Advanced Mechanical Vibration
EGME 511
California State University, Fullerton

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Contents

Chapter 1

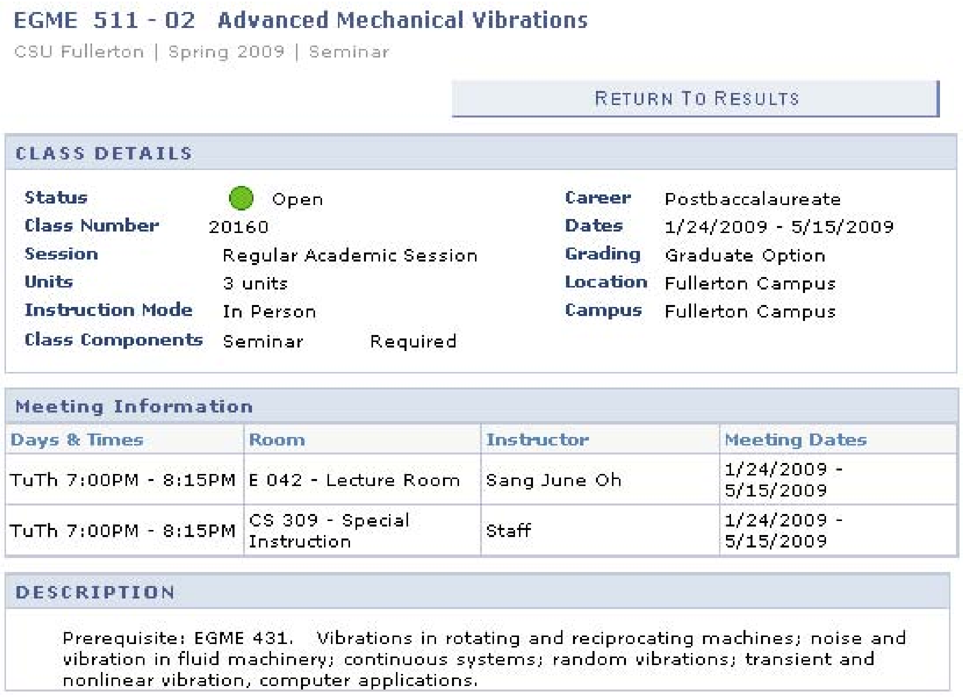
Introduction

Local contents

I took this course in Spring 2009 at CSUF. Not part of a degree program

1.1 course description

course description from catalog:



EGME 511 - 02 Advanced Mechanical Vibrations
CSU Fullerton | Spring 2009 | Seminar

RETURN TO RESULTS

CLASS DETAILS

Status	Open	**Career**	Postbaccalaureate
Class Number	20160	**Dates**	1/24/2009 - 5/15/2009
Session	Regular Academic Session	**Grading**	Graduate Option
Units	3 units	**Location**	Fullerton Campus
Instruction Mode	In Person	**Campus**	Fullerton Campus
Class Components	Seminar Required		

Meeting Information

Days & Times	Room	Instructor	Meeting Dates
TuTh 7:00PM - 8:15PM	E 042 - Lecture Room	Sang June Oh	1/24/2009 - 5/15/2009
TuTh 7:00PM - 8:15PM	CS 309 - Special Instruction	Staff	1/24/2009 - 5/15/2009

DESCRIPTION

Prerequisite: EGME 431. Vibrations in rotating and reciprocating machines; noise and vibration in fluid machinery; continuous systems; random vibrations; transient and nonlinear vibration, computer applications.

Figure 1.1: class info

1.2 Textbook

Figure 1.2: Text book

Chapter 2

sheetsheet

$$\int_a^b e^{c\tau}\sin(t-\tau)\,d\tau = \frac{c[e^{c\tau}\sin(t-\tau)]_a^b + [e^{c\tau}\cos(t-\tau)]_a^b}{(1+c^2)}$$

$$\int_a^b e^{c\tau}\cos(t-\tau)\,d\tau = \frac{c[e^{c\tau}\cos(t-\tau)]_a^b - [e^{c\tau}\sin(t-\tau)]_a^b}{1+c^2}$$

$$\int \cos at = \frac{\sin at}{a}$$

$$\int \sin at = \frac{-\cos at}{a}$$

F(t)	Guess
ke^{bt}	Ae^{bt}
kt^n	$A_n t^n + \cdots + A_0$
$\cos\omega t$ or $\sin\omega$	$c_1\cos\omega t + c_2\sin\omega t$
$ke^{at}\cos\omega t$	$e^{at}(c_1\cos\omega t + c_2\sin\omega t)$
$x'' + x' + x = f(t)$	$[s^2X - sx(0) - x'(0)]$
$L(t) = \int_0^\infty x(t)\,e^{-st}dt$	$+[sX - x(0)] + X = F$

roots $\int udv = uv - \int vdu$	$x(t)$
real and distinct	$Ae^{\lambda_1 t} + Be^{\lambda_2 t}$
double real	$Ae^{\lambda t} + Bte^{\lambda t}$
complex $\alpha \pm j\beta$	$e^{\alpha t}(A\cos\beta t + B\sin\beta t)$
$x(t) = A\cos\omega_n t + B\sin\omega_n t$	$x(t) = C\sin(\omega_n t + \theta)$
$A = u(0)\; B = \frac{v(0)}{\omega_n}, \theta = \arctan\left(\frac{A}{B}\right)$	$C = \sqrt{A^2+B^2}\;\; P = [v_1 v_2]$

$\begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix}$

modal: $x = M^{-\frac{1}{2}}q$

Find eigenvalus of

$\tilde{k} = M^{-\frac{1}{2}}kM^{-\frac{1}{2}}$

$r'' = \Lambda r, q = P\; r,$

$P^T = P^{-1}(orthon.)$

$r(0) = P^T M^{\frac{1}{2}}x(0)$

$x(t) = M^{-\frac{1}{2}}P\; r(t)$

$\omega_n^2 = \frac{k}{m}$
$r = \frac{\omega}{\omega_n}$
$\zeta = \frac{c}{c_{cr}}$
$c_{cr} = 2\sqrt{km} = 2\omega_n m$
$\omega_d = \omega_n\sqrt{1-\zeta^2}\quad \zeta < 1$

$mx''(t)+kx(t)=0 \rightarrow x(t)=e^{-\zeta\omega_n t}(A\cos\omega_d t+B\sin\omega_d t)$ or $x(t)=Ce^{-\zeta\omega_n t}\sin(\omega_d t-\theta)$
$A=u(0)\,,B=\frac{v(0)+x(0)\zeta\omega_n}{\omega_d},C=\sqrt{A^2+B^2},\theta=\tan^{-1}\left(\frac{B}{A}\right)$
$L=T-U,\ \frac{d}{dt}\frac{\partial L}{\partial q_i'}-\frac{\partial L}{\partial q_i}=Q_i,$ $\boxed{\text{Rayleigh } \frac{d}{dt}\frac{\partial L}{\partial q'}-\frac{\partial L}{\partial q}+\frac{\partial R}{\partial q'}=0 \text{ where } R=\frac{1}{2}c(q')^2}$ $\boxed{\delta W=Q_i\delta q_i}$
$mx''(t)+kx(t)=F_0\sin\omega t \rightarrow x(t)=A\cos\omega_n t+B\sin\omega_n t+\frac{F_0}{k}\frac{1}{1-r^2}\sin\omega t, A=x(0)\,,\ B=\frac{v(0)}{\omega_n}-\frac{r}{1-r^2}$
$mx''(t)+cx'(t)+kx(t)=F_0\sin\omega t \rightarrow x(t)=e^{-\zeta\omega_n t}(A\cos\omega_d t+B\sin\omega_d t)+\frac{F_0}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t-\theta)$
$\theta=\arctan\left(\frac{2\zeta r}{1-r^2}\right),\ \lambda_{1,2}=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$ where $ax^2+bx+c=0,$

$\sin 2A$	$2\sin A\cos A$
$\cos 2A$	$2\cos^2 A-1$
$\sin A\sin B$	$\frac{1}{2}(\cos(A-B)-\cos(A+B))$
$\cos A\cos B$	$\frac{1}{2}(\cos(A-B)+\cos(A+B))$
$\sin A\cos B$	$\frac{1}{2}(\sin(A-B)+\sin(A+B))$
$h=v_i t+\frac{1}{2}gt^2$	$h=\frac{v_i+v_f}{2}t$
$v_f^2=v_i^2+2gh$	$v_f=v_i+gt$
speed is $\sqrt{2gh}$	$h_u(t)=\frac{\hat{F}}{m\omega_n}\sin\omega_n t$
	$\hat{F}=F\Delta t=mv$

phase roots λ_1 and $\lambda_2>0$	Unstable, repelling
phase roots λ_1 and $\lambda_2<0$	stable, attracting
both real, one >0 and one <0	unstable saddle point
equal roots and >0	unstable, degenrate
equal roots and <0	stable, degenrate
complex, real part>0	unstabe, spiral out
complex, real part<0	stable, spiral in
pure complex conjugrates	marginaly stable, cirlce

time betwen$\frac{x_2^2}{2}-\frac{g}{l}\cos x_1=c$
$\frac{dx_1}{dt}=\pm\sqrt{c_1+\frac{2g}{l}\cos x_1}$
$t=t_0+\int_{x_1(t_0)}^{x_1(t)}\frac{dx_1}{\sqrt{c_1+\frac{2g}{l}\cos x_1}}$
convert: $x''+kx=0$
$\frac{dx_2}{dt}+kx_1=0,\frac{dx_2}{dx_1}\frac{dx_1}{dt}=-kx_1$
$\frac{dx_2}{dx_1}x_2=-kx_1$
$\frac{x_2^2}{2}=-k\frac{x_1^2}{2}+C$
$h_d(t)=\frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$

$\sin(a\pm b)$	$\sin a\cos b\pm\cos a\sin b$
$\cos(a\pm b)$	$\cos a\cos b\mp\sin a\sin b$
$\sin^2 a$	$\frac{1}{2}(1-\cos 2a)$

$\cos^2 a=\frac{1}{2}(1+\cos 2a)$	$sin(A\pm 90)=\cos A$
$\cos\left(a\pm 90^0\right)=\mp\sin a$	$sin(A\pm 180)=\mp\sin A$
$\cos\left(a\pm 90^0\right)=\cos a$	$\cos(A\pm 180)=-\cos A$

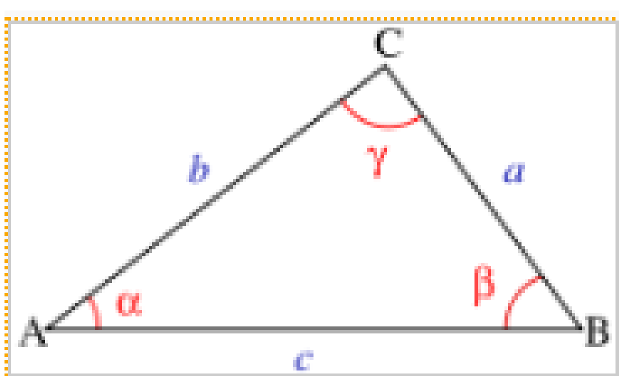


Figure 2.1: laws of cosine

$c^2=a^2+b^2-2ab\cos(\lambda)$
$\lim_{t->0^+} f(t)=\lim_{s->\infty} sF(s)$
$\lim_{s->0} sF(s)=\lim_{t->\infty} f(t)$

$M\ddot{x}+kx=0$, assume $x_i=A_i\cos\left(\omega t+\phi_i\right)$,Plug in, rewrite as $[sys]\,[A]=0$, find eigens of sys,each ω_i,find $r_1=\left(\frac{A_2^{(1)}}{A_1^{(1)}}\right)$, r_2
$x_1=A_1^{(1)}\cos\left(\omega_1 t+\phi_1\right)+A_1^{(2)}\cos\left(\omega_2 t+\phi_2\right),\quad x_2=A_2^{(1)}\cos\left(\omega_1 t+\phi_1\right)+A_2^{(2)}\cos\left(\omega_2 t+\phi_2\right)$, use $A_2^{(1)}=r_1A_1^{(1)}$, $A_2^{(2)}=r_2$
$x_1=A_1^{(1)}\cos\left(\omega_1 t+\phi_1\right)+A_1^{(2)}\cos\left(\omega_2 t+\phi_2\right),\quad x_2=r_1A_1^{(1)}\cos\left(\omega_1 t+\phi_1\right)+r_2A_1^{(2)}\cos\left(\omega_2 t+\phi_2\right)$

$g(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n(2\pi f)\,t+b_n\sin n(2\pi f)\,t$
$a_0=\frac{1}{T/2}\int_0^T f(t)\qquad b_n=\frac{1}{T/2}\int_0^T f(t)\sin n(2\pi f)\,tdt$
$a_n=\frac{1}{T/2}\int_0^T f(t)\cos n(2\pi f)\,tdt\qquad T=period\ of\ f(t)$

$h_{over}(t)=\frac{1}{2m\omega_n\sqrt{\xi^2-1}}e^{-\xi\omega_n t}\left(e^{\omega_n\sqrt{\xi^2-1}t}-e^{-\omega_n\sqrt{\xi^2-1}t}\right)$
$h_{critical}(t)=\frac{1}{m}te^{-\xi\omega_n t}$
$f(t)$ =impulse= $F\Delta t=\left[mv(0^-)-mv(0^+)\right]\delta(t)$

solid disk, around center $I=\frac{mr^2}{2}$

thin loop, around center $I=mr^3$

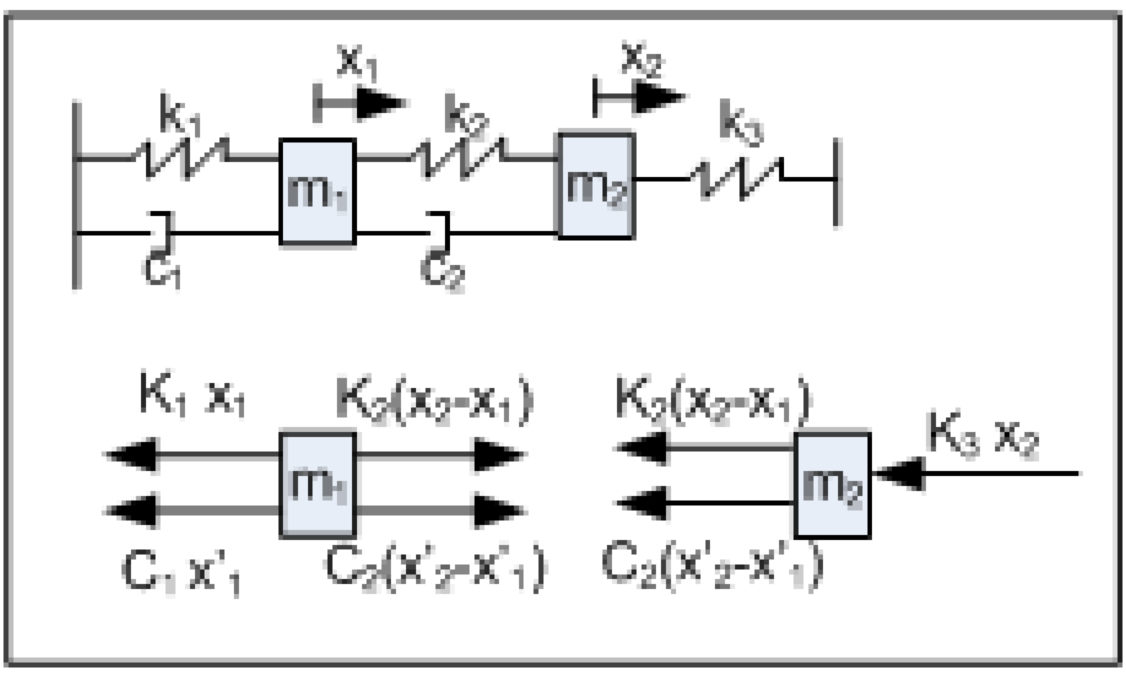
solid sphere $I=\frac{2}{5}mr^2$

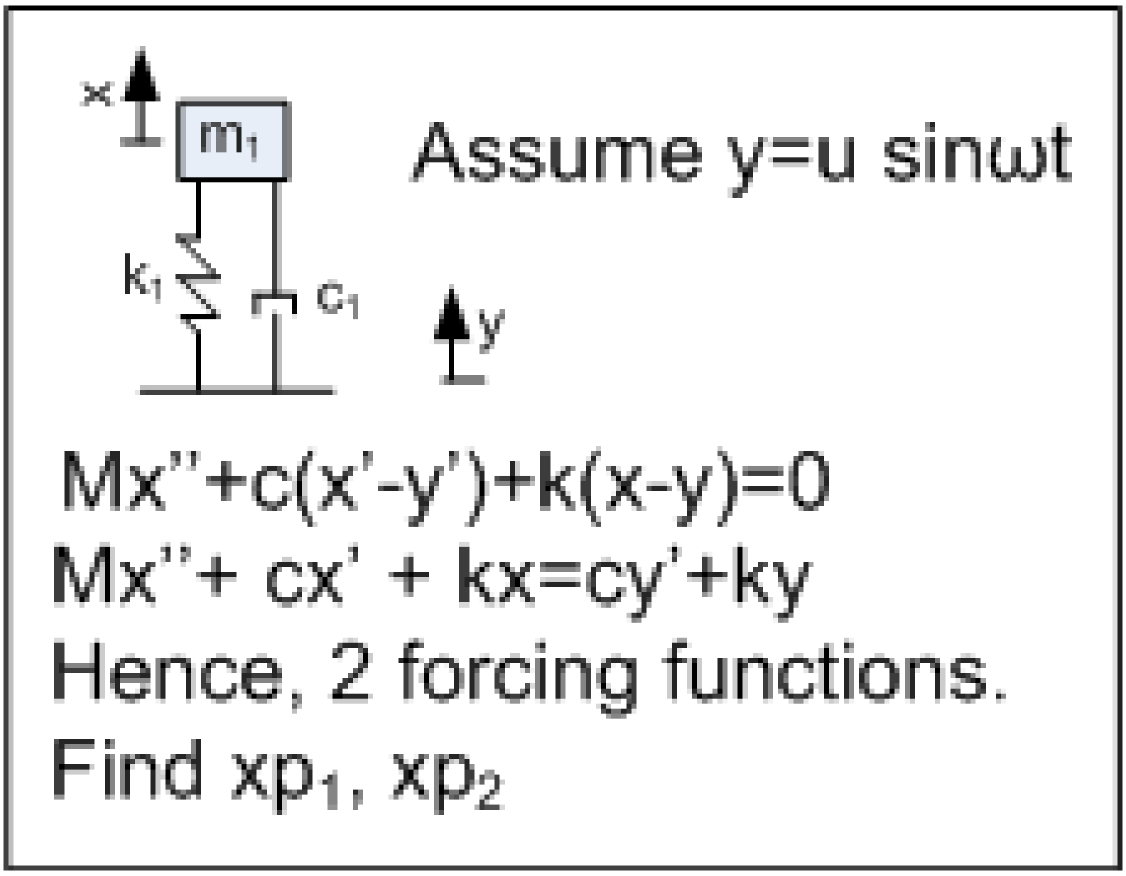
rod, axis at center of rod $I=\frac{ML^2}{12}$

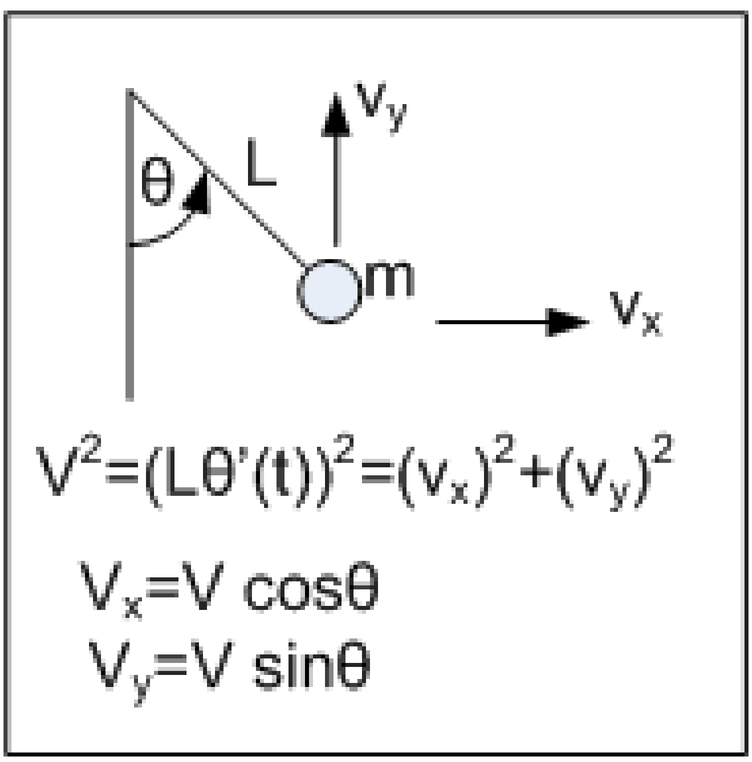
rod, axis at end of rod $I=\frac{ML^2}{3}$

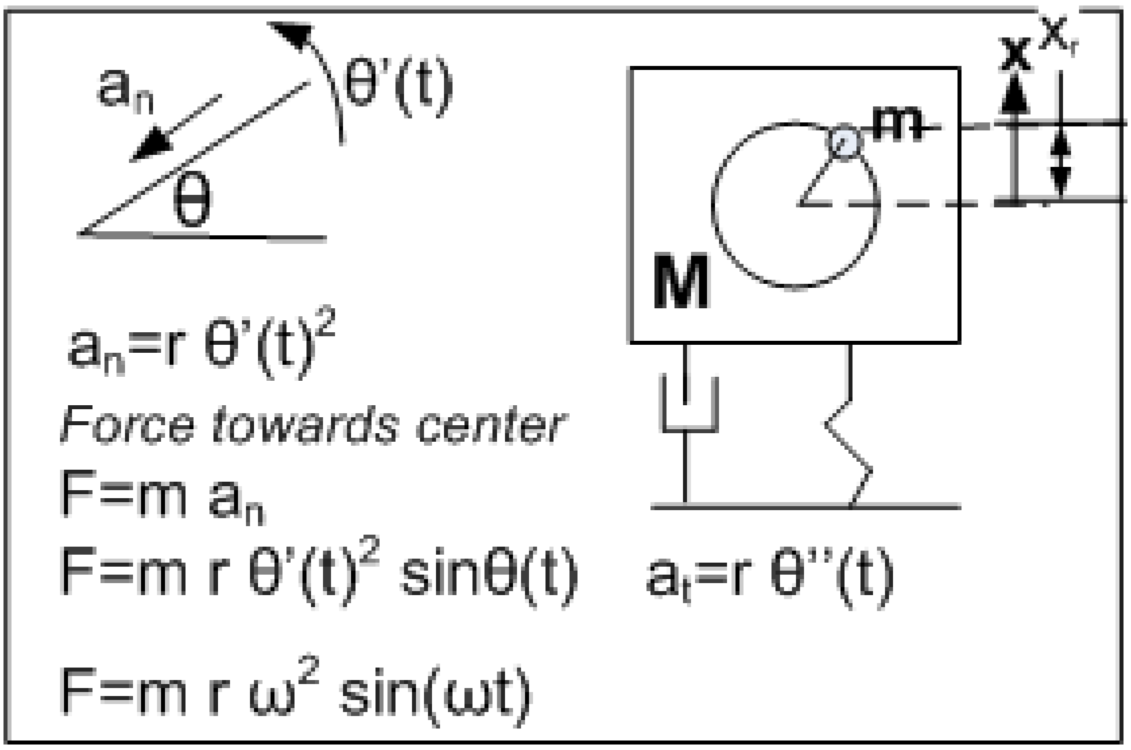
series: $\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2}$par $k=k_1+k_2$

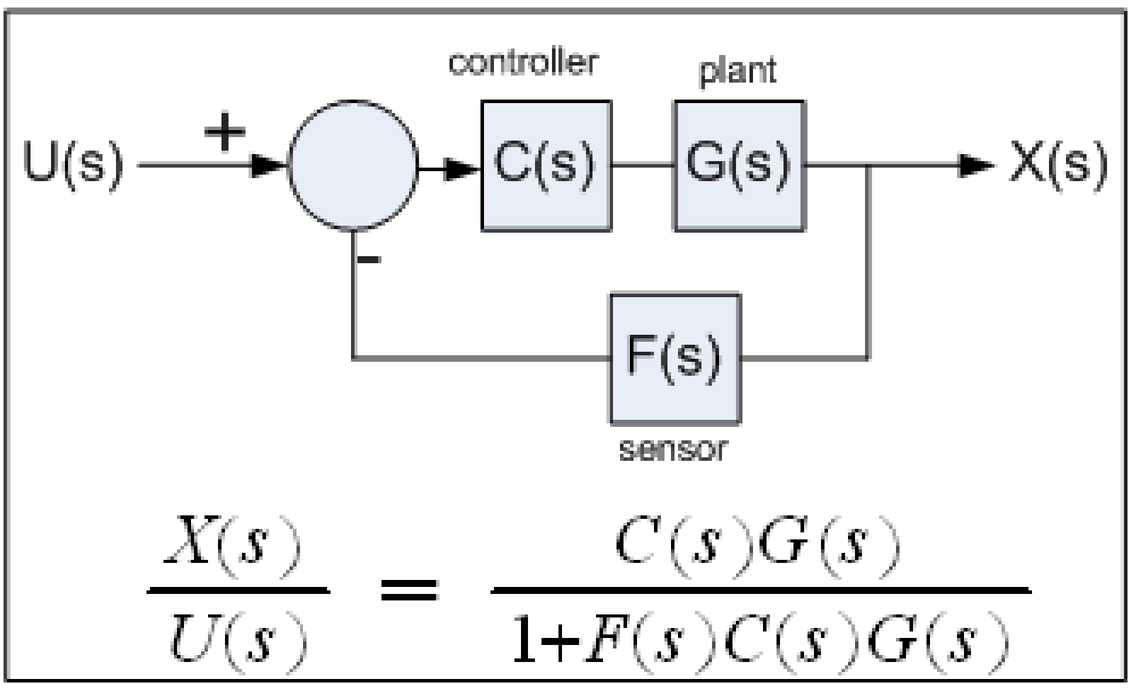
$\int_a^b \tau\sin\omega(t-\tau)\,d\tau=\frac{-\sin(\omega(t-a))-a\omega\cos(\omega(t-a))+\sin(\omega(t-b))+b\omega\cos(\omega(t-b))}{\omega}$




x
m_1
Assume y=u sinωt
k_1
c_1
y
Mx''+c(x'-y')+k(x-y)=0
Mx''+ cx' + kx=cy'+ky
Hence, 2 forcing functions.
Find xp_1, xp_2


v_y
θ
L
m
v_x
$V^2=(L\theta'(t))^2=(v_x)^2+(v_y)^2$
$V_x=V\cos\theta$
$V_y=V\sin\theta$





2 equations of motions for unbalanced: $(M-m)\,\ddot{x}+c\dot{x}+kx=F_r$ and $m(\ddot{x}+\ddot{x}_r)=-F_r$, where $x_r=e\sin\omega t$, eq for M is

$M\ddot{x}+c\dot{x}+kx=me\omega^2\sin\omega t$, guess $X_p=X\sin(\omega t-\theta)$, we obtain $X=\frac{Me}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\xi r)^2}},\theta=\tan^{-1}\frac{2\xi r}{1-r^2}$ perturbation: $x''+\omega_0^2x+\alpha x^3=0\rightarrow x=x_0+\alpha x_1+\alpha^2x_2+\cdots,\omega^2=\omega_0^2+\alpha\omega_1^2(A)+\alpha\omega_2^2(A)+\cdots$,hence $\omega_0^2=\omega^2-\alpha\omega_1^2(A)$. Sub in ODE, generate 2 ODE's and solve for x_0 and use result to find x_1.watch for IC and resonanse. For system ID, set up $|G(j\omega)|=\frac{1}{\sqrt{(c\omega)^2+(k-m\omega^2)^2}}$ and from the spectrum, find m,c,k

Chapter 3

HWs

Local contents

3.1 HW1

Local contents

3.1.1 Description of HW

1. Solve 2nd order ODE
2. Calculate maximum value of the peak response (magnification factor) for a system with some damping ratio given (Quadrature peak picking method)
3. Solve for the forced response of a single-degree-of-freedom system to a harmonic excitation
4. Discuss the stability of 2nd order ODE
5. Find range of values for PD controller in feedback for stability
6. Compute a feedback law with full state feedback
7. Find the equilibrium points of the nonlinear pendulum equation

3.1.2 Problem 1.4

Solve $\ddot{x} - \dot{x} + x = 0$ with $x_0 = 1$ and $v_0 = 0$ for $x(t)$ and sketch the solution

Answer

$$x = x_h + x_p$$

Since there is no forcing function, x_p do not exist, hence $x = x_h$. To determine x_h we first find the characteristic equation and find its root. The characteristic equation is $\lambda^2 - \lambda + 1 = 0$ which has solutions

$$\lambda_1 = \frac{1}{2} + j\frac{\sqrt{3}}{2}$$
$$\lambda_2 = \frac{1}{2} - j\frac{\sqrt{3}}{2}$$

This is of the form $\lambda = \alpha \pm \beta j$ (complex conjugates) which has the solution

$$x(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$$

Hence

$$\boxed{x(t) = e^{\frac{1}{2}t}\left(A\cos\tfrac{\sqrt{3}}{2}t + B\sin\tfrac{\sqrt{3}}{2}t\right)}$$

To find A and B we use the initial conditions. At $t = 0$, $x(0) = 1$, hence

$$\boxed{A = 1}$$

Now

$$\dot{x}(t) = \frac{1}{2}e^{\frac{1}{2}t}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right) + e^{\frac{1}{2}t}\left(-A\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t + B\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t\right)$$

At $t = 0$, $v_0 = 0$, hence the above becomes

$$0 = \frac{1}{2}A + B\frac{\sqrt{3}}{2}$$

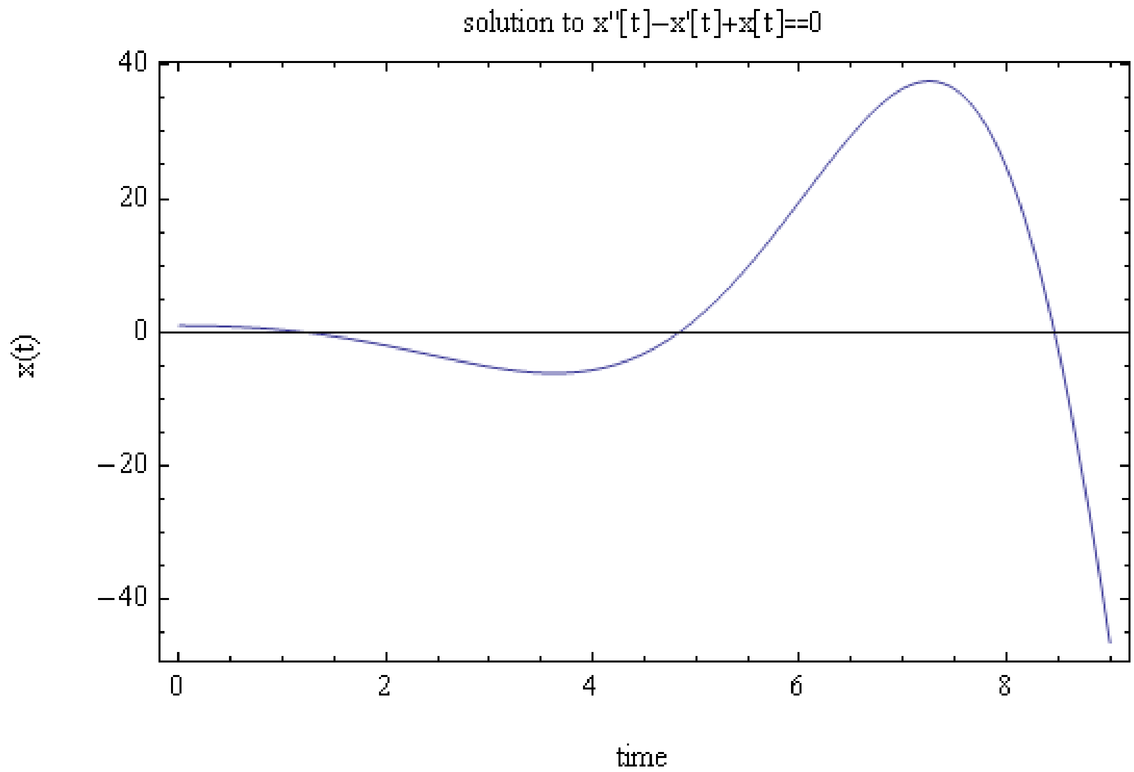
But $A = 1$, hence

$$\boxed{B = -\tfrac{1}{\sqrt{3}}}$$

Then the solution is

$$\boxed{x(t) = e^{\frac{1}{2}t}\left(\cos\tfrac{\sqrt{3}}{2}t - \tfrac{1}{\sqrt{3}}\sin\tfrac{\sqrt{3}}{2}t\right)}$$

The solution will blow up in oscillatory fashion due to the exponential term at the front. This is a plot for up to $t = 10$



3.1.3 Problem 1.9

Calculate the maximum value of the peak response (magnification factor) for the system in figure 1.18 with $\zeta = \frac{1}{\sqrt{2}}$

Solution

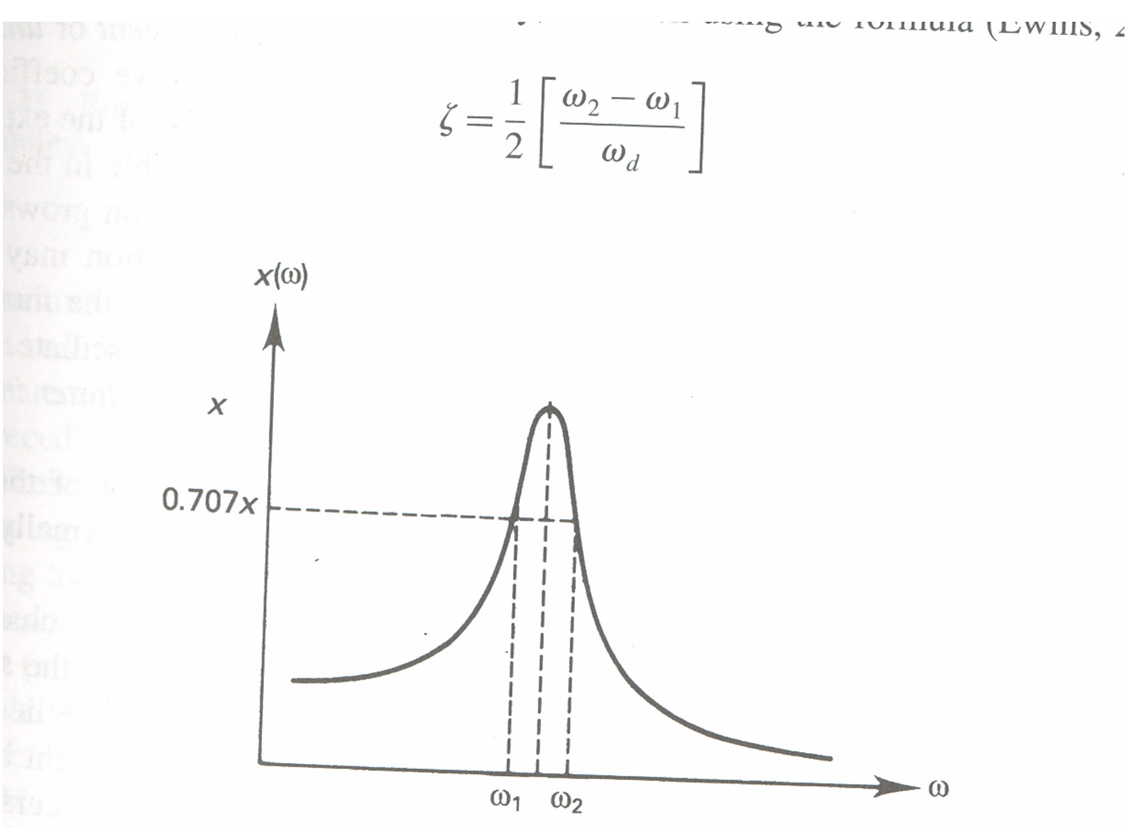


Figure 1.18 Quadrature peak picking method.

In this figure, the y-axis is the magnitude of the frequency response of the second order system. Hence we must first calculate the frequency response of the system

$$m\ddot{x} + c\dot{x} + kx = u(t)$$

or

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{u(t)}{m}$$

Take Laplace transform

$$\begin{aligned} s^2X(s)+2\xi\omega_n sX(s)+\omega_n^2X(s) &= \frac{1}{m}U(s) \\ X(s)\left(s^2+2\xi\omega_n s+\omega_n^2\right) &= \frac{1}{m}U(s) \end{aligned}$$

Hence the transfer function is

$$Z(s)=\frac{X(s)}{U(s)}=\frac{1}{m}\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}$$

Let $s=j\omega$, the above becomes the frequency response

$$\begin{aligned} Z(j\omega) &= \frac{1}{m}\left(\frac{1}{-\omega^2+2j\xi\omega_n\omega+\omega_n^2}\right) \\ &= \frac{1}{m\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}+2j\xi\frac{\omega}{\omega_n}\right)} \end{aligned}$$

But $\omega_n^2=\frac{k}{m}$, hence

$$Z(j\omega)=\frac{1/k}{1-\frac{\omega^2}{\omega_n^2}+2j\xi\frac{\omega}{\omega_n}}$$

Introduce $G(j\omega)\equiv kZ(j\omega)$ and let $r=\frac{\omega}{\omega_n}$

$$\boxed{G(j\omega)=\frac{1}{1-r^2+2j\xi r}}$$

Now we can determine the magnitude of the frequency response

$$\begin{aligned} |G(j\omega)| &= \sqrt{G(j\omega)\,G^*(j\omega)} \\ &= \left[\left(\frac{1}{1-r^2+2j\xi r}\right)\left(\frac{1}{1-r^2-2j\xi r}\right)\right]^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{\left(1-r^2\right)^2+\left(2\xi r\right)^2}} \end{aligned}$$

The maximum of $|G(j\omega)|$ occurs when $\frac{d|G(j\omega)|}{d\omega}=0$ But

$$\frac{d|G(j\omega)|}{d\omega}=-\frac{1}{2}\frac{2\left(1-r^2\right)(-2r)+4\xi^2(2r)}{\left[\left(1-r^2\right)^2+\left(2\xi r\right)^2\right]^{\frac{3}{2}}}$$

Hence for the above to be zero, set the numerator to zero, we obtain

$$\begin{aligned} 2\left(1-r^2\right)(-2r)+4\xi^2(2r) &= 0 \\ -\left(1-r^2\right)r+2\xi^2 r &= 0 \\ -1+r^2+2\xi^2 &= 0 \end{aligned}$$

Hence the maximum of $|G(j\omega)|$ occurs at

$$\boxed{r_{\max}=\frac{\omega}{\omega_n}=\sqrt{1-2\xi^2}}$$

The above is valid only when $1-2\xi^2>0$ which means $\xi<\frac{1}{\sqrt{2}}$.

Now substitute $r_{\max}$ value into $|G(j\omega)|$ we obtain

$$\begin{aligned} |G(j\omega)|_{\max} &= \left(\frac{1}{\sqrt{\left(1-r^2\right)^2+\left(2\xi r\right)^2}}\right)_{r=r_{\max}} \\ &= \frac{1}{\sqrt{\left(1-\left(1-2\xi^2\right)\right)^2+\left(2\xi\sqrt{1-2\xi^2}\right)^2}} \\ &= \frac{1}{\sqrt{4\xi^4+4\xi^2\left(1-2\xi^2\right)}} \\ &= \frac{1}{\sqrt{4\xi^4+4\xi^2-8\xi^4}} \\ &= \frac{1}{\sqrt{4\xi^2-4\xi^4}} \\ &= \boxed{\frac{1}{2\xi\sqrt{1-\xi^2}}} \end{aligned}$$

We are given $\xi = \frac{1}{\sqrt{2}}$, hence from the above

$$|G(j\omega)|_{\max} = \frac{\sqrt{2}}{2\sqrt{1-\frac{1}{2}}} = \frac{\sqrt{2}}{2\sqrt{\frac{1}{2}}} = \frac{\sqrt{2}}{\sqrt{2}}$$

Hence

$$\boxed{|G(j\omega)|_{\max} = 1}$$

But $G(j\omega) = kZ(j\omega)$, hence

$$\boxed{|Z(j\omega)|_{\max} = \frac{1}{k}}$$

Note that $|G(j\omega)|_{\max}$ is called the quality factor. Hence for different values of ξ there will be a different quality factor value.

3.1.4 Problem 1.12

Solve for the forced response of a single-degree-of-freedom system to a harmonic excitation with $\xi = 1.1$ and $\omega_n^2 = 4$. Plot the magnitude of the steady state response versus the driving frequency. For what values of ω_n is the response maximum?

