EGME 541. Finite Elements
Audit course
California State University, Fullerton. Spring 2010

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Contents

Chapter 1

Introduction

1.1 Course syllabus

CALIFORNIA STATE UNIVERSITY, FULLERTON
Department of Mechanical Engineering

EGME 541

FINITE ELEMENT METHOD FOR MECHANICAL ENGINEERS
(Revised Outline)

Spring 2010

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Office Hours:	Monday & Wednesday 5:30-7:00 PM Friday 1:30-2:30 PM	*URL:*	http://blackboard.fullerton.edu

TEXTBOOK

Concepts and Applications of Finite Element Analysis, 4th Ed., by R.D. Cook, et al., Wiley, 2002.

COURSE OBJECTIVE

This course offers an advanced coverage for the subject of finite element analysis and its applications. The main objective of the course is to emphasize mathematical formulations that constitute the foundation of the finite element method. The course also covers applications of the method to a sample of disciplines within engineering for demonstration of its versatility. It also develops skills needed for using general purpose software to conduct an accurate analysis of structural systems and mechanical devices. Overall, the materials and exercises in this course will provide you with a strong foundation that is needed for your continuous learning in this field throughout your professional career.

COURSE CONTENT AND ASSIGNMENTS

The list of topics to be covered in this course is attached. The course will *approximately* follow the schedule as given, and consists of three principle parts: 1) The finite element method and algorithms; 2) Software available at CSUF; and 3) Applications and case studies.

This is a graduate-level course and students are expected to actively participate in their learning. Accordingly, you must read the assigned text and handouts completely, and participate in class discussions. Also, you have to independently search for and study materials beyond those covered in the class lectures. Homework and computer exercises will be assigned during each class meeting. Assignments will be usually due on the first meeting of the following week and will be collected/graded on a random basis. The solutions for selected homework sets can be accessed through the *Blackboard* system at http://blackboard.fullerton.edu/student/default.htm. These solutions will be available at the time the graded assignments are returned to the students, usually one week after the due date for each assignment. Audio/video recording of the class

From time to time projects will be assigned which may require more time in compare to those of regular homework problems. The required project report format will be provided by the instructor at the time of project assignment. ***No late homework/project report will be accepted***. Opportunities for extra credit will be offered to everyone in the class. Extra credit activity will be announced during the semester.

TESTS AND FINAL EXAM

There will be one midterm test and a comprehensive final examination. Unannounced quizzes may also be given. If you miss the midterm or the final exam for a medical or other legitimate reason, you must present a medical report or a legal document for justification. Otherwise, a zero grade will be assigned to the test or the final. Make-up midterm test will not be given. Instead, the weight of the missed test will be shifted to the final exam. You are required to completely shut off your mobile phones during the quiz/exam periods.

GRADING POLICY

Relative Weight of Course Activities		**Tentative Basis for Assigning Course Letter Grades***	
Assignments	10%	*A*	85-100%
Midterm Exam & Quizzes	40%	*B*	70-84%
Final Exam	50%	*C*	55-69%
		D	40-54%

* The above ranges for the course letter grades may be adjusted based on the overall performance of the class that will be determined at the end of semester. +/- grading will not be used in this course.

IMPORTANT INFORMATION

- During the current semester, the faculty at CSUF is subjected to a furlough (work reduction) in order to cut costs, prevent excessive course cancellations, and reduce the number of faculty layoffs. No lectures will be conducted and no office hours will be held for this course on those days that coincide with the following tentative furlough schedule that applies only to the instructor for this course:

  February: 12; March: 26; April: 5, 19; May: 3

  Please note that the above dates are subject to change. Updated information will be provided as these become available. **During the faculty furlough days students are expected to continue working on their assignments.**

- The University requires students with disabilities to register with the Office of Disabled Student Services, located in UH-101 and at (657) 278-3112, in order to receive accommodations appropriate to their disability. Students requesting accommodations should inform the instructor during the first week of classes about any disability or special needs that may require specific arrangements related to attending class sessions, completing course assignments, and taking quizzes/examinations.

- During an emergency it is necessary for students to have a basic understanding of their personal responsibilities and the University's emergency response procedures. Please review these procedures that are posted at *http://www.fullerton.edu/emergencypreparedness/ep_students.html*.
- Disruptive activities that cause distractions for other students in the classroom are not allowed. Such disruptive behaviors include: Late arrival to the classroom; early exit from the classroom; using mobile phones/pagers; eating; and conversations outside the discussions led by the instructor.
- Academic dishonesty will not be tolerated. According to the CSUF's University Policy, "Academic dishonesty includes such things as cheating, inventing false information or citations, plagiarism, and helping someone else commit an act of academic dishonesty. It usually involves an attempt by a student to show possession of a level of knowledge or skill which he/she does not possess."

Disruptive behavior and academic dishonesty will be dealt with according to the University regulations. The details of the "University Regulations" can be found in the University Catalog or through the CSUF website at www.fullerton.edu.

LEARNING GOALS FOR THIS COURSE

- Ability to apply advanced mathematics, science & engineering to analyze and design complex systems
- Ability to develop and follow a systematic procedure to analyze or design a system, component, or process
- Ability to learn about contemporary issues through self-education and continuous learning
- Ability to identify emerging technologies
- Ability to use the techniques, skills and modern engineering tools necessary for engineering practice

EGME 541

FINITE ELEMENT METHOD FOR MECHANICAL ENGINEERS

LIST OF TOPICS

Week	**Topics**	**Reading Assignments**
1	Basic Concepts, Mathematical Review	Chapter 1, App. A & B
2-3	Finite Element Formulations for 1-D Problems	Chapters 2, 3, 4
4	Coordinate Transformations	Chapter 2
5	Mechanical and Thermal Loadings	Chapter 2
6	Finite Element Formulations for 2-D Problems	Chapter 3
7	General Purpose FEA Software; Stress Calculations	On-Line Resources
8	Natural Coordinates/Shape Functions; Midterm Test	Chapter 3
9	Hands-on Activity: FEA Commercial Software	On-Line Resources
10-11	Isoparametric Elements - Transformations	Chapters 6 & 8
12-13	Applications: Vibration Problems	Chapter 11, App. C
14	Practical Issues: Modeling, Errors, and Accuracy	Chapter 10
15	Applications: Heat Transfer & Fluid Flow Problems	Chapter 12
16	Final Exam: Wednesday May 19, 2010 at 7:30 PM	

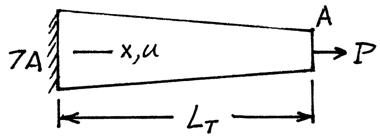
Chapter 2

HWs

Local contents

2.1 HW 1

$\boxed{1.3\text{-}1}$



(a)

