

Computing Assignment, Part 2. A follow up. Derivation for the $Ax=b$ and FEM
program for illustration

Mathematics 503, Mathematical Modeling

California State University, Fullerton

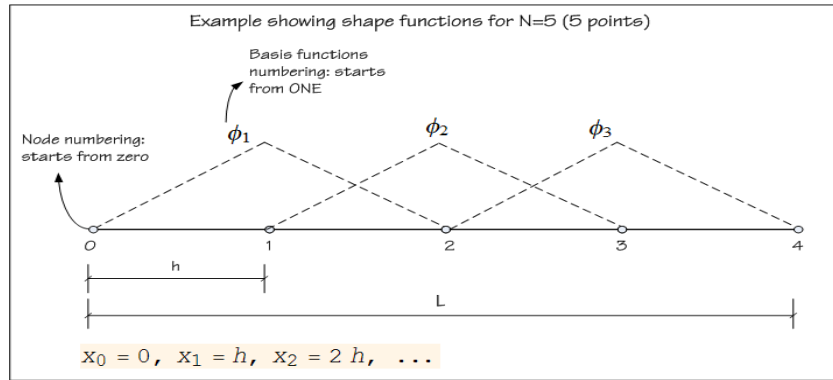
July 29 2007.

BY

Nasser Abbasi

This is a supplement to the report for the computer project for Math 503. This includes the symbolic derivation of the A matrix and the b vector for the problem of $Ax = b$ which is generated from the FEM formulation for this project. I also include a very short Mathematica program which implements the FEM solution.

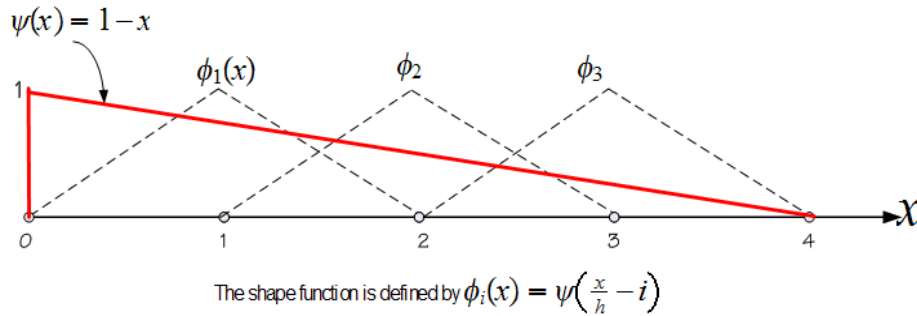
For $x = [0, L]$ where L is the length, we define the shape functions (called tent function in this case) as shown below



The shape function is defined by $\phi_i(x) = \psi\left(\frac{x}{h} - i\right)$ where

$$\psi(z) = \begin{cases} 1 - z & 0 < z < 1 \\ 0 & z > 1 \end{cases} \quad (1)$$

and $\psi(z) = \psi(-z)$ as shown in this diagram



Now the derivative of $\phi'_i(x)$ is given by

$$\phi'_i(x) = \begin{cases} \frac{1}{h} & (i-1)h < x \leq i h \\ -\frac{1}{h} & i h < x < (i+1) h \\ 0 & \text{otherwise} \end{cases}$$

Now we write the weak form in terms of the above shape function (which is our admissible direction). From part 1 we had

$$I = \int_0^L y'(x) \phi'(x) + q y(x) \phi(x) - f \phi(x) \, dx = 0$$

And Let

$$y(x) = \sum_{j=1}^N c_j \phi_j(x)$$

$$y'(x) = \sum_{j=1}^N c_j \phi'_j(x)$$

Hence, now we pick one admissible direction at a time, and need to satisfy the above integral for each of these. Hence we write

$$I_j = \int_0^L \left(\sum_{i=1}^N c_i \phi'_i(x) \right) \phi'_j(x) + q \left(\sum_{i=1}^N c_i \phi_i(x) \right) \phi_j(x) - f \phi_j(x) \, dx = 0 \quad j = 1, 2, \dots, N$$