Answer Since the excitation is harmonic, assume it has the form $F\sin\omega t$ where ω is the deriving frequency. Then the equation of motion for the SDOF system is

$$m\ddot{x} + c\dot{x} + kx = F\cos\omega t$$

Dividing by m and using $\omega_n^2 = \sqrt{\frac{k}{m}}$ and $\xi = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}}$ the above becomes

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = f_0\cos\omega t \tag{1}$$

Where $f_0 = \frac{F}{m}$

Since this is an overdamped system ($\xi > 1$), then the transient solution is

$$x_h(t) = e^{-\xi\omega_n t}\left(Ae^{-\omega_n t\sqrt{\xi^2-1}} + Be^{\omega_n t\sqrt{\xi^2-1}}\right)$$

But we need only consider the particular solution since we are asked to plot the steady state solution. Assume

$$\boxed{x_p(t) = c_1\cos\omega t + c_2\sin\omega t}$$

Then

$$\dot{x}_p(t) = -\omega c_1\sin\omega t + c_2\omega\cos\omega t$$
$$\ddot{x}_p(t) = -\omega^2 c_1\cos\omega t - c_2\omega^2\sin\omega t$$

Substitute $x_p(t), \dot{x}_p(t), \ddot{x}_p(t)$ in (1) we obtain

$$\left(-\omega^2 c_1\cos\omega t - c_2\omega^2\sin\omega t\right) + 2\xi\omega_n(-\omega c_1\sin\omega t + c_2\omega\cos\omega t) + \omega_n^2(c_1\cos\omega t + c_2\sin\omega t) = f_0\cos\omega t$$
$$\left(-c_2\omega^2 - 2\xi\omega_n\omega c_1 + c_2\omega_n^2\right)\sin\omega t + \left(-\omega^2 c_1 + 2\xi\omega_n c_2\omega + \omega_n^2 c_1\right)\cos\omega t = f_0\cos\omega t$$

Hence by comparing coefficients in the LHS and RHS we obtain 2 equations to solve for c_1 and c_2

$$-c_2\omega^2 - 2\xi\omega_n\omega c_1 + c_2\omega_n^2 = 0$$
$$-\omega^2 c_1 + 2\xi\omega_n c_2\omega + \omega_n^2 c_1 = f_0$$

or

$$c_1(-2\xi\omega_n\omega) + c_2\left(\omega_n^2 - \omega^2\right) = 0 \tag{2}$$
$$c_1\left(\omega_n^2 - \omega^2\right) + c_2(2\xi\omega_n\omega) = f_0 \tag{3}$$

From (2) we obtain $c_1 = \frac{c_2\left(\omega_n^2-\omega^2\right)}{2\xi\omega_n\omega}$, and substitute this into (3)

$$\left(\frac{c_2\left(\omega_n^2-\omega^2\right)}{2\xi\omega_n\omega}\right)\left(\omega_n^2-\omega^2\right) + c_2(2\xi\omega_n\omega) = f_0$$
$$c_2\left[\frac{\left(\omega_n^2-\omega^2\right)^2}{2\xi\omega_n\omega} + 2\xi\omega_n\omega\right] = f_0$$
$$c_2\left[\left(\omega_n^2-\omega^2\right)^2 + 4\xi^2\omega_n^2\omega^2\right] = 2\xi\omega_n\omega f_0$$

Hence

$$\boxed{c_2 = \frac{2\xi\omega_n\omega f_0}{(\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2}}$$

Substitute the above into (2) we solve for c_1

$$c_1(-2\xi\omega_n\omega) + \left(\frac{2\xi\omega_n\omega f_0}{(\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2}\right)(\omega_n^2-\omega^2) = 0$$

or

$$\boxed{c_1 = \frac{f_0(\omega_n^2-\omega^2)}{(\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2}}$$

Hence, since

$$x_p(t) = c_1\cos\omega t + c_2\sin\omega t$$

Then

$$\boxed{x_p(t) = \frac{f_0(\omega_n^2-\omega^2)}{(\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2}\cos\omega t + \frac{2\xi\omega_n\omega f_0}{(\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2}\sin\omega t}$$

We can convert the above to the form $x_p(t) = c\cos(\omega t-\theta)$ by using the relation $c = \sqrt{c_1^2+c_2^2}$ and $\tan\theta = \frac{c_2}{c_1}$, hence

$$\begin{aligned}
c &= \sqrt{\left(\frac{f_0(\omega_n^2-\omega^2)}{(\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2}\right)^2 + \left(\frac{2\xi\omega_n\omega f_0}{(\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2}\right)^2} \\
&= f_0\sqrt{\frac{(\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2}{\left((\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2\right)^2}} \\
&= \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2+(2\xi\omega_n\omega)^2}}
\end{aligned}$$

The last equation can be written as

$$\begin{aligned}
c &= \frac{F/m}{\omega_n^2\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}} \\
&= \boxed{\frac{F/k}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}}
\end{aligned}$$

And

$$\begin{aligned}
\tan\theta = \frac{c_2}{c_1} &= \frac{\left(\frac{2\xi\omega_n\omega f_0}{(\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2}\right)}{\left(\frac{f_0(\omega_n^2-\omega^2)}{(\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2}\right)} \\
&= \frac{2\xi\omega_n\omega}{(\omega_n^2-\omega^2)} \\
&= \frac{2\xi\frac{\omega}{\omega_n}}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}
\end{aligned}$$

Hence

$$\begin{aligned}
x_p(t) &= c\cos(\omega t-\theta) \\
&= \overbrace{\frac{F/k}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}}^{\text{magnitude}}\cos\left(\omega t-\tan^{-1}\left(\frac{2\xi\frac{\omega}{\omega_n}}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}\right)\right) \qquad (4)
\end{aligned}$$

Let $r = \frac{\omega}{\omega_n}$, then the above becomes

$$\boxed{x_p(t) = \frac{F/k}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\cos\left(\omega t-\tan^{-1}\left(\frac{2\xi r}{(1-r^2)}\right)\right)}$$

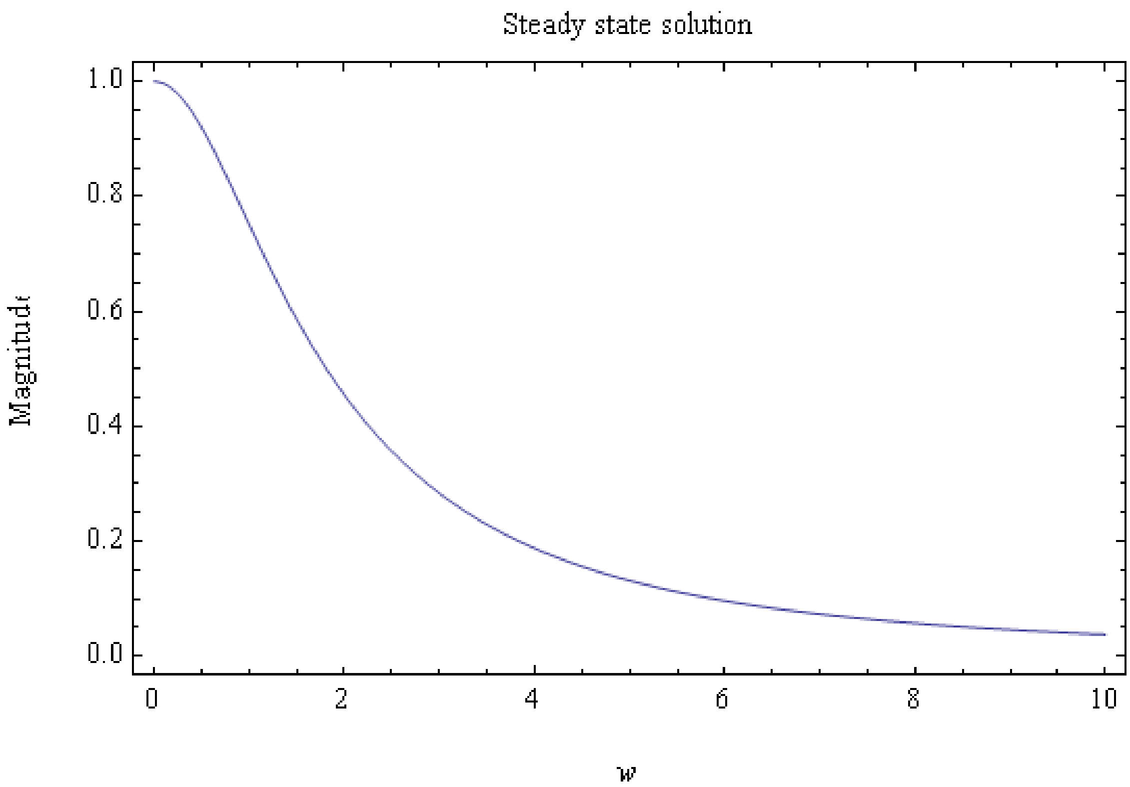
For the supplied values for $\omega_n^2 = 4$ and $\xi = 1.1$then the above steady state solution becomes

$$x_p(t)=\overbrace{\frac{F/k}{\sqrt{\left(1-\frac{\omega^2}{4}\right)^2+1.21\omega^2}}}^{X}\cos\left(\omega t-\tan^{-1}\left(\frac{1.1\omega}{\left(1-\frac{\omega^2}{4}\right)}\right)\right)$$

To plot the magnitude, use a normalized $F=1$, and let $k=1$, use the supplied values for $\omega_n^2=4$ and $\xi=1.1$, hence magnitude X of steady state response is

$$X=\frac{1}{\sqrt{\left(1-\frac{\omega^2}{4}\right)^2+1.21\omega^2}}$$

Plot the expression for the magnitude X against the driving frequency ω



To answer the final question about the resonance. Looking at the steady state solution in equation (4), we see that the amplitude of the x_p is $\frac{F/k}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$ which is maximum when the denominator is minimum which occurs as ω approaches ω_n, but in this problem since the system is overdamped, hence no oscillation will occur and $\boxed{\text{the maximum response occurs when } \omega=0}$ (i.e. input is non oscillatory).

3.1.5 Problem 1.18

Discuss the stability of following system $2\ddot{x} - 3\dot{x} + 8x = -3\dot{x} + \sin 2t$

Answer

The system can be rewritten as

$$2\ddot{x} + 8x = \sin 2t$$

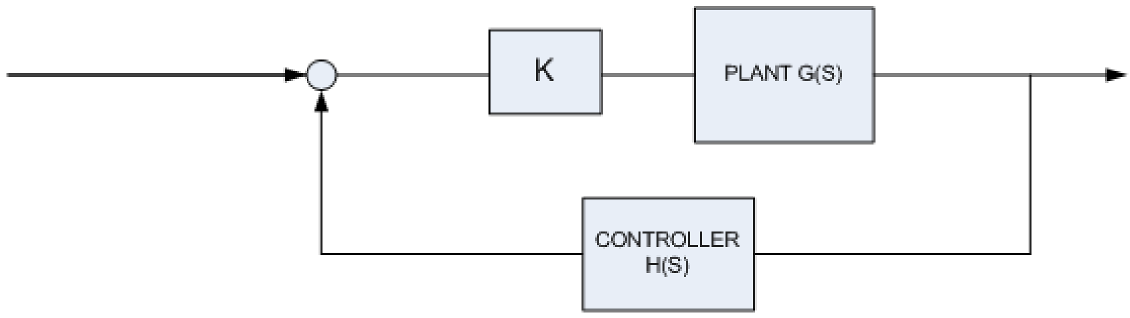
We need to consider only the transient response (homogeneous solution). Hence the characteristic equation is

$$2\lambda^2 + 8 = 0$$

which has roots $\pm\sqrt{2}j$. Since the roots are on the j axis, then this is a $\boxed{\text{marginally unstable}}$ system

3.1.6 Problem 1.20

Calculate an allowable range of values for the gains K, g_1, g_2 for the system $2\ddot{x} + 0.8\dot{x} + 8x = f(t)$ such that the closed-loop system is stable and the formulae for the overshoot and peak time of an underdamped system are valid



Closed loop system

The transfer function of the controller (a P.D. controller) is $H(s) = sg_1 + g_2$ and for the plant (the system) the transfer function is $G(s) = \frac{1}{2s^2+0.8s+8}$, hence the closed loop transfer function, which we call $C(s)$, is

$$\begin{aligned}
C(s) &= \frac{kG(s)}{1 + H(s)KG(s)} \\
&= \frac{k\frac{1}{2s^2+0.8s+8}}{1 + k\frac{sg_1+g_2}{2s^2+0.8s+8}} \\
&= \frac{k}{2s^2 + 0.8s + 8 + (sg_1 + g_2)k} \\
&= \frac{k}{2s^2 + (0.8 + kg_1)s + 8 + kg_2}
\end{aligned}$$

The characteristic equation is the denominator of the above transfer function. Hence

$$f(s) = \overbrace{2}^{a} s^2 + \overbrace{(0.8+kg_1)}^{b} s + \overbrace{8+kg_2}^{c}$$

This has roots at

$$\begin{aligned}
\lambda &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-0.8 - kg_1}{4} \pm \frac{\sqrt{(0.8 + kg_1)^2 - 8(8 + kg_2)}}{4} \\
&= -0.2 - \frac{kg_1}{4} \pm \frac{\sqrt{k^2g_1^2 + 1.6kg_1 - 8kg_2 - 15.36}}{4} \\
&= -0.2 - \frac{kg_1}{4} \pm \sqrt{\frac{k^2g_1^2}{16} + 0.1kg_1 - 0.5kg_2 - 0.96}
\end{aligned}$$

The system is stable if the real part of the roots is in the left hand side of the imaginary axis. Hence we require that

$$-0.2 - \frac{kg_1}{4} < 0$$

Which implies $\frac{-kg_1}{4} < 0.2$ or $\frac{kg_1}{4} > -0.2$ or

$$kg_1 > -0.8 \tag{1}$$

and we require that

$$\frac{k^2g_1^2}{16} + 0.1kg_1 - 0.5kg_2 - 0.96 < 0 \tag{2}$$

Using the minimum value for kg_1 which is -0.8 and substitute that in above equation,

$$\begin{aligned}
\frac{0.8^2}{16} + 0.1(0.8) - 0.5kg_2 - 0.96 &< 0 \\
0.04 + 0.08 - 0.96 - 0.5kg_2 &< 0 \\
-0.84 - 0.5kg_2 &< 0 \\
-0.5kg_2 &< 0.84 \\
0.5kg_2 &> -0.84
\end{aligned}$$

Hence

$$\boxed{kg_2 > -0.42}$$

And $k > 0$ (positive gain is assumed). In summary, these are the allowed ranges

$$\begin{aligned} kg_2 &> -0.42 \\ kg_1 &> -0.8 \\ k &> 0 \end{aligned}$$

3.1.7 Problem 1.21

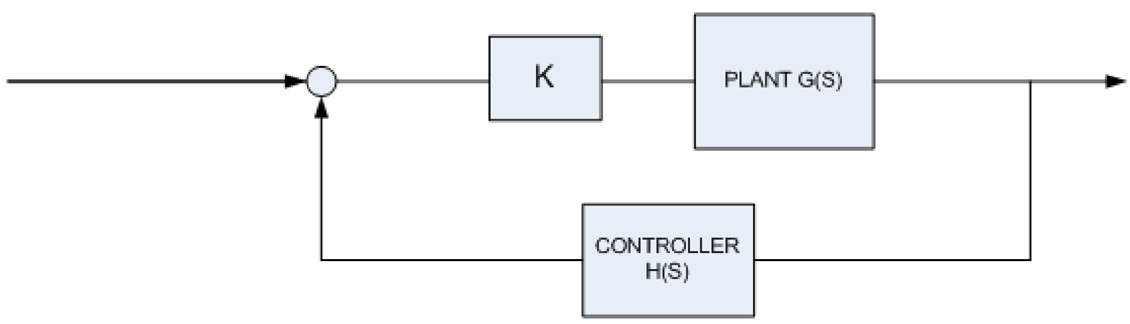
Compute a feedback law with full state feedback (of the form given in equation 1.62 in the book that stabilizes the system $4\ddot{x} + 16x = 0$ and causes the closed loop setting time to be 1 second.

Answer

Equation 1.62 in the book is

$$m\ddot{x} + \left(c + \bar{k}g_1\right)\dot{x} + \left(k + \bar{k}g_2\right)x = \bar{k}f(t)$$

Notice that I modified the notation in this equation, where the lower case k is the stiffness and $\bar{k}$ is the gain, this is to reduce ambiguity in notations



Closed loop system

Using the controller required, the equation $4\ddot{x} + 16x = 0$ becomes

$$m\ddot{x} + \bar{k}g_1\dot{x} + \left(k + \bar{k}g_2\right)x = 0$$

Notice that there is no damping in $4\ddot{x} + 16x = 0$, $(c = 0)$, but now $\bar{k}g_1$term acts in place of the damping. From the original equation $m = 4$ and $k = 16$, hence we can write the above as

$$4\ddot{x} + \bar{k}g_1\dot{x} + \left(16 + \bar{k}g_2\right)x = 0$$

The characteristic equation is

$$4\lambda^2 + \bar{k}g_1\lambda + \left(16 + \bar{k}g_2\right) = 0$$

Hence

$$\begin{aligned} \lambda_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\bar{k}g_1 \pm \sqrt{\left(\bar{k}g_1\right)^2 - 16\left(16 + \bar{k}g_2\right)}}{8} \\ &= \frac{-\bar{k}g_1}{8} \pm \frac{1}{8}\sqrt{\bar{k}^2 g_1^2 - 256 - 16\bar{k}g_2} \end{aligned}$$

Hence for stability, the real part of the root must be negative, hence $\frac{-\bar{k}g_1}{8} < 0$ or $\frac{\bar{k}g_1}{8} > 0$ or $\boxed{\bar{k}g_1 > 0}$

And we require that $\bar{k}^2 g_1^2 - 256 - 16\bar{k}g_2 < 0$ (for oscillation to occur). This implies

$$\bar{k}^2 g_1^2 - 16\bar{k}g_2 < 256 \tag{1}$$

Now settling time is given by

$$t_s = \frac{3.2}{\omega_n \zeta} = \frac{3.2}{\omega_n \frac{c}{c_{cr}}} = \frac{3.2}{\omega_n \frac{c}{2\omega_n m}} = \frac{3.2(2m)}{c}$$

But in this system (modified) $m = 4$ and $c = \bar{k}g_1$, hence the above becomes

$$t_s = \frac{3.2 \times 8}{\bar{k}g_1}$$

But $t_s = 1$sec., hence

$$\boxed{\bar{k}g_1 = 25.6}$$

Substitute the above into (1) we obtain

$$\begin{aligned} 25.6^2 - 16\,\bar{k}g_2 &< 256 \\ 655.36 - 16\,\bar{k}g_2 &< 256 \\ 2.56 - 0.0625\,\bar{k}g_2 &< 1 \\ -0.0625\,\bar{k}g_2 &< -1.56 \\ \bar{k}g_2 &> \frac{1.56}{0.0625} \end{aligned}$$

Hence

$$\boxed{\bar{k}g_2 > 24.96}$$

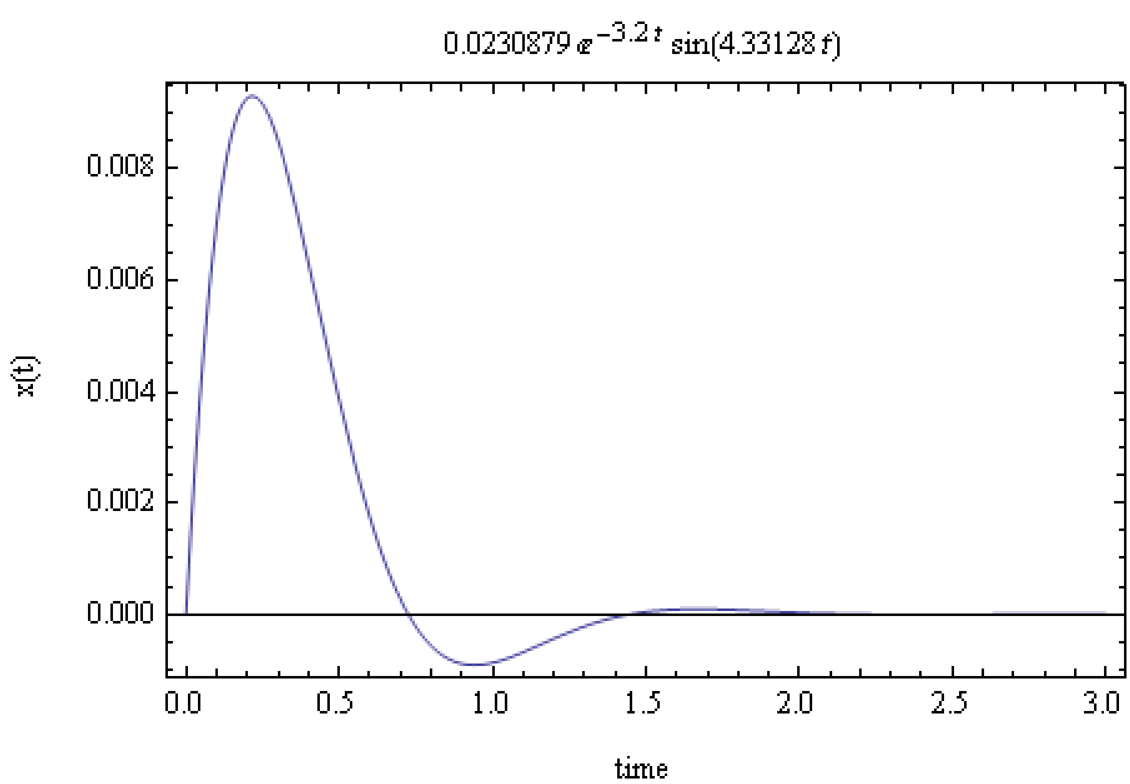
To plot the solution, choose $\bar{k}g_2$ say100 and since $\bar{k}g_1 = 25.6$, then since the loop back equation of motion is

$$m\ddot{x} + \bar{k}g_1\dot{x} + \left(k + \bar{k}g_2\right)x = 0$$

Then plugging in the above values for $\bar{k}g_2$ and $\bar{k}g_1$ we obtain

$$\begin{aligned} 4\ddot{x} + 25.6\dot{x} + (16 + 100)\,x &= 0 \\ 4\ddot{x} + 25.6\dot{x} + 116x &= 0 \end{aligned}$$

To confirm the result, I plot the solution to the above equation (which is now stable) using some initial condition such as $v_0 = 0.5$ and $x_0 = 0$ (arbitrary I.C.). The result is the following



3.1.8 Problem 1.22

Find the equilibrium points of the nonlinear pendulum equation $ml^2\ddot{\theta}+mgl\sin\theta=0$

Answer

The equation of motion can be simplified to be

$$\ddot{\theta}+\frac{g}{l}\sin\theta=0$$

Convert to state space format.

$$\begin{bmatrix}x_1=\theta\\x_2=\dot{\theta}\end{bmatrix}\rightarrow\begin{bmatrix}\dot{x}_1=\dot{\theta}=x_2\\\dot{x}_2=\ddot{\theta}=-\frac{g}{l}\sin\theta=-\frac{g}{l}\sin x_1\end{bmatrix}$$

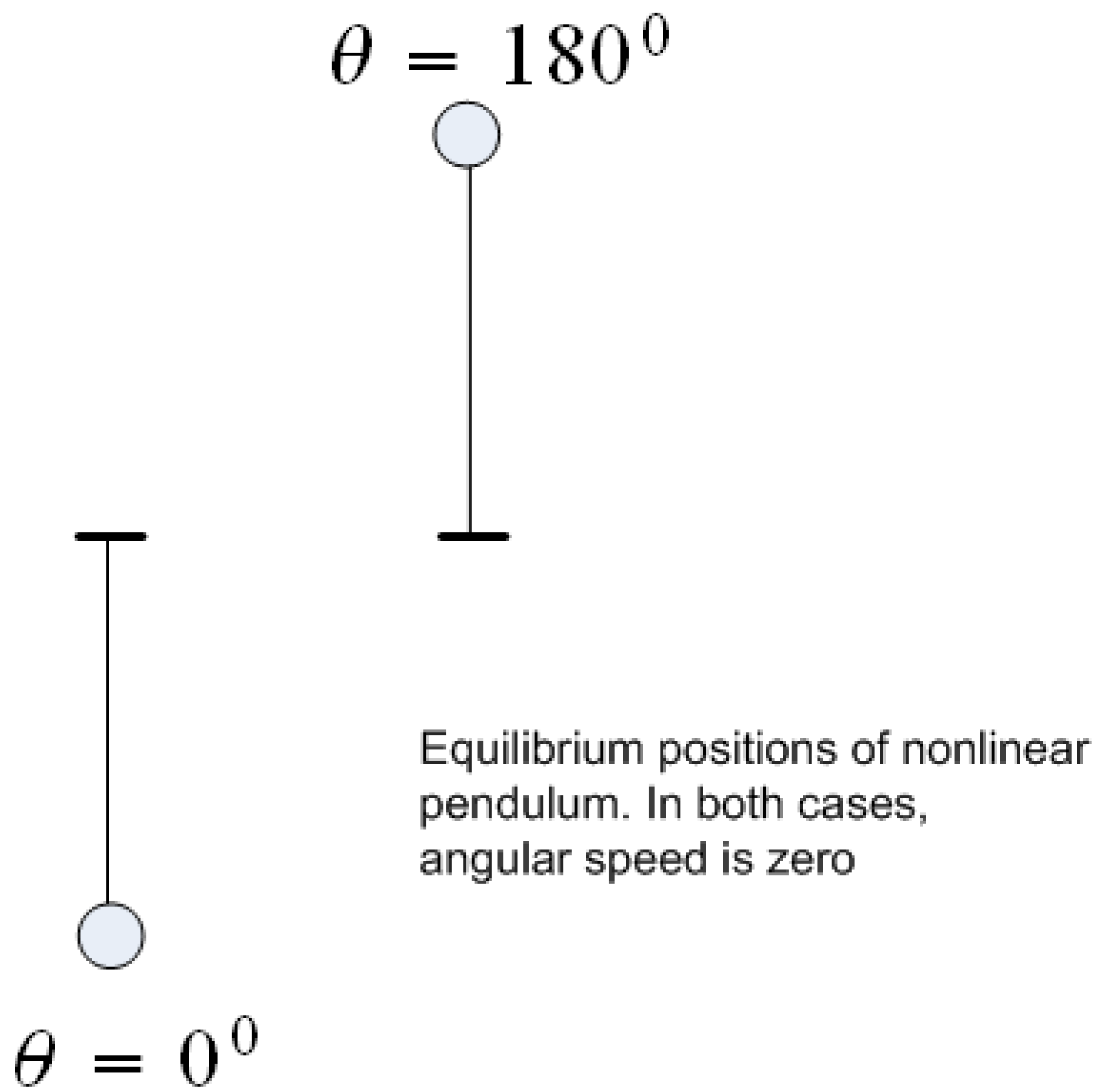
Hence

$$\overbrace{\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}}^{\dot{X}}=\begin{bmatrix}x_2\\-\frac{g}{l}\sin x_1\end{bmatrix}$$

For equilibrium of a nonlinear system, we require that $\dot{X}=0$, hence $x_2=0$ and $-\frac{g}{l}\sin x_1=0$

But $-\frac{g}{l}\sin x_1=0$ implies that $x_1=n\pi$ for $n=0,\pm1,\pm2,\cdots$

Since $x_1=\theta$, and θ is assumed to be zero when the pendulum is hanging in the vertical direction. Hence the equilibrium positions are as shown below (showing the first stable and the first unstable points)



In both cases, $\dot{\theta}=0$. Notice that at $\theta=n\pi$ for $n=\pm1,\pm3,\pm5,\cdots$ the pendulum in a marginally stable equilibrium position, while at $n=0,\pm2,\pm4,\cdots$ it is at a stable equilibrium position.

3.1.9 Key for HW1

EGME 511
HW1 SOLUTION

1.4)

Solution:

$\ddot{x} - \dot{x} + x = 0 \quad x_0 = 1, \;\; v_0 = 0$ (1)

Let $x(t) = Ce^{\lambda t}$

$Ce^{\lambda t}(\lambda^2 - \lambda + 1) = 0$

$\lambda_{1,2} = \frac{1}{2}\left(1 \pm j\sqrt{3}\right)$

$x(t) = A_1' e^{1/2(1+j\sqrt{3})t} + A_2' e^{1/2(1-j\sqrt{3})t}$ (2)

Using Euler's equation,

$x(t) = e^{\frac{1}{2}t}\left(A_1 \cos\sqrt{3}t + A_2 \sin\sqrt{3}t\right)$

Apply the initial conditions and obtain,

$1 = A_1$

$0 = \frac{1}{2}A_1 + \sqrt{3}A_2, \;\; A_2 = -\frac{1}{2\sqrt{3}}$

To obtain the solution in the form of Eq.(3),

$x(t) = A_3 e^{\frac{1}{2}t} \sin\left(\sqrt{3}t + \phi\right)$

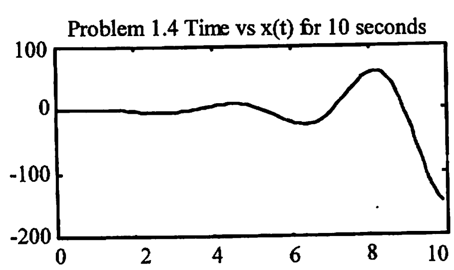
Using the trigonometric identities from problem (1) and get,

$A_3 = 1.041$

$\phi = 0.281$ rad.

$x(t) = 1.041e^{0.5t} \sin\left(\sqrt{3}t + 0.281)\right)$

The graph of the response of the system is shown below.



1.9)

Solution: Using equation (1.22) directly yields:

$$\zeta = \frac{1}{\sqrt{2}}, M_p = \frac{1}{2\frac{1}{\sqrt{2}}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}} = 1$$

1.12)

Solutions: $m\ddot{x} + c\dot{x} + kx = F\sin\omega t$, $\zeta = 1.1$ and $\omega_n^2 = 4$ (1)

Assume $x_p(t) = X\sin(\omega t - \phi)$ and substitute into (1),

$$X = \frac{F/k}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right)$$

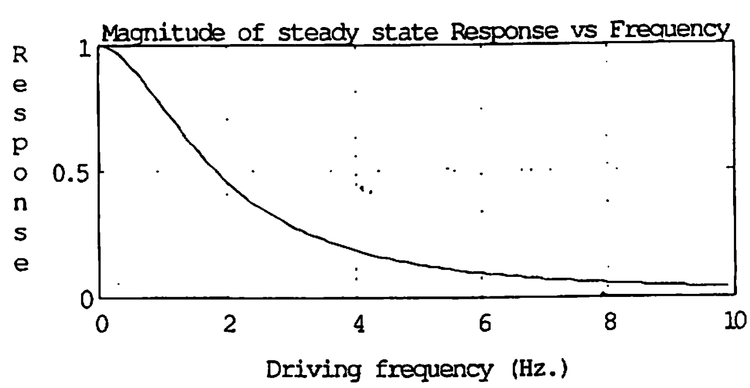
The particular solution is

$$x_p(t) = \frac{F/k}{\sqrt{\left(1-0.25\omega^2\right)^2 + 1.21\omega^2}}\sin(\omega t - \phi).$$

where

$$\phi = \tan^{-1}\left(\frac{1.1\omega}{1-0.25\omega^2}\right)$$

The response is plotted in the figure shown below, where the y-axis represents $\frac{Xk}{F}$.



Resonance does not occur because the system is overdamped.

1.18)

Solution: Gathering up terms, the equation of motion can be written as

$$\ddot{x}(t) + 4x(t) = 0.5\sin 2t$$

In this form it is clear that the natural frequency and the driving frequency are both 2 rad/s, hence the system is in resonance. The homogeneous form is undamped, hence stable. However, with a bounded input of 0.5sin2*t*, the response becomes unbounded and, hence the forced response is unstable. The solution can be computed as (see Inman 2001, page 96):

$$x(t) = \frac{v_o}{2}\sin 2t + x_0\cos 2t + 0.125t\sin 2t$$

which clearly grows without bound as it oscillates.

1.20)

Solution:

The equation of motion with controller is:

$$2\ddot{x} + (0.8 + Kg_1)\dot{x} + (8 + Kg_2)x = Ku(t) \qquad (1)$$

$$m = 2,\ c = 0.8 + Kg_1,\ k = 8 + Kg_2 \qquad (2)$$

The design expressions for overshoot and settling time are only valid for underdamped systems, if $1-\zeta^2 > 0$. Substitution for the value of ζ yields that

$$1 - \frac{c^2}{4km} > 0 \Rightarrow 4km > c^2 \qquad (3)$$

Substituting (2) into (3) and simplifying,

$$\underline{64 + 8Kg_2 > (0.8 + Kg_1)^2}$$

To insure BIBO stability of the closed loop system, the equivalent open loop system must be asymptotically stable. This requires the coefficients to be positive: $\underline{Kg_2 > -8 \text{ and } Kg_1 \geq -0.8}$. Note that in general, negative feedback is used so that Kg_1 and Kg_2 are usually positive. However, in order to obtain a specified settling time and overshoot, it may be that the gains g_i could be negative, hence stability must be checked.

1.21)

Solution: Since the open loop system is already stable, only a damping term needs to be added by the controller. Choosing, $K = 1$, and $g_2 = 0$ yields:

$$4\ddot{x}(t) + g_1\dot{x}(t) + 16x(t) = f(t)$$

This yields that $\omega_n = 2$ and

$$2\zeta\omega_n = \frac{g_1}{4} \Rightarrow g_1 = 16\zeta$$

However from equation (1.34) the settling time is

$$t_s = \frac{3.2}{\omega_n\zeta} \Rightarrow \zeta = \frac{3.2}{2t_s}$$

Combining these last two expression yields

$$g_1 = 16\frac{3.2}{2t_s} = \underline{25.6}$$

1.22) Compute the equilibrium positions of the pendulum equation: $m\ell^2\ddot{\theta}(t) + mg\ell\sin\theta(t) = 0$.

Solution: First put the system into first order form by defining the two states of position and velocity: $x_1 = \theta$, $x_2 = \dot{\theta} = \dot{x}_1$, and writing the equations of motion in state space, or first order form (diving through by the leading coefficient):

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{g}{\ell}\sin(x_1(t))$$

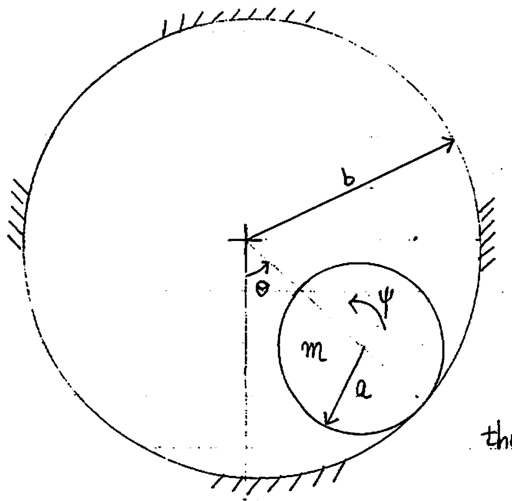
$$\Rightarrow \mathbf{F}(t) = \begin{bmatrix} x_2 \\ \sin x_1 \end{bmatrix}$$

Setting $\mathbf{F} = \mathbf{0}$ yields the equilibrium points:

$$\mathbf{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}, \begin{bmatrix} 3\pi \\ 0 \end{bmatrix}, \dots \begin{bmatrix} n\pi \\ 0 \end{bmatrix}, \dots$$

Example of Lagrangian Dynamics

Gear Problem

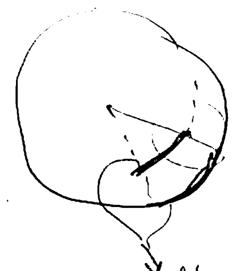


Assumptions

1) Outer gear is fixed
2) Inner gear rolls due to gravity force
3) Radius of gyration of small gear = K
4) One degree of freedom system [θ]

$$a\dot{\psi} = (b-a)\dot{\theta}$$

thus $V = (b-a)\dot{\theta}$



then 2 must be equal for no slippage

Kinetic Energy: $T = \frac{1}{2}mV^2 + \frac{1}{2}I\dot{\psi}^2$

where V = linear velocity of center of mass = $(b-a)\dot{\theta}$

I = rotational inertia of center of mass = mk^2

thus $T = \frac{1}{2}m[\underbrace{(b-a)\dot{\theta}}_{V}]^2 + \frac{1}{2}(\underbrace{mk^2}_{I})\left[\frac{b-a}{a}\dot{\theta}\right]^2$

Potential Energy

$$V = mg(b-a)(1-\cos\theta)$$

$$-\frac{\partial V}{\partial \theta} = -mg(b-a)\sin\theta$$

From Lagrange's Equation for conservative system,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}$$

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2}m(b-a)^2(2)\dot{\theta} + \frac{1}{2}mk^2\left(\frac{b-a}{a}\right)^2 2\dot{\theta}$$

$$\frac{\partial T}{\partial \dot{\theta}} = m(b-a)^2\dot{\theta} + mk^2\left(\frac{b-a}{a}\right)^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = m(b-a)^2\ddot{\theta} + mk^2\left(\frac{b-a}{a}\right)^2\ddot{\theta}$$

$$\frac{\partial T}{\partial \theta} = 0 \quad \text{and} \quad -\frac{\partial V}{\partial \theta} = -mg(b-a)\sin\theta$$

Hence $\frac{d}{dt}\frac{\partial T}{\partial \dot\theta} - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}$ becomes

$$m(b-a)^2\ddot\theta + mk^2\left(\frac{b-a}{a}\right)^2\ddot\theta + mg(b-a)\sin\theta = 0$$

Also by dividing by $(b-a)$

$$(b-a)\ddot\theta + \frac{k^2(b-a)}{a^2}\ddot\theta + g\sin\theta = 0$$

$$(b-a)\ddot\theta\left[1 + \frac{k^2}{a^2}\right] + g\sin\theta = 0$$

$$(b-a)\left[\frac{a^2+k^2}{a^2}\right]\ddot\theta + g\sin\theta = 0$$

$$\ddot\theta + \frac{g}{(b-a)\left[\frac{a^2+k^2}{a^2}\right]}\sin\theta = 0$$

3.2 HW2

Local contents

3.2.1 Description of HW

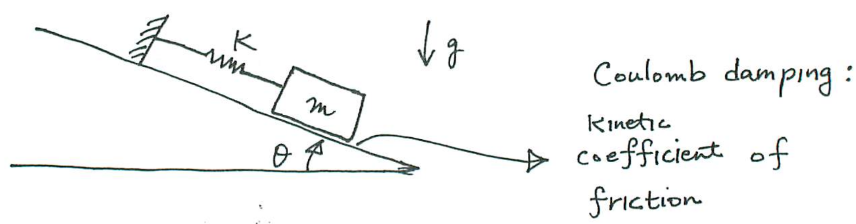
1. Find EQM for mass-spring with dynmaic friction on incline (this is nonlinear EQM due to columb friction)
2. Modal analysis problem on 2 by 2 system
3. Find EQM using lagrangian, 2 pendulums attached by one spring between them
4. Another Modal analysis problem on 2 by 2 system
5. 2nd order system, subject to 2 impulses, find response using convolution
6. Convolution problem. Underdamped system, force is half sin

3.2.2 Problems to solve

HW #2

Advanced Engineering Vibration 511
due March 17 2009

1. $x(0)=x_0$, $\dot{x}(0)=0$



Compute the equation of motion for the system

2. Given

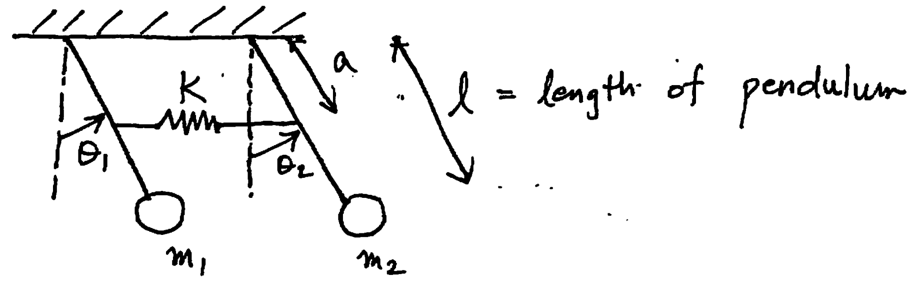
$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \ddot{X} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} X = 0$$

m : Kg
K : N/m

Use modal analysis to calculate the solution of this given $X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ mm , $\dot{X}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ mm/sec

Also calculate the eigenvalues of the system.
And calculate the eigenvectors and normalize.

3.



$K = 20\ N/m \qquad \ell = 0.5\ m$

$m_1 = m_2 = 10\,kg \qquad a = 0.1\ m$ along the pendulum

Determine the natural frequencies and mode shapes.

4. $$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \ddot{X} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} X = 0$$

Calculate the response of the system to IC

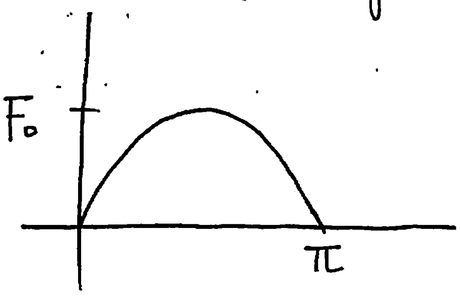
$$X_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \qquad \dot{X}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

5 Calculate the response of the system

$$3\ddot{x}(t) + 6\dot{x}(t) + 12x(t) = 3\delta(t) - \delta(t-1)$$

Subject to IC $x(0) = 0.01\,m \qquad V(0) = 1\ m/sec$

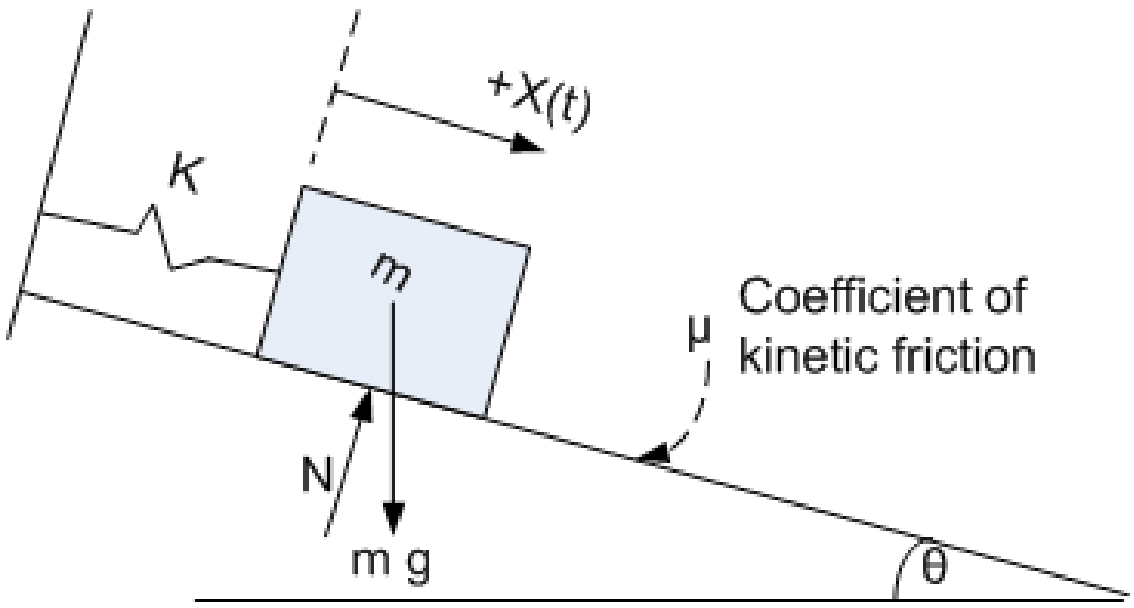
6 Calculate the response of an underdamped system to the excitation given below



$f(t) = F_0 \sin t$

3.2.3 Problem 1

Find the equation of motion for the following system



Solution

Assume initial conditions are $x(0) = x_0$ and $\dot{x}(0) = 0$. Assume that x_0 was positive (i.e. to the right of the static equilibrium position, and also assume that $kx_0 > N\ \mu_{static}$). This second requirement is needed to enable the mass to undergo motion by overcoming static friction. The normal force N is given by

$$N = mg\cos\theta$$

And the dynamic friction force f_c due to the dynamic friction is defined as follows

$$f_c = \begin{cases} -\mu N & \dot{x} > 0 \\ 0 & \dot{x} = 0 \\ \mu N & \dot{x} < 0 \end{cases}$$

But since $N = mg\cos\theta$, then the above becomes

$$f_c = \begin{cases} -\mu mg\cos\theta & \dot{x} > 0 \\ 0 & \dot{x} = 0 \\ \mu mg\cos\theta & \dot{x} < 0 \end{cases} \tag{1}$$

Where μ is the coefficient of dynamic friction. Now we can obtain the Lagrangian

$$\begin{aligned} L &= T - U \\ T &= \frac{1}{2}m\dot{x}^2 \\ U &= \frac{1}{2}kx^2 \end{aligned}$$

Hence

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

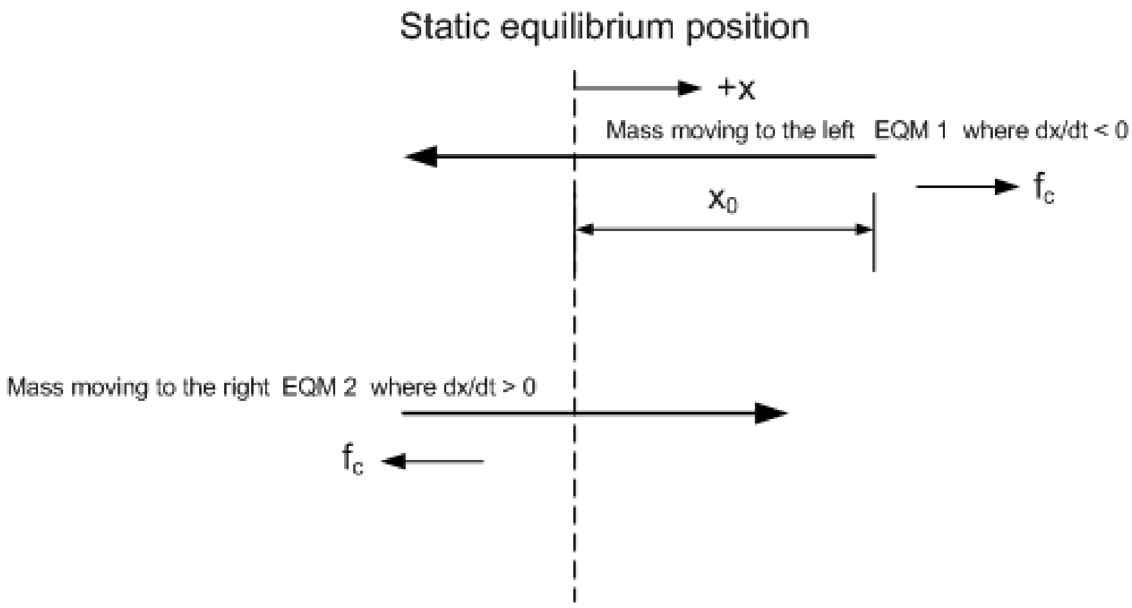
and

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= m\dot{x} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= m\ddot{x} \\ \frac{\partial L}{\partial x} &= -kx \end{aligned}$$

Then the EQM is

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= f_c \\ m\ddot{x} + kx &= f_c \end{aligned}$$

Where f_c is given by (1). Since f_c sign depends in the mass is moving to the left or to the right, we will generate 2 equation of motions, one for each case.