For $0 \le x \le L_T$,

$$A_x = A\left(7 - \frac{6x}{L_T}\right)$$

and $u = u(x) = \int_0^x \frac{P\,dx}{A_x E} = \frac{P}{EA}\int_0^x \frac{dx}{7 - \frac{6x}{L_T}}$

$$u = \frac{P}{EA}\left[-\frac{\ln(7 - 6x/L_T)}{6/L_T}\right]_0^x$$

$$u = \frac{PL_T}{6EA}\left[\ln 7 - \ln\left(7 - \frac{6x}{L_T}\right)\right]$$

$$\begin{array}{cccc} & \underline{x = L_T/3} & \underline{x = 2L_T/3} & \underline{x = L_T} \\ u & \underbrace{0.0561 \qquad 0.1412 \qquad 0.3243}_{*\,PL_T/EA} \end{array}$$

By FEA:

At $x = L_T/3$, $u = \frac{P(L_T/3)}{6EA} = 0.0556\frac{PL_T}{EA}$

At $x = 2L_T/3$, $u = 0.0556\frac{PL_T}{EA} + \frac{P(L_T/3)}{4EA}$

$= 0.1389\frac{PL_T}{EA}$

At $x = L_T$, $u = 0.1389\frac{PL_T}{EA} + \frac{P(L_T/3)}{2EA}$

$= 0.3056\frac{PL_T}{EA}$

(b) FEA stresses at element midpoints:

$$\sigma_{1\text{-}2} = \frac{E}{L_T/3}\,0.0556\frac{PL_T}{EA} = 0.167\frac{P}{A}$$

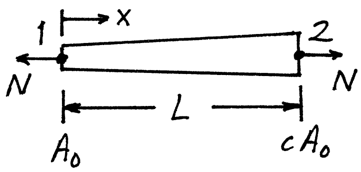
$$\sigma_{2\text{-}3} = \frac{E}{L_T/3}(0.1389 - 0.0556)\frac{PL_T}{EA} = 0.250\frac{P}{A}$$

$$\sigma_{3\text{-}4} = \frac{E}{L_T/3}(0.3056 - 0.1389)\frac{PL_T}{EA} = 0.500\frac{P}{A}$$

These stresses are exact.

2.2 HW 2

$\boxed{2.2\text{-}1}$



(a) Eq. 2.2-1:

$$[\underset{\sim}{k}] = \frac{A_{ave}E}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{(1+c)A_0}{2}\frac{E}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Exact: let δ = elongation

$$\delta = \int_0^L \frac{N\,dx}{AE} \quad \text{where } A = A_0 + (c-1)\frac{x}{L}A_0$$

$$\delta = \frac{N}{A_0E}\,\frac{L}{c-1}\left(1+(c-1)\frac{x}{L}\right)\Big|_0^L = \frac{NL\ln c}{A_0E(c-1)}$$

For $\delta = 1$, $N = \dfrac{A_0E(c-1)}{L\ln c}$

$$[\underset{\sim}{k}] = \frac{A_0E(c-1)}{L\ln c}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}.$$ Now let $c=2$:

Conventional: $[\underset{\sim}{k}] = 1.5\dfrac{A_0E}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

Exact: $[\underset{\sim}{k}] = 1.443\dfrac{A_0E}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

(b) Exact: $\delta = \dfrac{PL_T}{2A_0E}\ln 3 = 0.5493\dfrac{PL_T}{A_0E}$

One el.:

$$\delta = \frac{PL_T}{(2A_0)E} = 0.5\frac{PL_T}{A_0E} \quad 8.98\%\text{ low}$$

Two els.:

$$\delta = \frac{P(L_T/2)}{A_0E}\left(\frac{1}{1.5}+\frac{1}{2.5}\right) = 0.5333\frac{PL_T}{A_0E}$$

2.91 % low

Three els.:

$$\delta = \frac{P(L_T/3)}{A_0E}\left(\frac{1}{1.333}+\frac{1}{2.000}+\frac{1}{2.667}\right)$$

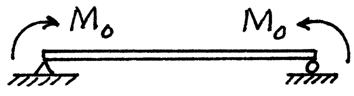
$$\delta = 0.5417\frac{PL_T}{A_0E} \qquad 1.39\%\text{ low}$$

Four els.:

$$\delta = \frac{P(L_T/4)}{A_0E}\left(\frac{1}{1.25}+\frac{1}{1.75}+\frac{1}{2.25}+\frac{1}{2.75}\right)$$

$$\delta = 0.5449\frac{PL_T}{A_0E} \qquad 0.81\%\text{ low}$$

$\boxed{2.3\text{-}7}$



(a)

From Eq. 2.3-6, with θ_{z1} and θ_{z2} the active d.o.f.,

$$\frac{EI_z}{(1+\phi_y)L}\begin{bmatrix} 4+\phi_y & 2-\phi_y \\ 2-\phi_y & 4+\phi_y \end{bmatrix}\begin{Bmatrix} \theta_{z1} \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} -M_o \\ M_o \end{Bmatrix}$$

From which $\theta_{z1} = -\theta_{z2} = -\dfrac{M_o L}{2EI_z}$

(b)

$$v_2 = \frac{P}{Y_1} = \frac{(1+\phi_y)PL^3}{12EI_z}$$

$$\phi_y = \frac{12EI_z k_y}{AGL^2} = \frac{12E[3(4^3)/12]1.2}{3(4)(E/2)L^2} = \frac{38.4}{L^2}$$

$$v_2 = \frac{(1+38.4/L^2)PL^3}{12E[3(4^3)/12]} = \frac{1+38.4/L^2}{192}\frac{PL^3}{E}$$

No trans. shear deformation: $v_2 = \dfrac{PL^3}{192E} = \bar{v}$

$L=8$: $v_2 = 1.6\bar{v}$; 37.5% from shear

$L=16$: $v_2 = 1.15\bar{v}$; 13.0% from shear

$L=32$: $v_2 = 1.0375\bar{v}$; 3.6% from shear

(c)

$$v_2 = \frac{(1+\phi_y)PL^3}{12EI_z} = \frac{PL^3}{12EI_z}\left[\frac{AGL^2+12EI_z k_y}{AGL^2}\right]$$

$$v_2 = \frac{PL}{AG}\left[\frac{AGL^2}{12EI_z} + k_y\right]$$

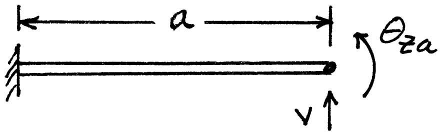
$L\to\infty$: $v_2 \to \dfrac{PL^3}{12EI_z}$

(as in elementary beam theory)