But due to sphere on influence of the $\phi_j(x)$ extending to only $x_{j-1} \dots x_{j+1}$ the above becomes

$$I_j = \int_{x_{j-1}}^{x_{j+1}} \left(\sum_{i=j-1}^{j+1} c_i \phi'_i(x) \right) \phi'_j(x) + q \left(\sum_{i=j-1}^{j+1} c_i \phi_i(x) \right) \phi_j(x) - f \phi_j(x) \, dx = 0 \quad j = 1, 2, \dots, N$$

Hence we obtain N equations which we solve for the N coefficients c_j

Now to evaluate I_j we write

$$\begin{aligned} I_j &= \int_{x_{j-1}}^{x_j} \dots \, dx + \int_{x_j}^{x_{j+1}} \dots \, dx \\ &= \int_{x_{j-1}}^{x_j} \left(\sum_{i=j-1}^j c_i \phi'_i(x) \right) \phi'_j(x) + q \left(\sum_{i=j-1}^j c_i \phi_i(x) \right) \phi_j(x) - f \phi_j(x) \, dx \\ &+ \\ &\int_{x_j}^{x_{j+1}} \left(\sum_{i=j}^{j+1} c_i \phi'_i(x) \right) \phi'_j(x) + q \left(\sum_{i=j}^{j+1} c_i \phi_i(x) \right) \phi_j(x) - f \phi_j(x) \, dx \end{aligned}$$

Now we will show the above for $j = 1$ which will be sufficient to build the A matrix due to symmetry.

For $j = 1$

$$I_1 = \int_0^{2h} \left(\sum_{i=1}^2 c_i \phi'_i(x) \right) \phi'_1(x) + q \left(\sum_{i=1}^2 c_i \phi_i(x) \right) \phi_1(x) - f \phi_1(x) \, dx$$

Hence breaking the interval into 2 parts we obtain

$$\begin{aligned} I_1 &= \int_0^h \left(\sum_{i=1}^1 c_i \phi'_i(x) \right) \phi'_1(x) + q \left(\sum_{i=1}^1 c_i \phi_i(x) \right) \phi_1(x) - f \phi_1(x) \, dx \\ &+ \\ &\int_h^{2h} \left(\sum_{i=1}^2 c_i \phi'_i(x) \right) \phi'_1(x) + q \left(\sum_{i=1}^2 c_i \phi_i(x) \right) \phi_1(x) - f \phi_1(x) \, dx \end{aligned}$$

Hence

$$\begin{aligned} I_1 &= \int_0^h (c_1 \phi'_1(x)) \phi'_1(x) + q (c_1 \phi_1(x)) \phi_1(x) - f \phi_1(x) \, dx \\ &+ \\ &\int_h^{2h} (c_1 \phi'_1(x) + c_2 \phi'_2(x)) \phi'_1(x) + q (c_1 \phi_1(x) + c_2 \phi_2(x)) \phi_1(x) - f \phi_1(x) \, dx \end{aligned} \quad (2)$$

Now set up a little table to do the above integral.

Range	ϕ'_1	ϕ'_2	ϕ_1	ϕ_2
$[0, h]$	$\frac{1}{h}$	N/A	$\psi\left(-\frac{x}{h} + 1\right) \rightarrow \frac{x}{h}$	N/A
$[h, 2h]$	$\frac{-1}{h}$	$\frac{1}{h}$	$\psi\left(\frac{x}{h} - 1\right) \rightarrow 2 - \frac{x}{h}$	$\psi\left(-\frac{x}{h} + 2\right) \rightarrow \frac{x}{h} - 1$

The above table was build by noting that for ϕ_j , it will have the equation $\psi\left(\frac{x}{h} - i\right)$ when x is under the left leg of tent. And it will have the equation $\psi\left(-\frac{x}{h} + i\right)$ when x is under the right leg of the tent. This is because for

$x < 0$, the argument to $\psi()$ is negative and so we flip the argument as per the definition for ψ shown in the top of this report.