When mass is moving to the left, EQM 1 is

$$m\ddot{x} + kx = \mu mg \cos\theta \tag{2}$$

When mass is moving to the right, EQM 2 is

$$m\ddot{x} + kx = -\mu mg \cos\theta \tag{3}$$

So, for the first move, starting from x_0 and moving to the left, we have

$$\ddot{x} + \frac{k}{m}x = \mu g \cos\theta$$
$$\ddot{x} + \omega_n^2 x = \mu g \cos\theta$$

$$x = x_h + x_p$$

Guess $x_p = X$, hence $\omega_n^2 X = \mu g \cos\theta$ or $X = \frac{\mu g \cos\theta}{\omega_n^2}$, and $x_h = A\cos\omega_n t + B\sin\omega_n t$, therefore, the solution to EQM 1 is

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{\mu g \cos\theta}{\omega_n^2}$$

$x(0) = x_0 = A + \frac{\mu g \cos\theta}{\omega_n^2}$ hence $A = x_0 - \frac{\mu g \cos\theta}{\omega_n^2}$, then

$$x(t) = \left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\cos\omega_n t + B\sin\omega_n t + \frac{\mu g \cos\theta}{\omega_n^2}$$

and

$$\dot{x}(t) = -\omega_n\left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\sin\omega_n t + \omega_n B\cos\omega_n t$$
$$\dot{x}(0) = v_0 = 0 = \omega_n B$$

Hence $B = 0$, then EQM is (for $0 < t < \frac{\pi}{\omega_n}$)

$$x_{left}(t) = \left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\cos\omega_n t + \frac{\mu g \cos\theta}{\omega_n^2} \tag{4}$$

The mass will move according to the above equation (4) until the velocity is zero, then it will turn and start moving to the right. To find the time this happens:

$$\dot{x}(t) = -\omega_n\left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\sin\omega_n t$$

Now solve for t when $\dot{x}(t) = 0$, i.e.,

$$0 = -\omega_n\left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\sin\omega_n t \tag{5}$$

Hence $\omega_n t = n\pi$, where $n = 0, \pm 1, \pm 2, \cdots$The case for $n = 0$ do not apply since this implies $t = 0$, then consider the next time this can happen, which is $n = 1$, which implies

$$t_1 = \frac{\pi}{\omega_n} \tag{6}$$

Now we need to determine $x(t)$ at this time t_1 since this will become the initial x for the second equation of motion going to the right in the second leg of the journey. Using (4) and (6) we obtain

$$\begin{aligned} x\left(\frac{\pi}{\omega_n}\right) &= \left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\cos\omega_n \frac{\pi}{\omega_n} + \frac{\mu g \cos\theta}{\omega_n^2} \\ &= \frac{2\mu g \cos\theta}{\omega_n^2} - x_0 \end{aligned}$$

Notice that in the above equation, x_0 is a positive number, since we assumed that the initial conditions x_0 was to the right of the static equilibrium position, and we are assume the right of the static equilibrium position to be positive. This also implied that $x\left(\frac{\pi}{\omega_n}\right)$ will be negative number (which is what we expect, as the mass will by the end of its first trip be on the left of the static equilibrium position).

Now we can use right equation of motion (EQM 2) to solve for the mass moving to the right. Notice that the initial conditions for this motion are $x_1 = \frac{2\mu g \cos\theta}{\omega_n^2} - x_0$ and $t_1 = \frac{\pi}{\omega_n}$

The equation of motion is now

$$\begin{aligned} m\ddot{x} + kx &= -\mu m g \cos\theta \\ \ddot{x} + \omega_n^2 x &= -\mu g \cos\theta \end{aligned}$$

With the general solution

$$x(t) = A\cos\omega_n t + B\sin\omega_n t - \frac{\mu g \cos\theta}{\omega_n^2} \tag{7}$$

At $t = \frac{\pi}{\omega_n}, x(t) = \frac{2\mu g \cos\theta}{\omega_n^2} - x_0$, hence from the above

$$\begin{aligned} \frac{2\mu g \cos\theta}{\omega_n^2} - x_0 &= A\cos\omega_n \frac{\pi}{\omega_n} + B\sin\omega_n \frac{\pi}{\omega_n} - \frac{\mu g \cos\theta}{\omega_n^2} \\ &= -A - \frac{\mu g \cos\theta}{\omega_n^2} \\ A &= x_0 - \frac{3\mu g \cos\theta}{\omega_n^2} \end{aligned}$$

Hence (7) becomes

$$x(t) = \left(x_0 - \frac{3\mu g \cos\theta}{\omega_n^2}\right)\cos\omega_n t + B\sin\omega_n t - \frac{\mu g \cos\theta}{\omega_n^2}$$

And

$$\dot{x}(t) = -\omega_n\left(x_0 - \frac{3\mu g \cos\theta}{\omega_n^2}\right)\sin\omega_n t + \omega_n B\cos\omega_n t$$

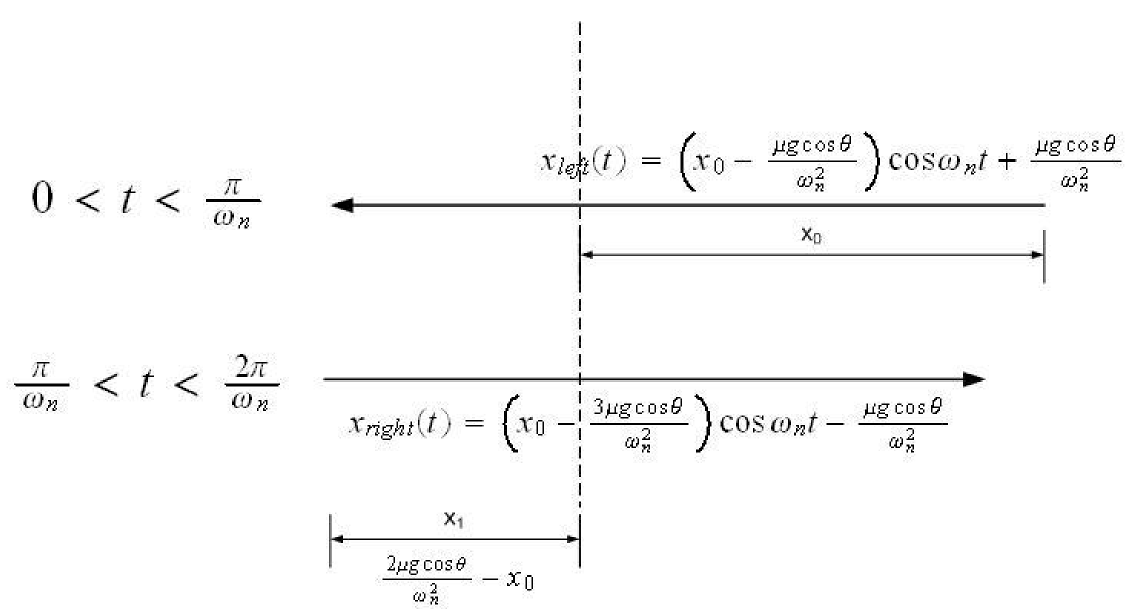
But $\dot{x}(t) = 0$ at $t = \frac{\pi}{\omega_n}$, hence the above becomes

$$\begin{aligned} 0 &= -\omega_n\left(x_0 - \frac{3\mu g \cos\theta}{\omega_n^2}\right)\sin\omega_n \frac{\pi}{\omega_n} + \omega_n B\cos\omega_n \frac{\pi}{\omega_n} \\ &= -\omega_n B \end{aligned}$$

Hence $B = 0$, then the EQM for the right move is, for $\frac{\pi}{\omega_n} < t < \frac{2\pi}{\omega_n}$

$$\boxed{x_{right}(t) = \left(x_0 - \frac{3\mu g \cos\theta}{\omega_n^2}\right)\cos\omega_n t - \frac{\mu g \cos\theta}{\omega_n^2}}$$

This diagram below summarize this



Now, we would like to have one equation to express the motion with for any time instance when the mass is moving to the left, or to the right. Looking at the above 2 equation of motion, we see immediately that we can write the equation of motion as follows

$$\boxed{x_n(t) = \left(x_0 - \frac{(2n-1)\mu g \cos\theta}{\omega_n^2}\right)\cos\omega_n t \ + (-1)^{n+1}\frac{\mu g \cos\theta}{\omega_n^2}}$$

Where n above is the number of the trip. So, the first trip, going from x_0 and moving to the left, will have $n = 1$, and then second trip, moving from x_1 and going to the right will have $n = 2$, and so on. As for the time during which trip travels, this is found by the following equation

$$\frac{(n-1)\,\pi}{\omega_n} < t_n < \frac{n\pi}{\omega_n}$$

What the above is saying is that for first trip $(n = 1)$, we have

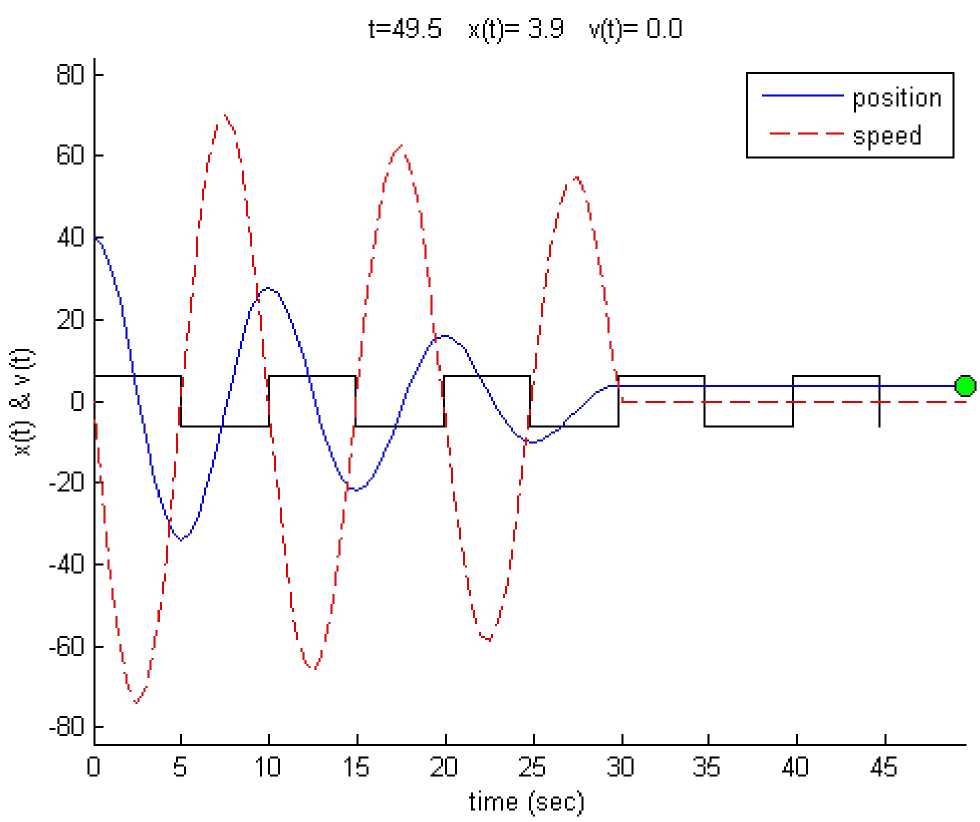
$$0 < t < \frac{\pi}{\omega_n}$$

And for the second trip, we have

$$\frac{\pi}{\omega_n} < t < \frac{2\pi}{\omega_n}$$

etc...

Now that we have one equation, and we have the time during which each equation is valid, we can now plot the equation of motion vs. time. The following is a plot for some values for k, g, m. Please see the appendix for the Matlab code which generated this simulation.



Observation found on this problem: Changing the angle of inclination θ causes no change in results. In other words, the same oscillation will occur for flat plane $(\theta = 0)$ or for $\theta = 45^0$ or any other angle. The reason is because x_0, the initial position, is measured from the static equilibrium position, and this static equilibrium position will be different as the angle changes, but the effect of the angle change is already accounted for by this change and will not be reflected in the actual displacement $x(t)$.

3.2.4 Problem 2

Given $\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}\ddot{\mathbf{X}}+\begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix}\mathbf{X}=\begin{bmatrix}0\\ 0\end{bmatrix}$, $m:kg,k:N/m$, use modal analysis to calculate the solution of this given $\mathbf{X}(0)=\begin{bmatrix}0\\ 1\end{bmatrix}mm$, $\dot{\mathbf{X}}(0)=\begin{bmatrix}0\\ 0\end{bmatrix}mm/\sec$ also calculate the eigenvalues of the system and the normalized eigenvectors.

Answer

Since this is a 2 ODE's that are coupled, we use modal analysis to de-couple the system first in order to obtain 2 separate ODE's which we can then solve easily.

Let

$M=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}$ and let $K=\begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix}$, then the above system becomes

$$M\ddot{\mathbf{X}}+K\mathbf{X}=\mathbf{0} \tag{1}$$

Let $\mathbf{X}=M^{-\frac{1}{2}}\mathbf{q}$, then $\ddot{\mathbf{X}}=M^{-\frac{1}{2}}\ddot{\mathbf{q}}$ and the above equation becomes

$$MM^{-\frac{1}{2}}\ddot{\mathbf{q}}+KM^{-\frac{1}{2}}\mathbf{q}=\mathbf{0}$$

premultiply by $M^{-\frac{1}{2}}$ we obtain

$$\begin{aligned}M^{-\frac{1}{2}}MM^{-\frac{1}{2}}\ddot{\mathbf{q}}+M^{-\frac{1}{2}}KM^{-\frac{1}{2}}\mathbf{q}&=\mathbf{0}\\ I\ddot{\mathbf{q}}+\tilde{K}\mathbf{q}&=\mathbf{0}\end{aligned} \tag{2}$$

Where $\tilde{K}=M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$

Let $\mathbf{q}=\mathbf{v}e^{i\omega t}$, then $\ddot{\mathbf{q}}=-\omega^2\mathbf{v}e^{i\omega t}$ and (2) becomes

$$\begin{aligned}-\omega^2e^{i\omega t}I\mathbf{v}+\tilde{K}\mathbf{v}e^{i\omega t}&=\mathbf{0}\\ \left(\tilde{K}-\omega^2I\right)\mathbf{v}&=\mathbf{0}\end{aligned}$$

Let $\lambda=\omega^2$ then we have

$$\left(\tilde{K}-\lambda I\right)\mathbf{v}=\mathbf{0} \tag{3}$$

For $\mathbf{v}\neq\mathbf{0}$, we requires that $\left|\tilde{K}-\lambda I\right|=0$ But

$$\begin{aligned}\tilde{K}&=M^{-\frac{1}{2}}KM^{-\frac{1}{2}}\\ &=\begin{bmatrix}1^{\frac{-1}{2}} & 0\\ 0 & 4^{\frac{-1}{2}}\end{bmatrix}\begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1^{\frac{-1}{2}} & 0\\ 0 & 4^{\frac{-1}{2}}\end{bmatrix}\\ &=\begin{bmatrix}3 & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{4}\end{bmatrix}\end{aligned}$$

Hence

$$\begin{aligned}\left|\tilde{K}-\lambda I\right|&=0\\ \left|\begin{bmatrix}3 & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{4}\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right|&=0\\ \left|\begin{bmatrix}3-\lambda & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{4}-\lambda\end{bmatrix}\right|&=0\\ (3-\lambda)\left(\frac{1}{4}-\lambda\right)-\frac{1}{4}&=0\\ \lambda^2-\frac{13}{4}\lambda+\frac{1}{2}&=0\end{aligned}$$

Hence

$$\begin{aligned}\lambda&=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}\\ &=\frac{13}{8}\pm\frac{\sqrt{\left(\frac{13}{4}\right)^2-2}}{2}\\ &=\frac{13}{8}\pm\frac{1}{8}\sqrt{137}\end{aligned}$$

Hence

$$\boxed{\lambda_{1,2}=\left\{\frac{13-\sqrt{137}}{8},\frac{13+\sqrt{137}}{8}\right\}=\{0.161\,91,3.088\,1\}}$$

From (3) we then have

$$\left(\tilde{K}-\lambda I\right)\mathbf{v}=\mathbf{0}$$
$$\left(\begin{bmatrix}3 & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{4}\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)\mathbf{v}=\mathbf{0}$$

When $\lambda=\lambda_1=0.161\,9$ we obtain

$$\left(\begin{bmatrix}3.0 & -0.5\\ -0.5 & 0.25\end{bmatrix}-\begin{bmatrix}0.161\,9 & 0\\ 0 & 0.161\,9\end{bmatrix}\right)\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}2.\,838\,1 & -0.5\\ -0.5 & 0.088\,1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Hence

$$2.\,838\,1a-0.5b=0$$
$$-0.5a-0.094\,1b=0$$

Let $a=1$, then $b=\frac{-2.\,838\,1}{-0.5}=5.\,676\,2$, hence the second eigenvector is

$$\mathbf{v}_1=\begin{bmatrix}1\\ 5.\,676\,2\end{bmatrix}$$

$\|\mathbf{v}_1\|=\sqrt{1+5.\,676\,2^2}=5.\,763\,6$, hence normalized $\mathbf{v}_1$ is

$$\mathbf{v}_1=\frac{1}{5.\,763\,6}\begin{bmatrix}1\\ 5.\,676\,2\end{bmatrix}$$
$$\boxed{\mathbf{v}_1=\begin{bmatrix}0.173\,5\\ 0.984\,84\end{bmatrix}}$$

When $\lambda=\lambda_2=3.\,088\,1$ we obtain

$$\left(\begin{bmatrix}3.0 & -0.5\\ -0.5 & 0.25\end{bmatrix}-\begin{bmatrix}3.\,088\,1 & 0\\ 0 & 3.\,088\,1\end{bmatrix}\right)\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}-0.088\,1 & -0.5\\ -0.5 & -2.\,838\,1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Hence

$$-0.088\,1a-0.5b=0$$
$$-0.5a-2.\,838\,1b=0$$

Let $a=1$ in the first equation above, then $b=\frac{-0.088\,1}{0.5}=-0.176\,2$, hence the first eigenvector is

$$\mathbf{v}_2=\begin{bmatrix}1\\ -0.176\,2\end{bmatrix}$$

$\|\mathbf{v}_2\|=\sqrt{1+0.176\,2^2}=1.\,015\,4$, hence normalized $\mathbf{v}_2$ is

$$\mathbf{v}_2=\frac{1}{1.\,015\,4}\begin{bmatrix}1\\ -0.176\,2\end{bmatrix}$$
$$\boxed{\mathbf{v}_2=\begin{bmatrix}0.984\,83\\ -0.173\,53\end{bmatrix}}$$

Then the P matrix

$$[P]=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}$$
$$=\boxed{\begin{bmatrix}0.173\,5 & 0.984\,83\\ 0.984\,84 & -0.173\,53\end{bmatrix}}$$

Now let $\mathbf{q}=P\mathbf{r}$, then equation (2) above becomes

$$I\ddot{\mathbf{q}}+\tilde{K}\mathbf{q}=\mathbf{0}$$
$$IP\ddot{\mathbf{r}}+\tilde{K}P\mathbf{r}=\mathbf{0}$$

Premultiply by P^T

$$P^TIP\ddot{\mathbf{r}}+P^T\tilde{K}P\mathbf{r}=\mathbf{0}$$
$$I\ddot{\mathbf{r}}+P^T\tilde{K}P\mathbf{r}=\mathbf{0}$$

Let $\Lambda = P^T\tilde{K}P$ then the above becomes

$$I\ddot{\mathbf{r}}+\Lambda\mathbf{r}=\mathbf{0} \tag{4}$$

Now find Λ[1]

$$\begin{aligned}\Lambda &= P^T\tilde{K}P\\ &= \begin{bmatrix}0.1735 & 0.98483\\ 0.98484 & -0.17353\end{bmatrix}^T \begin{bmatrix}3.0 & -0.5\\ -0.5 & 0.25\end{bmatrix} \begin{bmatrix}0.1735 & 0.98483\\ 0.98484 & -0.17353\end{bmatrix}\\ &= \boxed{\begin{bmatrix}0.16191 & 0\\ 0 & 3.0881\end{bmatrix}}\end{aligned}$$

Hence (4) becomes

$$I\ddot{\mathbf{r}} + \begin{bmatrix}0.16191 & 0\\ 0 & 3.0881\end{bmatrix}\mathbf{r}=\mathbf{0}$$

Which can be written as 2 equations

$$\begin{bmatrix}\ddot{r}_1\\ \ddot{r}_2\end{bmatrix} + \begin{bmatrix}0.16191r_1\\ 3.0881r_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

or

$$\begin{aligned}\ddot{r}_1 + 0.16191r_1 &= 0\\ \ddot{r}_2 + 3.0881r_2 &= 0\end{aligned} \tag{5}$$

With IC given as

$$\mathbf{X}(0) = \begin{bmatrix}0\\ 1\end{bmatrix}$$

and

$$\dot{\mathbf{X}}(0) = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now $\mathbf{X}$=$M^{-\frac{1}{2}}\mathbf{q}$ and $\mathbf{q}$=$P\mathbf{r}$, hence $\mathbf{X}$=$M^{-\frac{1}{2}}P\mathbf{r}$, then

$$\begin{aligned}\mathbf{r}(0) &= P^TM^{\frac{1}{2}}\mathbf{X}(0)\\ \begin{bmatrix}r_1(0)\\ r_2(0)\end{bmatrix} &= \begin{bmatrix}0.1735 & 0.98483\\ 0.98484 & -0.17353\end{bmatrix}^T \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\\ \begin{bmatrix}r_1(0)\\ r_2(0)\end{bmatrix} &= \begin{bmatrix}1.9697\\ -0.34706\end{bmatrix}\end{aligned}$$

now need to find $\dot{\mathbf{r}}(0)$,but since $\ddot{\mathbf{X}}(0) = \mathbf{0}$, then $\dot{\mathbf{r}}(0) = \mathbf{0}$ as well.

Now we can solve for $r_1(t)$ and $r_2(t)$ since we have the IC. From (5) above

$$\begin{aligned}\ddot{r}_1 + 0.16191r_1 &= 0\\ r_1(t) &= A\cos\omega_{n_1}t + B\sin\omega_{n_1}t\end{aligned}$$

At $t = 0, r_1(0) = 1.9696$, hence $1.9696 = A$, then

$$\begin{aligned}r_1(t) &= 1.9696\cos\omega_{n_1}t + B\sin\omega_{n_1}t\\ \dot{r}_1(t) &= -1.9696\omega_{n_1}\sin\omega_{n_1}t + \omega_{n1}B\cos\omega_{n_1}t\end{aligned}$$

At $t = 0$

$$\dot{r}_1(t) = 0 = \omega_{n_1}B$$

Hence $B = 0$, then

$$r_1(t) = 1.9696\cos\omega_{n_1}t$$

But $\omega_{n_1} = \sqrt{0.16191} = 0.40238$, hence

$$\boxed{r_1(t) = 1.9696\cos(0.40238t)}$$

Similarly we find $r_2(t)$

$$\begin{aligned}\ddot{r}_2 + 3.0881r_2 &= 0\\ r_2(t) &= A\cos\omega_{n_2}t + B\sin\omega_{n_2}t\end{aligned}$$

[1]This can also be found more quickly by noting that $\Lambda = diag(\lambda_1, \lambda_2)$

At $t = 0, r_2(0) = -0.346\,98$, hence $-0.346\,98 = A$, then

$$r_2(t) = -0.346\,98 \cos \omega_{n_2} t + B \sin \omega_{n_2} t$$
$$\dot{r}_2(t) = 0.346\,98 \omega_{n_2} \sin \omega_{n_2} t + \omega_{n_2} B \cos \omega_{n_2} t$$

At $t = 0$

$$\dot{r}_2(t) = 0 = \omega_{n_2} B$$

Hence $B = 0$, then

$$r_2(t) = -0.346\,98 \cos \omega_{n_2} t$$

But $\omega_{n_2} = \sqrt{3.088\,1} = 1.757\,3$, hence

$$\boxed{r_2(t) = -0.346\,98 \cos(1.757\,3t)}$$

Now that we found the solution in the r space, we switch back to the original x space

$$\mathbf{X}(t) = M^{-\frac{1}{2}} P \mathbf{r}(t)$$

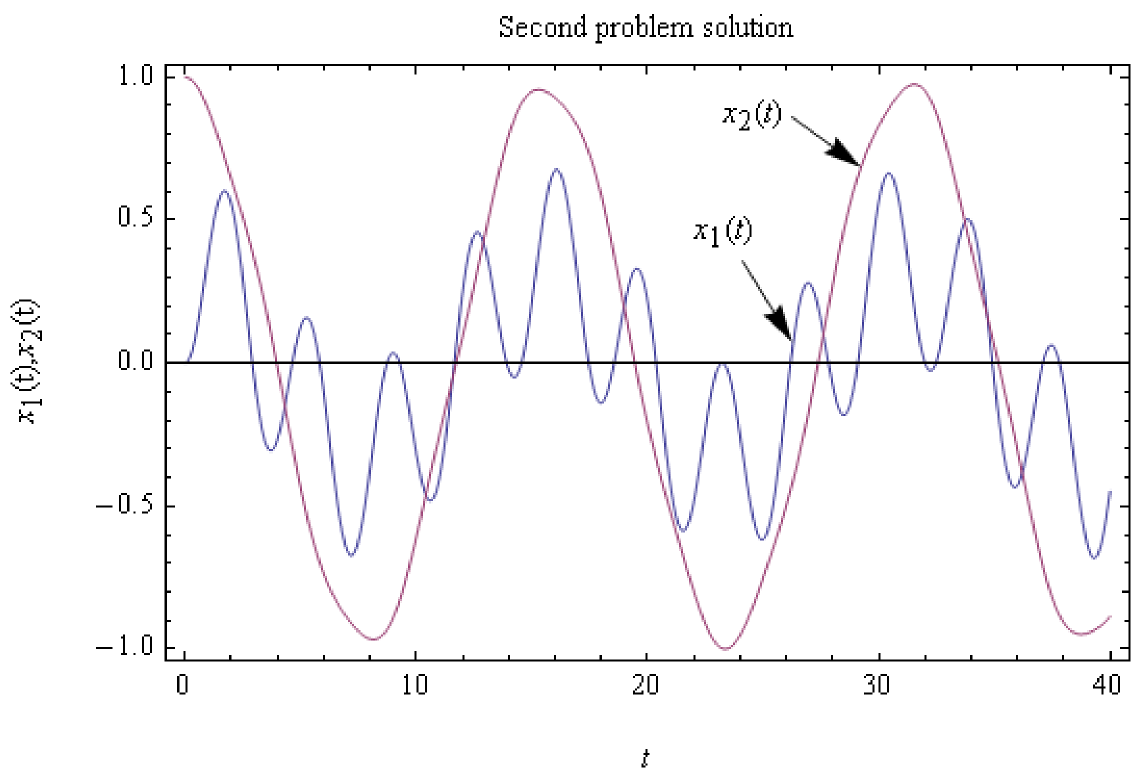
Then

$$\mathbf{X}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.173\,5 & 0.984\,83 \\ 0.984\,84 & -0.173\,53 \end{bmatrix} \begin{bmatrix} 1.969\,6 \cos(0.402\,38t) \\ -0.346\,98 \cos(1.757\,3t) \end{bmatrix}$$

Hence

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.341\,73 \cos 0.402\,38t - 0.341\,72 \cos 1.757\,3t \\ 0.969\,87 \cos 0.402\,38t + 0.03010\,6 \cos 1.757\,3t \end{bmatrix}}$$

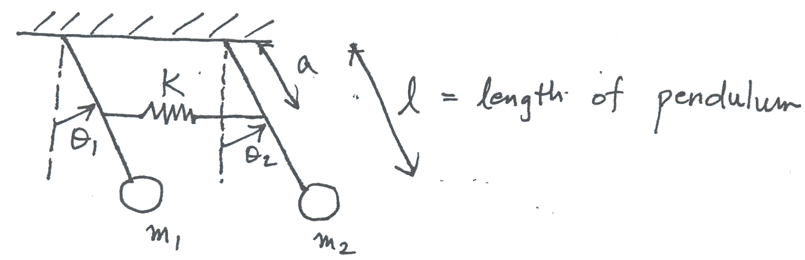
This is a plot of the solutions



Observation on final result: Notice that power of the harmonic $\omega_n = 1.757\,3$ rad/sec. in the motion $x_2(t)$ is small (amplitude is only 0.03) hence the dominant harmonic present in $x_2(t)$ is $\omega_n = 0.402\,38$ rad/sec. and this reflects in the plot where it appears that $x_2(t)$ contain one harmonic. In the case of $x_1(t)$ we see from the solution that both frequencies contribute equal amount of power, hence the plot for $x_1(t)$ reflects this.

3.2.5 Problem 3

3.



K = 20 N/m $\quad l = 0.5$ m

$m_1 = m_2 = 10$ kg $\quad a = 0.1$ m along the pendulum

Determine the natural frequencies and mode shapes.

Solution Use as generalized coordinates θ_1, θ_2,. Assume that the spring remain horizontal, and assume that $\theta_2 > \theta_1$

$$
\begin{aligned}
L &= T - U \\
T &= \frac{1}{2} m_1 \left(L\dot{\theta}_1\right)^2 + \frac{1}{2} m_2 \left(L\dot{\theta}_2\right)^2 \\
U_{gravity} &= m_1 g L(1 - \cos\theta_1) + m_2 g L(1 - \cos\theta_2) \\
U_{spring} &= \frac{1}{2} k (a \sin\theta_2 - a \sin\theta_1)^2
\end{aligned}
$$

Hence

$$
L = \frac{1}{2} m_1 \left(L\dot{\theta}_1\right)^2 + \frac{1}{2} m_2 \left(L\dot{\theta}_2\right)^2 - \left(m_1 g L(1 - \cos\theta_1) + m_2 g L(1 - \cos\theta_2) + \frac{1}{2} k (a \sin\theta_2 - a \sin\theta_1)^2 \right)
$$

Now determine the Lagrangian equation

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_1} &= m_1 L^2 \dot{\theta}_1 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} &= m_1 L^2 \ddot{\theta}_1 \\
\frac{\partial L}{\partial \dot{\theta}_2} &= m_2 L^2 \dot{\theta}_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} &= m_2 L^2 \ddot{\theta}_2 \\
\frac{\partial L}{\partial \theta_1} &= -m_1 g L \sin\theta_1 + a k (a \sin\theta_2 - a \sin\theta_1) \cos\theta_1 \\
\frac{\partial L}{\partial \theta_2} &= -m_1 g L \sin\theta_2 - a k (a \sin\theta_2 - a \sin\theta_1) \cos\theta_2
\end{aligned}
$$

Hence the EQM for m_1 is

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} &= 0 \\
m_1 L^2 \ddot{\theta}_1 + m_1 g L \sin\theta_1 - a k (a \sin\theta_2 - a \sin\theta_1) \cos\theta_1 &= 0
\end{aligned}
$$

Now apply small angle approximation. $\sin\theta \approx \theta$ and $\cos\theta \approx 1$hence

$$
\begin{aligned}
m_1 L^2 \ddot{\theta}_1 + m_1 g L \theta_1 - a k (a\theta_2 - a\theta_1) &= 0 \\
m_1 L^2 \ddot{\theta}_1 + m_1 g L \theta_1 - a^2 k \theta_2 + a^2 k \theta_1 &= 0 \\
m_1 L^2 \ddot{\theta}_1 + \left(m_1 g L + a^2 k\right) \theta_1 - a^2 k \theta_2 &= 0 \qquad (1)
\end{aligned}
$$

And the EQM for m_2 is

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} &= 0 \\
m_2 L^2 \dot{\theta}_2 + m_1 g L \sin\theta_2 + a k (a \sin\theta_2 - a \sin\theta_1) \cos\theta_2 &= 0
\end{aligned}
$$

Now apply small angle approximation. $\sin\theta \approx \theta$ and $\cos\theta \approx 1$hence

$$
\begin{aligned}
m_2 L^2 \dot{\theta}_2 + m_1 g L \theta_2 + a k (a\theta_2 - a\theta_1) &= 0 \\
m_2 L^2 \dot{\theta}_2 + m_1 g L \theta_2 + a^2 k \theta_2 - a^2 k \theta_1 &= 0
\end{aligned}
$$

Therefore

$$m_2L^2\ddot{\theta}_2 + \theta_2\left(m_2gL + a^2k\right) - a^2k\theta_1 = 0$$

Now we write the system as $M\ddot{\ } + K\grave{\ } = 0$

$$\begin{bmatrix} m_1L^2 & 0 \\ 0 & m_2L^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} m_1gL + a^2k & -a^2k \\ -a^2k & m_2gL + a^2k \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Substitute numerical values for the above quantities, we obtain

$$\begin{bmatrix} 10\times 0.5^2 & 0 \\ 0 & 10\times 0.5^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 10\times 9.8\times 0.5 + 0.1^2\times 20 & -a^2\times 20 \\ -0.1^2\times 20 & 10\times 9.8\times 0.5 + 0.1^2\times 20 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 49.2 & -0.2 \\ -0.2 & 49.2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The above can be written as

$$M\ddot{\ } + K\grave{\ } = \mathbf{0}$$

Let $\grave{\ } = M^{-\frac{1}{2}}\mathbf{q}$, then $\ddot{\ } = M^{-\frac{1}{2}}\ddot{\mathbf{q}}$ and the above equation becomes

$$MM^{-\frac{1}{2}}\ddot{\ } + KM^{-\frac{1}{2}}\grave{\ } = \mathbf{0}$$

premultiply by $M^{-\frac{1}{2}}$ we obtain

$$M^{-\frac{1}{2}}MM^{-\frac{1}{2}}\ddot{\ } + M^{-\frac{1}{2}}KM^{-\frac{1}{2}}\grave{\ } = \mathbf{0}$$

$$I\ddot{\ } + \tilde{K}\grave{\ } = \mathbf{0} \tag{2}$$

Where $\tilde{K} = M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$

Let $\mathbf{q} = \mathbf{v}e^{i\omega t}$, then $\ddot{\mathbf{q}} = -\omega^2\mathbf{v}e^{i\omega t}$ and (2) becomes

$$-\omega^2e^{i\omega t}I\mathbf{v} + \tilde{K}\mathbf{v}e^{i\omega t} = \mathbf{0}$$

$$\left(\tilde{K} - \omega^2I\right)\mathbf{v} = \mathbf{0}$$

Let $\lambda = \omega^2$ then we have

$$\left(\tilde{K} - \lambda I\right)\mathbf{v} = \mathbf{0} \tag{3}$$

For $\mathbf{v} \neq \mathbf{0}$, we requires that $\left|\tilde{K} - \lambda I\right| = 0$ But

$$\begin{aligned}
\tilde{K} &= M^{-\frac{1}{2}}KM^{-\frac{1}{2}} \\
&= \begin{bmatrix} 2.5^{-\frac{1}{2}} & 0 \\ 0 & 2.5^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} 49.2 & -0.2 \\ -0.2 & 49.2 \end{bmatrix}\begin{bmatrix} 2.5^{-\frac{1}{2}} & 0 \\ 0 & 2.5^{-\frac{1}{2}} \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 19.68 & -0.08 \\ -0.08 & 19.68 \end{bmatrix}}
\end{aligned}$$

Hence

$$\left|\tilde{K} - \lambda I\right| = 0$$

$$\left|\begin{bmatrix} 19.68 & -0.08 \\ -0.08 & 19.68 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 19.68 - \lambda & -0.08 \\ -0.08 & 19.68 - \lambda \end{bmatrix}\right| = 0$$

$$(19.68 - \lambda)(19.68 - \lambda) - 0.08^2 = 0$$

Hence the characteristic equation is

$$\boxed{\lambda^2 - 39.36\,\lambda + 387.30 = 0}$$

Hence

$$\boxed{\lambda_{1,2} = 19.6, 19.76}$$

Hence the natural frequencies are

$$\begin{aligned}
\omega_n &= \left\{\sqrt{19.6}, \sqrt{19.76}\right\} \\
&= \boxed{\{4.4272, 4.4452\}\text{ rad/sec}}
\end{aligned}$$

From (3) we then have

$$\left(\tilde{K} - \lambda I\right)\mathbf{v} = \mathbf{0}$$

$$\left(\begin{bmatrix} 19.68 & -0.08 \\ -0.08 & 19.68 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\mathbf{v} = \mathbf{0}$$

When $\lambda = \lambda_1 = 19.6$ we obtain

$$\left(\begin{bmatrix} 19.68 & -0.08 \\ -0.08 & 19.68 \end{bmatrix} - \begin{bmatrix} 19.6 & 0 \\ 0 & 19.6 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0.08 & -0.08 \\ -0.08 & 0.08 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence

$$0.08a - 0.08b = 0$$
$$-0.08a + 0.08b = 0$$

Hence $a = b$ then

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.707\,11 \\ 0.707\,11 \end{bmatrix}$$

When $\lambda = \lambda_2 = 19.76$ we obtain

$$\left(\begin{bmatrix} 19.68 & -0.08 \\ -0.08 & 19.68 \end{bmatrix} - \begin{bmatrix} 19.76 & 0 \\ 0 & 19.76 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -0.08 & -0.08 \\ -0.08 & -0.08 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence $a = -b$, then

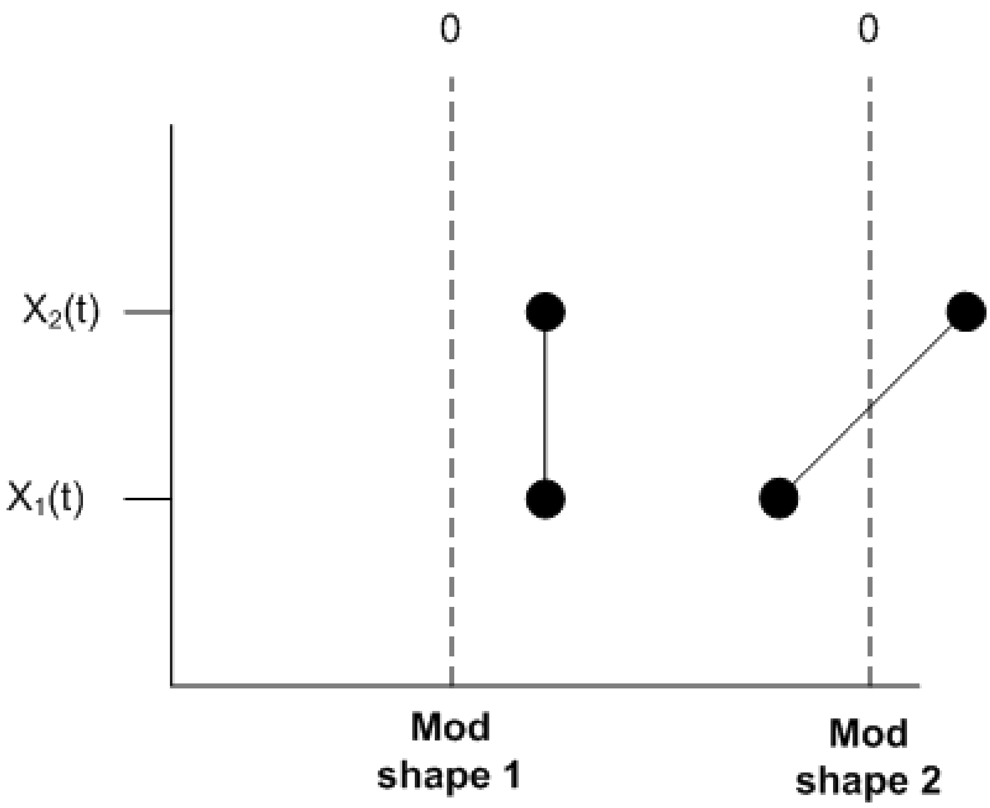
$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.707\,11 \\ 0.707\,11 \end{bmatrix}$$

Now that we have obtained the eigenvectors of the de-coupled system, we can plot the mode shapes[2]. I will use a diagram similar to that shown in the textbook Engineering Vibration by Inman on page 313)

[2]The book also calls the S matrix as the shape matrix, so I better show this as well, which is defined as $S = M^{-\frac{1}{2}} P$, hence

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{aligned} S &= \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}^{-\frac{1}{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 2.5^{-\frac{1}{2}} & 0 \\ 0 & 2.5^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \sqrt{2} \begin{bmatrix} 0.316\,23 & -0.316\,23 \\ 0.316\,23 & 0.316\,23 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 0.447\,22 & -0.447\,22 \\ 0.447\,22 & 0.447\,22 \end{bmatrix}} \end{aligned}$$



$$\begin{bmatrix} 0.70711 \\ 0.70711 \end{bmatrix} \begin{bmatrix} -0.70711 \\ 0.70711 \end{bmatrix}$$

3.2.6 Problem 4

4. $\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \ddot{X} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} X = 0$

Calculate the response of the system to IC

$X_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \quad \dot{X}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$M\ddot{\mathbf{x}}+K\mathbf{x} = \mathbf{0}$$

Where $K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}, M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$ Let $\mathbf{X} = M^{-\frac{1}{2}}\mathbf{q}$, then $\ddot{\mathbf{X}} = M^{-\frac{1}{2}}\ddot{\mathbf{q}}$ and the above equation becomes

$$MM^{-\frac{1}{2}}\ddot{\mathbf{q}} + KM^{-\frac{1}{2}}\mathbf{q} = \mathbf{0}$$

premultiply by $M^{-\frac{1}{2}}$ we obtain

$$\begin{aligned} M^{-\frac{1}{2}}MM^{-\frac{1}{2}}\ddot{\mathbf{q}} + M^{-\frac{1}{2}}KM^{-\frac{1}{2}}\mathbf{q} &= \mathbf{0} \\ I\ddot{\mathbf{q}} + \tilde{K}\mathbf{q} &= \mathbf{0} \end{aligned} \tag{2}$$

Where $\tilde{K} = M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$

Let $\mathbf{q} = \mathbf{v}e^{i\omega t}$, then $\ddot{\mathbf{q}} = -\omega^2\mathbf{v}e^{i\omega t}$ and (2) becomes

$$\begin{aligned} -\omega^2 e^{i\omega t} I\mathbf{v} + \tilde{K}\mathbf{v}e^{i\omega t} &= \mathbf{0} \\ \left(\tilde{K} - \omega^2 I\right)\mathbf{v} &= \mathbf{0} \end{aligned}$$