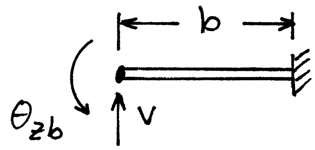
$L\to 0$: $v_2 \to \dfrac{PL}{AG}k_y$

$\boxed{2.5\text{-}3}$

No rotational connection at the hinge — retain two θ_z d.o.f. there, so $\{\underset{\sim}{D}\} = \lfloor v \quad \theta_{za} \quad \theta_{zb} \rfloor^T$



$$[\underset{\sim}{k}]_a = EI_z \begin{array}{c} \begin{bmatrix} 12/a^3 & -6/a^2 & 0 \\ -6/a^2 & 4/a & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} v \\ \theta_{za} \\ \theta_{zb} \end{array}$$



$$[\underset{\sim}{k}]_b = EI_z \begin{bmatrix} 12/b^3 & 0 & -6/b^2 \\ 0 & 0 & 0 \\ -6/b^2 & 0 & 4/b \end{bmatrix} \begin{array}{l} v \\ \theta_{za} \\ \theta_{zb} \end{array}$$

$$[K] = EI_z \begin{bmatrix} 12/a^3 + 12/b^3 & -6/a^2 & -6/b^2 \\ -6/a^2 & 4/a & 0 \\ -6/b^2 & 0 & 4/b \end{bmatrix} \begin{array}{l} v \\ \theta_{za} \\ \theta_{zb} \end{array}$$

2.3 HW 2 correction

The result of integration is:

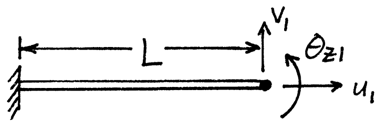
$$\delta = \frac{N}{A_0 E} \frac{L}{c-1} \left[\ln \left[1 + (c-1) \frac{x}{L} \right] \right]_0^L$$

Which, when compared with the solution that is posted, it shows that the posted solution has the natural logarithmic symbol (ln) missing. After you plug in the limits and do proper cancellations you get:

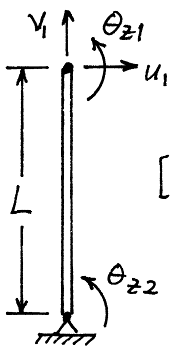
$$\delta = \frac{N}{A_0 E} \frac{L \ln c}{c-1}$$

2.4 HW 3

$\boxed{2.5\text{-}4}$ $\quad a = AE/L \qquad b = EI/L^3$



$$[\underset{\sim}{k}] = \begin{array}{c} \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 12b & -6bL & 0 \\ 0 & -6bL & 4bL^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} u_1 \\ v_1 \\ \theta_{z1} \\ \theta_{z2} \end{array}$$



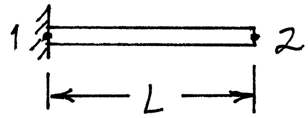
$$[\underset{\sim}{k}] = \begin{array}{c} \begin{bmatrix} 12b & 0 & 6bL & 6bL \\ 0 & a & 0 & 0 \\ 6bL & 0 & 4bL^2 & 2bL^2 \\ 6bL & 0 & 2bL^2 & 4bL^2 \end{bmatrix} \end{array} \begin{array}{l} u_1 \\ v_1 \\ \theta_{z1} \\ \theta_{z2} \end{array}$$

Add; get

$$[\underset{\sim}{K}] = \begin{bmatrix} a+12b & 0 & 0 & 0 \\ 0 & a+12b & -6bL & 0 \\ 6bL & -6bL & 8bL^2 & 2bL^2 \\ 6bL & 0 & 2bL^2 & 4bL^2 \end{bmatrix}$$

2.7-1

(a)



From Eq. 2.3-5,

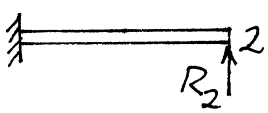
$$EI_z\begin{bmatrix}12/L^3 & -6/L^2\\ -6/L^2 & 4/L\end{bmatrix}\begin{Bmatrix}v_2\\ \theta_{z2}\end{Bmatrix}=\begin{Bmatrix}R_2\\ M_2\end{Bmatrix} \qquad (A)$$

Set $v_2 = \bar{v}_2$

$$EI_z\begin{bmatrix}1 & 0\\ 0 & 4/L\end{bmatrix}\begin{Bmatrix}v_2\\ \theta_{z2}\end{Bmatrix}=\begin{Bmatrix}EI_z\bar{v}_2\\ 6EI_z\bar{v}_2/L^2\end{Bmatrix} \text{ gives } \theta_{z2}=\frac{3\bar{v}_2}{2L},\ v_2=\bar{v}_2$$

Eq. (A) then gives $R_2 = EI_z\left[\frac{12}{L^3}\bar{v}_2 - \frac{6}{L^2}\,\frac{3\bar{v}_2}{2L}\right] = \frac{3EI_z\bar{v}_2}{L^3}$

Beam theory:

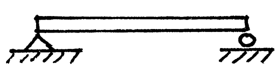


$$\underbrace{v_2=\frac{R_2L^3}{3EI_z},\quad \theta_{z2}=\frac{R_2L^2}{2EI_z}}$$

$$\theta_{z2}=\frac{L^2}{2EI_z}\,\frac{3EI_z}{L^3}v_2=\frac{3v_2}{2L}$$

$$R_2=\frac{2EI_z}{L^2}\theta_{z2}=\frac{3EI_z}{L^3}v_2$$

(b)



From Eq. 2.3-5,

$$\frac{2EI_z}{L}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\theta_{z1}\\ \theta_{z2}\end{Bmatrix}=\begin{Bmatrix}M_1\\ M_2\end{Bmatrix} \qquad (B)$$

Set $\theta_{z1}=\bar{\theta}_{z1}$

$$\frac{2EI_z}{L}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}\theta_{z1}\\ \theta_{z2}\end{Bmatrix}=\begin{Bmatrix}2EI_z\bar{\theta}_{z1}/L\\ -2EI_z\bar{\theta}_{z1}/L\end{Bmatrix} \text{ gives } \theta_{z1}=\bar{\theta}_{z1},\ \theta_{z2}=-\frac{\bar{\theta}_{z1}}{2}$$

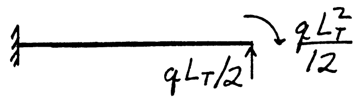
Eq. (B) then gives $M_1=\frac{2EI_z}{L}\left[2\bar{\theta}_{z1}-\frac{\bar{\theta}_{z1}}{2}\right]=\frac{3EI_z}{L}\bar{\theta}_{z1}$

These results appear in tables of beam deflections.