Hence we obtain for the integral in (2)

$$I_1 = \int_0^h \left[c_1 \left(\frac{1}{h} \right) \right] \left(\frac{1}{h} \right) + q \left(c_1 \frac{x}{h} \right) \frac{x}{h} - f \frac{x}{h} dx$$

$$+ \int_h^{2h} \left[c_1 \left(\frac{-1}{h} \right) + c_2 \left(\frac{1}{h} \right) \right] \left(\frac{-1}{h} \right) + q \left(c_1 \left(2 - \frac{x}{h} \right) + c_2 \left(\frac{x}{h} - 1 \right) \right) \left(2 - \frac{x}{h} \right) - f \left(2 - \frac{x}{h} \right) dx$$

so the above becomes integral becomes

$$I_1 = \int_0^h \frac{c_1}{h^2} + q c_1 \left(\frac{x^2}{h^2} \right) - f \frac{x}{h} dx$$

$$+ \int_h^{2h} \frac{c_1}{h^2} - \frac{c_2}{h^2} + q c_1 \left(4 - 4 \frac{x}{h} + \frac{x^2}{h^2} \right) + q c_2 \left(3 \frac{x}{h} - \frac{x^2}{h^2} - 2 \right) - 2f + f \frac{x}{h} dx$$

Hence

$$I_1 = \frac{c_1}{h^2} \int_0^h dx + \frac{q}{h^2} c_1 \int_0^h x^2 dx - \frac{f}{h} \int_0^h x dx$$

$$+ \frac{c_1}{h^2} \int_h^{2h} dx - \frac{c_2}{h^2} \int_h^{2h} dx + q c_1 \int_h^{2h} \left(4 - 4 \frac{x}{h} + \frac{x^2}{h^2} \right) dx + q c_2 \int_h^{2h} \left(3 \frac{x}{h} - \frac{x^2}{h^2} - 2 \right) dx - 2f \int_h^{2h} dx + \frac{f}{h} \int_h^{2h} x dx$$

Which becomes

$$I_1 = \frac{c_1}{h^2} h + \frac{q}{h^2} c_1 \left(\frac{1}{3} h^3 \right) - \frac{f}{2h} h^2$$

$$+ \frac{c_1}{h^2} h - \frac{c_2}{h^2} h + q c_1 \left(4h - 2 \frac{3h^2}{h} + \frac{1}{3} \frac{7h^3}{h^2} \right) + q c_2 \left(\frac{3}{2h} (3h^2) - \frac{1}{3h^2} (7h^3) - 2h \right) - 2fh + \frac{f}{2h} 3h^2$$

or

$$I_1 = \frac{c_1}{h} + c_1 \frac{qh}{3} - \frac{f}{2} h$$

$$+ \frac{c_1}{h} - \frac{c_2}{h} + q c_1 \left(4h - 6h + \frac{7}{3} h \right) + q c_2 \left(\frac{9h}{2} - \frac{7}{3} h - 2h \right) - 2fh + \frac{3f}{2} h$$

Therefore

$$I_1 = \frac{c_1}{h} + c_1 \frac{qh}{3} + \frac{c_1}{h} - \frac{c_2}{h} + q c_1 \left(\frac{1}{3} h \right) + q c_2 \left(\frac{1}{6} h \right) - fh$$

Hence

$$I_1 = c_1 \left(\frac{2}{h} + \frac{2}{3} qh \right) + c_2 \left(-\frac{1}{h} + \frac{1}{6} qh \right) - fh = 0$$

Multiply by h we obtain

$$\boxed{I_1 = c_1 \left(2 + \frac{2}{3}h^2q\right) + c_2 \left(-1 + \frac{1}{6}h^2q\right) - fh^2 = 0} \quad (2)$$

Hence we now can set up the $Ax = b$ system using only the above equation by taking advantage that A will be tridiagonal and there is symmetry along the diagonal.

$$\begin{bmatrix} \left(2 + \frac{2}{3}h^2q\right) & \left(-1 + \frac{1}{6}h^2q\right) & 0 & 0 & \cdots & 0 \\ \left(-1 + \frac{1}{6}h^2q\right) & \left(2 + \frac{2}{3}h^2q\right) & \left(-1 + \frac{1}{6}h^2q\right) & 0 & \cdots & 0 \\ 0 & \left(-1 + \frac{1}{6}h^2q\right) & \left(2 + \frac{2}{3}h^2q\right) & \left(-1 + \frac{1}{6}h^2q\right) & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \left(-1 + \frac{1}{6}h^2q\right) & \left(2 + \frac{2}{3}h^2q\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_N \end{bmatrix} = fh^2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

