

Computing Assignment, Part 1

Mathematics 503, Mathematical Modeling

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problem:

Let V be the space of continuously differentiable functions y such that $y(0) = 0, y(1) = 0$. On this space consider the functional

$$J(y) = \int_0^1 [y'(x)]^2 + q[y(x)]^2 - 2fy(x) \, dx$$

where $q > 0$ and f are given constants.

(a) Show that J achieves a minimum at y iff

$$I = \int_0^1 y'(x)\phi'(x) + qy(x)\phi(x) - f\phi(x) \, dx = 0$$

for all $\phi(x) \in V$

(b) Show that if $y \in V$ is twice continuously differentiable and satisfies this optimality condition then y satisfies the differential equation $-y''(x) + qy(x) = f, y(0) = 0, y(1) = 0$

(c) Conversely show that if $y \in V$ is twice continuously differentiable, and satisfies the differential equation above, then y satisfies the optimality condition of part (a)

answer:

part(a). Let $A = \{y \in C^1[0, 1], y(0) = y(1) = 0\}$, and let $A_y = \{\phi \in C^1[0, 1], \phi(0) = \phi(1) = 0\}$. The set A is called the set of admissible functions (from which the function which will minimize the functional will be found), and A_y is called the set of admissible directions. We require also that $y + t\phi \in A$ where t is some small scalar.

Since this is an IFF, then we need to show the forward and the backward case. Start with the *forwards case*:

Given: $J(y) = \int_0^1 y'^2 + qy^2 - 2fy \, dx$ show that

$$\text{if } I = \int_0^1 y'\phi' + qy\phi - f\phi \, dx = 0 \text{ then } J(y) \text{ is minimum}$$

Consider

$$\begin{aligned} J(y + \phi) &= \int_0^1 [(y + \phi)']^2 + q[(y + \phi)]^2 - 2f(y + \phi) \, dx \\ &= \int_0^1 (y'^2 + \phi'^2 + 2y'\phi') + q(y^2 + \phi^2 + 2y\phi) - 2fy - 2f\phi \, dx \\ &= \int_0^1 y'^2 + qy^2 - 2fy \, dx + \int_0^1 \phi'^2 + q\phi^2 \, dx + \int_0^1 2y'\phi' + 2qy\phi - 2f\phi \, dx \\ &= J(y) + Q + \int_0^1 2y'\phi' + 2qy\phi - 2f\phi \, dx \end{aligned}$$

Where $Q = \int_0^1 \phi'^2 + q\phi^2 \, dx \geq 0$ since $q > 0$ and the other functions are squared. Hence this implies from the above that if $\int_0^1 2y'\phi' + 2qy\phi - 2f\phi \, dx = 0$ then y is a minimizer of $J(y)$. In other words

$$J'(y; \phi) = \int_0^1 y'\phi' + qy\phi - f\phi \, dx = 0$$

(This is because any change from y along any of the admissible directions ϕ will result in a functional $J(y + \phi)$ which is larger than it was at $J(y)$).

Now to show the *backward* case:

If $J'(y; \phi) = \int_0^1 y' \phi' + qy\phi - f\phi \, dx \neq 0$ then $J(y)$ is not a minimum.

Assume for some $\phi = \phi_0$ we have $J'(y; \phi_0) < 0$

$$\begin{aligned} J(y + t\phi_0) &= \int_0^1 [(y + t\phi_0)']^2 + q[(y + t\phi_0)]^2 - 2f(y + t\phi_0) \, dx \\ &= \int_0^1 (y'^2 + t^2\phi_0'^2 + 2ty'\phi_0') + q(y^2 + t^2\phi_0^2 + 2ty\phi) - 2fy - 2tf\phi_0 \, dx \\ &= \int_0^1 y'^2 + qy^2 - 2fy \, dx + \int_0^1 t^2\phi_0'^2 + qt^2\phi_0^2 \, dx + \int_0^1 2ty'\phi_0' + 2tqy\phi_0 - 2tf\phi_0 \, dx \\ &= J(y) + t^2Q + t \int_0^1 2y'\phi_0' + 2qy\phi_0 - 2f\phi_0 \, dx \end{aligned}$$

Where $Q = \int_0^1 \phi_0'^2 + q\phi_0^2 \geq 0$. Hence we have

$$J(y + t\phi_0) = J(y) + t \left[2 \int_0^1 y'\phi_0' + qy\phi_0 - f\phi_0 \, dx + tQ \right]$$

Now, no matter how large Q is, we can make t small enough so that tQ is smaller than the absolute value of $|2J'(y; \phi_0)|$. But since $J'(y; \phi_0) < 0$, then $\left[2 \int_0^1 y'\phi_0' + qy\phi_0 - f\phi_0 \, dx + tQ \right]$ will be a *negative quantity*. Hence, since $t \times$ negative quantity is also negative quantity, then we conclude that

$$J(y + t\phi_0) < J(y)$$

Hence y is not a minimizer of $J(y)$. So no matter which y we hope it is our minimum, we can find an admissible direction ϕ_0 such that if move very slightly away from this y in this admissible direction, we find that $J(y + t\phi_0)$ is smaller (this will always be the case if $J'(y; \phi_0) < 0$)

Part (b)

Given $y \in C^2[0, 1]$, and $J'(y; \phi) = 0$ for all admissible directions ϕ . Show that y satisfies the differential equation $-y''(x) + qy(x) = f$, $y(0) = 0, y(1) = 0$

Since $J'(y; \phi) = 0$, in other words $\int_0^1 y' \phi' + q y \phi - f \phi \, dx = 0$. Then now we do integration by parts.

$$\int_0^1 \overbrace{y' \phi'}^{\text{"u"} \text{ "dv"}} \, dx = [y' \phi]_0^1 - \int_0^1 y'' \phi \, dx$$
$$\int_0^1 y' \phi' \, dx = - \int_0^1 y'' \phi \, dx$$

Since $\phi(0) = \phi(1) = 0$. Now substitute the above back into $\int_0^1 y' \phi' + q y \phi - f \phi \, dx$ and take ϕ as common factor, we obtain

$$\int_0^1 (-y'' + qy - f) \phi \, dx = 0$$

Now we apply the fundamental theory of variational calculus (which Lemma 3.13 is special case) and argue that since ϕ is arbitrary admissible direction, then for the above to be zero for every ϕ , we must have

$$-y'' + qy - f = 0$$

or

$$\boxed{-y'' + qy = f}$$

with $y(0) = y(1) = 0$

Part (c)

Given $y \in C^2[0, 1]$ and $-y'' + qy = f, y(0) = y(1) = 0$ show that $\int_0^1 y' \phi' + q y \phi - f \phi \, dx = 0$
Let $\phi \in A_d$. Now

$$\begin{aligned}\frac{d}{dx} (y' \phi) &= y'' \phi + y' \phi' \\ \int_0^1 \frac{d}{dx} (y' \phi) \, dx &= \int_0^1 y'' \phi + y' \phi' \, dx \\ [y' \phi]_0^1 &= \int_0^1 y'' \phi + y' \phi' \, dx\end{aligned}$$

But $[y' \phi]_0^1 = 0$ since $\phi = 0$ at both ends. Hence the above becomes

$$0 = \int_0^1 y'' \phi + y' \phi' \, dx$$

But $y'' = -f + qy$ since y satisfies the differential equation. Hence the above becomes

$$\begin{aligned}0 &= \int_0^1 (-f + qy) \phi + y' \phi' \, dx \\ &= \int_0^1 y' \phi' + qy \phi - f \phi \, dx\end{aligned}$$

Which is the optimality condition (weak form) of part (a) we are asked to show.