2.5 HW 4

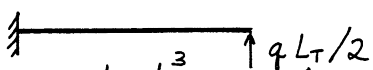
$\boxed{2.9\text{-}4}$

(a) Exact: $v = \dfrac{qL_T^4}{8EI}$, $M = \dfrac{qL_T^2}{2}$



$$v = \frac{qL_T}{2}\frac{L_T^3}{3EI} - \frac{qL_T^2}{12}\frac{L_T^2}{2EI} = \frac{qL_T^4}{8EI} \quad \text{exact}$$

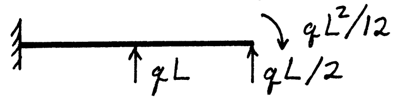
$$M = \frac{qL_T}{2}L_T - \frac{qL_T^2}{12} = 0.417qL_T^2 \quad -16.7\%$$



$$v = \frac{qL_T}{2}\frac{L_T^3}{3EI} = \frac{qL_T^4}{6EI} \quad \text{———} \quad +33.3\%$$

$$M = \frac{qL_T}{2}L_T = \frac{qL_T^2}{2} \quad \text{———} \quad \text{exact}$$

(b)



$L = \dfrac{L_T}{2}$

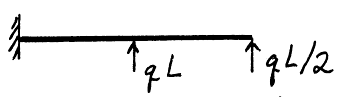
$$v = v_{\text{center force}} + v_{\text{end force}} + v_{\text{end moment}}$$

$$v = \left(qL\frac{L^3}{3EI} + qL\frac{L^2}{2EI}L\right) + \left(\frac{qL}{2}\frac{(2L)^3}{3EI}\right) - \left(\frac{qL^2}{12}\frac{(2L)^2}{2EI}\right) = \frac{qL^4}{EI}\left(\frac{5}{6} + \frac{4}{3} - \frac{1}{6}\right)$$

$$v = \frac{2qL^4}{EI} = \frac{qL_T^4}{8EI} \quad \text{————} \quad \text{exact}$$

$$M = qL(L) + \frac{qL}{2}(2L) - \frac{qL^2}{12}$$

$$M = \frac{23qL^2}{12} = \frac{23qL_T^2}{48} = 0.479qL_T^2 \quad -4.2\%$$



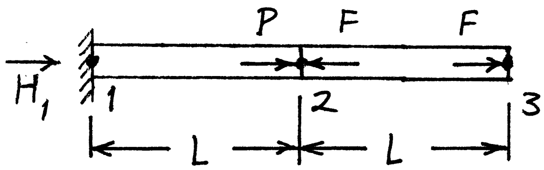
$L = \dfrac{L_T}{2}$

Omit $v_{\text{end moment}}$ from foregoing calculation.

$$v = \frac{qL^4}{EI}\left(\frac{5}{6} + \frac{4}{3}\right) = \frac{13qL^4}{6EI} = 0.135\frac{qL_T^4}{EI} \quad +8.3\%$$

$$M = qL(L) + \frac{qL}{2}2L = 2qL^2 = \frac{qL_T^2}{2} \quad \text{exact}$$

2.10-1



$\sigma_0 = -E\alpha \dfrac{T_2 + T_3}{2}$ (right element only)

$F = |A\sigma_0| = AE\alpha \dfrac{T_2 + T_3}{2}$

In Eq. 2.10-3, set $u_1 = 0$ and $H_3 = 0$. Thus

$$\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P - EA\alpha(T_2+T_3)/2 \\ EA\alpha(T_2+T_3)/2 \end{Bmatrix}$$

from which $u_2 = \dfrac{PL}{AE}$, $u_3 = \dfrac{PL}{AE} + \alpha L \dfrac{T_2 + T_3}{2}$

1st of Eqs. 2.10-3 then gives $H_1 = -P$

Stresses:

$$\sigma_{1-2} = E\frac{u_2}{L} + (\text{zero}) = \frac{P}{A} \quad ✓$$

$$\sigma_{2-3} = E\frac{u_3 - u_2}{L} + \sigma_0 = E\alpha\frac{T_2+T_3}{2} + \left(-E\alpha\frac{T_2+T_3}{2}\right) = 0 \quad ✓$$

2.6 HW 5

$\boxed{3.3\text{-}1}$

(a) $u = \lfloor 1 \quad x \rfloor \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$, $\lfloor \underset{\sim}{B}_a \rfloor = \dfrac{\partial}{\partial x} \lfloor 1 \quad x \rfloor = \lfloor 0 \quad 1 \rfloor$

$$[\underset{\sim}{k}_a] = \int_0^L \lfloor \underset{\sim}{B}_a \rfloor^T \lfloor \underset{\sim}{B}_a \rfloor AE\, dx = AEL \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $x_1 = 0$, $x_2 = L$ in Eq. 3.2-4. Then $[\underset{\sim}{A}] = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}$, $[\underset{\sim}{A}]^{-1} = \begin{bmatrix} 1 & 0 \\ -1/L & 1/L \end{bmatrix}$

$$[\underset{\sim}{k}] = [\underset{\sim}{A}]^{-T} [\underset{\sim}{k}_a] [\underset{\sim}{A}]^{-1} = \begin{bmatrix} 1 & -1/L \\ 0 & 1/L \end{bmatrix} \left(AEL \begin{bmatrix} 0 & 0 \\ -1/L & 1/L \end{bmatrix} \right) = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(b) $v = \lfloor 1 \quad x \quad x^2 \quad x^3 \rfloor \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$, $\dfrac{d^2 v}{dx^2} = \lfloor 0 \quad 0 \quad 2 \quad 6x \rfloor \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \lfloor \underset{\sim}{B}_a \rfloor \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$

$$[\underset{\sim}{k}_a] = \int_0^L \lfloor \underset{\sim}{B}_a \rfloor^T \lfloor \underset{\sim}{B}_a \rfloor EI\, dx = EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4L & 6L^2 \\ 0 & 0 & 6L^2 & 12L^3 \end{bmatrix}$$

$[\underset{\sim}{A}]^{-1}$ is calculated in Problem 3.2-4

$$[\underset{\sim}{k}] = [\underset{\sim}{A}]^{-1} [\underset{\sim}{k}_a] [\underset{\sim}{A}]^{-1} = \begin{bmatrix} 1 & 0 & -3/L^2 & 2/L^3 \\ 0 & 1 & -2/L & 1/L^2 \\ 0 & 0 & 3/L^2 & -2/L^3 \\ 0 & 0 & -1/L & 1/L^2 \end{bmatrix} \left(EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 6 & 0 & -6 & 6L \end{bmatrix} \right)$$

$$[\underset{\sim}{k}] = EI \begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & 4/L & -6/L^2 & 2/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ 6/L^2 & 2/L & -6/L^2 & 4/L \end{bmatrix}$$