The following is the FEM program to implement the above, with few plots showing how close it gets to the real solution as N increases.

```

In[1815]= Remove["Global`*"];
(*by Nasser Abbasi. FEM program for Math 503*)

ψ[x_] := 1 - x /; 0 < x < 1
ψ[x_] := 0 /; x > 1
ψ[x_] := ψ[-x] /; x < -1
φ[x_, i_, h_] := ψ[ $\frac{x}{h} - i$ ];

yApprox[x_, c_, h_, n_] := Module[{}, Sum[c[[i]] φ[x, i, h], {i, 1, n}]]

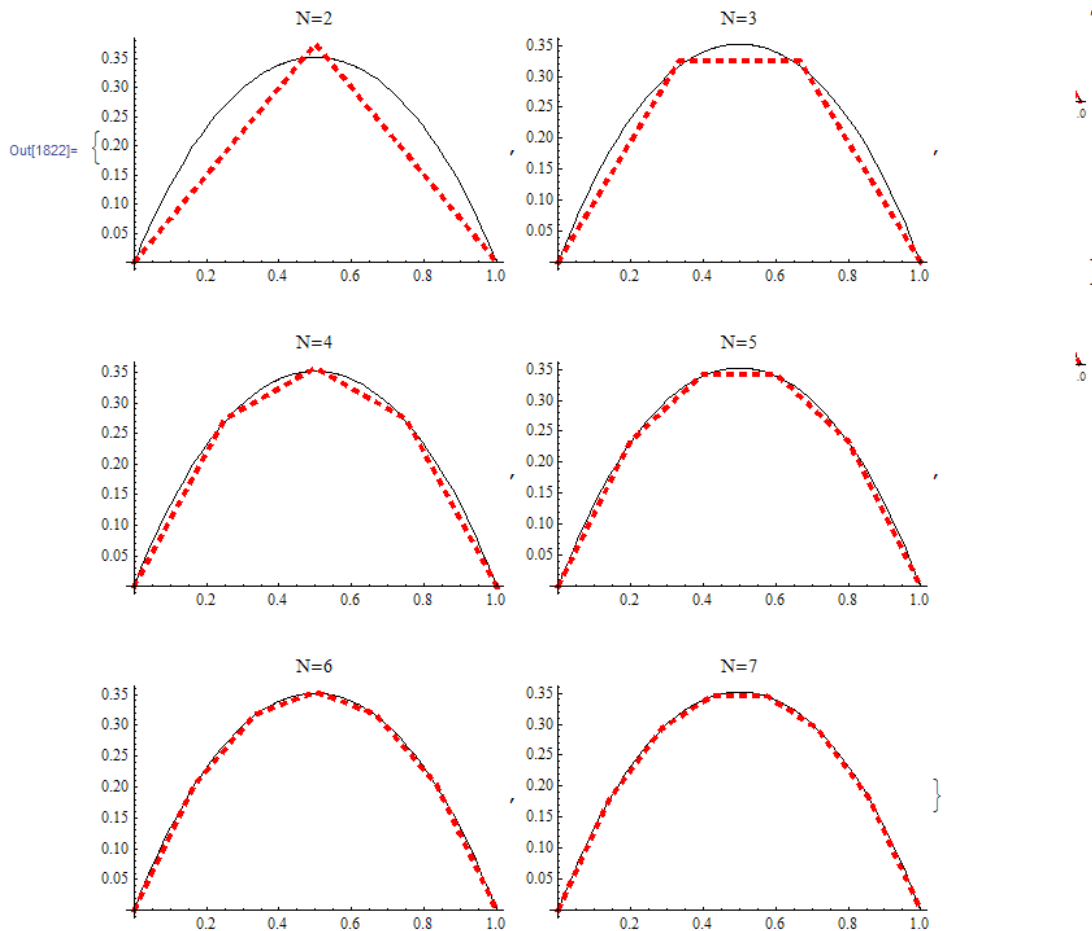
plotIt[msg_, n_] := Module[{A, b, c, nElements, nShapeFunctions, i, j, h, q, f, L},
  q = 4; f = 4; L = 1; nElements = n; nShapeFunctions = nElements - 1; h =  $\frac{L}{nElements}$ ;
  exactSol = y[x] /. First@DSolve[{-y''[x] + q y[x] == f, y[0] == 0, y[L] == 0}, y[x], x];
  A = Table[If[i == j,  $2 + \frac{2}{3} q h^2$ , If[j == i - 1 || j == i + 1,  $-1 + \frac{1}{6} q h^2$ , 0]],
    {i, nShapeFunctions}, {j, nShapeFunctions}];
  b = Table[h^2 f, {nShapeFunctions}];
  c = LinearSolve[A, b];
  Plot[{exactSol, yApprox[x, c, h, nShapeFunctions]}, {x, 0, L}, PlotRange -> All,
    PlotLabel -> msg, ImageSize -> 250, PlotStyle -> {Black, {Dashed, Red, Thickness -> .01}}]
]