Let $\lambda = \omega^2$ then we have

$$\left(\tilde{K} - \lambda I\right)\mathbf{v} = \mathbf{0} \tag{3}$$

For $\mathbf{v} \neq \mathbf{0}$, we requires that $\left|\tilde{K} - \lambda I\right| = 0$ But

$$\begin{aligned}
\tilde{K} &= M^{-\frac{1}{2}} K M^{-\frac{1}{2}} \\
&= \begin{bmatrix} 9^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 9^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3.0 & -1.0 \\ -1.0 & 3.0 \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\left|\tilde{K} - \lambda I\right| &= 0 \\
\left|\begin{bmatrix} 3.0 & -1.0 \\ -1.0 & 3.0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| &= 0 \\
\left|\begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix}\right| &= 0 \\
(3-\lambda)^2 - 1 &= 0
\end{aligned}$$

Hence the characteristic equation is

$$\boxed{\lambda^2 - 6\lambda + 8 = 0}$$

Hence

$$\boxed{\lambda_{1,2} = \{2, 4\}}$$

Then the natural frequencies are

$$\omega_n = \left\{\sqrt{2}, 2\right\}$$

From (3) we then have

$$\begin{aligned}
\left(\tilde{K} - \lambda I\right)\mathbf{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 3.0 & -1.0 \\ -1.0 & 3.0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\mathbf{v} &= \mathbf{0}
\end{aligned}$$

When $\lambda = \lambda_1 = 2$ we obtain

$$\begin{aligned}
\left(\begin{bmatrix} 3.0 & -1.0 \\ -1.0 & 3.0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
a - b &= 0 \\
-a + b &= 0
\end{aligned}$$

Then $a = b$, hence

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.707\,11 \\ 0.707\,11 \end{bmatrix}$$

When $\lambda = \lambda_2 = 4$ we obtain

$$\begin{aligned}
\left(\begin{bmatrix} 3.0 & -1.0 \\ -1.0 & 3.0 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

Hence $a = -b$, then

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.707\,11 \\ -0.707\,11 \end{bmatrix}$$

Then the matrix

$$\begin{aligned}
[P] &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 0.707\,11 & 0.707\,11 \\ 0.707\,11 & -0.707\,11 \end{bmatrix}}
\end{aligned}$$

Now let $\mathbf{q} = P\mathbf{r}$, then equation (2) above becomes

$$\begin{aligned}
I\ddot{\mathbf{q}} + \tilde{K}\mathbf{q} &= \mathbf{0} \\
IP\ddot{\mathbf{r}} + \tilde{K}P\mathbf{r} &= \mathbf{0}
\end{aligned}$$

Premultiply by P^T

$$P^T I P\ddot{\mathbf{r}} + P^T \tilde{K} P\mathbf{r} = \mathbf{0}$$
$$I\ddot{\mathbf{r}} + P^T \tilde{K} P\mathbf{r} = \mathbf{0}$$

Let $\Lambda = P^T \tilde{K} P$ then the above becomes

$$I\ddot{\mathbf{r}} + \Lambda\mathbf{r} = \mathbf{0} \tag{4}$$

Now find Λ

$$\begin{aligned} \Lambda &= P^T \tilde{K} P \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}} \end{aligned}$$

Hence (4) becomes

$$I\ddot{\mathbf{r}} + \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{r} = \mathbf{0}$$

Which can be written as 2 equations

$$\begin{bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \end{bmatrix} + \begin{bmatrix} 2r_1 \\ 4r_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\ddot{r}_1 + 2r_1 = 0 \tag{5}$$
$$\ddot{r}_2 + 4r_2 = 0 \tag{6}$$

With IC given as $\mathbf{X}(0) = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$, $\dot{\mathbf{X}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but

$\mathbf{X} = M^{-\frac{1}{2}}\mathbf{q}$ and $\mathbf{q} = P\mathbf{r}$, hence $\mathbf{X} = M^{-\frac{1}{2}} P\mathbf{r}$, then

$$\begin{aligned} \mathbf{r}(0) &= P^T M^{\frac{1}{2}} \mathbf{X}(0) \\ \mathbf{r}(0) &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} 9^{\frac{1}{2}} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \\ \begin{bmatrix} r_1(0) \\ r_2(0) \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

And since $\dot{\mathbf{X}}(0) = \mathbf{0}$, then $\dot{\mathbf{r}}(0) = 0$, now we have found IC for $r(t)$ we can solve the ODEs

$$\begin{aligned} r_1(t) &= A_1 \cos \sqrt{2}t + B_1 \sin \sqrt{2}t \\ r_2(t) &= A_2 \cos 2t + B_2 \sin 2t \end{aligned}$$

$r_1(0) = 1$ hence $A_1 = 1$, and $B_1 = 0$, similarly, $A_2 = 0$, and $B_1 = 0$, hence

$$\begin{aligned} r_1(t) &= \cos \sqrt{2}t \\ r_2(t) &= 0 \end{aligned}$$

But

$$\mathbf{X}(t) = M^{-\frac{1}{2}} P\mathbf{r}(t)$$

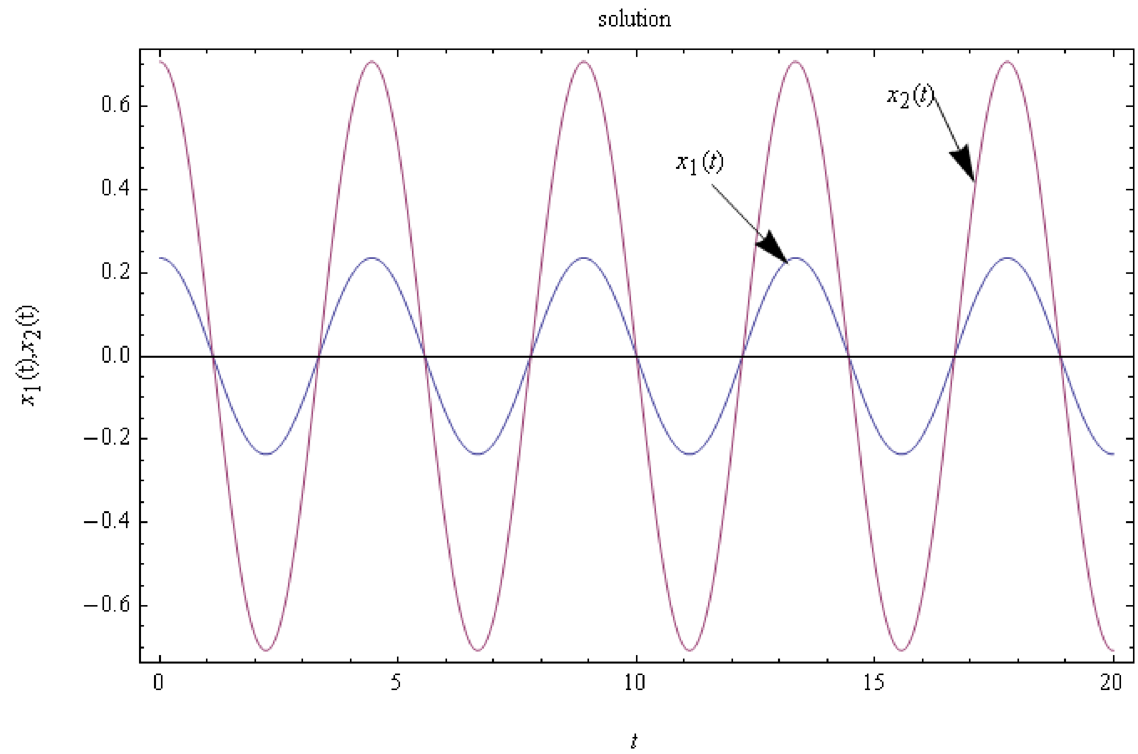
Then

$$\begin{aligned} \mathbf{X}(t) &= \begin{bmatrix} 9^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{r}(t) \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos \sqrt{2}t \\ 0 \end{bmatrix} \end{aligned}$$

Hence

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3\sqrt{2}} (\cos \sqrt{2}t) \\ \frac{1}{\sqrt{2}} (\cos \sqrt{2}t) \end{bmatrix}}$$

Here is a plot of the solution



3.2.7 Problem 5 (not correct, left here to check something)

5 Calculate the response of the system

$3\ddot{x}(t) + 6\dot{x}(t) + 12x(t) = 3\delta(t) - \delta(t-1)$

Subject to IC $x(0) = 0.01\,m$ $V(0) = 1\,m/sec$

$m = 3, c = 6, k = 12$, hence $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12}{3}} = 2$ rad/sec and $\xi = \frac{c}{c_{cr}} = \frac{c}{2\omega_n m} = \frac{6}{2\times 2\times 3} = \frac{1}{2}$, hence the system is underdamped and $\omega_d = \omega_n\sqrt{1-\xi^2} = 2\sqrt{1-\frac{1}{2}^2} = \sqrt{3}$ rad/sec

Let the response to $3\delta(t)$ be $x_{p_1}(t)$ and let the response to $\delta(t-1)$ be $x_{p_2}(t)$ hence the response of the system becomes

$$x(t) = x_h(t) + x_{p_1}(t) - x_{p_2}(t) \tag{1}$$

Where

$$x_h = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) \tag{2}$$

And

$$x_{p_1}(t) = \frac{3}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t \tag{3}$$

and

$$x_{p_2}(t) = \frac{1}{m\omega_d}e^{-\xi\omega_n(t-1)}\sin\omega_d(t-1)\,\Phi(t-1)$$

Hence, substitute (2),(3) into (1)

$$x(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{3}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t + \frac{1}{m\omega_d}e^{-\xi\omega_n(t-1)}\sin\omega_d(t-1)\,\Phi(t-1) \tag{4}$$

Now using IC to find A, B. Note, we use only $x(t) = x_h(t) + x_{p_1}(t)$ for the purpose of finding A, B from I.C's since the response to the delayed impulse is not active at $t = 0$. We find

$$x(0) = \frac{1}{100} = A$$

And for the derivative

$$\begin{aligned}\dot{x}(t) &= \dot{x}_h(t) + \dot{x}_{p_1}(t)\\ &= -\xi\omega_n e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\xi\omega_n t}(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t)\\ &+ \frac{3}{m\omega_d}e^{-\xi\omega_n t}\omega_d\cos\omega_d t - \frac{3\xi\omega_n}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t\end{aligned}$$

Hence

$$\begin{aligned}\dot{x}(0) = 1 &= -\xi\omega_n A + B\omega_d + \frac{3}{m}\\ 1 &= -\frac{1}{100} + B\omega_d + 1\end{aligned}$$

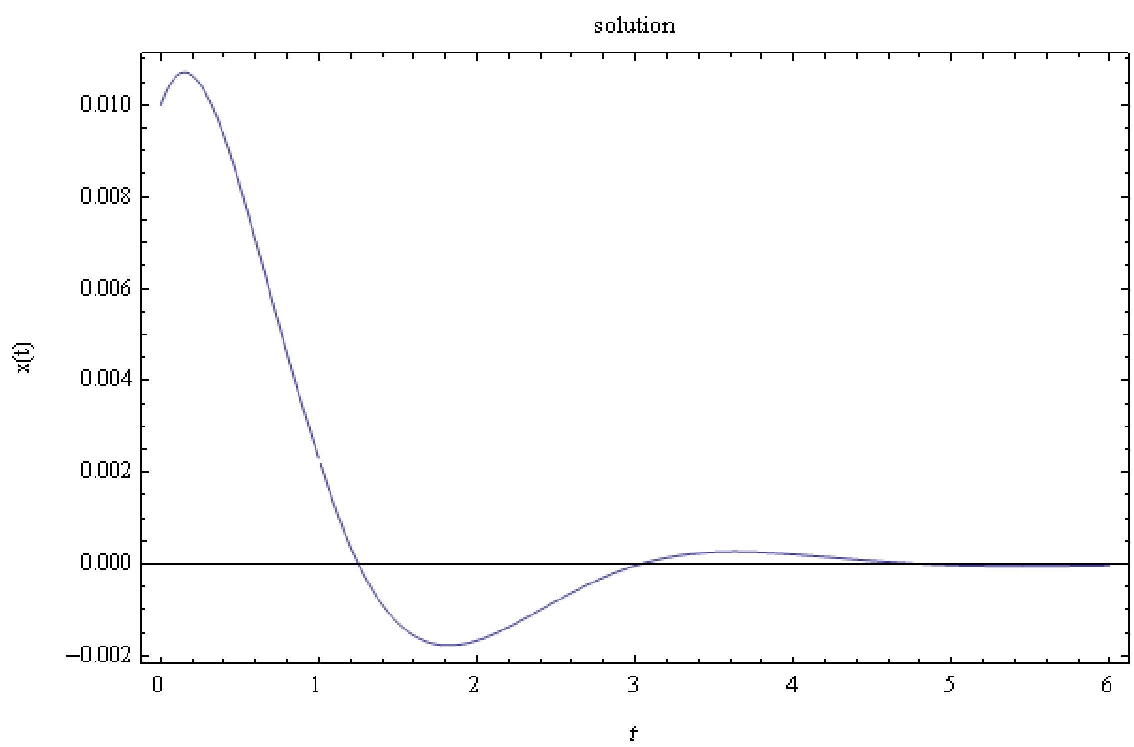
Hence

$$B = \frac{1}{100\sqrt{3}}$$

Therefore the solution is, by substituting values found for A, B into the general solution from above equation (4), we obtain

$$x(t) = \frac{e^{-t}}{100}\left(\cos\sqrt{3}t + \frac{1}{100\sqrt{3}}\sin\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t - \left(\frac{1}{3\sqrt{3}}e^{-(t-1)}\sin\sqrt{3}(t-1)\,\Phi(t-1)\right) \tag{5}$$

The following is a plot of the solution for up to $t = 6$



3.2.8 Problem 5 (again, correct solution)

5 Calculate the response of the system
$3\ddot{x}(t) + 6\dot{x}(t) + 12x(t) = 3\delta(t) - \delta(t-1)$
Subject to IC $x(0) = 0.01$ m $V(0) = 1$ m/sec

$m = 3, c = 6, k = 12$, hence $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12}{3}} = 2$ rad/sec and $\xi = \frac{c}{c_{cr}} = \frac{c}{2\omega_n m} = \frac{6}{2\times 2\times 3} = \frac{1}{2}$, hence the system is underdamped and $\omega_d = \omega_n\sqrt{1-\xi^2} = 2\sqrt{1-\frac{1}{2}^2} = \sqrt{3}$ rad/sec

Let the response to $3\delta(t)$ be $x_{p_1}(t)$ and let the response to $\delta(t-1)$ be $x_{p_2}(t)$ hence the response of the system becomes

$$x(t) = x_h(t) + x_{p_1}(t) - x_{p_2}(t) \tag{1}$$

Where

$$x_h = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) \tag{2}$$

And

$$x_{p_1}(t) = \frac{3}{m\omega_d} e^{-\xi\omega_n t}\sin\omega_d t \tag{3}$$

and

$$x_{p_2}(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n (t-1)}\sin\omega_d (t-1)\,\Phi(t-1)$$

To find A, B use only $x_h(t)$ .At $t=0$. We find

$$x(0) = \frac{1}{100} = A$$

And for the derivative

$$\begin{aligned}\dot{x}(t) &= \dot{x}_h(t)\\ &= -\xi\omega_n e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\xi\omega_n t}(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t)\end{aligned}$$

Hence

$$\begin{aligned}\dot{x}(0) = 1 &= -\xi\omega_n A + B\omega_d\\ 1 &= -\frac{1}{100} + B\omega_d\end{aligned}$$

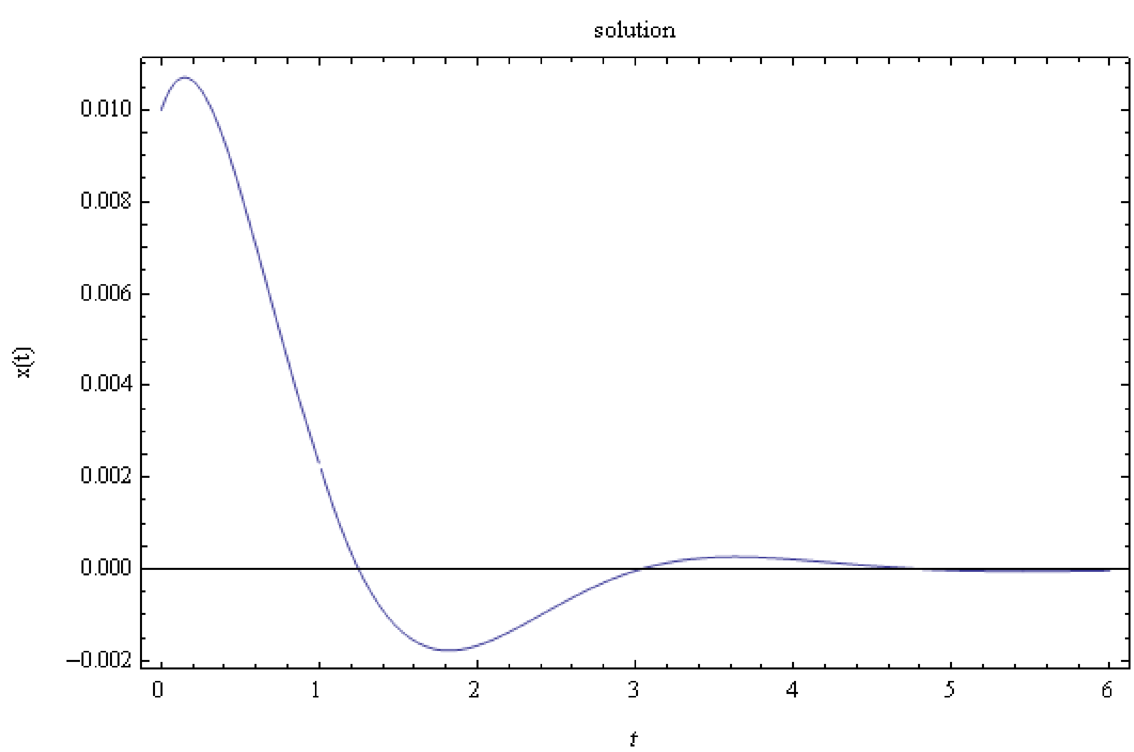
Hence

$$\begin{aligned}B &= \frac{1+\frac{1}{100}}{\sqrt{3}}\\ &= \frac{101}{100\sqrt{3}}\end{aligned}$$

Therefore the solution is, by substituting values found for A, B into the general solution from above equation (4), we obtain

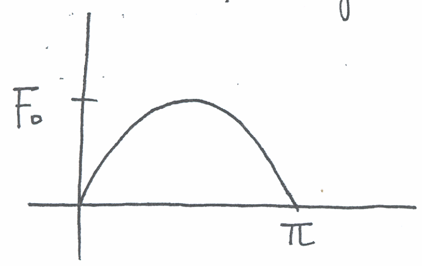
$$x(t) = \frac{e^{-t}}{100}\left(\cos\sqrt{3}t + \frac{101}{100\sqrt{3}}\sin\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t - \left(\frac{1}{3\sqrt{3}}e^{-(t-1)}\sin\sqrt{3}(t-1)\,\Phi(t-1)\right) \tag{5}$$

The following is a plot of the solution for up to $t=6$



3.2.9 Problem 6

6 Calculate the response of an underdamped system to the excitation given below



$f(t) = F_0 \sin t$

Let the response by $x(t)$. Hence $x(t) = x_h(t) + x_p(t)$, where $x_p(t)$ is the particular solution, which is the response due the the above forcing function. Using convolution

$$x_p(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

Where $h(t)$ is the unit impulse response of a second order underdamped system which is

$$h(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n t} \sin \omega_d t$$

hence

$$\begin{aligned} x_p(t) &= \frac{F_0}{m\omega_d} \int_0^t \sin(\tau)\, e^{-\xi\omega_n (t-\tau)} \sin(\omega_d(t-\tau))\, d\tau \\ &= \frac{F_0 e^{-\xi\omega_n t}}{m\omega_d} \int_0^t e^{\xi\omega_n \tau} \sin(\tau) \sin(\omega_d(t-\tau))\, d\tau \end{aligned}$$

Using $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ then

$$\sin(\tau)\sin(\omega_d(t-\tau)) = \frac{1}{2}[\cos(\tau - \omega_d(t-\tau)) - \cos(\tau + \omega_d(t-\tau))]$$

Then the integral becomes

$$x_p(t) = \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d} \left(\int_0^t e^{\xi\omega_n \tau} \cos(\tau - \omega_d(t-\tau))\, d\tau - \int_0^t e^{\xi\omega_n \tau} \cos(\tau + \omega_d(t-\tau))\, d\tau \right)$$

Consider the first integral I_1 where

$$I_1 = \int_0^t e^{\xi\omega_n \tau} \cos(\tau - \omega_d(t-\tau))\, d\tau$$

Integrate by parts, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n \tau} \rightarrow v = \frac{e^{\xi\omega_n \tau}}{\xi\omega_n}$ and let $u = \cos(\tau - \omega_d(t-\tau)) \rightarrow du = -(1+\omega_d)\sin(\tau - \omega_d(t-\tau))$, hence

$$\begin{aligned} I_1 &= \left[\cos(\tau - \omega_d(t-\tau)) \frac{e^{\xi\omega_n \tau}}{\xi\omega_n}\right]_0^t - \int_0^t \frac{e^{\xi\omega_n \tau}}{\xi\omega_n} [-(1+\omega_d)\sin(\tau - \omega_d(t-\tau))]\, d\tau \\ &= \left[\cos(t - \omega_d(t-t)) \frac{e^{\xi\omega_n t}}{\xi\omega_n} - \cos(0 - \omega_d(t-0)) \frac{1}{\xi\omega_n}\right] + \frac{(1+\omega_d)}{\xi\omega_n} \int_0^t e^{\xi\omega_n \tau} \sin(\tau - \omega_d(t-\tau))\, d\tau \\ &= \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{\xi\omega_n} \int_0^t e^{\xi\omega_n \tau} \sin(\tau - \omega_d(t-\tau))\, d\tau \qquad (1) \end{aligned}$$

Integrate by parts again the last integral above, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n \tau} \rightarrow v = \frac{e^{\xi\omega_n \tau}}{\xi\omega_n}$ and let $u = \sin(\tau - \omega_d(t-\tau)) \rightarrow du = (1+\omega_d)\cos(\tau - \omega_d(t-\tau))$, hence

$$\begin{aligned} \int_0^t e^{\xi\omega_n \tau} \sin(\tau - \omega_d(t-\tau))\, d\tau &= \left[\sin(\tau - \omega_d(t-\tau)) \frac{e^{\xi\omega_n \tau}}{\xi\omega_n}\right]_0^t - \int_0^t \frac{e^{\xi\omega_n \tau}}{\xi\omega_n} (1+\omega_d)\cos(\tau - \omega_d(t-\tau))\, d\tau \\ &= \frac{1}{\xi\omega_n}\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right] - \frac{(1+\omega_d)}{\xi\omega_n} \int_0^t e^{\xi\omega_n \tau} \cos(\tau - \omega_d(t-\tau))\, d\tau \qquad (2) \end{aligned}$$

Substitute (2) into (1) we obtain

$$
\begin{aligned}
I_1 &= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t}-\cos(\omega_d t)\right]+\\
&\frac{(1+\omega_d)}{\xi\omega_n}\left(\frac{1}{\xi\omega_n}\left[\sin(t)\,e^{\xi\omega_n t}+\sin(\omega_d t)\right]-\frac{(1+\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos(\tau-\omega_d(t-\tau))\,d\tau\right)\\
&= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t}-\cos(\omega_d t)\right]+\frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t}+\sin(\omega_d t)\right]-\frac{(1+\omega_d)^2}{(\xi\omega_n)^2}\int_0^t e^{\xi\omega_n\tau}\cos(\tau-\omega_d(t-\tau))\,d\tau\\
&= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t}-\cos(\omega_d t)\right]+\frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t}+\sin(\omega_d t)\right]-\frac{(1+\omega_d)^2}{(\xi\omega_n)^2}I_1
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_1+\frac{(1+\omega_d)^2}{(\xi\omega_n)^2}I_1 &= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t}-\cos(\omega_d t)\right]+\frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t}+\sin(\omega_d t)\right]\\
I_1\left(\frac{(\xi\omega_n)^2+(1+\omega_d)^2}{(\xi\omega_n)^2}\right) &= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t}-\cos(\omega_d t)\right]+\frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t}+\sin(\omega_d t)\right]\\
I_1 &= \left(\frac{(\xi\omega_n)^2}{(\xi\omega_n)^2+(1+\omega_d)^2}\right)\left(\frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t}-\cos(\omega_d t)\right]+\frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t}+\sin(\omega_d t)\right]\right)\\
&= \frac{\xi\omega_n\left[\cos(t)\,e^{\xi\omega_n t}-\cos(\omega_d t)\right]+(1+\omega_d)\left[\sin(t)\,e^{\xi\omega_n t}+\sin(\omega_d t)\right]}{(\xi\omega_n)^2+(1+\omega_d)^2}
\end{aligned}
$$

Now consider the second integral I_2 where

$$
I_2=\int_0^t e^{\xi\omega_n\tau}\cos(\tau+\omega_d(t-\tau))\,d\tau
$$

Integrate by parts, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n\tau}\rightarrow v=\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u=\cos(\tau+\omega_d(t-\tau))\rightarrow du=-(1-\omega_d)\sin(\tau+\omega_d(t-\tau))$, hence

$$
\begin{aligned}
I_2 &= \left[\cos(\tau+\omega_d(t-\tau))\,\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t-\int_0^t\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\left[-(1-\omega_d)\sin(\tau+\omega_d(t-\tau))\right]d\tau\\
&= \left[\cos(t+\omega_d(t-t))\,\frac{e^{\xi\omega_n t}}{\xi\omega_n}-\cos(0+\omega_d(t-0))\,\frac{1}{\xi\omega_n}\right]+\frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin(\tau+\omega_d(t-\tau))\,d\tau\\
&= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t}-\cos(\omega_d t)\right]+\frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin(\tau+\omega_d(t-\tau))\,d\tau \qquad (3)
\end{aligned}
$$

Integrate by parts again the last integral above, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n\tau}\rightarrow v=\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u=\sin(\tau+\omega_d(t-\tau))\rightarrow du=(1-\omega_d)\cos(\tau+\omega_d(t-\tau))$, hence

$$
\begin{aligned}
\int_0^t e^{\xi\omega_n\tau}\sin(\tau+\omega_d(t-\tau))\,d\tau &= \left[\sin(\tau+\omega_d(t-\tau))\,\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t-\int_0^t\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}(1-\omega_d)\cos(\tau+\omega_d(t-\tau))\,d\tau\\
&= \frac{1}{\xi\omega_n}\left[\sin(t)\,e^{\xi\omega_n t}-\sin(\omega_d t)\right]-\frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos(\tau+\omega_d(t-\tau))\,d\tau \qquad (4)
\end{aligned}
$$

Substitute (4) into (3) we obtain

$$
\begin{aligned}
I_2 &= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t}-\cos(\omega_d t)\right]+\\
&\frac{(1-\omega_d)}{\xi\omega_n}\left(\frac{1}{\xi\omega_n}\left[\sin(t)\,e^{\xi\omega_n t}-\sin(\omega_d t)\right]-\frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos(\tau+\omega_d(t-\tau))\,d\tau\right)\\
&= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t}-\cos(\omega_d t)\right]+\frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t}-\sin(\omega_d t)\right]-\frac{(1-\omega_d)^2}{(\xi\omega_n)^2}\int_0^t e^{\xi\omega_n\tau}\cos(\tau+\omega_d(t-\tau))\,d\tau\\
&= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t}-\cos(\omega_d t)\right]+\frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t}+\sin(\omega_d t)\right]-\frac{(1-\omega_d)^2}{(\xi\omega_n)^2}I_2
\end{aligned}
$$

Hence

$$\begin{aligned}
I_2 + \frac{(1-\omega_d)^2}{(\xi\omega_n)^2} I_2 &= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right] \\
I_2\left(\frac{(\xi\omega_n)^2 + (1-\omega_d)^2}{(\xi\omega_n)^2}\right) &= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right] \\
I_2 &= \left(\frac{(\xi\omega_n)^2}{(\xi\omega_n)^2 + (1-\omega_d)^2}\right)\left(\frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]\right) \\
&= \frac{\xi\omega_n\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2}
\end{aligned}$$

Using the above expressions for I_1, I_2, we find (and multiplying the solution by $(\Phi(t) - \Phi(t-\pi))$ since the force is only active from $t = 0$ to $t = \pi$, we obtain

$$x_p(t) = \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}(I_1 - I_2)\left(\Phi(t) - \Phi(t-\pi)\right)$$

$$\begin{aligned}
&= (\Phi(t) - \Phi(t-\pi)) * \\
&\frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1+\omega_d)\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1+\omega_d)^2} \\
&- \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2}
\end{aligned} \tag{5}$$

Hence $x_p(t) = (\Phi(t) - \Phi(t-\pi))$

$$\left[\frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\left(\frac{\xi\omega_n\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1+\omega_d)\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1+\omega_d)^2} - \frac{\xi\omega_n\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2}\right)\right]$$

And

$$x_h(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

Hence the overall solution is

$$x(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + x_p(t)$$

The above solution is a bit long due to integration by parts. I will not solve the same problem using Laplace transformation method. The differential equation is

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) = f(t)$$

Take Laplace transform, we obtain (assuming $x(0) = x_0$ and $\dot{x}(0) = v_0$)

$$\begin{aligned}
\left(s^2 X - s x(0) - \dot{x}(0)\right) + 2\xi\omega_n(sX - x(0)) + \omega_n^2 X &= F(s) \\
\left(s^2 X - s x_0 - v_0\right) + 2\xi\omega_n(sX - x_0) + \omega_n^2 X &= F(s)
\end{aligned} \tag{7}$$

Now we find Laplace transform of $f(t)$

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \int_0^\pi e^{-st} F_0 \sin t\ dt \\
&= F_0\left[\int_0^\pi e^{-st}\sin t\ dt\right]
\end{aligned}$$

Integration by parts gives

$$F(s) = F_0\left[\frac{1 + e^{-\pi s}}{1 + s^2}\right] \tag{8}$$

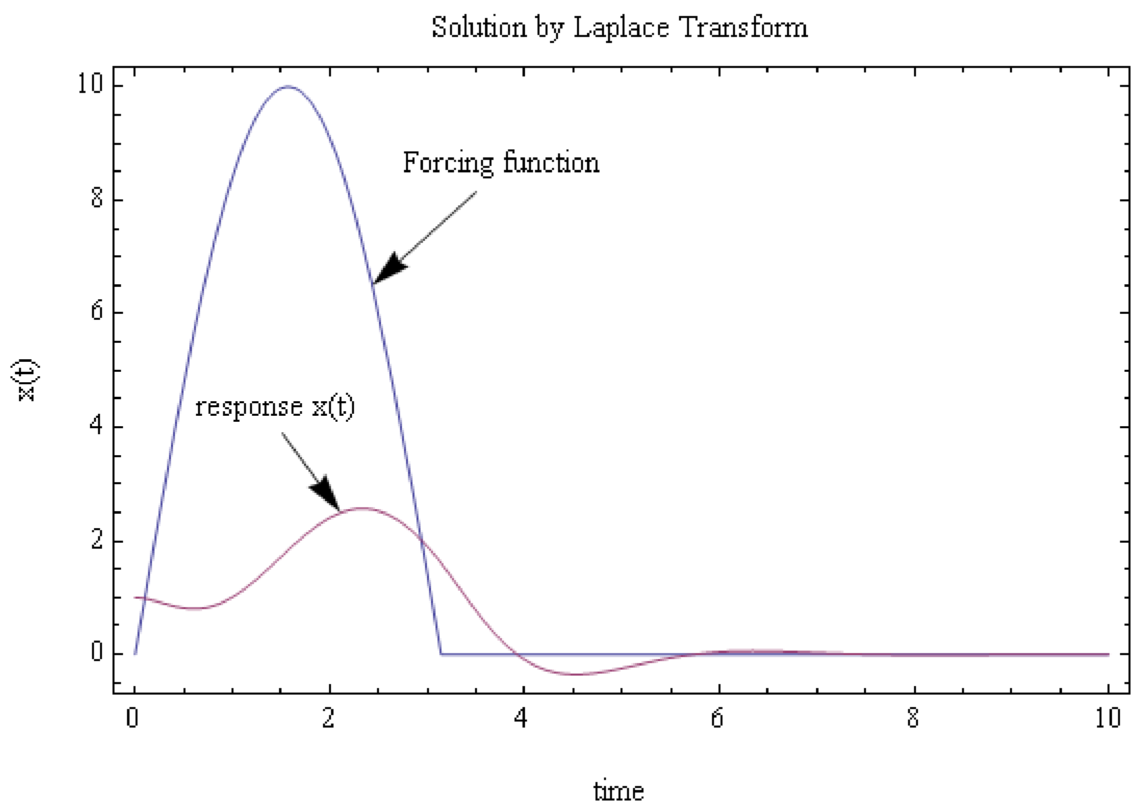
Substitute (8) into (7) we obtain

$$\begin{aligned}
\left(s^2 X - s x_0 - v_0\right) + 2\xi\omega_n(sX - x_0) + \omega_n^2 X &= F_0\left[\frac{1 + e^{-\pi s}}{1 + s^2}\right] \\
X\left(s^2 + 2\xi\omega_n s + \omega_n^2\right) - s x_0 - v_0 - 2\xi\omega_n x_0 &= \frac{F_0(1 + e^{-\pi s})}{1 + s^2} \\
X\left(s^2 + 2\xi\omega_n s + \omega_n^2\right) &= \frac{F_0(1 + e^{-\pi s})}{1 + s^2} + s x_0 + v_0 + 2\xi\omega_n x_0 \\
&= \frac{F_0(1 + e^{-\pi s}) + \left(1 + s^2\right) s x_0 + v_0\left(1 + s^2\right) + 2\xi\omega_n x_0\left(1 + s^2\right)}{1 + s^2}
\end{aligned}$$

Hence

$$X = \frac{F_0(1+e^{-\pi s}) + (1+s^2)\, s x_0 + v_0 (1+s^2) + 2\xi\omega_n x_0 (1+s^2)}{(1+s^2)\,(s^2+2\xi\omega_n s+\omega_n^2)}$$
$$= \frac{F_0 + v_0 + \frac{F_0}{e^{\pi s}} + s x_0 + s^2 v_0 + s^3 x_0 + 2\xi\omega_n x_0 + 2s^2\xi\omega_n x_0}{(1+s^2)\,(s^2+2\xi\omega_n s+\omega_n^2)}$$

Now we can use inverse Laplace transform on the above. It is easier to do partial fraction decomposition and use tables. I used CAS to do this and this is the result. I plot the solution $x(t)$. I used the following values to be able to obtain a plot $\xi = 0.5, \omega_n = 2, F_0 = 10, x_0 = 1, v_0 = 0$



3.2.10 Solving problem shown in class for Vibration 431, CSUF, Spring 2009

Problem

Solve $\ddot{x} + 2\dot{x} + 4x = \delta(t) - \delta(t-4)$ with the IC's $x(0) = 1mm, \dot{x}(0) = -1mm$

Answer

$m = 1, c = 2, k = 4$, hence $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{4} = 2$ rad/sec and $\xi = \frac{c}{c_{cr}} = \frac{c}{2\omega_n m} = \frac{2}{2\times 2\times 1} = \frac{1}{2}$, hence the system is underdamped and $\omega_d = \omega_n\sqrt{1-\xi^2} = 2\sqrt{1-\frac{1}{2}^2} = \sqrt{3}$ rad/sec

Let the response to $\delta(t)$ be $x_1(t)$ and let the response to $\delta(t-4)$ be $x_2(t)$ hence the response of the system becomes

$$x(t) = x_h(t) + x_1(t) - x_2(t) \tag{1}$$

Where

$$x_h = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) \tag{1}$$

And

$$x_1(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n t}\sin\omega_d t \tag{3}$$

and

$$x_2(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n (t-4)}\sin\omega_d (t-4)\,\Phi(t-4)$$

Hence, substitute (2),(3) ,(4) into (1)

$$x(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{1}{m\omega_d} e^{-\xi\omega_n t}\sin\omega_d t - \frac{1}{m\omega_d} e^{-\xi\omega_n (t-4)}\sin\left(\omega_d (t-4)\right)\Phi(t-4) \tag{4}$$

Now using IC to find A, B

$$x(0) = 1 = A$$

and

$$\dot{x}(t) = -\xi\omega_n e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\xi\omega_n t}(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t) +$$
$$\frac{1}{m\omega_d}\left(-\xi\omega_n e^{-\xi\omega_n t}\sin\omega_d t + \omega_d e^{-\xi\omega_n t}\cos\omega_d t\right) -$$
$$\frac{e^{-\xi\omega_n(t-4)}}{m\omega_d}\left(\omega_d\cos\left(\omega_d(t-4)\right)\Phi(t-4) + \delta(t-4)\sin\left(\omega_d(t-4)\right) - \xi\omega_n\omega_d\sin\left(\omega_d(t-4)\right)\Phi(t-4)\right)$$

At $t = 0, \dot{x}(0) = -1$, Hence the above becomes (terms with $\delta(t-4)$ and $\Phi(t-4)$ vanish at $t = 0$ by definition)

$$-1 = -\xi\omega_n A + B\omega_d + \frac{1}{m}$$
$$B = \frac{-1}{\sqrt{3}}$$

Hence (1) becomes

$$x(t) = e^{-\xi\omega_n t}\left(\cos\omega_d t - \frac{1}{\sqrt{3}}\sin\omega_d t\right) + \frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t - \frac{1}{m\omega_d}e^{-\xi\omega_n(t-4)}\sin\left(\omega_d(t-4)\right)\Phi(t-4)$$

If we substitute the numerical values for the problem parameters, the above becomes

$$x(t) = e^{-t}\left(\cos\sqrt{3}t - \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right) + \frac{e^{-t}}{\sqrt{3}}\sin\sqrt{3}t - \frac{1}{\sqrt{3}}e^{-(t-4)}\sin\left(\sqrt{3}(t-4)\right)\Phi(t-4)$$
$$= \boxed{e^{-t}\cos\sqrt{3}t - \tfrac{1}{\sqrt{3}}e^{-(t-4)}\sin\left(\sqrt{3}(t-4)\right)\Phi(t-4)}$$

Compare the above with the solution given in class, which is

$$x(t) = \boxed{e^{-t}\left(\cos\sqrt{3}t + \tfrac{1}{\sqrt{3}}\sin\sqrt{3}t\right) - \tfrac{1}{\sqrt{3}}e^{-(t-4)}\sin\left(\sqrt{3}(t-4)\right)\Phi(t-4)}$$

3.2.11 Solving problem shown in class for Vibration 431, CSUF, Spring 2009. Version 2

Problem

Solve $\ddot{x} + 2\dot{x} + 4x = \delta(t) - \delta(t-4)$ with the IC's $x(0) = 1mm, \dot{x}(0) = -1mm$

Answer

$m = 1, c = 2, k = 4$, hence $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{4} = 2$ rad/sec and $\xi = \frac{c}{c_{cr}} = \frac{c}{2\omega_n m} = \frac{2}{2\times2\times1} = \frac{1}{2}$, hence the system is underdamped and $\omega_d = \omega_n\sqrt{1-\xi^2} = 2\sqrt{1-\frac{1}{2}^2} = \sqrt{3}$ rad/sec

Let the response to $\delta(t)$ be $x_{p_1}(t)$ and let the response to $\delta(t-4)$ be $x_{p_2}(t)$ hence the response of the system becomes

$$x(t) = x_h(t) + x_{p_1}(t) - x_{p_2}(t) \tag{1}$$

Where

$$x_h = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) \tag{1}$$

And

$$x_{p_1}(t) = \frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t \tag{3}$$

and

$$x_{p_2}(t) = \frac{1}{m\omega_d}e^{-\xi\omega_n(t-4)}\sin\omega_d(t-4)\,\Phi(t-4)$$

Hence, substitute (2),(3) ,(4) into (1)

$$x(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t - \frac{1}{m\omega_d}e^{-\xi\omega_n(t-4)}\sin\left(\omega_d(t-4)\right)\Phi(t-4) \tag{4}$$

Now using IC to find A, B

$$x(0) = 1$$

Hence

$$\boxed{A = 1}$$

Now take the derivative of the above and evaluate at zero to find B. In doing so, we need to consider only the x_h. The reason is that the particular solution $x_{p_2}(t)$ of the delayed pulse (the second pulse) will have no effect at $t = 0$ and the first pulse particular solution $x_{p_1}(t)$ will also have no contribution, since its response is assume to occur at 0^+, i.e. an infitismal time after $t = 0$. Therefore, since we intend to evaluate $\dot{x}(t)$ at $t = 0$, we only need to take x_h derivative at this point

$$\dot{x}(t) = -\xi\omega_n e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\xi\omega_n t}(-A\omega_d \sin\omega_d t + B\omega_d \cos\omega_d t)$$

At $t = 0, \dot{x}(0) = -1$, Hence the above becomes

$$\begin{aligned} -1 &= -\xi\omega_n A + B\omega_d \\ -1 &= -1 + B\sqrt{3} \\ &\boxed{B = 0} \end{aligned}$$

Hence (1) becomes

$$x(t) = e^{-\xi\omega_n t}\cos\omega_d t + \frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t - \frac{1}{m\omega_d}e^{-\xi\omega_n (t-4)}\sin\left(\omega_d(t-4)\right)\Phi(t-4)$$

If we substitute the numerical values for the problem parameters, the above becomes

$$\begin{aligned} x(t) &= e^{-t}\cos\sqrt{3}t + \frac{e^{-t}}{\sqrt{3}}\sin\sqrt{3}t - \frac{1}{\sqrt{3}}e^{-(t-4)}\sin\left(\sqrt{3}(t-4)\right)\Phi(t-4) \\ &= \boxed{e^{-t}\left(\cos\sqrt{3}t + \tfrac{1}{\sqrt{3}}\sin\sqrt{3}t\right) - \tfrac{1}{\sqrt{3}}e^{-(t-4)}\sin\left(\sqrt{3}(t-4)\right)\Phi(t-4)} \end{aligned}$$

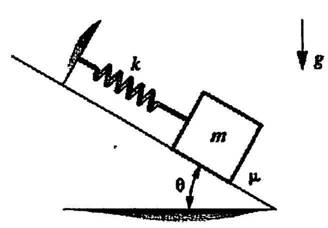
Which now matches the solution given in class

3.2.12 Key for HW2

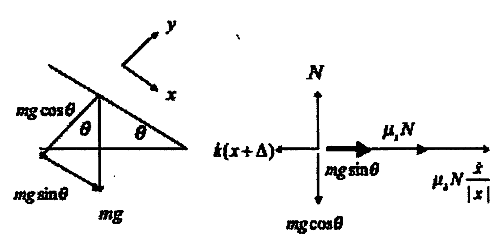
EGME 511

HW #2
SOLUTIONS

1.



Solution: Choose the x y coordinate system to be along the incline and perpendicular to it. Let μ_s denote the static friction coefficient, μ_k the coefficient of kinetic friction and Δ the static deflection of the spring. A drawing indicating the angles and a free-body diagram is given in the figure:



For the static case

$$\sum F_x = 0 \Rightarrow k\Delta = \mu_s N + mg\sin\theta, \text{ and } \sum F_y = 0 \Rightarrow N = mg\cos\theta$$

For the dynamic case

$$\sum F_x = m\ddot{x} = -k(x+\Delta) + \mu_s N + mg\sin\theta - \mu_k N\frac{\dot{x}}{|\dot{x}|}$$

Combining these three equations yields

$$m\ddot{x} + \mu_k mg\cos\theta\frac{\dot{x}}{|\dot{x}|} + kx = 0$$

Note that as the angle θ goes to zero the equation of motion becomes that of a spring mass system with Coulomb friction on a flat surface as it should.