$\boxed{3.4\text{-}3}$

(a) The only nonzero d.o.f. are u_3 and v_3. From Eq. 3.4-10,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\underset{\sim}{B}]\{\underset{\sim}{d}\} = \begin{bmatrix} 0 & 0 \\ 0 & 1/a \\ 1/a & 0 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} \quad , \quad \text{For } \nu = 0,$$

$$[\underset{\sim}{k}] = Et\,\frac{a^2}{2} \begin{bmatrix} 0 & 0 & 1/a \\ 0 & 1/a & 0 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[\underset{\sim}{k}] = \frac{Et}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix} = \frac{Et}{2} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

2.7 HW 6

$\boxed{3.6\text{-}3}$ Beam theory will give $u_F = -u_D$, $v_F = v_D$

(a) From Eq. 3.6-6,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{4ab} \begin{bmatrix} \cdots & b-y & 0 & b+y & 0 & \cdots \\ \cdots & 0 & -(a+x) & 0 & a+x & \cdots \\ \cdots & -(a+x) & b-y & a+x & b+y & \cdots \end{bmatrix} \begin{Bmatrix} \vdots \\ u_D \\ v_D \\ -u_D \\ v_D \\ \vdots \end{Bmatrix}$$

From beam theory, Prob. 3.4-1, $\quad u_D = -\dfrac{3ML}{2Etc^2}$, $\quad v_D = -\dfrac{3ML^2}{4Etc^3}$

Hence

$$\epsilon_x = \frac{3MLy}{4Etabc^2}, \quad \epsilon_y = 0, \quad \gamma_{xy} = \frac{3ML(a+x)}{4ab\,Etc^2} - \frac{3ML^2}{8aEtc^3}$$

For $\nu = 0$, $\quad \sigma_x = E\epsilon_x = \dfrac{3MLy}{4tabc^2}$, $\quad \sigma_y = E\epsilon_y = 0$,

$$\tau_{xy} = \frac{E}{2}\gamma_{xy} = \frac{3ML(a+x)}{8abtc^2} - \frac{3ML^2}{16atc^3}$$

But $L = 2a$ and $c = b$, so $\quad \sigma_x = \dfrac{3My}{2tb^3}$, $\quad \sigma_y = 0$, $\quad \tau_{xy} = \dfrac{3Mx}{4b^3t}$

(in the coordinate system of Fig. 3.6-1). In this system,

$\sigma_x = \dfrac{My}{I} = \dfrac{3My}{2tb^3}$, $\quad \sigma_y = \tau_{xy} = 0$ according to beam theory.

$\boxed{3.6\text{-}5}$

<u>Method 1</u>: Evaluate $u = a_1 + a_2 x + a_3 y + a_4 xy$ at nodes, solve for a's, gather coefficients of u_1, u_2, u_3, u_4.

$u_1 = a_1$ Node 1

$u_2 = u_1 + a_2(2a)$; $a_2 = \dfrac{u_2 - u_1}{2a}$ Node 2

$u_4 = u_1 + a_3(2b)$; $a_3 = \dfrac{u_4 - u_1}{2b}$ Node 4

$$u_3 = u_1 + \frac{u_2 - u_1}{2a}2a + \frac{u_4 - u_1}{2b}2b + a_4(2a)(2b)$$

from which $a_4 = \dfrac{u_1 - u_2 + u_3 - u_4}{4ab}$

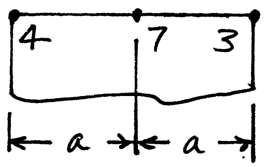
$$u = u_1 + \frac{u_2 - u_1}{2a}x + \frac{u_4 - u_1}{2b}y + \frac{u_1 - u_2 + u_3 - u_4}{4ab}xy$$

$$u = \left(1 - \frac{x}{2a} - \frac{y}{2b} + \frac{xy}{4ab}\right)u_1 + \left(\frac{x}{2a} - \frac{xy}{4ab}\right)u_2 + \left(\frac{xy}{4ab}\right)u_3 + \left(\frac{y}{2b} - \frac{xy}{4ab}\right)u_4$$

Coefficients of the u_i are the N_i.

$\boxed{3.11\text{-}1}$

Eq. 3.11-6 :
$$\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = \underbrace{\frac{a}{15}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}}_{[\underset{\sim}{H}]} \begin{Bmatrix} q_4 \\ q_7 \\ q_3 \end{Bmatrix}$$



Unit thickness

(a)
$$\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = [\underset{\sim}{H}] \begin{Bmatrix} \sigma \\ 0 \\ -\sigma \end{Bmatrix} = \frac{\sigma a}{3} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

Couple-moment: $M = F_4 (2a) = \dfrac{2\sigma a^2}{3}$

Flexure formula :
$$M = \frac{\sigma I}{c} = \frac{\sigma (2a)^3/12}{a} = \frac{2\sigma a^2}{3}$$

(b)
$$\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = [\underset{\sim}{H}] \begin{Bmatrix} 0 \\ \sigma/2 \\ \sigma \end{Bmatrix} = \frac{\sigma a}{3} \begin{Bmatrix} 0 \\ 2 \\ 1 \end{Bmatrix}$$

Moment : $M = 2aF_3 + aF_7 = \dfrac{4\sigma a^2}{3}$

Flexure formula, for section 4a units deep :
$$M = \frac{\sigma I}{c} = \frac{\sigma (4a)^3/12}{2a} = \frac{8\sigma a^2}{3}$$

Contribution of half the section is
$$\frac{M}{2} = \frac{4\sigma a^2}{3}$$

(c)
$$\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = [\underset{\sim}{H}] \begin{Bmatrix} 0 \\ \tau \\ 0 \end{Bmatrix} = \frac{\tau a}{15} \begin{Bmatrix} 2 \\ 16 \\ 2 \end{Bmatrix}$$

Shear force : $F_4 + F_7 + F_3 = \dfrac{4\tau a}{3}$

Beam theory (parabolic distribution):
$$\tau = \frac{3}{2}\frac{V}{2a}, \quad V = \frac{4\tau a}{3}$$

