotRange -> All, PlotStyle -> {Red}, DisplayFunction -> Identity]
```

```
mmaDispErr[pt_] := Abs[(numericalSolution[pt] - mmaAnalyticSol /. x -> pt)/(mmaAnalyticSol /. x -> .5)]

mmaMomentErr[pt_] := Abs[((D[numericalSolution[x], {x, 2}] - mmaAnalyticMoment) /. x -> pt)/(mmaAnalyticMoment /. x -> .5)]

pp3 = Plot[mmaDispErr[x], {x, 0, 1},
  PlotLabel -> "mma Displacment Error %", DisplayFunction -> Identity]
pp4 = Plot[mmaMomentErr[x], {x, 0, 1}, PlotLabel -> "mma Moment Error %",
  PlotRange -> All, DisplayFunction -> Identity]
```

Print the Exact solution and the numerical solution

```
FullSimplify[mmaAnalyticSol]
```

$$\frac{e^{-\frac{x}{\sqrt{2}}}\left(e^{\frac{x}{\sqrt{2}}}\left(1+e^{\sqrt{2}}+2\,e^{\frac{1}{\sqrt{2}}}\,\mathrm{Cos}\left[\frac{1}{\sqrt{2}}\right]\right)-e^{\frac{1}{\sqrt{2}}}\left(1+e^{\sqrt{2}\,x}\right)\mathrm{Cos}\left[\frac{-1+x}{\sqrt{2}}\right]-\left(e^{\sqrt{2}}+e^{\sqrt{2}\,x}\right)\mathrm{Cos}\left[\frac{x}{\sqrt{2}}\right]\right)}{1+e^{\sqrt{2}}+2\,e^{\frac{1}{\sqrt{2}}}\,\mathrm{Cos}\left[\frac{1}{\sqrt{2}}\right]}$$

Print the numerical solution

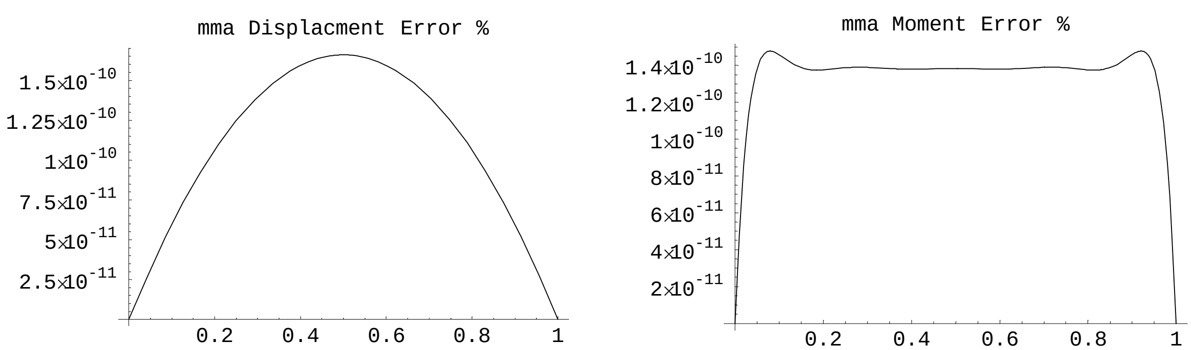
```
FullSimplify[N[numericalSolution[x]]]
```

$$-3.28259\times 10^{-8}\,(-7.02822+x)\,(-1.61671+x)\,(-1.+x)\,x\,(0.61671+x)\,(6.02822+x)\,(750.815+(-1.+x)\,x)\,(39.6206+(-1.+x)\,x)$$

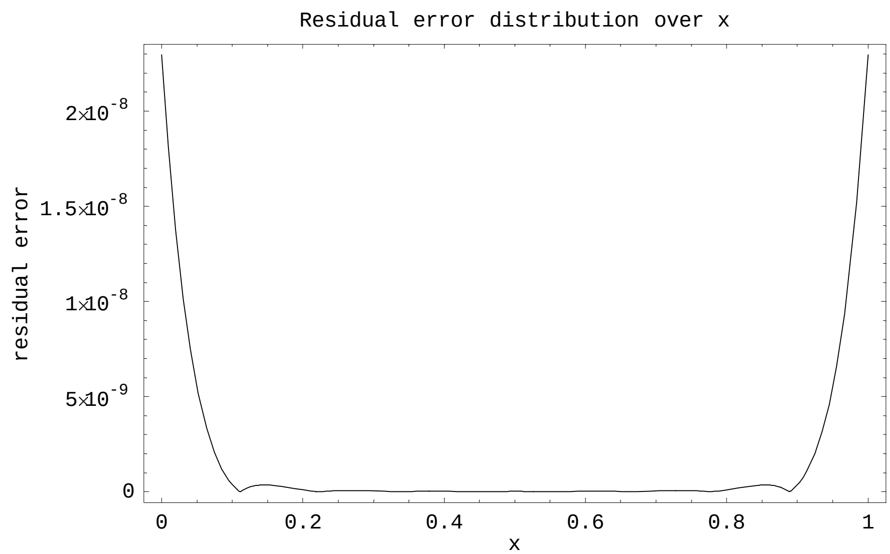
```
(*Verify DSolve solution matches textbook solution*)
mmaAnalyticSol - textAnalyticSol[x] // FullSimplify
```

0

```
Show[GraphicsArray[{pp3, pp4}], DisplayFunction → $DisplayFunction]
```



```
Plot[Evaluate[Abs[numericalSolutionDiff4[x] + numericalSolution[x] - 1]],
 {x, 0, 1}, PlotRange → All, PlotLabel -> "Residual error distribution over x",
 FrameLabel → {"x", "residual error"}, Frame → True]
```



Reference

1. Methods of computer modeling in engineering & the sciences. Volume 1
   by Professor Atluri, S.N.
2. Lecture notes. MAE207. Instructor: Professor Atluri. UCI. SPring 2006