2 **Solution:** Given:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = 0$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & -0.5 \\ -0.5 & 0.25-\lambda \end{vmatrix} = \lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_{1,2} = 0.162, 3.088$$

The spectral matrix is

$$\Lambda = \text{diag}\left(\lambda_i\right) = \begin{bmatrix} 0.162 & 0 \\ 0 & 3.088 \end{bmatrix}$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 0.162$$

$$\begin{bmatrix} 2.838 & -0.5 \\ -0.5 & 0.088 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \Rightarrow v_{11} = 1.762 v_{21}$$

$$\|v_1\| = \sqrt{v_{11}^2 + v_{21}^2} = \sqrt{(0.1762)^2 v_{21}^2 + v_{21}^2} = 1.015 v_{21} = 1$$

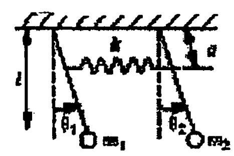
$$v_{21} = 0.9848 \text{ and } v_{11} = 0.1735 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.1735 \\ 0.9848 \end{bmatrix}$$

$$\lambda_2 = 3.088$$

$$\begin{bmatrix} -0.088 & -0.5 \\ -0.5 & -2.838 \end{bmatrix}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 0 \Rightarrow v_{12} = 1.762 v_{22}$$

$$\|v_2\| = \sqrt{v_{12}^2 + v_{22}^2} = \sqrt{(-5.676)^2 v_{22}^2 + v_{22}^2} = 5.764 v_{22} = 1$$

$$\Rightarrow v_{22} = 0.1735 \text{ and } v_{12} = -0.9848 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.9848 \\ 0.1735 \end{bmatrix}$$



3 **Solution:** Given:

$$k = 20 \text{ N/m} \quad m_1 = m_2 = 10 \text{ kg}$$
$$a = 0.1 \text{ m} \qquad l = 0.5 \text{ m}$$

For gravity use $g = 9.81 \text{ m/s}^2$. For a mass on a pendulum, the inertia is: $I = ml^2$

Calculate mass and stiffness matrices (for small θ). The equations of motion are:

$$\begin{array}{l} I_1\ddot{\theta}_1 = ka^2(\theta_2 - \theta_1) - m_1 g l \theta_1 \\ I_2\ddot{\theta}_2 = -ka^2(\theta_2 - \theta_1) - m_2 g l \theta_2 \end{array} \Rightarrow ml^2\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} mgl + ka^2 & -ka^2 \\ -ka^2 & mgl + ka^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Substitution of the given values yields:

$$\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}\ddot{\theta} + \begin{bmatrix} 49.05 & -0.2 \\ -0.2 & 49.05 \end{bmatrix}\theta = 0$$

Natural frequencies:

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 19.7 & -0.08 \\ -0.08 & 19.7 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 19.54 \text{ and } \lambda_2 = 19.7 \Rightarrow \omega_1 = 4.42 \text{ rad/s and } \omega_2 = 4.438 \text{ rad/s}$$

Eigenvectors:

$\lambda_1 = 19.54$

$$\begin{bmatrix} 0.08 & -0.08 \\ -0.08 & 0.08 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 19.7$

$$\begin{bmatrix} -0.08 & -0.08 \\ -0.08 & -0.08 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

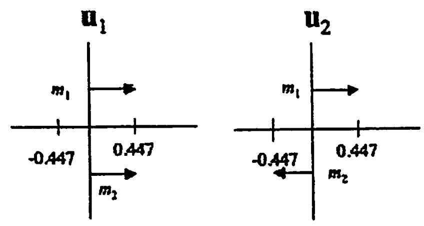
Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Mode shapes:

$$\mathbf{u}_1 = M^{-1/2}\mathbf{v}_1 = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix}$$

$$\mathbf{u}_2 = M^{-1/2}\mathbf{v}_2 = \begin{bmatrix} 0.4472 \\ -0.4472 \end{bmatrix}$$

A plot of the mode shapes is simply



This shows the first mode vibrates in phase and in the second mode the masses vibrate out of phase.

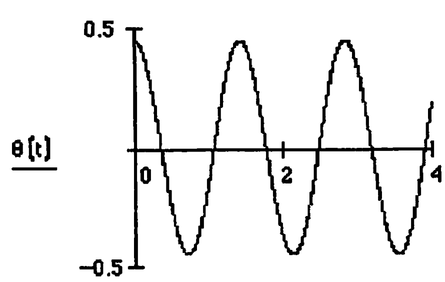
$$\theta(0) = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix} \quad \dot{\theta}(0) = 0, \; S = M^{-1/2}P = \begin{bmatrix} 0.4472 & 0.4472 \\ 0.4472 & -0.4472 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 1.118 & 1.118 \\ 1.118 & -1.118 \end{bmatrix}, \; \mathbf{r}(0) = S^{-1}\theta(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \dot{\mathbf{r}}(0) = 0$$

$$r_1(t) = \sin\left(4.42t + \frac{\pi}{2}\right) = \cos 4.45t, \; r_2(t) = 0$$

Convert to physical coordinates: $\theta(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.4472\cos 4.42t \\ 0.4472\cos 4.42t \end{bmatrix}$ rad

$\theta(t) := 0.4472\cdot\cos(4.429\cdot t)$



4

Solution:

$\omega_1 = \sqrt{\lambda_1} = 1.414$ rad/s, $\omega_2 = \sqrt{\lambda_2} = 2$ rad/s

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow S = M^{-1/2}P = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \text{ and } S^{-1} = P^T M^{1/2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

Next compute the modal initial conditions

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solution ror

$$\mathbf{r}(t) = \begin{bmatrix} \cos 1.414t \\ 0 \end{bmatrix}$$

Note that the second coordinate modal coordinate has zero initial conditions and is hence not vibrating. Convert this solution back into physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \cos 1.414t \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} 0.236\cos 1.414t \\ 0.707\cos 1.414t \end{bmatrix}$$

The unique feature about the solution is that both masses are vibrating at only one frequency. That is the frequency of the first mode shape. This is because the system is excited with a position vector equal to the first mode of vibration.

5

Solution: First compute the natural frequency and damping ratio:

$$\omega_n = \sqrt{\frac{12}{3}} = 2 \text{ rad/s}, \ \zeta = \frac{6}{2\cdot 2\cdot 3} = 0.5, \ \omega_d = 2\sqrt{1-0.5^2} = 1.73 \text{ rad/s}$$

so that the system is underdamped. Next compute the responses to the two impulses:

$$x_1(t) = \frac{\hat{F}}{m\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t = \frac{3}{3(1.73)} e^{-(t-1)}\sin 1.73(t-1) = 0.577 e^{-t}\sin 1.73t, t > 0$$

$$x_2(t) = \frac{\hat{F}}{m\omega_d} e^{-\zeta\omega_n (t-1)}\sin\omega_d (t-1) = \frac{1}{3(1.73)} e^{-t}\sin 1.73t = 0.193 e^{-(t-1)}\sin 1.73(t-1), t > 1$$

Now compute the response to the initial conditions

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}, \quad \phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] = 0.071 \text{ rad}$$

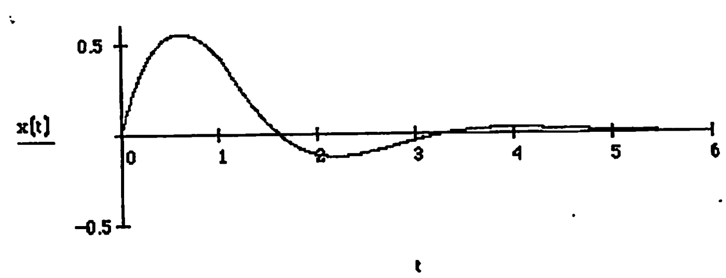
$$\Rightarrow x_h(t) = 0.5775 e^{-t}\sin(t + 0.017)$$

Using the Heaviside function the total response is

$$x(t) = 0.577 e^{-t}\sin 1.73t + 0.583 e^{-t}\sin(t + 0.017) + 0.193 e^{-(t-1)}\sin 1.73(t-1)\Phi(t-1)$$

This is plotted below in Mathcad:

$$x(t) := \left(\frac{e^{-\zeta\cdot\omega n\cdot t}}{\omega d}\sin(\omega d\cdot t) + A\cdot e^{-\zeta\cdot\omega n\cdot t}\cdot\sin(\omega d\cdot t + \phi)\right) + \left[\frac{e^{-\zeta\cdot\omega n\cdot(t-1)}}{-3\cdot\omega d}\sin[\omega d\cdot(t-1)]\right]\cdot\Phi(t-1)$$



Note the slight bump in the response at $t = 1$ when the second impact occurs.

6

Solution:

$$x(t)=\frac{1}{m\omega_d}e^{-\zeta\omega_n t}\int_0^t\left[F(\tau)e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\right]d\tau$$

$$F(t)=F_0\sin(t)\quad , t<\pi\quad \text{(From Figure P3.16)}$$

For $t\le\pi$, $$x(t)=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\int_0^t\left(\sin\tau e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\right)d\tau$$

$$x(t)=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\times$$

$$\left[\frac{1}{2\left[1+2\omega_d+\omega_n^2\right]}\left\{e^{\zeta\omega_n t}\left[(\omega_d-1)\sin t-\zeta\omega_n\cos t\right]-(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\right\}\right.$$

$$\left.+\frac{1}{2\left[1+2\omega_d+\omega_n^2\right]}\left\{e^{\zeta\omega_n t}\left[(\omega_d-1)\sin t-\zeta\omega_n\cos t\right]+(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\right\}\right]$$

For $\tau>\pi$,: $\int_0^t f(\tau)h(t-\tau)d\tau=\int_0^\pi f(\tau)h(t-\tau)d\tau+\int_\pi^t(0)h(t-\tau)d\tau$

$$x(t)=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\int_0^{\pi}\left(\sin\tau e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\right)d\tau$$

$$=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\times$$

$$\left[\frac{1}{2\left[1+2\omega_d+\omega_n^2\right]}\left\{\begin{matrix} e^{\zeta\omega_n t}\left[(\omega_d-1)\sin\left[\omega_d(t-\pi)\right]-\zeta\omega_n\cos\left[\omega_d(t-\pi)\right]\right] \\ -(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\end{matrix}\right\}\right.$$

$$\left.+\frac{1}{2\left[1+2\omega_d+\omega_n^2\right]}\left\{\begin{matrix} e^{\zeta\omega_n t}\left[(\omega_d+1)\sin\left[\omega_d(t-\tau)\right]+\zeta\omega\cos\left[\omega_d(t-\pi)\right]\right] \\ +(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\end{matrix}\right\}\right]$$

Alternately, one could take a Laplace Transform approach and assume the under-damped system is a mass-spring-damper system of the form

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t)$$

The forcing function given can be written as

$$F(t)=F_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

Normalizing the equation of motion yields

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2x(t)=f_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

where $f_0=\frac{F_0}{m}$ and m, c and k are such that $0<\zeta<1$.

Assuming initial conditions, transforming the equation of motion into the Laplace domain yields

$$X(s)=\frac{f_0\left(1+e^{-\pi s}\right)}{\left(s^2+1\right)\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

The above expression can be converted to partial fractions

$$X(s)=f_0\left(1+e^{-\pi s}\right)\left(\frac{As+B}{s^2+1}\right)+f_0\left(1+e^{-\pi s}\right)\left(\frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

where A, B, C, and D are found to be

$$A = \frac{-2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$
$$B = \frac{\omega_n^2 - 1}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$
$$C = \frac{2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$
$$D = \frac{\left(1-\omega_n^2\right) + \left(2\zeta\omega_n\right)^2}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

Notice that $X(s)$ can be written more attractively as

$$\begin{aligned} X(s) &= f_0\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right) + f_0 e^{-\pi s}\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right) \\ &= f_0\left(G(s) + e^{-\pi s}G(s)\right) \end{aligned}$$

Performing the inverse Laplace Transform yields

$$x(t) = f_0\left(g(t) + H(t-\pi)g(t-\pi)\right)$$

where $g(t)$ is given below

$$g(t) = A\cos(t) + B\sin(t) + Ce^{-\zeta\omega_n t}\cos(\omega_d t) + \left(\frac{D - C\zeta\omega_n}{\omega_d}\right)e^{-\zeta\omega_n t}\sin(\omega_d t)$$

ω_d is the damped natural frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

Let m=1 kg, c=2 kg/sec, k=3 N/m, and F_0=2 N. The system is solved numerically. Both exact and numerical solutions are plotted below

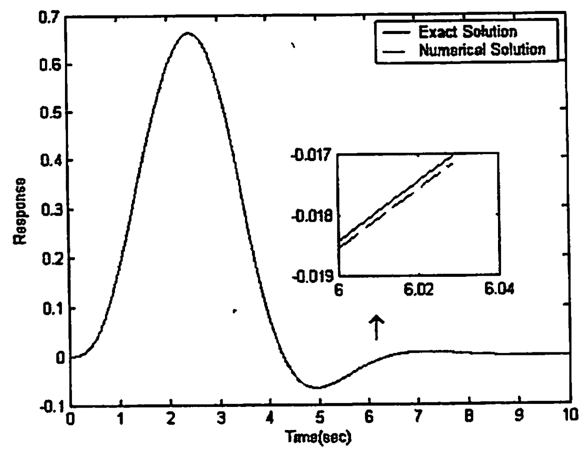


Figure 1 Analytical vs. Numerical Solutions

Below is the code used to solve this problem

```
% Establish a time vector
t=[0:0.001:10];

% Define the mass, spring stiffness and damping coefficient
m=1;
c=2;
k=3;

% Define the amplitude of the forcing function
F0=2;

% Calculate the natural frequency, damping ratio and normalized force amplitude
zeta=c/(2*sqrt(k*m));
wn=sqrt(k/m);
f0=F0/m;

% Calculate the damped natural frequency
wd=wn*sqrt(1-zeta^2);

% Below is the common denominator of A, B, C and D (partial fractions
% coefficients)
dummy=(1-wn^2)^2+(2*zeta*wn)^2;

% Hence, A, B, C, and D are given by
A=-2*zeta*wn/dummy;
B=(wn^2-1)/dummy;
C=2*zeta*wn/dummy;
```

```
D=((1-wn^2)+(2*zeta*wn)^2)/dummy;

% EXACT SOLUTION
%
************************************************************************
*
%
************************************************************************
*
for i=1:length(t)
   % Start by defining the function g(t)
   g(i)=A*cos(t(i))+B*sin(t(i))+C*exp(-zeta*wn*t(i))*cos(wd*t(i))+((D-
C*zeta*wn)/wd)*exp(-zeta*wn*t(i))*sin(wd*t(i));
   % Before t=pi, the response will be only g(t)
   if t(i)<pi
      xe(i)=f0*g(i);
      % d is the index of delay that will correspond to t=pi
      d=i;
   else
      % After t=pi, the response is g(t) plus a delayed g(t). The amount
      % of delay is pi seconds, and it is d increments
      xe(i)=f0*(g(i)+g(i-d));
   end;
end;

% NUMERICAL SOLUTION
%
************************************************************************
*
%
************************************************************************
*

% Start by defining the forcing function
for i=1:length(t)
   if t(i)<pi
      f(i)=f0*sin(t(i));
   else
      f(i)=0;
   end;
end;

% Define the transfer functions of the system
% This is given below
%        1
% ----------------------
```

```
% s^2+2*zeta*wn+wn^2

% Define the numerator and denominator
num=[1];
den=[1 2*zeta*wn wn^2];
% Establish the transfer function
sys=tf(num,den);

% Obtain the solution using lsim
xn=lsim(sys,f,t);

% Plot the results
figure;
set(gcf,'Color','White');
plot(t,xe,t,xn,'--');
xlabel('Time(sec)');
ylabel('Response');
legend('Forcing Function','Exact Solution','Numerical Solution');
text(6,0.05,'\uparrow','FontSize',18);
axes('Position',[0.55 0.3/0.8 0.25 0.25])
plot(t(6001:6030),xe(6001:6030),t(6001:6030),xn(6001:6030),'--');
```

3.3 HW3

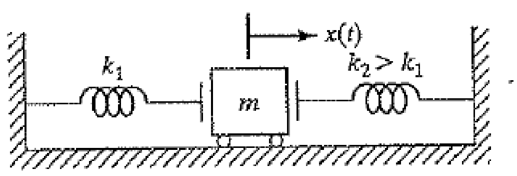
Local contents

3.3.1 Description of HW

1. Find EQM, one mass, 2 springs, different k, springs only attached when hit
2. Find EQM using Lagrangian, pendulum, but string is rubber band with some stiffness.
3. Find exact solution to nonlinear pendulum EQM
4. nonlinear second order ODE. Find equilibrium points and stability at these.
5. nonlinear 2nd order. Find stability around equilibrium
6. similar to above, but find stability conditions based on damping sign
7. columb damping and phase plane
8. Given phase plane equation (i.e. dy/dx), determine stability. i.e. go back from phase plane to the system matrix
9. Solve Van Der Pol using perurbation

3.3.2 Problem 1

1 Two springs, having different stiffnesses k_1 and k_2 with $k_2 > k_1$, are placed on either side of a mass m, as shown When the mass is in its equilibrium position, no spring is in contact with the mass. However, when the mass is displaced from its equilibrium position, only one spring will be compressed. If the mass is given an initial velocity $\dot{x}_0$ at $t = 0$, determine (a) the maximum deflection and (b) the period of vibration of the mass.



3.3.2.1 Part (a)

Initially, when mass is given velocity v_0 then the equation of motion is

$$m\ddot{x} + k_2 x = 0$$

with IC $\dot{x}(0) = v_0, x(0) = 0$, hence the solution is

$$x(t) = A\cos\omega_n t + B\sin\omega_n t$$

Where $\omega_n = \sqrt{\frac{k_2}{m}}$ in this case.

From IC $x(0) = 0$ we obtain that $A = 0$ and now

$$\dot{x}(t) = B\sqrt{\frac{k_2}{m}}\cos\sqrt{\frac{k_2}{m}}t$$

Hence from IC $\dot{x}(0) = v_0$ we obtain that $B = \frac{v_0}{\sqrt{\frac{k_2}{m}}}$ and then we write the solution as

$$x(t) = \sqrt{\frac{m}{k_2}}v_0\sin\sqrt{\frac{k_2}{m}}t$$

The above is the $\boxed{\text{solution for EQM of the mass when it is attached to } k_2\text{spring}}$.

Now the mass will move to the right, losing its kinetic energy to the potential energy of the spring until it stops at the maximum displacement on the right, which will be $\sqrt{\frac{m}{k_2}}v_0$. Then the mass will starts to move to the left again towards the static equilibrium position, gaining speed as it does and the spring losing potential energy until the mass is back to $x = 0$ where it will have speed of v_0 but in the left direction. When it hits the left spring k_1, it will move in an EQM given by

$$m\ddot{x} + k_1 x = 0$$

With initial x given by static equilibrium position (i.e. $x = 0$) and initial velocity of v_0 but to the left direction. Hence as before, we obtain

$$x(t) = \sqrt{\frac{m}{k_1}}v_0\sin\sqrt{\frac{k_1}{m}}t$$

The above is the solution for EQM of the mass when it is attached to spring k_1. We see that the maximum displacement will be $x(t) = \sqrt{\frac{m}{k_1}} v_0$ in this case.

Therefore, we conclude the following:

Mass will move to the right of the static equilibrium position a maximum distance of $\sqrt{\frac{m}{k_2}} v_0$

and

Mass will move to the left of the static equilibrium position a maximum distance of $\sqrt{\frac{m}{k_1}} v_0$

And since $k_2 > k_1$, then it will move the left a longer distance than to the right.

3.3.2.2 Part(b)

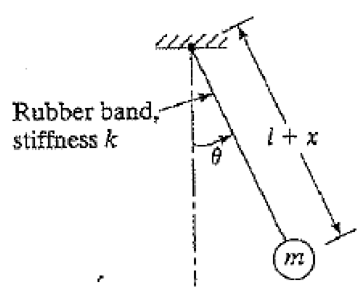
From above, the period of motion when the mass is attached to k_2 is found by setting $\sqrt{\frac{k_2}{m}} t = 2\pi f t$ hence $f = \frac{1}{2\pi}\sqrt{\frac{k_2}{m}}$, therefore $T = 2\pi\sqrt{\frac{m}{k_2}}$ sec

The period of motion when the mass is attached to k_1 is found by setting $\sqrt{\frac{k_1}{m}} t = 2\pi f t$ hence $f = \frac{1}{2\pi}\sqrt{\frac{k_1}{m}}$, therefore $T = 2\pi\sqrt{\frac{m}{k_1}}$ sec

We see that the period when the mass is attached to k_1 is longer than the period when the mass is attached to k_2.

3.3.3 Problem 2

2. A mass m, connected to an elastic rubber band of unstretched length l and stiffness k, is permitted to swing as a pendulum bob, as shown. Derive the nonlinear equations of motion of the system using x and θ as coordinates. Linearize the equations of motion and determine the natural frequencies of vibration of the system.



The Lagrangian which I will call Γ (since I am using L for the current length of the band) is given by $T - U$, where T is the kinetic energy of the system and U is the potential energy of the system.

We take x to be from the unstretched length of the rubber band along the length of the band.

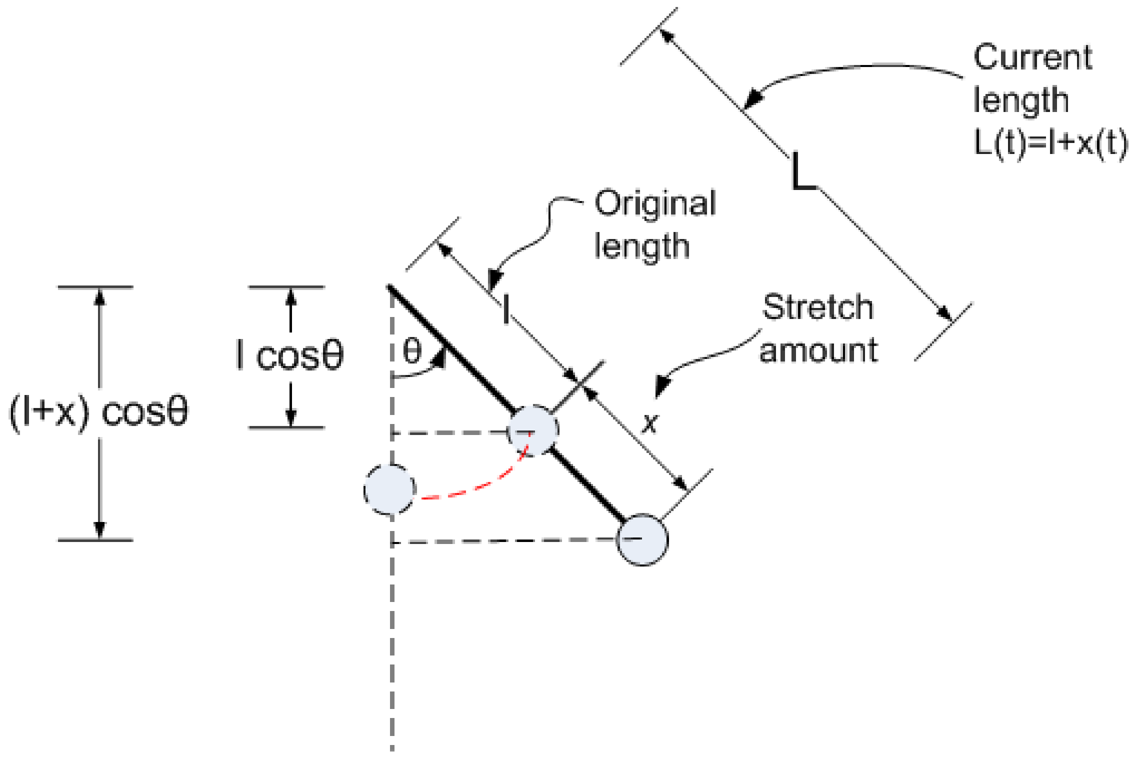
First, we determine the velocity of mass m. Assume that the length of the rubber band at any point time is given by $L(t)$, then

$$
\begin{aligned}
v^2 &= v_{horizontal}^2 + v_{vertical}^2 \\
&= \left[\frac{d}{dt}(s_{horizontal})\right]^2 + \left[\frac{d}{dt}(s_{vertical})\right]^2 \\
&= \left[\frac{d}{dt}(L(t)\sin\theta)\right]^2 + \left[\frac{d}{dt}(L(t)\cos\theta)\right]^2 \\
&= \left[\dot{L}(t)\sin\theta + L(t)\cos(\theta)\dot{\theta}\right]^2 + \left[\dot{L}(t)\cos\theta - L(t)\sin(\theta)\dot{\theta}\right]^2 \\
&= \dot{L}^2(t)\sin^2\theta + L^2(t)\cos^2(\theta)\dot{\theta}^2 + 2\dot{L}(t)\sin(\theta)L(t)\cos(\theta)\dot{\theta} \\
&+ \dot{L}^2(t)\cos^2\theta + L^2(t)\sin^2(\theta)\dot{\theta}^2 - 2\dot{L}(t)\cos(\theta)L(t)\sin(\theta)\dot{\theta} \\
&= \dot{L}^2(t)\left[\sin^2\theta + \cos^2\theta\right] + L^2(t)\dot{\theta}^2\left[\cos^2(\theta) + \sin^2(\theta)\right] \\
&= \dot{L}^2(t) + L^2(t)\dot{\theta}^2
\end{aligned}
$$

Therefore, the system kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\dot{L}^2(t) + L^2(t)\,\dot{\theta}^2\right) \end{aligned}$$

Now we find U, the potential energy for the mass, with the help of this diagram



$$U_{mass} = -mg(L\cos\theta - l)$$

Where the minus sign at the front since the mass has lost PE as it is assume x has stretched the band and hence the mass is lower than its static position.

And the potential energy for the band is

$$U_{band} = \frac{1}{2}kx^2(t)$$

Hence, the Lagrangian Γ is

$$\begin{aligned} \Gamma &= T - U \\ &= \frac{1}{2}m\left(\dot{L}^2(t) + L^2(t)\,\dot{\theta}^2\right) - \left(\frac{1}{2}kx^2(t) - mg(L\cos\theta - l)\right) \end{aligned}$$

But $L = l + x(t)$, hence the above becomes

$$\Gamma = \frac{1}{2}m\left[\left(\frac{d}{dt}(l + x(t))\right)^2 + (l + x(t))^2\,\dot{\theta}^2\right] - \left(\frac{1}{2}kx^2(t) - mg[(l + x(t))\cos\theta - l]\right)$$

Hence

$$\boxed{\Gamma = \tfrac{1}{2}m\left[\dot{x}^2(t) + \left(l^2 + x^2(t) + 2lx(t)\right)\dot{\theta}^2\right] - \tfrac{1}{2}kx^2(t) + mg[l(\cos\theta - 1) + x(t)\cos\theta]}$$

Hence EQM is now found. For θ we have

$$\begin{aligned} \frac{d}{dt}\frac{\partial\Gamma}{\partial\dot{\theta}} - \frac{\partial\Gamma}{\partial\theta} &= 0 \\ \frac{d}{dt}\left(\frac{1}{2}m\left[2\left(l^2 + x^2(t) + 2lx(t)\right)\dot{\theta}\right]\right) - mg[-l\sin\theta - x(t)\sin\theta] &= 0 \\ m\left[\left(2x(t)\,\dot{x}(t) + 2l\dot{x}(t)\right)\dot{\theta} + \left(l^2 + x^2(t) + 2lx(t)\right)\ddot{\theta}\right] + mg[l\sin\theta + x(t)\sin\theta] &= 0 \\ \left(l^2 + x^2(t) + 2lx(t)\right)\ddot{\theta} + \left(2x(t)\,\dot{x}(t) + 2l\dot{x}(t)\right)\dot{\theta} + g\sin\theta[l + x(t)] &= 0 \\ \left(l^2 + x^2(t) + 2lx(t)\right)\ddot{\theta} + (l + x(t))\,2\dot{x}(t)\,\dot{\theta} + g\sin\theta[l + x(t)] &= 0 \end{aligned}$$

The above can be simplified more if we observer that $\left(l^2 + x^2(t) + 2lx(t)\right) = [l + x(t)]^2 = L^2$ and $l + x(t) = L$, hence EQM becomes

$$L^2\ddot{\theta} + 2L\dot{x}\dot{\theta} + gL\sin\theta = 0$$

Or

$$\boxed{L\ddot{\theta}+2\dot{x}\dot{\theta}+g\sin\theta=0}$$

Using small angle approximation, $\sin\theta\simeq\theta$ and $\dot{\theta}$ can be neglected, we obtain

$$L\ddot{\theta}+g\theta=0$$
$$\ddot{\theta}+\frac{g}{L}\theta=0$$

Hence, the effective stiffness is $\frac{g}{L}$ and $\omega_{n_\theta}=\sqrt{\frac{g}{L}}=\sqrt{\frac{g}{l+x(t)}}$Hence we observe that as the band is stretched more, ω_n becomes smaller and the period becomes longer. Now we derive the EQM in the x direction

$\frac{1}{2}m\left[\dot{x}^2(t)+\left(l^2+x^2(t)+2lx(t)\right)\dot{\theta}^2\right]-\frac{1}{2}kx^2(t)+mg[l(\cos\theta-1)+x(t)\cos\theta]$

$$\frac{d}{dt}\frac{\partial\Gamma}{\partial\dot{x}}-\frac{\partial\Gamma}{\partial x}=0$$
$$\frac{d}{dt}(m\dot{x}(t))+kx(t)=0$$

Hence EQM is

$$\boxed{\ddot{x}(t)+\frac{k}{m}x(t)=0}$$

Hence, the effective stiffness is $\frac{k}{m}$ and $\omega_{n_x}=\sqrt{\frac{k}{m}}$ The solutions can now be given easily as

$$\theta(t)=A\cos\omega_{n_\theta}t+B\sin\omega_{n_\theta}t$$
$$x(t)=C\cos\omega_{n_x}t+D\sin\omega_{n_x}t$$

or

$$\theta(t)=A\cos\sqrt{\frac{g}{l+x\left(t\right)}}t+B\sin\sqrt{\frac{g}{l+x\left(t\right)}}t$$
$$L(t)=l+C\cos\sqrt{\frac{k}{m}}t+D\sin\sqrt{\frac{k}{m}}t$$

Where A,B,C,D can be obtained from initial conditions.

3.3.4 Problem 3

3 Find the exact solution of the nonlinear pendulum equation

$$\ddot{\theta}+\omega_0^2\left(\theta-\frac{\theta^3}{6}\right)=0$$

with $\dot{\theta}=0$ when $\theta=\theta_0$, where θ_0 denotes the maximum angular displacement.

EQM is given by

$$\ddot{\theta}+\omega_0^2\left(\theta-\frac{\theta^3}{6}\right)=0$$

The above can be put in the form

$$\ddot{\theta}=f(\theta) \tag{1}$$

Where

$$f(\theta)=\omega_0^2\left(\frac{\theta^3}{6}-\theta\right)$$

Hence, this is an autonomous differential equation since $f(\theta)$ does not depend on the independent variable t explicitly.

Now, Let $x_1=\theta$ and $x_2=\dot{\theta}$, then $\frac{dx_1}{dt}=x_2$ and using the new state variables we can rewrite the differential equation as

$$\frac{dx_2}{dt}+\omega_0^2\left(x_1-\frac{x_1^3}{6}\right)=0$$
$$\frac{dx_2}{dx_1}\frac{dx_1}{dt}=-\omega_0^2\left(x_1-\frac{x_1^3}{6}\right)$$
$$\frac{dx_2}{dx_1}x_2=-\omega_0^2\left(x_1-\frac{x_1^3}{6}\right)$$
$$x_2dx_2=-\omega_0^2\left(x_1-\frac{x_1^3}{6}\right)dx_1$$

Integrate both side

$$\frac{x_2^2}{2} = -\omega_0^2 \int \left(x_1 - \frac{x_1^3}{6} \right) dx_1 + C_1$$

$$x_2^2 = -2\omega_0^2 \left[\frac{x_1^2}{2} - \frac{x_1^4}{24} \right] + C_1$$

But $x_2 = \dot{\theta}$ and $x_1 = \theta$, then the above becomes

$$\dot{\theta}^2 = -2\omega_0^2 \left[\frac{\theta^2}{2} - \frac{\theta^4}{24} \right] + C_1 \tag{2}$$

We are told that when $\theta = \theta_0$ then $\dot{\theta} = 0$, hence from the above

$$0 = -2\omega_0^2 \left[\frac{\theta_0^2}{2} - \frac{\theta_0^4}{24} \right] + C_1$$

$$C_1 = \omega_0^2 \left[\theta_0^2 - \frac{\theta_0^4}{12} \right]$$

Then (2) becomes

$$\dot{\theta}^2 = -2\omega_0^2 \left[\frac{\theta^2}{2} - \frac{\theta^4}{24} \right] + \omega_0^2 \left[\theta_0^2 - \frac{\theta_0^4}{12} \right]$$

or

$$\dot{\theta}^2 = \omega_0^2 \left[\frac{1}{12}\theta^4 - \theta^2 \right] + \omega_0^2 \left[\theta_0^2 - \frac{\theta_0^4}{12} \right]$$

Therefore

$$\dot{\theta} = \omega_0 \sqrt{\frac{1}{12}\theta^4 - \theta^2 + \theta_0^2 - \frac{\theta_0^4}{12}}$$

$$\frac{d\theta}{dt} = \frac{\omega_0}{\sqrt{12}} \sqrt{\theta^4 - 12\theta^2 + 12\theta_0^2 - \theta_0^4}$$

$$= \frac{\omega_0}{2\sqrt{3}} \sqrt{\theta^2 \left(\theta^2 - 12 \right) + \theta_0^2 \left(12 - \theta_0^2 \right)}$$

Hence integrating the above we obtain

$$\int \frac{1}{\sqrt{\theta^2 \left(\theta^2 - 12 \right) + \theta_0^2 \left(12 - \theta_0^2 \right)}} d\theta = \frac{\omega_0}{2\sqrt{3}} \int dt + C_2$$

$$\int \frac{1}{\sqrt{\theta^2 \left(\theta^2 - 12 \right) + \theta_0^2 \left(12 - \theta_0^2 \right)}} d\theta = \frac{\omega_0}{2\sqrt{3}} t + C_2$$

We can stop here. What remains is to evaluate the integral above by some analytical method to obtain an expression for $\theta(t)$. The constant C_2 can be found if we are given the position initial condition.

3.3.5 Problem 3, different method

EQM is given by

$$\ddot{\theta} + \omega_0^2 \left(\theta - \frac{\theta^3}{6} \right) = 0$$

The above can be put in the form

$$\ddot{\theta} = f(\theta) \tag{1}$$

Where

$$f(\theta) = \omega_0^2 \left(\frac{\theta^3}{6} - \theta \right)$$

Hence, this is an autonomouse differential equation since $f(\theta)$ does not depend on the independent variable t explicility.

To solve (1), we first write

$$\ddot{\theta} = \frac{d}{dt}\left(\frac{d\theta}{dt} \right)$$

$$= \left[\frac{d}{d\theta}\left(\frac{d\theta}{dt} \right) \right] \frac{d\theta}{dt}$$

$$= \left[\frac{d}{d\theta}\left(\left(\frac{dt}{d\theta} \right)^{-1} \right) \right] \left(\frac{dt}{d\theta} \right)^{-1} \tag{2}$$

But

$$\frac{d}{d\theta}\left(\left(\frac{dt}{d\theta}\right)^{-1}\right)=-\left(\frac{dt}{d\theta}\right)^{-2}\frac{d^2t}{d\theta^2}\left(\frac{dt}{d\theta}\right)^{-1}$$

Substitute the above into (2) we obtain

$$\begin{aligned}\ddot{\theta}&=-\left(\frac{dt}{d\theta}\right)^{-2}\frac{d^2t}{d\theta^2}\left(\frac{dt}{d\theta}\right)^{-1}\\&=-\left(\frac{dt}{d\theta}\right)^{-3}\frac{d^2t}{d\theta^2}\end{aligned}\tag{3}$$

But

$$\frac{1}{2}\frac{d}{d\theta}\left(\left(\frac{dt}{d\theta}\right)^{-2}\right)=-\left(\frac{dt}{d\theta}\right)^{-3}\frac{d^2t}{d\theta^2}\tag{4}$$

Compare (4) and (3) we see that (3) can be written as

$$\ddot{\theta}=\frac{1}{2}\frac{d}{d\theta}\left(\left(\frac{dt}{d\theta}\right)^{-2}\right)$$

THerefore, we use this expression for $\ddot{\theta}$ in (1) and obtain

$$\frac{1}{2}\frac{d}{d\theta}\left(\left(\frac{dt}{d\theta}\right)^{-2}\right)=f(\theta)$$

Substitute the expression for $f(\theta)$ we obtain

$$\boxed{\frac{1}{2}\frac{d}{d\theta}\left(\left(\frac{dt}{d\theta}\right)^{-2}\right)=\omega_0^2\left(\frac{\theta^3}{6}-\theta\right)}$$

Integrate we obtain

$$\begin{aligned}\frac{1}{2}\left(\frac{dt}{d\theta}\right)^{-2}&=\int\omega_0^2\left(\frac{\theta^3}{6}-\theta\right)d\theta+C_1\\\left(\frac{dt}{d\theta}\right)^{2}&=\frac{1}{2\int\omega_0^2\left(\frac{\theta^3}{6}-\theta\right)d\theta+C_1}\\&=\frac{1}{2\omega_0^2\left(\frac{\theta^4}{4\times 6}-\frac{\theta^2}{2}\right)+C_1}\end{aligned}$$

Hence

$$\frac{dt}{d\theta}=\frac{1}{\sqrt{2\omega_0^2\left(\frac{\theta^4}{4\times 6}-\frac{\theta^2}{2}\right)+C_1}}$$

Integrate again, we obtain

$$\begin{aligned}t&=\int\left(2\omega_0^2\left(\frac{\theta^4}{24}-\frac{\theta^2}{2}\right)+C_1\right)^{\frac{-1}{2}}d\theta+C_2\\&=C_2+\int\frac{1}{\sqrt{2\omega_0^2\left(\frac{\theta^4}{24}-\frac{\theta^2}{2}\right)+C_1}}d\theta\\&=C_2+\int\frac{1}{\sqrt{\frac{\theta^4}{12}-\omega_0^2\theta^2+C_1}}d\theta\\&=C_2+\int\frac{\sqrt{12}}{\sqrt{\theta^4-12\omega_0^2\theta^2+C_3}}d\theta\end{aligned}$$

Where $C_3 = 12C_1$, a new constant. Hence

$$\boxed{t = C_2 + 2\sqrt{3}\int \frac{1}{\sqrt{\theta^4 - 12\omega_0^2\theta^2 + C_3}}d\theta}$$

second approach

Let $\left.\begin{array}{l} x_1 = \theta \\ x_2 = \dot{\theta} \end{array}\right\} \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega_0^2\left(x_1 - \frac{x_1^3}{6}\right) \end{array}$, hence using the new state variables we can rewrite the differential equation as

$$\begin{aligned}
\ddot{\theta} + \omega_0^2\left(\theta - \frac{\theta^3}{6}\right) &= 0 \\
\frac{dx_2}{dt} + \omega_0^2\left(x_1 - \frac{x_1^3}{6}\right) &= 0 \\
\frac{dx_2}{dx_1}\frac{dx_1}{dt} &= -\omega_0^2\left(x_1 - \frac{x_1^3}{6}\right) \\
\frac{dx_2}{dx_1}x_2 &= -\omega_0^2\left(x_1 - \frac{x_1^3}{6}\right) \\
x_2 dx_2 &= -\omega_0^2\left(x_1 - \frac{x_1^3}{6}\right)dx_1
\end{aligned}$$

Integrate both side

$$\begin{aligned}
\frac{x_2^2}{2} &= -\omega_0^2\int\left(x_1 - \frac{x_1^3}{6}\right)dx_1 + C_1 \\
x_2^2 &= -2\omega_0^2\left[\frac{x_1^2}{2} - \frac{x_1^4}{24}\right] + C_1
\end{aligned}$$

But $x_2 = \dot{\theta}$ and $x_1 = \theta$, then the above becomes

$$\dot{\theta}^2 = -2\omega_0^2\left[\frac{\theta^2}{2} - \frac{\theta^4}{24}\right] + C_1 \tag{2}$$

We are told that when $\theta = \theta_0$ then $\dot{\theta} = 0$, hence from the above

$$\begin{aligned}
0 &= -2\omega_0^2\left[\frac{\theta_0^2}{2} - \frac{\theta_0^4}{24}\right] + C_1 \\
C_1 &= \omega_0^2\left[\theta_0^2 - \frac{\theta_0^4}{12}\right]
\end{aligned}$$

Then (2) becomes

$$\dot{\theta}^2 = -2\omega_0^2\left[\frac{\theta^2}{2} - \frac{\theta^4}{24}\right] + \omega_0^2\left[\theta_0^2 - \frac{\theta_0^4}{12}\right]$$

or

$$\begin{aligned}
\dot{\theta}^2 &= -\omega_0^2\left[\theta^2 - \frac{1}{12}\theta^4\right] + \omega_0^2\left[\theta_0^2 - \frac{\theta_0^4}{12}\right] \\
\dot{\theta} &= \sqrt{\omega_0^2\left[\frac{1}{12}\theta^4 - \theta^2\right] + \omega_0^2\left[\theta_0^2 - \frac{\theta_0^4}{12}\right]} \\
&= \omega_0\sqrt{\frac{1}{12}\theta^4 - \theta^2 + \theta_0^2 - \frac{\theta_0^4}{12}} \\
&= \frac{\omega_0}{2\sqrt{3}}\sqrt{\theta^4 - 12\theta^2 + 12\theta_0^2 - \theta_0^4}
\end{aligned}$$

Hence integrating the above we obtain

$$\theta(t) = \frac{\omega_0}{2\sqrt{3}} \int \sqrt{\theta^4 - 12\theta^2 + 12\theta_0^2 - \theta_0^4}\,dt$$
$$= \left(\frac{\omega_0}{2\sqrt{3}} \sqrt{\theta^4 - 12\theta^2 + 12\theta_0^2 - \theta_0^4} \right) t + C_2$$

3.3.6 Problem 4

4. Find the equilibrium position and plot the trajectories in the neighborhood of the equilibrium position corresponding to the following equation:

$$\ddot{x} + 0.1(x^2 - 1)\dot{x} + x = 0$$

The nonlinear equation is

$$\ddot{x} + 0.1\left(x^2 - 1\right)\dot{x} + x = 0$$

Let

$$\left.\begin{matrix} x_1 = x \\ x_2 = \dot{x} \end{matrix}\right\} \left.\begin{matrix} \dot{x}_1 = \dot{x} \\ \dot{x}_2 = -0.1\left(x^2 - 1\right)\dot{x} - x \end{matrix}\right\} \begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = -0.1\left(x_1^2 - 1\right)x_2 - x_1 \end{matrix}$$

Hence

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -0.1\left(x_1^2 - 1\right)x_2 - x_1 \end{pmatrix} = \begin{pmatrix} g(x_1, x_2) \\ f(x_1, x_2) \end{pmatrix}$$

Solve for $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for equilibrium. Hence $x_2 = 0$ and therefore $x_1 = 0$ as well. Now we obtain the linearized state matrix A at the equilibrium point found. First we note that $\frac{\partial g}{\partial x_1} = 0$, $\frac{\partial g}{\partial x_2} = 1$, $\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}\left(-0.1x_1^2 x_2 + 0.1x_2 - x_1\right) = -0.2x_1x_2 - 1$ and $\frac{\partial f}{\partial x_2} = -0.1x_1^2 + 0.1$, hence

$$A = \begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix}_{x_1=0, x_2=0}$$
$$= \begin{pmatrix} 0 & 1 \\ -0.2x_1x_2 - 1 & -0.1x_1^2 + 0.1 \end{pmatrix}_{x_1=0, x_2=0}$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0.1 \end{pmatrix}$$

Find the eigenvalues, we obtain

$$\begin{vmatrix} -\lambda & 1 \\ -1 & 0.1 - \lambda \end{vmatrix} = 0$$
$$-0.1\lambda + \lambda^2 + 1 = 0$$

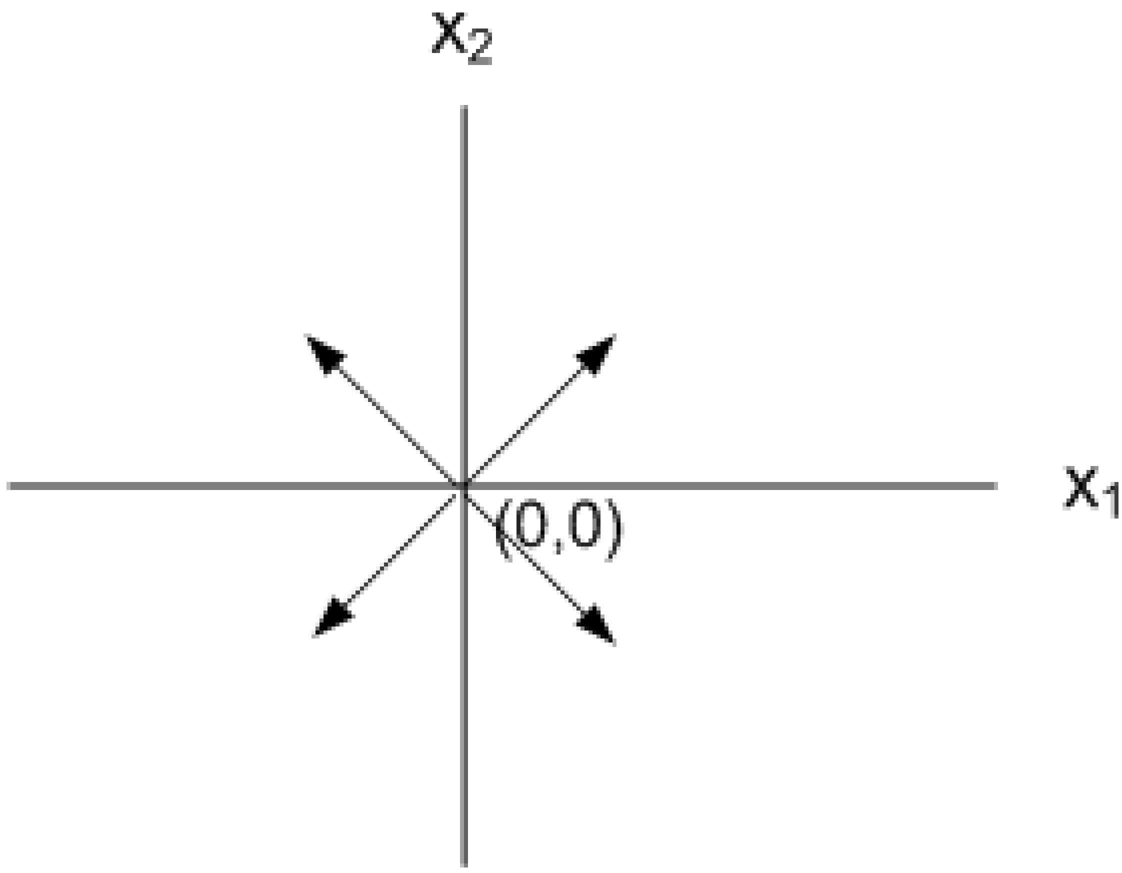
Hence

$$\lambda_{1,2} = \{0.05 + 0.998\,75i, 0.05 - 0.998\,75i\}$$

This is of the form

$$\lambda = \alpha \pm \beta i$$

With $\alpha > 0$, hence $\boxed{\text{unstable, and spiral out.}}$ So. now we can draw the phase portrait near $(0,0)$ as shown below.