$\boxed{3.11-2}$

With $[\underset{\sim}{N}]$ from Eq. 3.11-5, and constant q,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \int_{-a}^{a} [\underset{\sim}{N}]\, q\, dx = \int_{-a}^{a} \frac{1}{2a^2} \begin{Bmatrix} x(x-a) \\ 2(a^2-x^2) \\ x(x+a) \end{Bmatrix} q\, dx$$

$$= \frac{q}{2a^2} \begin{Bmatrix} \frac{x^3}{3} - \frac{ax^2}{2} \\ 2a^2x - \frac{2x^3}{3} \\ \frac{x^3}{3} + \frac{ax^2}{2} \end{Bmatrix}_{-a}^{a} = \frac{q}{2a^2} \begin{Bmatrix} 2a^3/3 \\ 8a^3/3 \\ 2a^3/3 \end{Bmatrix} = q \begin{Bmatrix} a/3 \\ 4a/3 \\ a/3 \end{Bmatrix} = F \begin{Bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{Bmatrix}$$

where $F = 2qa$

2.8 HW 7

$\boxed{6.1\text{-}4}$

$$x = \lfloor \underset{\sim}{N} \rfloor \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad u = \lfloor \underset{\sim}{N} \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \textit{where}$$

$$\lfloor \underset{\sim}{N} \rfloor = \frac{1}{2} \lfloor 1-\xi, \ 1+\xi \rfloor. \quad J = \frac{dx}{d\xi} = \frac{x_2 - x_1}{2} = \frac{L}{2}$$

$$\epsilon_x = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{1}{J}\frac{du}{d\xi} = \frac{2}{L}\left\lfloor -\frac{1}{2} \quad \frac{1}{2} \right\rfloor \{\underset{\sim}{d}\} = \underbrace{\frac{1}{L}\lfloor -1 \quad 1 \rfloor}_{\lfloor \underset{\sim}{B} \rfloor} \{\underset{\sim}{d}\}$$

$$[\underset{\sim}{k}] = \int_{-1}^{1} \lfloor \underset{\sim}{B} \rfloor^T \lfloor \underset{\sim}{B} \rfloor AE \, J \, d\xi$$

$$[\underset{\sim}{k}] = \frac{AE}{2L} \int_{-1}^{1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\boxed{6.2\text{-}7}$

Use Eq. 6.2-6. Define the 2 by 8 matrix in Eq. 6.2-6 as $[\underset{\sim}{D}_N]$ and write it in the form

$$[\underset{\sim}{D}_N] = \frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \eta & -\eta & \eta & -\eta \\ \xi & -\xi & \xi & -\xi \end{bmatrix}$$

(a) $$[\underset{\sim}{D}_N]\begin{bmatrix} -3 & -2 \\ 3 & -2 \\ 3 & 2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \ J=6$$

(b) $$[\underset{\sim}{D}_N]\begin{bmatrix} 3 & -2 \\ 3 & 2 \\ -3 & 2 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}, \ J=6$$

(c) $$[\underset{\sim}{D}_N]\begin{bmatrix} 0 & -2 \\ 3 & 0 \\ 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & -\eta \\ 0 & 1-\xi \end{bmatrix}, \ J = \frac{3}{2}(1-\xi)$$

(d) $$[\underset{\sim}{D}_N]\begin{bmatrix} -3 & -2 \\ 0 & -2 \\ 3 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 3/2 & 2 \end{bmatrix}, \ J = 3$$

Area ratios & J		(a)	(b)	(c)	(d)
	$A_{el} \div A_{2\times2}$	6	6	$3/2$	3
	J	6	6	$\frac{3}{2}(1-\xi)$	3

J is ratio $\dfrac{dx\,dy}{d\xi\,d\eta}$ & is unity if $\xi = x$ & $\eta = y$, which is the case of 2×2 square.

$\boxed{6.3-6}$

(a) Exact: $\int_{-1}^{1}(\xi^2+\xi^3)\,d\xi = \frac{2}{3} = I$

1 pt. $I_1 = 2\,(0+0) = 0$ 100% low

2 pts. Let $a = \sqrt{3}/3$, then

$I_2 = (a^2-a^3)+(a^2+a^3) = 2a^2 = \frac{2}{3}$ exact

3 pts. Let $b = \sqrt{0.6}$, then

$I_3 = \frac{5}{9}(b^2-b^3)+\frac{8}{9}(0)+\frac{5}{9}(b^2+b^3)$

$I_3 = \frac{10}{9}b^2 = \frac{2}{3}$ exact

(b) $\int_{-1}^{1}\cos 1.5\xi\,d\xi = \left.\frac{\sin 1.5\xi}{1.5}\right|_{-1}^{1} = 1.3300 = I$

1 pt. $I_1 = 2\,(1) = 2$ +50.4%

2 pts. $I_2 = \cos\left(-\frac{1.5}{\sqrt{3}}\right)+\cos\left(\frac{1.5}{\sqrt{3}}\right) = 1.2957$ −2.58%

3 pts. $I_3 = \frac{5}{9}\cos\left(-1.5\sqrt{0.6}\right)+\frac{8}{9}\cos(0)$

$+\frac{5}{9}\cos\left(1.5\sqrt{0.6}\right) = 1.3307$ +0.05%

(c) $\int_{-1}^{1}\frac{1-\xi}{2+\xi}\,d\xi = \int_{-1}^{1}\frac{d\xi}{2+\xi} - \int_{-1}^{1}\frac{\xi\,d\xi}{2+\xi}$

$= \ln(2+\xi)\Big|_{-1}^{1} - \left[2+\xi-2\ln(2+\xi)\right]_{-1}^{1}$

$= \ln 3 - 2 + 2\ln 3 = 3\ln 3 - 2 = 1.2958 = I$

1 pt. $I_1 = 2\,\frac{1}{2} = 1$ −22.8%

2 pts. $I_2 = \frac{1+\frac{1}{\sqrt{3}}}{2-\frac{1}{\sqrt{3}}}+\frac{1-\frac{1}{\sqrt{3}}}{2+\frac{1}{\sqrt{3}}} = 1.2727$ −1.8%

3 pts. $I_3 = \frac{5}{9}\,\frac{1+\sqrt{0.6}}{2-\sqrt{0.6}}+\frac{8}{9}\,\frac{1}{2}+\frac{5}{9}\,\frac{1-\sqrt{0.6}}{2+\sqrt{0.6}}$

$= 1.2941$ −0.13%

(d) $x = \left\lfloor \frac{1-\xi}{2} \quad \frac{1+\xi}{2} \right\rfloor \begin{Bmatrix}1\\7\end{Bmatrix} = \frac{8+6\xi}{2} = 4+3\xi$

$I = \int_{1}^{7}\frac{dx}{x} = \ln 7 = 1.94591 = \int_{-1}^{1}\frac{3\,d\xi}{4+3\xi}$

$I_1 = 2\,\frac{3}{4} = 1.5$ −22.9%

$I_2 = \frac{3}{4-\sqrt{3}}+\frac{3}{4+\sqrt{3}} = 1.84615$ −5.1%

$I_3 = \frac{5}{9}\left(\frac{3}{4-3\sqrt{.6}}+\frac{3}{4+3\sqrt{.6}}\right)+\frac{8}{9}\,\frac{3}{4} = 1.92453$ −1.1%