```

```

In[1822]= p = Table[plotIt["N=" <> ToString[i], i], {i, 2, 7}]

```



I also written a small Manipulate program to simulate the above. Here it is

```

In[345]= Remove["Global`*"];
(*by Nasser Abbasi. FEM program for Math 503*)

ψ[x_, L_] := L - x /; 0 ≤ x ≤ L
ψ[x_, L_] := 0 /; x > L
ψ[x_, L_] := ψ[-x, L] /; x < L
φ[x_, i_, h_, L_] := ψ[ $\frac{x}{h}$  - i, L]

yApprox[x_, c_, h_, n_, L_] := Module[{i}, Sum[c[[i]] φ[x, i, h, L], {i, 1, n}]]

plotIt[n_] :=
Module[{A, b, c, nElements, nShapeFunctions, i, j, h, q, f, t1, t2, t3, L, y,
  exactSol, grid, p, p2, nPoints, rmserror, data},
  L = 1; q = 4; f = 4; nElements = n; nShapeFunctions = nElements - 1; h =  $\frac{L}{nElements}$ ;
  nPoints = nElements + 1;
  exactSol = y[x] /. First@DSolve[{-y''[x] + q y[x] == f, y[0] == 0, y[L] == 0}, y[x], x];
  (*t1 = ( $\frac{2}{h} + \frac{2h(1-3L+3L^2)}{3L^2}q$ ); t2 = ( $-\frac{1}{h} + \frac{1}{6}h(6 + \frac{1}{L^2} - \frac{6}{L})q$ ); t3 = -f h (-2 +  $\frac{1}{L}$ );*)
  t1 = 2 +  $\frac{2}{3}qh^2$ ; t2 = -1 +  $\frac{1}{6}qh^2$ ; t3 = h2f;
  A = Table[If[i == j, t1, If[j == i - 1 || j == i + 1, t2, 0]], {i, nShapeFunctions},
    {j, nShapeFunctions}];
  b = Table[t3, {nShapeFunctions}];
  c = LinearSolve[A, b];
  grid = Range[0, L, h] // N;
  rmserror =  $\sum_{i=1}^{nPoints} (\text{exactSol} /. x \rightarrow \text{grid}[[i]] - \text{yApprox}[\text{grid}[[i]], c, h, nShapeFunctions, L])^2$ ;
  rmserror = Sqrt[rmserror] / nPoints;
  p = Plot[{exactSol, yApprox[x, c, h, nShapeFunctions, L]}, {x, 0, L},
    PlotRange → All,
    PlotLabel → "N=" <> ToString[n] <> "\nRMS error=" <> ToString[rmserror],
    PlotStyle → {Black, {Dashed, Red}}, PlotRange → {{0, 1}, {0, .5}}, AxesOrigin → {0, 0}];

  data = Table[{grid[[i]], yApprox[grid[[i]], c, h, nShapeFunctions, L]}, {i, 1, nPoints}];
  p2 = Graphics[{PointSize[Large], Point[data]}];
  Show[{p, p2}]
]

```

```

In[353]= demo = Manipulate[plotIt[i], {i, 2, 30, 1}, AutorunSequencing → {{1, 80}},
  FrameLabel →
  "Finite Element solution, hardcoded A/b method. Math 503, summer 2007, CSUF
  by Nasser Abbasi"]

```