Side QUESTION:

If I wanted to draw the phase plot itself, I am getting this. How to finish this last step? It is not separable?

To obtain phase plane plot, we need to express x_2 as function of x_1. Looking at the original nonlinear differential equation again and rewrite using the state variables, we obtain

$$\begin{aligned}
\ddot{x} + 0.1\left(x^2 - 1\right)\dot{x} + x &= 0 \\
\frac{dx_2}{dt} + 0.1\left(x_1^2 - 1\right)x_2 + x_1 &= 0 \\
\frac{dx_2}{dx_1}\frac{dx_1}{dt} + 0.1\left(x_1^2 - 1\right)x_2 + x_1 &= 0 \\
\frac{dx_2}{dx_1}x_2 + 0.1\left(x_1^2 - 1\right)x_2 + x_1 &= 0 \\
\frac{dx_2}{dx_1}x_2 + 0.1x_1^2x_2 - 0.1x_2 + x_1 &= 0 \\
\frac{dx_2}{dx_1} &= 0.1 - 0.1x_1^2 - \frac{x_1}{x_2}
\end{aligned}$$

3.3.7 Problem 5

5 The equation of motion of a simple pendulum, subjected to external force, is given by

$$\ddot{\theta} + 0.5\,\dot{\theta} + \sin\theta = 0.8$$

Find the nature of singularity at $\theta = \sin^{-1}(0.8)$.

The equation is

$$\ddot{\theta} + 0.5\dot{\theta} + \sin\theta = 0.8$$

Let

$$\left.\begin{matrix} x_1 = \theta \\ x_2 = \dot{\theta} \end{matrix}\right\} \left.\begin{matrix} \dot{x}_1 = \dot{\theta} \\ \dot{x}_2 = 0.8 - 0.5\dot{\theta} - \sin\theta \end{matrix}\right\} \begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0.8 - 0.5x_2 - \sin x_1 \end{matrix}$$

Hence

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0.8 - 0.5x_2 - \sin x_1 \end{pmatrix} = \begin{pmatrix} g(x_1, x_2) \\ f(x_1, x_2) \end{pmatrix}$$

Solve for $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for equilibrium. Hence $x_2 = 0$ and therefore $x_1 = \sin^{-1}(0.8)$.

Now we obtain the linearized state matrix A at the equilibrium point found. First we note that $\frac{\partial g}{\partial x_1} = 0$, $\frac{\partial g}{\partial x_2} = 1$, $\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}(0.8 - 0.5x_2 - \sin x_1) = -\cos x_1$ and $\frac{\partial f}{\partial x_2} = -0.5$, hence

$$\begin{aligned} A &= \begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_1} \end{pmatrix}_{x_1 = \sin^{-1}(0.8), x_2 = 0} \\ &= \begin{pmatrix} 0 & 1 \\ -\cos x_1 & -0.5 \end{pmatrix}_{x_1 = \sin^{-1}(0.8), x_2 = 0} \\ &= \begin{pmatrix} 0 & 1 \\ -\cos\left(\sin^{-1}(0.8)\right) & -0.5 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -\cos(0.927295) & -0.5 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -0.6 & -0.5 \end{pmatrix} \end{aligned}$$

Hence find the eigenvalues, we obtain

$$\begin{vmatrix} -\lambda & 1 \\ -0.6 & -0.5 - \lambda \end{vmatrix} = 0$$
$$\lambda^2 + 0.5\lambda + 0.6 = 0$$

Hence

$$\lambda_{1,2} = \{-0.25 + 0.733\,14i, -0.25 - 0.733\,14i\}$$

This is of the form

$$\lambda = \alpha \pm \beta i$$

With $\alpha < 0$, , Hence stable, spiral in.

3.3.8 Problem 6

6 The equation of motion of a simple pendulum subject to viscous damping can be expressed as

$$\ddot{\theta} + c\dot{\theta} + \sin\theta = 0$$

If the initial conditions are $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, show that the origin in the phase plane diagram represents (a) a stable focus for $c > 0$ and (b) an unstable focus for $c < 0$.

The equation is

$$\ddot{\theta} + c\dot{\theta} + \sin\theta = 0$$

With IC $\theta(0) = \theta_0$,and $\dot{\theta}(0) = 0$ Let

$$\left.\begin{array}{l} x_1 = \theta \\ x_2 = \dot{\theta} \end{array}\right\} \begin{array}{l} \dot{x}_1 = \dot{\theta} \\ \dot{x}_2 = -c\dot{\theta} - \sin\theta \end{array}\left.\right\} \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -cx_2 - \sin x_1 \end{array}$$

Hence

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -cx_2 - \sin x_1 \end{pmatrix} = \begin{pmatrix} g(x_1, x_2) \\ f(x_1, x_2) \end{pmatrix}$$

Now, we are told to consider the initial condition $\dot{\theta} = 0$, but this is the same as $\dot{x}_1 = 0$. But if speed is zero, then acceleration must also be zero, hence $\ddot{\theta} = 0$ or $\dot{x}_2 = 0$. Therefore we need to solve for $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or

$$\begin{pmatrix} x_2 \\ -cx_2 - \sin x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore $x_2 = 0$ and then $\sin x_1 = 0$ or $x_1 = n\pi$ for $n = 0, \pm 1, \pm 2, \cdots$.

Now we obtain the linearized state matrix A at the equilibrium point found. First we note that $\frac{\partial g}{\partial x_1} = 0$, $\frac{\partial g}{\partial x_2} = 1$, $\frac{\partial f}{\partial x_1} = -\cos x_1$ and $\frac{\partial f}{\partial x_2} = -c$, hence

$$\begin{aligned} A &= \begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix}_{x_1 = n\pi, x_2 = 0} \\ &= \begin{pmatrix} 0 & 1 \\ -\cos x_1 & -c \end{pmatrix}_{x_1 = n\pi, x_2 = 0} \\ &= \begin{pmatrix} 0 & 1 \\ -\cos(n\pi) & -c \end{pmatrix} \end{aligned}$$

Hence find the eigenvalues, we obtain

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 \\ -\cos(n\pi) & -c - \lambda \end{vmatrix} &= 0 \\ -\lambda(-c - \lambda) + \cos(n\pi) &= 0 \\ \lambda^2 + c\lambda + \cos(n\pi) &= 0 \end{aligned}$$

Now, we are asked to evaluate this at the center of the phase portrait, which means at $x_1 = 0$ and $x_2 = 0$, in other words, when $n = 0$ (since when $n = 0$, then $x_1 = 0$). Hence, when $n = 0$, the characteristic equation becomes

$$\lambda^2 + c\lambda + 1 = 0$$

Hence

$$\lambda_{1,2} = \left\{ \begin{array}{c} -\frac{c}{2} + \sqrt{\frac{c^2}{4} - 1} \\ -\frac{c}{2} - \sqrt{\frac{c^2}{4} - 1} \end{array} \right\}$$

We now consider all the possible values of c and see its effect on the roots of the characteristic equation. This is done using a table

c value	roots form	Location of roots	type of stability at $(0,0)$
$c < 0$ and $\lvert c\rvert < 2$	$\alpha \pm i\beta$ where $\alpha > 0$	In RHS complex plane	Spiral out, UNSTABLE
$c < 0$ and $\lvert c\rvert > 2$	$\alpha \pm \beta$ where $\alpha > 0$ and $\beta < \alpha$	In RHS on the real line	Repelling, UNSTABLE
$c > 0$ and $\lvert c\rvert < 2$	$\alpha \pm i\beta$ where $\alpha < 0$	In LHS complex plane	Spiral in, STABLE
$c > 0$ and $\lvert c\rvert > 2$	$\alpha \pm \beta$ where $\alpha < 0$ and $\beta < \alpha$	In LHS on the real line	Attracting, STABLE

Therefore, we conclude that for $c < 0$ the system is unstable at equilibrium point $(0,0)$ and for $c > 0$ the system is stable at equilibrium point $(0,0)$.

Notice that we did not use the initial condition on the position at all. i.e. knowing that $\theta(0) = \theta_0$ was not needed to solve this problem.

3.3.9 Problem 7

7 A single degree of freedom system is subjected to Coulomb friction so that the equation of motion is given by

$$\ddot{x} + f\frac{\dot{x}}{|\dot{x}|} + \omega_n^2 x = 0$$

Construct the phase plane trajectories of the system using the initial conditions $x(0) = 10(f/\omega_n^2)$ and $\dot{x}(0) = 0$.

EQM is

$$\ddot{x} + f\frac{\dot{x}}{|\dot{x}|} + \omega_n^2 x = 0$$

We need to determine the phase plane trajectories. The term $\frac{\dot{x}}{|\dot{x}|}$ will be either $+1$ or -1 depending on the sign of $\dot{x}$

Hence for $\dot{x} > 0$ we have

$$\begin{aligned}\ddot{x} + f + \omega_n^2 x &= 0\\ \ddot{x} + \omega_n^2 x &= -f\end{aligned}$$

And for $\dot{x} < 0$ we have

$$\begin{aligned}\ddot{x} - f + \omega_n^2 x &= 0\\ \ddot{x} + \omega_n^2 x &= f\end{aligned}$$

Analyze each case separately. For $\dot{x} > 0$ we have

$$\begin{aligned}\frac{d}{dt}\dot{x} + \omega_n^2 x_1 &= -f\\ \frac{dx_2}{dt} + \omega_n^2 x_1 &= -f\\ \frac{dx_2}{dx_1}\frac{dx_1}{dt} + \omega_n^2 x_1 &= -f\\ \frac{dx_2}{dx_1}x_2 + \omega_n^2 x_1 &= -f\\ \frac{dx_2}{dx_1}x_2 &= -f - \omega_n^2 x_1\\ dx_2 x_2 &= \left(-f - \omega_n^2 x_1\right) dx_1\end{aligned}$$

Integrating both sides, we obtain

$$\frac{x_2^2}{2} = \left(-fx_1 - \frac{\omega_n^2 x_1^2}{2}\right) + C$$

Using IC given by $x_1(0) = 10\left(\frac{f}{\omega_n^2}\right)$ and $x_2(0) = 0$, then the above becomes

$$\begin{aligned}0 &= \left(-f \times 10\left(\frac{f}{\omega_n^2}\right) - \frac{\omega_n^2\left[10\left(\frac{f}{\omega_n^2}\right)\right]^2}{2}\right) + C\\ 0 &= -\frac{10f^2}{\omega_n^2} - 50\omega_n^2\left(\frac{f^2}{\omega_n^4}\right) + C\\ C &= \frac{10f^2}{\omega_n^2} + 50\frac{f^2}{\omega_n^2}\end{aligned}$$

Hence

$$\boxed{C = 60\left(\frac{f}{\omega_n}\right)^2}$$

Therefore, the phase portrait is

$$x_2^2 = -2fx_1 - \omega_n^2 x_1^2 + 120\left(\frac{f}{\omega_n}\right)^2$$

Hence

$$\boxed{x_2 = \pm\sqrt{120\left(\frac{f}{\omega_n}\right)^2 - 2fx_1 - \omega_n^2 x_1^2}}$$

Given f and ω_n we can plot the phase plane. For $\dot{x} < 0$ we have

$$\frac{d}{dt}\dot{x} + \omega_n^2 x_1 = f$$
$$\frac{dx_2}{dx_1} x_2 = f - \omega_n^2 x_1$$
$$dx_2 x_2 = \left(f - \omega_n^2 x_1\right) dx_1$$

Integrating both sides, we obtain

$$\frac{x_2^2}{2} = \left(f x_1 - \frac{\omega_n^2 x_1^2}{2}\right) + C$$

Using IC given by $x_1(0) = 10\left(\frac{f}{\omega_n^2}\right)$ and $x_2(0) = 0$, then the above becomes

$$0 = \left(f \times 10\left(\frac{f}{\omega_n^2}\right) - \frac{\omega_n^2\left[10\left(\frac{f}{\omega_n^2}\right)\right]^2}{2}\right) + C$$
$$C = -\frac{10f^2}{\omega_n^2} + 50\frac{f^2}{\omega_n^2}$$

Hence

$$\boxed{C = 40\left(\frac{f}{\omega_n}\right)^2}$$

Therefore, the phase portrait is

$$\frac{x_2^2}{2} = \left(f x_1 - \frac{\omega_n^2 x_1^2}{2}\right) + 40\left(\frac{f}{\omega_n}\right)^2$$
$$x_2^2 = 2f x_1 - \omega_n^2 x_1^2 + 80\left(\frac{f}{\omega_n}\right)^2$$

Hence

$$\boxed{x_2 = \pm\sqrt{80\left(\frac{f}{\omega_n}\right)^2 + 2f x_1 - \omega_n^2 x_1^2}}$$

Given f and ω_n we can plot the phase plane.

3.3.10 Problem 8

The phase plane equation of a single degree of freedom system is given by

$$\frac{dy}{dx} = \frac{-cy - (x - 0.1x^3)}{y}$$

Investigate the nature of singularity at $(x, y) = (0, 0)$ for $c > 0$.

$$\frac{dy}{dx} = \frac{-cy - (x - 0.1x^3)}{y}$$

From the above phase plane, obtain the differential equation, and then convert back to state space and obtain the system matrix.

Writing it in state space, where we take $y = x_2$ and $x = x_1$, we obtain

$$\frac{dx_2}{dx_1} = \frac{-cx_2 - (x_1 - 0.1x_1^3)}{x_2}$$
$$\frac{dx_2}{dx_1} x_2 = -cx_2 - (x_1 - 0.1x_1^3)$$
$$\frac{dx_2}{dx_1}\frac{dx_1}{dt} = -cx_2 - (x_1 - 0.1x_1^3)$$
$$\frac{dx_2}{dt} = -cx_2 - (x_1 - 0.1x_1^3)$$
$$\ddot{x} = -cx_2 - (x_1 - 0.1x_1^3)$$

Hence the ODE is

$$\ddot{x} + c\dot{x} + (x - 0.1x^3) = 0$$

Therefore

$$\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \begin{array}{l} \dot{x}_1 = \dot{x} \\ \dot{x}_2 = -c\dot{x} - \left(x - 0.1x^3\right) \end{array} \left.\right\} \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -cx_2 - \left(x_1 - 0.1x_1^3\right) \end{array}$$

Hence

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -cx_2 - \left(x_1 - 0.1x_1^3\right) \end{pmatrix} = \begin{pmatrix} g(x_1, x_2) \\ f(x_1, x_2) \end{pmatrix}$$

Hence, the linearized system matrix is, which we evaluate at $(0,0)$ is

$$A = \begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix}_{x_1=0, x_2=0}$$

But $\frac{\partial g}{\partial x_1} = 0$, $\frac{\partial g}{\partial x_2} = 1$, $\frac{\partial f}{\partial x_1} = -1 + 0.3x_1^2$, $\frac{\partial f}{\partial x_2} = -c$, hence

$$A = \begin{pmatrix} 0 & 1 \\ -1 + 0.3x_1^2 & -c \end{pmatrix}_{x_1=0, x_2=0}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$$

Hence

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -c - \lambda \end{vmatrix} = 0$$

$$(-\lambda)(-c - \lambda) + 1 = 0$$

$$\lambda^2 - c\lambda + 1 = 0$$

Hence

$$\lambda_{1,2} = \begin{pmatrix} \frac{c}{2} + \sqrt{\frac{c^2}{4} - 1} \\ \frac{c}{2} - \sqrt{\frac{c^2}{4} - 1} \end{pmatrix}$$

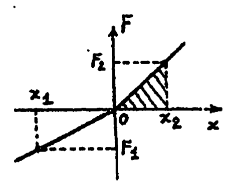
We set up the following table

c value	roots form	Location of roots	type of stability at $(0,0)$
$c > 0$ and $\lvert c \rvert < 2$	$\alpha \pm i\beta$ where $\alpha > 0$	In RHS complex plane	Spiral out, UNSTABLE
$c > 0$ and $\lvert c \rvert > 2$	$\alpha \pm \beta$ where $\alpha > 0$ and $\beta < \alpha$	In RHS on the real line	Repelling, UNSTABLE
$c < 0$ and $\lvert c \rvert < 2$	$\alpha \pm i\beta$ where $\alpha < 0$	In LHS complex plane	Spiral in, STABLE
$c < 0$ and $\lvert c \rvert > 2$	$\alpha \pm \beta$ where $\alpha < 0$ and $\beta < \alpha$	In LHS on the real line	Attracting, STABLE

We see that for $c > 0$, system is UNSTABLE and depending on value of c, it is either Spiral out or Repelling

3.3.11 Key for HW3

1.



T = kinetic energy at time zero $= \frac{1}{2} m (\dot{x}_0)^2$

Let x_2 = maximum displacement on right side.

V = potential energy in spring at displacement $x_2 = \frac{1}{2} k_2 x_2^2$ ($\dot{x}$ is zero at x_2)

Since T = V, $x_2 = \sqrt{\frac{m(\dot{x}_0)^2}{k_2}} = \sqrt{\frac{m}{k_2}}\, \dot{x}_0$

Let x_1 = maximum displacement to left side. $V = \frac{1}{2} k_1 x_1^2$

T = V gives $x_1 = \sqrt{\frac{m}{k_1}(\dot{x}_0)^2} = \sqrt{\frac{m}{k_1}}\, \dot{x}_0$

(a) Since $k_1 < k_2$, maximum deflection $= x_1 = \sqrt{\frac{m}{k_1}}\, \dot{x}_0$

(b) Period of vibration for a spring-mass system is $\tau_n = 2\pi \sqrt{\frac{m}{k}}$.

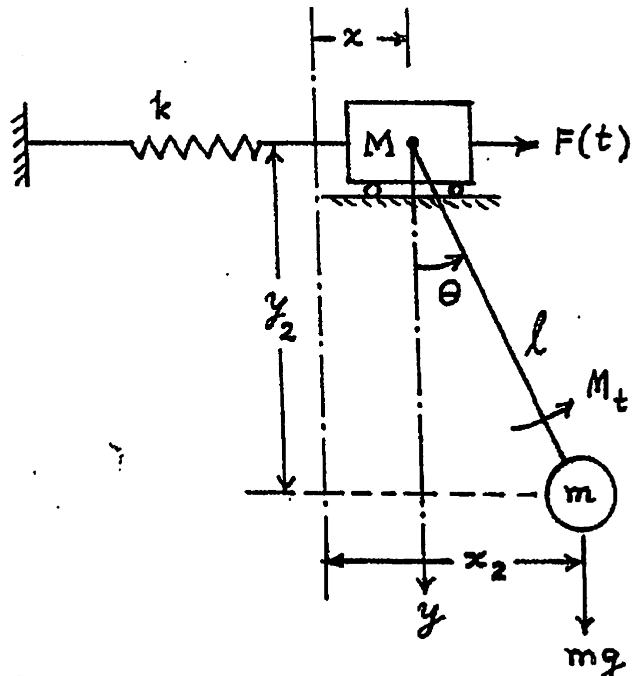
In the present case, τ_n = (time for m to go to $x = x_1$ from $x = 0$ and return to $x = 0$) + (time for m to go to $x = x_2$ from $x = 0$ and return to $x = 0$)

$$\therefore \tau_n = \pi\left(\sqrt{\frac{m}{k_1}} + \sqrt{\frac{m}{k_2}}\right)$$

P1

Key solution
HW #3
511
4/27/09

2.



$x_2 = x + \ell \sin\theta$; $\dot{x}_2 = \dot{x} + \ell\,\dot{\theta}\cos\theta$

$y_2 = \ell\cos\theta$; $\dot{y}_2 = -\ell\,\dot{\theta}\sin\theta$

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[(\dot{x} + \ell\,\dot{\theta}\cos\theta)^2 + (-\ell\,\dot{\theta}\sin\theta)^2\right]$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2} m \ell^2 \dot{\theta}^2 + m \ell \dot{x} \dot{\theta}\cos\theta$$

$$V = \frac{1}{2} k x^2 + m g \ell (1 - \cos\theta)$$

$Q_x = F(t)$; $Q_\theta = M_t(t)$

Equations of motion:

$$(M+m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta + kx = F(t) \quad (1)$$

$$m\ell^2\ddot{\theta} + m\ell\ddot{x}\cos\theta - m\ell\dot{x}\dot{\theta}\sin\theta + mg\ell\sin\theta = M_t(t) \quad (2)$$

P2

Using the approximations

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \;;\; \sin\theta \approx \theta - \frac{\theta^3}{6}$$

Eqs. (1) and (2) can be expressed as

$$(M+m)\ddot{x} + m\ell\ddot{\theta} - \frac{1}{2}m\ell\theta^2\ddot{\theta} - m\ell\theta\dot{\theta}^2 + \frac{1}{6}m\ell\theta^3\dot{\theta}^2 + kx = F(t) \quad (3)$$

$$m\ell^2\ddot{\theta} + m\ell\ddot{x} - \frac{1}{2}m\ell\theta^2\ddot{x} - m\ell\theta\dot{\theta}\dot{x} + \frac{1}{6}m\ell\theta^3\dot{\theta}\dot{x} + mg\ell\theta - \frac{1}{6}mg\ell\theta^3 = M_t(t) \quad (4)$$

By neglecting the nonlinear terms, the linearized equations of motion can be written as

$$(M+m)\ddot{x} + m\ell\ddot{\theta} + kx = F(t) \quad (5)$$

$$m\ell^2\ddot{\theta} + m\ell\ddot{x} + mg\ell\theta = M_t(t) \quad (6)$$

P3

3.

$$\ddot{\theta} + \omega_0^2 \left(\theta - \frac{1}{6}\theta^3\right) = 0 \qquad (E_1)$$

This equation is similar to Eq. (13.9) with

$x = \theta$, $a = \omega_0$, $F(x) = F(\theta) = \theta - \frac{1}{6}\theta^3$.

Eq. (E_1) can be rewritten as

$$\frac{d}{d\theta}(\dot{\theta}^2) + 2\omega_0^2 \left(\theta - \frac{1}{6}\theta^3\right) = 0 \qquad (E_2)$$

which upon integration gives

$$\dot{\theta}^2 = 2\omega_0^2 \int_\theta^{\theta_0} F(\eta)\cdot d\eta = 2\omega_0^2 \int_\theta^{\theta_0} \left(\eta - \frac{1}{6}\eta^3\right)\cdot d\eta$$

$$= 2\omega_0^2 \left(\frac{1}{2}\eta^2 - \frac{1}{24}\eta^4\right)_\theta^{\theta_0} = \omega_0^2\left(\theta_0^2 - \frac{1}{12}\theta_0^4 - \theta^2 + \frac{1}{12}\theta^4\right) \qquad (E_3)$$

$$= \omega_0^2 (\theta_0^2 - \theta^2)\left\{1 - \frac{1}{12}(\theta_0^2 + \theta^2)\right\} \qquad (E_4)$$

Since the maximum value of θ is θ_0, we assume

$$\theta(t) = \theta_0 \sin\beta \qquad (E_5)$$

Thus $\theta_0^2 - \theta^2 = \theta_0^2 - \theta_0^2 \sin^2\beta = \theta_0^2 \cos^2\beta \qquad (E_6)$

$$\theta_0^2 + \theta^2 = \theta_0^2 (1 + \sin^2\beta) \qquad (E_7)$$

and $\dot{\theta} = A_0 \cos\beta \, \frac{d\beta}{dt} \qquad (E_8)$

Substitution of Eqs. (E_6) to (E_8) into (E_4) gives

$$\theta_0^2 \cos^2\beta \left(\frac{d\beta}{dt}\right)^2 = \omega_0^2 \theta_0^2 \cos^2\beta \left\{1 - \frac{1}{12}\theta_0^2 (1 + \sin^2\beta)\right\}$$

i.e.,

$$\left(\frac{d\beta}{dt}\right)^2 = \omega_0^2 \left(1 - \frac{1}{12}\theta_0^2\right)\left\{1 - \frac{\theta_0^2 \sin^2\beta}{12\left(1 - \frac{1}{12}\theta_0^2\right)}\right\} \qquad (E_9)$$

Defining

$$a^2 = \frac{\theta_0^2}{12\left(1 - \frac{1}{12}\theta_0^2\right)} \qquad (E_{10})$$

Eq. (E_9) can be used to express (taking positive root):

$$\frac{d\beta}{dt} = \omega_0 \left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}} \left(1 - a^2 \sin^2\beta\right)^{\frac{1}{2}} \qquad (E_{11})$$

i.e.,

$$\omega_0 \left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}} dt = \int \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} \qquad (E_{12})$$

Integration of (E_{12}) yields

$$\omega_0 \left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}} (t - t_0) = \int_{\beta_0}^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} \qquad (E_{13})$$

P4

Using the initial conditions $\beta_0 = 0$ at $t_0 = 0$, Eq. (E_{13}) can be reduced to

$$\omega_0 \left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}} \cdot t = \int_0^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} = F(a, \beta) \qquad (E_{14})$$

where $F(a, \beta)$ is an incomplete elliptic integral of the first kind. Using $\beta = \frac{\pi}{2}$ when $\theta = \theta_0$ and $\beta = 0$ when $\theta = 0$, we get for one-quarter period,

$$\frac{\tau}{4} = t = \frac{1}{\omega_0 \left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \qquad (E_{15})$$

Thus the time period of the pendulum is given by

$$\tau = \frac{4}{\omega_0 \left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \qquad (E_{16})$$

P5

4.

$\ddot{x} + 0.1(x^2-1)\dot{x} + x = 0$ or $\ddot{x} = -[0.1(x^2-1)\dot{x} + x]$

Let $x = x_1$. $\dot{x}_1 = x_2 = f_1(x_1, x_2)$

$\dot{x}_2 = -[0.1(x_1^2-1)x_2 + x_1] = f_2(x_1, x_2)$

For equilibrium,

$f_1 = 0 \Rightarrow x_2 = 0$; $f_2 = 0 \Rightarrow x_1 = 0$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

where

$$a_{11} = \frac{\partial f_1}{\partial x_1}\bigg|_{(0,0)} = 0, \quad a_{12} = \frac{\partial f_1}{\partial x_2}\bigg|_{(0,0)} = 1,$$

$$a_{21} = \frac{\partial f_2}{\partial x_1}\bigg|_{(0,0)} = -\left(0.2\, x_1 x_2 + 1\right)\bigg|_{(0,0)} = -1,$$

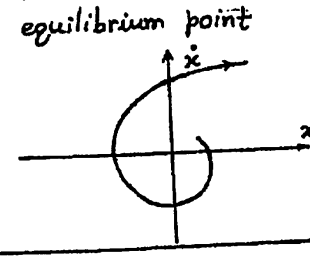
$$a_{22} = \frac{\partial f_2}{\partial x_2}\bigg|_{(0,0)} = -\left[0.1(x_1^2-1)\right]\bigg|_{(0,0)} = 0.1$$

we find $p = 0.1$, $q = 1$

$\lambda_1, \lambda_2 = \frac{1}{2}\left(0.1 \pm \sqrt{0.01-4}\right)$ = complex with positive real parts

Since $p > 0$, the system is unstable at the equilibrium point $(x, \dot{x}) = (0, 0)$.

Hence the phase-plane trajectory in the neighborhood of the equilibrium position appears as shown in the figure.



[EQ 4A]

[EQ 4B]

[EQ 4C]

P6

5

Equation of motion: $\ddot{x} + f\,\frac{x}{|\dot{x}|} + \omega_n^2\, x = 0$ (E1)

i.e. $\ddot{x} + \omega_n^2\,(x + a) = 0$ for $\dot{x} > 0$ (E2)

and $\ddot{x} + \omega_n^2\,(x - a) = 0$ for $\dot{x} < 0$ (E3)

where $a = f/\omega_n^2$ (E4)

Multiplying by $2\dot{x}$ and integrating, (E2) and (E3) yield

$$\dot{x}^2 + \omega_n^2\,(x+a)^2 = R_j^2 \quad \text{for } \dot{x} > 0 \qquad (E_5)$$

$$\dot{x}^2 + \omega_n^2\,(x-a)^2 = R_{j+1}^2 \quad \text{for } \dot{x} < 0 \qquad (E_6)$$

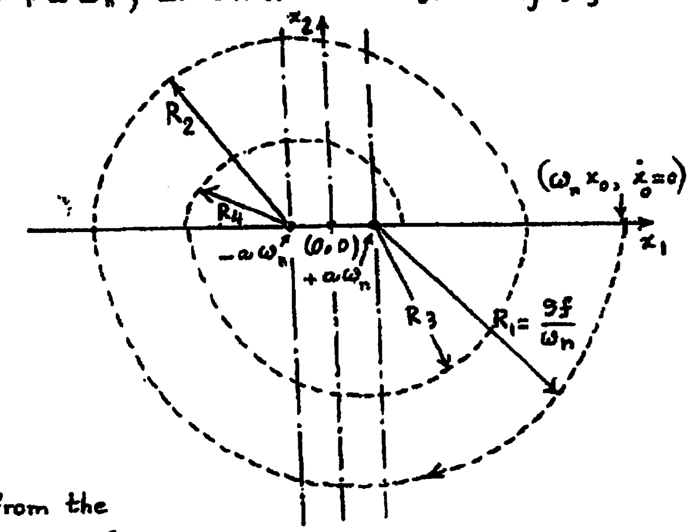
P7

where R_j^2 and R_{j+1}^2 are integration constants which are to be computed at each switching of sign of $\dot{x}$.

We can plot the trajectories of a representative point whose coordinates are

$$x_1 = \omega_n x \quad , \quad x_2 = \dot{x} \qquad (E_7)$$

Eqs. (E_5) and (E_6) show that the trajectory is made of semicircles whose centers are located at $x = -a$ (or $x_1 = -a\omega_n$) and $x = +a$ (or $x_1 = +a\,\omega_n$) as shown in the following figure.



R_1 can be obtained from the initial conditions using Eq. (E_6) as:

$$R_1^2 = 0^2 + \omega_n^2 \left(\frac{10f}{\omega_n^2} - \frac{f}{\omega_n^2}\right)^2 = \left(\frac{9f}{\omega_n}\right)^2 \quad ; \quad R_1 = \frac{9f}{\omega_n} \qquad (E_8)$$

Notice that the radii of the circles $R_1, R_2, \ldots$ decrease according to the relation

$$R_j = R_{j-1} - 2a\,\omega_n \quad ; \quad j = 1, 2, \ldots$$

and the system will stop when

$$R_k \leq 2a\,\omega_n$$

Here $R_1 = \frac{9f}{\omega_n}$, $R_2 = R_1 - \frac{2f}{\omega_n} = \frac{7f}{\omega_n}$, $R_3 = R_2 - \frac{2f}{\omega_n} = \frac{5f}{\omega_n}$,

$R_4 = R_3 - \frac{2f}{\omega_n} = \frac{3f}{\omega_n}$, $R_5 = R_4 - \frac{2f}{\omega_n} = \frac{f}{\omega_n}$,

and the motion stops at this point (after five half-cycles) since $R_6 < 2a\,\omega_n = \frac{2f}{\omega_n}$.

P8

6

$\ddot{\theta} + c\dot{\theta} + \sin\theta = 0$ or $\ddot{\theta} = -c\dot{\theta} - \sin\theta$

Let $x = \theta$ and $y = \frac{dx}{dt} = \dot{\theta}$

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - \sin x \qquad (E_1)$$

Equilibrium or critical point (where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$) of this system is $(x = 0, y = 0)$. Linearization of Eqs. (E_1) about the equilibrium point (origin) leads to

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - x$$

or

$$\begin{Bmatrix} dx/dt \\ dy/dt \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \qquad (E_2)$$

The eigenvalues of this system are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

i.e. $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda - c \end{vmatrix} = 0$ i.e., $\lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$

i.e., $\lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - 1}$ (E_3)

If $c = 0$: $p = 0$; $q = 1$; $\lambda_{1,2} = \pm\sqrt{-1}$

The origin will be a center.

If $0 < c < 2$: $p > 0$; $q > 0$; $\lambda_{1,2}$ = complex conjugates

The origin will be a stable focal point (spiral point).

If $c = 2$: $p > 0$; $q > 0$; $\lambda_{1,2}$ = negative and equal.

The origin will be a stable nodal point.

If $c > 2$: $p > 0$; $q > 0$; If $\lambda_{1,2}$ = negative real, the origin will be a stable nodal point.

If $-2 < c < 0$: $p < 0$; $q > 0$; $\lambda_{1,2}$ = complex conjugates.

The origin will be an unstable focal point (spiral point).

7

Equation of motion: $\ddot{\theta} + 0.5\dot{\theta} + \sin\theta = 0.8$ (E_1)

Let $x = \theta$ and $y = dx/dt$

$\therefore \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x - 0.5y + 0.8$ P9 (E_2)

$$\frac{dy}{dx} = \frac{-\sin x - 0.5\,y + 0.8}{y} \qquad (E_3)$$

At $(x = \sin^{-1} 0.8,\ y = 0)$, $\frac{dy}{dx} = \frac{0}{0}$ and hence it will be an equilibrium point. To investigate the nature of singularity, we rewrite Eqs. (E_2) in linearized form as

$$\left.\begin{aligned} \frac{dx}{dt} &= (0)x + (1)y \\ \frac{dy}{dt} &= (0)x - 0.5\,y \end{aligned}\right\} \qquad (E_4)$$

Thus the eigenvalues of the system are given by

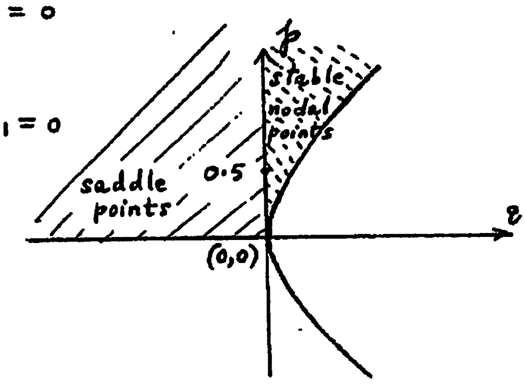
$$\left| \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & 1 \\ 0 & -0.5-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 + 0.5\lambda \equiv \lambda^2 + p\lambda + q = 0$

$\therefore\ \lambda_1 = 0,\ \lambda_2 =$ negative

Here $p =$ positive, $q = 0$, $\lambda_1 = 0$ and $\lambda_2 =$ negative.

Thus the equilibrium point falls on the border of saddle points and stable nodal points as shown in the adjacent figure.



8

$$\frac{dx}{dt} = (0)\,x + (1)\,y \qquad (E_1)$$

$$\frac{dy}{dt} = -1.x - c\,y + (0.1)\,x^3 \qquad (E_2)$$

Eqs. (E_1) and (E_2) are zero at $(x = 0,\ y = 0)$. Hence the origin $(0,0)$ will be equilibrium point (singularity). The eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$

i.e., $\lambda_{1,2} = \left\{ \frac{-c \pm \sqrt{c^2 - 4}}{2} \right\}$

P10

For $c > 0$ and $c < 2$:

$p > 0$, $q > 0$ and $\lambda_{1,2}$ = complex conjugates.

Hence the origin will be a stable focus (or spiral point).

For $c \geq 2$:

$p > 0$, $q > 0$; $\lambda_{1,2}$ = negative real.

Hence the origin will be a stable nodal point.

P11

9

Van der Pol's equation: $\ddot{x} - \alpha(1-x^2)\dot{x} + x = 0, \quad \alpha > 0$ (E_1)

Assume $x(t) = x_0(t) + \alpha\, x_1(t) + \alpha^2 x_2(t)$ (E_2)

$\omega_0^2 = 1 = \omega^2 - \alpha\,\omega_1 - \alpha^2 \omega_2$ (E_3)

where $\omega_0^2 = 1 =$ coefficient of x in Eq. (E_1).

Substitution of (E_2) and (E_3) into (E_1) gives

$$\alpha^0 \left[\ddot{x}_0 + \omega^2 x_0\right] + \alpha^1 \left[\ddot{x}_1 - \dot{x}_0 + \dot{x}_0 x_0^2 - \omega_1 x_0 + \omega^2 x_1\right]$$
$$+ \alpha^2\left[\ddot{x}_2 - \dot{x}_1 + \dot{x}_1 x_0^2 + 2 x_0 \dot{x}_0 x_1 - \omega_2 x_0 - \omega_1 x_1 + \omega^2 x_2\right]$$
$$+ \alpha^3[\cdots] + \cdots = 0 \qquad (E_4)$$

Setting coefficient of α^0 in (E_4) to zero, we obtain

$$\ddot{x}_0 + \omega^2 x_0 = 0 \quad , \quad \text{i.e.,} \quad x_0(t) = A_1 \cos\omega t + A_2 \sin\omega t \qquad (E_5)$$

Assuming the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$, we get $A_1 = A$ and $A_2 = 0$. Thus (E_5) reduces to

$$x_0(t) = A \cos\omega t \qquad (E_6)$$

Setting coefficient of α^1 to zero, in Eq. (E_4),

$$\ddot{x}_1 + \omega^2 x_1 = \dot{x}_0 - \dot{x}_0 x_0^2 + \omega_1 x_0$$
$$= -A\omega \sin\omega t + A^3\omega \sin\omega t \cdot \cos^2\omega t + \omega_1 A \cos\omega t$$
$$= \left(-A\omega + \frac{1}{4} A^3\omega\right) \sin\omega t + \omega_1 A \cos\omega t + \frac{A^3\omega}{4} \sin 3\omega t \leftarrow \qquad (E_7)$$

The coefficients of $\sin\omega t$ and $\cos\omega t$ must be zero in Eq. (E_7) to avoid secular terms. This gives

$$A = \pm 2\ , \quad \omega_1 = 0 \qquad (E_8)$$

Thus the particular solution of (E_7) can be expressed as

$$x_1(t) = A_3 \sin 3\omega t + A_4 \cos 3\omega t \qquad (E_9)$$

Substitution of (E_8) and (E_9) into E_7 gives

$$A_3 = \frac{1}{32}\frac{A^3}{\omega} \quad \text{and} \quad A_4 = 0 \qquad (E_{10})$$

Thus $x_1(t) = \frac{1}{32}\frac{A^3}{\omega} \sin 3\omega t$ (E_{11})

Finally, setting coefficient of α^2 in (E_4) to zero, we get

$$\ddot{x}_2 + \omega^2 x_2 = \dot{x}_1 - \dot{x}_1 x_0^2 - 2x_0 \dot{x}_0 x_1 + \omega_2 x_0 + \omega_1 x_1 \qquad (E_{12})$$

Substitution of (E_{11}), (E_6) and (E_8) into (E_{12}) leads to

$$\ddot{x}_2 + \omega^2 x_2 = \frac{3}{32} A^3 \cos 3\omega t - \left(\frac{3}{32} A^3 \cos 3\omega t\right) A^2 \cos^2\omega t$$
$$- 2(A\cos\omega t)(-A\omega \sin\omega t)\left(\frac{A^3}{32\omega} \sin 3\omega t\right) + \omega_2 A \cos\omega t$$
$$= \left(-\frac{3}{128} A^5 + \frac{1}{64} A^5 + A\omega_2\right)\cos\omega t + \left(\frac{3}{32} A^3 - \frac{3}{64} A^5\right) \cos 3\omega t$$
$$+ \left(-\frac{3}{128} A^5 - \frac{1}{64} A^5\right) \cos 5\omega t \qquad (E_{13})$$

P12

To avoid secular terms, the coefficient of $\cos \omega t$ in (E_{13}) must be zero. This gives

$$\omega_2 = \frac{1}{128} A^4 \qquad (E_{14})$$

With this, and using $A = 2$, Eq. (E_{13}) reduces to

$$\ddot{x}_2 + \omega^2 x_2 = -\frac{3}{4} \cos 3\omega t - \frac{5}{4} \cos 5\omega t \qquad (E_{15})$$

Assuming

$$x_2(t) = A_5 \cos 3 s t + A_6 \cos 5\omega t \qquad (E_{16})$$

we find, from Eq. (E_{15}),

$$A_5 = \frac{3}{32} \cdot \frac{1}{\omega^2}, \qquad A_6 = \frac{5}{96} \cdot \frac{1}{\omega^2} \qquad (E_{17})$$

$$\therefore x_2(t) = \frac{3}{32\omega^2} \cos 3\omega t + \frac{5}{96\omega^2} \cos 5\omega t \qquad (E_{18})$$

Thus the complete solution, Eqs. (E_2) and (E_3), become

$$x(t) = 2 \cos \omega t + \frac{\alpha}{4\omega} \sin 3\omega t + \frac{3\alpha^2}{32\omega^2} \cos 3\omega t + \frac{5\alpha^2}{96\omega^2} \cos 5\omega t$$

and

$$\omega^2 = 1 + \frac{\alpha^2}{8}.$$

P13

Correct 2

Using x and θ as the coordinates, the kinetic and potential energies of the system can be expressed as

$$T = \frac{1}{2}\, m\, \dot{x}^2 + \frac{1}{2}\, J_0\, \dot{\theta}^2 \tag{1}$$

where $J_0 = m\,(\ell + x)^2$ and

$$V = \frac{1}{2}\, k\,(x + \delta_{st})^2 - m\, g\,(\ell + x)\cos\theta \tag{2}$$

where $\delta_{st} = \frac{mg}{k}$. Equations (1) and (2) give

$$\frac{\partial T}{\partial \dot{x}} = m\,\dot{x} \quad ; \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\,\ddot{x}$$

$$\frac{\partial T}{\partial \dot{\theta}} = J_0\,\dot{\theta} \quad ; \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \dot{J}_0\,\dot{\theta} + J_0\,\ddot{\theta} = 2\, m\,(\ell + x)\,\dot{x}\,\dot{\theta} + J_0\,\ddot{\theta}$$

$$\frac{\partial T}{\partial x} = m\,(\ell + x)\,\dot{\theta}^2 \quad ; \quad \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial V}{\partial x} = k\,(x + \delta_{st}) - m\, g\cos\theta \quad ; \quad \frac{\partial V}{\partial \theta} = m\, g\,(\ell + x)\sin\theta$$

The equations of motion can be derived using Lagrange's equations, Eq. (6.44), as:

$$m\,\ddot{x} - m\,(\ell + x)\,\dot{\theta}^2 + k\, x + m\, g - m\, g\cos\theta = 0 \tag{3}$$

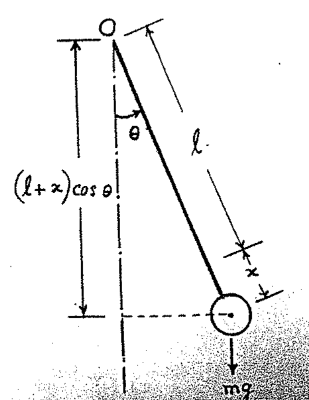
$$m\,(\ell + x)^2\,\ddot{\theta} + 2\, m\,(\ell + x)\,\dot{x}\,\dot{\theta} + m\, g\,(\ell + x)\sin\theta = 0 \tag{4}$$

Using $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and neglecting nonlinear terms involving $x^2\,\ddot{\theta}$, $\dot{\theta}^2$, $x\,\theta$, and $\dot{x}\,\dot{\theta}$, Eqs. (3) and (4) can be reduced (linearized) to obtain:

$$m\,\ddot{x} + k\, x = 0 \tag{5}$$

$$m\,\ell^2\,\ddot{\theta} + m\, g\,\ell\,\theta = 0 \tag{6}$$

Equations (5) and (6) correspond to the natural frequencies:

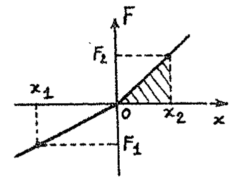


$$\omega_{n1} = \sqrt{\frac{k}{m}} \tag{7}$$

$$\omega_{n2} = \sqrt{\frac{g}{\ell}} \tag{8}$$

1.