$\boxed{6.7\text{-}1}$

$$\begin{bmatrix} 12 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 24 \\ 0 \end{Bmatrix}, \text{ or } \begin{bmatrix} 6 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 12 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_3 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 24 \\ 24 \end{Bmatrix}$$

$$[\underset{\sim}{k}_{cc}] = \begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix}, \quad [\underset{\sim}{k}_{cc}]^{-1} = \frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[\underset{\sim}{k}]_{cond.} = 6 - \lfloor -6 \quad 0 \rfloor \frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} -6 \\ 0 \end{Bmatrix} = 6 - 4 = 2$$

$$\{\underset{\sim}{r}\}_{cond.} = 0 - \lfloor -6 \quad 0 \rfloor \frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 24 \\ 24 \end{Bmatrix} = \frac{6}{18} 72 = 24$$

i.e. $2u_4 = 24$ so $u_4 = 12$

Recover $\{\underset{\sim}{d}_c\}$:

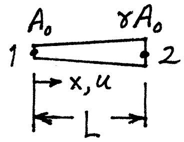
$$\begin{Bmatrix} u_3 \\ u_2 \end{Bmatrix} = -\frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{Bmatrix} -6 \\ 0 \end{Bmatrix} 12 - \begin{Bmatrix} 24 \\ 24 \end{Bmatrix} \right) = \begin{Bmatrix} 12 \\ 8 \end{Bmatrix}$$

Check: (order of d.o.f. here as originally written)

$$\begin{bmatrix} 12 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 6 \end{bmatrix} \begin{Bmatrix} 8 \\ 12 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 24 \\ 0 \end{Bmatrix} \quad \checkmark$$

2.9 HW 8

$\boxed{11.3\text{-}3}$



$$u = \lfloor \underset{\sim}{N} \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad A = \lfloor \underset{\sim}{N} \rfloor \begin{Bmatrix} A_0 \\ \gamma A_0 \end{Bmatrix}$$

where $\lfloor \underset{\sim}{N} \rfloor = \left\lfloor \dfrac{L-x}{L} \quad \dfrac{x}{L} \right\rfloor$

$$[\underset{\sim}{m}] = \int \rho \lfloor \underset{\sim}{N} \rfloor^T \lfloor \underset{\sim}{N} \rfloor A \, dx, \quad A = \frac{A_0}{L}\left[L + (\gamma - 1)x\right]$$

$$[\underset{\sim}{m}] = \frac{\rho A_0}{L^3} \int_0^L \begin{bmatrix} (L-x)^2 & x(L-x) \\ x(L-x) & x^2 \end{bmatrix} \left[L + (\gamma-1)x\right] dx$$

$$\int_0^L (L-x)^2 dx = \int_0^L x^2 dx = \frac{L^3}{3}, \quad \int_0^L x(L-x)dx = \frac{L^3}{6}$$

$$\int_0^L (L-x)^2 x \, dx = \frac{L^4}{12}, \quad \int_0^L x^2(L-x)dx = \frac{L^4}{12}, \quad \int_0^L x^3 dx = \frac{L^4}{4}$$

$$[\underset{\sim}{m}] = \rho A_0 L \left(\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} + (\gamma-1) \begin{bmatrix} 1/12 & 1/12 \\ 1/12 & 1/4 \end{bmatrix} \right)$$

$$[\underset{\sim}{m}] = \frac{\rho A_0 L}{12} \begin{bmatrix} 4 + (\gamma-1) & 2 + (\gamma-1) \\ 2 + (\gamma-1) & 4 + 3(\gamma-1) \end{bmatrix}$$

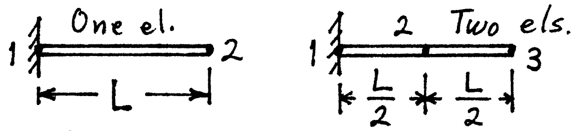
(c)

$$\{\ddot{\underset{\sim}{d}}\} = \begin{Bmatrix} a \\ 0 \\ a \\ 0 \end{Bmatrix}, \quad [\underset{\sim}{m}]\{\ddot{\underset{\sim}{d}}\} = \frac{ma}{6} \begin{Bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{Bmatrix} = \{\underset{\sim}{r}\}$$

$$\sum F_x = \lfloor 1 \quad 0 \quad 1 \quad 0 \rfloor \{\underset{\sim}{r}\} = \frac{ma}{6} 6 = ma \checkmark$$

$$[\underset{\sim}{m}]\{\ddot{\underset{\sim}{d}}\} = \frac{ma}{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} = \{\underset{\sim}{r}\}, \quad \sum F_x = \frac{ma}{2} 2 = ma \checkmark$$

$\boxed{11.4\text{-}3}$



(a) One el. $\left(\frac{AE}{L} - \omega_1^2 \frac{m}{3}\right)\bar{u}_2 = 0, \ \omega_1^2 = \frac{3AE}{mL}$

Two els. $\left(\frac{AE}{L/2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\frac{m}{2}\omega^2}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$7\lambda^2 - 10\lambda + 1 = 0$, where $\lambda = \frac{m\omega^2 L}{24AE}$

$\lambda_1 = 0.1082, \ \omega_1^2 = 2.597\,(AE/mL)$

(b) One el. $\left(\frac{AE}{L} - \omega_1^2 \frac{m}{2}\right)\bar{u}_2 = 0, \ \omega_1^2 = \frac{2AE}{mL}$

Two els. $\left(\frac{AE}{L/2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\frac{m}{2}\omega^2}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$2\lambda^2 - 4\lambda + 1 = 0$, where $\lambda = \frac{m\omega^2 L}{8AE}$

$\lambda_1 = 0.2929, \ \omega_1^2 = 2.343\,(AE/mL)$

(c)

$$[\underset{\sim}{m}] = \frac{1}{2}\left(\frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{m}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{m}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

One el. $\left(\frac{AE}{L} - \omega_1^2 \frac{5m}{12}\right)\bar{u}_2 = 0, \ \omega_1^2 = \frac{2.4AE}{mL}$

Two els. $\left(\frac{AE}{L/2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\frac{m}{2}\omega^2}{12}\begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix}\right)\begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$49\lambda^2 - 22\lambda + 1 = 0$, where $\lambda = \frac{m\omega^2 L}{48AE}$

$\lambda_1 = 0.05132, \ \omega_1^2 = 2.463\,(AE/mL)$

(d) $\left(\frac{AE}{L} - \omega_1^2 m \frac{3-\beta}{6}\right)\bar{u}_2 = 0, \quad \omega_1^2 = \frac{6AE}{(3-\beta)mL}$

Exact: $\omega_1^2 = \left(\frac{\pi}{2}\right)^2 \frac{AE}{mL}$

Equate these ω_1^2 values; thus

$\frac{6}{3-\beta} = \left(\frac{\pi}{2}\right)^2$ and $\beta = 0.568$

2.10 HW 9

$\boxed{11.6\text{-}3}$

$$[\underset{\sim}{K}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}, \quad [\underset{\sim}{M}] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$K_{ss} = k$, $K_{ms} = -k$, $[\underset{\sim}{T}] = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ $\quad [\underset{\sim}{T}]^T[\underset{\sim}{K}][\underset{\sim}{T}] = 0$

$[\underset{\sim}{T}]^T[\underset{\sim}{M}][\underset{\sim}{T}] = 2m$

$\left[0 - \omega^2(2m)\right]\bar{u}_1 = 0$

Hence $\omega = 0$; the rigid body mode; OK.

$\boxed{11.6-4}$

(a) With c a constant, $\frac{M_{11}}{K_{11}} = c\,\frac{156}{12}$ and $\frac{M_{22}}{K_{22}} = c\,\frac{4L^2}{4L^2} = c$. $\frac{M_{11}}{K_{11}} > \frac{M_{22}}{K_{22}}$, therefore the choice is proper.

(b)
$$[\underset{\sim}{T}] = \begin{bmatrix} -\frac{L^3}{12EI}\,\frac{6EI}{L^2} \\ 1 \end{bmatrix} = \begin{Bmatrix} -\frac{L}{2} \\ 1 \end{Bmatrix}$$

$$[\underset{\sim}{T}]^T[\underset{\sim}{K}][\underset{\sim}{T}] = \begin{bmatrix} -\frac{L}{2} & 1 \end{bmatrix} \frac{EI}{L^3} \begin{Bmatrix} -6L+6L \\ -3L^2+4L^2 \end{Bmatrix} = \frac{EI}{L}$$

$$[\underset{\sim}{T}]^T[\underset{\sim}{M}][\underset{\sim}{T}] = \begin{bmatrix} -\frac{L}{2} & 1 \end{bmatrix} \frac{m}{420} \begin{Bmatrix} -78L-13L \\ 6.5L^2+4L^2 \end{Bmatrix} = \frac{14mL^2}{105}$$

$$\left(\frac{EI}{L} - \omega^2 \frac{14mL^2}{105}\right)\bar{\theta}_2 = 0, \quad \omega^2 = 7.5\,\frac{EI}{mL^3}$$

11.7-2

Let $\bar{\underset{\sim}{D}}_i^*$ be a vector before normalization.

Evaluate c in $(\bar{\underset{\sim}{D}}_i^*)^T \underset{\sim}{M} \bar{\underset{\sim}{D}}_i^* = c$

Scaled vector $\bar{\underset{\sim}{D}}_i = \frac{1}{\sqrt{c}} \bar{\underset{\sim}{D}}_i^*$ will yield $\bar{\underset{\sim}{D}}_i^T \underset{\sim}{M} \bar{\underset{\sim}{D}}_i = 1$

From Prob. 11.4-2a, $\omega_1^2 = 1$, $\omega_2^2 = 6$, and $c = \sqrt{5}$

Now use Eqs. 11.7-5 and 11.7-6

$$[\underset{\sim}{\phi}] = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{1}{\sqrt{5}} \begin{Bmatrix} 2R_1 + R_2 \\ R_1 - 2R_2 \end{Bmatrix}$$

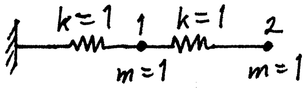
Eq. 13.6-5 yields

$$\ddot{z}_1 + z_1 = (2R_1 + R_2)/\sqrt{5}$$

$$\ddot{z}_2 + 6z_2 = (R_1 - 2R_2)/\sqrt{5}$$

$\boxed{11.7\text{-}4}$

(a)



$$\left(\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix} - \omega^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}\bar{u}_1\\ \bar{u}_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Solution of this eigenproblem yields

$\lambda_1 = 0.381966$, $\omega_1 = 0.618034$

$\lambda_2 = 2.61803$, $\omega_2 = 1.618034$

Eigenvectors, respectively un-normalized and normalized, are

$$\begin{Bmatrix}1\\ 1.61803\end{Bmatrix} \rightarrow \begin{Bmatrix}0.52573\\ 0.85065\end{Bmatrix}, \quad \begin{Bmatrix}1\\ -0.618034\end{Bmatrix} \rightarrow \begin{Bmatrix}0.85065\\ -0.52573\end{Bmatrix}$$

Hence $[\underset{\sim}{\phi}] = \begin{bmatrix}0.52573 & 0.85065\\ 0.85065 & -0.52573\end{bmatrix}$

Initial conditions, using Eq. 11.7-4, are

$$\begin{Bmatrix}z_1\\ z_2\end{Bmatrix}_0 = \begin{Bmatrix}0\\ 0\end{Bmatrix}, \quad \begin{Bmatrix}\dot{z}_1\\ \dot{z}_2\end{Bmatrix}_0 = [\underset{\sim}{\phi}]^T\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}0\\ 1\end{Bmatrix} = \begin{Bmatrix}0.85065\\ -0.52573\end{Bmatrix}$$

With $\xi_i = p_i = 0$, Eq. 11.7-6 has the solution

$$z_i = A_i \sin\omega_i t + B_i \cos\omega_i t$$

$B_i = 0$ because $z_i = 0$ at $t = 0$.

Next, $\dot{z}_i = A_i\omega_i\cos\omega_i t$, and at $t = 0$

$$\left.\begin{aligned}\dot{z}_1 &= 0.85065 = A_1(0.618)\\ \dot{z}_2 &= -0.52573 = A_2(1.618)\end{aligned}\right\} \begin{aligned}A_1 &= 1.3764\\ A_2 &= -0.3249\end{aligned}$$

$z_1 = 1.3764 \sin 0.618t$

$z_2 = -0.3249 \sin 1.618t$

By Eq. 11.7-4, $\{\underset{\sim}{D}\} = [\underset{\sim}{\phi}]^T\{\underset{\sim}{z}\}$ yields

$$\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}0.7236 \sin 0.618t - 0.2764 \sin 1.618t\\ 1.171 \sin 0.618t + 0.1708 \sin 1.618t\end{Bmatrix}$$

	t=0	t=1	t=2	t=3	t=4	t=5
u_1	0	.1432	.7095	.9684	.3972	-.2315
u_2	0	.8491	1.090	.9552	.7589	.2262

(c) Compare greatest magnitudes, regardless of the times at which they appear.

$$\frac{0.7236}{0.7236 + 0.2764} = 0.7236 \quad (27.6\,\%\text{ error in } u_1)$$

$$\frac{1.171}{1.171 + 0.1708} = 0.8727 \quad (12.7\,\%\text{ error in } u_2)$$