T = kinetic energy at time zero $= \frac{1}{2} m (\dot{x}_0)^2$



Let x_2 = maximum displacement on right side.

V = potential energy in spring at displacement $x_2 = \frac{1}{2} k_2 x_2^2$ ($\dot{x}$ is zero at x_2)

Since $T = V$, $\quad x_2 = \sqrt{\frac{m(\dot{x}_0)^2}{k_2}} = \sqrt{\frac{m}{k_2}}\ \dot{x}_0$

Let x_1 = maximum displacement to left side. $\quad V = \frac{1}{2} k_1 x_1^2$

$T = V$ gives $\quad x_1 = \sqrt{\frac{m}{k_1} (\dot{x}_0)^2} = \sqrt{\frac{m}{k_1}}\ \dot{x}_0$

(a) Since $k_1 < k_2$, maximum deflection $= x_1 = \sqrt{\frac{m}{k_1}}\ \dot{x}_0$

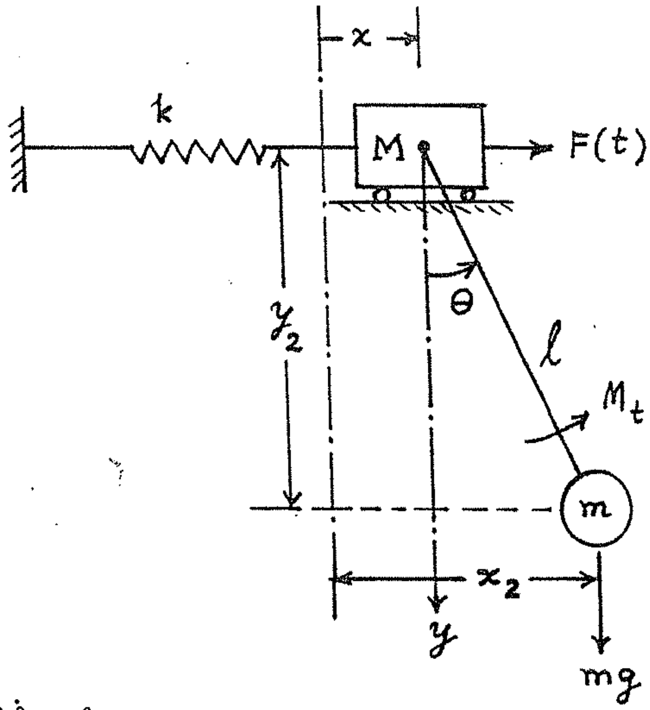
(b) Period of vibration for a spring-mass system is $\tau_n = 2\pi\sqrt{\frac{m}{k}}$.

In the present case, τ_n = (time for m to go to $x = x_1$ from $x = 0$ and return to $x = 0$) + (time for m to go to $x = x_2$ from $x = 0$ and return to $x = 0$)

$$\therefore\ \tau_n = \pi\left(\sqrt{\frac{m}{k_1}} + \sqrt{\frac{m}{k_2}}\right)$$

P1

2.



$x_2 = x + \ell \sin\theta$; $\dot{x}_2 = \dot{x} + \ell\,\dot{\theta}\cos\theta$

$y_2 = \ell\cos\theta$; $\dot{y}_2 = -\ell\,\dot{\theta}\sin\theta$

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m\,(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[(\dot{x} + \ell\,\dot{\theta}\cos\theta)^2 + (-\ell\,\dot{\theta}\sin\theta)^2\right]$$

$$= \frac{1}{2}(M+m)\,\dot{x}^2 + \frac{1}{2} m\,\ell^2\,\dot{\theta}^2 + m\,\ell\,\dot{x}\,\dot{\theta}\cos\theta$$

$$V = \frac{1}{2} k\,x^2 + m\,g\,\ell\,(1-\cos\theta)$$

$Q_x = F(t)$; $Q_\theta = M_t(t)$

Equations of motion:

$$(M+m)\,\ddot{x} + m\,\ell\,\ddot{\theta}\cos\theta - m\,\ell\,\dot{\theta}^2\sin\theta + k\,x = F(t) \qquad (1)$$

$$m\,\ell^2\,\ddot{\theta} + m\,\ell\,\ddot{x}\cos\theta - m\,\ell\,\dot{x}\,\dot{\theta}\sin\theta + m\,g\,\ell\sin\theta = M_t(t) \qquad (2)$$

P2

Using the approximations

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \quad ; \quad \sin\theta \approx \theta - \frac{\theta^3}{6}$$

Eqs. (1) and (2) can be expressed as

$$(M+m)\,\ddot{x} + m\,\ell\,\ddot{\theta} - \frac{1}{2}\,m\,\ell\,\theta^2\,\ddot{\theta} - m\,\ell\,\theta\,\dot{\theta}^2 + \frac{1}{6}\,m\,\ell\,\theta^3\,\dot{\theta}^2 + k\,x = F(t) \tag{3}$$

$$\begin{aligned} m\,\ell^2\,\ddot{\theta} + m\,\ell\,\ddot{x} - \frac{1}{2}\,m\,\ell\,\theta^2\,\ddot{x} - m\,\ell\,\theta\,\dot{\theta}\,\dot{x} + \frac{1}{6}\,m\,\ell\,\theta^3\,\dot{\theta}\,\dot{x} \\ + m\,g\,\ell\,\theta - \frac{1}{6}\,m\,g\,\ell\,\theta^3 = M_t(t) \end{aligned} \tag{4}$$

By neglecting the nonlinear terms, the linearized equations of motion can be written as

$$(M+m)\,\ddot{x} + m\,\ell\,\ddot{\theta} + k\,x = F(t) \tag{5}$$

$$m\,\ell^2\,\ddot{\theta} + m\,\ell\,\ddot{x} + m\,g\,\ell\,\theta = M_t(t) \tag{6}$$

P3

3.

$$\ddot{\theta} + \omega_0^2 \left(\theta - \tfrac{1}{6}\,\theta^3\right) = 0 \qquad (E_1)$$

This equation is similar to Eq. (13.9) with

$x = \theta$, $a = \omega_0$, $F(x) = F(\theta) = \theta - \frac{1}{6}\theta^3$.

Eq. (E_1) can be rewritten as

$$\frac{d}{d\theta}\left(\dot{\theta}^2\right) + 2\omega_0^2\left(\theta - \tfrac{1}{6}\,\theta^3\right) = 0 \qquad (E_2)$$

which upon integration gives

$$\dot{\theta}^2 = 2\omega_0^2 \int_{\theta}^{\theta_0} F(\eta)\cdot d\eta = 2\omega_0^2 \int_{\theta}^{\theta_0} \left(\eta - \tfrac{1}{6}\eta^3\right)\cdot d\eta$$

$$= 2\omega_0^2 \left(\tfrac{1}{2}\eta^2 - \tfrac{1}{24}\eta^4\right)_{\theta}^{\theta_0} = \omega_0^2\left(\theta_0^2 - \tfrac{1}{12}\theta_0^4 - \theta^2 + \tfrac{1}{12}\theta^4\right) \qquad (E_3)$$

$$= \omega_0^2\left(\theta_0^2 - \theta^2\right)\left\{1 - \tfrac{1}{12}\left(\theta_0^2 + \theta^2\right)\right\} \qquad (E_4)$$

Since the maximum value of θ is θ_0, we assume

$$\theta(t) = \theta_0 \sin\beta \qquad (E_5)$$

Thus

$$\theta_0^2 - \theta^2 = \theta_0^2 - \theta_0^2 \sin^2\beta = \theta_0^2 \cos^2\beta \qquad (E_6)$$

$$\theta_0^2 + \theta^2 = \theta_0^2\left(1 + \sin^2\beta\right) \qquad (E_7)$$

and

$$\dot{\theta} = A_0 \cos\beta \,\frac{d\beta}{dt} \qquad (E_8)$$

Substitution of Eqs. (E_6) to (E_8) into (E_4) gives

$$\theta_0^2 \cos^2\beta \left(\frac{d\beta}{dt}\right)^2 = \omega_0^2\,\theta_0^2 \cos^2\beta\left\{1 - \tfrac{1}{12}\theta_0^2\left(1 + \sin^2\beta\right)\right\}$$

i.e.,

$$\left(\frac{d\beta}{dt}\right)^2 = \omega_0^2\left(1 - \tfrac{1}{12}\theta_0^2\right)\left\{1 - \frac{\theta_0^2 \sin^2\beta}{12\left(1 - \frac{1}{12}\theta_0^2\right)}\right\} \qquad (E_9)$$

Defining

$$a^2 = \frac{\theta_0^2}{12\left(1 - \frac{1}{12}\theta_0^2\right)} \qquad (E_{10})$$

Eq. (E_9) can be used to express (taking positive root):

$$\frac{d\beta}{dt} = \omega_0\left(1 - \tfrac{1}{12}\theta_0^2\right)^{\frac{1}{2}}\left(1 - a^2 \sin^2\beta\right)^{\frac{1}{2}} \qquad (E_{11})$$

i.e.,

$$\omega_0\left(1 - \tfrac{1}{12}\theta_0^2\right)^{\frac{1}{2}} dt = \int \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} \qquad (E_{12})$$

Integration of (E_{12}) yields

$$\omega_0\left(1 - \tfrac{1}{12}\theta_0^2\right)^{\frac{1}{2}}\left(t - t_0\right) = \int_{\beta_0}^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} \qquad (E_{13})$$

P4

Using the initial conditions $\beta_0 = 0$ at $t_0 = 0$, Eq. (E_{13}) can be reduced to

$$\omega_0 \left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}} \cdot t = \int_0^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} = F(a, \beta) \qquad (E_{14})$$

where $F(a, \beta)$ is an incomplete elliptic integral of the first kind. Using $\beta = \frac{\pi}{2}$ when $\theta = \theta_0$ and $\beta = 0$ when $\theta = 0$, we get for one-quarter period,

$$\frac{\tau}{4} = t = \frac{1}{\omega_0 \left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \qquad (E_{15})$$

Thus the time period of the pendulum is given by

$$\tau = \frac{4}{\omega_0 \left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \qquad (E_{16})$$

P5

4.

$$\ddot{x} + 0.1(x^2-1)\dot{x} + x = 0 \quad \text{or} \quad \ddot{x} = -[0.1(x^2-1)\dot{x} + x]$$

Let $x = x_1$. $\dot{x}_1 = x_2 = f_1(x_1, x_2)$

$$\dot{x}_2 = -[0.1(x_1^2-1)x_2 + x_1] = f_2(x_1, x_2)$$

For equilibrium,

$$f_1 = 0 \Rightarrow x_2 = 0 \; ; \; f_2 = 0 \Rightarrow x_1 = 0$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

where

$$a_{11} = \left.\frac{\partial f_1}{\partial x_1}\right|_{(0,0)} = 0, \quad a_{12} = \left.\frac{\partial f_1}{\partial x_2}\right|_{(0,0)} = 1,$$

$$a_{21} = \left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0)} = -\left(0.2\, x_1 x_2 + 1\right)\Big|_{(0,0)} = -1,$$

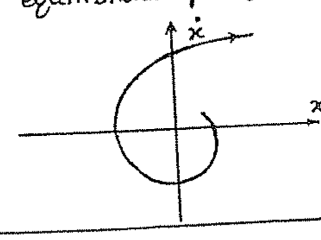
$$a_{22} = \left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0)} = -\left[0.1(x_1^2-1)\right]\Big|_{(0,0)} = 0.1$$

we find $p = 0.1$, $q = 1$

$\lambda_1, \lambda_2 = \frac{1}{2}\left(0.1 \pm \sqrt{0.01-4}\right)$ = complex with positive real parts

Since $p > 0$, the system is unstable at the equilibrium point $(x, \dot{x}) = (0, 0)$.

Hence the phase-plane trajectory in the neighborhood of the equilibrium position appears as shown in the figure.



[EQ 4A]

[EQ 4B]

[EQ 4C]

P6

5

Equation of motion: $\ddot{x} + f\dfrac{\dot{x}}{|\dot{x}|} + \omega_n^2 x = 0$ (E1)

i.e. $\ddot{x} + \omega_n^2 (x + a) = 0$ for $\dot{x} > 0$ (E2)

and $\ddot{x} + \omega_n^2 (x - a) = 0$ for $\dot{x} < 0$ (E3)

where $a = f/\omega_n^2$ (E4)

Multiplying by $2\dot{x}$ and integrating, (E2) and (E3) yield

$\dot{x}^2 + \omega_n^2 (x+a)^2 = R_j^2$ for $\dot{x} > 0$ (E5)

$\dot{x}^2 + \omega_n^2 (x-a)^2 = R_{j+1}^2$ for $\dot{x} < 0$ (E6)

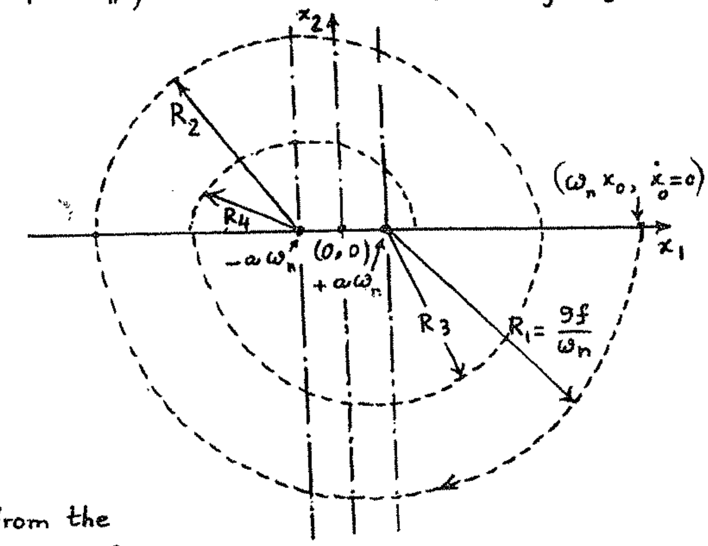
P7

where R_j^2 and R_{j+1}^2 are integration constants which are to be computed at each switching of sign of $\dot{x}$.

We can plot the trajectories of a representative point whose coordinates are

$$x_1 = \omega_n x \quad , \quad x_2 = \dot{x} \qquad (E_7)$$

Eqs. (E_5) and (E_6) show that the trajectory is made of semicircles whose centers are located at $x = -a$ (or $x_1 = -a\omega_n$) and $x = +a$ (or $x_1 = +a\omega_n$) as shown in the following figure.



R_1 can be obtained from the initial conditions using Eq. (E_6) as:

$$R_1^2 = 0^2 + \omega_n^2 \left(\frac{10f}{\omega_n^2} - \frac{f}{\omega_n^2} \right)^2 = \left(\frac{9f}{\omega_n} \right)^2 \quad ; \quad R_1 = \frac{9f}{\omega_n} \qquad (E_8)$$

Notice that the radii of the circles $R_1, R_2, \ldots$ decrease according to the relation

$$R_j = R_{j-1} - 2a\omega_n \quad ; \quad j = 1, 2, \ldots$$

and the system will stop when

$$R_k \le 2a\omega_n$$

Here $R_1 = \frac{9f}{\omega_n}$, $R_2 = R_1 - \frac{2f}{\omega_n} = \frac{7f}{\omega_n}$, $R_3 = R_2 - \frac{2f}{\omega_n} = \frac{5f}{\omega_n}$,

$R_4 = R_3 - \frac{2f}{\omega_n} = \frac{3f}{\omega_n}$, $R_5 = R_4 - \frac{2f}{\omega_n} = \frac{f}{\omega_n}$,

and the motion stops at this point (after five half-cycles) since $R_6 < 2a\omega_n = \frac{2f}{\omega_n}$.

P8

6

$\ddot{\theta} + c\dot{\theta} + \sin\theta = 0$ or $\ddot{\theta} = -c\dot{\theta} - \sin\theta$

Let $x = \theta$ and $y = \frac{dx}{dt} = \dot{\theta}$

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - \sin x \qquad (E_1)$$

Equilibrium or critical point (where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$) of this system is $(x = 0, y = 0)$. Linearization of Eqs. (E_1) about the equilibrium point (origin) leads to

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - x$$

or

$$\begin{Bmatrix} dx/dt \\ dy/dt \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \qquad (E_2)$$

The eigenvalues of this system are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

i.e. $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda - c \end{vmatrix} = 0$ i.e., $\lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$

i.e., $\lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - 1}$ (E_3)

If $c = 0$: $p = 0$; $q = 1$; $\lambda_{1,2} = \pm\sqrt{-1}$
The origin will be a center.

If $0 < c < 2$: $p > 0$; $q > 0$; $\lambda_{1,2}$ = complex conjugates
The origin will be a stable focal point (spiral point).

If $c = 2$: $p > 0$; $q > 0$; $\lambda_{1,2}$ = negative and equal.
The origin will be a stable nodal point.

If $c > 2$: $p > 0$; $q > 0$; If $\lambda_{1,2}$ = negative real, the origin will be a stable nodal point.

If $-2 < c < 0$: $p < 0$; $q > 0$; $\lambda_{1,2}$ = complex conjugates.
The origin will be an unstable focal point (spiral point).

7

Equation of motion: $\ddot{\theta} + 0.5\dot{\theta} + \sin\theta = 0.8$ (E_1)

Let $x = \theta$ and $y = dx/dt$

$\therefore \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x - 0.5y + 0.8$ P9 (E_2)

$$\frac{dy}{dx} = \frac{-\sin x - 0.5y + 0.8}{y} \qquad (E_3)$$

At $(x = \sin^{-1} 0.8,\ y = 0)$, $\frac{dy}{dx} = \frac{0}{0}$ and hence it will be an equilibrium point. To investigate the nature of singularity, we rewrite Eqs. (E_2) in linearized form as

$$\left.\begin{aligned} \frac{dx}{dt} &= (0)x + (1)y \\ \frac{dy}{dt} &= (0)x - 0.5y \end{aligned}\right\} \qquad (E_4)$$

Thus the eigenvalues of the system are given by

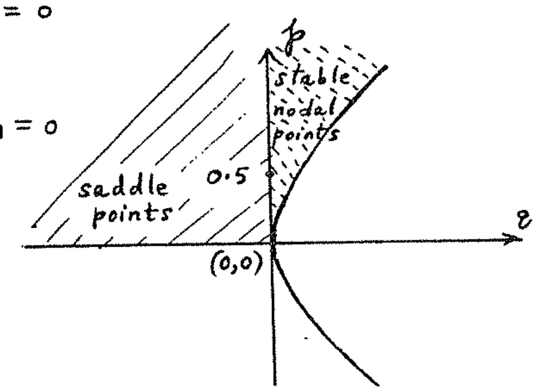
$$\left| \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & 1 \\ 0 & -0.5-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 + 0.5\lambda \equiv \lambda^2 + p\lambda + q = 0$

$\therefore\ \lambda_1 = 0,\ \lambda_2 = \text{negative}$

Here p = positive ; $q = 0$, $\lambda_1 = 0$ and λ_2 = negative.

Thus the equilibrium point falls on the border of saddle points and stable nodal points as shown in the adjacent figure.



8

$$\frac{dx}{dt} = (0)x + (1)y \qquad (E_1)$$

$$\frac{dy}{dt} = -1.x - cy + (0.1)x^3 \qquad (E_2)$$

Eqs. (E_1) and (E_2) are zero at $(x = 0,\ y = 0)$. Hence the origin $(0,0)$ will be equilibrium point (singularity). The eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$

i.e., $\lambda_{1,2} = \left\{ \frac{-c \pm \sqrt{c^2 - 4}}{2} \right\}$

P10

For $c > 0$ and $c < 2$:

$p > 0$, $q > 0$ and $\lambda_{1,2}$ = complex conjugates.

Hence the origin will be a stable focus (or spiral point).

For $c \geq 2$:

$p > 0$, $q > 0$; $\lambda_{1,2}$ = negative real.

Hence the origin will be a stable nodal point.

P11

9

Van der Pol's equation: $\ddot{x} - \alpha(1-x^2)\dot{x} + x = 0, \quad \alpha > 0$ (E1)

Assume $x(t) = x_0(t) + \alpha\, x_1(t) + \alpha^2 x_2(t)$ (E2)

$\omega_0^2 = 1 = \omega^2 - \alpha\,\omega_1 - \alpha^2 \omega_2$ (E3)

where $\omega_0^2 = 1 =$ coefficient of x in Eq. (E1).

Substitution of (E2) and (E3) into (E1) gives

$$\alpha^0 \left[\ddot{x}_0 + \omega^2 x_0\right] + \alpha^1\left[\ddot{x}_1 - \dot{x}_0 + \dot{x}_0 x_0^2 - \omega_1 x_0 + \omega^2 x_1\right]$$
$$+ \alpha^2\left[\ddot{x}_2 - \dot{x}_1 + \dot{x}_1 x_0^2 + 2 x_0 \dot{x}_0 x_1 - \omega_2 x_0 - \omega_1 x_1 + \omega^2 x_2\right] \quad (E_4)$$
$$+ \alpha^3[\cdots] + \cdots = 0$$

Setting coefficient of α^0 in (E4) to zero, we obtain

$\ddot{x}_0 + \omega^2 x_0 = 0$, i.e., $x_0(t) = A_1 \cos\omega t + A_2 \sin\omega t$ (E5)

Assuming the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$, we get $A_1 = A$ and $A_2 = 0$. Thus (E5) reduces to

$x_0(t) = A \cos\omega t$ (E6)

Setting coefficient of α^1 to zero, in Eq. (E4),

$$\ddot{x}_1 + \omega^2 x_1 = \dot{x}_0 - \dot{x}_0 x_0^2 + \omega_1 x_0$$
$$= -A\omega \sin\omega t + A^3\omega \sin\omega t.\cos^2\omega t + \omega_1 A \cos\omega t$$
$$= \left(-A\omega + \tfrac{1}{4}A^3\omega\right)\sin\omega t + \omega_1 A \cos\omega t + \frac{A^3\omega}{4}\sin 3\omega t \quad (E_7)$$

The coefficients of $\sin\omega t$ and $\cos\omega t$ must be zero in Eq. (E7) to avoid secular terms. This gives

$A = \pm 2$, $\omega_1 = 0$ (E8)

Thus the particular solution of (E7) can be expressed as

$x_1(t) = A_3 \sin 3\omega t + A_4 \cos 3\omega t$ (E9)

Substitution of (E8) and (E9) into E7 gives

$A_3 = \frac{1}{32}\frac{A^3}{\omega}$ and $A_4 = 0$ (E10)

Thus $x_1(t) = \frac{1}{32}\frac{A^3}{\omega}\sin 3\omega t$ (E11)

Finally, setting coefficient of α^2 in (E4) to zero, we get

$\ddot{x}_2 + \omega^2 x_2 = \dot{x}_1 - \dot{x}_1 x_0^2 - 2x_0\dot{x}_0 x_1 + \omega_2 x_0 + \omega_1 x_1$ (E12)

Substitution of (E11), (E6) and (E8) into (E12) leads to

$$\ddot{x}_2 + \omega^2 x_2 = \frac{3}{32}A^3\cos 3\omega t - \left(\frac{3}{32}A^3\cos 3\omega t\right)A^2\cos^2\omega t$$
$$- 2(A\cos\omega t)(-A\omega\sin\omega t)\left(\frac{A^3}{32\omega}\sin 3\omega t\right) + \omega_2 A\cos\omega t$$
$$= \left(-\frac{3}{128}A^5 + \frac{1}{64}A^5 + A\omega_2\right)\cos\omega t + \left(\frac{3}{32}A^3 - \frac{3}{64}A^5\right)\cos 3\omega t$$
$$+ \left(-\frac{3}{128}A^5 - \frac{1}{64}A^5\right)\cos 5\omega t \quad (E_{13})$$

P12

To avoid secular terms, the coefficient of $\cos \omega t$ in (E_{13}) must be zero. This gives
$$\omega_2 = \frac{1}{128} A^4 \qquad (E_{14})$$
With this, and using $A = 2$, Eq. (E_{13}) reduces to
$$\ddot{x}_2 + \omega^2 x_2 = -\frac{3}{4} \cos 3\omega t - \frac{5}{4} \cos 5\omega t \qquad (E_{15})$$
Assuming
$$x_2(t) = A_5 \cos 3\omega t + A_6 \cos 5\omega t \qquad (E_{16})$$
we find, from Eq. (E_{15}),
$$A_5 = \frac{3}{32} \cdot \frac{1}{\omega^2}, \qquad A_6 = \frac{5}{96} \cdot \frac{1}{\omega^2} \qquad (E_{17})$$
$$\therefore\ x_2(t) = \frac{3}{32\omega^2} \cos 3\omega t + \frac{5}{96\omega^2} \cos 5\omega t \qquad (E_{18})$$
Thus the complete solution, Eqs. (E_2) and (E_3), become
$$x(t) = 2 \cos \omega t + \frac{\alpha}{4\omega} \sin 3\omega t + \frac{3\alpha^2}{32\omega^2} \cos 3\omega t + \frac{5\alpha^2}{96\omega^2} \cos 5\omega t$$
and
$$\omega^2 = 1 + \frac{\alpha^2}{8}.$$

P13

Chapter 4

Projects

Local contents

4.1 Impulse response of second order system which is not under-damped

Abstract

The impulse response $h(t)$ for second order single degree of freedom system which is under-damped is well known. In this note, the derivation to the impulse response of critically damped and over-damped systems are given.

4.1.1 Impulse response for over-damped system

Given the system

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) = \delta(t) \tag{1}$$

Where $\delta(t)$ is an impulse. We seek to find $x(t)$, the response of the above system to this impulse.

Assume the system is initially at rest. Due to the action of this impulse, the system will obtain an initial speed which is found as follows. Let $\delta(t) \equiv \hat{F} = F\Delta t$ where Δt is the duration of the impulse and F is the magnitude (in Newtons) of the impulse (hence units of $\hat{F}$ is N sec). This impulse will impart a momentum on the mass being hit which we use to determine the initial speed

$$\hat{F} = mv_0$$
$$v_0 = \frac{\hat{F}}{m}$$

Hence, the system will now have initial conditions of $x(0) = 0$ and $\dot{x}(0) = v_0 = \frac{\hat{F}}{m}$. Now, the response of (1), when $\xi > 1$ is known and given by

$$x(t) = e^{-\xi\omega_n t}\left(Ae^{\omega_n\sqrt{\xi^2-1}t} + Be^{-\omega_n\sqrt{\xi^2-1}t}\right) \tag{2}$$

Apply $x(0) = 0$, we obtain that $0 = A + B$ or $B = -A$. Now

$$\begin{aligned}\dot{x}(t) &= -\xi\omega_n e^{-\xi\omega_n t}\left(Ae^{\omega_n\sqrt{\xi^2-1}t} + Be^{-\omega_n\sqrt{\xi^2-1}t}\right)\\ &+ e^{-\xi\omega_n t}\left(A\omega_n\sqrt{\xi^2-1}e^{\omega_n\sqrt{\xi^2-1}t} - B\omega_n\sqrt{\xi^2-1}e^{-\omega_n\sqrt{\xi^2-1}t}\right)\end{aligned}$$

Apply $\dot{x}(0) = \frac{\hat{F}}{m}$ to the above, we obtain

$$\frac{\hat{F}}{m} = \left(A\omega_n\sqrt{\xi^2-1} - B\omega_n\sqrt{\xi^2-1}\right)$$

But $B = -A$, hence $\frac{\hat{F}}{m} = 2A\omega_n\sqrt{\xi^2-1}$ or $A = \frac{\hat{F}}{2m\omega_n\sqrt{\xi^2-1}}$

Hence (2) becomes

$$\begin{aligned}x(t) &= e^{-\xi\omega_n t}\left(\frac{\hat{F}}{2m\omega_n\sqrt{\xi^2-1}}e^{\omega_n\sqrt{\xi^2-1}t} - \frac{\hat{F}}{2m\omega_n\sqrt{\xi^2-1}}e^{-\omega_n\sqrt{\xi^2-1}t}\right)\\ &= \frac{\hat{F}}{2m\omega_n\sqrt{\xi^2-1}}e^{-\xi\omega_n t}\left(e^{\omega_n\sqrt{\xi^2-1}t} - e^{-\omega_n\sqrt{\xi^2-1}t}\right)\end{aligned}$$

When the magnitude of the impulse is unity, i.e. a unit impulse, hence $\hat{F} = 1$, then we obtain the unit impulse response

$$\boxed{h(t) = \frac{1}{2m\omega_n\sqrt{\xi^2-1}}e^{-\xi\omega_n t}\left(e^{\omega_n\sqrt{\xi^2-1}t} - e^{-\omega_n\sqrt{\xi^2-1}t}\right)}$$

4.1.2 Impulse response for critically damped system

The response of (1), when $\xi = 1$ is given by

$$x(t) = Ae^{-\xi\omega_n t} + Bte^{-\xi\omega_n t} \tag{3}$$

Apply $x(0) = 0$, we obtain that $0 = A$ Now

$$\dot{x}(t) = Be^{-\xi\omega_n t} - \xi\omega_n Bte^{-\xi\omega_n t}$$

Apply $\dot{x}(0) = \frac{\hat{F}}{m}$ to the above, we obtain

$$\frac{\hat{F}}{m} = B$$

Hence (3) becomes

$$x(t) = \frac{\hat{F}}{m}te^{-\xi\omega_n t}$$

When the magnitude of the impulse is unity, i.e. a unit impulse, hence $\hat{F} = 1$, then we obtain the unit impulse response

$$h(t) = \frac{1}{m}te^{-\xi\omega_n t}$$

4.2 Final Project. Stabilization of an inverted pendulum on moving cart using feedback control

4.2.1 Introduction

Given the following system

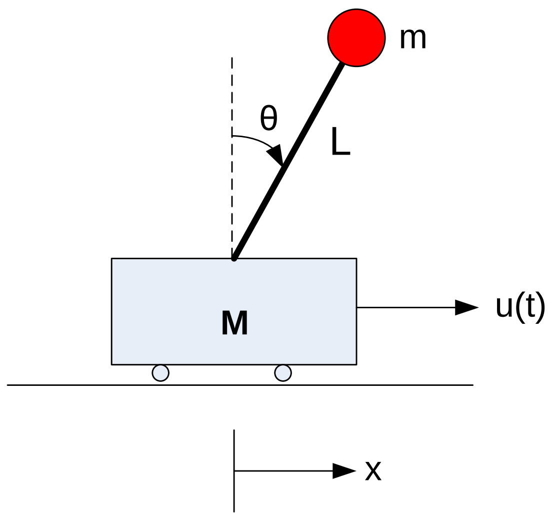


Figure 4.1: System to control

Need to find control law $u(t)$ to stabilize the inverted pendulum. First we need to obtain the equations of motions.

4.2.2 Analysis

Let the Lagrangian coordinates be θ and x as shown. Let L be the Lagrangian. Let T be the kinetic energy of the system and let U be the potential energy. Hence

$$L = T - U$$

and

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}mv^2$$

Where v is the linear velocity of the blob m relative to the inertial system.

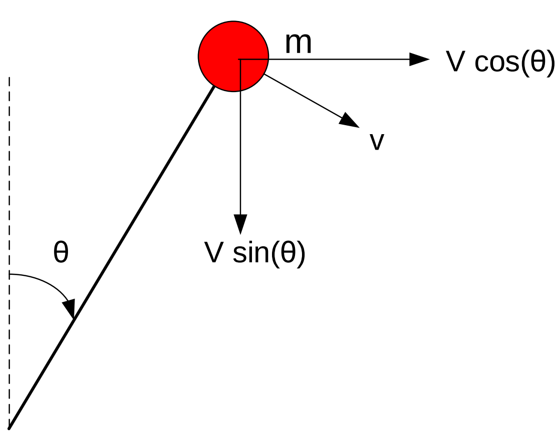


Figure 4.2: Velocity diagram

Hence, since $v = l\dot{\theta}$, we obtain

$$\begin{aligned} v^2 &= (\dot{x} + v_x)^2 + v_y^2 \\ &= (\dot{x} + v\cos\theta)^2 + (v\sin\theta)^2 \\ &= \left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \left(l\dot{\theta}\sin\theta\right)^2 \\ &= \dot{x}^2 + l^2\dot{\theta}^2\cos^2\theta + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\sin^2\theta \\ &= \dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}\cos\theta \end{aligned}$$

Hence T becomes

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}\cos\theta\right)$$

And since the blob is losing potential energy as it move downwards, we obtain U as (assuming zero potential energy is the ground level)

$$U = mgl\cos\theta$$

Therefore the Lagrangian is

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}\cos\theta\right) - mgl\cos\theta \end{aligned}$$

To obtain the equation of motions, we need to evaluate $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$ for each Lagrangian coordinate q_i and Q_i is the generalized force for that coordinate. Hence for θ we obtain

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}} &= \frac{1}{2}m\left(2l^2\dot{\theta} + 2\dot{x}l\cos\theta\right) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{1}{2}m\left(2l^2\ddot{\theta} + 2\ddot{x}l\cos\theta - 2\dot{x}l\sin(\theta)\,\dot{\theta}\right) \\ \frac{\partial L}{\partial \theta} &= \frac{1}{2}m\left(-2\dot{x}l\dot{\theta}\sin\theta\right) + mgl\sin\theta \end{aligned}$$

Hence EQM for θ is

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= Q \\ \frac{1}{2}m\left(2l^2\ddot{\theta} + 2\ddot{x}l\cos\theta - 2\dot{x}l\sin(\theta)\,\dot{\theta}\right) - \left(\frac{1}{2}m\left(-2\dot{x}l\dot{\theta}\sin\theta\right) + mgl\sin\theta\right) &= Q \\ ml\ddot{\theta} + m\ddot{x}\cos\theta - m\dot{x}\sin\theta\dot{\theta} + m\dot{x}\dot{\theta}\sin\theta - mg\sin\theta &= \frac{Q}{l} \\ ml\ddot{\theta} + m\ddot{x}\cos\theta - mg\sin\theta &= \frac{Q}{l} \end{aligned} \tag{1}$$

Now we need to obtain Q for the coordinate θ. Apply a virtual displacement $\delta\theta$ and determine the work done by $u(t)$

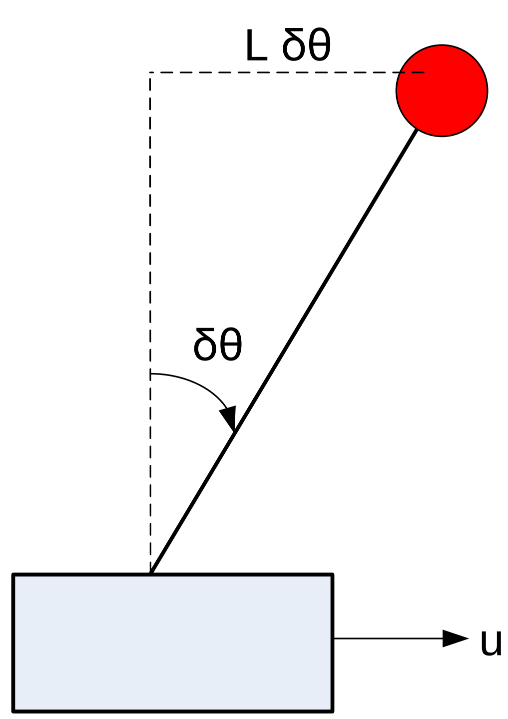


Figure 4.3: virtual work for θ

Hence the work done by u is making virtual displacement $\delta\theta$ is zero, since u is not in the line of force along this displacement. Therefore, the EQM for θ is from Eq (1) above

$$ml\ddot{\theta} + m\ddot{x}\cos\theta - mg\sin\theta = 0 \tag{2}$$

Now we find EQM for coordinate x

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}} &= M\dot{x}+\frac{1}{2}m\left(2\dot{x}+2l\dot{\theta}\cos\theta\right)\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= M\ddot{x}+\frac{1}{2}m\left(2\ddot{x}+2l\ddot{\theta}\cos\theta-2l\dot{\theta}\sin\left(\theta\right)\dot{\theta}\right)\\ &= M\ddot{x}+m\left(\ddot{x}+l\ddot{\theta}\cos\theta-l\dot{\theta}^2\sin\theta\right)\\ \frac{\partial L}{\partial x} &= 0\end{aligned}$$

Hence EQM for x is

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x} &= Q\\ \left(M+m\right)\ddot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta &= Q\end{aligned}$$

Now we need to find Q for x. Apply virtual displacement in the x direction, and find work done by u

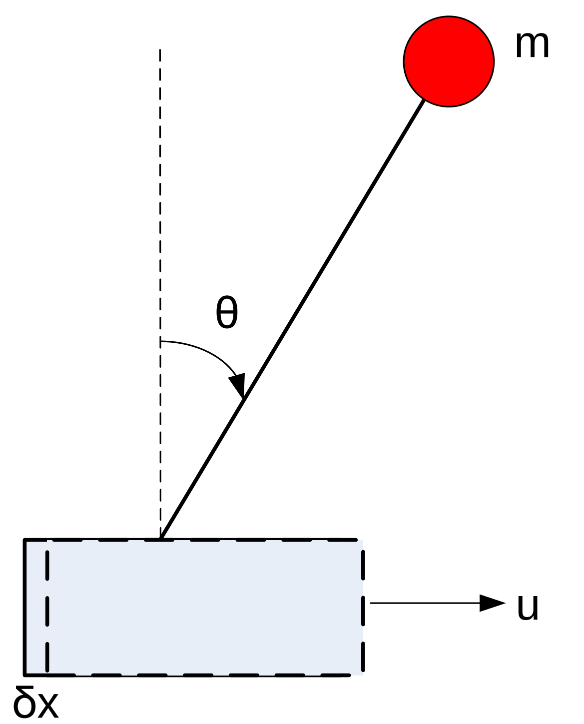


Figure 4.4: virtual displacement in the x direction

$$\delta W = u(\delta x)$$

But $Q=\frac{\delta W}{\delta x}$, hence we see that $Q=u$, therefore, the EQM becomes

$$\left(M+m\right)\ddot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta = u \tag{3}$$

<u>Conclusion</u>: The two equations of motion are

$$\begin{aligned}ml\ddot{\theta}+m\ddot{x}\cos\theta-mg\sin\theta &= 0\\ \left(M+m\right)\ddot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta &= u\end{aligned}$$

Assuming small angle approximation gives

$$l\ddot{\theta}+\ddot{x}-g\theta = 0 \tag{4}$$

$$\left(M+m\right)\ddot{x}+ml\ddot{\theta} = u \tag{5}$$

Now we solve for $\ddot{x}$ and $\ddot{\theta}$ from Eqs (4) and (5). From Eq (5)

$$\ddot{x}=\frac{u-ml\ddot{\theta}}{\left(M+m\right)}$$

Substituting the above into Eq (4) gives

$$\begin{aligned}l\ddot{\theta}+\left(\frac{u-ml\ddot{\theta}}{\left(M+m\right)}\right)-g\theta &= 0\\ \left(M+m\right)l\ddot{\theta}+u-ml\ddot{\theta}-\left(M+m\right)g\theta &= 0\\ \ddot{\theta}Ml-\left(M+m\right)g\theta &= -u\end{aligned}$$

$$\ddot{\theta}=\frac{-u+\left(M+m\right)g\theta}{Ml} \tag{6}$$

Using result for $\ddot{\theta}$ found in Eq (6) and substituting into (5) gives

$$\begin{aligned}\ddot{x} &= \frac{u-ml\ddot{\theta}}{(M+m)}\\ \ddot{x} &= \frac{u-ml\left(\frac{-u+(M+m)g\theta}{Ml}\right)}{(M+m)}\\ &= \frac{uM+mu-mMg\theta-m^2g\theta}{M\,(M+m)}\\ &= \frac{-gm\theta(M+m)}{M\,(M+m)}+\frac{u}{M}\\ &= \frac{-gm\theta}{M}+\frac{u}{M}\end{aligned}$$

To summarize what we have so far. We have obtained two linearized equations of motion for θ and x and they are the following

$$\begin{aligned}l\ddot{\theta}+\ddot{x}-g\theta &= u\\ (M+m)\,\ddot{x}+ml\ddot{\theta} &= u\end{aligned}$$

Now we convert the equations to state space. Let $x_1=x, x_2=\dot{x}, x_3=\theta, x_4=\dot{\theta}$, hence

$$\left.\begin{matrix}x_1=x\\ x_2=\dot{x}\\ x_3=\theta\\ x_4=\dot{\theta}\end{matrix}\right\}\rightarrow\left.\begin{matrix}\dot{x}_1=\dot{x}=x_2\\ \dot{x}_2=\ddot{x}\\ \dot{x}_3=\dot{\theta}=x_4\\ \dot{x}_4=\ddot{\theta}\end{matrix}\right\}\rightarrow\left.\begin{matrix}\dot{x}_1=\dot{x}=x_2\\ \dot{x}_2=\frac{-gm\theta}{M}+\frac{u}{M}\\ \dot{x}_3=\dot{\theta}=x_4\\ \dot{x}_4=\frac{-u+(M+m)g\theta}{Ml}\end{matrix}\right\}$$

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{-gmx_3}{M}+\frac{u}{M}\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= x_3\frac{g(M+m)}{Ml}-\frac{u}{Ml}\end{aligned}$$

Writing the above in the form $\dot{X}=AX+Bu$ we obtain

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix}=\begin{pmatrix}0&1&0&0\\ 0&0&\frac{-gm}{M}&0\\ 0&0&0&1\\ 0&0&\frac{g(M+m)}{Ml}&0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}+\begin{pmatrix}0\\ \frac{1}{M}\\ 0\\ \frac{-1}{Ml}\end{pmatrix}u$$

$$\mathbf{y}=\begin{pmatrix}1&0&1&0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}$$

4.2.2.1 Stability of open loop system

To determine the stability of the above system (now that it is a linear system since we have linearized it), we first find the equilibrium point. This is found by setting $\dot{\mathbf{x}}=\mathbf{0}$, and this results in $x_2=0, x_3=0, x_4=0$, i.e. $\dot{x}=0, \theta=0$, and $\dot{\theta}=0$. Notice that the value of x is not important for the equilibrium point. Now we need to determine if this point is stable or not.

$$\begin{aligned}\det\begin{pmatrix}-\lambda&1&0&0\\ 0&-\lambda&\frac{-gm}{M}&0\\ 0&0&-\lambda&1\\ 0&0&\frac{g(M+m)}{Ml}&-\lambda\end{pmatrix}&=0\\ -\frac{1}{Ml}\left(Mg\lambda^2-Ml\lambda^4+gm\lambda^2\right)&=0\\ \lambda^2\left(Ml\lambda^2-g(m+M)\right)&=0\end{aligned}$$

Hence

$$\lambda=\left\{0,0,\frac{1}{Ml}\sqrt{Mgl\,(M+m)},-\frac{1}{Ml}\sqrt{Mgl\,(M+m)}\right\}$$

Since M,l,m,g are all positive, we see that one root will be in the RHS of the complex plane. Therefore the open loop system is unstable.

To stabilize it, we need to supply a control law u to force the roots of the new A matrix to be all in the LHS of the complex plane.

Let

$$\begin{aligned}\mathbf{u} &= \mathbf{Fx} \qquad (7)\\ &= (f_1,f_2,f_3,f_4)\,(x_1,x_2,x_3,x_4)^T\\ &= f_1x_1+f_2x_2+f_3x_3+f_4x_4\end{aligned}$$

Hence Eq (7) becomes

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \left\{\begin{pmatrix}0&1&0&0\\0&0&\frac{-gm}{M}&0\\0&0&0&1\\0&0&\frac{g(M+m)}{Ml}&0\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{M}\\ 0\\ \frac{-1}{Ml}\end{pmatrix}\begin{pmatrix}f_1 & f_2 & f_3 & f_4\end{pmatrix}\right\}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \left\{\begin{pmatrix}0&1&0&0\\0&0&\frac{-gm}{M}&0\\0&0&0&1\\0&0&\frac{g(M+m)}{Ml}&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\ \frac{1}{M}f_1&\frac{1}{M}f_2&\frac{1}{M}f_3&\frac{1}{M}f_4\\0&0&0&0\\-\frac{1}{Ml}f_1&-\frac{1}{Ml}f_2&-\frac{1}{Ml}f_3&-\frac{1}{Ml}f_4\end{pmatrix}\right\}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \begin{pmatrix}0&1&0&0\\ \frac{1}{M}f_1&\frac{1}{M}f_2&\frac{1}{M}f_3-\frac{1}{M}gm&\frac{1}{M}f_4\\0&0&0&1\\-\frac{1}{Ml}f_1&-\frac{1}{Ml}f_2&\frac{1}{M}\frac{g}{l}(M+m)-\frac{1}{Ml}f_3&-\frac{1}{Ml}f_4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$

$$\mathbf{y} = \begin{pmatrix}1&0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$

Therefore

$$\det\begin{pmatrix}-\lambda&1&0&0\\ \frac{1}{M}f_1&\frac{1}{M}f_2-\lambda&\frac{1}{M}f_3-\frac{1}{M}gm&\frac{1}{M}f_4\\0&0&-\lambda&1\\-\frac{1}{Ml}f_1&-\frac{1}{Ml}f_2&\frac{1}{M}\frac{g}{l}(M+m)-\frac{1}{Ml}f_3&-\frac{1}{Ml}f_4-\lambda\end{pmatrix} = 0$$

$$\frac{1}{Ml}\left(gf_1+\lambda^2 f_3+\lambda^3 f_4+g\lambda f_2-Mg\lambda^2+Ml\lambda^4-gm\lambda^2-l\lambda^2 f_1-l\lambda^3 f_2\right) = 0$$

Hence

$$\lambda^4+\lambda^3\frac{(f_4-l\ f_2)}{M\ l}+\lambda^2\frac{(f_3-M\ g-g\ m-l\ f_1)}{M\ l}+\lambda\frac{g\ f_2}{M\ l}+\frac{g\ f_1}{M\ l}=0 \tag{8}$$

We now need to determine f_1, f_2, f_3 and f_4. Assume we require that the closed loop poles be located at

$$\lambda = \{-1, -2, -1+i, -1-i\}$$

Hence, the characteristic polynomial is

$$\begin{aligned}\Delta(\lambda) &= (\lambda+1)(\lambda+2)(\lambda+1-i)(\lambda+1+i)\\ &= \lambda^4+5\lambda^3+10\lambda^2+10\lambda+4\end{aligned} \tag{9}$$

Compare Eqs (8,9) we obtain the following

$$\begin{aligned}\frac{(f_4-l\ f_2)}{M\ l} &= 5\\ \frac{(f_3-M\ g-g\ m-l\ f_1)}{M\ l} &= 10\\ \frac{g\ f_2}{M\ l} &= 10\\ \frac{g\ f_1}{M\ l} &= 4\end{aligned}$$

Or

$$\begin{aligned}f_4-l\ f_2 &= 5Ml\\ f_3-l\ f_1 &= 10Ml+g(M+m)\\ f_2 &= 10\frac{Ml}{g}\\ f_1 &= 4\frac{Ml}{g}\end{aligned}$$

Hence

$$\begin{aligned}f_4 &= 5Ml\left(1+\frac{2l}{g}\right)\\ f_3 &= 2Ml\left(5+\frac{2l}{g}\right)+g(M+m)\\ f_2 &= 10\frac{Ml}{g}\\ f_1 &= 4\frac{Ml}{g}\end{aligned}$$

Therefore, given M, m, g, l we can find f_1, f_2, f_3, f_4 which will generate force $u(t)$ which will keep the poles of the closed loop system in the LHS of the complex plane, and keep the inverted pendulum stable. For example, using $M = 1kg, m = 0.1kg, l = 1, g = 10m/s^2$ gives

$$\begin{aligned} f_4 &= 5\left(1+\frac{2}{10}\right) = 6 \\ f_3 &= 2\left(5+\frac{2}{10}\right) + 10(1.1) = 21.4 \\ f_2 &= 10\frac{1}{10} = 1 \\ f_1 &= \frac{4}{10} = 0.4 \end{aligned}$$

4.2.3 Comparing solution with and without stabilizing control law

We will now generate the solution $x(t), \theta(t)$ for some initial conditions and plot these solutions against time. In the first case, we assume $u(t)$ is zero. Hence we will observe that the system is unstable, i.e. $\theta(t)$ will grow away from the marginally stable position which is $\theta = 0^0$ and will not return back. Next, we will introduce $u(t)$ as determined in the previous section, and observe the new solution to see that it remains near or at the $\theta = 0^0$ position.

First, we need to decide on some initial conditions. These must be such that $\theta(0)$ close to zero and for $x(0)$ we can use *zero*. Hence, let

$$\begin{aligned} \theta(0) &= \theta_0 \\ \dot{\theta}(0) &= \dot{\theta}_0 \\ x(0) &= 0 \\ \dot{x}(0) &= \dot{x}_0 \end{aligned}$$

To determine $\mathbf{y}$, which is the solution of the system, we first must solve equation (7) and (8) for the above IC.

The solution to (7) is given by solution to

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{10}$$

Which is

$$\mathbf{x}(t) = \mathbf{x}(0)\, e^{At}$$

Where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}$$

And

$$\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{x}_0 \\ \theta_0 \\ \dot{\theta}_0 \end{pmatrix}$$

Taking Laplace transform of (10) results in

$$\begin{aligned} sX(s) - \mathbf{x}(0) &= AX(s) \\ X(s) &= (sI - A)^{-1}\,\mathbf{x}(0) \end{aligned}$$

Hence

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]\mathbf{x}(0)$$

Therefore, the solution to

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-gm}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g(M+m)}{Ml} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Is

$$\mathbf{x}(t)=\mathcal{L}^{-1}\left[\left(\begin{pmatrix}s&0&0&0\\0&s&0&0\\0&0&s&0\\0&0&0&s\end{pmatrix}-\begin{pmatrix}0&1&0&0\\0&0&\frac{-gm}{M}&0\\0&0&0&1\\0&0&\frac{g(M+m)}{Ml}&0\end{pmatrix}\right)^{-1}\right]\begin{pmatrix}0\\\dot{x}_0\\\theta_0\\\dot{\theta}_0\end{pmatrix}$$

$$=\mathcal{L}^{-1}\left[\begin{pmatrix}s&-1&0&0\\0&s&\frac{1}{M}gm&0\\0&0&s&-1\\0&0&-\frac{1}{M}\frac{g}{l}(M+m)&s\end{pmatrix}^{-1}\right]\begin{pmatrix}0\\\dot{x}_0\\\theta_0\\\dot{\theta}_0\end{pmatrix}$$

$$=\mathcal{L}^{-1}\left[\begin{pmatrix}\frac{1}{s}&\frac{1}{s^2}&gl\frac{m}{Mgs-Mls^3+gms}&gl\frac{m}{Mgs^2-Mls^4+gms^2}\\0&\frac{1}{s}&gl\frac{m}{-Mls^2+Mg+gm}&gl\frac{m}{Mgs-Mls^3+gms}\\0&0&-Ml\frac{s}{-Mls^2+Mg+gm}&-M\frac{l}{-Mls^2+Mg+gm}\\0&0&-\frac{Mg+gm}{-Mls^2+Mg+gm}&-Ml\frac{s}{-Mls^2+Mg+gm}\end{pmatrix}\right]\begin{pmatrix}0\\\dot{x}_0\\\theta_0\\\dot{\theta}_0\end{pmatrix}$$

$$=\begin{pmatrix}1&t&glm\left(\frac{1}{Mg+gm}-\frac{\cosh t\sqrt{\frac{1}{Ml}(Mg+gm)}}{Mg+gm}\right)&glm\left(\frac{t}{Mg+gm}-\frac{\sinh t\sqrt{\frac{1}{Ml}(Mg+gm)}}{(Mg+gm)\sqrt{\frac{1}{Ml}(Mg+gm)}}\right)\\0&1&-\frac{1}{M}gm\frac{\sinh t\sqrt{\frac{1}{Ml}(Mg+gm)}}{\sqrt{\frac{1}{Ml}(Mg+gm)}}&glm\left(\frac{1}{Mg+gm}-\frac{\cosh t\sqrt{\frac{1}{Ml}(Mg+gm)}}{Mg+gm}\right)\\0&0&\cosh t\sqrt{\frac{1}{Ml}(Mg+gm)}&\frac{\sinh t\sqrt{\frac{1}{Ml}(Mg+gm)}}{\sqrt{\frac{1}{Ml}(Mg+gm)}}\\0&0&\frac{1}{Ml}\left(\sinh t\sqrt{\frac{1}{Ml}(Mg+gm)}\right)\frac{Mg+gm}{\sqrt{\frac{1}{Ml}(Mg+gm)}}&\cosh t\sqrt{\frac{1}{Ml}(Mg+gm)}\end{pmatrix}\begin{pmatrix}0\\\dot{x}_0\\\theta_0\\\dot{\theta}_0\end{pmatrix}$$

$$=\begin{pmatrix}t\dot{x}_0+glm\theta_0\left(\frac{1}{Mg+gm}-\frac{\cosh t\sqrt{\frac{1}{Ml}(Mg+gm)}}{Mg+gm}\right)+\dot{\theta}_0glm\left(\frac{t}{Mg+gm}-\frac{\sinh t\sqrt{\frac{1}{Ml}(Mg+gm)}}{(Mg+gm)\sqrt{\frac{1}{Ml}(Mg+gm)}}\right)\\\dot{x}_0+\dot{\theta}_0glm\left(\frac{1}{Mg+gm}-\frac{\cosh t\sqrt{\frac{1}{Ml}(Mg+gm)}}{Mg+gm}\right)-\frac{1}{M}gm\theta_0\frac{\sinh t\sqrt{\frac{1}{Ml}(Mg+gm)}}{\sqrt{\frac{1}{Ml}(Mg+gm)}}\\\theta_0\cosh t\sqrt{\frac{1}{Ml}(Mg+gm)}+\dot{\theta}_0\frac{\sinh t\sqrt{\frac{1}{Ml}(Mg+gm)}}{\sqrt{\frac{1}{Ml}(Mg+gm)}}\\\dot{\theta}_0\cosh t\sqrt{\frac{1}{Ml}(Mg+gm)}+\frac{1}{Ml}\theta_0\left(\sinh t\sqrt{\frac{1}{Ml}(Mg+gm)}\right)\frac{Mg+gm}{\sqrt{\frac{1}{Ml}(Mg+gm)}}\end{pmatrix}$$

Therefore, the solution to $x_3(t)$ which is $\theta(t)$ is given by

$$\theta(t)=\theta_0\cosh t\sqrt{\frac{1}{Ml}(Mg+gm)}+\dot{\theta}_0\frac{\sinh t\sqrt{\frac{1}{Ml}(Mg+gm)}}{\sqrt{\frac{1}{Ml}(Mg+gm)}}$$

Let $\theta_0=\frac{\pi}{10},\dot{\theta}_0=1rad/\text{sec}$, we plot the above solution for $t=0$ up to 10 seconds

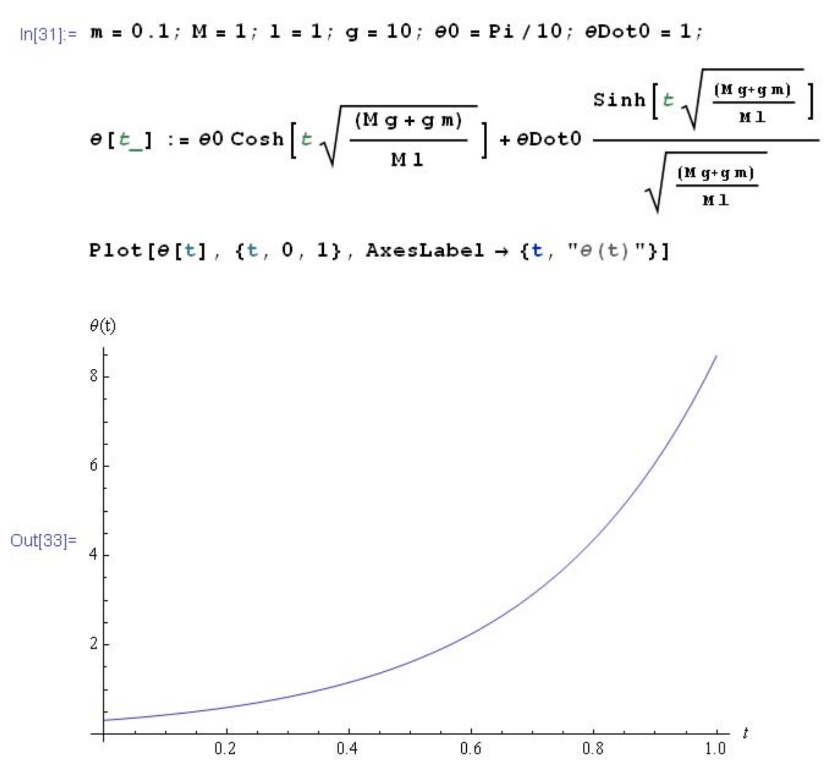


Figure 4.5: unstable θ

We plot the solution to (8), which is the state space equation with the stabilizing control law derived above,which is the following

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0\\ \frac{1}{M}f_1 & \frac{1}{M}f_2 & \frac{1}{M}f_3 - \frac{1}{M}gm & \frac{1}{M}f_4\\ 0 & 0 & 0 & 1\\ -\frac{1}{Ml}f_1 & -\frac{1}{Ml}f_2 & \frac{1}{M}\frac{g}{l}(M+m) - \frac{1}{Ml}f_3 & -\frac{1}{Ml}f_4 \end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}$$

Where

$$\begin{aligned} f_4 &= 5Ml\left(1+\frac{2l}{g}\right)\\ f_3 &= 2Ml\left(5+\frac{2l}{g}\right)+g(M+m)\\ f_2 &= 10\frac{Ml}{g}\\ f_1 &= 4\frac{Ml}{g} \end{aligned}$$

Where the above values determined to cause the closed loop poles to be located at

$$\{-1,-2,-1+i,-1-i\}$$

Hence

$$\begin{aligned} \mathbf{x}(t) &= \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]\mathbf{x}(0)\\ &= \mathcal{L}^{-1}\left(\begin{pmatrix} s & 0 & 0 & 0\\ 0 & s & 0 & 0\\ 0 & 0 & s & 0\\ 0 & 0 & 0 & s\end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0\\ \frac{1}{M}f_1 & \frac{1}{M}f_2 & \frac{1}{M}f_3 - \frac{1}{M}gm & \frac{1}{M}f_4\\ 0 & 0 & 0 & 1\\ -\frac{1}{Ml}f_1 & -\frac{1}{Ml}f_2 & \frac{1}{M}\frac{g}{l}(M+m) - \frac{1}{Ml}f_3 & -\frac{1}{Ml}f_4 \end{pmatrix}\right)^{-1}\begin{pmatrix}0\\ \dot{x}_0\\ \theta_0\\ \dot{\theta}_0\end{pmatrix}\\ &= \mathcal{L}^{-1}\begin{pmatrix} s & -1 & 0 & 0\\ -\frac{1}{M}f_1 & s-\frac{1}{M}f_2 & \frac{1}{M}gm - \frac{1}{M}f_3 & -\frac{1}{M}f_4\\ 0 & 0 & s & -1\\ \frac{1}{Ml}f_1 & \frac{1}{Ml}f_2 & \frac{1}{Ml}f_3 - \frac{1}{M}\frac{g}{l}(M+m) & s+\frac{1}{Ml}f_4 \end{pmatrix}^{-1}\begin{pmatrix}0\\ \dot{x}_0\\ \theta_0\\ \dot{\theta}_0\end{pmatrix} \end{aligned}$$

To make the computation easier, we now substitute numerical values for all the above parameters, which are $M = 1kg, m = 0.1kg, l = 1, g = 10m/s^2$, and we obtain

$$\mathbf{x}(t) = \mathcal{L}^{-1}\begin{pmatrix} s & -1 & 0 & 0\\ -f_1 & s-f_2 & 1-f_3 & -f_4\\ 0 & 0 & s & -1\\ f_1 & f_2 & f_3 - 10(1.1) & s+f_4 \end{pmatrix}^{-1}\begin{pmatrix}0\\ \dot{x}_0\\ \theta_0\\ \dot{\theta}_0\end{pmatrix}$$

And

$$\begin{aligned} f_4 &= 5\left(1+\frac{2}{10}\right) = 6\\ f_3 &= 2\left(5+\frac{2}{10}\right)+10(1.1) = 21.4\\ f_2 &= 10\frac{1}{10} = 1\\ f_1 &= \frac{4}{10} = 0.4 \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{x}(t) &= \mathcal{L}^{-1}\begin{pmatrix} s & -1 & 0 & 0\\ -0.4 & s-1 & 1-21.4 & -6\\ 0 & 0 & s & -1\\ 0.4 & 1 & 21.4-10(1.1) & s+6 \end{pmatrix}^{-1}\begin{pmatrix}0\\ \dot{x}_0\\ \theta_0\\ \dot{\theta}_0\end{pmatrix}\\ &= \mathcal{L}^{-1}\begin{pmatrix} s & -1 & 0 & 0\\ -0.4 & s-1 & -20.4 & -6\\ 0 & 0 & s & -1\\ 0.4 & 1 & 21.4-10(1.1) & s+6 \end{pmatrix}^{-1}\begin{pmatrix}0\\ \dot{x}_0\\ \theta_0\\ \dot{\theta}_0\end{pmatrix} \end{aligned}$$

Using $\theta_0 = \frac{\pi}{10}, \dot{\theta}_0 = 1rad/\text{sec}, \dot{x}_0 = 1m/\text{sec}$, and solving for $\theta(t)$ gives

$$\mathbf{x}(t) = \mathcal{L}^{-1}\begin{pmatrix} s & -1 & 0 & 0\\ -0.4 & s-1 & -20.4 & -6\\ 0 & 0 & s & -1\\ 0.4 & 1 & 21.4-10(1.1) & s+6 \end{pmatrix}^{-1}\begin{pmatrix}0\\ 1\\ \frac{\pi}{10}\\ 1\end{pmatrix}$$

Using CAS system to matrix inverse the above and obtain the inverse Laplace transform, and pick the $\theta(t)$ solution and plot it, we observe that now the system becomes stable as expected.

```
In[95]:= A = {{s, -1, 0, 0}, {-.4, s-1, 1-21.4, -6}, {0, 0, s, -1},
           {0.4, 1, 21.4-10 (1.1), s+6}}
         inv = Inverse[A]
         Chop[Simplify[InverseLaplaceTransform[%, s, t]]];
         MatrixForm[sol = %.{0, 1, Pi/10, 1}];
         FullSimplify[sol[[3]]];
         ExpToTrig[%]
         Plot[%, {t, 0, 10}, AxesLabel → {t, "θ(t)"}]
```

Out[95]=

$$\begin{pmatrix} s & -1 & 0 & 0 \\ -0.4 & s-1 & -20.4 & -6 \\ 0 & 0 & s & -1 \\ 0.4 & 1 & 10.4 & s+6 \end{pmatrix}$$

Out[96]=

$$\begin{pmatrix} \frac{s^3+5s^2+10.4s+10.}{s^4+5s^3+10.s^2+10.s+4.} & \frac{s^2+6s+10.4}{s^4+5s^3+10.s^2+10.s+4.} & \frac{20.4s+60.}{s^4+5s^3+10.s^2+10.s+4.} & \frac{6s+20.4}{s^4+5s^3+10.s^2+10.s+4.} \\ \frac{0.4s^2+0.s-4.}{s^4+5s^3+10.s^2+10.s+4.} & \frac{s^3+6s^2+10.4s}{s^4+5s^3+10.s^2+10.s+4.} & \frac{20.4s^2+60.s}{s^4+5s^3+10.s^2+10.s+4.} & \frac{6s^2+20.4s}{s^4+5s^3+10.s^2+10.s+4.} \\ \frac{0.-0.4s}{s^4+5s^3+10.s^2+10.s+4.} & \frac{-s-0.4}{s^4+5s^3+10.s^2+10.s+4.} & \frac{s^3+5s^2-0.4s+0.}{s^4+5s^3+10.s^2+10.s+4.} & \frac{s^2-s-0.4}{s^4+5s^3+10.s^2+10.s+4.} \\ \frac{0.s-0.4s^2}{s^4+5s^3+10.s^2+10.s+4.} & \frac{-s^2-0.4s}{s^4+5s^3+10.s^2+10.s+4.} & \frac{-10.4s^2-10.s-4.}{s^4+5s^3+10.s^2+10.s+4.} & \frac{s^3-s^2-0.4s}{s^4+5s^3+10.s^2+10.s+4.} \end{pmatrix}$$

Out[100]=

$$-(2.14823-1.17124\,i)\sin((1.+1.\,i)t)+(1.17124+2.14823\,i)\cos((1.+1.\,i)t)-(1.17124-2.14823\,i)\sinh((1.+1.\,i)t)+(1.17124-2.14823\,i)\cosh((1.+1.\,i)t)-3.5823\sinh(1.t)+5.61062\sinh(2.t)+3.5823\cosh(1.t)-5.61062\cosh(2.t)$$

Out[101]=

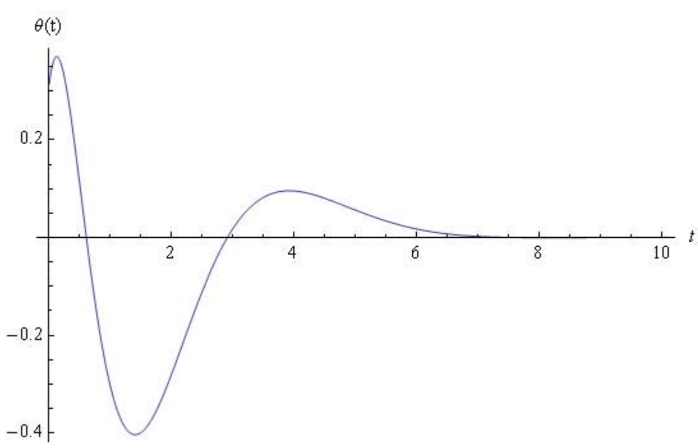


Figure 4.6: Using CAS system to find inverse Laplace

4.2.4 Conclusion

We observe from the above plots and the plots shown in the computation section that with the control law derived to force the poles of the closed loop to be stable, the inverted pendulum has been stabilized.

The final angle θ that the inverted pendulum makes with the vertical does go to zero.

From the plot of the position $x(t)$, we see that the cart moves to the right and away from the $x=0$ position, then it return back to $x=0$ position, while in the same time, the pendulum swings back and forth about the $\theta=0$ position before it finally settles down at the stable position.

This shows the using pole placement resulted in an effective control law which stabilized the system. Small angle approximation was used and the initial angle used was also assumed to be small.

4.3 Eigen modal analysis

In[271]:=
```
mMat = {{m, 0}, {0, m}}
```

Out[271]= $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$

In[272]:=
```
kmat = {{2 k, -k}, {-k, 2 k}}
```

Out[272]= $\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$

In[273]:=
```
MatrixForm[sys = mMat.{-A1 w^2, -A2 w^2} + kmat.{A1, A2}]
```

Out[273]//MatrixForm=

$$\begin{pmatrix} 2\,\text{A1}\,k - \text{A1}\,m\,w^2 - \text{A2}\,k \\ -\text{A1}\,k + 2\,\text{A2}\,k - \text{A2}\,m\,w^2 \end{pmatrix}$$

In[274]:=
```
syseq = sys[[1]] == 0
```

Out[274]= $2\,\text{A1}\,k - \text{A1}\,m\,w^2 - \text{A2}\,k = 0$

In[275]:=
```
eq = CoefficientArrays[sys, {A1, A2}] // Normal;
eq = eq[[2]]
```

Out[276]= $\begin{pmatrix} 2k - m\,w^2 & -k \\ -k & 2k - m\,w^2 \end{pmatrix}$

In[277]:=
```
Det[eq]
```

Out[277]= $3k^2 - 4\,k\,m\,w^2 + m^2\,w^4$

In[278]:=
```
sol = Solve[% == 0, w]
```

Out[278]= $\left\{\left\{w \to -\frac{\sqrt{k}}{\sqrt{m}}\right\}, \left\{w \to \frac{\sqrt{k}}{\sqrt{m}}\right\}, \left\{w \to -\frac{\sqrt{3}\,\sqrt{k}}{\sqrt{m}}\right\}, \left\{w \to \frac{\sqrt{3}\,\sqrt{k}}{\sqrt{m}}\right\}\right\}$

In[279]:=
```
sol = {sol[[1]], sol[[3]]}
```

Out[279]= $\left\{\left\{w \to -\frac{\sqrt{k}}{\sqrt{m}}\right\}, \left\{w \to -\frac{\sqrt{3}\,\sqrt{k}}{\sqrt{m}}\right\}\right\}$

In[280]:=
```
updatedSys = syseq /. sol[[1]]
```

Out[280]= $\text{A1}\,k - \text{A2}\,k = 0$

In[281]:=
```
First[Solve[updatedSys, A1]]
```

Out[281]= $\{\text{A1} \to \text{A2}\}$

In[282]:=
```
r1 = A2 / (A1 /. %)
```

Out[282]= 1

In[283]:=
```
updatedSys = syseq /. sol[[2]]
```

Out[283]= $-\text{A1}\,k - \text{A2}\,k = 0$

In[284]:=
```
First[Solve[updatedSys, A1]]
```

Out[284]= $\{\text{A1} \to -\text{A2}\}$

In[285]:=
```
r2 = A2 / (A1 /. %)
```

Out[285]= -1

In[286]:=
```
x1 = A11 Cos[(w /. sol[[1]]) t + θ1] + A12 Cos[(w /. sol[[2]]) t + θ2]
```

Out[286]= $\text{A11}\cos\left(\frac{\sqrt{k}\ t}{\sqrt{m}} - \theta 1\right) + \text{A12}\cos\left(\frac{\sqrt{3}\ \sqrt{k}\ t}{\sqrt{m}} - \theta 2\right)$

In[287]:=
```
x2 = A21 Cos[(w /. sol[[1]]) t + θ1] + A22 Cos[(w /. sol[[2]]) t + θ2]
```

Out[287]= $\text{A21}\cos\left(\frac{\sqrt{k}\ t}{\sqrt{m}} - \theta 1\right) + \text{A22}\cos\left(\frac{\sqrt{3}\ \sqrt{k}\ t}{\sqrt{m}} - \theta 2\right)$

In[288]:=
```
x2 = x2 /. {A21 → r1 A11, A22 → r2 A12}
```

Out[288]= $\text{A11}\cos\left(\frac{\sqrt{k}\ t}{\sqrt{m}} - \theta 1\right) - \text{A12}\cos\left(\frac{\sqrt{3}\ \sqrt{k}\ t}{\sqrt{m}} - \theta 2\right)$

In[289]:=
```
icx1 = {1, 0}
icx2 = {1, 0}
```

Out[289]= $\{1, 0\}$

Out[290]= $\{1, 0\}$

In[291]:=
```
eq1 = icx1[[1]] == x1 /. t → 0
```

Out[291]= $1 = \text{A11}\cos(\theta 1) + \text{A12}\cos(\theta 2)$

In[292]:=
```
eq2 = icx1[[2]] == D[x1, t] /. t → 0
```

Out[292]= $0 = \frac{\text{A11}\sqrt{k}\ \sin(\theta 1)}{\sqrt{m}} + \frac{\sqrt{3}\ \text{A12}\sqrt{k}\ \sin(\theta 2)}{\sqrt{m}}$

In[293]:=
```
eq3 = icx2[[1]] == x2 /. t → 0
```

Out[293]= $1 = \text{A11}\cos(\theta 1) - \text{A12}\cos(\theta 2)$

In[294]:=
```
eq4 = icx2[[2]] == D[x2, t] /. t → 0
```

Out[294]= $0 = \frac{\text{A11}\sqrt{k}\ \sin(\theta 1)}{\sqrt{m}} - \frac{\sqrt{3}\ \text{A12}\sqrt{k}\ \sin(\theta 2)}{\sqrt{m}}$

In[300]:=
```
MatrixForm[{eq1, eq2, eq3, eq4} /. {k → 1, m → 1}]
```

Out[300]//MatrixForm=

$$\begin{pmatrix} 1 = \text{A11}\cos(\theta 1) + \text{A12}\cos(\theta 2) \\ 0 = \text{A11}\sin(\theta 1) + \sqrt{3}\ \text{A12}\sin(\theta 2) \\ 1 = \text{A11}\cos(\theta 1) - \text{A12}\cos(\theta 2) \\ 0 = \text{A11}\sin(\theta 1) - \sqrt{3}\ \text{A12}\sin(\theta 2) \end{pmatrix}$$

In[301]:=
```
Solve[{eq1, eq2, eq3, eq4} /. {k → 1, m → 1}, {A11, A12, θ1, θ2}]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[301]= $\{\{A11 \to -1,\ A12 \to 0,\ \theta 1 \to -\pi\},\ \{A11 \to -1,\ A12 \to 0,\ \theta 1 \to \pi\},\ \{A11 \to 1,\ A12 \to 0,\ \theta 1 \to 0\}\}$

notebook

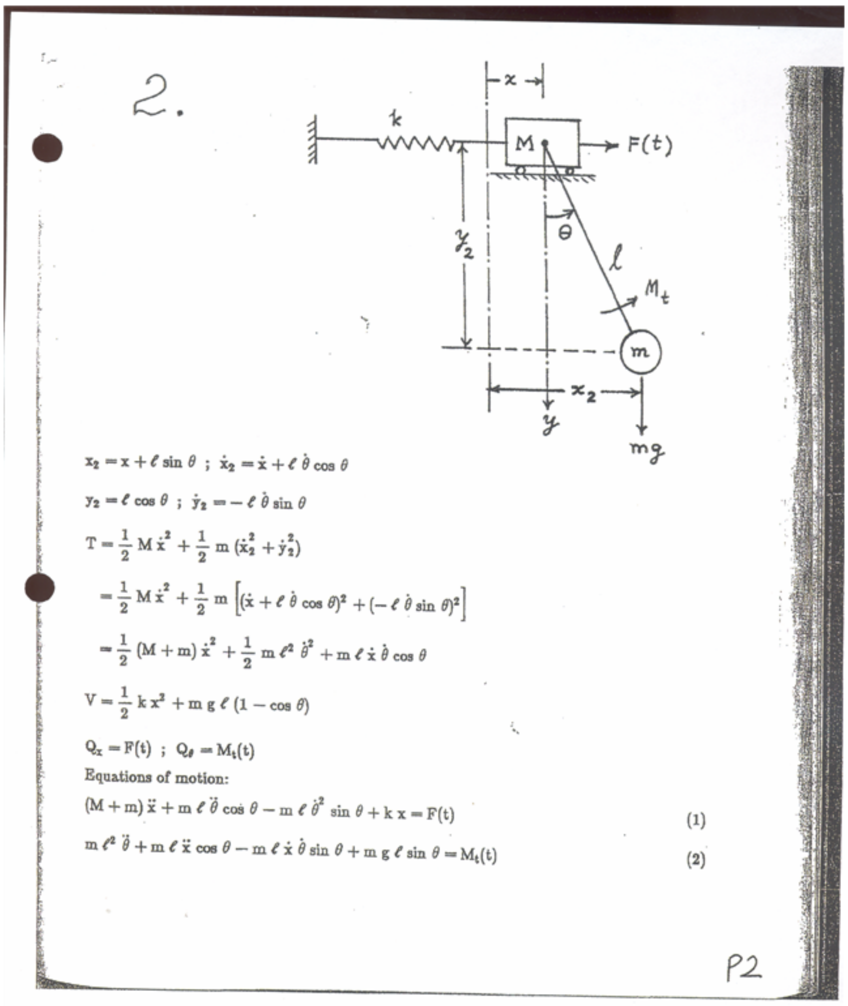
Chapter 5

some notes

Local contents

5.1 possible error in key

possible error in key solution:


2.

$x_2 = x + \ell \sin\theta \; ; \; \dot{x}_2 = \dot{x} + \ell\dot{\theta}\cos\theta$

$y_2 = \ell\cos\theta \; ; \; \dot{y}_2 = -\ell\dot{\theta}\sin\theta$

$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_2^2 + \dot{y}_2^2)$

$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[(\dot{x} + \ell\dot{\theta}\cos\theta)^2 + (-\ell\dot{\theta}\sin\theta)^2 \right]$

$= \frac{1}{2} (M+m)\dot{x}^2 + \frac{1}{2} m \ell^2 \dot{\theta}^2 + m\ell\dot{x}\dot{\theta}\cos\theta$

$V = \frac{1}{2} k x^2 + m g \ell (1-\cos\theta)$

$Q_x = F(t) \; ; \; Q_\theta = M_t(t)$

Equations of motion:

$$(M+m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta + kx = F(t) \quad (1)$$

$$m\ell^2\ddot{\theta} + m\ell\ddot{x}\cos\theta - m\ell\dot{x}\dot{\theta}\sin\theta + mg\ell\sin\theta = M_t(t) \quad (2)$$

P2

$$T = \frac{1}{2}(M+m)\,\dot{x}^2 + \frac{1}{2} m L^2 \dot{\theta}^2 + m L \dot{x}\dot{\theta}\cos\theta$$
$$V = \frac{1}{2} k x^2 + m g l (1 - \cos\theta)$$

Now

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}(M+m)\,\dot{x}^2 + \frac{1}{2}mL^2\dot{\theta}^2 + mL\dot{x}\dot{\theta}\cos\theta - \left(\frac{1}{2}kx^2 + mgl(1-\cos\theta)\right) \\
&= \frac{1}{2}(M+m)\,\dot{x}^2 + \frac{1}{2}mL^2\dot{\theta}^2 + mL\dot{x}\dot{\theta}\cos\theta - \frac{1}{2}kx^2 - mgl(1-\cos\theta)
\end{aligned}
$$

EQM for θ show is WRONG. Proof:

$$
\begin{aligned}
\frac{\partial L}{\partial\dot{\theta}} &= mL^2\dot{\theta} + mL\dot{x}\cos\theta \\
\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} &= mL^2\ddot{\theta} + mL\ddot{x}\cos\theta - mL\dot{x}\dot{\theta}\sin\theta
\end{aligned}
$$

and

$$
\frac{\partial L}{\partial\theta} = -mL\dot{x}\dot{\theta}\sin\theta - mgl(\sin\theta)
$$

Hence, EQM is

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} &= M_t \\
mL^2\ddot{\theta} + mL\ddot{x}\cos\theta - mL\dot{x}\dot{\theta}\sin\theta - \left(-mL\dot{x}\dot{\theta}\sin\theta - mgl(\sin\theta)\right) &= M_t \\
mL^2\ddot{\theta} + mL\ddot{x}\cos\theta - mL\dot{x}\dot{\theta}\sin\theta + mL\dot{x}\dot{\theta}\sin\theta + mgl(\sin\theta) &= M_t \\
mL^2\ddot{\theta} + mL\ddot{x}\cos\theta + mgl(\sin\theta) &= M_t
\end{aligned}
$$

Which is NOT the same as shown in the key solution

5.2 note on solving wave equation

note on solving wave equation.

step 1 $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ —— (1)

let $u = X(x)\,T(t)$. $\Rightarrow \dfrac{\partial^2 u}{\partial t^2} = X(x)\dfrac{d^2T}{dt^2}$, $\dfrac{\partial^2 u}{\partial x^2} = T\dfrac{d^2X}{dx^2}$

so (1) becomes $X(x)\dfrac{d^2T}{dt^2} = c^2 T \dfrac{d^2X}{dx^2}$

or $\dfrac{1}{c^2 T}\dfrac{d^2T}{dt^2} = \dfrac{1}{X(x)}\dfrac{d^2X}{dx^2}$

hence each must equal a constant, say $-K$.
we can find by K must be > 0. hence let $K = \lambda^2$ for $\lambda > 0$.

so we obtain 2 ODE's

$$\frac{d^2X}{dx^2} + \lambda^2 X(x) = 0 \quad \text{—— (2)}$$

$$\frac{d^2T}{dt^2} + c^2\lambda^2 T(t) = 0 \quad \text{—— (3)}$$

step 2
from (2) $\Rightarrow X(x) = A\cos\lambda x + B\sin\lambda x$
from (3) $\Rightarrow T(t) = D\cos c\lambda t + E\sin c\lambda t$

from B.C. $u(0,t)=0$, $\Rightarrow \boxed{A=0}$
from B.C. $u(L,t)=0 \Rightarrow B\sin\lambda L = 0 \Rightarrow \boxed{\lambda = \dfrac{n\pi}{L}}$, $n=1,2,\ldots$

step 3
$X_n(x) = B_n \sin\lambda_n x$
$T_n(t) = D_n\cos c\lambda_n t + E_n \sin c\lambda_n t$

step 4 apply superposition

$$u = \sum_{n=1}^{\infty} X_n T_n = \sum \left(D_n\cos c\lambda_n t + E_n\sin c\lambda_n t\right) B_n \sin\lambda_n x$$

$$u = \sum \left(\bar{D}_n\cos c\lambda_n t + \bar{E}_n\sin c\lambda_n t\right)\sin\lambda_n x$$

where B_n is absorbed into $\bar{D}_n$, $\bar{E}_n$.

now apply time initial condition

$u(x,0) = f(x) \Rightarrow f(x) = \sum \bar{D}_n \sin\lambda_n x \rightarrow$ Find $\bar{D}_n$

$\dfrac{\partial u}{\partial t}(x,0) = g(x) \Rightarrow g(x) = \sum c\lambda_n \bar{E}_n \sin\lambda_n x \rightarrow$ Find $c\lambda_n\bar{E}_n$

Done!

5.3 note on eigenvalue modal analysis

let $\theta_1 = A_1 \cos(\omega t + \phi)$

let $\theta_2 = A_2 \cos(\omega t + \phi)$

Note*

For eigenvalue Modal analysis

given $M\ddot{x} + Kx = 0$ —— (1)

(1) assume $x_1 = A_1 \cos(\omega t + \phi)$, $x_2 = A_2 \cos(\omega t + \phi)$

(2) Plug in (1) and obtain

$$[M]\begin{pmatrix} A_1\omega^2 \\ A_2\omega^2 \end{pmatrix} + [K]\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(3) Rewrite as

$$\underbrace{\Big(\qquad\qquad\Big)}_{sys}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{—— (2)}$$

(4) find characteristic equation by $Det(sys) = 0$

(5) solve for ω_1, ω_2 from 4.

(6) For each ω_1, ω_2, go to (2), plug in, and ~~obtain~~ using first equation (or second, it does not matter), and find $\left(\frac{A_2}{A_1}\right)^{(1)} = r_1$ and $\left(\frac{A_2}{A_1}\right)^{(2)} = r_2$

(7) $x_1 = A_1^{(1)} \cos(\omega_1 t + \phi_1) + A_1^{(2)} \cos(\omega_2 t + \phi_2)$

$x_2 = A_2^{(1)} \cos(\omega_1 t + \phi_1) + A_2^{(2)} \cos(\omega_2 t + \phi_2)$

(8) using $A_2^{(1)} = r_1 A_1^{(1)}$ and $A_2^{(2)} = r_2 A_1^{(2)}$, rewrite above as

$$x_1 = A_1^{(1)} \cos(\omega_1 t + \phi_1) + r_1 A_1^{(1)} \cos(\omega_2 t + \phi_2)$$
$$x_2 = A_2^{(1)} \cos(\omega_1 t + \phi_1) + r_2 A_1^{(2)} \cos(\omega_2 t + \phi_2)$$

2 equations, 4 unknowns, 4 Initial Conditions.